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ABSTRACT We define diagrammatic extremal weight projectors for \mathfrak{gl}_N ($N > 2$), a refinement of Jones–Wenzl projectors and Kuperberg’s clasps. As by-products, we obtain compatible diagrammatic presentations of the representation categories of \mathfrak{gl}_N and its Cartan subalgebra, and a categorification of power-sum symmetric polynomials.

1. INTRODUCTION

The topological motivation for this article is the search for an extension of Khovanov–Rozansky link homologies [15, 17] to invariants of links in 3-manifolds other than \mathbb{R}^3 . Since quantum link homologies, in their mode of definition and computation, currently depend on the presentation of links as 2-dimensional projections, the most accessible 3-manifolds in this endeavor are thickened surfaces $\Sigma \times I$. Just as Khovanov homology categorifies the Jones polynomial, the surface link homologies should categorify surface skein modules [24, 32, 1]. These admit an algebra structure induced by stacking links, with distinguished bases (conjecturally) satisfying strong integrality and positivity properties [12, 30, 21, 22, 6]. They provide quantizations of surface character varieties that play an important role in quantum Teichmüller theory [5]. Both aspects make such *skein algebras* prime targets for categorification via link homology technology.

In order to categorify quantum invariants, it is useful to have explicit, combinatorial or diagrammatic descriptions of underlying representation categories. For example, Khovanov homology can be built from a categorification of the Temperley–Lieb category, which describes the representation category of $U_q(\mathfrak{sl}_2)$, and all incarnations of Khovanov–Rozansky homology implicitly employ a categorification of the *MOY* or *web calculus* for the representation category of $U_q(\mathfrak{gl}_N)$ [23, 9].

The main purpose of the present paper is to provide representation-theoretic tools for categorifying skein algebras. The key novelty when working with skein algebras is that their proposed distinguished bases are obtained from links colored not by irreducible representations of the corresponding quantum group, but colored only by the sums of their *extremal weight spaces*, i.e. the weight spaces for weights in the Weyl group orbit of the highest weight. In [28, Section 1.3] we have proposed a strategy for lifting these colorings to the categorified level of toric link homologies, which is inspired by Khovanov’s categorification of the colored Jones polynomial [16]. The main tool necessary in this approach is a diagrammatic presentation of a certain subcategory

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of representations of a Cartan subalgebra $U(\mathfrak{h}) \subset U(\mathfrak{gl}_N)$, which is the first result in this paper:

RESULT 1.1. *In Corollary 2.39 we prove that there exists a diagrammatic presentation for a suitable full subcategory of the representation category of $U(\mathfrak{h})$, given by an affine extension $N\mathbf{AWeb}^{\text{ess}}$ of the web calculus for $U(\mathfrak{gl}_N)$ in which $\bigwedge^k(V)$ -labeled essential circles are set to zero for $0 < k < N$.*

We emphasize that we deal with universal enveloping algebras, rather than their quantizations. The former are intended to control the \mathbb{C} -linear *morphism spaces* in our skein module categorifications [28, Section 1.3], whereas the expected quantum parameter q is promoted to a grading on *objects*. This is related to the fact that *annular Khovanov homology* [13] has a natural action of $U(\mathfrak{sl}_2)$, not of $U_q(\mathfrak{sl}_2)$. An additional quantization seems possible, see Remark 1.5, but will not play a role here. For a detailed discussion of the relation between our diagrammatic presentation and one that follows from the work of Cautis–Kamnitzer [8, Section 2.6], we refer to the paper by Lacabanne–Tubbenhauer–Vaz [20] that appeared during revisions on this paper.

In [28], we have constructed a diagrammatic presentation for the case of \mathfrak{sl}_2 , and identified idempotent morphisms that encode the projections onto sums of extremal weight spaces in finite-dimensional $U(\mathfrak{sl}_2)$ -representations. These *extremal weight projectors* are analogous to, but finer than Jones–Wenzl projectors [14, 33] (every Jones–Wenzl projector splits into a non-negative linear combination of extremal weight projectors), and they can also be defined recursively. In this article, we identify and study extremal weight projectors for \mathfrak{gl}_N .

RESULT 1.2. *In Section 2.8 we give an explicit recursive description of the idempotents in the diagrammatic category $N\mathbf{AWeb}^{\text{ess}}$ that correspond to projections onto the extremal weight spaces in the $U(\mathfrak{gl}_N)$ -representations $\text{Sym}^k(V)$. We call them *extremal weight projectors*.*

In fact we prove a slightly stronger version of this result in a central extension of $N\mathbf{AWeb}^{\text{ess}}$, which has an additional grading by winding number, that will be important for categorifying skein modules, see Theorem 2.28.

The \mathfrak{sl}_2 extremal weight projectors can be considered as categorifications of Chebyshev polynomials of the first kind by decategorifying their images to elements of the representation ring $K_0(\mathbf{Rep}(\mathfrak{sl}_2)) \cong \mathbb{Z}[X]$. Analogously, the extremal weight projectors for \mathfrak{gl}_N categorify power-sum symmetric polynomials in the representation ring of \mathfrak{gl}_N . Such categorifications of classical orthogonal polynomials are of independent interest, see e.g. [18]. Motivated by this, we prove a categorified Newton identity.

RESULT 1.3. *In Theorem 3.5 we prove that the extremal weight projectors satisfy a categorified version of the Newton identity relating power-sum symmetric and elementary symmetric polynomials.*

The main application for our diagrammatic extremal weight projectors, however, is in categorifying toric skein modules. In [29], we construct a categorification of the \mathfrak{gl}_2 skein module of the thickened torus via a toric \mathfrak{gl}_2 foam category, whose homotopy category is the target of a toric link homology functor. (The use of \mathfrak{gl}_2 foams, as opposed to Bar-Natan cobordisms [2], is necessary to guarantee the functoriality of the resulting link homology, see [4, 10].) The category $2\mathbf{AWeb}^{\text{ess}}$ describes morphism spaces in this foam category, with affine webs corresponding to rotationally symmetric foams. In particular, the rotation foams generated by extremal weight projectors represent indecomposable objects in the Karoubi completed toric foam category. In

Section 4, we prove a delooping lemma for \mathfrak{gl}_2 webs and decomposition formulas for tensor products of \mathfrak{gl}_2 extremal weight projectors, which are used in [29] to show that the toric \mathfrak{gl}_2 foams indeed categorify the \mathfrak{gl}_2 skein module of the thickened torus. We expect that rotation foams generated by the extremal weight projectors for \mathfrak{gl}_N give indecomposable objects in a candidate categorification of the \mathfrak{gl}_N skein algebra of the torus based on \mathfrak{gl}_N foam categories.

REMARK 1.4. Affine web categories have appeared before in work of the first-named author [25] on skein modules, and of Cautis–Kamnitzer [8] on a K-theoretic version of the derived geometric Satake correspondence for SL_N . The main differences are that here we work at $q = 1$, which makes the affine web categories symmetric monoidal, and that we take a quotient by $\bigwedge^k(V)$ -labeled essential circles for $0 < k < N$. It is unclear to us how to define an analogous quotient for generic q that would admit extremal weight projectors.

Affine web categories for \mathfrak{gl}_N can also be seen as (idempotent completions of) quotients of HOMFLYPT skein categories as studied, for example in [7, Section 4].

REMARK 1.5. Affine web categories at generic q describe morphism spaces in quantized toric foam categories, which can be defined using a quantized horizontal trace construction. This is analogous to the quantized annular Bar-Natan cobordisms of Beliakova–Putyra–Wehrli [3]. However, such a quantization involves a non-canonical choice of a simple closed curve on the torus, that breaks a natural mapping class group action which is desirable for categorified skein modules. In [29], we thus proceed with affine webs at $q = 1$ and unquantized toric foam categories.

2. AFFINE \mathfrak{gl}_N WEBS AND EXTREMAL WEIGHT PROJECTORS

We start by recalling the diagrammatic calculus of \mathfrak{gl}_N webs, which describes the category of representations of $U_q(\mathfrak{gl}_N)$ that is monoidally generated by exterior powers of the vector representation and their duals.

2.1. THE CATEGORY OF \mathfrak{gl}_N WEBS. The category $N\mathbf{Web}_q$ of \mathfrak{gl}_N webs is the $\mathbb{C}(q)$ -linear pivotal tensor category with objects generated by points on the line \mathbb{R} that are labeled with integers in the set $\{1, \dots, N\}$ and that carry an orientation *up* or *down*. We may consider the tensor unit as a 0-labeled point without orientation. The morphisms in $N\mathbf{Web}_q$ are spanned by *webs* that are properly embedded in the strip $\mathbb{R} \times [0, 1]$, and composition is given by (the bilinear extension) of stacking in the interval direction. Webs are trivalent graphs, with edges oriented and labeled by elements of $\{1, \dots, N\}$, with a flow condition at each vertex imposing that the sum of incoming labels equals the sum of the outgoing ones. When properly embedded in the strip, these graphs are interpreted as mapping from the bottom sequence of boundary points (with labels and orientations) to that at the top. The morphisms in $N\mathbf{Web}_q$, represented by $\mathbb{C}(q)$ -linear combinations of webs, are considered modulo isotopy relative to the boundary and subject to certain local relations. To describe the relations, it is convenient to temporarily allow edges labeled by 0 and trivalent vertices involving such edges. The relations make use of *quantum integers* and *quantum binomial coefficients* defined for $m \in \mathbb{N}$, $n \in \mathbb{Z}$ as $[n] := \frac{q^n - q^{-n}}{q - q^{-1}}$, $[n]! = \prod_{i=1}^n [i]$, and $\begin{bmatrix} n \\ m \end{bmatrix} := \frac{[n]!}{[m]![n-m]!}$. Locally, the following relations hold:

$$(1) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} = \begin{array}{c} \left[\begin{array}{c} k+l \\ k \end{array} \right]_{k+l} \\ \sum_t \left[\begin{array}{c} k-l+a-b \\ t \end{array} \right] \\ \left[\begin{array}{c} 1 \\ 1 \end{array} \right]_{k+l} \end{array}, \quad \begin{array}{c} \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \end{array} = \begin{array}{c} \left[\begin{array}{c} N-k \\ l \end{array} \right]_k \\ \left[\begin{array}{c} 1 \\ 1 \end{array} \right]_{k+l} \\ [N-k-l] \end{array}, \quad \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} = \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array}$$

as well as their orientation reversals. The following relations are useful consequences of the ones above:

$$(2) \quad \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} = \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} = \left[\begin{array}{c} N \\ l \end{array} \right], \quad \begin{array}{c} \uparrow \\ N \end{array} \begin{array}{c} \downarrow \\ N \end{array} = \begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array}, \quad \begin{array}{c} \uparrow \\ k \end{array} \begin{array}{c} \downarrow \\ N \end{array} = \begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array}$$

The first equation is a special case of the second relation in (1), obtained by setting $k = 0$ and erasing 0-labeled edges.

2.2. LINK WITH $U_q(\mathfrak{gl}_N)$ -REPRESENTATION THEORY. Let $\mathbf{Rep}(U_q(\mathfrak{gl}_N))$ denote the $\mathbb{C}(q)$ -linear pivotal tensor category of $U_q(\mathfrak{gl}_N)$ -representations that is monoidally generated by quantum exterior powers of the vector representation and their duals. The main purpose of the diagrammatic calculus of \mathfrak{gl}_N webs is to describe this category.

THEOREM 2.1. *There exists an equivalence of $\mathbb{C}(q)$ -linear pivotal tensor categories*

$$\phi: N\mathbf{Web}_q \rightarrow \mathbf{Rep}(U_q(\mathfrak{gl}_N))$$

that sends k -labeled upward points to k -fold exterior powers of the vector representation of $U_q(\mathfrak{gl}_N)$.

Essentially, this theorem is due to Cautis–Kamnitzer–Morrison [9], although they state it for $U_q(\mathfrak{sl}_N)$ in [9, Theorem 3.3.1]. Versions for $U_q(\mathfrak{gl}_N)$ have appeared as special cases of [27, Main Theorem A] and [31, Theorem 3.20]. We now describe the functor ϕ explicitly.

Recall that $U_q(\mathfrak{gl}_N)$ is the $\mathbb{C}(q)$ -algebra generated by E_i, F_i for $1 \leq i \leq N - 1$ and $L_j^{\pm 1}$ for $1 \leq j \leq N$ subject to the following relations:

$$(3) \quad L_i E_i = q E_i L_i, \quad L_i F_i = q^{-1} F_i L_i, \quad L_{i+1} E_i = q^{-1} E_i L_{i+1}, \quad L_{i+1} F_i = q F_i L_{i+1}$$

$$(4) \quad [E_i, F_j] = \delta_{i,j} \frac{L_i L_{i+1}^{-1} - L_{i+1} L_i^{-1}}{q - q^{-1}}, \quad [L_i, L_j] = 0.$$

$$(5) \quad E_i^2 E_j - [2] E_i E_j E_i + E_j E_i^2 = 0 \text{ if } |i - j| = 1 \text{ and } [E_i, E_j] = 0 \text{ otherwise;}$$

analogously for F_i s.

It is a Hopf algebra with coproduct, antipode and counit as follows:

$$\begin{aligned} \Delta(E_i) &= E_i \otimes L_i L_{i+1}^{-1} + 1 \otimes E_i, & \Delta(F_i) &= F_i \otimes 1 + L_i^{-1} L_{i+1} \otimes F_i \\ \Delta(L_i^{\pm 1}) &= L_i^{\pm 1} \otimes L_i^{\pm 1} \\ S(L_i^{\pm 1}) &= L_i^{\mp 1}, & S(E_i) &= -E_i L_i^{-1} L_{i+1}, & S(F_i) &= -L_i L_{i+1}^{-1} F_i \\ \epsilon(L_i^{\pm 1}) &= 1, & \epsilon(E_i) &= 0, & \epsilon(F_i) &= 0 \end{aligned}$$

Let $V = \mathbb{C}(q)\langle v_1, v_2, \dots, v_N \rangle$ denote the vector representation of $U_q(\mathfrak{gl}_N)$ and $V^* = \mathbb{C}(q)\langle v_1^*, v_2^*, \dots, v_N^* \rangle$ its dual.

To leftward oriented cups and caps, the functor ϕ associates the natural evaluation and co-evaluation maps for duals:

$$\text{cup} \xrightarrow{\phi} \begin{cases} \mathbb{C} \rightarrow V \otimes V^* \\ 1 \mapsto \sum_{k=1}^N v_k \otimes v_k^* \end{cases}, \quad \text{cap} \xrightarrow{\phi} \begin{cases} V^* \otimes V \rightarrow \mathbb{C} \\ v_k^* \otimes v_l \mapsto \delta_{k,l} \end{cases}$$

Let $\wedge^k V$ denote the k -th quantum exterior power of V . This has a basis indexed by subsets $S = \{i_1, \dots, i_k\} \subset \{1, 2, \dots, N\}$ of size k . If $1 \triangleleft i_1 < \dots < i_k \triangleleft N$, we use the following notation for the corresponding basis vector: $v_S := v_{i_1} \wedge \dots \wedge v_{i_k}$. The dual $(\wedge^k V)^* \cong \wedge^k V^*$ then has the dual basis given by vectors v_S^* and we have corresponding thickness k cap and cup morphisms as above.

$$\text{thick cup} \xrightarrow{\phi} \begin{cases} \mathbb{C} \rightarrow \wedge^k V \otimes \wedge^k V^* \\ 1 \mapsto \sum_{|S|=k} v_S \otimes v_S^* \end{cases}, \quad \text{thick cap} \xrightarrow{\phi} \begin{cases} \wedge^k V^* \otimes \wedge^k V \rightarrow \mathbb{C} \\ v_S^* \otimes v_T \mapsto \delta_{S,T} \end{cases}$$

The other duality maps, that is, rightward oriented caps and cups, are perturbed by powers of q .

$$\text{perturbed cup} \xrightarrow{\phi} \begin{cases} \mathbb{C} \rightarrow \wedge^k V^* \otimes \wedge^k V \\ 1 \mapsto \sum_{|S|=k} q^{-\epsilon_S} v_S^* \otimes v_S \end{cases}, \quad \text{perturbed cap} \xrightarrow{\phi} \begin{cases} \wedge^k V \otimes \wedge^k V^* \rightarrow \mathbb{C} \\ v_S \otimes v_T^* \mapsto \delta_{S,T} q^{\epsilon_S} \end{cases}$$

Here $\epsilon_S = \sum_{i \in S} (N + 1 - 2i)$.

Merges of thick strands act as (q -deformed) exterior product:

$$(6) \quad \text{merge} \xrightarrow{\phi} \begin{cases} \wedge^k V \otimes \wedge^l V \rightarrow \wedge^{k+l} V \\ v_S \otimes v_T \mapsto 0 & \text{if } S \cap T \neq \emptyset \\ v_S \otimes v_T \mapsto (-q)^{\epsilon_{S,T}} v_{S \cup T} & \text{otherwise} \end{cases}$$

Here $\epsilon_{S,T}$ is the number of inversions in the concatenation of the ordered lists of elements of S and T . The split vertex acts as follows:

$$(7) \quad \text{split} \xrightarrow{\phi} \begin{cases} \wedge^{k+l} V \rightarrow \wedge^k V \otimes \wedge^l V \\ v_S \mapsto (-1)^{kl} \sum_{T \subset S, |T|=k} (-q)^{-\epsilon_{S \setminus T, T}} v_T \otimes v_{S \setminus T} \end{cases}$$

Analogous formulas hold for merges and splits of duals, which implies that merges and splits can be slid around caps and cups:

$$(8) \quad \text{dual merge} \xrightarrow{\phi} \begin{cases} \wedge^k V^* \otimes \wedge^l V^* \rightarrow \wedge^{k+l} V^* \\ v_S^* \otimes v_T^* \mapsto 0 & \text{if } S \cap T \neq \emptyset \\ v_S^* \otimes v_T^* \mapsto (-1)^{kl} (-q)^{-\epsilon_{S,T}} v_{S \cup T}^* & \text{otherwise} \end{cases}$$

and:

$$(9) \quad \text{dual split} \xrightarrow{\phi} \begin{cases} \wedge^{k+l} V^* \rightarrow \wedge^k V^* \otimes \wedge^l V^* \\ v_S^* \mapsto \sum_{T \subset S, |T|=k} (-q)^{\epsilon_{S \setminus T, T}} v_T^* \otimes v_{S \setminus T}^* \end{cases}$$

This concludes the description of the functor ϕ .

The category $\mathbf{Rep}(U_q(\mathfrak{gl}_N))$ is braided, and by virtue of Theorem 2.1, so is $N\mathbf{Web}_q$. The diagrammatic description of the braiding of two fundamental $U_q(\mathfrak{gl}_N)$ -representations in $N\mathbf{Web}_q$ is given as follows:

$$(10) \quad \text{crossing} = (-q)^{kl} \sum_{\substack{a, b \geq 0 \\ b-a=k-l}} (-q)^{b-k} \text{braiding diagram}$$

In particular, a crossing of two 1-labeled strands is given by:

$$\begin{array}{c} \nearrow \\ \searrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array} - q \begin{array}{c} \nearrow \\ \searrow \end{array}$$

For negative crossings, one uses the above formulas with q inverted. (Setting $q = 1$ recovers the ordinary braiding given by swapping tensor factors.)

LEMMA 2.2. *The following analogs of Reidemeister moves hold in $N\mathbf{Web}_q$, where strands can carry all possible orientations and labels.*

$$q^{-k(N-1)} \begin{array}{c} \circlearrowleft \\ k \end{array} = \begin{array}{c} | \\ k \end{array} = q^{k(N-1)} \begin{array}{c} \circlearrowright \\ k \end{array},$$

$$\begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} | \\ | \\ | \end{array}, \quad \begin{array}{c} \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array}, \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \uparrow \end{array}$$

We will refer to the last relation as a forklide move.

Proof. The Reidemeister II, III and forklide moves follow from the property of a braiding, and our braiding convention is only a minor rescaling of the one in [9, Corollary 6.2.3], see also [31, Section 2.4]. The Reidemeister I moves can be verified inductively as in [25, Lemma 2.9].

DEFINITION 2.3. *We denote by $N\mathbf{Web}^+$ the full subcategory of $N\mathbf{Web}$ with objects given by arbitrary sequences with exclusively upward pointing orientations.*

In the following we will use the same superscript $+$ to indicate analogous full subcategories of other web categories, consisting of those objects with upward (or outward) pointing orientations. The next lemma is a well-known consequence the proof of Theorem 2.1 using quantum skew Howe duality, see [9, Section 4.3 (fullness) and Section 5 (ladder webs)] and analogously in [27, Prop 6.8] and [31].

LEMMA 2.4. *The morphism spaces of $N\mathbf{Web}^+$ are spanned by upward-pointing webs, i.e. webs whose edges admit oriented parametrisations with derivative having a positive vertical component everywhere.*

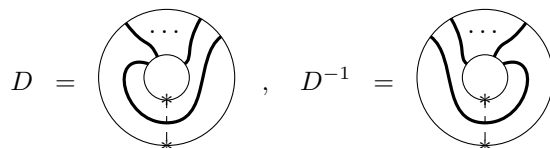
In the following, we will consider *skain modules* of isotopy classes of webs as in Section 2.1, but embedded in different surfaces, modulo the local relations from (1), and we will also vary the ground ring. In the following sections we deal with webs over \mathbb{C} , whose defining relations are obtained from (1) by specializing $q = 1$. We indicate categories of webs at $q = 1$ by the omission of the q -subscript, e.g. $N\mathbf{Web}$ instead of $N\mathbf{Web}_q$. The functor ϕ also specializes to $q = 1$ and then relates $N\mathbf{Web}$ to the symmetric monoidal category of $U(\mathfrak{gl}_N)$ -representations.

Note that setting $q = 1$ identifies the evaluation of positive and negative crossings in terms of webs in (10), and so we sometimes do not display any over- or under-crossing information in graphics. In particular, the braid group action induced by 1-labeled crossings becomes a symmetric group action.

2.3. AFFINIZATION AT $q = 1$. In [28], we considered an affine extension of the Temperley-Lieb category, and extended an analog of the functor ϕ to this more general category. Just as in this simpler \mathfrak{sl}_2 case, we will consider a more general affine web category that will give us the freedom to extend the diagrammatic presentation of the representation category of $U(\mathfrak{gl}_N)$ to a Cartan subalgebra.

We define the category $N\mathbf{AWeb}$ to be the \mathbb{C} -linear category with morphisms spaces spanned by webs properly embedded in the annulus $[0, 1] \times [0, 1] / \{(0, s) \sim (1, s) \mid s \in [0, 1]\}$, subject to the same local relations as in $N\mathbf{Web}$, i.e. relations (1) at $q = 1$. We remember the *seam* $\alpha = \{(0, s) \sim (1, s) \mid s \in [0, 1]\}$, which will be drawn as a dashed line in illustrations. The endpoints of a web are on the two boundary circles $[0, 1] \times \{0\} \cap \{(0, 0)\}$ and $[0, 1] \times \{1\} \cap \{(0, 1)\}$, so that the web can be interpreted as a mapping from the configuration of points on the first (inner) circle to the second (outer) one. We further require of webs that their endpoints are disjoint from the endpoints of the seam.

Just as before, one can compose annular webs by stacking the annuli. It is easy to see that the morphisms of $N\mathbf{AWeb}$ can be generated from those morphisms that are supported in the strip $(0, 1) \times [0, 1]$ (which may be considered as specifying morphisms in $N\mathbf{Web}$), and the additional new *wrapping morphisms*:

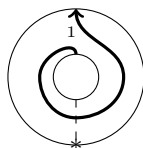


When starting from an abstract annulus rather than our concrete model, a choice of seam endpoints is necessary to make the composition in $N\mathbf{AWeb}$ well-defined and a choice of seam is necessary to distinguish identity morphisms from all other (invertible) wrapping morphisms.

We also stress that in $N\mathbf{AWeb}$ webs can come with any orientation on the boundary.

2.4. LINK WITH REPRESENTATION THEORY. We will extend the domain of the functor ϕ from $N\mathbf{Web}$ to $N\mathbf{AWeb}$ by sending the wrapping morphisms to maps between $U(\mathfrak{gl}_N)$ -representations, which respect the weight space decomposition but break the $U(\mathfrak{gl}_N)$ -action. This will allow us to build new diagrammatic projectors, and we will now explain how to choose this preferred extension.

We first consider a single counterclockwise wrap morphism $D = D_1$ of a single 1-labeled outward pointing strand.



The requirement that $\phi(D)$ respects the weight space decomposition of V implies that $\phi(D)(v_k) = \gamma_k v_k$ for some $\gamma_k \in \mathbb{C}$ and the desired invariance under ambient isotopy forces these scalars to be invertible. In fact, this choice of scalars determines the action of $\phi(D_k)$, the k -labeled version of the wrap. To see this, note that the first relation in (1) allows one to open a digon in the k -edge. Iterating this procedure, one can open a k -blister in the k -edge, i.e. a configuration of nested digons with k -many parallel 1-edges in the innermost part (see the proof of Lemma 2.9 below for an illustration). Sliding one half of the k -blister around the wrap D_k , the eigenvalues of $\phi(D_k)$ can be seen to be k -fold products of the eigenvalues of $\phi(D)$: $\phi(D_k)(v_S) = (\prod_{i \in S} \gamma_i) v_S$. Furthermore, inverse wraps have inverse eigenvalues: $\phi(D_k^{-1})(v_S) = (\prod_{i \in S} \gamma_i)^{-1} v_S$. Next, we would like to have relations of the form:

$$(11) \quad \phi \left(\text{Diagram 1} \right) = \phi \left(\text{Diagram 2} \right)$$

To ensure that ϕ respects such isotopy relations for sliding cups and caps around the annulus, we need to have $\phi(D_k)(v_S^*) = (\prod_{i \in S} \gamma_i)^{-1} v_S^*$ and $\phi(D_k^{-1})(v_S^*) = (\prod_{i \in S} \gamma_i) v_S^*$, which determine the maps assigned to inward pointing versions of D and D^{-1} .

In order to be able to project onto the 1-dimensional spaces spanned by specific standard basis vectors in V , we would like $\phi(D)$ to have distinct eigenvalues on the v_k . Furthermore, we would like to find a set of diagrammatic relations in the annular web category that enforces a choice of $\phi(D)$ with distinct eigenvalues, or in other words, with a separable characteristic polynomial $\prod_{i=1}^N (X - \gamma_i) = \sum_{k=0}^N X^{N-k} (-1)^k e_k(\vec{\gamma})$. Here $e_k(\vec{\gamma}) = e_k(\{\gamma_1, \dots, \gamma_N\})$ denotes the k -th elementary symmetric polynomial evaluated at the complex numbers $\gamma_1, \dots, \gamma_N$.

LEMMA 2.5. *Suppose that ϕ is a functor from $N\mathbf{AWeb}$ to complex vector spaces, which agrees on the subcategory $N\mathbf{Web}$ with the functor from Theorem 2.1 and such that $\phi(D)$ respects the weight space decomposition of V . Then the coefficients of the characteristic polynomial of $\phi(D)$ are determined by the image of ϕ on endomorphisms of the empty object. More precisely:*

$$(12) \quad \phi \left(\text{Diagram 3} \right) = \sum_{|S|=k} \left(\prod_{i \in S} \gamma_i \right) \text{id}_C = e_k(\vec{\gamma})$$

The annular web shown on the left-hand side will be called an essential circle in the annulus.

Proof. The morphism can be written as the composition of a k -cup, a k -wrap and a k -cap. The sum $\sum_{|S|=k}$ comes from the cup and the factors from the action of the wrap on v_S .

We now prescribe $\phi(D): V \rightarrow V$ to have the separable characteristic polynomial $X^N - 1$, and we may index the roots as $\gamma_k = \zeta^k = e^{k2\pi i/N}$. This choice of relation is homogeneous with respect to a Z/NZ -grading by winding number, see Definition 2.15, and it has the effect of setting $e_k(\vec{\gamma}) = 0$ for $1 \leq k < N$ and $e_N(\vec{\gamma}) = (-1)^{N-1}$ in (12). (More general evaluations are studied in [20].) We now extend the definition of ϕ to the new generators D in the general case, that is, allowing more than one strand.

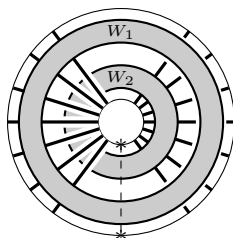
DEFINITION 2.6. *Let $V \otimes W$ be the image under ϕ of the domain of D and $W \otimes V$ its co-domain. Then we define $\phi(D)$ to be the linear map determined by $v_k \otimes w \mapsto \zeta^k w \otimes v_k$ for $v_k \in V$ and any $w \in W$. Furthermore we set $\phi(D^{-1}) = \phi(D)^{-1}$ and analogously for the duals.*

Let \mathfrak{h} denote the Cartan subalgebra of diagonal matrices in \mathfrak{gl}_N , h_1, \dots, h_N its standard basis elements, and consider $U(\mathfrak{h}) \subset U(\mathfrak{gl}_N)$. We denote by $\mathbf{Rep}(\mathfrak{h})$ the category of finite-dimensional $U(\mathfrak{h})$ -representations of integral weights, i.e. where the h_i act semisimply with integral eigenvalues. Note that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{gl}_N$ induces a restriction functor $\mathbf{Rep}(\mathfrak{gl}_N) \rightarrow \mathbf{Rep}(\mathfrak{h})$ and that $\phi(D)$ and $\phi(D^{-1})$ are morphisms in $\mathbf{Rep}(\mathfrak{h})$.

LEMMA 2.7. *The functor $\phi: \mathbf{NAWeb} \rightarrow \mathbf{Rep}(\mathfrak{h})$ is well-defined.*

Proof. We will show that ϕ respects all relations defining the morphism spaces of \mathbf{NAWeb} . All morphisms in \mathbf{NAWeb} are linear combinations of compositions of caps or cups between adjacent strands, vertices, as well as the morphisms D and D^{-1} , considered modulo local linear relations and isotopy. As usual in skein theory, it suffices to impose the local linear relations in a fixed disk, e.g. away from the seam α . These relations are respected by ϕ because we extended ϕ from $\mathbf{NWeb} \subset \mathbf{NAWeb}$. It thus remains to check that ϕ respects the isotopy relations of sliding caps, cups and vertices along wraps around the annulus. However, the images under ϕ of the wrap morphisms were precisely chosen for these relations to hold (see Equation (11)).

2.5. THE TENSOR PRODUCT ON ANNULAR WEBS. Let $\otimes: \mathbf{NAWeb} \times \mathbf{NAWeb} \rightarrow \mathbf{NAWeb}$ denote the bi-functor given on objects by concatenation and on morphisms by superimposing a pair of annular webs (W_1, W_2) as follows:



and resolving all crossings via (10). In words, we stack the annular web W_1 on top of the annular web W_2 , making sure that the “legs” of W_1 meet the inner (source) boundary of the annulus to the left of the legs of W_2 . Similarly, the “arms” of W_1 meet the outside (target) boundary of the annulus to the left of all arms of W_2 . This operation creates new crossings, as the legs of W_1 might cross strings from W_2 , and arms of W_2 might cross strings from W_1 . The operation is well-defined thanks to Lemma 2.2. In the above picture we have drawn W_1 stacked on top of W_2 , with strands leaving the former crossing over the latter. However, since we are working at $q = 1$, over- and under-crossings are equal and using isotopy relations it is straightforward to check that \otimes together with the symmetric braiding from \mathbf{NWeb} yield a symmetric monoidal structure on \mathbf{NAWeb} such that both the inclusion $\mathbf{NWeb} \rightarrow \mathbf{NAWeb}$ and $\phi: \mathbf{NAWeb} \rightarrow \mathbf{Rep}(\mathfrak{h})$ become symmetric monoidal functors. This also justifies why we have described the action \otimes to be “superimposing” annular webs.

DEFINITION 2.8. *For $m > 0$ we denote by $\mathbf{NAWeb}(m)$ the endomorphism algebra in \mathbf{NAWeb} of the object consisting of a sequence of m points with label 1 and outward orientation. We denote by s_i for $1 \leq i \leq m$ the element of this endomorphism algebra that is given by the crossing between the strands in positions i and $i + 1$, with positions understood modulo m . We also write $u_i = \text{id}_2 - s_i$ for the corresponding dumbbell web.*

More generally, for two objects \vec{k} and \vec{l} we use the shorthand $\mathbf{NAWeb}(\vec{k}, \vec{l}) := \text{Hom}_{\mathbf{NAWeb}}(\vec{k}, \vec{l})$. We will also use these notation conventions for other categories.

The following lemma will allow us to freely express webs in terms of images of 1-labeled tangles, which will be very useful in a number of proofs.

LEMMA 2.9. *Every element of $\mathbf{NAWeb}(m)$ for $m > 0$ can be written as a \mathbb{C} -linear combination of 1-labeled annular tangles.*

As we will see later, we may assume that the closed components are essential circles (possibly carrying higher labels) and the non-closed components are oriented 1-labeled arcs from the inner to the outer boundary circle, that are everywhere outward pointing.

Proof. We include this classical proof for completeness. It is well-known that every closed web can be written as a \mathbb{C} -linear combination of webs W with only 1-labeled edges, interacting at most in 2-labeled dumbbells, see e.g. [31, Proof of Lemma 4.1]. Indeed, the argument can be inductively built from the following two operations:

$$\begin{array}{c} | \\ \dots \\ | \end{array} \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \begin{array}{c} \dots \\ \dots \\ \dots \end{array} = \frac{1}{k!} \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \quad \text{and} \quad \begin{array}{c} \overbrace{\dots}^k \\ \dots \\ \underbrace{\dots}_k \end{array} = (-1)^k \begin{array}{c} \overbrace{\dots}^{k-1} \\ \dots \\ \underbrace{\dots}_{k-1} \end{array} + (-1)^{k+1} \begin{array}{c} \overbrace{\dots}^{k-1} \\ \dots \\ \underbrace{\dots}_{k-1} \end{array}$$

The remaining 2-labeled dumbbells can now be expanded in terms of crossings and their oriented resolutions, resulting in a linear combination of 1-labeled annular tangles.

An analogous result is true at generic q over $\mathbb{C}(q)$.

LEMMA 2.10. *The endomorphism algebra of the empty object in $N\mathbf{AWeb}$ is isomorphic to $\mathbb{C}[c_1, \dots, c_{N-1}, c_N^{\pm 1}]$, where c_i denotes the counter-clockwise oriented i -labeled essential circle in the annulus and c_N^{-1} the N -labeled clockwise oriented essential circle.*

Proof. The proof proceeds in two steps. First we show that any closed web in the annulus can be written as a \mathbb{C} -linear combination of collections of essential circles from the set $\{c_1, c_2, \dots, c_N, c_N^{-1}\}$. The essential circles commute, which is easily seen by applying Reidemeister II moves, and c_N and c_N^{-1} are indeed mutually inverse by (2). Second, we check that the counter-clockwise oriented circles are algebraically independent.

By Lemma 2.9 any closed web may be expressed as a linear combination of 1-labeled annular links. Recall that we are working at $q = 1$, so crossings signs are irrelevant and the components of the link can be pulled apart into a disjoint union. Moreover, each of the components can be assumed to wrap exclusively clockwise or counter-clockwise around the annulus. (By this we mean that upon choosing an orientation of the seam α , all intersection points of α with the link component have the same sign.) The annular evaluation algorithm from [26, Lemma 5.2] implies that any such web can be resolved in terms of concentric essential circles of various labels. Hence, the same is true for the original web. To finish the first part, note that the clockwise-oriented circle of label i is equal to $c_{N-i}c_N^{-1}$. This follows from the last relation in (2).

In order to prove algebraic independence, we use an extension of the functor ϕ from Lemma 2.7. Let $R = \mathbb{C}[X^{\pm 1}] = \mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]$ be a Laurent polynomial ring in N variables and consider the category $R\mathbf{Rep}(\mathfrak{h})$ obtained by tensoring all morphism spaces in $\mathbf{Rep}(\mathfrak{h})$ by R . Then $\phi_R: N\mathbf{AWeb} \rightarrow R\mathbf{Rep}(\mathfrak{h})$ can be defined just as ϕ was above, except that the X_i now play the role of the eigenvalues of the 1-labeled counter-clockwise wrap: $\phi_R(D)(v_k) := X_k v_k$.

For a closed annular web W , it follows that $\phi(W) \in R$. In particular, counter-clockwise oriented circles c_i evaluate to elementary symmetric polynomials $e_i(X)$ in the variables X_k and their clockwise counterparts evaluate to $e_i(X^{-1})$. As a consequence of the first part of this proof, the evaluation of closed annular webs $\phi_R(W)$ takes values in the symmetric part $R^{\mathfrak{S}_N} \cong \mathbb{C}[e_1(X), \dots, e_{N-1}(X), e_N(X)^{\pm 1}]$ of R . Now the

algebraic independence of the c_i follows from the algebraic independence of their images $e_i(X)$ under ϕ_R .

An analogous version of this result holds for $N\mathbf{AWeb}_q$ over $\mathbb{C}(q)$. The second part in its $q = 1$ version is already sufficient to establish algebraic independence of the counter-clockwise essential circles in $N\mathbf{AWeb}_q$.

Just as in the non-annular case, we denote by $N\mathbf{AWeb}^+$ the full subcategory of $N\mathbf{AWeb}$ with objects given by all upward, or *outward*, pointing boundary sequences. The following is the analog of Lemma 2.4.

LEMMA 2.11. *The morphism spaces of $N\mathbf{AWeb}^+$ are spanned by webs with all edges outward oriented (i.e. admitting an oriented parametrisation with derivative having a positive radial component everywhere), but potentially superimposed with essential circles.*

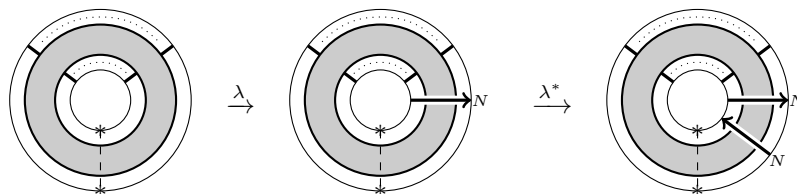
Note that Lemma 2.10 allows us to restrict to counter-clockwise essential circles, except for the N -labeled ones.

Proof. We will prove the claim for webs W whose source and target objects are only 1-labeled. The general claim follows by a usual merging argument.

Lemma 2.9 allows us to write W as a linear combination of annular tangles. The closed components of these tangles evaluate to essential circles thanks to Lemma 2.10, while the non-closed components can be isotoped to be outward oriented arcs. (This shortcut is possible since we work at $q = 1$.) The superposition of such outward arcs is itself a linear combination of outward webs, as desired.

2.6. EQUIVALENCES BETWEEN BLOCKS. The categories $N\mathbf{Web}$, $N\mathbf{AWeb}$ and all of their specializations and quotients considered in the following decompose into blocks (i.e. full subcategories) indexed by $m \in \mathbb{Z}$, which consist of those objects, whose signed sum of labels equals m . Here we count upward oriented boundary points positively, and downward pointing ones negatively. We indicate such blocks by the subscript m , e.g. $N\mathbf{Web}_m$.

We will use the notation λ for the endofunctors of these categories that act on objects by tensoring with a single N -labeled upward boundary point on the right and on morphisms by superimposing with an upward-oriented N -labeled edge parallel to the seam and with boundary points to the right of all other boundary points. We denote by λ^* the analogous operation with downward orientations. We give an example for $N\mathbf{AWeb}$:

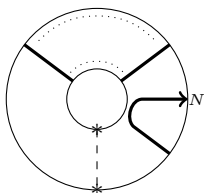


Here we display the N -labeled strand as crossing over the remaining web for better visibility, even though this has no significance at $q = 1$.

LEMMA 2.12. *The endofunctors λ and λ^* establish an equivalence between the blocks $N\mathbf{AWeb}_m$ and $N\mathbf{AWeb}_{m+N}$.*

Proof. Using isotopy relations, it is easy to see that the following types of webs provide natural isomorphisms between the identity functor on $N\mathbf{AWeb}_m$ and the endofunctor

given by the composition $\lambda^* \circ \lambda$:



Analogously, there are natural transformations between $\lambda \circ \lambda^*$ and the identity functor on \mathbf{NAWeb}_m .

2.7. THE QUOTIENTS BY ESSENTIAL CIRCLES. It is a key observation that the functor $\phi: \mathbf{NAWeb} \rightarrow \mathbf{Rep}(\mathfrak{h})$ is not faithful.

PROPOSITION 2.13. We have the following identities in $\mathbf{Rep}(\mathfrak{h})$:

$$(13) \quad \phi \left(\begin{array}{c} \text{outer circle with 4 dots} \\ \text{inner circle with 3 dots} \\ \text{strand labeled } k \text{ crossing inner circle} \\ \text{strand labeled } N \text{ crossing outer circle} \\ \text{dashed line with asterisks} \end{array} \right) = \phi \left(\begin{array}{c} \text{outer circle with 4 dots} \\ \text{inner circle with 3 dots} \\ \text{strand labeled } k \text{ crossing inner circle} \\ \text{strand labeled } N \text{ crossing outer circle} \\ \text{dashed line with asterisks} \end{array} \right) = \begin{cases} (-1)^{N-1} \text{id} & \text{if } k = N \\ 0 & \text{otherwise} \end{cases} .$$

Proof. First consider the case where there are no strands mapping between the inner and outer circles. In this case, the result follows by specializing (12) at $\gamma_l = e^{k2\pi i/N}$.

In the case where there are strands crossing the essential k -labeled circle, the result follows similarly: a crossing is sent under ϕ to the transposition $u \otimes v \rightarrow v \otimes u$. Then the computation in the proof of Lemma 2.5 can be performed even in the presence of extra strands, yielding the same scalar.

DEFINITION 2.14. We let $\overline{\mathbf{NAWeb}}^{\text{ess}}$ denote the quotient of \mathbf{NAWeb} by the tensor ideal generated by the k -labeled essential circles c_k for $k < N$. We define $\mathbf{NAWeb}^{\text{ess}}$ to be quotient of \mathbf{NAWeb} by the tensor ideal generated by the c_k for $k < N$ as well as $c_N + (-1)^N$.

Note that the monoidal structure \otimes and the functor ϕ descend to the quotients $\overline{\mathbf{NAWeb}}^{\text{ess}}$ and $\mathbf{NAWeb}^{\text{ess}}$. One of the key results of this paper will be to prove that this category $\mathbf{NAWeb}^{\text{ess}}$ is equivalent to the full subcategory of $\mathbf{Rep}(\mathfrak{h})$ generated by the images of the objects of $\mathbf{Rep}(\mathfrak{gl}_N)$.

Upon reinterpreting linear categories as locally unital algebras, $\overline{\mathbf{NAWeb}}^{\text{ess}}$ can be thought of as a central extension of $\mathbf{NAWeb}^{\text{ess}}$. For every object there exists an automorphism, always denoted c_N , given by the N -labeled counter-clockwise oriented essential circle superimposed onto the identity morphism of the object, which “commutes” with all morphisms. Then $\mathbf{NAWeb}^{\text{ess}}$ is obtained from $\overline{\mathbf{NAWeb}}^{\text{ess}}$ by setting $c_N = (-1)^{N-1}$.

DEFINITION 2.15. Let W be a web in \mathbf{NAWeb} . Then the flow winding number $w_f(W)$ of W is given by the algebraic intersection number of the web with the segment α (assuming no trivalent vertex occurs on it), where k -labeled edges crossing α count as $\pm k$.

It is clear that all web relations in \mathbf{NAWeb} and $\overline{\mathbf{NAWeb}}^{\text{ess}}$ preserve the flow winding number. This \mathbb{Z} -grading on the morphism spaces of these categories descends to a $\mathbb{Z}/N\mathbb{Z}$ grading on $\mathbf{NAWeb}^{\text{ess}}$.

The following corollaries are implied by Lemma 2.10 and Lemma 2.11.

COROLLARY 2.16. *The endomorphism algebra of the empty object in $\mathbf{NAWeb}^{\text{ess}}$ is \mathbb{C} and in $\overline{\mathbf{NAWeb}}^{\text{ess}}$ it is isomorphic to the Laurent polynomial ring over \mathbb{C} generated by an essential N -labeled circle.*

COROLLARY 2.17. *The morphism spaces of $\mathbf{NAWeb}^{\text{ess},+}$ are spanned by outward pointing webs. The same is true in $\overline{\mathbf{NAWeb}}^{\text{ess},+}$ up to superposition with an integer power of the N -labeled essential circle c_N .*

LEMMA 2.18. *In $\overline{\mathbf{NAWeb}}^{\text{ess}}(1)$ we have $D^N = (-1)^{N-1}c_N$, where we abuse notation by writing c_N for the superposition of the identity on 1 with the counterclockwise N -labeled essential circle. In $\mathbf{NAWeb}^{\text{ess}}(1)$ this specializes to $D^N = \text{id}_1$.*

Proof. We compute:

$$\begin{aligned}
 (-1)^{N-1}c_N &= (-1)^{N-1} \left(\text{Diagram 1} \right) = \left(\text{Diagram 2} \right) = \left(\text{Diagram 3} \right) \circ D \\
 &= (-1)^{N-1} \left(\text{Diagram 4} \right) \circ D + \left(\text{Diagram 5} \right) \circ D = \left(\text{Diagram 6} \right) \circ D^2 \\
 &= \dots = - \left(\text{Diagram 7} \right) \circ D^{N-1} + D^N = D^N
 \end{aligned}$$

The second equality arises by resolving the crossing while the third is an isotopy. The equalities then alternate between such which hold by expanding a crossing and such that use isotopies and (13).

REMARK 2.19. An analogous argument shows $\sum_{i=0}^N D^{N-i}(-1)^i c_i = 0$ in \mathbf{NAWeb} , c.f. [8, Section 8.2].

Consider the algebra $\mathbb{C}[D^{\pm 1}]/\langle D^N - 1 \rangle$. Lemma 2.18 implies that this surjects onto the subalgebra of $\mathbf{NAWeb}^{\text{ess}}(1)$ generated by the wrapping morphisms D and D^{-1} , and the flow winding grading implies that the surjection is an isomorphism. The Chinese remainder theorem implies $\mathbb{C}[D^{\pm 1}]/\langle D^N - 1 \rangle \cong \bigoplus_{j=1}^N \mathbb{C}[D]/\langle D - e^{j2\pi i/N} \rangle$.

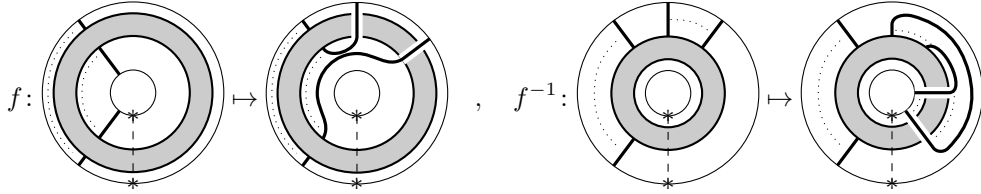
DEFINITION 2.20. For $1 \leq k \leq N$ we denote by $P_k \in \mathbb{C}[D]$ a chosen representative for the idempotent that projects onto the direct summand $\mathbb{C}[D]/\langle D - e^{k2\pi i/N} \rangle$ of $\mathbb{C}[D^{\pm 1}]/\langle D^N - 1 \rangle$.

By abuse of notation we also write P_k for the corresponding orthogonal idempotents in $\mathbf{NAWeb}^{\text{ess}}(1)$. It is a straightforward but crucial observation that $\phi(P_k(D))$ is the projection $V \rightarrow \mathbb{C}\langle v_k \rangle \hookrightarrow V$.

THEOREM 2.21. *The functor $\phi: \mathbf{NAWeb} \rightarrow \mathbf{Rep}(\mathfrak{h})$ is full.*

Proof. We show that the induced functor on the quotient $\mathbf{NAWeb}^{\text{ess}}$ is full. For this, let \vec{k} and \vec{l} be objects in $\mathbf{NAWeb}^{\text{ess}}$ and \vec{k}^* be the dual of \vec{k} , which is obtained from \vec{k} by inverting orientations and the order of the sequence. We consider the duality

isomorphism $f: \mathbf{NAWeb}^{\text{ess}}(\vec{k}, \vec{l}) \rightarrow \mathbf{NAWeb}^{\text{ess}}(\vec{?}, \vec{l} \otimes \vec{k})$ and its inverse, which can be explicitly described as the following operations on diagrams.



Since these operations are given by tensoring with an identity morphism and then pre-composing with cups, or post-composing with caps, there are corresponding isomorphisms $\phi(f^{\pm 1})$ between the relevant morphism spaces in the target category $\mathbf{Rep}(\mathfrak{h})$.

To prove the theorem, it now suffices to check that ϕ restricts to a surjective map from the morphism space $\mathbf{NAWeb}^{\text{ess}}(\vec{?}, \vec{l} \otimes \vec{k})$ to $\mathbf{Rep}(\mathfrak{h})(\mathbb{C}, \phi(\vec{l} \otimes \vec{k}))$, the latter of which is isomorphic to the zero weight space in $\phi(\vec{l} \otimes \vec{k})$ via the map that evaluates elements of the morphism space at $1 \in \mathbb{C}$.

Next we claim that it is enough to prove surjectivity in the case where $\vec{l} \otimes \vec{k}$ consists entirely of entries 1, the first n of which point outward and the last n inward. Indeed, if all weight zero vectors in this tensor product are in the image (upon evaluation at $1 \in \mathbb{C}$), then composing with the images of merge webs and the symmetric braiding, one may find any weight zero vector of a tensor product of fundamentals and their duals in the respective image.

Actually, it is sufficient to find for every $1 \leq k \leq N$ a morphism $L_k \in \mathbf{NAWeb}^{\text{ess}}(\vec{?}, (1, 1^*))$, such $v_k \otimes v_k^* = \phi(L_k)(1)$, as then we can take diagrammatic tensor products of such morphisms L_{k_i} , composed with permutations, to find any standard basis vector of weight zero in the image. To show that $v_k \otimes v_k^*$ is in the image, we invert the bending process via the isomorphisms $f^{\pm 1}$ and $\phi(f^{\pm 1})$, and the problem becomes equivalent to finding the projection $V \rightarrow \mathbb{C}\langle v_k \rangle \hookrightarrow V$ in the image of ϕ . But this we have already seen; it is given by $\phi(P_k)$.

Later, we will prove that ϕ induces a functor from $\mathbf{NAWeb}^{\text{ess}}$ to $\mathbf{Rep}(\mathfrak{h})$ that is not only full, but also faithful (see Theorem 2.35). Nevertheless, we will continue to work in the more general framework of $\overline{\mathbf{NAWeb}}^{\text{ess}}$ whenever possible.

2.8. EXTREMAL WEIGHT PROJECTORS. In [28], we defined the concept of extremal weight projectors in the context of (affine) \mathfrak{sl}_2 skein theory. This involved finding a suitable quotient of the affine Temperley-Lieb category, in which we identified a family of idempotents akin to Jones-Wenzl projectors and corresponding, on the representation-theoretic side, to projections onto the direct sum of the top and bottom weight spaces in the tensor powers of the vector representation of $U(\mathfrak{sl}_2)$. The same question naturally extends beyond the \mathfrak{sl}_2 case, and the definition can be adapted and generalized to the \mathfrak{gl}_N case as follows.

DEFINITION 2.22. *The elements $T_m \in \overline{\mathbf{NAWeb}}^{\text{ess}}(m)$ are recursively defined via:*

- $T_1 = \text{id}_1$,
- $T_2 = \frac{1}{N} \sum_{k=0}^{N-1} D^{-k} \otimes D^k$,
- $T_{m+1} = (\text{id}_{m-1} \otimes T_2)(T_m \otimes \text{id}_1)$ for $m > 2$.

In the following we abbreviate the notation for standard basis elements of V^m to $v_{a_1 a_2 \dots a_m} := v_{a_1} \otimes v_{a_2} \otimes \dots \otimes v_{a_m}$.

THEOREM 2.23. *The element $\phi(T_m)$ is the endomorphism of $V^{\otimes m}$ projecting onto the sum of extremal weight spaces $\mathbb{C}\langle v_{i_1 \dots i_m} \mid i \in \{1, \dots, N\} \rangle$ in $\text{Sym}^m(V) \subset V^{\otimes m}$.*

Proof. For $m = 1$ this is tautological. For $m = 2$ we compute $\phi(D^{-1} \otimes \text{id}_1)(v_{ij}) = \zeta^{-i}v_{ij}$ and $\phi(\text{id}_1 \otimes D)(v_{ij}) = \zeta^j v_{ij}$. Thus we have:

$$\phi(T_2)(v_{ij}) = \frac{1}{N} \sum_{k=0}^{N-1} \zeta^{k(j-i)} v_{ij} = \begin{cases} v_{ii} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

For the vanishing, recall that $0 = (X^N - 1) = (X - 1)(1 + X + \dots + X^{N-1})$, so if $X^N = 1$ but $X \neq 1$, then X is a zero of the cyclotomic polynomial. In particular, this holds for $X = \zeta^{(j-i)}$ when $i \neq j$.

For the induction step, we see immediately from the recursion that $\phi(T_{m+1})$ annihilates $v_{\epsilon_1 \epsilon_2 \dots \epsilon_{m+1}}$ unless $\epsilon_1 = \dots = \epsilon_m =: \epsilon$. In the remaining cases we have:

$$\phi(\text{id}_{m-1} \otimes T_2)\phi(T_m \otimes \text{id}_1)(v_{\epsilon \dots \epsilon k}) = \phi(\text{id}_{m-1} \otimes T_2)(v_{\epsilon \dots \epsilon k}) = \begin{cases} v_{\epsilon \dots \epsilon \epsilon} & \text{if } k = \epsilon \\ 0 & \text{if } k \neq \epsilon \end{cases}$$

So $\phi(T_{m+1})$ is the extremal weight projector.

In the following we show that the T_m are idempotents that satisfy a number of properties analogous to the extremal weight projectors $\phi(T_m)$. This will lead to a proof that ϕ is indeed faithful on $\text{NAWeb}^{\text{ess}}$. We start by studying properties of T_2 .

LEMMA 2.24. *The endomorphism T_2 is an idempotent in $\overline{\text{NAWeb}}^{\text{ess}}$ and thus also in the quotient $\text{NAWeb}^{\text{ess}}$.*

Proof. We compute:

$$T_2^2 = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} D^{-k-l} \otimes D^{k+l} = \frac{1}{N} \sum_{l=0}^{N-1} D^{-l} \otimes D^l = T_2$$

We are using that $D^{-N-i} \otimes D^{N+i} = (-1)^{2N-2} c_N^{-1} c_N (D^{-i} \otimes D^i) = D^{-i} \otimes D^i$.

LEMMA 2.25. *In $\overline{\text{NAWeb}}^{\text{ess}}$ (3) we have that $\text{id}_1 \otimes T_2$ and $T_2 \otimes \text{id}_1$ commute.*

Proof. $(\text{id}_1 \otimes T_2)(T_2 \otimes \text{id}_1) = \frac{1}{N^2} \sum_{k,l=0}^{N-1} D^{-k} \otimes D^{k-l} \otimes D^l = (\text{id}_1 \otimes T_2)(T_2 \otimes \text{id}_1)$

The following lemma states a relation that is trivially satisfied on the representation-theoretic side, i.e. after applying ϕ , but which is non-obvious in $\overline{\text{NAWeb}}^{\text{ess}}$ and $\text{NAWeb}^{\text{ess}}$. For the latter, the result can be deduced from [8, Equation (37)].

LEMMA 2.26. *The idempotent T_2 absorbs the crossing $s = s_1$ between its two strands. More precisely $sT_2 = T_2s = T_2$ in $\overline{\text{NAWeb}}^{\text{ess}}$ (2) and thus also in $\text{NAWeb}^{\text{ess}}$ (2).*

The diagrammatic proof is involved and we postpone it until Section 2.10. The reason for the result to hold after applying ϕ is because $\phi(T_2)$ projects onto the span of the vectors $v_{i,i}$ (see Theorem 2.23), and a crossing acts as the identity on this subspace.

LEMMA 2.27. *In $\overline{\text{NAWeb}}^{\text{ess}}$ (2) we have $D^{-1}T_2D = T_2$.*

Proof. For this we rewrite T_2 in terms of $D^{-1} \otimes \text{id}_1 = D^{-1}s$ and $\text{id}_1 \otimes D = Ds$. Using Lemma 2.26 and $(D^{-1}s)^N (Ds)^N = D^{-N} \otimes D^N = 1$, we compute:

$$D^{-1}T_2D = D^{-1}sT_2D = \frac{1}{N} \sum_{k=0}^{N-1} D^{-1}s(D^{-1}s)^k (Ds)^k Ds = T_2s = T_2.$$

We can now state and prove the following theorem, establishing that the T_m 's are indeed idempotents, and that they satisfy desirable properties with respect to crossings, turnbacks, and wrapping morphisms.

THEOREM 2.28. *The elements T_m of $\overline{N\mathbf{AWeb}}^{\text{ess}}$ satisfy the following properties:*

- (1) $T_m^2 = T_m$;
- (2) $T_m(\text{id}_k \otimes T_n \otimes \text{id}_{m-n-k}) = (\text{id}_k \otimes T_n \otimes \text{id}_{m-n-k})T_m = T_m$ for $1 \leq n < m$ and $0 \leq k \leq m - n$;
- (3) $(T_k \otimes \text{id}_{m-k})(\text{id}_{m-l} \otimes T_l) = (\text{id}_{m-l} \otimes T_l)(T_k \otimes \text{id}_{m-k}) = T_m$ for $k + l > m$;
- (4) $T_m s_i = s_i T_m = T_m$ for $m > 2$;
- (5) $T_m u_i = u_i T_m = 0$ for $m > 2$;
- (6) $D^{-1}T_m D = T_m$.

Here, s_i and u_i again refer to crossings and dumbbell webs between the strands in position i and $i + 1$, see Definition 2.8.

Proof. We have already checked in Lemma 2.24 that T_2 is idempotent. From the definition, it is clear that T_m is a product of $m - 1$ distinct factors of the form $\text{id}_k \otimes T_2 \otimes \text{id}_{m-2-k}$ for $0 \leq k \leq m - 2$. Lemma 2.25 implies that these factors commute and so T_m is an idempotent (1) that absorbs smaller T_n , i.e. (2). It is also clear that overlapping projectors T_l and T_k combine as in (3). The crossing absorption property of T_2 now implies the one for T_m and crossings s_i for $1 \leq i \leq m - 1$.

Using crossing absorption, we obtain the rotation conjugation invariance (6) from the T_2 case:

$$\begin{aligned} D^{-1}T_m D &= D^{-1}(T_{m-1} \otimes \text{id}_1)(\text{id}_{m-2} \otimes T_2)(T_{m-1} \otimes \text{id}_1)D \\ &= (\text{id}_1 \otimes T_{m-1})D^{-1}s_{m-1} \cdots s_2 s_1 (T_2 \otimes \text{id}_{m-2})s_1 s_2 \cdots s_{m-1} D(\text{id}_1 \otimes T_{m-1}) \\ &= (\text{id}_1 \otimes T_{m-1})((D^{-1}T_2 D) \otimes \text{id}_{m-2})(\text{id}_1 \otimes T_{m-1}) \\ &= (\text{id}_1 \otimes T_{m-1})(T_2 \otimes \text{id}_{m-2})(\text{id}_1 \otimes T_{m-1}) = T_m \end{aligned}$$

This implies the missing crossing absorption relation (4)

$$T_m s_m = T_m D^{-1}s_{m-1} D = D^{-1}T_m s_{m-1} D = D^{-1}T_m D = T_m.$$

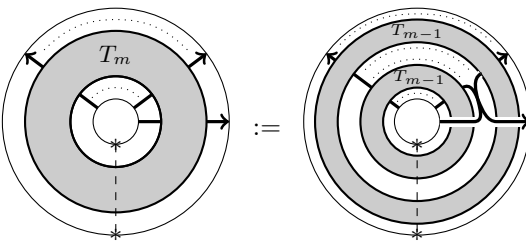
Finally, the u_i annihilation property (5) is equivalent to s_i crossing absorption.

Now we can give an alternative recursion relation for T_m for $m > 3$. This is the direct generalization of the defining recursive relation in [28, Definition 15], and it is reminiscent of the Jones-Wenzl projectors.

COROLLARY 2.29. *The idempotents T_m satisfy the following recursion for $m > 3$:*

$$T_m = (T_{m-1} \otimes \text{id}_1)s_{m-1}(T_{m-1} \otimes \text{id}_1)$$

Graphically, we write this as:

(14) 

Proof. We check this identity as follows.

$$\begin{aligned} (T_{m-1} \otimes \text{id}_1)s_{m-1}(T_{m-1} \otimes \text{id}_1) &= (T_{m-1} \otimes \text{id}_1)s_{m-2} \cdots s_2 s_1 (\text{id}_1 \otimes T_{m-1})s_1 s_2 \cdots s_{m-1} \\ &= (T_{m-1} \otimes \text{id}_1)(\text{id}_1 \otimes T_{m-1})s_1 s_2 \cdots s_{m-1} \\ &= T_m s_1 s_2 \cdots s_{m-1} = T_m \end{aligned}$$

The first equation holds by isotopy, the third by item (3) of Theorem 2.28, and the others follows from crossing absorption.

LEMMA 2.30. *Let $m, n \in \mathbb{N}$ with $m + n > 3$, then $(T_m \otimes T_n)s_m(T_m \otimes T_n) = T_{m+n}$. This means, crossing-connected projectors can be combined.*

Proof. We may assume that $m > 2$ and compute:

$$\begin{aligned} & (\text{id}_m \otimes T_n)(T_m \otimes \text{id}_n)s_m(T_m \otimes \text{id}_n)(\text{id}_m \otimes T_n) \\ &= (\text{id}_m \otimes T_n)(T_{m+1} \otimes \text{id}_{n-1})(\text{id}_m \otimes T_n) \\ &= T_{m+n} \end{aligned}$$

Here we have used the projector recursion (14) and the fact that overlapping projectors can be combined, i.e. (3) in Theorem 2.28.

Next we consider the images of the idempotents T_m in the quotient category $N\mathbf{AWeb}^{\text{ess}}$. Recall the morphisms $\{P_a\}_{a \in \{1, \dots, N\}}$ that were introduced just before Theorem 2.21 as diagrammatic versions of projectors on eigenspaces. We will see in Lemma 2.34 that they can be combined to give an alternate definition of the extremal weight projectors, which amounts to saying that in the quotient category $N\mathbf{AWeb}^{\text{ess}}$, the sum of extremal weight spaces can be broken into individual weight spaces.

LEMMA 2.31. *In $N\mathbf{AWeb}^{\text{ess}}$ we have $(P_a \otimes P_b) \circ T_2 = T_2 \circ (P_a \otimes P_b) = \delta_{a,b}(P_a \otimes P_b)$.*

Proof. The \mathbb{C} -algebra $R := \mathbb{C}[X^{\pm 1}, Y^{\pm 1}] / \langle X^N - 1, Y^N - 1 \rangle$ surjects onto the subalgebra of $N\mathbf{AWeb}^{\text{ess}}(2)$ generated by wraps and their inverses via the map $1 \mapsto \text{id}_2$, $X \mapsto D \otimes \text{id}_1 = sD$ and $Y \mapsto \text{id}_1 \otimes D = Ds$. We will check the desired equalities in R , where T_2 is represented by $\frac{1}{N} \sum_{k=0}^{N-1} X^{-k} Y^k$ and $P_a \otimes P_b$ is represented by $P_a(X)P_b(Y)$, which then implies that these equalities also hold in $N\mathbf{AWeb}^{\text{ess}}(2)$.

Note that the idempotents $P_a(X)P_b(Y)$ decompose $R \cong \bigoplus_{a,b} P_a(X)P_b(Y)R$ into 1-dimensional summands, which precisely consist of simultaneous eigenvectors for multiplication by X and Y with eigenvalues ζ^a and ζ^b respectively. Thus we can compute the action of T_2 on such an idempotent as:

$$\begin{aligned} P_a(X)P_b(Y)T_2 &= P_a(X)P_b(Y) \frac{1}{N} \sum_{k=0}^{N-1} X^{-k} Y^k = P_a(X)P_b(Y) \frac{1}{N} \sum_{k=0}^{N-1} \zeta^{k(b-a)} \\ &= \delta_{a,b} P_a(X)P_b(Y). \end{aligned}$$

COROLLARY 2.32. *In $N\mathbf{AWeb}^{\text{ess}}$ we have $T_2 = \sum_{k=1}^N P_k \otimes P_k$.*

COROLLARY 2.33. *In $N\mathbf{AWeb}^{\text{ess}}$ we have $s(P_k \otimes P_k) = (P_k \otimes P_k)s = P_k \otimes P_k$.*

Proof. We only consider composing with s on the left:

$$s(P_k \otimes P_k) = sT_2(P_k \otimes P_k) = T_2(P_k \otimes P_k) = P_k \otimes P_k$$

Here we have used Lemma 2.31 twice and Lemma 2.26 in between.

LEMMA 2.34. *In $N\mathbf{AWeb}^{\text{ess}}$ we have $T_m = \sum_{k=1}^N P_k^{\otimes m}$.*

Proof. We have observed this for $m = 2$ in Corollary 2.32. For $m = 1$ this follows from the decomposition $1 = \sum_{k=1}^N P_k$ in $\mathbb{C}[D] / \langle D^N - 1 \rangle$. Also, it is not hard to see that the P_k 's slide through crossings, e.g. $s \circ (P_k \otimes \text{id}_1) = (\text{id}_1 \otimes P_k)s$ in $N\mathbf{AWeb}^{\text{ess}}(2)$.

Now we proceed inductively for $m > 2$:

$$\begin{aligned} T_{m+1} &= (T_m \otimes \text{id}_1) s_m(T_m \otimes \text{id}_1) \\ &= \sum_{k=1}^N \sum_{l=1}^N (P_k \otimes \cdots \otimes P_k \otimes \text{id}_1) s_m(P_l \otimes \cdots \otimes P_l \otimes \text{id}_1) \\ &= \sum_{k=1}^N (\text{id}_{m-1} \otimes P_k \otimes \text{id}_1) s_m(P_k \otimes \cdots \otimes P_k \otimes \text{id}_1) \\ &= \sum_{k=1}^N s_m(P_k \otimes \cdots \otimes P_k \otimes P_k) = \sum_{k=1}^N P_k^{\otimes m} \end{aligned}$$

Here we have used the orthogonality of the idempotents P_k to proceed to the second line and the sliding property to proceed to the third line. The final crossing absorption follows from Corollary 2.33.

2.9. FAITHFULNESS OF THE DIAGRAMMATIC PRESENTATION. We will now combine the previous results to prove the following theorem.

THEOREM 2.35. *The functor $\phi : \mathbf{NAWeb}^{\text{ess}} \mapsto \mathbf{Rep}(\mathfrak{h})$ is faithful.*

We partition the proof of the theorem into three parts.

PROPOSITION 2.36. *The restriction of ϕ to the endomorphism algebra $\mathbf{NAWeb}^{\text{ess}}(n)$ is injective.*

Proof. To see this result, we will exhibit a spanning set in $\mathbf{NAWeb}^{\text{ess}}(n)$ that is sent under ϕ to a linear basis. Consider $\epsilon, \epsilon' \in \{1, \dots, N\}^n$ so that $|\{i, \epsilon_i = k\}| = |\{i, \epsilon'_i = k\}|$ for all $k = 1, \dots, N$. Choose $\sigma_\epsilon \in \mathfrak{S}_n$ to be the smallest in length so that $\epsilon'_{\sigma(i)} = \epsilon_i$. For example, it can be inductively defined by assigning to 1 the smallest r so that $\epsilon'_r = \epsilon_1$, etc. Recall the notation P_k for the polynomial such that $P_k(D) \in \mathbf{NAWeb}^{\text{ess}}(1)$ is the projector onto the ζ^k eigenspace of D . In $\mathbf{NAWeb}^{\text{ess}}(n)$, denote $w_i = \text{id}_{i-1} \otimes D \otimes \text{id}_{n-i}$ the complete wrap of the i -th strand. We define:

$$\phi_\epsilon^{\epsilon'} := \sigma P_{\epsilon_n}(w_n) \cdots P_{\epsilon_1}(w_1).$$

It is easy to see that the set $\{\phi_\epsilon^{\epsilon'}\}$ is linearly independent, because it is so under ϕ .

On the other hand, we can deduce from Lemma 2.11 and its proof that this set spans $\mathbf{NAWeb}^{\text{ess}}(n)$. Indeed, given the essential circle relations, we first deduce that any element $W \in \mathbf{NAWeb}^{\text{ess}}(n)$ is made of a composition of elements from $\mathfrak{S}_n \rightarrow \mathbf{NAWeb}^{\text{ess}}(n)$ and the wraps w_i . From there, using far-commutation and the formulas:

$$w_i s_{i-1} = s_{i-1} w_{i-1}, \quad w_i s_i = s_i w_{i+1},$$

one can see that this gives an algebra epimorphism $\langle w_i \rangle \circ \mathfrak{S}_n \rightarrow \mathbf{NAWeb}^{\text{ess}}(n)$.

Now, by construction, the polynomials $\{P_k(w)\}_{k \in \{0, \dots, N-1\}}$ form a basis of $\mathbb{C}[w]/(w^N - 1)$, and since the w_i 's commute, it follows that the elements $P_{\epsilon_n}(w_n) \cdots P_{\epsilon_1}(w_1)$ span $\langle w_i \rangle_{i \in \{1, \dots, n-1\}} \subset \mathbf{NAWeb}^{\text{ess}}(n)$. Thus, any element in $\mathbf{NAWeb}^{\text{ess}}(n)$ can be written as a linear combination of terms of the kind $\tilde{\sigma} \phi_\epsilon^\epsilon$ with $\tilde{\sigma}$ in the image of \mathfrak{S}_n . It remains to see that $\tilde{\sigma}$ can be assumed to be of minimal length, which is equivalent to saying that no two strands corresponding to the same value in ϵ cross. Via isotopies, this reduces to the identities proven in Corollary 2.33.

This proves that the set $\{\phi_\epsilon^{\epsilon'}\}$ spans $\mathbf{NAWeb}^{\text{ess}}(n)$ and concludes the proof.

PROPOSITION 2.37. *The restriction of ϕ to any morphism space in $\mathbf{NAWeb}^{\text{ess},+}$ is injective.*

Proof. Suppose a linear combination $\sum_i c_i W_i$ of webs W_i in such a morphism space is sent to zero under ϕ . Then for each web in this linear combination we pre-compose with a merge web M and post-compose with a splitter web S (they depend only on the common source resp. target object of all webs W_i) in order to obtain an endomorphism $\sum_i c_i S W_i M$ in $\mathbf{NAWeb}^{\text{ess}}$ whose source and target is a sequence of 1s. Then we have

$$\phi\left(\sum_i c_i S W_i M\right) = \sum_i c_i \phi(S)\phi(W_i)\phi(M) = 0$$

and Proposition 2.36 implies that $\sum_i c_i S W_i M = 0$. Now we consider the merge (split) web M' (S') obtained by reflecting S (M) in a horizontal line and reversing orientations. Then we compute $\sum_i c_i W_i = c \sum_i c_i M' S W_i M S' = 0$ where $c \neq 0$ is a scalar resulting from opening bigons; see the first relation in (1).

Proof of Theorem 2.35. Since ϕ is a braided monoidal functor and since $\mathbf{NAWeb}^{\text{ess}}$ has duals, the statement follows from Proposition 2.37. To make this argument more explicit, let W be a linear combination of webs in some morphism space of $\mathbf{NAWeb}^{\text{ess}}$, which is sent to zero under ϕ . There exists a composition of invertible bending and braiding operations similar to those used in the proof of Theorem 2.21 that transforms W into a linear combination W' of webs in a morphism space as in Proposition 2.37, which is also sent to zero under ϕ . The proposition then implies $W' = 0$ and, by invertibility of the operations, $W = 0$.

The final result of this section is best expressed in terms of Karoubi envelopes, the definition of which we recall now.

DEFINITION 2.38. *The Karoubi completion of a category \mathcal{C} is the category $\text{Kar}(\mathcal{C})$ with objects given by pairs (X, e) , where X is an object of \mathcal{C} and $e \in \text{Hom}_{\mathcal{C}}(X, X)$ an idempotent. Morphisms between (X, e) and (Y, f) are of the form $f \circ g \circ e$ with $g \in \text{Hom}_{\mathcal{C}}(X, Y)$. We will modify this classical definition in the following more specialized cases:*

- *If \mathcal{C} is linear but not yet additive, then we pass to the additive closure before taking the idempotent completion as above. The resulting additive and linear category will also be denoted $\text{Kar}(\mathcal{C})$.*
- *If the morphism spaces of \mathcal{C} furthermore admit a \mathbb{Z} -grading deg , i.e. \mathcal{C} is \mathbb{Z} -pre-graded, then in the definition of the Karoubi completion we only consider homogeneous idempotents. The resulting category is denoted $\text{Kar}(\mathcal{C})^*$; it is again pre-graded. We reserve the notation $\text{Kar}(\mathcal{C})$ for the \mathbb{Z} -graded, additive, linear category whose objects are generated by formal grading shifts $w^k(X, e)$ of the objects (X, e) in $\text{Kar}(\mathcal{C})^*$ and morphism are required to be of degree zero. I.e. $g: w^k(X, e) \rightarrow w^l(Y, f)$ is required to satisfy $\text{deg}(g) = l - k$.*

Note that if \mathcal{C} is additive or monoidal, then the Karoubi completion $\text{Kar}(\mathcal{C})$ inherits these structures.

COROLLARY 2.39. *The functor ϕ induces an equivalence of additive, \mathbb{C} -linear pivotal categories:*

$$\text{Kar}(\mathbf{NAWeb}^{\text{ess}})^* \simeq \mathbf{Rep}(\mathfrak{h}).$$

Proof. This follows from Theorems 2.21 and 2.35 since $\mathbf{Rep}(\mathfrak{h})$ is already idempotent complete and any of its objects can be written as the direct sum of ϕ -images of idempotents in $\mathbf{NAWeb}^{\text{ess}}$ since P_k is sent to the projection onto the span of v_k , see the observation following Definition 2.20.

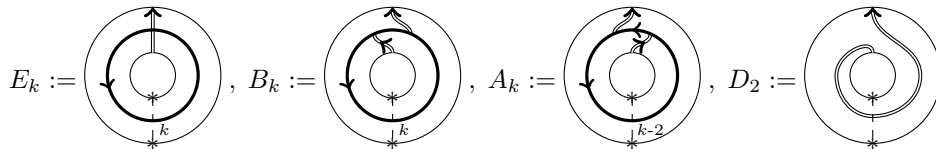
Let $\mathbf{Rep}^+(\mathfrak{h})$ denote the full subcategory of $\mathbf{Rep}(\mathfrak{h})$ containing only those integral \mathfrak{h} -representations whose weights have non-negative entries.

REMARK 2.40. The functor ϕ restricts to a fully faithful functor $N\mathbf{AWeb}^{\text{ess},+} \rightarrow \mathbf{Rep}^+(\mathfrak{h})$ that induces an equivalence of \mathbb{C} -linear monoidal categories

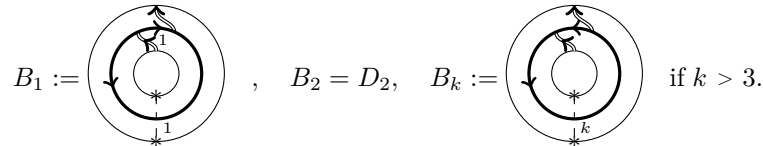
$$\text{Kar}(N\mathbf{AWeb}^{\text{ess},+})^* \simeq \mathbf{Rep}^+(\mathfrak{h}).$$

2.10. PROOF OF LEMMA 2.26. This section contains a proof of the fact that the T_2 projector absorbs crossings. It can be safely skipped on a first read-through.

In order to prove Lemma 2.26 we study the endomorphism algebra of the object 2 in $\overline{N\mathbf{AWeb}}^{\text{ess}}$. For $k > 1$ we introduce the following notation:



Here we set $A_1 = 0$, $B_2 = D_2$ and $A_2 = \text{id}$, and doubled edges stand for 2-labeled edges. Note that in the definition of B_k , we haven't depicted the orientation of one of the strands: this is because it depends on k . More precisely, we have:



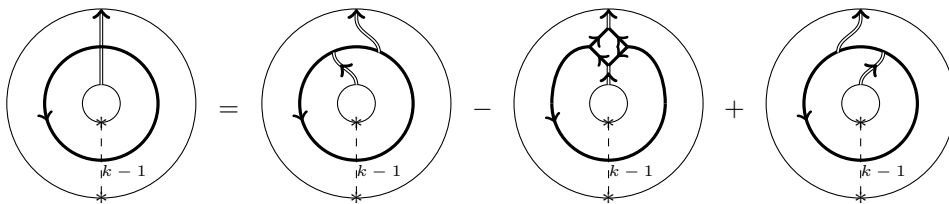
Note also that $A_k = B_k = E_k = 0$ for $k > N$.

LEMMA 2.41. *The following statements hold in the endomorphism algebra of the 2-labeled upward point in $\overline{N\mathbf{AWeb}}^{\text{ess}}$:*

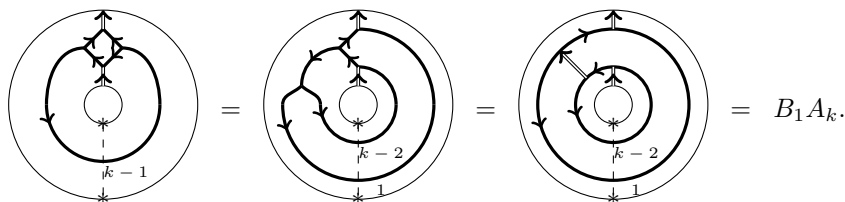
- (1) $E_k = \delta_{k,N} c_N$,
- (2) D_2 is invertible and central,
- (3) $B_k = A_k D_2$ for $k > 2$,
- (4) $B_1 A_k = -E_{k-1} + A_{k-1} D_2 + A_{k+1}$ for $k > 2$,
- (5) $B_N = E_N = c_N$ and thus $A_N = c_N D_2^{-1}$.

Note that only (1) and (5) depend on the value of N .

Proof. (1) holds by definition of $\overline{N\mathbf{AWeb}}^{\text{ess}}$, (2) and (3) follow from isotopies. For (4) we resolve the crossing in E_{k-1} to obtain:



Here the first and third summands are $B_{k-1} = A_{k-1} D_2$ and A_{k+1} respectively. The web in the second summand simplifies as follows:



As a corollary, we get that the elements A_k can be written in terms of powers of B_1 and D_2 :

COROLLARY 2.42. *The elements A_k for $k < N + 2$ satisfy the recursion $A_k := B_1 A_{k-1} - A_{k-2} D_2$ for $k > 5$ with initial conditions $A_3 = B_1$ and $A_4 = B_1^2 - D_2$.*

Proof. We induct on k . For $k = 1 < N$ we use Lemma 2.41 to obtain $B_1 = B_1 A_2 = -E_1 + A_1 D_2 + A_3 = A_3$. Similarly, for $k = 2 < N$, we get from Lemma 2.41 that $B_1^2 - D_2 = B_1 A_3 - A_2 D_2 = -E_2 + A_4 = A_4$. For the recurrence relation we compute: $B_1 A_{k-1} - A_{k-2} D_2 = -E_{k-2} + A_k = A_k$ if $5 \leq k < N + 2$.

REMARK 2.43. We will now find expressions for T_2 which are more convenient in the following proof. First we introduce the notation $t = D^{-1} s D$ for the rotation conjugate of the crossing. Now we use the facts that D^2 is central in $\overline{N\mathbf{AWeb}}^{\text{ess}}(2)$ and that $(sD)^{-1}$ and (Ds) commute to write $(sD)^{-k} (Ds)^k = (ts)^k$, which gives:

$$T_2 = \frac{1}{N} \sum_{k=0}^{N-1} (ts)^k$$

Further, we resolve the crossings as $s = \text{id}_2 - u$ and $t = \text{id}_2 - v$ where $u = u_1$ is the 2-labeled dumbbell and v its conjugate. So we have:

$$(15) \quad T_2 = \frac{1}{N} \sum_{k=0}^{N-1} ((\text{id}_2 - v)(\text{id}_2 - u))^k$$

If we write M and S for the merge and split vertices on two strands, such that $SM = u$, we get the following equality for $k > 2$:

$$\underbrace{\cdots vuvu}_{k \text{ factors}} = D^{1-k} S B_1^{k-1} M$$

We will now attempt to rewrite the expressions $X_n := D^{1-n} S A_{n+1} M$ in terms of powers of B_1 . We can then use the relation $X_N = 0$ to deduce a relation between compositions of the webs u and v . To this end we define $R_{2k-1} := \underbrace{(\text{id}_2 - v)(\text{id}_2 - u) \cdots (\text{id}_2 - v)}_{2k-1 \text{ factors}}$ and $S_x := \sum_{k=1}^x R_{2k-1}$.

LEMMA 2.44. *For $2 \leq n \leq N$ we have:*

$$X_n = \begin{cases} u - R_{n-1} u - u S_{n/2-1} u & \text{even } n \\ u - u S_{(n-1)/2} u & \text{odd } n \end{cases}$$

Proof. We will use the notation Y_n for the entries on the right-hand side of the equation in the statement of the lemma. The proof of $X_n = Y_n$ proceeds by induction on n . For $n = 2$ we have $X_2 = vu = u - (\text{id}_2 - v)u = Y_2$. Similarly, for $n = 3$, we have

$$X_3 = D^{-2} S A_4 M = D^{-2} S (B_1^2 - D_2) M = uvu - u = u - u(1 - v)u = Y_3$$

since $u^2 = 2u$ by the bigon relation. We prove the remaining cases recursively. For this, note that the elements $X_n = D^{1-n} S A_{n+1} M$ inherit a recurrence relation from the elements A_{n+1} :

$$(16) \quad \begin{aligned} X_n &= D^{1-n} S A_{n+1} M = D^{1-n} S B_1 A_n M - D^{1-n} S A_{n-1} D_2 M \\ &= D^{1-n} S B_1 A_n M - D^{3-n} S A_{n-1} M \\ &= \begin{cases} v X_{n-1} - X_{n-2} & \text{even } n \\ u X_{n-1} - X_{n-2} & \text{odd } n \end{cases} \end{aligned}$$

Here we have used $D^{1-n}SB_1 = vD^{2-n}S$ for even n and $D^{1-n}SB_1 = uD^{2-n}S$ for odd n . Now it remains to check that the Y_n also satisfy this recurrence (16). Indeed, for odd $N > 4$ we can verify:

$$\begin{aligned} Y_n - uY_{n-1} + Y_{n-2} &= (u - uS_{(n-1)/2}u) - u(u - R_{n-2}u - uS_{(n-3)/2}u) \\ &\quad + (u - uS_{(n-3)/2}u) \\ &= u(-S_{(n-1)/2} + R_{n-2} + S_{(n-3)/2})u = 0 \end{aligned}$$

Here we have used that $S_x - S_{x-1} = R_{2x-1}$. In order to check the recurrence for Y_n in the case of even n we need an auxiliary computation. For odd $x > 1$ we have

$$\begin{aligned} S_x &= (\text{id}_2 - v) + (\text{id}_2 - v)(\text{id}_2 - u)S_{x-1} \\ &= (\text{id}_2 - v) + (\text{id}_2 - v)S_{x-1} - uS_{x-1} + vuS_{x-1} \\ &= 2\text{id}_2 - v + (\text{id}_2 - u)S_{x-2} - uS_{x-1} + vuS_{x-1} \end{aligned}$$

which implies:

$$\begin{aligned} vuS_{n/2-1}u &= -2u + vu + S_{n/2}u - (\text{id}_2 - u)S_{n/2-2}u + uS_{n/2-1}u \\ &= -2u + vu + R_{n-1}u + R_{n-3}u + uS_{n/2-2}u + uS_{n/2-1}u \end{aligned}$$

Here we have used $(\text{id}_2 - v)S_x = \text{id}_2 + (\text{id}_2 - u)S_{x-1}$. Now we check the recurrence for even $n > 3$:

$$\begin{aligned} Y_n - vY_{n-1} + Y_{n-2} &= (u - R_{n-1}u - uS_{n/2-1}u) - v(u - uS_{n/2-1}u) \\ &\quad + (u - R_{n-3}u - uS_{n/2-2}u) \\ &= 2u - vu - R_{n-1}u - R_{n-3}u - uS_{n/2-2}u - uS_{n/2-1}u \\ &\quad + vuS_{n/2-1}u \\ &= 0 \end{aligned}$$

This completes the proof of the Lemma.

Proof of Lemma 2.26. We only prove $T_2s = T_2$, which is equivalent to $NT_2u = 0$ by expanding the crossing and multiplying by N . Using (15) we compute:

$$\begin{aligned} NT_2u &= \sum_{k=0}^{N-1} ((\text{id}_2 - v)(\text{id}_2 - u))^k u = u - \sum_{k=1}^{N-1} ((\text{id}_2 - v)(\text{id}_2 - u))^{k-1} (\text{id}_2 - v)u \\ &= u - \sum_{k=1}^{N-1} R_{2k-1}u \end{aligned}$$

Now note that $\text{id}_2 = (sD)^{-N}(Ds)^N = ((\text{id}_2 - v)(\text{id}_2 - u))^N$ implies that

$$(17) \quad R_{2k-1} = ((\text{id}_2 - u)R_{2N-2k-1}(\text{id}_2 - u))^{-1} = (\text{id}_2 - u)R_{2N-2k-1}(\text{id}_2 - u).$$

Now we distinguish two cases. For even N we expand:

$$\begin{aligned} NT_2u &= u - \sum_{k=1}^{N/2} R_{2k-1}u - \sum_{k=N/2+1}^{N-1} R_{2k-1}u \\ &= u - \sum_{k=1}^{N/2} R_{2k-1}u - (\text{id}_2 - u) \sum_{l=1}^{N/2-1} R_{2l-1}(\text{id}_2 - u)u \\ &= u - \sum_{k=1}^{N/2} R_{2k-1}u + (\text{id}_2 - u) \sum_{l=1}^{N/2-1} R_{2l-1}u = u - R_{N-1}u - uS_{N/2-1}u = X_N \end{aligned}$$

Here we have used (17) for the second equality and Lemma 2.44 for the last equality. For odd N we expand analogously:

$$\begin{aligned} NT_2u &= u - \sum_{k=1}^{(N-1)/2} R_{2k-1}u - \sum_{k=(N+1)/2+1}^{N-1} R_{2k-1}u \\ &= u - \sum_{k=1}^{(N-1)/2} R_{2k-1}u + (\text{id}_2 - u) \sum_{l=1}^{(N-1)/2} R_{2k-1}u = u - uS_{(N-1)/2}u = X_N \end{aligned}$$

We conclude the proof by noting that $X_N = D^{1-N}SA_{N+1}M = 0$ in $\overline{N\mathbf{AWeb}}^{\text{ess}}$ since $A_{N+1} = 0$.

REMARK 2.45. The expression X_N in the rewritten form in Lemma 2.44 expresses the longest Kazhdan–Lusztig basis element $\underline{H}_{sts\dots} = \underline{H}_{tst\dots}$ in the type $I_2(N)$ Hecke algebra in terms of products of $\underline{H}_s := u$ and $\underline{H}_t := v$, see [11, Section 2.3]. In particular, the relation $X^N = 0$ suggests that $\overline{N\mathbf{AWeb}}^{\text{ess}}$ (2) is related to a quotient of the Hecke algebra by the 2-cell containing the basis element associated to the longest word.

3. CATEGORIFICATION OF POWER-SUM SYMMETRIC POLYNOMIALS

Before turning to the topological applications of our work, we will in this section focus on identifying more precisely the structures that are categorified by the categories defined before. The main result of this section consists in a categorification of Newton’s identities for power-sum and elementary symmetric polynomials (see Theorem 3.5).

Let $X = \{X_1, \dots, X_N\}$ be an alphabet of N variables and denote by $\text{Sym}(X) := \mathbb{C}[X_1, \dots, X_N]^{\mathfrak{S}_N}$ the ring of symmetric polynomials in X . Recall that $\text{Sym}(X) \cong \mathbb{C}[e_1(X), \dots, e_N(X)]$, where $e_j(X)$ denotes the j^{th} elementary symmetric polynomial in X . We use the notation $h_j(X)$ for the j^{th} complete symmetric polynomial.

DEFINITION 3.1. *The split Grothendieck group of an additive category \mathcal{C} is the abelian group $K_0(\mathcal{C})$ defined as the quotient of the free abelian group spanned by the isomorphism classes $[X]$ of objects X of \mathcal{C} , modulo the ideal generated by relations of the form $[A \oplus B] = [A] + [B]$ for objects A, B of \mathcal{C} .*

If \mathcal{C} is monoidal, then $K_0(\mathcal{C})$ inherits a unital ring structure with multiplication $[A] \cdot [B] := [A \otimes B]$.

The following lemma is classical.

LEMMA 3.2. *There is an isomorphism*

$$K_0(\mathbf{Rep}^+(\mathfrak{gl}_N)) \otimes \mathbb{C} \cong K_0(\text{Kar}(\mathbf{Rep}^+(\mathfrak{gl}_N))) \otimes \mathbb{C} \cong \text{Sym}(X) \cong \mathbb{C}[e_1(X), \dots, e_N(X)]$$

sending the classes of the fundamental representations $[\wedge^k V]$ to the elementary symmetric polynomials $e_k(X)$. The class of the simple representation indexed by the partition λ is then given by the Schur polynomial $\pi_\lambda(X)$. If one includes duals, one obtains

$$\begin{aligned} K_0(\mathbf{Rep}(\mathfrak{gl}_N)) \otimes \mathbb{C} &\cong K_0(\text{Kar}(\mathbf{Rep}(\mathfrak{gl}_N))) \otimes \mathbb{C} \cong \mathbb{C}[X^{\pm 1}]^{\mathfrak{S}_N} \\ &\cong \mathbb{C}[e_1(X), \dots, e_{N-1}(X), e_N^{\pm 1}(X)]. \end{aligned}$$

For example, the classes of the symmetric and anti-symmetric power representations are related as follows.

$$(18) \quad h_{m+1}(X) = \sum_{i=1}^{m+1} (-1)^i h_{m+1-i}(X) e_i(X)$$

This can also be seen in the Grothendieck group of $N\mathbf{Web}$, at the cost of passing to the Karoubi envelope. For this, we recall the symmetric clasps [19], which are higher-rank analogs of Jones–Wenzl projectors, and their anti-symmetric counterparts.

DEFINITION 3.3. *The symmetric and anti-symmetric clasps $H_m \in N\mathbf{Web}$ and $V_m \in N\mathbf{Web}$ are defined by $H_1 = V_1 = \text{id}_1$ and then:*

$$\begin{array}{c} \uparrow \uparrow \\ \boxed{H_{m+1}} \\ \vdots \\ \underbrace{\quad}_{m+1} \end{array} := \begin{array}{c} \uparrow \uparrow \\ \boxed{H_m} \\ \vdots \\ \underbrace{\quad}_m \end{array} \uparrow - \frac{m}{m+1} \begin{array}{c} \uparrow \uparrow \\ \boxed{H_m} \\ \vdots \\ \underbrace{\quad}_m \end{array} \uparrow, \quad \begin{array}{c} \uparrow \uparrow \\ \boxed{V_{m+1}} \\ \vdots \\ \underbrace{\quad}_{m+1} \end{array} := \frac{1}{(m+1)!} \begin{array}{c} \uparrow \uparrow \\ \vdots \\ \underbrace{\quad}_{m+1} \end{array}$$

Note that the clasps are related by:

$$\begin{array}{c} \uparrow \uparrow \\ \boxed{H_{m+1}} \\ \vdots \\ \underbrace{\quad}_{m+1} \end{array} = \begin{array}{c} \uparrow \uparrow \\ \boxed{H_m} \\ \vdots \\ \underbrace{\quad}_m \end{array} \uparrow - \frac{2m}{m+1} \begin{array}{c} \uparrow \uparrow \\ \boxed{H_m} \\ \vdots \\ \underbrace{\quad}_m \end{array} \uparrow \uparrow, \quad \begin{array}{c} \uparrow \uparrow \\ \boxed{V_{m+1}} \\ \vdots \\ \underbrace{\quad}_{m+1} \end{array} = \begin{array}{c} \uparrow \uparrow \\ \boxed{V_m} \\ \vdots \\ \underbrace{\quad}_m \end{array} \uparrow - \frac{2m}{m+1} \begin{array}{c} \uparrow \uparrow \\ \boxed{V_m} \\ \vdots \\ \underbrace{\quad}_m \end{array} \uparrow \uparrow$$

It is well-known that ϕ sends H_m and V_m to the projections onto simple representations in $V^{\otimes m}$ given by the m -fold symmetric and anti-symmetric powers of the vector representation respectively. These formulas indeed match Young symmetrizers. The formula for H_m is the $q = 1$ specialization of the Jones–Wenzl recursion, see [33]. Matching the definition of V_m to the projection onto the m -fold exterior power directly follows from Equations 6 and 7. We will also use the symbols H_m and V_m to refer to the objects of the Karoubi envelope $\text{Kar}(N\mathbf{Web})$ that correspond to these idempotents.

THEOREM 3.4. *In $\text{Kar}(N\mathbf{Web})$ there is an isomorphism*

$$\bigoplus_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (k, V_{k-1-2i} \otimes H_{2i+1}) \simeq \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} (k, V_{k-2i} \otimes H_{2i})$$

which categorifies (18). Here H_m and V_m refer to the objects of the Karoubi envelope corresponding to the clasps from Definition 3.3.

The proof of this is similar to but easier than the proof of Theorem 3.5 below, and thus omitted.

3.1. CATEGORIFIED NEWTON’S IDENTITIES. We now explicitly show that the projectors T_m categorify the power-sum symmetric polynomials $p_m(X) = X_1^m + \dots + X_N^m$ in the same sense as the clasps H_m categorify the complete symmetric polynomials. To this end, we prove that the projectors T_m satisfy categorified versions of the classical Newton identities:

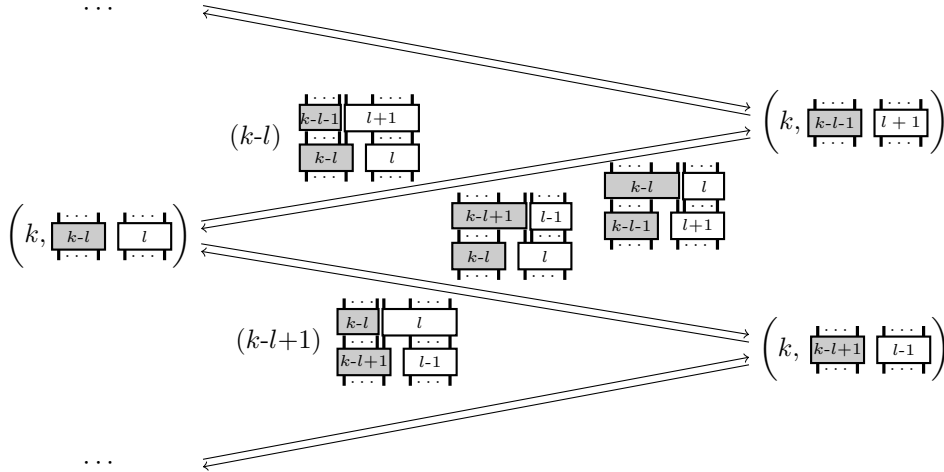
$$(19) \quad p_k(X) = (-1)^{k-1} k e_k(X) - \sum_{j=1}^{k-1} (-1)^{k-j} e_{k-j}(X) p_j(X) \quad \text{for } 1 \triangleleft k$$

THEOREM 3.5. *In $\text{Kar}(N\mathbf{AWeb}^{\text{ess}})^*$, there is an isomorphism*

$$\bigoplus_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (k, V_{k-1-2i} \otimes T_{2i+1}) \simeq \bigoplus_k (k, V_k) \oplus \bigoplus_{i=1}^{\lfloor \frac{k}{2} \rfloor} (k, V_{k-2i} \otimes T_{2i})$$

whose nonzero components are described in the proof; they connect the i -indexed summand on the left to the i - and $(i + 1)$ -indexed terms on the right, where we declare $\bigoplus_k (k, V_k)$ to be indexed by 0. Here V_m and T_m refer to the objects of the Karoubi envelope corresponding to the anti-symmetric clasps from Definition 3.3 and the extremal weight projectors from Section 2.8, respectively.

Proof. The desired isomorphism takes the shape of a “zig-zag”, i.e. each direct summand maps non-trivially to at most two direct summands (including $\oplus_k(k, V_k)$) on the other side. A typical segment of the zig-zag looks as follows:



Above, gray-colored boxes stand for the anti-symmetric clasps V_m , as pictured in Definition 3.3, while the other rectangles are our extremal weight projectors T_m . These formulas are only valid when $l > 2$, and we will deal with the $l = 1$ term at the end.

We check that the composite of the maps to the right with the maps to the left induce the identity on the components of the left. For this, we compute

$$\begin{aligned}
 (k-l+1) \begin{array}{|c|c|} \hline \dots & l \\ \hline k-l+1 & l \\ \hline \dots & l \\ \hline \end{array} &= -(k-l-1) \begin{array}{|c|c|} \hline \dots & l \\ \hline k-l & l \\ \hline \dots & l \\ \hline \end{array} + (k-l) \begin{array}{|c|c|} \hline \dots & l \\ \hline k-l & l \\ \hline \dots & l \\ \hline \end{array} \\
 (k-l) \begin{array}{|c|c|} \hline \dots & l+1 \\ \hline k-l & l+1 \\ \hline \dots & l+1 \\ \hline \end{array} &= (k-l) \begin{array}{|c|c|} \hline \dots & l \\ \hline k-l & l \\ \hline \dots & l \\ \hline \end{array} - (k-l) \begin{array}{|c|c|} \hline \dots & l \\ \hline k-l & l \\ \hline \dots & l \\ \hline \end{array}
 \end{aligned}$$

by expanding the middle projectors on the left-hand side using the recursions in (14) and Definition 3.3. Adding both equations, we obtain the desired equality:

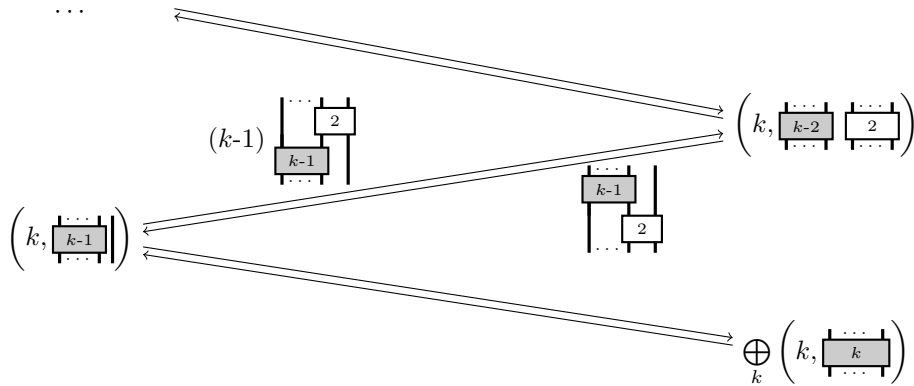
$$(k-l+1) \begin{array}{|c|c|} \hline \dots & l \\ \hline k-l+1 & l \\ \hline \dots & l \\ \hline \end{array} + (k-l) \begin{array}{|c|c|} \hline \dots & l+1 \\ \hline k-l & l+1 \\ \hline \dots & l+1 \\ \hline \end{array} = \begin{array}{|c|c|} \hline \dots & l \\ \hline k-l & l \\ \hline \dots & l \\ \hline \end{array}$$

We also check that all other components of this endomorphism of the left-hand side are zero:

$$(k-l) \begin{array}{|c|c|} \hline \dots & l+2 \\ \hline k-l-1 & l+1 \\ \hline \dots & l+1 \\ \hline k-l & l \\ \hline \dots & l \\ \hline \end{array} = 0 = (k-l+1) \begin{array}{|c|c|} \hline \dots & l-2 \\ \hline k-l+1 & l-1 \\ \hline \dots & l-1 \\ \hline k-l & l \\ \hline \dots & l \\ \hline \end{array}$$

These expressions are zero because the middle projectors can be absorbed into the top and bottom projectors respectively. The results have anti-symmetric clasps and extremal weight projectors that share two strands, which forces them to equal zero.

To finish the proof, we need to look at the top and bottom ends of the zig-zag. The top end, which contains (k, T_k) is treated precisely as in the generic case. The bottom end is more interesting, as it involves k copies of the object (k, V_k) .



In order to obtain the desired isomorphism, one needs to express the following term as a sum of k orthogonal projections onto k copies of the exterior power:

$$\begin{array}{|c|} \hline \dots \\ \hline k-1 \\ \hline \dots \\ \hline \end{array} - (k-1) \begin{array}{|c|} \hline \dots \\ \hline k-1 \\ \hline \dots \\ \hline 2 \\ \hline \dots \\ \hline k-1 \\ \hline \dots \\ \hline \end{array}$$

To this end, we introduce the following diagram as corresponding to taking the polynomial P_i ($i = 0 \dots N - 1$) in the wrap (*i.e.* $P_i(D)$ from Definition 2.20), thus projecting onto the i -th eigenspace of the wrap:



and more generally in the case of k parallel strands, for a tuple $I = (i_1, \dots, i_k)$ we use the diagrams

$$\begin{array}{|c|} \hline \dots \\ \hline I \\ \hline \dots \\ \hline \end{array} := \begin{array}{|c|} \hline \dots \\ \hline i_1 \\ \hline \dots \\ \hline i_k \\ \hline \dots \\ \hline \end{array}$$

which correspond to the projectors $P_{i_1} \otimes \dots \otimes P_{i_k}$. In the following we consider subsets $A \subset \{0, \dots, N-1\}$ as tuples of distinct elements, with the usual ordering. We write sA for the tuple obtained from this by acting with a permutation s . The corresponding projectors satisfy:

$$\begin{array}{|c|} \hline \dots \\ \hline k \\ \hline \dots \\ \hline I \\ \hline \dots \\ \hline k \\ \hline \dots \\ \hline \end{array} = 0 \quad \text{if } i_l = i_m \text{ for } l \neq m$$

Now one can write:

$$(20) \quad \begin{array}{|c|} \hline \dots \\ \hline k-1 \\ \hline \dots \\ \hline \end{array} = \begin{array}{|c|} \hline \dots \\ \hline k-1 \\ \hline \dots \\ \hline k-1 \\ \hline \dots \\ \hline \end{array} = \sum_{\substack{A \subset \{0, \dots, N-1\} \\ |A|=k \\ s \in \mathfrak{S}_k}} \begin{array}{|c|} \hline \dots \\ \hline k-1 \\ \hline \dots \\ \hline sA \\ \hline \dots \\ \hline k-1 \\ \hline \dots \\ \hline \end{array} + \sum_{\substack{B \subset \{0, \dots, N-1\} \\ |B|=k-1, x \in B \\ s \in \mathfrak{S}_{k-1}}} \begin{array}{|c|} \hline \dots \\ \hline k-1 \\ \hline \dots \\ \hline sB \\ \hline \dots \\ \hline k-1 \\ \hline \dots \\ \hline x \\ \hline \end{array}$$

Here we have expanded the identity web between the two anti-symmetric clasps into a sum of projectors, noting that the clasps will kill k -tuples that have two equal elements in the first $k - 1$ entries. The two summands correspond to the two alternatives of having a last entry x that is distinct from the first $k - 1$ ones, or a repeat.

Now, we make use of the following equality:

to rewrite the two summands in (20). The first sum is partitioned into k terms, the j -th of which contains those projectors sA that project to the j -th largest element of A on the right-most strand. For a fixed A and j , there are $(k - 1)!$ permutations s such that sA is of this type, each of which produces an identical summand by the previous equation.

In the second summand, we similarly reorder the x in sB to the right-most strand, which collects together $k - 1$ identical terms.

(21)

$$\begin{aligned}
 \left[\begin{array}{c} \dots \\ k-1 \\ \dots \end{array} \right] &= \left(\sum_{j=1}^k \sum_{\substack{A \subset \{0, \dots, N-1\} \\ |A|=k, \\ x \text{ } j\text{-th largest entry of } A}} (k-1)! \left[\begin{array}{c} \dots \\ k-1 \\ \dots \\ A \text{ } \gamma \text{ } \phi \\ \dots \\ k-1 \\ \dots \end{array} \right] \right) + (k-1) \left(\sum_{\substack{x \in \{0, \dots, N-1\} \\ C \subset \{0, \dots, N-1\} \\ |C|=k-2 \\ s \in \mathfrak{S}_{k-2}}} \left[\begin{array}{c} \dots \\ k-1 \\ \dots \\ sC, x \\ \dots \\ k-1 \\ \dots \end{array} \right] \right)
 \end{aligned}$$

The remaining summation in the second term is over $(k - 2)$ -element subsets C and their permutations. Faithfulness of ϕ implies that the anti-symmetric clasps absorb this sum of projectors and so we get:

$$(k-1) \left(\sum_{\substack{x \in \{0, \dots, N-1\} \\ C \subset \{0, \dots, N-1\} \\ |C|=k-2 \\ s \in \mathfrak{S}_{k-2}}} \left[\begin{array}{c} \dots \\ k-1 \\ \dots \\ sC, x \\ \dots \\ k-1 \\ \dots \end{array} \right] \right) = (k-1) \left(\sum_{x \in \{0, \dots, N-1\}} \left[\begin{array}{c} \dots \\ k-1 \\ \dots \\ x \\ \dots \\ k-1 \\ \dots \end{array} \right] \right) = (k-1) \left[\begin{array}{c} \dots \\ k-1 \\ \dots \\ 2 \\ \dots \\ k-1 \\ \dots \end{array} \right]$$

The k terms in the first summand in (21) are clearly orthogonal to each other and to the second summand, from which it follows that they are idempotents. It remains to argue that they are isomorphic to anti-symmetric clasps in the Karoubi envelope. To this end, we fix a j and argue that

$$\sum_{\substack{A \subset \{0, \dots, N-1\} \\ |A|=k, \\ x \text{ } j\text{-th largest entry of } A}} (k-1)! \left[\begin{array}{c} \dots \\ k-1 \\ \dots \\ A \text{ } \gamma \text{ } \phi \\ \dots \\ k \\ \dots \end{array} \right] \quad \text{and} \quad \sum_{\substack{A \subset \{0, \dots, N-1\} \\ |A|=k, \\ x \text{ } j\text{-th largest entry of } A}} k! \left[\begin{array}{c} \dots \\ k \\ \dots \\ A \text{ } \gamma \text{ } \phi \\ \dots \\ k-1 \\ \dots \end{array} \right]$$

give the desired inverse isomorphisms. So we simplify both composites:

$$\begin{aligned}
 \sum_{\substack{A, B \subset \{0, \dots, N-1\} \\ |A|=|B|=k, \\ x \text{ } j\text{-th largest entry of } A \\ y \text{ } j\text{-th largest entry of } B}} k!(k-1)! \quad & \begin{array}{c} \cdots \\ \boxed{k} \\ \circlearrowleft \text{ } \gamma \text{ } \circlearrowright \\ \boxed{k-1} \\ \circlearrowleft \text{ } \gamma \text{ } \circlearrowright \\ \boxed{k} \\ \cdots \end{array} = \sum_{\substack{A \subset \{0, \dots, N-1\} \\ |A|=k, \\ x \text{ } j\text{-th largest entry of } A}} k! \quad \begin{array}{c} \cdots \\ \boxed{k} \\ \circlearrowleft \text{ } \gamma \text{ } \circlearrowright \\ \boxed{k} \\ \cdots \end{array} \\
 & = \sum_{\substack{A \subset \{0, \dots, N-1\} \\ |A|=k}} k! \quad \begin{array}{c} \cdots \\ \boxed{k} \\ \circlearrowleft \text{ } \gamma \text{ } \circlearrowright \\ \boxed{k} \\ \cdots \end{array} = \begin{array}{c} \cdots \\ \boxed{k} \\ \cdots \end{array}
 \end{aligned}$$

Summands in the composite of the left-hand side are zero unless $x = y$ and $A \cap x = B \cap y$ and the anti-symmetric clasp is absorbed at the cost of dividing by $(k - 1)!$. The second equality is just a reordering, while the third uses that the anti-symmetric projectors absorb the sum of all A -projectors at the cost of dividing by $k!$. For the other composite we get:

$$\begin{aligned}
 \sum_{\substack{A, B \subset \{0, \dots, N-1\} \\ |A|=|B|=k, \\ x \text{ } j\text{-th largest entry of } A \\ y \text{ } j\text{-th largest entry of } B}} k!(k-1)! \quad & \begin{array}{c} \cdots \\ \boxed{k-1} \\ \circlearrowleft \text{ } \gamma \text{ } \circlearrowright \\ \boxed{k} \\ \circlearrowleft \text{ } \gamma \text{ } \circlearrowright \\ \boxed{k-1} \\ \cdots \end{array} = \sum_{\substack{A \subset \{0, \dots, N-1\} \\ |A|=k, \\ x \text{ } j\text{-th largest entry of } A}} (k-1)! \quad \begin{array}{c} \cdots \\ \boxed{k-1} \\ \circlearrowleft \text{ } \gamma \text{ } \circlearrowright \\ \boxed{k-1} \\ \cdots \end{array}
 \end{aligned}$$

Here we have used the same absorption property as in the first step in the computation of the other composite.

REMARK 3.6. In the case $k = N$ we can give an alternative characterization of the more mysterious part of the isomorphism in Theorem 3.5, which involves the N -fold direct sum of objects (N, V_N) . Indeed, after (21) and the following displayed equation, the goal was to decompose

$$\begin{array}{c} \cdots \\ \boxed{N-1} \\ \cdots \end{array} \Big| - (N-1) \begin{array}{c} \cdots \\ \boxed{N-1} \\ \boxed{2} \\ \boxed{N-1} \\ \cdots \end{array} = \sum_{\substack{A = \{0, \dots, N-1\} \\ s \in \mathcal{G}_N}} \begin{array}{c} \cdots \\ \boxed{N-1} \\ \circlearrowleft \text{ } \gamma \text{ } \circlearrowright \\ \boxed{N-1} \\ \cdots \end{array}$$

into a sum of N orthogonal idempotents, which are individually isomorphic to (N, V_N) . It is easy to check that the above is equal to

$$\sum_{x=1}^N (\text{id}_{N-1} \otimes D^{-x}) V_N (\text{id}_{N-1} \otimes D^x),$$

which is manifestly a sum of idempotents, which are orthogonal since $V_N (\text{id}_{N-1} \otimes D^{k-l}) V_N = \delta_{k,l} V_N$.

QUESTION 3.7. Can the extremal weight projectors and symmetric clasps be used to give categorifications of the following identities?

$$(22) \quad kh_k(X) = \sum_{j=1}^k h_{k-j}(X) p_j(X) \quad \text{for } 1 \leq k \leq N$$

An isomorphism categorifying this identity for $k = 2$ is easy to construct. For $k > 3$ such an isomorphism cannot be of zig-zag shape as for (19).

3.2. CATEGORIFICATION OF THE SYMMETRIC POLYNOMIAL RING. An easy consequence of Corollary 2.39 is the following.

LEMMA 3.8. *There is an algebra isomorphism*

$$K_0(\text{Kar}(N\mathbf{AWeb}^{\text{ess},+})^*) \otimes \mathbb{C} \cong K_0(\mathbf{Rep}^+(\mathfrak{h})) \otimes \mathbb{C} \cong \mathbb{C}[X]$$

sending the class of the object $(1, P_i)$ to $[\mathbb{C}\langle v_i \rangle]$ and further to X_i .

We have seen that the extremal weight projectors in $N\mathbf{AWeb}^{\text{ess},+}$ categorify the power sum symmetric polynomials. However, by Lemma 3.8 the Grothendieck group of the Karoubi envelope of $N\mathbf{AWeb}^{\text{ess},+}$ is larger than the symmetric polynomial ring $\text{Sym}(X) \cong K_0(\text{Kar}(N\mathbf{Web}^+)) \cong K_0(\mathbf{Rep}^+(\mathfrak{gl}_N))$. To see this, recall that the objects in $\mathbf{Rep}^+(\mathfrak{h})$ are direct sums of non-negative integral \mathfrak{gl}_N weight spaces. However, in the Grothendieck group, the classes of such direct sums can be written as linear combinations of classes of \mathfrak{gl}_N -representations (if and) only if the corresponding polynomials are invariant under the Weyl group \mathfrak{S}_N . In this section, we identify a sub-category of $N\mathbf{AWeb}^{\text{ess},+}$ that is \mathfrak{S}_N -equivariant, that contains the extremal weight projectors and has $\text{Sym}(X)$ as Grothendieck group.

To this end, note that \mathfrak{S}_N acts by (outer) automorphisms on \mathfrak{h} and thus by linear automorphisms on every object of $\mathbf{Rep}(\mathfrak{h})$, which permute weight spaces. With respect to these actions we make the following definition.

DEFINITION 3.9. *We let $\mathbf{Rep}(\mathfrak{h})^{\mathfrak{S}_N}$ denote the subcategory of $\mathbf{Rep}(\mathfrak{h})$ with objects that are stable under \mathfrak{S}_N and morphisms that are \mathfrak{S}_N -equivariant.*

LEMMA 3.10. *The category $\mathbf{Rep}(\mathfrak{h})^{\mathfrak{S}_N}$ is semi-simple and the homomorphism*

$$\text{Sym}(X) \cong K_0(\mathbf{Rep}^+(\mathfrak{gl}_N)) \otimes \mathbb{C} \rightarrow K_0(\mathbf{Rep}(\mathfrak{h})^{\mathfrak{S}_N}) \otimes \mathbb{C}$$

induced by the inclusion is an isomorphism.

Proof. The indecomposable objects in $\mathbf{Rep}(\mathfrak{h})^{\mathfrak{S}_N}$ are of the form $\mathbb{C}\langle v_{s(\epsilon_1)}, \dots, v_{s(\epsilon_n)} \rangle | s \in \mathfrak{S}_N$. In other words, the span of the extremal weight vectors in the \mathfrak{S}_N -orbit of a (highest weight) vector $v_{0, \dots, 0, 1, \dots, 1, \dots, N-1}$ with multiplicities n_i of the weights i determined by a partition $\lambda: n_0 > n_1 > \dots > n_{N-1}$ of n . There are no morphisms between distinct indecomposables and their endomorphism algebras are 1-dimensional over \mathbb{C} . This shows that $\mathbf{Rep}(\mathfrak{h})^{\mathfrak{S}_N}$ is semi-simple. The isomorphism follows since the classes of these indecomposables can be expressed as linear combinations of the classes of tensor products of fundamental representations in the same way as monomial symmetric polynomials can be expressed as polynomials in elementary symmetric polynomials.

We aim to describe the subcategory $\mathbf{Rep}(\mathfrak{h})^{\mathfrak{S}_N}$ of $\mathbf{Rep}(\mathfrak{h})$ by a subcategory of $N\mathbf{AWeb}^{\text{ess},+}$.

DEFINITION 3.11. *Let $N\mathbf{AWeb}_s^{+, \text{ess}}$ denote the symmetric monoidal \mathbb{C} -linear subcategory of $N\mathbf{AWeb}^{\text{ess},+}$ with the same objects, but with morphisms spaces generated (under tensor product and composition) by morphisms in $N\mathbf{Web}^+$ and the extremal weight projectors T_m for $m > 1$.*

Note that the restriction of ϕ to the subcategory $N\mathbf{AWeb}_s^{+, \text{ess}}$ has image contained in $\mathbf{Rep}(\mathfrak{h})^{\mathfrak{S}_N}$.

PROPOSITION 3.12. *The functor $\phi: N\mathbf{AWeb}_s^{+, \text{ess}} \rightarrow \mathbf{Rep}(\mathfrak{h})^{\mathfrak{S}_N}$ is fully faithful and induces an equivalence of \mathbb{C} -linear monoidal categories $\text{Kar}(N\mathbf{AWeb}_s^{+, \text{ess}}) \simeq \mathbf{Rep}(\mathfrak{h})^{\mathfrak{S}_N}$.*

Proof. Faithfulness is inherited from Theorem 2.35. We shall prove fullness by showing that the image of ϕ contains the projections onto the simple objects in $\mathbf{Rep}(\mathfrak{h})^{\mathfrak{S}_N}$ as identified in the proof of Lemma 3.10. Indeed, if $\lambda: n_0 > n_1 > \dots > n_{N-1}$ is a partition of n , then we will construct an idempotent morphism in $\mathbf{NAWeb}_s^{+, \text{ess}}$ that projects onto the span of the \mathfrak{S}_N -orbit of the vector $v_{0, \dots, 0, 1, \dots, 1, \dots, N-1}$ with weights i appearing with multiplicities n_i .

To this end, we first define an auxiliary projector O_n in $\mathbf{NAWeb}_s^{+, \text{ess}}$ for the case where $n_i \in \{0, 1\}$ for $1 \leq i \leq N$. We set $O_1 = \text{id}_1$ and $O_2 = \text{id}_2 - T_2$. For $n > 2$ we inductively define:

$$O_{n+1} := s_1(\text{id}_1 \otimes O_n) s_1(\text{id}_1 \otimes O_n)(O_n \otimes \text{id}_1)$$

It is easy to check that the image of O_n under ϕ is the desired projection, and so the O_n are the desired diagrammatic idempotents by faithfulness of ϕ . It is also clear that the O_n are contained in $\mathbf{NAWeb}_s^{+, \text{ess}}$.

Now let $\lambda: n_0 > n_1 > \dots > n_k$ be a partition of n with k non-zero parts n_i . Then consider the projector built as the composite of $T_{n_1} \otimes \dots \otimes T_{n_k}$, the permutation given by the product of the transpositions $(n_i, n-i)$ for $1 \leq i \leq k$, the projector $\text{id}_{n-k} \otimes O_k$, the inverse permutation, and again $T_{n_1} \otimes \dots \otimes T_{n_k}$.

The image of this element under ϕ is the idempotent projecting onto the span of the \mathfrak{S}_N -orbit of the vector $v_{0, \dots, 0, 1, \dots, 1, \dots, N-1}$ with weights i of multiplicities n_i , and by faithfulness of ϕ it is itself an idempotent in $\mathbf{NAWeb}_s^{+, \text{ess}}$.

COROLLARY 3.13. *There is an algebra isomorphism*

$$K_0(\text{Kar}(\mathbf{NAWeb}_s^{+, \text{ess}}) \otimes \mathbb{C}) \cong \text{Sym}(X)$$

sending the the class of the object (m, T_m) to the m -th power-sum symmetric polynomial.

REMARK 3.14. The isomorphisms of Theorem 3.5, which categorify the Newton identities, holds in $\mathbf{NAWeb}_s^{+, \text{ess}}$, although the direct sum decomposition $\bigoplus_k(k, V_k)$ on the right-hand side is not \mathfrak{S}_N -equivariant.

4. SPECIAL PROPERTIES OF THE \mathfrak{gl}_2 CASE

We now review some of the special properties of the extremal weight projectors in the $N = 2$ case. This special case is of course very close to the \mathfrak{sl}_2 one that was studied in [28], but the subtle difference between \mathfrak{sl}_2 and \mathfrak{gl}_2 is needed in topological applications. The idea of using the present construction to categorify the Frohman-Gelca formula for the skein algebra of the torus [12] motivates two major aspects of this section. Lemma 4.4 in particular describes some of the starting cases of the inductive proof in [12], and Lemma 4.15 is key in our next paper [29] to check that the categorical construction does decategorify to the skein module.

In this context, we encode 2-labeled edges in webs as double edges and henceforth omit the labels. For convenience, we list the \mathfrak{gl}_2 web relations for generic q separately.

$$(23) \quad \text{circle with arrow} = (q + q^{-1}) \text{?} = \text{circle with arrow} , \quad \text{circle with arrow} = \text{?} = \text{circle with arrow}$$

$$(24) \quad \text{circle with arrow} = (q + q^{-1}) \uparrow , \quad \text{circle with arrow} = \uparrow = \text{circle with arrow}$$

$$(25) \quad \text{web} = \uparrow \uparrow , \quad \text{web} = \uparrow \uparrow , \quad \uparrow \downarrow = \text{U} , \quad \downarrow \uparrow = \text{U}$$

In fact, \mathfrak{gl}_2 webs satisfy generalizations of the 1-labeled circle relation in (23) that we now describe.

LEMMA 4.1 (The delooping lemma). *Let W be a \mathfrak{gl}_2 -web (in a disc or some other surface) and denote by $c(W)$ the unoriented multi-curve (i.e. a possibly non-connected curve) obtained by erasing all 2-labeled edges. Suppose that $c(W)$ contains a circle c which bounds a disc \mathbf{D} in the complement of $c(W)$. Then $W = (q + q^{-1})V$, where V is a web that agrees with W outside a neighborhood of the disc \mathbf{D} and with underlying curve $c(V)$ obtained by removing the circle in question from $c(W)$.*

Proof. We only consider W in a neighborhood of the disc \mathbf{D} bounded by c . We will find a sequence of web relations which reduce the interaction of 2-labeled edges with c until c can be removed via a the first relation in (23). There are three types of interaction of c with 2-labeled edges to consider in sequence (see Figure 1):

- (1) Any 2-labeled circle contained in \mathbf{D} (see, for example, the first picture in Figure 1) can be removed using one of the relations in (23), starting with an innermost one.
- (2) Suppose there exists a 2-labeled edge in the interior of \mathbf{D} with boundary on c . We take an innermost such edge, i.e. one which encloses a region in the disc with no other 2-labeled edges in the interior (see, for example, the 2-labeled edge at the bottom left of the second picture in Figure 1). Such an intersection edge can be removed via the bigon relations in (24), provided there are no 2-labeled edges hitting the boundary of \mathbf{D} from the outside in the relevant region. Otherwise, jump to (3) to remove external edges first. Note that they always come in pairs for orientation reasons.
- (3) There is a pair of 2-labeled edges, hitting c from outside \mathbf{D} , which are adjacent in the sense that an arc along c connects them without hitting other 2-labeled edges. Then one application of the saddle relations in (25) creates a 2-labeled edge connecting two points on c from the outside (see the right side of Figure 1), which can be removed as in (2).

This algorithm relates W to a web that contains c as an oriented 1-labeled circle that can be removed via (23).

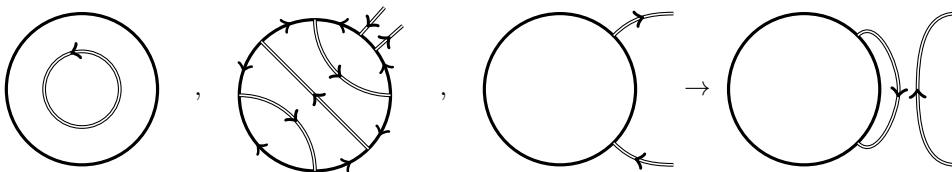


FIGURE 1. Types of interaction of 2-labeled edges with a 1-labeled circle bounding a disc: internal circles, internal and external edges.

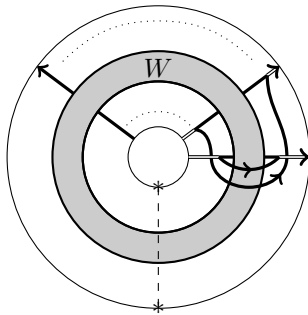
In the following, we again work in the $q = 1$ specialization.

LEMMA 4.2. *The morphism spaces in $\overline{2\mathbf{AWeb}}^{\text{ess},+}$ are spanned by outward pointing webs, except for the endomorphism space of the empty object, which is isomorphic to $\mathbb{C}[c_2^{\pm 1}]$.*

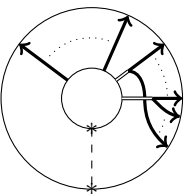
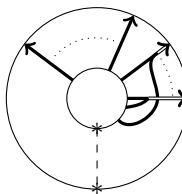
Proof. This follows from Corollaries 2.16 and 2.17.

4.1. DECOMPOSING THE TENSOR PRODUCT OF EXTREMAL WEIGHT PROJECTORS. We have seen that the tensor product of extremal weight projectors $T_m \otimes T_n$ decomposes as a sum of orthogonal idempotents, one of which is T_{m+n} . For \mathfrak{gl}_2 we will explicitly describe the difference $T_m \otimes T_n - T_{m+n}$ in terms of the projector $T_{|m-n|}$. The situation here is very similar to the \mathfrak{sl}_2 -case investigated in [28].

Let pTr_1 denote the linear maps on the morphism spaces of $2\mathbf{AWeb}^+$ that acts on a web W by first tensoring with id_1 and then pre- and post-composing the result with splitter and merge webs (the $k = l = 1$ specialization of the webs from Equations (7) and (6)) between the new strand and the two rightmost 1-labeled bottom and top boundary strands if they exist—otherwise we declare the result to be zero. We use the shorthand $\text{pTr}_n := (\text{pTr}_1)^n$. The following is an example of pTr_2 applied to a web W :



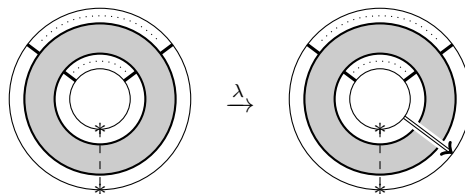
We can decompose $\text{pTr}_n(W) = M_n(W \otimes \text{id}_n)S_n$ where S_n is a splitter web and M_n is a merge web:

(26) $S_n =$  and $M_n =$ 

The diagram shows two circular webs, S_n and M_n . S_n is a splitter web where multiple strands on the left merge into a single strand on the right. M_n is a merge web where a single strand on the left splits into multiple strands on the right. Both diagrams include a dashed vertical line with asterisks at the top and bottom.

Above, the dots on the left in S_n mean that the first strands in S_n are just parallel copies of strands (1 or 2 labeled depending on the labeling of W). On the right there are n 2-labeled strands emerging from the source side. The leftmost splits first, with one newly created 1-labeled strand being braided to the right, crossing other 2-labeled strands. This process is iterated on all of the remaining 2-labeled strands on the right. The picture to have in mind is that if one erases the 2-labeled strands, one sees n nested cups on the right of the picture. M_n is obtained by the reverse process.

Recall that λ denotes the endofunctor of $\overline{2\mathbf{AWeb}}^{\text{ess}}$ given on morphisms by tensoring with a 2-labeled strand on the right as shown in the following:



LEMMA 4.3. *The extremal weight projectors in $\overline{2\mathbf{AWeb}}^{\text{ess}}$ satisfy $\text{pTr}_n(T_m) = \lambda^n(T_{m-n})$ for $1 \leq n < m$ and also for $n = m$ if we set $T_0 = 2$.*

Proof. The cases $m = 1$ and $m = 2$ are easily checked, so we assume $m > 3$. Then we proceed by induction on n . For $n = 1$ we start by expanding

$$\text{pTr}_1(T_m) = \text{pTr}_1((T_{m-1} \otimes \text{id}_1) s_{m-1} (T_{m-1} \otimes \text{id}_1)) = \lambda(T_{m-1}) \text{pTr}_1(s_{m-1}) \lambda(T_{m-1})$$

The result follows from the Reidemeister 1 type move $\text{pTr}_1(s_{m-1}) = \lambda(\text{id}_{m-1})$ and idempotency of T_{m-1} . The induction step $n \rightarrow n + 1$ is analogous, except that it additionally involves Reidemeister 2 moves between 1- and 2-labeled strands.

LEMMA 4.4. *For $m, n > 1$ we have an orthogonal decomposition of idempotents $T_m \otimes T_n = T_{m+n} + e_{m,n}$ in $\overline{2\mathbf{AWeb}}^{\text{ess}}$ where $e_{1,1} = u_1/2 + D^{-1}u_1D/2$ and $e_{m,n} = (T_m \otimes T_n)u_m(T_m \otimes T_n)$ otherwise.*

Proof. For $m = n = 1$ this follows from the explicit description of T_2 . Otherwise we use Lemma 2.30:

$$T_{m+n} = (T_m \otimes T_n) s_m (T_m \otimes T_n) = (T_m \otimes T_n) - (T_m \otimes T_n) u_m (T_m \otimes T_n).$$

Since $e_{m,n}$ contains u_m , it is orthogonal to T_{m+n} . This implies that $e_{m,n}$ is an idempotent as well.

LEMMA 4.5. *For $1 \leq n, m$ and $n + m > 3$, the idempotent $e_{m,n}$ can alternatively be written as*

$$e_{m,n} = (T_m \otimes T_n) (T_{m-r} \otimes (S_r M_r) \otimes T_{n-r}) (T_m \otimes T_n)$$

where $1 \leq r \leq \min(m, n)$ and M_r and S_r are the merge and splitter webs introduced in Equation (26).

Proof. By using forklifts we can write $S_r M_r = \beta u_1 \cdots u_r \beta^{-1}$ where β is the permutation $(1, 2r, 2, 2r - 1, \dots, r, r + 1)$. After replacing u_i by $\text{id} - s_i$, we see that $S_r M_r$ can be expressed as a signed sum of 2^r permutations, with precisely 2^{r-1} terms carrying minus signs. The identity appears only once with positive sign, and all other permutations γ satisfy $\gamma \in S_{2r} \cap (S_r \times S_r)$ and thus $(T_m \otimes T_n) (T_{m-r} \otimes \gamma \otimes T_{n-r}) (T_m \otimes T_n) = T_{m+n}$ by Lemma 2.30 and crossing absorption. This implies:

$$\begin{aligned} & (T_m \otimes T_n) (T_{m-r} \otimes (S_r M_r) \otimes T_{n-r}) (T_m \otimes T_n) \\ &= (T_m \otimes T_n) + (2^{r-1} - 1) T_{m+n} - 2^{r-1} T_{m+n} = e_{m,n}. \end{aligned}$$

LEMMA 4.6. *For $1 \leq n \leq m$ we have $(\text{id}_{m-n} \otimes M_n) (T_m \otimes T_n) (\text{id}_{m-n} \otimes S_n) = \lambda^n (T_{m-n})$ in $\overline{2\mathbf{AWeb}}^{\text{ess}}$.*

Proof. This follows from Lemma 4.3 once we have proved that $M_n(T_n \otimes T_n) = M_n(T_n \otimes \text{id}_n)$. The case $n = 1$ is trivial, whereas for $n = 2$ we have $M_2(T_2 \otimes T_2) - M_2(T_2 \otimes \text{id}_2) = M_2(T_2 \otimes e_{1,1})$ and it is not hard to check that the latter is zero. For the induction step we compute

$$\begin{aligned} M_n(T_n \otimes T_n) &= M_n(\text{id}_{n-2} \otimes T_2 \otimes T_2 \otimes \text{id}_{n-2}) (T_{n-1} \otimes \text{id}_2 \otimes T_{n-1}) \\ &= M_n(\text{id}_{n-2} \otimes T_2 \otimes \text{id}_n) (T_{n-1} \otimes \text{id}_2 \otimes T_{n-1}) \\ &= M_n(T_{n-1} \otimes \text{id}_2 \otimes T_{n-1}) (\text{id}_{n-2} \otimes T_2 \otimes \text{id}_n) \\ &= M_n(T_{n-1} \otimes \text{id}_{n+1}) (\text{id}_{n-2} \otimes T_2 \otimes \text{id}_n) = M_n(T_n \otimes \text{id}_n) \end{aligned}$$

In the first and last line, we use (3) in Theorem 2.28. For the second and last line we use the case $n = 2$ and the induction hypothesis for $n - 1$. The third line arises from projector commutation.

PROPOSITION 4.7. *For $1 \leq n < m$ the idempotents $e_{m,n}$, $e_{n,m}$ and $\lambda^n (T_{m-n})$ represent isomorphic objects in $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess}})$.*

Proof. We use Lemma 4.5 to write $e_{m,n} = (T_m \otimes T_n)(T_{m-n} \otimes S_n)(T_{m-n} \otimes M_n)(T_m \otimes T_n)$. Then it is immediate from Lemma 4.6 that the maps $(T_m \otimes T_n)(T_{m-n} \otimes S_n)$ and $(T_{m-n} \otimes M_n)(T_m \otimes T_n)$ are inverse isomorphisms between the elements of the Karoubi element represented by the idempotents $e_{m,n}$ and $\lambda^n(T_{m-n})$. The proof for $e_{n,m}$ is similar.

PROPOSITION 4.8. *The idempotent $e_{m,m}$ is isomorphic to $\lambda^m(?) \oplus w\lambda^m(?)$ in $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess},+})$.*

Proof. For $m = 1$ we have $e_{1,1} = u_1/2 + D^{-1}u_1D/2$. The two summands are orthogonal idempotents. The first is isomorphic to $\lambda(?)$ in $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess},+})$, while the conjugation by D in the second summand makes it isomorphic to $w\lambda(?)$. For $m > 1$ we rewrite

$$\begin{aligned} e_{m,m} &= (T_m \otimes T_m)(\text{id}_1 \otimes (S_{m-1}M_{m-1}) \otimes \text{id}_1)(T_m \otimes T_m) \\ &= (T_m \otimes T_m)S_mM_m(T_m \otimes T_m)/2 \\ &\quad + (T_m \otimes T_m)D^{-1}u_{2m-1}D(\text{id}_1 \otimes (S_{m-1}M_{m-1}) \otimes \text{id}_1)(T_m \otimes T_m)/2 \\ &= \underbrace{(T_m \otimes T_m)S_m}_{\phi_1} \circ \underbrace{M_m(T_m \otimes T_m)}_{\psi_1} / 2 \\ &\quad + \underbrace{(T_m \otimes T_m)D^{-1}(S_{m-1} \otimes S_1)}_{\phi_2} \circ \underbrace{(M_{m-1} \otimes M_1)D(T_m \otimes T_m)}_{\psi_2} / 2 \end{aligned}$$

The equality in the second line can be verified by inserting $D^{-1}(S_{m-1}M_{m-1} \otimes T_2)D$ between two factors of $T_m \otimes T_m$ and realizing that the result is zero. To prove the proposition it remains to verify that $\psi_i\phi_j = \delta_{i,j}\lambda^m(?)$. We give one example for orthogonality:

$$\begin{aligned} \psi_1\phi_2 &= M_m(T_m \otimes T_m)D^{-1}(S_{m-1} \otimes S_1)/2 \\ &= M_m(T_m \otimes \text{id}_m)D^{-1}(S_{m-1} \otimes S_1)/2 = \lambda^{m-1}(?) \otimes (M_1D^{-1}S_1) = 0 \end{aligned}$$

Here we have used the proof of Lemma 4.6, an isotopy and the essential torus relation. The proof of $\psi_2\phi_1 = 0$ is analogous. $\psi_i\phi_1 = \lambda^m(?)$ follows from Lemma 4.3. It remains to check

$$\psi_2\phi_2 = (M_{m-1} \otimes M_1)D(T_m \otimes T_m)D^{-1}(S_{m-1} \otimes S_1)/2 = \lambda^m(?)$$

For $m = 2$ this follows by expanding the left copy of T_2 and seeing that all terms except the identity term die. The result is evaluated using Lemma 4.3. For $m > 3$, we use the recursion on the left copy of T_m , absorb the resulting copies of T_{m-1} as the proof of Lemma 4.6 and then simplify via Lemma 4.3. The result is equal to $\lambda^{m-2}(?)$ superimposed with the $m = 2$ case, which we have already checked.

4.2. SKELETA. Recall that we put basepoints on the boundary components of the annulus and fix a connecting arc α between them, which cuts the annulus into a square – this is drawn as a dashed line above.

LEMMA 4.9. *The endomorphism algebra of T_m in $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess}})^*$ is isomorphic to $C[D^{\pm 1}]$ if $m > 1$ and isomorphic to $C[c_2^{\pm 1}]$ if $m = 0$. In $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess}})$ both are given by C .*

Proof. Any endomorphism of T_m is represented by a linear combination of outward pointing webs with m 1-labeled input and output strands. Such webs factor into dumbbells u_i and wraps $D^{\pm 1}$. Since T_m kills all u_i , the endomorphism algebra is generated by the isomorphisms $D^{\pm 1}$. Since all web relations in $\overline{2\mathbf{AWeb}}^{\text{ess}}$ preserve

the flow winding number of the web around the annulus, it is clear that the elements D^m for $m \in \mathbb{Z}$ are linearly independent.

LEMMA 4.10. *The endomorphism algebra of $\lambda^k(T_m)$ in $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess}})^*$ is isomorphic to $\mathbb{C}[\lambda^k(D^{\pm 1})]$ if $m > 1$ and isomorphic to $\mathbb{C}[D_2^{\pm 1}]$ if $m = 0$. In $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess}})$ both types of endomorphism algebras are isomorphic to \mathbb{C} . Furthermore, in both versions of the Karoubi envelope, there are no non-zero morphisms between $\lambda^k(T_m)$ and $\lambda^l(T_n)$ (or their w -shifts) unless $m = n$ and $k = l$.*

Proof. The first part is an immediate corollary of Lemma 4.9, due to the fact that λ is an auto-equivalence, which guarantees that it induces isomorphisms on endomorphism algebras. For the second part, it is clear that we need $m + 2k = n + 2l$ to have both objects in the same block. Now suppose, without loss of generality, that $k < l$. Then an application of the essential inverse $(\lambda^*)^k$ provides an isomorphism between the morphism space in question and $\text{Hom}(T_m, \lambda^{l-k}(T_n))$. However, any web representing a morphism in that space necessarily contains a merge vertex, which is killed by T_m . Thus $\text{Hom}(T_m, \lambda^{l-k}(T_n)) = 0$.

LEMMA 4.11. *For $m > 1$ and any a, b and k , we have*

$$\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess}})(w^a \lambda^k(T_m), w^b \lambda^k(T_m)) \cong \mathbb{C}\langle \lambda^k(D^{b-a}) \rangle.$$

On the other hand:

$$\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess}})(w^a \lambda^k(?), w^b \lambda^k(?)) \cong \begin{cases} \mathbb{C}\langle D_2^{(b-a)/2} \rangle & \text{if } a - b \text{ is even and } k > 1 \\ \mathbb{C}\langle c_2^{(b-a)/2} \rangle & \text{if } a - b \text{ is even and } k = 0 \\ 0 & \text{if } a - b \text{ is odd} \end{cases}$$

Clearly, all such non-zero morphisms are isomorphisms.

In particular, this implies, that all objects of the form $w^a \lambda^k(T_m)$ are actually isomorphic to unshifted objects $\lambda^k(T_m)$ if $m > 1$. The objects $w^a \lambda^k(?)$, on the other hand, are isomorphic to their versions with $a = 1$ or $a = 0$.

LEMMA 4.12. *Any object in $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess},+})$ is isomorphic to a direct sum of objects $\lambda^k(T_m)$ for $m > 1, k > 0$ or $\lambda^k(?)$ or $w\lambda^k(?)$ for $k > 0$.*

Proof. It suffices to decompose the objects in $\overline{2\mathbf{AWeb}}^{\text{ess},+}$ into a formal direct sum of objects of the above type. Moreover, since 2-labeled objects are isomorphic to idempotents on 1-labeled objects in the Karoubi envelope, we only need to decompose id_m . Then any idempotent endomorphism of id_m will give rise to an idempotent endomorphism of the decomposition, which is necessarily block-diagonal (there are no morphisms between distinct objects of the form $\lambda^k(T_{m-2k})$ or $w\lambda^{m/2}(?)$) and has entries in \mathbb{C} . Such idempotent matrices can be diagonalized, and thus decompose into objects of type $\lambda^k(T_{m-2k})$ or $w\lambda^{m/2}(?)$.

The decomposition for id_m follows inductively from the parallel product formulas in Propositions 4.7 and 4.8. More precisely, if we already know that id_m is isomorphic to a direct sum of terms $\lambda^{k_i}(T_{m-2k_i})$ and possibly $w\lambda^{m/2}(?)$ if m is even, then id_{m+1} can be decomposed into summands

$$\begin{aligned} \lambda^{k_i}(T_{m-2k_i}) \otimes \text{id}_1 &\cong \lambda^{k_i}(T_{m-2k_i} \otimes T_1) \\ &\cong \begin{cases} \lambda^{k_i}(T_{m-2k_i+1}) \oplus \lambda^{k_i+1}(T_{m-2k_i-1}) & m - 2k_i > 1 \\ \lambda^{k_i}(T_2) \oplus \lambda^{k_i+1}(?) \oplus w\lambda^{k_i+1}(?) & m - 2k_i = 1 \end{cases} \end{aligned}$$

as well as $\lambda^{m/2}(?) \otimes \text{id}_1 \cong w\lambda^{m/2}(?) \otimes \text{id}_1 \cong \lambda^{m/2}(T_1)$.

Since orientation of the boundary can be reversed by means of the auto-equivalences λ and λ^* , one easily extends the previous result to the whole category:

COROLLARY 4.13. *Any object in $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess}})$ is isomorphic to a direct sum of objects $\lambda^k(T_m)$ for $m > 1$, $k \in \mathbb{Z}$ or $\lambda^k(?)$ or $w\lambda^k(?)$ for $k \in \mathbb{Z}$. Here we identify $\lambda^{-1} = \lambda^*$.*

DEFINITION 4.14. *A subcategory D of a category C is a skeleton of C if the inclusion of D into C is an equivalence of categories and additionally no two distinct objects of D are isomorphic.*

LEMMA 4.15. *The full subcategory of $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess}})$ containing (lexicographically ordered direct sums of) the objects $\lambda^k(T_m)$ and $s\lambda^k(?)$ for $k, m > 0$ is a skeleton of $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess},+})$. Moreover, this skeleton is semisimple.*

Proof. The inclusion of this full subcategory is essentially surjective by Lemma 4.12. Moreover, by Lemma 4.10, the decomposition of an object of $\text{Kar}(\overline{2\mathbf{AWeb}}^{\text{ess},+})$ into a lexicographically ordered direct sum of such simples is essentially unique. In particular, there are also no isomorphisms between distinct direct sums of simples. We have also seen that the endomorphism algebras of simples are isomorphic to \mathbb{C} , which implies that the skeleton is semisimple.

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