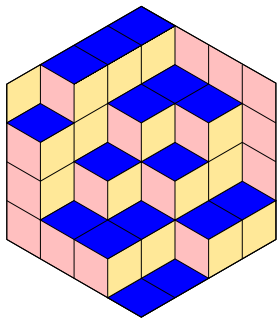

ALGEBRAIC COMBINATORICS

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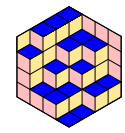

Combinatorics of the Delta conjecture at $q = -1$

Sylvie Corteel, Matthieu Josuat-Vergès & Anna Vanden Wyngaerd

Abstract In the context of the shuffle theorem, many classical integer sequences appear with a natural refinement by two statistics q and t: for example the Catalan and Schröder numbers. In particular, the bi-graded Hilbert series of diagonal harmonics is a q,t-analog of $(n+1)^{n-1}$ (and can be written in terms of symmetric functions via the nabla operator). The motivation for this work is the observation that at $q=-1$, this q,t-analog becomes a t-analog of Euler numbers, a famous integer sequence that counts alternating permutations. We prove this observation via a more general statement, that involves the Delta operator on symmetric functions (on one side), and new combinatorial statistics on permutations involving peaks and valleys (on the other side). An important tool are the schedule numbers of a parking function first introduced by Hicks; and expanded upon by Haglund and Sergel. Other empirical observations suggests that non negativity at $q=-1$ holds in far greater generality.

1. Introduction

In the early 2000s, Haglund, Haiman, Remmel, Loehr and Ulyanov stated the *shuffle conjecture* [14]: a combinatorial formula for the symmetric function ∇e_n in terms on *labeled Dyck paths*. The interest in the symmetric function ∇e_n (where ∇ is the Macdonald eigenoperator introduced in [1]) stems from it being the bi-graded Frobenius characteristic of the diagonal harmonic representation of the symmetric group [17]. More than a decade after its statement, Carlsson and Mellit proved the full shuffle conjecture, which thus became a theorem [3]. By then, many special cases were known: for example $\langle \nabla e_n, e_n \rangle$ gives the famous *q,t-Catalan numbers* [11] and $\langle \nabla e_n, h_d e_{n-d} \rangle$ the *q,t-Schröder numbers* [12]. A consequence of the full shuffle theorem is that the bi-graded Hilbert series $\langle \nabla e_n, h_1^n \rangle$ gives a q,t-analogue of $(n+1)^{n-1}$. It can be described combinatorially as the generating function of length n parking functions with respect to area and number of diagonal inversions.

The famous *Euler numbers* $(E_n)_{n\geqslant 0}$ can be defined by their generating series:

$$\sum_{n\geqslant 0} E_n \frac{z^n}{n!} = \tan(z) + \sec(z). \tag{1}$$

They answer various enumeration problems, the most famous one being that E_n is the number of alternating permutations in $\mathfrak{S}_n$, that is, those σ with $\sigma_1 > \sigma_2 < \sigma_3 > \cdots$.

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Keywords. Delta conjecture, Macdonald polynomials, permutations, schedule numbers.

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Furthermore, they appear in Arnold's theory of singularity, and in number theory via their relation with Bernoulli numbers. See [25, A000111] and the many references therein. Seeing them in the context of Macdonald q,t-combinatorics is new, and is the motivation for this project. We will show that specializing $q=-1$ in the q,t-analog of $(n+1)^{n-1}$ (the bi-graded Hilbert series of diagonal harmonics) gives:

$$\langle \nabla e_n, h_1^n \rangle|_{q=-1} = t^{\lfloor n^2/4 \rfloor} E_n(t) \tag{2}$$

where $E_n(t)$ is a t-analogue of E_n appearing in [18]. This specialization at $q=-1$ is a t-refinement of the identity

$$\sum_{P \in \mathsf{PF}_n} (-1)^{\mathsf{area}(P)} = E_n, \tag{3}$$

where PF_n are the parking functions of size n. The history of this identity can be found in [21, 24, 26]. For the definition of parking functions and their correspondence to standardly labeled Dyck paths, see [13, Chapter 5].

We will establish Equation (2) as a corollary of a more general statement involving a generalization of the shuffle theorem: the *valley version* of the *Delta conjecture* [15]. This is a combinatorial formula for the symmetric function $\Delta'_{e_{n-k-1}} e_n$. We will mainly use the following consequence of the Delta conjecture:

$$\langle \Delta'_{e_{n-k-1}} e_n, h_1^n \rangle = \sum_{P \in \mathsf{stLD}^{\bullet k}(n)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P, \tag{4}$$

where $\mathsf{stLD}(n)^{\bullet k}$ denotes the set of *standardly labeled Dyck paths* with k *decorated valleys* and dinv and area are combinatorial statistics on this set. See Section 2.1 for the precise combinatorial definitions. At $k=0$, we have $\Delta'_{e_{n-1}} e_n = \nabla e_n$, and the Delta conjecture reduces to the shuffle theorem.

Specializations of the shuffle theorem and Delta conjecture at $q=0$ or $q=1$ have been extensively studied (see [10] and [28], respectively). To our knowledge, apart from (3), nothing much was known about the specialization at $q=-1$.

We were inspired by the following remarkable symmetric function identity, which first appeared as the case $m=0$ in [6, Theorem 4.11]:

$$\sum_{k=0}^{n-1} (-q)^k \Delta'_{e_{n-k-1}} e_n = \nabla e_n|_{q=0}\,. \tag{5}$$

Taking the scalar product with h_1^n and evaluating at $q=-1$, we obtain

$$\sum_{k=0}^{n-1} \langle \Delta'_{e_{n-k-1}} e_n, h_1^n \rangle \Big|_{q=-1} = \langle \nabla e_n|_{q=0}\,, h_1^n \rangle = [n]_t!, \tag{6}$$

where the second equality is an easy consequence of the shuffle theorem.

Our main result is a combinatorial interpretation of the terms of this sum, conditional on (4).

Theorem 1.1. *For all $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n-1} \left(\sum_{P \in \mathsf{stLD}(n)^{\bullet k}} t^{\mathsf{area}(P)} (-1)^{\mathsf{dinv}(P)} \right) z^k = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{inv}_3(\sigma)} z^{\mathsf{monot}(\sigma)},$$

where inv_3 *is a new statistic on permutations generalizing Chebikin's notion of* alternating descents [4] *and* $\mathsf{monot}(\sigma)$ *is the number of double ascents or descents of σ (see Section 4 for the precise definitions).*

Thus, if the Delta conjecture is proven to be true, we will have the following symmetric function interpretation.

Corollary 1.2. *If Equation* (4) *is true, then for all $n \in \mathbb{N}$ we have:*

$$\sum_{k=0}^{n-1} \langle \Delta'_{e_{n-k-1}} e_n, h_1^n \rangle \Big|_{q=-1} z^k = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{inv}_3(\sigma)} z^{\mathsf{monot}(\sigma)}.$$

Notice that at $z = 1$ our theorem agrees with Equation (6).

The specialization at $z = 0$ of our theorem, and the fact that at $k = 0$ the Delta conjecture reduces to the shuffle theorem will imply our formula (2).

Our proof relies on the schedule formula decomposition of the combinatorial side of the valley Delta conjecture provided in [16]. We use this schedule framework to identify the valley decorated Dyck paths that do not cancel out when specializing to $q = -1$. We then provide a bijection between these paths and permutations. This map will be defined via specific generating trees of the objects and will send area to inv_3 and the number of decorations to monot. In this way the paths with no decorations ($k = 0$) get sent to permutations with no double ascents or descents, that is, alternating permutations.

2. The valley Delta conjecture

In this section, we give the definitions needed to state the valley Delta conjecture.

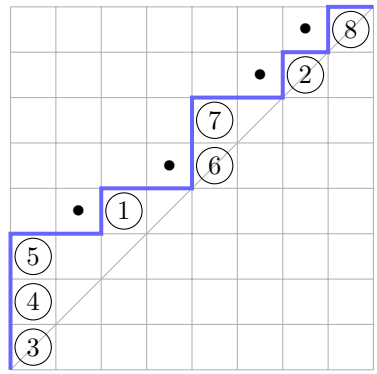

Figure 1. An element of $\mathsf{stLD}^{\bullet 4}(8)$.

2.1. Valley-decorated labeled Dyck paths.

Definition 2.1. *A* Dyck path *of size n is a lattice path going from $(0, 0)$ to (n, n) consisting of east or north unit steps, always staying weakly above the line $x = y$, called the* main diagonal. *The set of Dyck paths is denoted by $\mathsf{D}(n)$.*

Definition 2.2. *A* labeled Dyck path *is a pair (π, w), where $\pi \in \mathsf{D}(n)$ and w its* labeling*: a word of positive integers whose i-th letter labels the i-th vertical step of π, placed in the square to the right of this step, such that the labels appearing in the same column are increasing from bottom to top. A labeling is said to be* standard *if its labels are exactly $1, 2, \ldots, n$. The set of (standardly) labeled Dyck paths of size n is denoted by $\mathsf{LD}(n)$ (respectively, $\mathsf{stLD}(n)$).*

Standardly labeled Dyck paths are in bijection with parking functions.

Definition 2.3. *The* area word *of a Dyck path $\pi \in \mathsf{D}(n)$ is the word a of n non-negative integers whose i-th letter is the number of whole squares between the i-th vertical step of π and the main diagonal $x = y$. The* area *of a Dyck path is the sum of the letters of its area word and is denoted by $\mathsf{area}(\pi)$.*

Definition 2.4. *Given $P := (\pi, w) \in \mathsf{LD}(n)$ with area word a, the i-th vertical step of P is called a* contractible valley *if*

- *either $a_{i-1} > a_i$,*
- *or $a_{i-1} = a_i$ and $w_{i-1} < w_i$.*

In other words, the i-th vertical step is a contractible valley if it is preceded by a horizontal step and the following holds: after replacing the two steps ⌟ with ⌜ (and accordingly shifting the i-th label one cell to the left), we still get a valid labeled path where labels are increasing in each column.

Definition 2.5. *A* (valley) decorated labeled Dyck path *is a triple (π, w, dv) where $(\pi, w) \in \mathsf{LD}(n)$ and dv is some subset of the contractible valleys of (π, w). The elements of dv are called* decorations*, and we visualize them by drawing a $\bullet$ to the left of these contractible valleys. The set $\mathsf{LD}(n)^{\bullet k}$ denotes the decorated labeled Dyck paths with exactly k decorations.*

Definition 2.6. *Given $P := (\pi, w, dv) \in \mathsf{LD}(n)^{\bullet k}$ with area word a, a pair (i, j) of indices of vertical steps with $1 \leqslant i < j \leqslant n$ is said to be a*

- primary diagonal inversion *if $a_i = a_j, w_i < w_j$ and $i \notin dv$,*
- secondary diagonal inversion *if $a_i = a_j + 1, w_i > w_j$ and $i \notin dv$.*

The dinv *of P is defined to be the total number of primary and secondary diagonal inversions minus the number of decorated valleys and is denoted by $\mathsf{dinv}(P)$.*

Remark 2.7. We note that the dinv of a decorated labeled path is always a non-negative integer. Indeed, upon some reflection, one notices that each contractible valley forces the existence of at least one dinv pair.

Definition 2.8. *Given $P := (\pi, w, dv) \in \mathsf{LD}(n)^{\bullet k}$, the* area *of P is simply defined as the area of the underlying Dyck path, disregarding the labels and decorations: $\mathsf{area}(P) := \mathsf{area}(\pi)$.*

Example 2.9. See Figure 1 for an example of an element of $\mathsf{stLD}(8)^{\bullet 4}$. Its labeling is 34516728, its area word 01210100 and its area is 5. Its primary dinv pairs are

$$(1,5), (1,8), (2,6),$$

and its secondary dinv pairs are

$$(2,7), (3,4), (6,7).$$

Thus, since there are 4 decorated valleys, the dinv is equal to 2.

2.2. Symmetric functions. For all the undefined notations and the unproven identities, we refer to [8], where definitions, proofs and/or references can be found.

We denote by Λ the graded algebra of symmetric functions with coefficients in $\mathbb{Q}(q,t)$. The standard bases of the symmetric functions are the monomial $\{m_\lambda\}_\lambda$, complete $\{h_\lambda\}_\lambda$, elementary $\{e_\lambda\}_\lambda$, power $\{p_\lambda\}_\lambda$ and Schur $\{s_\lambda\}_\lambda$ bases. We denote by $\langle , \rangle$ the *Hall scalar product* on Λ, defined by declaring that the Schur functions form an orthonormal basis.

For a partition $\mu \vdash n$, we denote by

$$\tilde{H}_\mu := \tilde{H}_\mu[X] = \tilde{H}_\mu[X; q, t] = \sum_{\lambda \vdash n} \widetilde{K}_{\lambda\mu}(q,t) s_\lambda$$

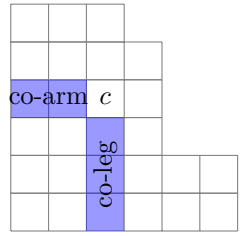

FIGURE 2. Co-arm and co-leg of a cell in a partition.

the *(modified) Macdonald polynomials*, where

$$\widetilde{K}_{\lambda\mu} \coloneqq \widetilde{K}_{\lambda\mu}(q,t) = K_{\lambda\mu}(q,1/t)t^{n(\mu)}$$

are the *(modified) Kostka coefficients* (see [13] for more details).

Macdonald polynomials form a basis of the ring of symmetric functions Λ. This is a modification of the basis introduced by Macdonald [23].

We identify the partition μ with its Ferrers diagram, i.e., with the collection of cells $\{(i,j) \mid 1 \leqslant i \leqslant \mu_j, 1 \leqslant j \leqslant \ell(\mu)\}$. For each cell $c \in \mu$ we define the *co-arm* and *co-leg* (denoted respectively as $a'_\mu(c), l'_\mu(c)$) as the number of cells in μ that are strictly to the left and below c in μ, respectively (see Figure 2). Define the following constant:

$$B_\mu \coloneqq B_\mu(q,t) = \sum_{c \in \mu} q^{a'_\mu(c)} t^{l'_\mu(c)}.$$

Let $f[g]$ denotes the *plethystic evaluation* of a symmetric function f in an expression g (see [13, Chapter 1 page 19]).

DEFINITION 2.10 ([1, 3.11]). *We define the linear operator $\nabla\colon \Lambda \to \Lambda$ on the eigenbasis of Macdonald polynomials as*

$$\nabla \widetilde{H}_\mu = e_{|\mu|}[B_\mu]\widetilde{H}_\mu = q^{n(\mu')}t^{n(\mu)}\widetilde{H}_\mu.$$

DEFINITION 2.11. *For $f \in \Lambda$, we define the linear operators $\Delta_f, \Delta'_f\colon \Lambda \to \Lambda$ on the eigenbasis of Macdonald polynomials as*

$$\Delta_f \widetilde{H}_\mu = f[B_\mu]\widetilde{H}_\mu, \qquad \Delta'_f \widetilde{H}_\mu = f[B_\mu - 1]\widetilde{H}_\mu.$$

2.3. THE STATEMENT. Now we have all the necessary definitions to state the valley Delta conjecture, which first appeared in [15].

CONJECTURE 2.12. *For all $n, k \in \mathbb{N}$,*

$$\Delta'_{e_{n-k-1}} e_n = \sum_{P \in \mathsf{LD}^{\bullet k}(n)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^D$$

where the sum is over the set of labeled Dyck paths *of size n with k decorations on* contractible valleys.

Taking the (Hall) scalar product with h_1^n (i.e., the Hilbert series), the Delta conjecture implies (4).

3. SCHEDULE FORMULA

In this section we discuss the schedule formula for the combinatorics of the valley Delta conjecture proved by Haglund and Sergel in [16]. Their formula is an extension of the first work on schedule numbers by Hicks in her thesis [19].

Definition 3.1. *The set of* decorated permutations $\mathfrak{S}_n^{\bullet k}$ *of* $[n] := \{1, \dots, n\}$ *is the set of permutations of* $[n]$ *where* k *of its* n *letters are* decorated, *represented as* $\dot{\sigma}_i$. *Set* $\mathfrak{S}_n^{\bullet} := \sqcup_{k\in[n]}\mathfrak{S}_n^{\bullet k}$. *For* $\sigma \in \mathfrak{S}_n^{\bullet}$, *we denote by* $\mathsf{dec}(\sigma)$ *its number of decorations.*

Definition 3.2. *Given* $P := (\pi, w, dv) \in \mathsf{stLD}(n)^{\bullet k}$ *with area word* a, *the* diagonal word *of a decorated labeled Dyck path is the decorated permutation of* n *obtained as follows. A label* w_i *of* P *is said to lie in the* j-th diagonal *if* $a_i = j$. *List all the labels* w_i *in the* 0*-th diagonal, in decreasing order, adding a decoration on the label if* $i \in dv$. *Then do the same for the* 1*-st diagonal,* 2*-nd diagonal, and so forth. Denote this diagonal word by* $\mathsf{dw}(P)$.

Example 3.3. The diagonal word of the path in Figure 1 is $\dot{8}\dot{6}\dot{3}\dot{2}7\dot{4}\dot{1}5$.

Definition 3.4. *For* $\sigma \in \mathfrak{S}_n$ *a permutation, its* major index *is defined to be the sum of the elements of the set* $\{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}$. *The* reverse major index *of a permutation is simply the major index of the reverse permutation* $\sigma^{\mathsf{rev}} := \sigma_n \cdots \sigma_1$. *For any marked permutation* $\sigma \in \mathfrak{S}_n^{\bullet}$, *denote by* $\mathsf{revmaj}(\sigma)$ *the reverse major index of its underlying permutation.*

The following is an easy consequence of the definitions.

Proposition 3.5. *For all* $P \in \mathsf{stLD}(n)^{\bullet k}$, *we have* $\mathsf{area}(P) = \mathsf{revmaj}(\mathsf{dw}(P))$.

Proof. Due to the condition of strictly increasing labels in the columns of labeled Dyck paths, each diagonal has at least one label which is bigger than some label of the previous diagonal. Thus, the labels in the j-th diagonal of P are exactly the numbers in the $(j+1)$-th decreasing run of $\mathsf{dw}(P)$, and they each contribute j units to the area. □

The following convention will greatly simplify definitions and proofs.

Convention 3.6. *Given* $\sigma \in \mathfrak{S}_n$, *we will implicitly consider that* σ *is preceded by a* 0*-th entry:* $\sigma_0 = 0$. *If* $\sigma \in \mathfrak{S}_n^{\bullet}$, *then* σ_0 *is never decorated.*

Definition 3.7 (Hicks [19]). *For* $\tau \in \mathfrak{S}_n^{\bullet}$ *define its* schedule numbers $\mathsf{sched}(\tau) = (s_i)_{1\leqslant i\leqslant n}$ *as follows. Take* $r_0, r_1, r_2, \dots$ *to be the decreasing runs of* $\tau_0 \cdots \tau_n$ *(we have* $r_0 = 0$*).*

- *If* τ_i *is undecorated and an element of* r_j, *let*
$$s_i = \#\{k \in r_j \mid k \text{ is undecorated and } k > \tau_i\} + \#\{k \in r_{j-1} \mid k \text{ is undecorated and } k < \tau_i\}.$$
- *If* τ_i *is decorated and an element of* r_j, *let*
$$s_i = \#\{k \in r_j \mid k \text{ is undecorated and } k < \tau_i\} + \#\{k \in r_{j+1} \mid k \text{ is undecorated and } k > \tau_i\}.$$

To reformulate this definition of schedule numbers, we introduce the following.

Definition 3.8. *Let* $\sigma \in \mathfrak{S}_n$. *A sequence of consecutive elements* $\sigma_i, \dots, \sigma_j$ *(with* $0 \leqslant i \leqslant j$*) in* σ *is a cyclic (decreasing) run if there exists an integer* k *such that* $\sigma_i + k \pmod{n+1}, \dots, \sigma_j + k \pmod{n+1}$ *is decreasing (where the modulo means we take the representative in* $\{0, \dots, n\}$*). Moreover, a cyclic run* $\sigma_i, \dots, \sigma_j$ *is left-maximal if either* $i = 0$ *or* $\sigma_{i-1}, \dots, \sigma_j$ *is not a cyclic run, and right-maximal if either* $j = n$ *or* $\sigma_i, \dots, \sigma_{j+1}$ *is not a cyclic run.*

Example 3.9. Let $\sigma = 0649751832$. Some right-maximal cyclic runs are 064, 6497, 7518. Some left-maximal cyclic runs are 1832, 6497.

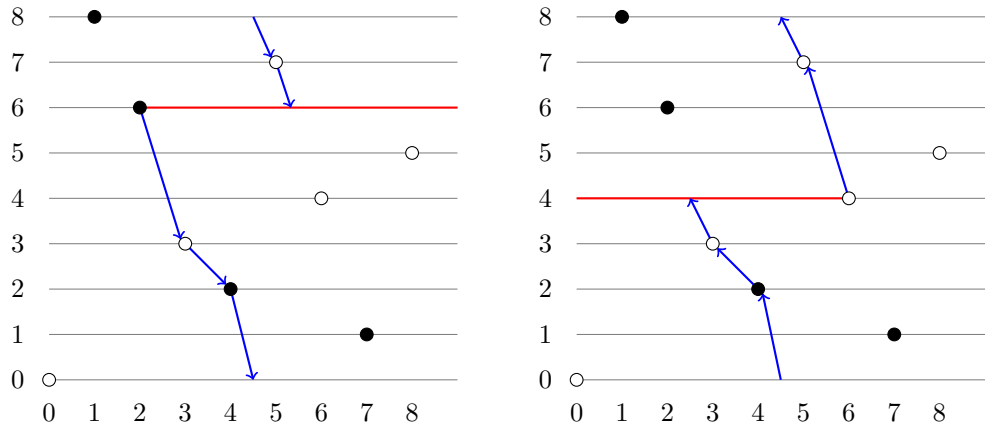

FIGURE 3. Illustration of cyclic runs.

Note that for each j, there is a unique left-maximal cyclic run $\sigma_i, \ldots, \sigma_j$ (obtained by choosing i to be minimal), and similarly for each i there is a unique right-maximal cyclic run $\sigma_i, \ldots, \sigma_j$ (obtained by choosing j to be maximal).

REMARK 3.10. The definition of schedules is rephrased as follows: if τ_i is undecorated (respectively, decorated), then s_i is the number of undecorated values (excluding τ_i) in the maximal decreasing cyclic run ending (respectively, starting) at τ_i.

EXAMPLE 3.11. If τ is the diagonal word of the path in Figure 1, then we have

$$\begin{array}{lccccccccc} \tau & 0 & \dot{8} & \dot{6} & 3 & \dot{2} & 7 & 4 & \dot{1} & 5 \\ s & & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1. \end{array}$$

In Figure 3, we visualize τ by placing dots at coordinates (i, τ_i): white dots for undecorated and black dots for decorated τ_i. Notice that if we view this picture as a cylinder, identifying the top and the bottom, the operation $\sigma_i \mapsto \sigma_i + k \pmod{n+1}$ can be seen as a rotation of this cylinder, hence the name "cyclic runs". The maximal cyclic run starting at $\dot{6}$ is $\dot{6}3\dot{2}7$ (Figure 3, left), so that the associated schedule number is $s_2 = 2$. The maximal cyclic run ending at 4 is $3\dot{2}74$ (Figure 3, right), so the associated schedule number is $s_6 = 2$.

Let us recall the following classical definitions.

DEFINITION 3.12. *For $n \in \mathbb{N}$, define its q-*analogue *by $[n]_q := 1 + q + \ldots + q^{n-1}$. The q-factorial of n is given by $[n]_q! = [n]_q[n-1]_q \ldots [1]_q$. Of course q is an arbitrary choice of variable, later in the text we will also encounter t-analogues.*

Now, the following result can be deduced from [16, Theorem 3.13].

THEOREM 3.13. *Given $\tau \in \mathfrak{S}_n^{\bullet k}$ and $(s_i)_{1 \leqslant i \leqslant n}$ its schedule numbers, we have:*

$$\sum_{\substack{P \in \mathsf{stLD}(n)^{\bullet k} \\ \mathsf{dw}(P) = \tau}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} = t^{\mathsf{revmaj}(\tau)} \prod_{i \in [n]} [s_i]_q.$$

A consequence of this theorem is that a marked permutation is the diagonal word of some permutation if and only if all its schedule numbers are strictly positive. At $q = -1$, more terms vanish as a consequence of the following.

LEMMA 3.14. *Let $\sigma \in \mathfrak{S}_n^{\bullet}$, and $(s_1, \ldots, s_n) = \mathsf{sched}(\sigma)$. If $s_i > 0$ for all $i \in [n]$, then there exists $j \geqslant 1$ such that the set of schedule numbers $\{s_i \mid 1 \leqslant i \leqslant n\}$ is equal to $[j]$.*

Proof. Suppose there is an index i such that $s_i > 1$. Our goal is to show that there exists i' such that $s_{i'} = s_i - 1$. We distinguish two cases, (which are very similar):

- σ_i is not decorated. Consider the left-maximal run ending at σ_i, denoted R, so $R = (\sigma_h, \dots, \sigma_i)$ for some h with $h < i$. Consider the maximal i' such that $\sigma_{i'}$ is not decorated, and $h \leqslant i' < i$. It exists because $s_i > 0$. Let $R' = (\sigma_{h'}, \dots, \sigma_{i'})$ be the left-maximal cyclic run ending at $\sigma_{i'}$. We have $h' \leqslant h$ (because we know that $(\sigma_h, \dots, \sigma_{i'})$ is a cyclic run), so R and R' overlap on every non-decorated element of R other than σ_i. We deduce that $s_{i'} \geqslant s_i - 1$. In case of equality, we are done. Otherwise, we iterate this construction to find i'' such that $i'' < i'$ and $s_{i''} \geqslant s_{i'} - 1$, and so on. The sequence $s_i, s_{i'}, \dots$ either increases arbitrarily or decreases by 1 at each step, and we can define the next element as long as the last element s_j satisfies $s_j > 1$. So this sequence of schedule numbers must eventually reach the value $s_i - 1$.
- σ_i is decorated. Consider the right-maximal run beginning at σ_i, denoted R, so $R = (\sigma_i, \dots, \sigma_j)$ for some j. Consider the maximal i' such that $\sigma_{i'}$ is not decorated, and $i < i' \leqslant j$. It exists because $s_i > 0$. Let $R' = \sigma_{h'}, \dots, \sigma_{i'}$ be the left-maximal cyclic run ending at $\sigma_{i'}$. We have $h' \leqslant i$ (because we know that $(\sigma_i, \dots, \sigma_{i'})$ is a cyclic run), so R and R' overlap on every non-decorated value of R other than $\sigma_{i'}$. We deduce that $s_{i'} \geqslant s_i - 1$. In case of equality, we are done. Otherwise, we are back to the situation of the previous case.

In either case, we eventually find i' as announced. The statement in the lemma follows by iteration. □

A consequence of this lemma is that the product $\prod_{i=1}^{n} [s_i]_q$ at $q = -1$ is 0 unless all schedule numbers are 1 (if $s_i > 1$ for some i, one of the factors is $[2]_q = 1 + q$). Therefore, the schedule formula of Theorem 3.13 becomes:

$$\sum_{\substack{P \in \mathsf{stLD}(n)^{\bullet k} \\ \mathsf{dw}(P) = \tau}} (-1)^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} = \begin{cases} t^{\mathsf{revmaj}(\tau)} & \text{if } \mathsf{sched}(\tau) = 1^n, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Notation. We denote by $\mathfrak{S}_n^\bullet(1^n)$ to be the subset of $\mathfrak{S}_n^\bullet$ of marked permutations with schedule 1^n.

Now, summing Equation (7) over all possible permutations with k decorations τ, we get the following interpretation of the combinatorics of the Hilbert series of the Delta conjecture at $q = -1$:

$$\sum_{P \in \mathsf{stLD}(n)^{\bullet k}} (-1)^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} = \sum_{\tau \in \mathfrak{S}_n^{\bullet k}(1^n)} t^{\mathsf{revmaj}(\tau)}. \tag{8}$$

Remark 3.15. We announced in the introduction that in the case $k = 0$ and $t = 1$, the left-hand side of (8) is the Euler number E_n. Comparing with the right-hand side, this means that undecorated permutations $\sigma \in \mathfrak{S}_n$ with schedule 1^n are counted by the Euler number E_n. On the other side, Ramassamy [27, Corollary 5] gave a new combinatorial interpretation of the Euler number E_n as the number of total cyclic orders on $\{0, \dots, n\}$ such that $(i, i+1, i+2)$ is clockwise oriented for each $i \in \{0, \dots, n-2\}$. Undecorated permutations $\sigma \in \mathfrak{S}_n$ with schedule 1^n are simply in bijection with Ramassamy's total cyclic orders via $\sigma \mapsto (0, \sigma^{-1}(1), \dots, \sigma^{-1}(n))$.

We close this section by providing a bijection between $\mathfrak{S}_n$ and $\mathfrak{S}_n^\bullet(1^n)$, whose inverse is given by simply removing the decorations. The existence of this bijection means that each permutation can be decorated in exactly one way, so that the result has schedule 1^n.

LEMMA 3.16. *Let $\sigma \in \mathfrak{S}_n^\bullet(1^n)$. Let $\sigma_k, \dots, \sigma_\ell$ be a left-maximal cyclic run, and assume that σ_ℓ is undecorated. Then σ_k is undecorated as well, and $\sigma_{k+1}, \dots, \sigma_{\ell-1}$ are decorated.*

Proof. Since $s_\ell = 1$, there is exactly one undecorated entry in $\sigma_k, \dots, \sigma_{\ell-1}$. Denote σ_u this undecorated entry. If $u \neq k$, σ_k is decorated and has at least two undecorated entries (namely σ_u and σ_ℓ) in the right-maximal cyclic run beginning at σ_k. This would give $s_k \geqslant 2$, which contradicts $s_k = 1$. Thus, $u = k$, and we get that σ_k in the only non-decorated entry in $\sigma_k, \dots, \sigma_{\ell-1}$. □

LEMMA 3.17. *For each permutation $\sigma \in \mathfrak{S}_n$, there exists exactly one decorated permutation with underlying permutation σ and schedule 1^n.*

Proof. Assuming that there exists a decoration of σ so that the schedule is 1^n, the previous lemma readily gives necessary conditions on how to find it (in particular, uniqueness will follow from existence): we proceed from right to left (starting from σ_n and ending at σ_1), noting than σ_n is necessarily undecorated (otherwise we have $s_n = 0$). We define a sequence of indices $i_1 > i_2 > \cdots > i_m$ for some $m \geqslant 1$ as follows;

- $i_1 = n$,
- knowing i_j, we find i_{j+1} by the condition that $\sigma_{i_{j+1}} \cdots \sigma_{i_j}$ is the left-maximal cyclic run ending at σ_{i_j},
- the sequence stops at $i_m = 0$.

We claim that decorating the indices not in $\{i_1, \dots, i_m\}$ yields the unique decoration such that the associated schedule is 1^n. It remains only to check that the schedule of this decorated permutation is indeed 1^n.

By construction, we have $s_k = 1$ if σ_k is undecorated. It remains to show $s_k = 1$ when σ_k is decorated. So, consider a right-maximal cyclic run $\sigma_k, \dots, \sigma_\ell$ where σ_k is decorated.

- Suppose $s_k \geqslant 2$. So, there are two (or more) undecorated entries in this run, say σ_i and σ_j with $k < i < j$. The left-maximal cyclic run ending at σ_j contains at least $\sigma_k, \dots, \sigma_j$, so σ_i being undecorated contradicts the construction of $i_1 > i_2 > \cdots > i_m$ as above.
- Suppose $s_k = 0$. So, there are no undecorated entries in this run. Let σ_i be an undecorated entry with $i > k$ and i minimal (this exists because σ_n is undecorated). We have $i > \ell$ (because $\sigma_{k+1}, \dots, \sigma_\ell$ are decorated). Consider the left-maximal cyclic run ending at σ_i, denoted $\sigma_{i'}, \dots, \sigma_i$. It cannot begin at $\sigma_{i'}$ with $i' \leqslant k$, because $\sigma_k, \dots, \sigma_i$ is not a cyclic run (the right-maximal cyclic run beginning at σ_k ends at σ_ℓ, and $i > \ell$). Thus, $i' > k$, so $\sigma_{i'}, \dots, \sigma_i$ does not contain any undecorated entry apart from σ_i. But this means $s_i = 0$, which is a contradiction.

Other cases being excluded, we thus have $s_k = 1$. This completes the proof of existence and uniqueness of the decoration with schedule 1^n. □

4. PERMUTATIONS

We continue to use Convention 3.6: for any $\sigma \in \mathfrak{S}_n$, we set $\sigma_0 = 0$.

DEFINITION 4.1. *Given a permutation $\sigma = \sigma_1 \cdots \sigma_n$ and an index $i \in \{2, \dots, n\}$, we say that σ_i is a*

- double ascent *if $\sigma_{i-2} < \sigma_{i-1} < \sigma_i$;*
- double descent *if $\sigma_{i-2} > \sigma_{i-1} > \sigma_i$;*
- peak *if $\sigma_{i-2} < \sigma_{i-1} > \sigma_i$;*
- valley *if $\sigma_{i-2} > \sigma_{i-1} < \sigma_i$.*

Definition 4.2. *For $\sigma \in \mathfrak{S}_n$, we define*

$$\mathsf{monot}(\sigma) = \#\big\{i \in \{2, \dots, n\} \;\big|\; \sigma_i \text{ is a double descent or a double ascent}\big\}.$$

Definition 4.3. *Let $\sigma \in \mathfrak{S}_n$, a pair (i, j) with $1 \leqslant i < j \leqslant n$ is said to be a* 3-inversion *if one of the following holds:*

- *σ_j is a double ascent and $\sigma_{j-1} < \sigma_i < \sigma_j$;*
- *σ_j is a double descent and $\sigma_{j-1} > \sigma_i > \sigma_j$;*
- *σ_j is a peak and $\sigma_i > \sigma_j$;*
- *σ_j is a valley and $\sigma_i < \sigma_j$.*

The number of 3-inversions of σ is denoted by $\mathsf{inv}_3(\sigma)$.

Example 4.4. For $n = 3$, 123 has zero 3-inversions, 132 and 321 have one 3-inversions, 231 and 312 have two 3-inversion and 213 has three 3-inversions.

Though the definition of the statistic might not seem very natural, we will see in the proof of Proposition 4.6 that it can be tracked via a rather simple insertion procedure on permutations (similar to the Lehmer codes).

We recall the following classical definitions.

Definition 4.5. *A statistic I on permutation is called* Mahonian *if*

$$\sum_{\sigma \in S_n} t^{I(\sigma)} = [n]_t!.$$

Two classical Mahonian statistics are the major index (Definition 3.4) and the inversion number defined by

$$\mathsf{inv}(\sigma) = \#\big\{(i, j) \;\big|\; 1 \leqslant i < j \leqslant n \text{ and } \sigma_i > \sigma_j\big\}.$$

From the main result in Section 5 (Theorem 5.1) and the fact that revmaj is Mahonian, we will be able to deduce the following.

Proposition 4.6. *The statistic inv_3 is Mahonian, that is:*

$$\sum_{\sigma \in S_n} t^{\mathsf{inv}_3(\sigma)} = [n]_t!.$$

In [4], Chebikin defines another variant of the inversion statistic.

Definition 4.7. *Let $\sigma \in \mathfrak{S}_n$, define $\hat{c}_i(\sigma)$ to be the number of indices $j > i$ such that*

- *i is odd and $\sigma_i > \sigma_j$; or*
- *i is even and $\sigma_i < \sigma_j$.*

Let $\hat{\imath}(\sigma) = \hat{c}_1(\sigma) + \ldots + \hat{c}_{n-1}(\sigma)$.

Proposition 4.8 ([4, Corollary 3.5]). *The statistic $\hat{\imath}$ is Mahonian. Indeed, we have*

$$\sum_{\sigma \in S_n} t^{\hat{\imath}(\sigma)} = [n]_t!.$$

Definition 4.9. *An* alternating permutation *of $[n]$ is a permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma_1 > \sigma_2 < \sigma_3 > \cdots$. We denote the set of such permutations by $\mathfrak{A}_n$. In other words,*

$$\mathfrak{A}_n \coloneqq \{\sigma \in \mathfrak{S}_n \mid \mathsf{monot}(\sigma) = 0\}.$$

The alternating permutations are counted by the Euler numbers E_n.

One can easily check that if σ is alternating then $\hat{\imath}(\sigma) = \mathsf{inv}_3(\sigma)$. But this is not true in general. For example if $\sigma = 123$, $\mathsf{inv}_3(\sigma) = 3$ and $\hat{\imath}(\sigma) = 1$.

Definition 4.10. *A* 31-2 pattern *in $\sigma \in \mathfrak{S}_n$ is a triple $1 \leqslant i < i + 1 < j \leqslant n$ such that $\sigma_i > \sigma_j > \sigma_{i+1}$. We denote by 31-2$(\sigma)$ the number of 31-2 patterns in σ.*

In [18], the authors introduced an interesting t-analogue to the Euler numbers that was subsequently studied in [4] and [20].

Definition 4.11. *For all $n \in \mathbb{N}$, define:*

$$E_n(t) \coloneqq \sum_{\sigma \in \mathfrak{A}_n} t^{31\text{-}2(\sigma)}. \tag{9}$$

This polynomial $E_n(t)$ has several beautiful properties including the facts that the generating functions $\sum_{n \geqslant 0} E_{2n}(t) z^n$ and $\sum_{n \geqslant 0} E_{2n+1}(t) z^n$ have nice continued fraction expressions [18, 20].

We study here a shift of this t-analogue, namely $t^{\lfloor n^2/4 \rfloor} E_n(t)$. This t-analogue is naturally connected to our 3-inversion statistic.

Proposition 4.12. *For all $n \in \mathbb{N}$,*

$$t^{\lfloor n^2/4 \rfloor} E_n(t) = \sum_{\sigma \in \mathfrak{A}_n} t^{\mathsf{inv}_3(\sigma)}. \tag{10}$$

Proof. This is a Corollary of Lemma 9.4 of [4]. In this lemma, Chebikin proves that if $\sigma \in \mathfrak{A}_n$ then $\lfloor n^2/4 \rfloor + 31\text{-}2(\sigma) = \hat{\imath}(\sigma)$, and we just remarked that if $\sigma \in \mathfrak{A}_n$ then $\mathsf{inv}_3(\sigma) = \hat{\imath}(\sigma)$. □

5. Generating trees

The goal of this section is to prove the following. Recall that we denote by $\mathfrak{S}_n^\bullet(1^n)$ the subset of $\mathfrak{S}_n^\bullet$ of decorated permutations with schedule 1^n.

Theorem 5.1. *There is a bijection $\phi : \mathfrak{S}_n^\bullet(1^n) \to \mathfrak{S}_n$ with the following properties:*

(i) $\mathsf{revmaj}(\tau) = \mathsf{inv}_3(\phi(\tau))$,
(ii) $\mathsf{dec}(\tau) = \mathsf{monot}(\phi(\tau))$.

We will exhibit this bijection by constructing two isomorphic generating trees.

5.1. The tree for decorated permutations of schedule 1^n. We continue to use Convention 3.6: for any $\sigma \in \mathfrak{S}_n$, we set $\sigma_0 = 0$.

Let $\tau \in \mathfrak{S}_n$. Using Definition 3.8, we know that $\tau_0 \dots \tau_j$ is a cyclic run if and only if $\tau_1 > \dots > \tau_j$. The generation of the tree will rely on the following manipulation.

Definition 5.2. *Let $\tau \in \mathfrak{S}_n$ and $1 \leqslant k \leqslant n+1$. Denote by $k + \tau$ the permutation σ of $\mathfrak{S}_{n+1}$ defined by*

$$\begin{cases} \sigma_0 = 0, \\ \sigma_{i+1} = (k + \tau_i) \mod n+1 \quad \text{for } 0 \leqslant i \leqslant n. \end{cases}$$

Example. If $n = 7$, $\tau = 01423657$, and $k = 3$, we have

$$k + \tau = 034756182.$$

Lemma 5.3. *Let $\tau \in \mathfrak{S}_n$ and $\sigma \coloneqq k + \tau$. For $0 \leqslant i \leqslant j \leqslant n$, $\tau_i \cdots \tau_j$ is a cyclic run of τ if and only if $\sigma_{i+1} \cdots \sigma_{j+1}$ is a cyclic run of σ. Moreover, for $1 \leqslant j \leqslant n$, ($\tau_0 \cdots \tau_j$ is a cyclic run and $\tau_j + k > n+1$) if and only if $\sigma_0 \cdots \sigma_{j+1}$ is a cyclic run.*

Proof. The first statement is a direct consequence of the definition of a cyclic run (Definition 3.8). Note that for any permutation τ, $\tau_0 \dots \tau_j$ is a cyclic run if and only if $\tau_1 > \dots > \tau_j$. Therefore, $\sigma_0 \dots \sigma_{j+1}$ is a cyclic run if and only if $k = \sigma_1 > \dots > \sigma_{j+1}$. As $\sigma_{j+1} = (\tau_j + k) \mod n+1$, we have that $\sigma_{j+1} < k$ if and only if $\tau_j + k > n+1$. □

The structure of the generating tree will closely depend upon the following quantity.

Definition 5.4. *Given $\tau \in \mathfrak{S}_n^\bullet(1^n)$, we define its* structural attribute*, $a(\tau)$ to be the value of its first undecorated letter (excluding $\sigma_0 = 0$).*

Definition 5.5. *Define a tree $\mathcal{T}_1$ of decorated permutations as follows. Take its root to be $01 \in \mathfrak{S}_1$. For $\tau \in \mathfrak{S}_n^\bullet(1^n)$ with $n \geqslant 1$ and $1 \leqslant k \leqslant n+1$, the k-th* descendant of τ in $\mathcal{T}_1$*, denoted $\delta_k(\tau)$ is the decorated permutation whose underlying permutation is $\sigma = k + \tau$. The decorations of σ are as follows:*

- *$\sigma_0 = 0$ is never decorated,*
- *σ_1 is decorated if and only if $k > n + 1 - a(\tau)$,*
- *for $i \geqslant 2$, σ_i is decorated if and only if τ_{i-1} is decorated.*

See the tree on the left in Figure 5 for an illustration of the first three levels of this tree.

Example 5.6. For example, take $n = 5$ and $\tau = 0\dot{5}34\dot{2}1$. Then $a(\tau) = 3$ and so $n + 1 - a(\tau) = 3$. We have

$$\delta_2(\tau) = 02\dot{1}56\dot{4}3, \qquad \delta_4(\tau) = 0\dot{4}\dot{3}12\dot{6}5.$$

Let us prove some observations about $\mathcal{T}_1$. Suppose that τ is a decorated permutation and its schedule is 1^n.

Lemma 5.7. *Take $\tau \in \mathfrak{S}_n^\bullet(1^n)$. If $\tau_0 \dots \tau_j$ is a right-maximal cyclic run then there is a unique ℓ such that $1 \leqslant \ell \leqslant j$ and τ_ℓ is undecorated.*

Proof. Call $r \coloneqq \tau_0 \dots \tau_j$. If r it contained at least two undecorated letters, the second largest one would have schedule 2. If r contains no undecorated letters there must be exactly one undecorated τ_i with $i > j$ in the maximal decreasing cyclic run starting at τ_k for all $1 \leqslant k \leqslant j$. It follows that the maximal decreasing run ending at τ_i contains only decorated letters and so its schedule would be 0 (see Remark 3.10). □

Proposition 5.8. *The n-th level of the tree $\mathcal{T}_1$ contains exactly all the decorated permutations whose schedule numbers are 1^n.*

Proof. Take $\tau \in \mathfrak{S}_n^\bullet(1^n)$. We have to show that for all $k \in [n+1]$, $\mathsf{sched}(\delta_k(\tau)) = 1^{n+1}$. The result will then follow from Lemma 3.17 and the fact that the tree in Definition 5.5 clearly generates all permutations of n at the n-th level.

Set $\tau_\ell \coloneqq a(\tau)$, the unique undecorated entry in the right-maximal cyclic run containing τ_0. Thanks to Lemma 5.7, this ℓ is unique.

We construct the decorated permutation $\sigma = \delta_k(\tau)$.

Thanks to Lemma 5.3 we know that if $\tau_0 \dots \tau_j$ is not a cyclic run then the cyclic runs that contain σ_{j+1} are in bijection with the cyclic runs that contain τ_j and the schedule of σ_{j+1} is equal to the schedule of τ_j, which is one.

Moreover, if $j > 0$ and $\tau_0 \dots \tau_j$ is a cyclic run and τ_j is decorated, the cyclic runs that start at σ_{j+1} are in bijection with the cyclic runs that start at τ_j and the schedule of σ_{j+1} is equal to the schedule of τ_j, which is one.

We just have to show that $\sigma_{\ell+1}$ and σ_1 have schedule 1.

If $\tau_\ell + k > n + 1$ then (by Lemma 5.3), $\sigma_0 \dots \sigma_{\ell+1}$ is a cyclic run. As $\sigma_{\ell+1}$ is not decorated, we must decorate σ_1 to force that $\sigma_{\ell+1}$ and σ_1 have schedule 1.

If $\tau_\ell + k \leqslant n + 1$ then $\sigma_0 \dots \sigma_{\ell+1}$ is not a cyclic run but $\sigma_1 \dots \sigma_{\ell+1}$ is. Therefore, $\sigma_1 \dots \sigma_{\ell+1}$ is a left maximal run and to force $\sigma_{\ell+1}$ to have schedule 1, σ_1 must be non-decorated (otherwise $\sigma_{\ell+1}$ has schedule 0). Moreover, if σ_1 is not decorated, its schedule is 1, and we can conclude the proof. □

Proposition 5.9. *For all decorated permutations τ, we have*

$$\mathsf{revmaj}(\delta_k(\tau)) = \mathsf{revmaj}(\tau) + (n + 1 - k).$$

Proof. This is clear for $k = n + 1$, as $\delta_{n+1}(\tau)$ is obtained by appending $n + 1$ at the beginning of τ (which creates no new descent in the reversed permutation, thus preserving the reverse major index).

Next, we show $\mathsf{revmaj}(\delta_k(\tau)) = \mathsf{revmaj}(\delta_{k-1}(\tau)) - 1$. To go from $\delta_{k-1}(\tau)$ to $\delta_k(\tau)$, we add 1 to every letter, and replace $n + 1$ with 1. Let m be the index of $n + 1$ in $\delta_{k-1}(\tau)^{\mathsf{rev}}$. Then we have:

$$\mathsf{Des}(\delta_k(\tau)^{\mathsf{rev}}) = \begin{cases} (\mathsf{Des}(\delta_{k-1}(\tau)^{\mathsf{rev}}) \setminus \{m\}) \cup \{m-1\} & \text{if } m > 1, \\ \mathsf{Des}(\delta_{k-1}(\tau)^{\mathsf{rev}}) \setminus \{m\} & \text{if } m = 1. \end{cases}$$

Indeed, the m-th letter of $\delta_{k-1}(\tau)^{\mathsf{rev}}$ was equal to $n + 1$ and so was a descent, but in $\delta_k(\tau)^{\mathsf{rev}}$ it equals 1 and so is not descent. Furthermore, if $m > 1$, the $(m-1)$-th letter of $\delta_k(\tau)^{\mathsf{rev}}$ is followed by 1 and so must be a descent. In any case

$$\mathsf{revmaj}(\delta_k(\tau)^{\mathsf{rev}}) = \mathsf{revmaj}(\delta_{k-1}(\tau)^{\mathsf{rev}}) - m + m - 1 = \mathsf{revmaj}(\delta_{k-1}(\tau)^{\mathsf{rev}}) - 1$$

and the result follows. □

Finally, the following property follows easily from Definitions 5.4 and 5.5.

Proposition 5.10. *Recall that* $\mathsf{dec}(\tau)$ *is the number of decorations of* $\tau \in \mathfrak{S}_n^\bullet(1^n)$. *We have:*

$$\mathsf{dec}(\delta_k(\tau)) = \mathsf{dec}(\tau) + \chi(n + 1 - k \leqslant a(\tau)).$$

Furthermore, we have:

$$a(\delta_k(\tau)) = \begin{cases} k & \text{if } n + 1 - k > a(\tau), \\ a(\tau) + k \pmod{n+1} & \text{otherwise.} \end{cases}$$

5.2. A second tree related to peaks and valleys.

Definition 5.11. *Define a tree* $\mathcal{T}_2$ *of permutations as follows. Take its root to be* $1 \in \mathfrak{S}_1$. *For* $\sigma \in \mathfrak{S}_n$ *and* $1 \leqslant l \leqslant n + 1$, *define* $\eta_l(\sigma) \in \mathfrak{S}_{n+1}$ *to be the unique permutation* σ' *such that* $\sigma'_{n+1} = l$ *and* $\sigma'_1, \ldots, \sigma'_n$ *are in the same relative order as* $\sigma_1, \ldots, \sigma_n$. *In other words, we have*

$$\sigma'_i = \sigma_i + \chi(\sigma_i \geqslant l)$$

for $1 \leqslant l \leqslant n+1$. *This permutation is called the* insertion of l in σ. *The permutations* $\eta_l(\sigma)$ *for* $1 \leqslant l \leqslant n + 1$ *will form the descendants of* σ *in* $\mathcal{T}_2$.

See the tree on the right in Figure 5. Later (Definition 5.14), we will define a total order on the descendants of a node in $\mathcal{T}_2$, in a way that will give the isomorphism with $\mathcal{T}_1$. This ordering will closely depend upon the following quantity.

Definition 5.12. *Given* $\sigma \in \mathfrak{S}_n$, *define its* structural attribute*:*

$$\tilde{a}(\sigma) = \begin{cases} n + 1 - \sigma_n & \text{if } \sigma_{n-1} < \sigma_n, \\ \sigma_n & \text{if } \sigma_{n-1} > \sigma_n, \end{cases}$$

where we consider $\sigma_0 = 0$ *in case* $n = 1$.

Proposition 5.13. *For* $\sigma \in \mathfrak{S}_n$, *the map* ψ *on* $\{1, 2, \ldots, n + 1\}$ *defined by*

$$\mathsf{inv}_3(\eta_l(\sigma)) = \mathsf{inv}_3(\sigma) + \psi(l)$$

is a bijection onto $\{0, 1, \ldots, n\}$.

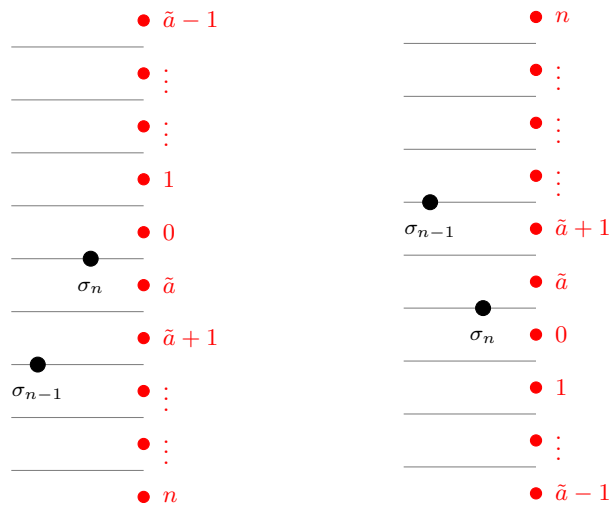

FIGURE 4. Contribution to inv_3 for the $n+1$ different insertions into a permutation of n.

Proof. Take $\sigma \in \mathfrak{S}_n$ and set $\sigma' = \eta_l(\sigma)$. Notice that for $j \leqslant n$ we have that (i,j) forms a 3-inversion in σ if and only if it forms a 3 inversion in σ' (see Definitions 4.3 and 5.11). It follows that $\mathsf{inv}_3(\sigma') = \mathsf{inv}_3(\sigma') + s$ where s is the number of 3-inversions of the form $(i, n+1)$.

We distinguish two cases. We have drawn a schematic representation of the proof in Figure 4.

(1) *Either σ ends with an ascent*, that is $\sigma_{n-1} < \sigma_n$. So we have $\tilde{a} := \tilde{a}(\sigma) = n+1-\sigma_n$. This is illustrated in the left part of Figure 4, where the values in red are the value of $\psi(l)$.
   - For $l > \sigma_n$, σ'_{n+1} is a double ascent. The number of i's such that $\sigma'_n < \sigma'_i < \sigma'_{n+1}$ is equal to $l-(\sigma_n+1)$. So for $l = \sigma_n+1, \sigma_n+2, \dots, n+1$ we have $\psi(l) = 0, 1, \dots, \tilde{a}-1$, respectively.
   - For $l \leqslant \sigma_n$, σ'_{n+1} is a peak. The number of i's such that $\sigma'_i > \sigma'_{n+1}$ is equal to $n+l-1$. So for $l = \sigma_n, \sigma_n-1, \dots, 1$ we have $\psi(l) = \tilde{a}, \tilde{a}+1, \dots, n$, respectively.

(2) *Or, σ ends with a descent*, that is $\sigma_{n-1} > \sigma_n$. So we have $\tilde{a} := \tilde{a}(\sigma) = \sigma_n$. This is illustrated in the right part of Figure 4.
   - For $l \leqslant \sigma_n$, σ'_{n+1} is a double descent. The number of i such that $\sigma'_n > \sigma'_i > \sigma'_{n+1}$ equals $\sigma_n - l$. So for $l = \sigma_n, \sigma_n-1, \dots, 1$ we have $\psi(l) = 0, 1, \dots, \tilde{a}-1$, respectively.
   - For $l > \sigma_n$, σ'_{n+1} is a valley. The number of i such that $\sigma'_i > \sigma'_{n+1}$ is equal to $l-1$. So for $l = \sigma_n+1, \dots, n+1$, we have $\psi(l) = \tilde{a}, \tilde{a}+1, \dots, n$, respectively.

In each of the two cases, we see that the values taken by $\psi(l)$ are exactly $0, \dots, n$. □

DEFINITION 5.14. *For $\sigma \in \mathfrak{S}_n$ we define the k-th* descendant of σ in $\mathcal{T}_2$, $\tilde{\delta}_k(\sigma)$, *to be the unique descendant $\eta_l(\sigma)$ such that $\psi(l) = n+1-k$, in other words $\mathsf{inv}_3(\tilde{\delta}_k(\sigma)) = \mathsf{inv}_3(\sigma) + (n+1-k)$.*

Finally, we can deduce the following from the proof of Proposition 5.13.

PROPOSITION 5.15. *Recall that* $\mathsf{monot}(\sigma)$ *denotes the number of double ascents and descents of* $\sigma \in \mathfrak{S}_n$*. We have:*

$$\mathsf{monot}(\eta_k(\sigma)) = \mathsf{monot}(\sigma) + \chi(n+1-k \leqslant \tilde{a}(\sigma)).$$

Furthermore, we have:

$$\tilde{a}(\eta_k(\sigma)) = \begin{cases} k & \text{if } k \leqslant n+1-\tilde{a}(\sigma), \\ \tilde{a}(\sigma) & \text{if } k > n+1-\tilde{a}(\sigma). \end{cases}$$

5.3. ISOMORPHISM OF THE TREES. To prove Theorem 5.1, we define ϕ recursively as follows:

$$\phi(1) = 1,$$
$$\phi(\delta_k(\sigma)) = \eta_k(\phi(\sigma)) \text{ for all } \sigma \in \biguplus_{n \geqslant 1} \mathfrak{S}_n^\bullet(1^n).$$

In Figure 5, the k-th descendant of each node is the k-th descendant from the bottom, so that the image by ϕ of an element in the left tree can be obtained by looking for the corresponding element in the right tree, if we were to "superpose" one tree on the other.

From this definition of ϕ and the definition of *k-th descendant* as being the descendant adding $n+1-k$ units to the relevant statistic in both trees (See Definition 5.5, Proposition 5.9 and Definition 5.14), we may conclude that

$$\mathsf{revmaj}(\sigma) = \mathsf{inv}_3(\phi(\sigma)).$$

From Proposition 5.10 and Proposition 5.15, we deduce

$$\mathsf{dec}(\sigma) = \mathsf{monot}(\phi(\sigma)).$$

Thus, we have now established Theorem 5.1.

6. CONCLUSION AND FUTURE DIRECTIONS

Let us now put the pieces together.

Proof of Theorem 1.1. From the bijection in Theorem 5.1, we may deduce that

$$\sum_{\tau \in \mathfrak{S}_n^\bullet(1^n)} t^{\mathsf{revmaj}(\tau)} z^{\mathsf{dec}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{inv}_3(\sigma)} z^{\mathsf{monot}(\sigma)}.$$

Recall Equation 8:

$$\sum_{P \in \mathsf{stLD}(n)^{\bullet k}} (-1)^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} = \sum_{\tau \in \mathfrak{S}_n^{\bullet k}(1^n)} t^{\mathsf{revmaj}(\tau)}.$$

Combining these last two equations and summing over k, we get:

$$\sum_{k=0}^{n-1} \left(\sum_{P \in \mathsf{stLD}(n)^{\bullet k}} t^{\mathsf{area}(P)} (-1)^{\mathsf{dinv}(P)} \right) z^k = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{inv}_3(\sigma)} z^{\mathsf{monot}(\sigma)},$$

which is exactly the statement in Theorem 1.1. □

Since the left-hand side of (8) is exactly the combinatorics of the Hilbert series of the Delta conjecture (Equation 4) at $q = -1$, the validity of the Delta conjecture would imply

$$\sum_{k=0}^{n-1} \langle \Delta'_{e_{n-k-1}} e_n, h_1^n \rangle \Big|_{q=-1} z^k = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{inv}_3(\sigma)} z^{\mathsf{monot}(\sigma)}, \tag{11}$$

as announced in Corollary 1.2.

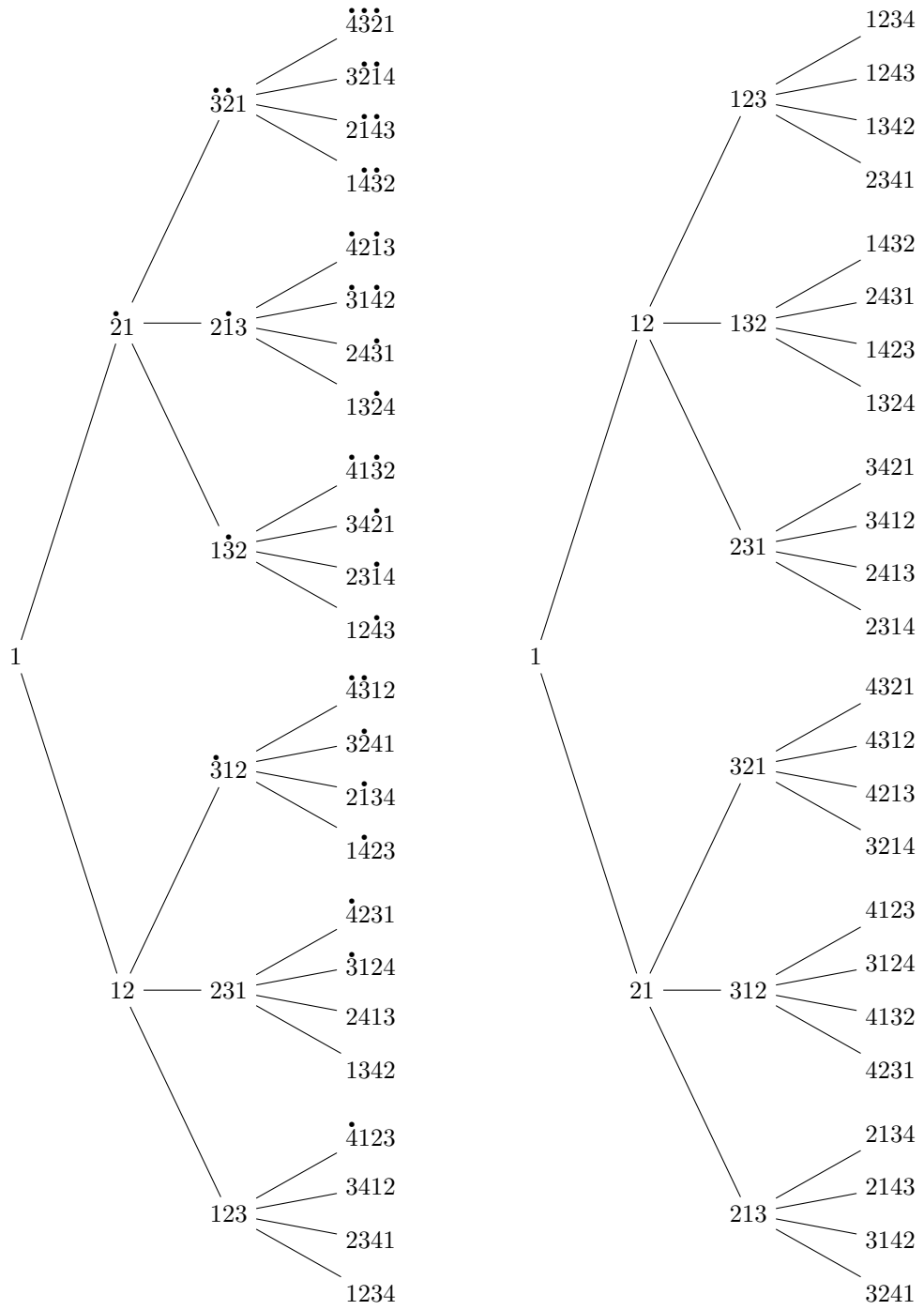

Figure 5. $\mathcal{T}_1$ and $\mathcal{T}_2$ up to level 4.

Since at $k = 0$ the Delta conjecture reduces to the shuffle theorem, our result implies that

$$\langle \nabla e_n, h_1^n \rangle|_{q=-1} = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{monot}(\sigma)=0}} t^{\mathsf{inv}_3(\sigma)} = \sum_{\sigma \in \mathfrak{A}_n} t^{\mathsf{inv}_3(\sigma)} = t^{\lfloor n^2/4 \rfloor} E_n(t),$$

where the last equality comes from Proposition 4.12. Thus, we have established (2).

6.1. On q, t-symmetry and the rise Delta. We remark that our results rely heavily on the schedule number formula. This formula breaks up the combinatorics of the valley Delta conjecture into sets of paths with the same diagonal word and thus the same area, and expresses the dinv as a product of q-analogues. Even though it is easy to see from the symmetric function that

$$\Delta'_{e_{n-k-1}} e_n[X; q, t] = \Delta'_{e_{n-k-1}} e_n[X; t, q],$$

this q, t-symmetry is not at all obvious on the combinatorial side. In particular, there is no schedule formula that fixes dinv and expresses area as a product of t-analogues. Thus, we have statements for $q = -1$ but not for $t = -1$. In the same vein, there is no known schedule formula for the rise version of the Delta conjecture (which is now a theorem [9]) and so (11) remains conditional on the valley Delta conjecture (Conjecture 2.12).

6.2. On refinements of the Delta conjecture. Let us now say a few words about how our results relate to the well-known refinements of the Delta conjecture.

The compositional (valley) Delta conjecture is a refinement of Conjecture 2.12, giving a formula for the symmetric function $\Theta_{e_k} \nabla C_\alpha$ in terms of (valley or rise) decorated Dyck paths, where α is a composition dictating at which points the path touches the diagonal $x = y$. Summing over all compositions, we have $\sum_{\alpha \vDash n-k} \Theta_{n-k} \nabla C_\alpha = \Delta'_{e_{n-k-1}} e_n$. See [7] for more details on the compositional Delta conjecture. The combinatorics in this paper suggests that the following symmetric function identities hold:

$$\sum_{\substack{\alpha \vDash n-k \\ \alpha \neq (n-k)}} \langle \Theta_{e_k} \nabla C_\alpha, h_1^n \rangle|_{q=-1} = 0,$$

$$\langle \Theta_{e_k} \nabla C_{(n-k)}, h_1^n \rangle|_{q=-1} = \langle \Delta'_{e_{n-k-1}} e_n, h_1^n \rangle|_{q=-1}.$$

Indeed, schedule 1^n paths always have composition $n - k$. If the Delta conjecture is true, this would prove these symmetric function identities. As it stands, only the $k = 0$ case is proven, via the shuffle theorem.

Similar identities can be found for the touching refinement $\Theta_k \nabla E_{n,l}$, of the Delta conjecture, for $q = -1$ and $\langle \cdot, h_1^n \rangle$:

$$\langle \Theta_{e_k} \nabla E_{n,l}, h_1^n \rangle|_{q=-1} = 0 \qquad \text{if } l > 1,$$

$$\langle \Theta_{e_k} \nabla E_{n,1}, h_1^n \rangle|_{q=-1} = \langle \Delta'_{e_{n-k-1}} e_n, h_1^n \rangle|_{q=-1}.$$

Furthermore, when setting $t = -1$ instead of q, we noticed that

$$(-1)^{n-l} \langle \nabla E_{n,l}, h_1^n \rangle|_{t=-1}$$

is a q-analogue of the number of ordered forests of k increasing unordered trees with n vertices and in which all outdegrees are $\leqslant 2$ (sequence [25, A185421]). The trees in these forests are counted by the Euler numbers. Using these combinatorial objects, we managed to interpret the following symmetric function identity at $q = 1$:

$$\sum_{l=1}^{n} \langle \nabla E_{n,l}, h_1^n \rangle|_{t=-1} = \langle \nabla e_n, h_1^n \rangle|_{t=-1}.$$

6.3. On representation-theoretic interpretations. Famously, the symmetric function of the shuffle theorem is the bi-graded Frobenius characteristic of the diagonal co-invariant module [17]. The symmetric function $\Delta'_{e_{n-k-1}} e_n$ is conjectured to be the character of the 2-bosonic, 1-fermionic diagonal module [30]. At present, we know of no representation-theoretic interpretation of the evaluation of at $q = -1$.

Some manipulation of Bergeron's conjecture for characters of general boson-fermion diagonal modules [2, Conjecture 1] might provide some hints in this direction.

6.4. More positivity at $q = -1$. Computational evidence suggests that the evaluation at $q = -1$ yields t-positive results for many other polynomials related to the shuffle theorem and Delta conjecture. For example

- $\langle \nabla e_n, h_\mu \rangle$ for any partition μ;
- $\langle \nabla \omega(p_n); h_1^n \rangle$, where $\nabla \omega(p_n)$ is the symmetric function related to the square theorem (conjectured in [22] and proved in [29]);
- $\langle \Theta_{e_k} \Theta_{e_l} \nabla e_{n-k-l}, h_1^n \rangle$, where $\Theta_{e_k} \Theta_{e_l} \nabla e_{n-k-l}$ is the symmetric function introduced in [7] that conjecturally unifies both versions of the Delta conjecture;
- $\langle \Theta_{e_\lambda} e_1, h_1^{|\lambda|+1} \rangle$, see the Theta trees conjecture in [5].

It would thus be interesting to study the $q = -1$ evaluation in a more general framework, for example in modified Macdonald polynomials.

6.5. An open (combinatorial) problem. Let us end this section with a combinatorial problem. Consider the sums

$$D_{n,j}(t) = \sum_{\substack{\sigma \in \mathfrak{S}_n, \\ \mathsf{monot}(\sigma) \leqslant j}} t^{\mathsf{inv}_3(\sigma)}. \tag{12}$$

In particular, this is 0 if $j < 0$, and $D_{n,j}(t) = D_{n,n-1}(t)$ if $j > n-1$.

Conjecture 6.1. *For $n > 1$ and $0 \leqslant j \leqslant n-1$, we have:*

$$D_{n,j}(t) = t^{n-j-1}[j+1]_t D_{n-1,j+1}(t) + [n-j-1]_t D_{n-1,j-1}(t). \tag{13}$$

It would be very interesting to give a combinatorial proof of (13). Of course, one might try to do to this starting from the other combinatorial interpretation:

$$D_{n,j}(t) = \sum_{\substack{\sigma \in \mathfrak{S}^\bullet(1^n), \\ \mathsf{dec}(\sigma) \leqslant j}} t^{\mathsf{revmaj}(\sigma)}.$$

In the case $t = 1$, it is possible to give a combinatorial proof of the conjecture, based on the combinatorial interpretation in (12). The idea is to consider the map $\sigma \mapsto \sigma'$ (where, for $\sigma \in \mathfrak{S}_n$, $\sigma' \in \mathfrak{S}_{n-1}$ is obtained by removing the entry n), and examine how monot is distributed among the n pre-images of a given $\sigma \in \mathfrak{S}_{n-1}$.

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