ALGEBRAIC COMBINATORICS

Hiraku Abe, Tatsuya Horiguchi, Hideya Kuwata & Haozhi Zeng

Geometry of Peterson Schubert calculus in type A and left-right diagrams

Volume 7, issue 2 (2024), p. 383-412.
https://doi.org/10.5802/alco.342

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 7, issue 2 (2024), p. 383–412
https://doi.org/10.5802/alco.342

Geometry of Peterson Schubert calculus in type A and left-right diagrams

Hiraku Abe, Tatsuya Horiguchi, Hideya Kuwata & Haozhi Zeng

Abstract We introduce an additive basis of the integral cohomology ring of the Peterson variety which reflects the geometry of certain subvarieties of the Peterson variety. We explain the positivity of the structure constants from a geometric viewpoint, and provide a manifestly positive combinatorial formula for them. We also prove that our basis coincides with the additive basis introduced by Harada–Tymoczko.

1. Introduction

Let n be a positive integer and Fl_n the full-flag variety of $\mathbb{C}^n$. As a set, Fl_n is the collection of nested sequences of linear subspaces of $\mathbb{C}^n$ given as follows:

$$Fl_n = \{V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i \ (1 \leqslant i \leqslant n)\}.$$

Let N be an $n \times n$ regular nilpotent matrix viewed as a linear map $N\colon \mathbb{C}^n \to \mathbb{C}^n$. The Peterson variety $Pet_n \subseteq Fl_n$ is defined by

$$Pet_n \coloneqq \{V_\bullet \in Fl_n \mid NV_i \subseteq V_{i+1} \ (1 \leqslant i \leqslant n-1)\},$$

where NV_i denotes the image of V_i under the map $N\colon \mathbb{C}^n \to \mathbb{C}^n$. It was introduced by Dale Peterson to study the quantum cohomology ring of Fl_n, and it has since appeared in several contexts (e.g. [6, 8, 19, 25, 31]).

For a permutation $w \in \mathfrak{S}_n$, let $X_w \subseteq Fl_n$ be the Schubert variety associated with w, and $\Omega_w \subseteq Fl_n$ the dual Schubert variety associated with w. We denote by $[n-1]$ the set of integers $1, 2, \ldots, n-1$. For a subset $J \subseteq [n-1]$, let $w_J \in \mathfrak{S}_n$ be the longest element of the Young subgroup $\mathfrak{S}_J$ of the permutation group $\mathfrak{S}_n$ associated with J (see Section 2.1 for details), and set

$$X_J \coloneqq X_{w_J} \cap Pet_n \quad \text{and} \quad \Omega_J \coloneqq \Omega_{w_J} \cap Pet_n.$$

Then X_J and Ω_J in Pet_n play roles analogous to those of Schubert varieties and dual Schubert varieties in Fl_n, and provide important information about the topology of Pet_n.

In this paper, we construct an additive basis $\{\varpi_J \mid J \subseteq [n-1]\}$ of the integral cohomology ring $H^*(Pet_n;\mathbb{Z})$ which reflects the geometry of X_J and Ω_J (Theorem 4.14). As a consequence, we may consider the structure constants for the multiplication rule:

$$\varpi_J \cdot \varpi_K = \sum_{L \subseteq [n-1]} d_{JK}^L \varpi_L, \qquad d_{JK}^L \in \mathbb{Z}. \tag{1.1}$$

Manuscript received 20th April 2023, revised 30th October 2023, accepted 31st October 2023.

Keywords. Peterson variety, Peterson Schubert calculus.

ISSN: 2589-5486

It turns out that all d_{JK}^L are non-negative integers, and we give a geometric proof of this positivity by using that of nef line bundles over Pet_n (Proposition 4.16). We also provide a manifestly positive combinatorial formula for d_{JK}^L (Theorem 5.6) in terms of *left-right diagrams*, which we introduce in this paper.

To find our formula for d_{JK}^L, we prove several properties of the cohomology classes ϖ_J which are inherited from the geometry of X_J and Ω_J. In particular, writing Ω_J as an intersection of divisors on Pet_n provides the geometric idea behind our formula for d_{JK}^L in terms of left-right diagrams.

We also show that our basis $\{\varpi_J \mid J \subseteq [n-1]\}$ coincides with the additive basis of the cohomology ring $H^*(Pet_n;\mathbb{C})$ with $\mathbb{C}$-coefficients introduced by Harada–Tymoczko [20]. Their basis is obtained by taking restriction of certain Schubert classes to Pet_n, and it is called the *Peterson Schubert basis*. It has been studied by Bayegan–Harada [9], Drellich [14] and Goldin–Gorbutt [18]. In [18], Goldin and Gorbutt gave combinatorial formulas for the structure constants of Harada–Tymoczko's basis (in a certain equivariant setting) which are manifestly positive and integral. Thus, after taking the non-equivariant limit, their formulas and ours both describe the same structure constants, but these formulas have different perspectives; their approach is mostly combinatorial whereas our approach is based on the geometry of X_J and Ω_J. We include a short comparison of their formulas and ours in Section 6.

Interestingly, our computations match with those of Berget–Spink–Tseng [10, Sect. 7] on a certain subring of the cohomology ring of a toric variety which is called the permutohedral variety. One of their results can be interpreted as a formula describing the structure constants d_{JK}^L as products of *mixed Eulerian numbers* which were introduced and studied by Postnikov [30]. With this connection in mind, our formula (Theorem 5.6) for d_{JK}^L can also be thought as computing some products of mixed Eulerian numbers by using the geometry of Pet_n. We explain this connection in Section 6 including the relations with the works of Nadeau–Tewari [28] and the second author [21].

Acknowledgements. We are grateful to Mikiya Masuda for his support and encouragement. We also thank the anonymous referees for reading this paper very carefully. This research is supported in part by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics): the topology and combinatorics of Hessenberg varieties. The first author is supported in part by JSPS Grant-in-Aid for Early-Career Scientists: 18K13413. The second author is supported in part by JSPS Grant-in-Aid for Young Scientists: 19K14508. The fourth author is supported in part by NSFC: 11901218.

2. Basic notations

In this section, we recall some terminology which will be used in this paper.

2.1. Combinatorics on the Dynkin diagram of type A. Let $n(\geqslant 2)$ be a positive integer. We use the notation $[n-1] \coloneqq \{1,2,\ldots,n-1\}$, and we regard it as the set of vertices of the Dynkin diagram of type A_{n-1} for the rest of the paper. Namely, two vertices $i,j \in [n-1]$ are connected by an edge if and only if $|i-j| = 1$. See Figure 1.

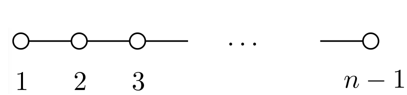

Figure 1. The Dynkin diagram of type A_{n-1}.

We regard each subset $J \subseteq [n-1]$ as a full-subgraph of the Dynkin diagram appearing above. We may decompose it into the connected components:

$$J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_m, \tag{2.1}$$

where each J_k $(1 \leqslant k \leqslant m)$ is the set of vertices of a maximal connected subgraph of J. To determine each J_k uniquely, we require that the maximal element of J_k is less than the minimal element of $J_{k'}$ when $k < k'$.

Example 2.1. Let $n = 10$ and $J = \{1, 2, 4, 5, 6, 9\}$. Then we have

$$J = \{1, 2\} \sqcup \{4, 5, 6\} \sqcup \{9\} = J_1 \sqcup J_2 \sqcup J_3.$$

For $J \subseteq [n-1]$, one has the Young subgroup given by

$$\mathfrak{S}_J := \mathfrak{S}_{J_1} \times \mathfrak{S}_{J_2} \times \cdots \mathfrak{S}_{J_m} \subseteq \mathfrak{S}_n,$$

where each $\mathfrak{S}_{J_k}$ is the subgroup of $\mathfrak{S}_n$ generated by the simple reflections s_i for all $i \in J_k$. Let w_J be the longest element of $\mathfrak{S}_J$, i.e.

$$w_J := w_0^{(J_1)} w_0^{(J_2)} \cdots w_0^{(J_m)} \in \mathfrak{S}_J, \tag{2.2}$$

where each $w_0^{(J_k)}$ is the longest element of the permutation group $\mathfrak{S}_{J_k}$ $(1 \leqslant k \leqslant m)$.

Example 2.2. Let $n = 10$ and $J = \{1, 2\} \sqcup \{4, 5, 6\} \sqcup \{9\}$ as above. By identifying the permutation w_J with its permutation matrix, we have

$$w_J = w_0^{(J_1)} w_0^{(J_2)} w_0^{(J_3)} = (s_1 s_2 s_1)(s_4 s_5 s_6 s_4 s_5 s_4)(s_9) = \left(\begin{array}{ccc|cccc|c|cc} & & 1 & & & & & & & \\ & 1 & & & & & & & & \\ 1 & & & & & & & & & \\ \hline & & & & & & 1 & & & \\ & & & & & 1 & & & & \\ & & & & 1 & & & & & \\ & & & 1 & & & & & & \\ \hline & & & & & & & 1 & & \\ \hline & & & & & & & & & 1 \\ & & & & & & & & 1 & \end{array}\right).$$

We can identify each permutation $w \in \mathfrak{S}_n$ with its permutation flag $V_\bullet \in Fl_n$ defined by $V_i = \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(i)} \rangle$ for $1 \leqslant i \leqslant n$, where $e_1, e_2, \ldots, e_n$ denotes the standard basis of $\mathbb{C}^n$, and the right hand side is the linear subspace of $\mathbb{C}^n$ spanned by $e_{w(1)}, e_{w(2)}, \ldots, e_{w(i)}$. Using this identification, we explain how the permutations w_J are related to the Peterson variety. Let $\mathrm{GL}_n(\mathbb{C})$ be the general linear group of invertible $n \times n$ complex matrices. Let $T \subseteq \mathrm{GL}_n(\mathbb{C})$ be the maximal torus consisting of diagonal matrices. Let us identify $\mathbb{C}^\times$ with a subgroup of T as follows:

$$\mathbb{C}^\times = \left\{ \begin{pmatrix} g & & & \\ & g^2 & & \\ & & \ddots & \\ & & & g^n \end{pmatrix} \in T \;\middle|\; g \in \mathbb{C}^\times \right\}. \tag{2.3}$$

The flag variety Fl_n admits a natural $\mathrm{GL}_n(\mathbb{C})$-action by regarding each element $g \in \mathrm{GL}_n(\mathbb{C})$ as an automorphism $g \colon \mathbb{C}^n \to \mathbb{C}^n$. Restricting this $\mathrm{GL}_n(\mathbb{C})$-action on Fl_n to the above subgroup $\mathbb{C}^\times$, it is well-known that the fixed point set $(Fl_n)^{\mathbb{C}^\times}$ is the set of the permutation flags, i.e. $(Fl_n)^{\mathbb{C}^\times} = \mathfrak{S}_n$ (e.g. [16, proof of Lemma 2 in Sect. 10.1]). It is straightforward to see that this $\mathbb{C}^\times$-action on Fl_n preserves Pet_n, and it was shown in [20] that the fixed point set $(Pet_n)^{\mathbb{C}^\times}$ is given by

$$(Pet_n)^{\mathbb{C}^\times} = \{w_J \in \mathfrak{S}_n \mid J \subseteq [n-1]\}. \tag{2.4}$$

Because of this relation, the combinatorics of w_J will be important to understand the structure of Pet_n.

For $1 \leqslant i \leqslant n-1$, let $s_i \in \mathfrak{S}_n$ be the simple reflection which interchanges i and $i+1$. We denote by $\leqslant$ the Bruhat order on $\mathfrak{S}_n$, that is, we have $u \leqslant v$ $(u, v \in \mathfrak{S}_n)$ if and only if a reduced expression of u is a subword of a reduced expression of v.

Lemma 2.3. *For $J \subseteq [n-1]$ and $1 \leqslant i \leqslant n-1$, we have $s_i \leqslant w_J$ if and only if $i \in J$.*

Proof. Recall that w_J is the product of longest elements in $\mathfrak{S}_{J_k}$ for $1 \leqslant k \leqslant m$:

$$w_J = w_0^{(J_1)} w_0^{(J_2)} \cdots w_0^{(J_m)}.$$

Since each $w_0^{(J_k)}$ $(1 \leqslant k \leqslant m)$ preserves the decomposition (2.1), it follows that the length of w_J is the same as the sum of the lengths of $w_0^{(J_k)}$ for $1 \leqslant k \leqslant m$. Thus, the products of reduced expressions of $w_0^{(J_k)}$ for $1 \leqslant k \leqslant m$ give a reduced expression of w_J. Here, an arbitrary reduced expression of $w_0^{(J_k)}$ contains a simple reflection s_i if and only if $i \in J_k$. Therefore, it follows that a reduced expression of w_J contains s_i if and only if $i \in J$ (see Example 2.2). This implies the desired claim. □

The following claim appears in [24, Lemma 6], but we give a proof for the reader.

Lemma 2.4. *For $J, J' \subseteq [n-1]$, we have*

$$w_{J'} \leqslant w_J \quad \text{if and only if} \quad J' \subseteq J. \tag{2.5}$$

Proof. We first prove that $J' \subseteq J$ under the assumption $w_{J'} \leqslant w_J$. For this, take an arbitrary element $i \in J'$. By the previous lemma, we have $s_i \leqslant w_{J'}$. Combining this with the assumption $w_{J'} \leqslant w_J$, we obtain that $s_i \leqslant w_J$. Thus, it follows that $i \in J$ by the previous lemma again.

We next prove that $w_{J'} \leqslant w_J$ under the assumption $J' \subseteq J$. Take the decomposition

$$J' = J'_1 \sqcup J'_2 \sqcup \cdots \sqcup J'_{m'}$$

into the connected components as in (2.1). Since each J'_ℓ $(1 \leqslant \ell \leqslant m')$ is connected, it is contained in some connected component J_k of J. This leads us to consider a map

$$\varphi\colon \{1, 2, \ldots, m'\} \to \{1, 2, \ldots, m\}$$

which we define by the conditions $J'_i \subseteq J_{\varphi(i)}$ for $1 \leqslant i \leqslant m'$. Then we have that

$$\bigsqcup_{\varphi(i)=k} J'_i \subseteq J_k$$

by the definition of the map φ. This implies that

$$\prod_{\varphi(i)=k} w_0^{(J'_i)} \leqslant w_0^{(J_k)} \qquad \text{in } \mathfrak{S}_{J_k}$$

since $w_0^{(J_k)}$ is the longest permutation in $\mathfrak{S}_{J_k}$. Recalling that each J_k is a connected component of J, these inequalities for $1 \leqslant k \leqslant m$ imply that $w_{J'} \leqslant w_J$. □

Example 2.5. *Let $n = 8$, $J' = \{1, 4, 5, 7\}$ and $J = \{1, 2, 4, 5, 6, 7\}$ so that we have $J' \subseteq J$. In this case, we have*

$$J' = \{1\} \sqcup \{4, 5\} \sqcup \{7\} = J'_1 \sqcup J'_2 \sqcup J'_3 \quad \text{and} \quad J = \{1, 2\} \sqcup \{4, 5, 6, 7\} = J_1 \sqcup J_2,$$

and hence

$$w_{J'} = (s_1)(s_4 s_5 s_4)(s_7) \leqslant (s_1 s_2 s_1)(s_4 s_5 s_6 s_7 s_4 s_5 s_6 s_4 s_5 s_4) = w_J.$$

2.2. Hessenberg varieties. In this subsection, we briefly recall the notion of Hessenberg varieties in type A. They are (possibly reducible) subvarieties of the flag variety Fl_n, and will appear in the next section. A function $h\colon [n] \to [n]$ is a *Hessenberg function* if it satisfies the following two conditions:

(i) $h(1) \leqslant h(2) \leqslant \cdots \leqslant h(n)$,
(ii) $h(j) \geqslant j$ for all $j \in [n]$.

Note that $h(n) = n$ by definition. We may identify a Hessenberg function h with a configuration of (shaded) boxes on a square grid of size $n \times n$ which consists of boxes in the i-th row and the j-th column satisfying $i \leqslant h(j)$ for $i, j \in [n]$, as we illustrate in the following example.

Example 2.6. *Let $n = 5$. The Hessenberg function $h\colon [5] \to [5]$ given by*

$$(h(1), h(2), h(3), h(4), h(5)) = (2, 3, 3, 5, 5)$$

corresponds to the configuration of the shaded boxes drawn in Figure 2.

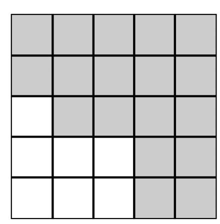

Figure 2. The configuration of the shaded boxes corresponding to h.

For an $n \times n$ matrix X considered as a linear map $X\colon \mathbb{C}^n \to \mathbb{C}^n$ and a Hessenberg function $h\colon [n] \to [n]$, the *Hessenberg variety* associated with X and h ([12, 33]) is defined as

$$\mathrm{Hess}(X, h) = \{V_\bullet \in Fl_n \mid XV_j \subseteq V_{h(j)} \text{ for all } j \in [n]\}. \tag{2.6}$$

We note that $\mathrm{Hess}(X, h) \cong \mathrm{Hess}(gXg^{-1}, h)$ for all $g \in \mathrm{GL}_n(\mathbb{C})$ so that we may always assume that the matrix X is in Jordan canonical form. Let N be the $n \times n$ regular nilpotent matrix in Jordan canonical form, i.e.

$$N = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}. \tag{2.7}$$

Then $\mathrm{Hess}(N, h)$ is called a *regular nilpotent Hessenberg variety*. It is known that $\mathrm{Hess}(N, h)$ is an irreducible projective variety of (complex) dimension $\sum_{i=1}^{n}(h(i) - i)$ ([7, Lemma 7.1] and [33]). When $h(i) = i + 1$ for all $1 \leqslant i \leqslant n - 1$, it is clear that $\mathrm{Hess}(N, h) = Pet_n$ by definition. In particular, we obtain the well-known formula

$$\dim_{\mathbb{C}} Pet_n = n - 1.$$

For Hessenberg functions $h, h'\colon [n] \to [n]$, it is clear that if $h(i) \leqslant h'(i)$ for $1 \leqslant i \leqslant n$ then $\mathrm{Hess}(N, h) \subseteq \mathrm{Hess}(N, h')$. For example, the Hessenberg function $h\colon [5] \to [5]$ given in Example 2.6 defines a 3-dimensional regular nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$ which is contained in $Pet_5 (\subseteq Fl_5)$.

3. Geometric constructions

In this section, we introduce two kinds of subvarieties X_J and Ω_J in Pet_n for each $J \subseteq [n-1]$, and we establish geometric properties of them. They will play important roles in constructing an additive basis of the integral cohomology ring $H^*(Pet_n;\mathbb{Z})$ in the next section.

3.1. Analogue of Schubert varieties in the Peterson variety.

For $w \in \mathfrak{S}_n$, let $X_w \subseteq Fl_n$ be the Schubert variety associated with w and $\Omega_w \subseteq Fl_n$ the dual Schubert variety associated with w ([16, Sect. 10]). We note that $\dim_{\mathbb{C}} X_w = \operatorname{codim}_{\mathbb{C}}(\Omega_w, Fl_n) = \ell(w)$ and $X_w \cap \Omega_w = \{w\}$, where $\ell(w)$ is the length of w.

Definition 3.1. *For $J \subseteq [n-1]$, we define*

$$X_J \coloneqq X_{w_J} \cap Pet_n \quad \text{and} \quad \Omega_J \coloneqq \Omega_{w_J} \cap Pet_n, \tag{3.1}$$

where $w_J \in \mathfrak{S}_n$ is the permutation defined in (2.2).

Peterson [29] studied a particular open affine subset of Ω_J to construct the quantum cohomology ring of Fl_n (c.f. [25, 31]). Also, Insko [23] and Insko–Tymoczko [24] studied X_J to show the injectivity of the homomorphism $H_*(Pet_n;\mathbb{Z}) \to H_*(Fl_n;\mathbb{Z})$. It turns out that X_J and Ω_J in Pet_n play roles analogous to those of Schubert varieties and dual Schubert varieties in Fl_n. As an illustrating property, we begin with the following claim. Recall that we have $X_w \cap \Omega_v \neq \varnothing$ in Fl_n if and only if $w \geqslant v$.

Proposition 3.2. *For $J, J' \subseteq [n-1]$, we have*

$$X_J \cap \Omega_{J'} \neq \varnothing \quad \textit{if and only if} \quad J \supseteq J'.$$

Moreover, when $J = J'$, we have $X_J \cap \Omega_J = \{w_J\}$.

Proof. If $X_J \cap \Omega_{J'} \neq \varnothing$, then we have $(X_{w_J} \cap \Omega_{w_{J'}}) \cap Pet_n \neq \varnothing$ by definition. Note that $(X_{w_J} \cap \Omega_{w_{J'}}) \cap Pet_n$ is complete, and it is preserved by the $\mathbb{C}^\times$-action on Pet_n described in Section 2. Thus, it follows that it contains a $\mathbb{C}^\times$-fixed point (e.g. [22, Chap. VIII, Sect. 21.2]). Since we have

$$((X_{w_J} \cap \Omega_{w_{J'}}) \cap Pet_n)^{\mathbb{C}^\times} = (X_{w_J} \cap \Omega_{w_{J'}})^{\mathbb{C}^\times} \cap (Pet_n)^{\mathbb{C}^\times},$$

we see that there exists a $\mathbb{C}^\times$-fixed point $w_K \in Pet_n$ (see (2.4)) such that $w_K \in X_{w_J}$ and $w_K \in \Omega_{w_{J'}}$. The former condition implies that $w_J \geqslant w_K$, and the latter condition implies that $w_K \geqslant w_{J'}$ ([16, Sect. 10.2 and 10.5]). Thus, we obtain $w_J \geqslant w_{J'}$, and it follows that $J \supseteq J'$ from Lemma 2.4.

If $J \supseteq J'$, then we have $w_J \geqslant w_{J'}$ by Lemma 2.4. This implies that $w_J \in X_{w_J} \cap \Omega_{w_{J'}}$, and hence $X_J \cap \Omega_{J'} \neq \varnothing$ follows. □

A distinguished property of X_J is that it is a regular nilpotent Hessenberg variety for a certain Hessenberg function. Let us explain this in the following. For each $J \subseteq [n-1]$, there is a natural Hessenberg function which is determined by J as follows. Let $h_J\colon [n] \to [n]$ be a function given by

$$h_J(i) = \begin{cases} i+1 & \text{if } i \in J \\ i & \text{if } i \notin J \end{cases} \tag{3.2}$$

for $1 \leqslant i \leqslant n$. Then h_J is a Hessenberg function, and we have $\mathrm{Hess}(N, h_J) \subseteq Pet_n$ since the Hessenberg function for Pet_n is given by $h(i) = i+1$ for $1 \leqslant i \leqslant n-1$ as we saw in Section 2.2.

EXAMPLE 3.3. Let $n = 10$ and $J = \{1,2\} \sqcup \{4,5,6\} \sqcup \{9\} = J_1 \sqcup J_2 \sqcup J_3$. Then the configuration of boxes of h_J is given in Figure 3. Compare the figure with the permutation matrix of w_J in Example 2.2.

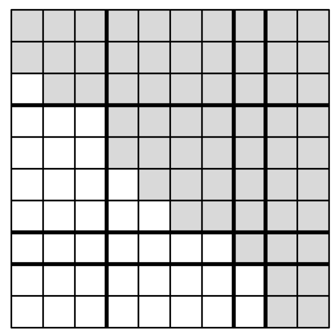

FIGURE 3. The Hessenberg function h_J.

As we mentioned above, Insko–Tymoczko [24] studied X_J, and they proved most of the following claim. Recall that X_J is defined to be the intersection $X_{w_J} \cap Pet_n$ where X_{w_J} is the Schubert variety associated with w_J.

PROPOSITION 3.4. *For $J \subseteq [n-1]$, we have*

$$X_J = \overline{X^\circ_{w_J} \cap Pet_n} = Hess(N, h_J) \tag{3.3}$$

where N is the regular nilpotent matrix given in (2.7)*, and $X^\circ_{w_J}$ is the Schubert cell associated with w_J. In particular, we have* $\dim_{\mathbb{C}} X_J = |J|$.

To prove this, we need the following lemma. Let $X^\circ_w \subseteq Fl_n$ be the Schubert cell associated with w and and $\Omega^\circ_w \subseteq Fl_n$ the dual Schubert cell associated with w.

LEMMA 3.5. *The following are equivalent.*

(1) $X^\circ_w \cap Pet_n \neq \varnothing$
(2) $\Omega^\circ_w \cap Pet_n \neq \varnothing$
(3) $w \in Pet_n$ (*i.e.* $w = w_J$ *for some* $J \subseteq [n-1]$)

Proof. It is clear that (3) implies (1). To see that (1) implies (3), take an element $z \in X^\circ_w \cap Pet_n \neq \varnothing$. Since $X^\circ_w \cap Pet_n \subseteq X^\circ_w$ is preserved under the $\mathbb{C}^\times$-action on X°_w, it follows that $t \cdot z \in X^\circ_w \cap Pet_n$ for all $t \in \mathbb{C}^\times$. Noticing that $X^\circ_w \cap Pet_n \subseteq X^\circ_w$ is a closed subset, we have

$$\lim_{t\to 0} t \cdot z \in X^\circ_w \cap Pet_n. \tag{3.4}$$

Under the standard identification $X^\circ_w = \mathbb{C}^{\ell(w)}$ (c.f. [16, Sect. 10.2]), the $\mathbb{C}^\times$-action on X°_w is identified with a linear action with positive weights. Thus it follows that $\lim_{t\to 0} t \cdot z = 0$, which corresponds to $w \in X^\circ_w$ (c.f. [24, proof of Lemma 5]). Thus it follows that $w \in Pet_n$ by (3.4). The equivalence of (2) and (3) follows by an argument similar to that for the equivalence of (1) and (3). ☐

Proof of Proposition 3.4. Let us first prove that

$$X^\circ_{w_J} \cap Pet_n \subseteq \mathrm{Hess}(N, h_J) \qquad \text{for each } J \subseteq [n-1]. \tag{3.5}$$

For this, take an arbitrary element $V_\bullet \in X^\circ_{w_J} \cap Pet_n$. Then we have

$$NV_i \subseteq V_{i+1} \qquad (1 \leqslant i \leqslant n-1).$$

To see that $V_\bullet \in \mathrm{Hess}(N, h_J)$, we need to show that

$$NV_p \subseteq V_p \qquad \text{for } p \notin J. \tag{3.6}$$

Since we are assuming $V_\bullet \in X^\circ_{w_J} \cap Pet_n$, the flag $V_\bullet$ lies in the Schubert cell $X^\circ_{w_J}$. Here, the permutation w_J is a product of the longest permutations of the symmetric group of smaller ranks as given in (2.2). Thus it follows (from e.g. [16, Sect. 10.2]) that

$$V_p = \langle e_1, e_2, \ldots, e_p \rangle \qquad \text{for } p \notin J,$$

where $e_1, e_2, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$. Since $Ne_1 = 0$ and $Ne_i = e_{i-1}$ for $2 \leqslant i \leqslant n$, it is clear that (3.6) follows. Thus we obtain (3.5).

Now, we prove the claim (3.3) of this proposition. Since we have $X_{w_J} = \bigsqcup_{v \leqslant w_J} X^\circ_v$, it follows that

$$X_J = X_{w_J} \cap Pet_n = \bigsqcup_{v \leqslant w_J} (X^\circ_v \cap Pet_n) = \bigsqcup_{J' \subseteq J} (X^\circ_{w_{J'}} \cap Pet_n), \tag{3.7}$$

where the last equality follows from Lemmas 2.4 and 3.5. For each intersection $X^\circ_{w_{J'}} \cap Pet_n$ in the right-most side, we have that $X^\circ_{w_{J'}} \cap Pet_n \subseteq \mathrm{Hess}(N, h_{J'})$ by (3.5). The condition $J' \subseteq J$ implies that $h_{J'}(i) \leqslant h_J(i)$ for $1 \leqslant i \leqslant n$, and hence we have $\mathrm{Hess}(N, h_{J'}) \subseteq \mathrm{Hess}(N, h_J)$. Combining this with the previous inclusion, we see that

$$X^\circ_{w_{J'}} \cap Pet_n \subseteq \mathrm{Hess}(N, h_J)$$

in (3.7). Thus it follows that

$$X_J \subseteq \mathrm{Hess}(N, h_J). \tag{3.8}$$

Note that $X^\circ_{w_J} \cap Pet_n \subseteq X_J (= X_{w_J} \cap Pet_n)$ by definition, and hence we have that $\overline{X^\circ_{w_J} \cap Pet_n} \subseteq X_J$ by taking the closure. Combining this with (3.8), we obtain that

$$\overline{X^\circ_{w_J} \cap Pet_n} \subseteq X_J \subseteq \mathrm{Hess}(N, h_J). \tag{3.9}$$

In this sequence, both sides have the same dimension. This is because we have $\dim_{\mathbb{C}} \overline{X^\circ_{w_J} \cap Pet_n} = |J|$ from [24, Lemma 9] and $\dim_{\mathbb{C}} \mathrm{Hess}(N, h_J) = |J|$ from [7, Lemma 7.1]. Since $\mathrm{Hess}(N, h_J)$ is irreducible, the two inclusions in (3.9) are equalities. This completes the proof. □

Combining Proposition 3.4 and a result of Drellich [13, Theorem 4.5], we may express X_J as a product of Peterson varieties of smaller ranks as follows. Let $J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_m$ be the decomposition into the connected components. Then, by definition, we have $w_J = w_0^{(J_1)} w_0^{(J_2)} \cdots w_0^{(J_m)}$, where each $w_0^{(J_k)}$ is a product of longest elements in $\mathfrak{S}_{J_k}$ for $1 \leqslant k \leqslant m$. Hence it follows that the Schubert variety $X_{w_J} \subseteq Fl_n$ associated with w_J is isomorphic to the product of the flag varieties of smaller ranks:

$$X_{w_J} = \prod_{k=1}^{m} X_{w_0^{(J_k)}} \cong \prod_{k=1}^{m} Fl_{n_k},$$

where we set

$$n_k \coloneqq |J_k| + 1 \qquad \text{for } 1 \leqslant k \leqslant m.$$

By restricting this isomorphism to $X_J = X_{w_J} \cap Pet_n$, it follows from Proposition 3.4 and [13, Theorem 4.5] that X_J is isomorphic to a product of Peterson varieties of smaller ranks.

COROLLARY 3.6. *For $J \subseteq [n-1]$, we have*

$$X_J = \prod_{k=1}^{m} X_{J_k} \cong \prod_{k=1}^{m} Pet_{n_k},$$

where $J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_m$ is the decomposition into the connected components and $n_k = |J_k| + 1$ $(1 \leqslant k \leqslant m)$.

EXAMPLE 3.7. Let $n = 10$ and $J = \{1,2\} \sqcup \{4,5,6\} \sqcup \{9\} = J_1 \sqcup J_2 \sqcup J_3$. The representation matrix of w_J is given in Example 2.2, and we have

$$X_{w_J} \cong Fl_3 \times Fl_4 \times Fl_2,$$
$$X_J \cong Pet_3 \times Pet_4 \times Pet_2.$$

Compared to Schubert varieties and dual Schubert varieties in Fl_n, the structures of X_J and Ω_J in Pet_n are rather simple as we explain below. To begin with, we make the following definition.

DEFINITION 3.8. *For $1 \leqslant i \leqslant n-1$, let*

$$D_i := X_{[n-1]\smallsetminus\{i\}} \quad \text{and} \quad E_i := \Omega_{\{i\}}. \tag{3.10}$$

where $X_{[n-1]\smallsetminus\{i\}}$ and $\Omega_{\{i\}}$ are defined in (3.1).

LEMMA 3.9. *For $1 \leqslant i \leqslant n-1$, the following hold.*

(1) *D_i and E_i have codimension 1 in Pet_n.*
(2) *$D_i \cap E_i = \varnothing$.*

Proof. For (1), we have $\dim_{\mathbb{C}} D_i = \dim_{\mathbb{C}} X_{[n-1]\smallsetminus\{i\}} = n-2$ by Proposition 3.4. We also have

$$E_i = \Omega_{\{i\}} = \Omega_{w_{\{i\}}} \cap Pet_n = \Omega_{s_i} \cap Pet_n.$$

It is well-known that Ω_{s_i} is irreducible and it has complex codimension 1 in Fl_n ([16, Sect. 10.2]). Hence Ω_{s_i} in Fl_n is locally cut out by a single function. We also know that $\Omega_{s_i} \cap Pet_n$ is a non-empty proper subset of Pet_n since we have $s_i \in \Omega_{s_i} \cap Pet_n$ and $\mathrm{id} = w_\varnothing \in Pet_n \smallsetminus \Omega_{s_i}$. Thus, it follows that $\dim_{\mathbb{C}} E_i = n-2$. For (2), the claim follows from Proposition 3.2. □

In the next subsection, we will see that D_i and E_i are divisors[(1)] on Pet_n. The following claim means that X_J and Ω_J can be described as intersections of divisors on Pet_n.

PROPOSITION 3.10. *For $J \subseteq [n-1]$, we have*

$$X_J = \bigcap_{i \notin J} D_i \quad \text{and} \quad \Omega_J = \bigcap_{i \in J} E_i. \tag{3.11}$$

Proof. By (3.7), we have

$$X_{w_{[n-1]\smallsetminus\{i\}}} \cap Pet_n = \bigsqcup_{J' \subseteq [n-1]\smallsetminus\{i\}} (X^\circ_{w_{J'}} \cap Pet_n).$$

This implies from the definition of D_i that

$$\bigcap_{i \notin J} D_i = \bigcap_{i \notin J} (X_{w_{[n-1]\smallsetminus\{i\}}} \cap Pet_n) = \bigsqcup_{J' \subseteq J} (X^\circ_{w_{J'}} \cap Pet_n).$$

---

(1) In this paper, a divisor on a variety Y always means the support of an effective Cartier divisor on Y, i.e. the zero locus of a section of a line bundle over Y.

Combining this with (3.7), we obtain the desired claim for X_J. An argument similar to this proves that

$$\bigcap_{i\in J} E_i = \bigcap_{i\in J}(\Omega_{w_{\{i\}}}\cap Pet_n) = \bigsqcup_{J''\supseteq J}(\Omega^\circ_{w_{J''}}\cap Pet_n) = \Omega_{w_J}\cap Pet_n = \Omega_J$$

by Lemmas 2.4 and 3.5. □

Example 3.11. Let $n=9$ and $J=\{2,3,4\}\sqcup\{7,8\}$ so that $[n-1]\smallsetminus J=\{1\}\sqcup\{5,6\}$. Then we have

$$X_J = D_1\cap D_5\cap D_6 \qquad \text{and} \qquad \Omega_J = E_2\cap E_3\cap E_4\cap E_7\cap E_8.$$

3.2. Defining equations of X_J and Ω_J. Let $B\subseteq \mathrm{GL}_n(\mathbb{C})$ be the Borel subgroup of upper-triangular matrices in $\mathrm{GL}_n(\mathbb{C})$. Then we have the standard identification $Fl_n=\mathrm{GL}_n(\mathbb{C})/B$. For a complex B-representation space V, we have the associated complex vector bundle[(2)] over $\mathrm{GL}_n(\mathbb{C})/B$:

$$\mathrm{GL}_n(\mathbb{C})\times^B V \to \mathrm{GL}_n(\mathbb{C})/B \quad ; \quad [g,v]\mapsto gB.$$

For a weight $\mu\colon T\to\mathbb{C}^\times$, we obtain $\mu\colon B\to\mathbb{C}^\times$ by composing that with the canonical projection $B\twoheadrightarrow T$, which we also denote by the same symbol. We denote by $\mathbb{C}_\mu=\mathbb{C}$ the corresponding 1-dimensional representation space of B. Set

$$L_\mu = \mathrm{GL}_n(\mathbb{C})\times^B \mathbb{C}^*_\mu = \mathrm{GL}_n(\mathbb{C})\times^B \mathbb{C}_{-\mu}, \tag{3.12}$$

where $\mathbb{C}^*_\mu$ is the dual representation space of $\mathbb{C}_\mu$. We also denote the restriction $L_\mu|_{Pet_n}$ by the same symbol L_μ when there is no confusion.

Let us introduce two representations of B associated with each $J\subseteq[n-1]$ as follows. For $1\leqslant i\leqslant n-1$, let $\varpi_i\colon T\to\mathbb{C}^\times$ be the i-th fundamental weight of T given by $\mathrm{diag}(t_1,t_2,\dots,t_n)\mapsto t_1t_2\cdots t_i$. For $J\subseteq[n-1]$, we obtain a representation space of T given by a direct sum

$$\bigoplus_{i\in J}\mathbb{C}^*_{\varpi_i}.$$

Through the canonical projection $B\twoheadrightarrow T$, we regard this as a representation of B. To introduce the other representation of B associated with J, let α_i ($1\leqslant i\leqslant n-1$) be the i-th simple root defined as a weight $\alpha_i\colon T\to\mathbb{C}^\times$ given by $\mathrm{diag}(t_1,t_2,\dots,t_n)\mapsto t_it_{i+1}^{-1}$. Let $H_J\subseteq\mathfrak{gl}_n(\mathbb{C})$ be the Hessenberg subspace (c.f. [33, Sect. 2]) corresponding to the Hessenberg function h_J defined in (3.2), that is,

$$H_J \coloneqq \mathfrak{b}\oplus\bigoplus_{i\in J}\mathfrak{g}_{-\alpha_i}\subseteq\mathfrak{gl}_n(\mathbb{C}), \tag{3.13}$$

where $\mathfrak{b}=\mathrm{Lie}(B)$ is the Lie algebra of B and each $\mathfrak{g}_{-\alpha_i}$ is the standard root space of $\mathfrak{gl}_n(\mathbb{C})$ associated with the i-th negative simple root $-\alpha_i$ ($1\leqslant i\leqslant n-1$). Since H_J is preserved by the adjoint action of B on $\mathfrak{gl}_n(\mathbb{C})$, the quotient space

$$H_{[n-1]}/H_J$$

is a representation space of B. Now, these two representations of B induce the following vector bundles over Fl_n:

$$U_J \coloneqq \mathrm{GL}_n(\mathbb{C})\times^B\left(H_{[n-1]}/H_J\right),$$
$$V_J \coloneqq \mathrm{GL}_n(\mathbb{C})\times^B\left(\bigoplus_{i\in J}\mathbb{C}^*_{\varpi_i}\right).$$

(2) We take the B-action on the product $\mathrm{GL}_n(\mathbb{C})\times V$ so that $[g,v]=[gb,b^{-1}\cdot v]$ in the quotient.

If there is no confusion, we denote the restrictions of U_J and V_J on Pet_n by the same symbol. Note that we have

$$\begin{aligned} \operatorname{rank} U_J &= (n-1) - |J|, \\ \operatorname{rank} V_J &= |J|. \end{aligned} \tag{3.14}$$

Recall that

$$Pet_n = \{gB \in \mathrm{GL}_n(\mathbb{C})/B \mid g^{-1}Ng \in H_{[n-1]}\}$$

(c.f. [24, 31] or [33, Sect. 2]). Thus, the following map gives a section of U_J over Pet_n:

$$\phi_J \colon Pet_n \to U_J \quad ; \quad \phi_J(gB) = [g, [g^{-1}Ng]],$$

where $[g^{-1}Ng] \in H_{[n-1]}/H_J$ is the class represented by $g^{-1}Ng \in H_{[n-1]}$. For $1 \leqslant i \leqslant n-1$, let

$$\det{}_i \colon \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}_{\varpi_i} (= \mathbb{C})$$

be the function which takes the leading principal minor of order i. This is a B-equivariant function with respect to the multiplication of B on $\mathrm{GL}_n(\mathbb{C})$ from the right. Thus, we have a well-defined section

$$\psi_J \colon Pet_n \to V_J \quad ; \quad gB \mapsto \Big[g, \sum_{i \in J} \det{}_i(g)\Big].$$

The following claim means that X_J and Ω_J in Pet_n are defined by the equation $\phi_J = 0$ and $\psi_J = 0$, respectively.

Proposition 3.12. *For $J \subseteq [n-1]$, we have*

$$X_J = Z(\phi_J) \quad \textit{and} \quad \Omega_J = Z(\psi_J),$$

where $Z(\phi_J)$ and $Z(\psi_J)$ denote the zero loci of the sections ϕ_J and ψ_J, respectively.

Proof. Since the defining condition of $\mathrm{Hess}(N, h_J)$ is precisely that $g^{-1}Ng \in H_J$ (e.g. [33, Sect. 2]), it is clear that we have $X_J = Z(\phi_J)$. It is known that

$$\Omega_{s_i} = \{gB \in \mathrm{GL}_n(\mathbb{C})/B \mid \det{}_i(g) = 0\}$$

as subsets of Fl_n (c.f. [16, Proposition 9 in Sect. 10.6]). Thus, it follows that

$$Z(\psi_{\{i\}}) = \Omega_{s_i} \cap Pet_n = E_i.$$

Now, we obtain that

$$Z(\psi_J) = \bigcap_{i \in J} Z(\psi_{\{i\}}) = \bigcap_{i \in J} E_i = \Omega_J$$

by Proposition 3.10. □

Corollary 3.13. *For $1 \leqslant i \leqslant n-1$, D_i and E_i are divisors on Pet_n.*

Proof. By (3.14), it follows that $U_{[n-1] \smallsetminus \{i\}}$ is a line bundle, and we have $D_i = X_{[n-1] \smallsetminus \{i\}} = Z(\phi_{[n-1] \smallsetminus \{i\}})$ by the previous proposition. This means that D_i is a divisor on Pet_n. Similarly, $E_i = \Omega_{\{i\}} = Z(\psi_{\{i\}})$ is a divisor on Pet_n since $V_{\{i\}}$ is a line bundle by (3.14). □

Corollary 3.14. *For $J \subseteq [n-1]$, we have $\operatorname{codim}_{\mathbb{C}} \Omega_J = |J|$ in Pet_n.*

Proof. Recall that $\Omega_J = Z(\psi_J)$ and $\operatorname{rank} V_J = |J|$. This means that Ω_J in Pet_n is locally cut out by $|J|$ functions, and it follows that each irreducible component of Ω_J has codimension at most $|J|$ in Pet_n ([17, Proposition 14.1]). This implies that

$$\dim_{\mathbb{C}} \Omega_J \geqslant (n-1) - |J| \tag{3.15}$$

since $\dim_{\mathbb{C}} Pet_n = n-1$ and $\Omega_J = \Omega_{w_J} \cap Pet_n \neq \varnothing$. We show that the equality holds by induction on $(n-1)-|J|$. When $J=[n-1]$, we have $\Omega_{[n-1]} = \Omega_{w_0} \cap Pet_n = \{w_0\}$ so that the claim is obvious. Let $J \subsetneq [n-1]$, and assume by induction that $\dim_{\mathbb{C}} \Omega_K = (n-1)-|K|$ for $J \subsetneq K \subseteq [n-1]$. We prove that $\dim_{\mathbb{C}} \Omega_J = (n-1)-|J|$. Take an element $i \in [n-1] \setminus J$, and set $K = J \sqcup \{i\}$. Then we have

$$\Omega_K = \Omega_J \cap E_i \subseteq \Omega_J \tag{3.16}$$

by Proposition 3.10. This means that Ω_K is the zero locus of the section $\psi_{\{i\}}$ of the line bundle $V_{\{i\}}$ restricted over Ω_J. Thus we see that

$$\dim_{\mathbb{C}} \Omega_K \geqslant \dim_{\mathbb{C}} \Omega_J - 1, \tag{3.17}$$

which follows by applying [17, Proposition 14.1] to each irreducible component of Ω_J. Namely, the dimension decreases at most by 1 in (3.16). Since we have $\dim_{\mathbb{C}} \Omega_K = (n-1)-|K|$ by the inductive hypothesis, we can rewrite (3.17) as

$$\dim_{\mathbb{C}} \Omega_J \leqslant \dim_{\mathbb{C}} \Omega_K + 1 = (n-1)-|K|+1 = (n-1)-|J|.$$

Combining this with (3.15), we obtain the desired equality. □

Remark 3.15. The vector bundles U_J and V_J are decomposed into line bundles as

$$U_J = \bigoplus_{i \notin J} L_{\alpha_i} \qquad \text{and} \qquad V_J = \bigoplus_{i \in J} L_{\varpi_i}.$$

The first equality follows since we have $H_{[n-1]}/H_J \cong \oplus_{i \notin J} \mathbb{C}_{-\alpha_i}$ as representations of B, where the right hand side is a direct sum representation. These decompositions can be viewed as the analogue of Proposition 3.10 in the language of vector bundles over Pet_n.

4. The cohomology ring of Pet_n

In this section, we construct an additive basis of the integral cohomology ring $H^*(Pet_n;\mathbb{Z})$ by incorporating the geometry established in the previous section. We also introduce the structure constants of the basis, and provide a geometric proof for their positivity.

For an algebraic variety X which admits a paving by complex affine spaces ([33, Definition 2.1]), an irreducible Zariski closed subset Y of X has its fundamental cycle (as a reduced scheme) in $H_{2d}(Y;\mathbb{Z})$, where $d = \dim_{\mathbb{C}} Y$. By abusing notation, we use the same symbol for its image in $H_{2d}(X;\mathbb{Z})$ under the induced map $i_* \colon H_{2d}(Y;\mathbb{Z}) \to H_{2d}(X;\mathbb{Z})$, where i is the inclusion map $i \colon Z \hookrightarrow X$. See [16, Appendix B] or [17, Chap. 19] for the details.

4.1. The homology group of Pet_n.

Recall that we have a decomposition of Fl_n by Schubert cells:

$$Fl_n = \bigsqcup_{w \in \mathfrak{S}_n} X_w^\circ.$$

This induces a set-theoretic decomposition

$$Pet_n = \bigsqcup_{J \subseteq [n-1]} (X_{w_J}^\circ \cap Pet_n),$$

by Lemmas 2.4 and 3.5. It is known from [29, 33] that each $X_{w_J}^\circ \cap Pet_n$ is isomorphic to an affine cell $\mathbb{C}^{|J|}$ and that these cells form a paving by affines. Recall also from Proposition 3.4 that we have

$$X_J = \overline{X_{w_J}^\circ \cap Pet_n}$$

and that $\dim_{\mathbb{C}} X_J = |J|$ for $J \subseteq [n-1]$. This implies that the cycles represented by X_J for $J \subseteq [n-1]$ form a $\mathbb{Z}$-basis of the homology group $H_*(Pet_n;\mathbb{Z})$.

PROPOSITION 4.1. ([29, 33]) *For each $J \subseteq [n-1]$, we have $[X_J] \in H_{2|J|}(Pet_n;\mathbb{Z})$, and the set $\{[X_J] \mid J \subseteq [n-1]\}$ is a $\mathbb{Z}$-basis of $H_*(Pet_n;\mathbb{Z})$;*

$$H_*(Pet_n;\mathbb{Z}) = \bigoplus_{J\subseteq[n-1]} \mathbb{Z}[X_J].$$

EXAMPLE 4.2. Let $n=4$ so that $[n-1]=\{1,2,3\}$. Then we have

$$\begin{aligned}H_*(Pet_n;\mathbb{Z}) = \mathbb{Z}[X_\varnothing] &\oplus (\mathbb{Z}[X_{\{1\}}] \oplus \mathbb{Z}[X_{\{2\}}] \oplus \mathbb{Z}[X_{\{3\}}]) \\ &\oplus (\mathbb{Z}[X_{\{1,2\}}] \oplus \mathbb{Z}[X_{\{1,3\}}] \oplus \mathbb{Z}[X_{\{2,3\}}]) \oplus \mathbb{Z}[X_{\{1,2,3\}}].\end{aligned}$$

REMARK 4.3. Compared to $X^\circ_{w_J} \cap Pet_n (\cong \mathbb{C}^{|J|})$, the geometry of $\Omega^\circ_{w_J} \cap Pet_n$ is known to encode the quantum cohomology ring of a partial flag variety specified by J (see [29, 31]).

4.2. THE COHOMOLOGY GROUP OF Pet_n. For each weight $\mu\colon T \to \mathbb{C}^\times$, we constructed a line bundle L_μ over Fl_n in Section 3.2. Recall also that α_i and ϖ_i are the i-th simple root and the i-th fundamental weight of T ($1 \leqslant i \leqslant n-1$), respectively. It is well-known that we have an isomorphism

$$\bigoplus_{i=1}^{n-1} \mathbb{Z}\varpi_i \xrightarrow{\cong} H^2(Fl_n;\mathbb{Z}) \quad ; \quad \mu \mapsto e(L_\mu),$$

where we regard each $\mu = a_1\varpi_1 + \cdots + a_{n-1}\varpi_{n-1}$ ($a_1,\ldots,a_{n-1} \in \mathbb{Z}$) as the weight $\mu\colon T \to \mathbb{C}^\times$ given by $\mathrm{diag}(t_1,\ldots,t_n) \mapsto t_1^{a_1}(t_1t_2)^{a_2}\cdots(t_1\cdots t_{n-1})^{a_{n-1}}$. Let $i\colon Pet_n \hookrightarrow Fl_n$ be the inclusion map. Insko [23] proved that the induced homomorphism $i_*\colon H_*(Pet_n;\mathbb{Z}) \to H_*(Fl_n;\mathbb{Z})$ is an injection whose image is a direct summand of $H_*(Fl_n;\mathbb{Z})$. This means that the map $i_*\colon H_2(Pet_n;\mathbb{Z}) \to H_2(Fl_n;\mathbb{Z})$ on degree 2 is an isomorphism since $\mathrm{rank}\, H_2(Fl_n;\mathbb{Z}) = \mathrm{rank}\, H_2(Pet_n;\mathbb{Z}) = n-1$ (see Proposition 4.1), and hence the restriction map

$$i^*\colon H^2(Fl_n;\mathbb{Z}) \xrightarrow{\cong} H^2(Pet_n;\mathbb{Z})$$

on degree 2 cohomology group is an isomorphism. By combining these isomorphisms, we obtain that $\bigoplus_{i=1}^{n-1}\mathbb{Z}\varpi_i \cong H^2(Pet_n;\mathbb{Z})$. In the rest of the paper, we identify $\bigoplus_{i=1}^{n-1}\mathbb{Z}\varpi_i$ with $H^2(Pet_n;\mathbb{Z})$ through this isomorphism, and we use the same symbol $\mu \in H^2(Pet_n;\mathbb{Z})$ for the element $e(L_\mu)$ by abusing notation. For example, we write

$$\alpha_i = e(L_{\alpha_i}) \qquad \text{and} \qquad \varpi_i = e(L_{\varpi_i})$$

as elements in $H^2(Pet_n;\mathbb{Z})$.

In Section 3.2, we constructed vector bundles U_J and V_J over Pet_n. Adopting the above notation, we may express the Euler class $e(V_J)$ as a monomial of ϖ_i for each $i \in J$. Namely, for $J \subseteq [n-1]$, we have

$$e(V_J) = \prod_{i\in J}\varpi_i \tag{4.1}$$

since the vector bundle V_J decomposes into line bundles as follows:

$$V_J = \mathrm{GL}_n(\mathbb{C}) \times^B \left(\bigoplus_{i\in J}\mathbb{C}^*_{\varpi_i}\right) = \bigoplus_{i\in J} L_{\varpi_i}.$$

For $J \subseteq [n-1]$, take the decomposition $J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_m$ into the connected components. We set

$$m_J := |J_1|!|J_2|!\cdots|J_m|!. \tag{4.2}$$

Definition 4.4. *For $J \subseteq [n-1]$, let*

$$\varpi_J \coloneqq \frac{1}{m_J} e(V_J) = \frac{1}{m_J} \prod_{i \in J} \varpi_i,$$

where m_J is defined in (4.2).

The cohomology class ϖ_J is defined to be an element of $H^{2|J|}(Pet_n; \mathbb{Q})$, but we will show that it belongs to the integral cohomology group $H^{2|J|}(Pet_n; \mathbb{Z})$.

Example 4.5. Let $n = 9$ and $J = \{2,3,4,7,8\}$ so that $J = \{2,3,4\} \sqcup \{7,8\}$ is the decomposition into the connected components. Then we have

$$\varpi_J = \frac{1}{3!2!}(\varpi_2 \varpi_3 \varpi_4)(\varpi_7 \varpi_8) = \frac{1}{12} \varpi_2 \varpi_3 \varpi_4 \varpi_7 \varpi_8.$$

Compare this with Example 3.11.

Remark 4.6. We also have

$$e(U_J) = \prod_{i \notin J} \alpha_i$$

(c.f. Remark 3.15). These decompositions of $e(U_J)$ and $e(V_J)$ can be viewed as the cohomological analogue of Proposition 3.10.

The main purpose of this subsection is to prove that the set of cohomology classes $\{\varpi_J \mid J \subseteq [n-1]\}$ forms a module basis of the integral cohomology group $H^*(Pet_n; \mathbb{Z})$. We will state this in Theorem 4.14, and we devote the rest of this subsection for its proof.

Lemma 4.7. *For $1 \leqslant i \leqslant n-1$, we have*

$$\alpha_i \varpi_i = 0 \quad \textit{in } H^4(Pet_n, \mathbb{Z}).$$

Proof. Notice that $\alpha_i \varpi_i$ is the Euler class of the rank 2 vector bundle $U_{[n-1] \setminus \{i\}} \oplus V_{\{i\}} = L_{\alpha_i} \oplus L_{\varpi_i}$ (c.f. Remark 3.15). From Section 3.2, we have the section $\phi_{[n-1] \setminus \{i\}} + \psi_{\{i\}}$ of this bundle whose zero locus is $Z(\phi_{[n-1] \setminus \{i\}}) \cap Z(\psi_{\{i\}}) = D_i \cap E_i$ as we saw in the proof of Corollary 3.13. Now, by Lemma 3.9 (2), this is the empty set. Thus, $\phi_{[n-1] \setminus \{i\}} + \psi_{\{i\}}$ on Pet_n is a nowhere-zero section, and hence the Euler class $\alpha_i \varpi_i$ vanishes (see [27, Property 9.7]). □

Remark 4.8. In [15, Corollary 3.4] and [19, Theorem 4.1], the equations $\alpha_i \varpi_i = 0$ for $1 \leqslant i \leqslant n-1$ appeared as the fundamental relations in the presentation of the cohomology ring $H^*(Pet_n; \mathbb{C})$.

For $1 \leqslant i \leqslant n$, let F_i be the i-th tautological vector bundle over Fl_n whose fiber at $V_\bullet \in Fl_n$ is V_i. As a convention, we denote by F_0 the trivial sub-bundle of F_1 of rank 0. The quotient line bundle $L_i \coloneqq F_i / F_{i-1}$ is called the i-th tautological line bundle, and we set

$$x_i \coloneqq c_1(L_i^*) \in H^2(Fl_n; \mathbb{Z}) \quad (1 \leqslant i \leqslant n), \tag{4.3}$$

where we note that $x_1 + \cdots + x_n = 0$. We will also denote by the same symbol the restriction of x_i to $H^2(Pet_n; \mathbb{Z})$. It is well-known that for $1 \leqslant i \leqslant n-1$, we have

$$\begin{aligned} \alpha_i &= x_i - x_{i+1}, \\ \varpi_i &= x_1 + x_2 + \cdots + x_i. \end{aligned} \tag{4.4}$$

For $1 \leqslant i < j \leqslant n$, let

$$\alpha_{i,j} \coloneqq \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} = x_i - x_j.$$

For a homology cycle $Z \in H_k(Fl_n;\mathbb{Z})$ of degree k, the Poincaré dual of Z is the (unique) cohomology class $\gamma \in H^{2d-k}(Fl_n;\mathbb{Z})$ ($d = \dim_{\mathbb{C}} Fl_n$) satisfying $\gamma \cap [Fl_n] = Z$. In the following lemma, we regard Fl_{n-1} as a subvariety of Fl_n whose flags are contained in the linear subspace of $\mathbb{C}^n$ generated by $e_1, e_2, \ldots, e_{n-1}$. The claim (2) of the following lemma seems to be well-known, but we provide a proof using Hessenberg varieties for the completeness of the paper.

Lemma 4.9. *The following hold:*

(1) *The Poincaré dual of* $[Pet_n] \in H_*(Fl_n;\mathbb{Z})$ *is* $\prod_{j-i\geqslant 2} \alpha_{i,j} \in H^*(Fl_n;\mathbb{Z})$.
(2) *The Poincaré dual of* $[Fl_{n-1}] \in H_*(Fl_n;\mathbb{Z})$ *is* $\frac{1}{n}\alpha_{1,n}\alpha_{2,n}\cdots\alpha_{n-1,n} \in H^*(Fl_n;\mathbb{Z})$.

Proof. We first prove the claim (1). Recall from (3.2) and (3.13) that $H_J \subseteq \mathfrak{gl}_n(\mathbb{C})$ for $J \subseteq [n-1]$ is the Hessenberg space corresponding to the Hessenberg function h_J. Consider the associated vector bundle

$$\mathcal{N} := \mathrm{GL}_n(\mathbb{C}) \times^B (\mathfrak{gl}_n(\mathbb{C})/H_{[n-1]})$$

over $Fl_n = \mathrm{GL}_n(\mathbb{C})/B$. By an argument similar to that used in Section 3.2, Pet_n can be written as the zero locus of a section of the vector bundle $\mathcal{N}$, and it is shown in [2, Corollary 3.9] that the Poincaré dual of $[Pet_n]$ is the Euler class $e(\mathcal{N}) \in H^*(Fl_n;\mathbb{Z})$. It is straightforward to verify that $e(\mathcal{N}) = \prod_{j-i\geqslant 2} \alpha_{i,j}$ by the same inductive argument as that in [2, Sect. 4] using short exact sequences of vector bundles.

Next we prove the claim (2). Let S be an $n \times n$ regular semisimple matrix in diagonal form (i.e. a diagonal matrix with distinct eigenvalues) and $\mathrm{Hess}(S, h_0)$ a regular semisimple Hessenberg variety, where h_0 is the Hessenberg function $h_0 \colon [n] \to [n]$ given by

$$h_0(i) := \begin{cases} n-1 & (1 \leqslant i \leqslant n-1) \\ n & (i = n). \end{cases}$$

It is shown in [7, Sect, 3 and Sect. 4] that the Poincaré dual of $[\mathrm{Hess}(S, h_0)]$ is

$$(x_1 - x_n)(x_2 - x_n)\cdots(x_{n-1} - x_n) \in H^*(Fl_n;\mathbb{Z}),$$

where the left hand side is equal to $\alpha_{1,n}\alpha_{2,n}\cdots\alpha_{n-1,n}$ by the definition of $\alpha_{i,j}$. It is also known that $\mathrm{Hess}(S, h_0)$ has n connected components and that all the connected components give the same cycle $[Fl_{n-1}]$ ([32, Sect. 3]). Thus the Poincaré dual of $n[Fl_{n-1}]$ is $\alpha_{1,n}\alpha_{2,n}\cdots\alpha_{n-1,n}$, which implies the claim (2). ☐

Remark 4.10. [7, Corollary 7.2] with the formula for the double Schubert polynomial associated to a dominant permutation given in [7, p.2613] provides a more general formula than that of Lemma 4.9 (1) for regular nilpotent Hessenberg *schemes*, which were not known to be reduced when it was published. After that, [1] proved that they are in fact reduced when they contain Pet_n (c.f. [1, Remark 3.8]), and the formula are now generalized in [2] for an arbitrary Lie type.

For an (irreducible) projective variety Y, we denote the fundamental cycle of Y as $[Y] \in H_{2d}(Y;\mathbb{Z})$, where $d = \dim_{\mathbb{C}} Y$. For a cohomology class $\beta \in H^{2d}(Y;\mathbb{Z})$, we write

$$\int_Y \beta := \langle [Y], \beta \rangle_Y \quad (\in \mathbb{Z}),$$

where the right hand side is the value of the standard paring

$$\langle\ ,\ \rangle_Y \colon H_{2d}(Y;\mathbb{Z}) \times H^{2d}(Y;\mathbb{Z}) \to \mathbb{Z}.$$

Proposition 4.11. *We have*

$$\int_{Pet_n} \varpi_1\varpi_2\cdots\varpi_{n-1} = (n-1)!.$$

Proof. Let us first prove that

$$\int_{Pet_n} \varpi_1\varpi_2\cdots\varpi_{n-1} = (n-1)\int_{Pet_{n-1}} \varpi_1\varpi_2\cdots\varpi_{n-2}. \tag{4.5}$$

For the ϖ_{n-1} in the left hand side of (4.5), notice that

$$\varpi_{n-1} = \frac{1}{n}\sum_{i=1}^{n-1} i\alpha_i,$$

which follows from (4.4). Since we have $\varpi_i\alpha_i = 0$ for $1 \leqslant i \leqslant n-1$ from Lemma 4.7, we see that

$$\int_{Pet_n} \varpi_1\varpi_2\cdots\varpi_{n-1} = \frac{n-1}{n}\int_{Pet_n} \varpi_1\varpi_2\cdots\varpi_{n-2}\alpha_{n-1}. \tag{4.6}$$

By Lemma 4.9 (1), the right hand side of (4.6) can be computed as the following integral over Fl_n:

$$\frac{n-1}{n}\int_{Fl_n} \varpi_1\varpi_2\cdots\varpi_{n-2}\alpha_{n-1}\prod_{j-i\geqslant 2}\alpha_{i,j}.$$

Since we have $\alpha_{n-1} = \alpha_{n-1,n}$, the last expression can be written as

$$\frac{n-1}{n}\int_{Fl_n} \varpi_1\varpi_2\cdots\varpi_{n-2}(\alpha_{1,n}\alpha_{2,n}\cdots\alpha_{n-1,n})\prod_{\substack{j-i\geqslant 2\\ j\neq n}}\alpha_{i,j}.$$

By Lemma 4.9 (2), we can rewrite this as the following integral over Fl_{n-1}:

$$(n-1)\int_{Fl_{n-1}} \varpi_1\varpi_2\cdots\varpi_{n-2}\prod_{\substack{j-i\geqslant 2\\ j\neq n}}\alpha_{i,j}.$$

Applying Lemma 4.9 (1) to $Pet_{n-1} \subseteq Fl_{n-1}$, this can be written as the following integral over Pet_{n-1}:

$$(n-1)\int_{Pet_{n-1}} \varpi_1\varpi_2\cdots\varpi_{n-2}.$$

Hence we proved (4.5).

Now using (4.5) repeatedly, we obtain that

$$\int_{Pet_n} \varpi_1\varpi_2\cdots\varpi_{n-1} = (n-1)!\int_{Pet_2} \varpi_1.$$

Noticing that $Pet_2 = Fl_2 = \mathbb{P}^1$, we see that $\varpi_1(= x_1)$ is the first Chern class of the dual of the standard tautological line bundle over $\mathbb{P}^1$ by (4.3). Thus the integral in the right hand side is equal to 1, which completes the proof. □

Lemma 4.12. *For* $J \subseteq [n-1]$*, we have*

$$\int_{X_J} e(V_J) = m_J,$$

where $m_J = |J_1|!|J_2|!\cdots|J_m|!$ *is defined in* (4.2).

Proof. The isomorphism given in Corollary 3.6 induces an isomorphism

$$H^*(X_J;\mathbb{Z}) \xrightarrow{\cong} \bigotimes_{k=1}^{m} H^*(Pet_{n_k};\mathbb{Z})$$

which sends $e(V_J)(=\prod_{i\in J}\varpi_i) \in H^{2|J|}(X_J;\mathbb{Z})$ to $\bigotimes_{k=1}^{m}\varpi_1\varpi_2\cdots\varpi_{|J_k|}$. It also induces an isomorphism

$$H_*(X_J;\mathbb{Z}) \xrightarrow{\cong} \bigotimes_{k=1}^{m} H_*(Pet_{n_k};\mathbb{Z})$$

which sends $[X_J]\in H_{2|J|}(X_J;\mathbb{Z})$ to $\bigotimes_{k=1}^{m}[Pet_{n_k}]$. Now the claim follows from Proposition 4.11. □

Proposition 4.13. *For $J,K\subseteq[n-1]$ such that $|J|=|K|$, the degree of the homology class $[X_J]$ is the same as the degree of the Euler class $e(V_K)$, and we have*

$$\langle [X_J], e(V_K)\rangle_{Pet_n} = \begin{cases} m_J & \text{if } J=K,\\ 0 & \text{if } J\neq K.\end{cases}$$

Proof. Note that we have

$$\langle [X_J], e(V_K)\rangle_{Pet_n} = \int_{X_J} e(V_K). \tag{4.7}$$

For the case $J=K$, the claim follows from the previous lemma. Let us consider the case $J\neq K$. This condition and $|J|=|K|$ imply that $J\not\supseteq K$. We prove that the right hand side of (4.7) is equal to zero due to the vanishing of the Euler class $e(V_K)$ on X_J. Recall from Proposition 3.12 that we have the section $\psi_K\colon Pet_n\to V_K$ satisfying $Z(\psi_K)=\Omega_K$. Thus, the vector bundle $V_K|_{X_J}$ restricted to X_J admits a nowhere-zero section given by $\psi_K|_{X_J}$ since we have $X_J\cap\Omega_K=\varnothing$ by $J\not\supseteq K$ and Proposition 3.2. Thus the Euler class $e(V_K)$ vanishes on X_J, and hence the right hand side of (4.7) is equal to 0 in this case. □

For $J\subseteq[n-1]$, recall from Definition 4.4 that

$$\varpi_J = \frac{1}{m_J}e(V_J) = \frac{1}{m_J}\prod_{i\in J}\varpi_i.$$

Theorem 4.14. *For each $J\subseteq[n-1]$, the cohomology class ϖ_J is an element of the integral cohomology $H^{2|J|}(Pet_n;\mathbb{Z})$, and the set*

$$\{\varpi_J\in H^*(Pet_n;\mathbb{Z}) \mid J\subseteq[n-1]\}$$

is a $\mathbb{Z}$-basis of $H^*(Pet_n;\mathbb{Z})$.

Proof. Recall from Proposition 4.1 that $\{[X_J]\mid J\subseteq[n-1]\}$ forms a $\mathbb{Z}$-basis of $H_*(Pet_n;\mathbb{Z})$. Since the paring between $H_*(Pet_n;\mathbb{Z})$ and $H^*(Pet_n;\mathbb{Z})$ is perfect, the previous proposition implies the desired claim. □

Example 4.15. Let $n=4$ so that $[n-1]=\{1,2,3\}$. The additive basis given in Theorem 4.14 is

$$\begin{aligned} H^*(Pet_n;\mathbb{Z}) = \mathbb{Z}\varpi_{\varnothing} &\oplus (\mathbb{Z}\varpi_{\{1\}}\oplus\mathbb{Z}\varpi_{\{2\}}\oplus\mathbb{Z}\varpi_{\{3\}})\\ &\oplus(\mathbb{Z}\varpi_{\{1,2\}}\oplus\mathbb{Z}\varpi_{\{1,3\}}\oplus\mathbb{Z}\varpi_{\{2,3\}})\oplus\mathbb{Z}\varpi_{\{1,2,3\}}.\end{aligned}$$

As we saw in Proposition 4.13, this is the dual basis of the basis of the homology group $H_*(Pet_n;\mathbb{Z})$ given in Example 4.2.

4.3. Structure constants and their positivity. By Theorem 4.14, we can study the cohomology ring $H^*(Pet_n;\mathbb{Z})$ in terms of the basis $\{\varpi_J\}_{J\subseteq[n-1]}$. Specifically, we expand the product of two classes ϖ_J and ϖ_K as a linear combination of the basis:

$$\varpi_J \cdot \varpi_K = \sum_{L\subseteq[n-1]} d_{JK}^L\, \varpi_L, \quad d_{JK}^L \in \mathbb{Z}. \tag{4.8}$$

The coefficients d_{JK}^L are called the structure constant for the basis $\{\varpi_J\}_{J\subseteq[n-1]}$. In the following, we explain a geometric interpretation of d_{JK}^L, and deduce their positivity. Note that the degree of ϖ_L in $H^*(Pet_n;\mathbb{Z})$ is $2|L|$ and that the degree of $\varpi_J\cdot\varpi_K$ in $H^*(Pet_n;\mathbb{Z})$ is $2(|J|+|K|)$. Thus we may assume that

$$|L| = |J| + |K| \tag{4.9}$$

for each summand of (4.8) since we have $d_{JK}^L = 0$ otherwise. Then by Proposition 4.13, we have

$$d_{JK}^L = \langle [X_L], \varpi_J\cdot\varpi_K\rangle_{Pet_n} = \frac{1}{m_J}\frac{1}{m_K}\int_{X_L}\left(\prod_{j\in J}\varpi_j\right)\left(\prod_{k\in K}\varpi_k\right), \tag{4.10}$$

where m_J and m_K are the positive integers defined in (4.2). Now recall that each $\varpi_i \in H^2(X_L;\mathbb{Z})$ is the Euler class of the line bundle $V_{\{i\}}$ corresponding to the divisor $E_i(=Z(\psi_{\{i\}}))$ on Pet_n. Hence it follows from (4.10) that the structure constant d_{JK}^L computes an intersection number of (possibly duplicate) divisors $E_i\cap X_L$'s on X_L up to a constant multiple given by $\frac{1}{m_J}\frac{1}{m_K}$ (c.f. [26, Sect. 1.1.C]). This provides a geometric interpretation of d_{JK}^L in (4.8), and it leads us to the following instance of positivity.

Proposition 4.16. *We have $d_{JK}^L \geqslant 0$ for all $J,K,L\subseteq[n-1]$.*

Proof. Recall that each line bundle L_{ϖ_i} over Fl_n is nef for $1\leqslant i\leqslant n-1$ (e.g. [11, proof of Proposition 1.4.1] or [3, Lemma 3.5]). Hence the restriction of L_{ϖ_i} over X_L is nef as well for $1\leqslant i\leqslant n-1$. Thus the claim $d_{JK}^L\geqslant 0$ follows from (4.10) and the positivity of intersection numbers of nef divisors [26, Example 1.4.16]. □

5. Structure constants and Left-right diagrams

Recall from the previous section that the structure constants d_{JK}^L are defined to be the coefficients of the expansion formula (4.8) for the product $\varpi_J\cdot\varpi_K$ for $J,K\subseteq[n-1]$:

$$\varpi_J \cdot \varpi_K = \sum_{L\subseteq[n-1]} d_{JK}^L\, \varpi_L, \quad d_{JK}^L \in \mathbb{Z}.$$

In this section, we provide a manifestly positive combinatorial formula which computes the structure constant d_{JK}^L for all $J,K,L\subseteq[n-1]$. We start with the following lemma which tells us how to expand a monomial of $\varpi_1,\dots,\varpi_{n-1}$ containing a square in the simplest case.

Lemma 5.1. *For $1\leqslant a\leqslant i\leqslant b\leqslant n-1$, we have*

$$\varpi_i\cdot(\varpi_a\varpi_{a+1}\cdots\varpi_b) = \frac{b-i+1}{b-a+2}\varpi_{a-1}\varpi_a\cdots\varpi_b + \frac{i-a+1}{b-a+2}\varpi_a\cdots\varpi_b\varpi_{b+1}$$

in $H^*(Pet_n;\mathbb{Z})$, where we take the convention $\varpi_0=\varpi_n=0$.

Proof. We prove the claim by induction on $b-a(\geqslant 0)$. When $b-a=0$, we have $a=i=b$ so that the left hand side is ϖ_a^2. Noticing that $\alpha_a=-\varpi_{a-1}+2\varpi_a-\varpi_{a+1}$ (with the above convention), we have that

$$\varpi_a(-\varpi_{a-1}+2\varpi_a-\varpi_{a+1})=0$$

by Lemma 4.7. Thus the claim in this case follows since this equality can be expressed as

$$\varpi_a^2=\frac{1}{2}\varpi_{a-1}\varpi_a+\frac{1}{2}\varpi_a\varpi_{a+1}.$$

We now prove the claim for the case $a<b$. Assume by induction that the claim holds for any $a'\leqslant i'\leqslant b'$ with $b'-a'<b-a$. When $i=a$, we have

$$\begin{aligned}
\varpi_a(\varpi_a\varpi_{a+1}\cdots\varpi_b) &= \Big(\varpi_a(\varpi_a\varpi_{a+1}\cdots\varpi_{b-1})\Big)\varpi_b \\
&= \Big(\frac{b-a}{b-a+1}\varpi_{a-1}\varpi_a\cdots\varpi_{b-1}+\frac{1}{b-a+1}\varpi_a\varpi_{a+1}\cdots\varpi_b\Big)\varpi_b \\
&\qquad\qquad\text{(by the inductive hypothesis)} \\
&= \frac{b-a}{b-a+1}\varpi_{a-1}\varpi_a\cdots\varpi_b+\frac{1}{b-a+1}\varpi_a\Big(\varpi_{a+1}\cdots\varpi_{b-1}\varpi_b^2\Big) \\
&= \frac{b-a}{b-a+1}\varpi_{a-1}\varpi_a\cdots\varpi_b \\
&\quad+\frac{1}{(b-a+1)^2}\varpi_a\Big(\varpi_a\varpi_{a+1}\cdots\varpi_b+(b-a)\varpi_{a+1}\varpi_{a+2}\cdots\varpi_{b+1}\Big) \\
&\qquad\qquad\text{(by the inductive hypothesis).}
\end{aligned}$$

Since the left hand side and the second summand of the right hand side are proportional, this equation can be written as

$$\begin{aligned}
\frac{(b-a+1)^2-1}{(b-a+1)^2}\varpi_a^2(\varpi_{a+1}\cdots\varpi_b) &= \frac{b-a}{b-a+1}\varpi_{a-1}\varpi_a\cdots\varpi_b \\
&\quad+\frac{b-a}{(b-a+1)^2}\varpi_a\varpi_{a+1}\cdots\varpi_{b+1}.
\end{aligned}$$

Noticing that $(b-a+1)^2-1=(b-a)(b-a+2)$ for the numerator of the coefficient of the left hand side, we obtain that

$$\varpi_a^2(\varpi_{a+1}\cdots\varpi_b)=\frac{b-a+1}{b-a+2}\varpi_{a-1}\varpi_a\cdots\varpi_b+\frac{1}{b-a+2}\varpi_a\varpi_{a+1}\cdots\varpi_{b+1} \tag{5.1}$$

which verifies the claim for the case $i=a$. Now suppose that $a<i(\leqslant b)$. We then have that

$$\begin{aligned}
&\varpi_i(\varpi_a\varpi_{a+1}\cdots\varpi_b) \\
&\quad=\varpi_a\Big(\varpi_i(\varpi_{a+1}\cdots\varpi_b)\Big) \\
&\quad=\varpi_a\Big(\frac{b-i+1}{b-a+1}\varpi_a\varpi_{a+1}\cdots\varpi_b+\frac{i-a}{b-a+1}\varpi_{a+1}\varpi_{a+2}\cdots\varpi_{b+1}\Big) \\
&\qquad\qquad\text{(by the induction hypothesis)} \\
&\quad=\frac{b-i+1}{b-a+1}\varpi_a^2(\varpi_{a+1}\cdots\varpi_b)+\frac{i-a}{b-a+1}\varpi_a\varpi_{a+1}\cdots\varpi_{b+1} \\
&\quad=\frac{b-i+1}{b-a+1}\Big(\frac{b-a+1}{b-a+2}\varpi_{a-1}\varpi_a\cdots\varpi_b+\frac{1}{b-a+2}\varpi_a\varpi_{a+1}\cdots\varpi_{b+1}\Big) \\
&\qquad\quad+\frac{i-a}{b-a+1}\varpi_a\varpi_{a+1}\cdots\varpi_{b+1} \qquad \text{(by (5.1))}
\end{aligned}$$

$$= \frac{b-i+1}{b-a+2}\varpi_{a-1}\varpi_a\cdots\varpi_b + \frac{i-a+1}{b-a+2}\varpi_a\varpi_{a+1}\cdots\varpi_{b+1}.$$

Thus we complete the proof by induction. □

Lemma 5.1 is the simplest case of expansions, but it turns out that it provides an effective way for computing the expansion of $\varpi_J \cdot \varpi_K$ for $J, K \subseteq [n-1]$ as we see in the following example.

Example 5.2. Let $n = 10$, and take $J = \{1,3,5,6,7\}$ and $K = \{3,6,8\}$. The product $\varpi_J \cdot \varpi_K$ can be computed by using Lemma 5.1 repeatedly as follows. We first extract ϖ_i's which produce squares:

$$\begin{aligned}\varpi_J \cdot \varpi_K &= \left(\frac{1}{1!1!3!}\varpi_1\varpi_3\varpi_5\varpi_6\varpi_7\right)\cdot\left(\frac{1}{1!1!1!}\varpi_3\varpi_6\varpi_8\right)\\ &= \frac{1}{3!}\varpi_6\cdot\varpi_3\cdot(\varpi_1\varpi_3\varpi_5\varpi_6\varpi_7\varpi_8).\end{aligned}$$

By applying Lemma 5.1 to ϖ_3^2, this can be computed as

$$\begin{aligned}&\frac{1}{3!}\varpi_6\cdot\varpi_3\cdot(\varpi_1\varpi_3\varpi_5\varpi_6\varpi_7\varpi_8)\\ &\quad= \frac{1}{3!}\varpi_6\cdot\left(\varpi_1\left(\frac{1}{2}\varpi_2\varpi_3+\frac{1}{2}\varpi_3\varpi_4\right)\varpi_5\varpi_6\varpi_7\varpi_8\right)\\ &\quad= \frac{1}{3!}\cdot\frac{1}{2}\varpi_6\cdot(\varpi_1\varpi_2\varpi_3\varpi_5\varpi_6\varpi_7\varpi_8) + \frac{1}{3!}\cdot\frac{1}{2}\varpi_6\cdot(\varpi_1\varpi_3\varpi_4\varpi_5\varpi_6\varpi_7\varpi_8).\end{aligned}$$

Then, by applying Lemma 5.1 to $\varpi_5\varpi_6^2\varpi_7\varpi_8$ in the first summand and to $\varpi_3\varpi_4\varpi_5\varpi_6^2\varpi_7\varpi_8$ in the second summand, we can continue our computation as

$$\begin{aligned}&\frac{1}{3!}\cdot\frac{1}{2}\varpi_6\cdot(\varpi_1\varpi_2\varpi_3\varpi_5\varpi_6\varpi_7\varpi_8) + \frac{1}{3!}\cdot\frac{1}{2}\varpi_6\cdot(\varpi_1\varpi_3\varpi_4\varpi_5\varpi_6\varpi_7\varpi_8)\\ &\quad= \frac{1}{3!}\cdot\frac{1}{2}\left(\varpi_1\varpi_2\varpi_3\left(\frac{3}{5}\varpi_4\varpi_5\varpi_6\varpi_7\varpi_8+\frac{2}{5}\varpi_5\varpi_6\varpi_7\varpi_8\varpi_9\right)\right)\\ &\qquad\quad+\frac{1}{3!}\cdot\frac{1}{2}\left(\varpi_1\left(\frac{3}{7}\varpi_2\varpi_3\varpi_4\varpi_5\varpi_6\varpi_7\varpi_8+\frac{4}{7}\varpi_3\varpi_4\varpi_5\varpi_6\varpi_7\varpi_8\varpi_9\right)\right)\\ &\quad= \frac{1}{3!}\cdot\left(\frac{1}{2}\cdot\frac{3}{5}+\frac{1}{2}\cdot\frac{3}{7}\right)\cdot 8!\,\varpi_{\{1,2,3,4,5,6,7,8\}}\\ &\qquad\quad+\frac{1}{3!}\cdot\frac{1}{2}\cdot\frac{2}{5}\cdot 3!\cdot 5!\,\varpi_{\{1,2,3,5,6,7,8,9\}}+\frac{1}{3!}\cdot\frac{1}{2}\cdot\frac{4}{7}\cdot 7!\,\varpi_{\{1,3,4,5,6,7,8,9\}}\\ &\quad= 3456\varpi_{\{1,2,3,4,5,6,7,8\}}+24\varpi_{\{1,2,3,5,6,7,8,9\}}+240\varpi_{\{1,3,4,5,6,7,8,9\}}.\end{aligned}$$

Thus we conclude that

$$\varpi_J\cdot\varpi_K = 3456\varpi_{\{1,2,3,4,5,6,7,8\}}+24\varpi_{\{1,2,3,5,6,7,8,9\}}+240\varpi_{\{1,3,4,5,6,7,8,9\}}$$

which gives a particular case of the expansion (4.8). As one can see, the geometric idea behind this computation is the realization of Ω_J by intersecting the divisors E_i; see (3.11) and (4.1).

Let J, K, L be subsets of $[n-1]$. By tracking the computations in the above example, it is straightforward to see that if ϖ_L appears in the expansion of the product $\varpi_J\cdot\varpi_K$, then L must contain $J\cup K$. Combining this with (4.9), we see that

$$d_{JK}^L = 0 \text{ unless } L \supseteq J\cup K \text{ and } |L| = |J|+|K|. \tag{5.2}$$

We now introduce a combinatorial object which effectively computes the structure constants d_{JK}^L. Because of (5.2), we always assume that $L \supseteq J\cup K$ and $|L| = |J|+|K|$ in what follows. We first prepare the following two steps.

(1) Write the elements of $[n-1]$ in increasing order, and draw a square grid of size $(1+|J\cap K|)\times|L|$ over the subset $L\subseteq[n-1]$. On the left side of the grid, write the elements of $J\cap K$ in increasing order from the second row to the bottom row.

For each box in the grid, we define the *row number* of the box as the number which is written beside the row containing the box, and define the *column number* of the box as the number which is written below the column containing the box.

(2) Shade the boxes in the first row whose column numbers belong to $J\cup K(\subseteq L)$. Mark each box with a cross $\times$ whose row number is the same as the column number.

Example 5.3. Let $n=10$ and take $J=\{1,3,5,6,7\}$ and $K=\{3,6,8\}$ as in the previous example. We depict the resulting grids after the steps (1) and (2) for the following two choices of L.

(i) If $L=\{1,2,3,4,5,6,7,8\}$, then the resulting grid is depicted in Figure 4. For example, the row number of the marked box in the second row is 3.

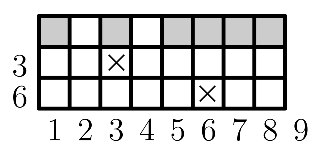

Figure 4. The resulting grid for $L=\{1,2,3,4,5,6,7,8\}$.

(ii) If $L'=\{1,2,3,5,6,7,8,9\}$, then the resulting grid is depicted in Figure 5.

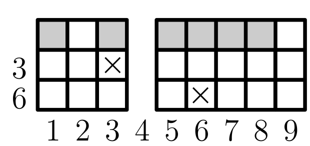

Figure 5. The resulting grid for $L'=\{1,2,3,5,6,7,8,9\}$.

We now play a combinatorial game on the grid prepared above. Let us explain the rule of the game inductively.
(The game):

> Assume that some boxes in the i-th row are shaded $(1\leqslant i\leqslant|J\cap K|)$. Then shade the boxes in the $(i+1)$-th row whose column numbers are the same as those of the shaded boxes in the i-th row. If there is a non-shaded box adjacent to the left (L) or the right (R) of the consecutive string of the shaded boxes in the $(i+1)$-th row containing the marked box, then shade one of them darkly. In this case, continue to the next row. If there are no such boxes, then we stop the game.

We say that the combinatorial game explained above is *successful* if we can continue the game to the bottom row. We define a *left-right diagram* associated with (J,K,L) as a configuration of boxes on a square grid of size $(1+|J\cap K|)\times|L|$ over $L(\subseteq[n-1])$ which is obtained as the resulting configuration of the shaded boxes of a successful game. We denote by Δ^L_{JK} the set of left-right diagrams associated with (J,K,L).

Example 5.4. We take the triples (J, K, L) and (J, K, L') given in Example 5.3.

(i) The left-right diagrams associated with (J, K, L) are P_1 and P_2 in Figure 6.

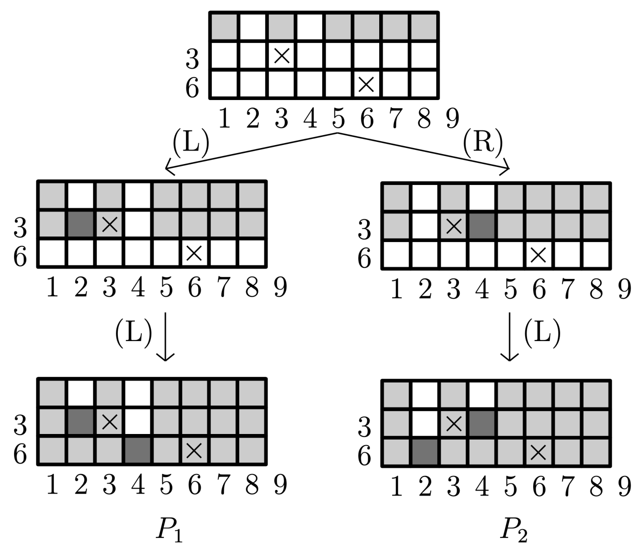

Figure 6. The games for the (J, K, L).

(ii) The left-right diagram associated with (J, K, L') is the P' in Figure 7.

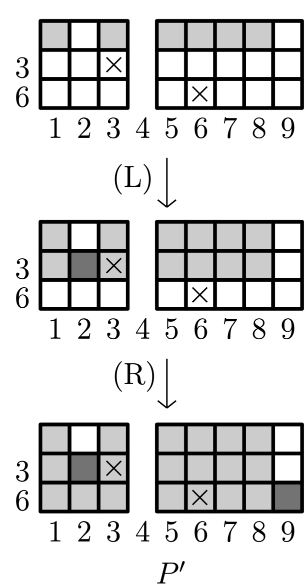

Figure 7. The (unique) game for the (J, K, L').

Next, we define the weight of a left-right diagram $P \in \Delta^L_{JK}$ as follows. For each row of P (except for the first row), we consider the consecutive string of the shaded boxes which contains the marked box in the focused row. Then the set of the column numbers for these boxes must be of the form $\{a, a+1, \ldots, b\}$ for some $a, b \in L$, and the column number i of the marked box satisfies $a \leqslant i \leqslant b$. Motivated by Lemma 5.1, we assign to this row a positive rational number given by

$$\frac{b-i+1}{b-a+2} \quad \text{if the additional box ■ is to the left of the marked box ⊠},$$
$$\frac{i-a+1}{b-a+2} \quad \text{if the additional box ■ is to the right of the marked box ⊠}.$$

Note that the column number of the additional box is $a-1$ in the former case and is $b+1$ in the latter case (c.f. Lemma 5.1). We may pictorially interpret this rational number as follows.

- The denominator is the number of the shaded boxes counted from the additional box ■ to the terminal box lying on the opposite side of the string of shaded boxes across the marked box ⊠.
- The numerator is the number of the shaded boxes counted from the marked box ⊠ to the same terminal box as above.

We define the *weight* of P as the product of these positive rational numbers assigned to the rows of P (except for the first row), and denote it by $\mathrm{wt}(P)$.

Example 5.5. Continuing with Example 5.4, the weights of the left-right diagrams P_1, P_2, P' can be computed as follows.

(i) The weights of the left-right diagrams P_1 and P_2 associated with the (J,K,L) are

$$\mathrm{wt}(P_1) = \frac{1}{2}\cdot\frac{3}{5} \quad \text{and } \mathrm{wt}(P_2) = \frac{1}{2}\cdot\frac{3}{7}$$

(See Figure 8). By construction, these weights appear in the computation of the coefficient of the $\varpi_L = \varpi_{\{1,2,3,4,5,6,7,8\}}$ in Example 5.2.

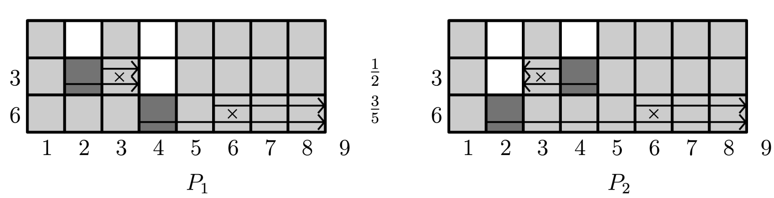

Figure 8. The computations of the weights of P_1 and P_2.

(ii) The weight of the left-right diagram $P' \in \Delta_{JK}^{L'}$ can be computed as

$$\mathrm{wt}(P') = \frac{1}{2}\cdot\frac{2}{5}.$$

(See Figure 9). This weight appears in the computation of the coefficient of the $\varpi_{L'} = \varpi_{\{1,2,3,5,6,7,8,9\}}$ in Example 5.2 as well.

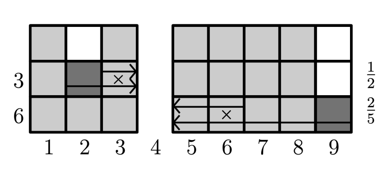

Figure 9. The computation of the weight of P'.

We now summarize our computation of the structure constants. For $J \subseteq [n-1]$, recall from Definition 4.4 that

$$\varpi_J = \frac{1}{m_J} e(V_J) = \frac{1}{m_J} \prod_{i \in J} \varpi_i.$$

Theorem 5.6. *Let J, K be subsets of $[n-1]$. In $H^*(Pet_n; \mathbb{Z})$, we have*

$$\varpi_J \cdot \varpi_K = \sum_{\substack{K \cup J \subseteq L \subseteq [n-1] \\ |L| = |J| + |K|}} d_{JK}^L \, \varpi_L, \quad d_{JK}^L \in \mathbb{Z}, \tag{5.3}$$

and the structure constant d_{JK}^L in this equality is given by

$$d_{JK}^L = \frac{m_L}{m_J m_K} \sum_{P \in \Delta_{JK}^L} \mathrm{wt}(P),$$

where Δ_{JK}^L is the set of left-right diagrams and $\mathrm{wt}(P)$ is the weight of P defined above. In particular, we have $d_{JK}^L = 0$ in (5.3) if and only if $\Delta_{JK}^L = \varnothing$.

Proof. Recall from Definition 4.4 that

$$\varpi_J \cdot \varpi_K = \frac{1}{m_J m_K} \left(\prod_{j \in J} \varpi_j \right) \cdot \left(\prod_{k \in K} \varpi_k \right). \tag{5.4}$$

If $J \cap K = \varnothing$, then this does not contain a square of $\varpi_1, \ldots, \varpi_{n-1}$, and it is clearly equal to

$$\frac{m_L}{m_J m_K} \varpi_L.$$

Thus we may assume that $J \cap K \neq \varnothing$ which implies that the right hand side of (5.4) contains some squares. By extracting the terms which produce the squares, we can express the product in the right hand side of (5.4) as

$$\left(\prod_{j \in J} \varpi_j \right) \cdot \left(\prod_{k \in K} \varpi_k \right) = \left(\prod_{i \in J \cap K} \varpi_i \right) \cdot \left(\prod_{q \in J \cup K} \varpi_q \right). \tag{5.5}$$

We compute the product in the right hand side of this equality. For this purpose, take the decomposition $J \cup K = M_1 \sqcup \cdots \sqcup M_s$ into the connected components. Let i be the smallest element of $J \cap K$. Then we have $i \in M_r$ for some r ($1 \leqslant r \leqslant s$). Since M_r is connected, we can express it as $M_r = \{a, a+1, \ldots, b\}$ for some $a, b \in J \cup K$ with $a \leqslant i \leqslant b$. Then, by Lemma 5.1 the product $\varpi_i \cdot \left(\prod_{q \in J \cup K} \varpi_q \right)$ can be expanded as

$$\varpi_i \cdot \left(\prod_{q \in J \cup K} \varpi_q \right) = \frac{b-i+1}{b-a+2} \prod_{q \in J \cup K \cup \{a-1\}} \varpi_q + \frac{i-a+1}{b-a+2} \prod_{q \in J \cup K \cup \{b+1\}} \varpi_q,$$

where we have no squares of ϖ_q's in the right hand side since M_r is a connected component of $J \cup K$. If $|J \cap K| \geqslant 2$, then let i' be the smallest element of $J \cap K \setminus \{i\}$. Multiplying $\varpi_{i'}$ to the right hand side of this equality, we can expand it by square-free monomials in $\varpi_1, \ldots, \varpi_{n-1}$ by Lemma 5.1 again (c.f. Example 5.2). Repeating this procedure for each element of $J \cap K$ in increasing order, we obtain that

$$\left(\prod_{i \in J \cap K} \varpi_i \right) \cdot \left(\prod_{q \in J \cup K} \varpi_q \right) = \sum_{\substack{L \supseteq J \cup K \\ |L| = |J| + |K|}} \left(\left(\sum_{P \in \Delta_{JK}^L} \mathrm{wt}(P) \right) \prod_{q \in L} \varpi_q \right)$$

by the construction of the left-right diagrams and their weights. Combining this with (5.4) and (5.5), we obtain that

$$\varpi_J \cdot \varpi_K = \sum_{\substack{L \supseteq J \cup K \\ |L|=|J|+|K|}} \left(\frac{m_L}{m_J m_K} \sum_{P \in \Delta_{JK}^L} \mathrm{wt}(P) \right) \varpi_L,$$

which implies the desired claim. □

Example 5.7. Let $n = 10$ and take $J = \{1,3,5,6,7\}$, $K = \{3,6,8\}$ as in Example 5.2. We compute the coefficients in (5.3) for the following two choices of L. Note that we have $m_J = 3!$ and $m_K = 1$.

(i) For $L = \{1,2,3,4,5,6,7,8\}$, we have $m_L = 8!$, and the weights of left-right diagrams associated with the (J,K,L) are computed in Example 5.5. Hence we obtain that

$$d_{JK}^L = \frac{m_L}{m_J m_K} \sum_{P \in \Delta_{JK}^L} \mathrm{wt}(P) = \frac{8!}{3!} \left(\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{3}{7} \right) = 3456$$

which coincides with the coefficient of $\varpi_L = \varpi_{\{1,2,3,4,5,6,7,8\}}$ in Example 5.2.

(ii) For $L' = \{1,2,3,5,6,7,8,9\}$, we have $m_{L'} = 3! \cdot 5!$, and the weight of the left-right diagram associated with the (J,K,L') are computed in Example 5.5. Hence we obtain that

$$d_{JK}^{L'} = \frac{m_{L'}}{m_J m_K} \sum_{P \in \Delta_{JK}^{L'}} \mathrm{wt}(P) = \frac{3! \cdot 5!}{3!} \left(\frac{1}{2} \cdot \frac{2}{5} \right) = 24$$

which coincides with the coefficient of $\varpi_{L'} = \varpi_{\{1,2,3,5,6,7,8,9\}}$ in Example 5.2.

Remark 5.8. Theorem 5.6 provides the combinatorial description of the computation demonstrated in Example 5.2. As we observed there, the geometric idea behind our computation is the realization of Ω_J by intersecting the divisors E_i.

6. Relations to other works

In this section, we clarify how the results in this paper are related to other works in existing literature. Especially, we explain the relations to the work of Goldin–Gorbutt [18] on *Peterson Schubert calculus* and to the works of Berget–Spink–Tseng [10], Nadeau–Tewari [28], and the second author [21] on *mixed Eulerian numbers*. We emphasize that [10, 18, 28] are announced earlier than this paper.

6.1. Relations to Peterson Schubert calculus.

We begin with reviewing the motivation of Peterson Schubert calculus from [9, 14, 18, 20]. We first note that these papers studied the equivariant cohomology ring of Pet_n with respect to the $\mathbb{C}^\times$-action explained in Section 2.1, but we focus on the ordinary cohomology ring to compare with our computation (see [9, 14, 18, 20] for the results in the equivariant cohomology). Recall that the dual Schubert variety Ω_w associated with $w \in \mathfrak{S}_n$ determines the homology cycle $[\Omega_w]$ in $H_*(Fl_n;\mathbb{Z})$. We denote by $\sigma_w \in H^{2\ell(w)}(Fl_n;\mathbb{Z})$ the Poincaré dual of $[\Omega_w]$, which is called the Schubert class associated with w. It is well-known that the set of Schubert classes $\{\sigma_w \mid w \in \mathfrak{S}_n\}$ forms an additive basis of $H^*(Fl_n;\mathbb{Z})$. Thus we may express the product $\sigma_u \cdot \sigma_v$ as a linear combination of the Schubert classes:

$$\sigma_u \cdot \sigma_v = \sum_{w \in \mathfrak{S}_n} c_{uv}^w \, \sigma_w, \quad c_{uv}^w \in \mathbb{Z}.$$

Computations of the structure constants c_{uv}^w fall under the umbrella of *Schubert calculus* on the flag variety Fl_n. Geometrically, c_{uv}^w is the intersection number $\int_{Fl_n}(\sigma_u \cdot \sigma_v \cdot \sigma_{w_0 w})$, and this implies the positivity for the structure constants, i.e. $c_{uv}^w \geqslant 0$ by Kleiman's transversality theorem (see e.g. [11, Sect. 1.3]).

Motivated by this, Harada and Tymoczko considered the following problem in [20]. Let $p_w \in H^*(Pet_n;\mathbb{C})$ denote the image of the Schubert class $\sigma_w \in H^*(Fl_n;\mathbb{C})$ under the restriction map $H^*(Fl_n;\mathbb{C}) \to H^*(Pet_n;\mathbb{C})$. They called p_w the *Peterson Schubert class* corresponding to w. Since the restriction map $H^*(Fl_n;\mathbb{C}) \to H^*(Pet_n;\mathbb{C})$ is surjective ([20, 23]), it is natural to ask whether there exists a natural subset of Peterson Schubert classes p_w which forms an additive basis of $H^*(Pet_n;\mathbb{C})$. They gave an answer to this question as follows. Let $J = \{j_1 < j_2 < \cdots < j_m\}$ be a subset of $[n-1]$. They defined the element $v_J \in \mathfrak{S}_n$ to be the product of simple transpositions whose indices are in J, in increasing order, that is,

$$v_J \coloneqq s_{j_1} s_{j_2} \cdots s_{j_m}. \tag{6.1}$$

Theorem 6.1 ([20, Theorem 4.12]). *The set $\{p_{v_J} \mid J \subseteq [n-1]\}$ forms a $\mathbb{C}$-basis of $H^*(Pet_n;\mathbb{C})$.*

By this theorem, we may expand the product $p_{v_J} \cdot p_{v_K}$ in terms of the Peterson Schubert classes p_{v_L}:

$$p_{v_J} \cdot p_{v_K} = \sum_{L \subseteq [n-1]} c_{JK}^L \, p_{v_L}, \quad c_{JK}^L \in \mathbb{C}. \tag{6.2}$$

The framework of computing the structure constants c_{JK}^L is called *Peterson Schubert calculus* in [18]. Harada and Tymoczko also gave Monk's formula for c_{JK}^L in [20, Theorem 6.12], which is the case for $|J| = 1$. Recently, Goldin and Gorbutt gave combinatorial formulas for the structure constants c_{JK}^L in [18, Theorems 1,4,6,7] which are manifestly positive and integral. In particular, their formulas imply the positivity for the structure constants.

Theorem 6.2. ([18, Corollary 8]) *The structure constants c_{JK}^L in* (6.2) *are non-negative integers for all $J, K, L \subseteq [n-1]$.*

This theorem ensures that all the coefficients c_{JK}^L in (6.2) are (non-negative) integers, but it is not obvious whether $\{p_{v_J} \in H^{2|J|}(Pet_n;\mathbb{Z}) \mid J \subseteq [n-1]\}$ forms a $\mathbb{Z}$-basis of $H^*(Pet_n;\mathbb{Z})$. Moreover, it is natural to seek a geometric reason for this positivity of the structure constants c_{JK}^L (c.f. [9, Remark 3.4] and [20, p.43, question (2)]). In what follows, we give an answer to this question. Recall from [9] that we have Giambelli's formula for the Peterson Schubert classes.

Theorem 6.3 (Giambelli's formula for the Peterson variety, [9, Theorem 3.2]). *For $J \subseteq [n-1]$, we have*

$$p_{v_J} = \frac{1}{|J_1|!|J_2|!\cdots|J_m|!} \prod_{i \in J} p_{s_i}, \tag{6.3}$$

where J_k $(1 \leqslant k \leqslant m)$ are the the connected components of J.

Remark 6.4. Drellich gave Giambelli's formula for arbitrary Lie types in [14].

As is well-known, the Schubert class σ_{s_i} can be written as $\sigma_{s_i} = x_1 + \cdots + x_i = \varpi_i$ in $H^2(Fl_n;\mathbb{Z})$, where $x_1, \ldots, x_n$ are defined in (4.3). This implies that

$$p_{s_i} = \varpi_i \quad \text{in } H^2(Pet_n;\mathbb{Z}),$$

for $1 \leqslant i \leqslant n-1$ by taking the restriction. Thus, the right hand side of (6.3) is nothing but ϖ_J in Definition 4.4. As a consequence of Theorems 4.14 and 6.3, we obtain the following result which explains the geometric background of Peterson Schubert calculus.

Corollary 6.5. *For $J \subseteq [n-1]$, we have $p_{v_J} = \varpi_J$. In particular, the set*

$$\{p_{v_J} \in H^{2|J|}(Pet_n;\mathbb{Z}) \mid J \subseteq [n-1]\}$$

forms a $\mathbb{Z}$-basis of $H^*(Pet_n;\mathbb{Z})$. Moreover, the structure constant c_{JK}^L in (6.2) *is equal to the structure constant d_{JK}^L in Theorem* 5.6.

Remark 6.6. This implies that Lemma 5.1 is essentially a special case of Monk's formula [20, Theorem 6.12].

By Corollary 6.5, the structure constants d_{JK}^L can also be computed by the formulas for c_{JK}^L proved earlier by Goldin–Gorbutt [18] in the $\mathbb{C}^\times$-equivariant setting (see Section 2.1). Their approach to the structure constants is mostly combinatorial whereas our approach is geometric and based on the properties of X_J and Ω_J. We end this subsection by giving a short observation on the difference of their formulas and ours.

Suppose that $J,K,L \subseteq [n-1]$ are all connected subsets such that $J \cup K \subseteq L$, $|L| = |J| + |K|$. Then, we may write $J = [a_1, a_2]$, $K = [b_1, b_2]$, $L = [c_1, c_2]$, and we may assume that $a_1 \leqslant b_1$ by interchanging the roles of J and K if necessary. In this case, their formula ([18, Corollary 2]) for c_{JK}^L is quite simple:

$$c_{JK}^L = \binom{a_2 - b_1 + 1}{a_1 - c_1}\binom{b_2 - a_1 + 1}{b_1 - c_1}.$$

For general $J,K,L \subseteq [n-1]$, their computation of c_{JK}^L consists of three (ordered) formulas ([18, Theorems 3, 5, 6]) each of which successively makes a reduction to the computations in the former case.

In contrast, our formula has several terms even when $J,K,L \subseteq [n-1]$ are all connected, however it provides a single formula which covers all the cases of general $J,K,L \subseteq [n-1]$.

6.2. Relations to mixed Eulerian numbers. We next explain the relations of the results in this paper to the works on mixed Eulerian numbers introduced and studied by Postnikov [30].

We briefly recall the definition of mixed Eulerian numbers. For $a_1,\dots,a_n \in \mathbb{R}^n$, the permutohedron $P_n(a_1,\dots,a_n)$ is defined to be the convex hull of the $\mathfrak{S}_n$-orbits of $(a_1,\dots,a_n)$ in $\mathbb{R}^n$:

$$P_n(a_1,\dots,a_n) = \mathrm{ConvexHull}\{(a_{w(1)},\dots,a_{w(n)}) \in \mathbb{R}^n \mid w \in \mathfrak{S}_n\}.$$

This is at most $(n-1)$-dimensional, and it sits inside of an affine hyperplane in $\mathbb{R}^n$. The $(n-1)$-dimensional volume (computed by projecting down to $\mathbb{R}^{n-1}$) of $P_n(a_1,\dots,a_n)$ in terms of $u_i = a_i - a_{i+1}$ for $1 \leqslant i \leqslant n-1$ can be written as

$$\mathrm{Vol}\, P_n(a_1,\dots,a_n) = \sum_{c_1,\dots,c_{n-1}} A_{c_1,\dots,c_{n-1}} \frac{u_1^{c_1}}{c_1!}\cdots\frac{u_{n-1}^{c_{n-1}}}{c_{n-1}!},$$

where the sum is taken over all non-negative integers $c_1,\dots,c_{n-1}$ with $c_1+\cdots+c_{n-1} = n-1$. The coefficients $A_{c_1,\dots,c_{n-1}}$ are called *mixed Eulerian numbers*, which are known to be non-negative integers (see [30] for details).

In [10], Berget–Spink–Tseng studied log-concavity of matroid h-vectors in relation to mixed Eulerian numbers. For that purpose, they considered the invariant subring of the Chow ring of the permutohedral variety with respect to the action of the symmetric group (which can be identified with Tymoczko's dot action on the cohomology ring). They introduced a basis δ_S of this invariant subring, and they proved that the structure constants of this basis can be written as products of mixed Eulerian numbers ([10, Proposition 7.7 and Corollary 7.9]). This invariant subring is known to be isomorphic to $H^*(Pet_n;\mathbb{Z})$ by [5, Theorem 1.1] (c.f. [4, Theorem B] for $\mathbb{Q}$-coefficients), and one can see that their basis corresponds to ϖ_J in $H^*(Pet_n;\mathbb{Z})$ (compare [10, Corollary 7.9] and Lemma 5.1 in this paper). Therefore, our formula (Theorem 5.6) can also be regarded as computing some products of mixed Eulerian numbers by using the geometry of Pet_n.

Nadeau–Tewari [28] also found a relation between mixed Eulerian numbers and intersection numbers of Schubert varieties and the permutohedral variety for an arbitrary Lie type. After [10] and [28], the second author of this paper investigated in [21] a connection between Peterson Schubert calculus and mixed Eulerian numbers. More precisely, it was shown that the mixed Eulerian numbers can be written as intersection numbers of Schubert divisors in Peterson variety for an arbitrary Lie type ([21, Theorem 1.1]). We remark that, for type A, this formula was proved in [10] and [28] independently. Including this paper, all of these works are done independently, and these established connections between Peterson Schubert calculus and mixed Eulerian numbers.

To end this paper, let us lastly deduce the formula for d_{JK}^L in terms of mixed Eulerian numbers in the context of Peterson Schubert calculus. For $J, K, L \subseteq [n-1]$, recall from (4.10) that we have

$$d_{JK}^L = \langle [X_L], \varpi_J \cdot \varpi_K \rangle_{Pet_n} = \frac{1}{m_J}\frac{1}{m_K}\int_{X_L}\left(\prod_{j\in J}\varpi_j\right)\left(\prod_{k\in K}\varpi_k\right)$$

if $|J|+|K| = |L|$ and that we have $d_{JK}^L = 0$ if $|J|+|K| \neq |L|$. Taking the decomposition $L = L_1 \sqcup \cdots \sqcup L_q$ into the connected components of L, we have $X_L = \prod_{i=1}^q X_{L_i}$ by Corollary 3.6. Hence, the integration over X_L above can be written as a product of integrations over X_{L_i} for $1 \leqslant i \leqslant q$;

$$\int_{X_L}\left(\prod_{j\in J}\varpi_j\right)\left(\prod_{k\in K}\varpi_k\right) = \prod_{i=1}^q \int_{X_{L_i}}\left(\prod_{j\in J\cap L_i}\varpi_j\right)\left(\prod_{k\in K\cap L_i}\varpi_k\right).$$

Denoting $\ell_i := |L_i| + 1$, we have $X_{L_i} \cong Pet_{\ell_i}$ by Corollary 3.6 again. Namely, each integration in the last equality is an intersection number of divisors on Pet_{ℓ_i}. We note that under this isomorphism $\varpi_r \in H^*(Pet_{\ell_i};\mathbb{Q})$ $(1 \leqslant r \leqslant |L_i|)$ corresponds to $\varpi_{r+\min L_i - 1} \in H^*(X_{L_i};\mathbb{Q})$ since we have $Pet_{\ell_i} \subseteq Fl(\mathbb{C}^{\ell_i})$ and $X_{L_i} \subseteq Pet_n \subseteq Fl(\mathbb{C}^n)$. As explained above, the second author gave a formula which computes those intersection numbers as mixed Eulerian numbers ([21, Theorem 1.1]). By applying it to the integrations above, we obtain the following formula for which we take the convention that $A_{c_1,\ldots,c_p} = 0$ unless $c_1 + \cdots + c_p = p$ for positive integers p.

Theorem 6.7. *For $J, K, L \subseteq [n-1]$, we have*

$$d_{JK}^L = \frac{1}{m_J}\frac{1}{m_K}\prod_{i=1}^q A_{c_1^{(i)},\ldots,c_{\ell_i-1}^{(i)}},$$

where $L = L_1 \sqcup \cdots \sqcup L_q$ is the decomposition into the connected components of L and $c_1^{(i)}, \ldots, c_{\ell_i-1}^{(i)}$ are the multiplicities of the product $(\prod_{j\in J\cap L_i}\varpi_j)(\prod_{k\in K\cap L_i}\varpi_k)$ given by

$$c_r^{(i)} := \begin{cases} 2 & \text{if } r+\min L_i - 1 \in J\cap K, \\ 1 & \text{if } r + \min L_i - 1 \in (J\cup K) - (J\cap K), \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leqslant i \leqslant q$ *and* $1 \leqslant r \leqslant |L_i|$ *(which means* $r + \min L_i - 1 \in L_i$*).*

Remark 6.8. As we noted above, this formula can also be deduced from [10, Proposition 7.7].

Remark 6.9. The indexes of the mixed Eulerian numbers appearing in Theorem 6.7 are always less than or equal to 2. In [10] and [21], mixed Eulerian numbers with arbitrary indexes are considered.

References

[1] Hiraku Abe, Lauren DeDieu, Federico Galetto, and Megumi Harada, *Geometry of Hessenberg varieties with applications to Newton-Okounkov bodies*, Selecta Math. (N.S.) **24** (2018), no. 3, 2129–2163.

[2] Hiraku Abe, Naoki Fujita, and Haozhi Zeng, *Geometry of regular Hessenberg varieties*, Transform. Groups **25** (2020), no. 2, 305–333.

[3] ———, *Fano and weak Fano Hessenberg varieties*, Michigan Math. J. **73** (2023), no. 3, 511–555.

[4] Hiraku Abe, Megumi Harada, Tatsuya Horiguchi, and Mikiya Masuda, *The cohomology rings of regular nilpotent Hessenberg varieties in Lie type A*, Int. Math. Res. Not. IMRN (2019), no. 17, 5316–5388.

[5] Hiraku Abe and Haozhi Zeng, *The integral cohomology rings of Peterson varieties in type A*, 2022, `https://arxiv.org/abs/2203.02629`.

[6] Takuro Abe, Tatsuya Horiguchi, Mikiya Masuda, Satoshi Murai, and Takashi Sato, *Hessenberg varieties and hyperplane arrangements*, J. Reine Angew. Math. **764** (2020), 241–286.

[7] Dave Anderson and Julianna Tymoczko, *Schubert polynomials and classes of Hessenberg varieties*, J. Algebra **323** (2010), no. 10, 2605–2623.

[8] Ana Bălibanu, *The Peterson variety and the wonderful compactification*, Represent. Theory **21** (2017), 132–150.

[9] Darius Bayegan and Megumi Harada, *A Giambelli formula for the* S^1*-equivariant cohomology of type A Peterson varieties*, Involve **5** (2012), no. 2, 115–132.

[10] Andrew Berget, Hunter Spink, and Dennis Tseng, *Log-concavity of matroid h-vectors and mixed Eulerian numbers*, Duke Math. J. **172** (2023), no. 18, 3475–3520.

[11] Michel Brion, *Lectures on the geometry of flag varieties*, in Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 33–85.

[12] F. De Mari, C. Procesi, and M. A. Shayman, *Hessenberg varieties*, Trans. Amer. Math. Soc. **332** (1992), no. 2, 529–534.

[13] Elizabeth Drellich, *Combinatorics of equivariant cohomology: Flags and regular nilpotent Hessenberg varieties*, Ph.D. thesis, University of Massachusetts, 2015.

[14] ———, *Monk's rule and Giambelli's formula for Peterson varieties of all Lie types*, J. Algebraic Combin. **41** (2015), no. 2, 539–575.

[15] Yukiko Fukukawa, Megumi Harada, and Mikiya Masuda, *The equivariant cohomology rings of Peterson varieties*, J. Math. Soc. Japan **67** (2015), no. 3, 1147–1159.

[16] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997.

[17] ———, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998.

[18] Rebecca Goldin and Brent Gorbutt, *A positive formula for type A Peterson Schubert calculus*, Matematica **1** (2022), no. 3, 618–665.

[19] Megumi Harada, Tatsuya Horiguchi, and Mikiya Masuda, *The equivariant cohomology rings of Peterson varieties in all Lie types*, Canad. Math. Bull. **58** (2015), no. 1, 80–90.

[20] Megumi Harada and Julianna Tymoczko, *A positive Monk formula in the* S^1*-equivariant cohomology of type A Peterson varieties*, Proc. Lond. Math. Soc. (3) **103** (2011), no. 1, 40–72.

[21] Tatsuya Horiguchi, *Mixed Eulerian Numbers and Peterson Schubert Calculus*, Int. Math. Res. Not. IMRN (2024), no. 2, 1422–1471.

[22] James E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 21, Springer-Verlag, New York-Heidelberg, 1975.

[23] Erik Insko, *Schubert calculus and the homology of the Peterson variety*, Electron. J. Combin. **22** (2015), no. 2, article no. 2.26 (12 pages).

[24] Erik Insko and Julianna Tymoczko, *Intersection theory of the Peterson variety and certain singularities of Schubert varieties*, Geom. Dedicata **180** (2016), 95–116.

[25] Bertram Kostant, *Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight ρ*, Selecta Math. (N.S.) **2** (1996), no. 1, 43–91.
[26] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004.
[27] John W. Milnor and James D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, vol. 76, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1974.
[28] Philippe Nadeau and Vasu Tewari, *The permutahedral variety, mixed Eulerian numbers, and principal specializations of Schubert polynomials*, Int. Math. Res. Not. IMRN (2023), no. 5, 3615–3670.
[29] Dale Peterson, *Quantum cohomology of G/P*, 1997, lecture course, M.I.T., spring term.
[30] Alexander Postnikov, *Permutohedra, associahedra, and beyond*, Int. Math. Res. Not. IMRN (2009), no. 6, 1026–1106.
[31] Konstanze Rietsch, *A mirror construction for the totally nonnegative part of the Peterson variety*, Nagoya Math. J. **183** (2006), 105–142.
[32] Nicholas Teff, *Representations on Hessenberg varieties and Young's rule*, in 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), Discrete Math. Theor. Comput. Sci. Proc., vol. AO, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011, pp. 903–914.
[33] Julianna S. Tymoczko, *Linear conditions imposed on flag varieties*, Amer. J. Math. **128** (2006), no. 6, 1587–1604.

Hiraku Abe, Okayama University of Science, Faculty of Science, Department of Applied Mathematics, 1-1 Ridai-cho, Kita-ku, Okayama, 700-0005, Japan
E-mail : hirakuabe@globe.ocn.ne.jp

Tatsuya Horiguchi, National institute of technology, Ube college, 2-14-1, Tokiwadai, Ube, Yamaguchi, 755-8555, Japan
E-mail : tatsuya.horiguchi0103@gmail.com

Hideya Kuwata, Kindai University Technical College, 7-1 Kasugaoka, Nabari, Mie 518-0459, Japan
E-mail : hideya0813@gmail.com

Haozhi Zeng, Huazhong University of Science and Technology, School of Mathematics and Statistics, Wuhan, 430074, P.R., China
E-mail : zenghaozhi@icloud.com