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Quasihomomorphisms from the integers into Hamming metrics

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Abstract A function $f : \mathbb{Z} \to \mathbb{Q}^n$ is a *c-quasihomomorphism* if the Hamming distance between $f(x+y)$ and $f(x)+f(y)$ is at most c for all $x, y \in \mathbb{Z}$. We show that any c-quasihomomorphism has distance at most some constant $C(c)$ to an actual group homomorphism; here $C(c)$ depends only on c and not on n or f. This gives a positive answer to a special case of a question posed by Kazhdan and Ziegler.

1. Introduction

Let c be a nonnegative real number. A c-quasihomomorphism from a group G to a group H with a left-invariant metric d is a map $f : G \to H$ such that $d(f(xy), f(x)f(y)) \leqslant c$ for all x, y in G. A central question in geometric group theory, raised by Ulam in [17, Chapter 6], is whether there exists an actual homomorphism $f' : G \to H$ such that $d(f(x), f'(x))$ is at most some constant C for all x. (Related questions were studied before Ulam, e.g. by Turing in his work on approximability of groups [16].) Different versions of Ulam's question are of interest: for example, C may be allowed to depend on $c, G, (H, d)$ but not on f; $G, (H, d)$ may be restricted to certain classes and C is only allowed to depend on c.

A well-known example where the answer to this question is negative is the case where $G = H = \mathbb{Z}$ with the standard metric. Here, quasihomomorphisms modulo bounded maps are a model of the real numbers [15, 1], and the answer is yes only for those quasihomomorphisms that correspond to integers. In fact, this construction can be extended to construct completions of fields in general [11].

Much literature in this area focusses on *quasimorphisms*, which are quasihomomorphisms into the real numbers $\mathbb{R}$ with the standard metric; we refer to [12] for a brief introduction. In particular, the concept of a quasimorphism features in bounded cohomology, see [13, 4, 6]. In another branch of the research on quasihomomorphisms H is assumed nonabelian, and one of the first positive results on the central question above is Kazhdan's theorem on ε-representations of amenable groups [9]. For more

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recent results on quasihomomorphisms into nonabelian groups we refer to [7, 8, 5, 2] and the references there.

The following instance of the central question was formulated by Kazhdan and Ziegler in their work on approximate cohomology [10].

Question 1.1. *Let $c \in \mathbb{N}$. Does there exist a constant $C = C(c)$ such that the following holds: For all $n \in \mathbb{N}$ and all functions $f : \mathbb{Z} \longrightarrow \mathbb{C}^{n\times n}$ such that*

$$\text{for all } x, y \in \mathbb{Z},\ \operatorname{rk}(f(x+y) - f(x) - f(y)) \leqslant c,$$

there exists a matrix g such that

$$\text{for all } x \in \mathbb{Z},\ \operatorname{rk}(f(x) - xg) \leqslant C(c)?$$

Here, G equals $\mathbb{Z}$ and H equals $\mathbb{C}^{n\times n}$, both with addition, and the metric on H is defined by $d(A, B) := \operatorname{rk}(A - B)$. In [10, p1], the function $R(\mathbb{Z}, c, \mathbb{C})$ denotes the minimal possible choice of $C(c)$. Our main result is an affirmative answer to Question 1.1 in the special case where all matrices $f(x)$ are assumed to be *diagonal*.

Definition 1.2. *Let $(Q, +)$ be an abelian group. For an element $v \in Q^n$, the* Hamming weight *$w_H(v)$ is the number of nonzero entries of v. For a pair of elements $u, v \in Q^n$, their* Hamming distance *is $w_H(v-u)$. This metric is clearly left-invariant, and indeed even bi-invariant.*

Definition 1.3. *Let A be another abelian group. A function $f : A \to Q^n$ is called a c-quasihomomorphism if*

$$\text{for all } x, y \in A,\ w_H(f(x+y) - f(x) - f(y)) \leqslant c. \tag{1}$$

Remark 1.4. The map $\operatorname{diag} : \mathbb{C}^n \to \mathbb{C}^{n\times n}$ is an isometric embedding from $\mathbb{C}^n$ with the Hamming metric to $\mathbb{C}^{n\times n}$ with the rank metric. This connects Definition 1.3 to Question 1.1.

Definition 1.5. *Let $C \in \mathbb{N}$ and let $f : A \to Q^n$ be a c-quasihomomorphism. A group homomorphism $h : A \to Q^n$ is a C-approximation of f if the Hamming distance between f and h satisfies*

$$\text{for all } x \in A,\ w_H(f(x) - h(x)) \leqslant C.$$

We are ready to state our main result.

Theorem 1.6 (Main Theorem). *Let $c \in \mathbb{N}$. Then there exists a constant $C = C(c) \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and c-quasihomorphisms $f : \mathbb{Z} \to \mathbb{Q}^n$, we have:*

$$\text{for all } x \in \mathbb{Z},\ w_H(f(x) - xf(1)) \leqslant C.$$

Moreover, we can take $C = 28c$.

Remark 1.7. The coefficient 28 is probably not optimal. However, we certainly have that $C(c) \geqslant c$. Indeed, any map $f : \mathbb{Z} \to \mathbb{Q}^n$ for which the only nonzero entries of $f(x)$ are among the first c, is automatically a c-quasihomomorphism.

Corollary 1.8. *Theorem 1.6 also holds with $\mathbb{Q}$ replaced by any torsion-free abelian group Q, with the same value of $C = C(c)$.*

Proof. Suppose, for a contradiction, that we have a c-quasihomomorphism $f : \mathbb{Z} \to Q^n$ but $w_H(f(y) - yf(1)) > C$ for some $y \in \mathbb{Z}$. Since Q is torsion-free, the natural map ι from Q into the $\mathbb{Q}$-vector space $V := \mathbb{Q} \otimes_{\mathbb{Z}} Q$ is injective. Consequently, $g := \iota^n \circ f$ is a c-quasihomomorphism $\mathbb{Z} \to V^n$ with $w_H(g(y) - yg(1)) > C$. Now choose any $\mathbb{Q}$-linear function $\xi : V \to \mathbb{Q}$ that is nonzero on the nonzero entries of $g(y) - yg(1)$. Then $h := \xi^n \circ g$ is a c-quasihomomorphism $\mathbb{Z} \to \mathbb{Q}^n$ with $w_H(h(y) - yh(1)) > C$, a contradiction to Theorem 1.6. □

Remark 1.9. As a referee kindly pointed out to us, our result fits in the broader context of $\mathcal{G}$-*stability* for a family $\mathcal{G}$ of groups endowed with a bi-invariant metric; this was first introduced in [9] and further studied in [3] under the name of *Ulam stability.* Let $\mathcal{G}$ be the family of groups $\{\mathrm{GL}_n(\mathbb{C})\}_{n\geqslant 1}$ with the normalized rank metric, i.e. $d(A,B)=\frac{1}{n}\,\mathrm{rk}(A-B)$. Let $\mathcal{G}_d$ be the subfamily of $\mathcal{G}$ consisting of diagonal matrices. Theorem 1.6 shows that the abelian group $\mathbb{Z}$ is uniformly $\mathcal{G}_d$-stable with a linear estimate.

Theorem 1.6 shows that for a c-quasihomomorphism $f:\mathbb{Z}\to\mathbb{Q}^n$, the group homomorphism $\tilde{f}:\mathbb{Z}\to\mathbb{Q}^n$ defined by $\tilde{f}(x)=xf(1)$ gives a C-approximation for some constant $C\in\mathbb{N}$ independent on n. However, $\tilde{f}$ need not be the homomorphism closest to f, as the next example shows.

Example 1.10. Let $c=1$ and $n\geqslant 3$. Define $f:\mathbb{Z}\to\mathbb{Q}^n$ to be

$$f(x)=\left(\left\lfloor\frac{2x}{5}\right\rceil,\left\lfloor\frac{x}{5}\right\rceil,\alpha_x,0,\dots,0\right), \tag{2}$$

where $\alpha_x\in\mathbb{Q}$ is arbitrary if $5\mid x$, and $\alpha_x=0$ otherwise. Here $\lfloor\,\rceil$ denotes rounding to the nearest integer. To check that f is a 1-quasihomomorphism (1) we work mod 5. For simplicity, restrict to the case $n=3$. Then, for $k\in\mathbb{Z}$,

$$\begin{aligned} f(5k)&=(2k,k,\alpha_{5k}), & f(5k+1)&=(2k,k,0),\\ f(5k+2)&=(2k+1,k,0), & f(5k+3)&=(2k+1,k+1,0),\\ f(5k+4)&=(2k+2,k+1,0). \end{aligned}$$

Let $x=5k+\ell_1$ and $y=5h+\ell_2$ with $0\leqslant\ell_1\leqslant\ell_2<5$. Then we can verify that

$$w_H(f(5(k+h)+(\ell_1+\ell_2))-f(5k+\ell_1)-f(5h+\ell_2))\leqslant c=1$$

in all cases. Roughly speaking, the check boils down to verifying that there are no cases where both $\lfloor\frac{x+y}{5}\rceil\neq\lfloor\frac{x}{5}\rceil+\lfloor\frac{y}{5}\rceil$ and $\left\lfloor\frac{2(x+y)}{5}\right\rceil\neq\lfloor\frac{2x}{5}\rceil+\lfloor\frac{2y}{5}\rceil$, and moreover that if $5\mid x+y$, then $f(x+y)-f(x)-f(y)=(0,0,\alpha_x)$ (because in this case, $\frac{x}{5}$ is rounded down if and only if $\frac{y}{5}$ is rounded up). Note that $w_H(f(x)-xf(1))\leqslant 3$ where equality is sometimes achieved (provided there is at least one $x\neq 0$ for which we chose $\alpha_x\neq 0$). However, there also exist 2-approximations of f. For instance, letting $v=(\frac{2}{5},\frac{1}{5},0,\dots,0)\in\mathbb{Q}^n$, one verifies that

$$w_H(f(x)-xv)\leqslant 2 \text{ for all } x\in\mathbb{Z}.$$

In [14], the authors show that for every 1-quasihomomorphism $f:\mathbb{Z}\to\mathbb{Q}^n$, and even for every 1-quasihomomorphism from $\mathbb{Z}$ into the space of *symmetric* $n\times n$-matrices with the rank metric, there is a 2-approximation.

(This result is consistent with the second paragraph of [10], where a proof of the corresponding statement for general matrices is sketched. However, the above example shows that that proof is incomplete: viewing f as a map to the diagonal matrices, and assuming $\alpha_0=0$ as is done in that paragraph, we obtain a counterexample to the statement in [10] that there exists either a subspace of codimension 1 living in the kernel of all matrices $f(n+m)-f(n)-f(m)$ or else a subspace of dimension 1 containing all their images.)

On the other hand, the following shows that the best possible approximation of a given quasihomomorphism f is at most twice as close as the homomorphism $x\mapsto xf(1)$.

Remark 1.11. Suppose that a map $f : \mathbb{Z} \to \mathbb{Q}^n$ has a C'-approximation h. Then $h(x) = xv$ for some $v \in \mathbb{Q}^n$, and

$$w_H(f(x) - xv) \leqslant C' \text{ for all } x \in \mathbb{N}.$$

Substituting $x = 1$ yields $w_H(f(1) - v) \leqslant C'$. Thus

$$w_H(f(x) - xf(1)) \leqslant w_H(f(x) - xv) + w_H(xv - xf(1)) \leqslant 2C'.$$

Remark 1.12. A result similar to Theorem 1.6 is easily proven in positive characteristic if we allow the constant C to depend on the characteristic. Let K be a field of characteristic $p > 0$, and let $f : \mathbb{Z} \to K^n$ be a c-quasihomomorphism. Then there exists a constant $C = C(p, c)$ such that $w_H(f(x) - xf(1)) \leqslant C$, for all $x \in \mathbb{Z}$.

To see this, we observe that for all $u, v \in \mathbb{Z}$ with $u \geqslant 1$, we have

$$w_H(f(uv) - uf(v)) \leqslant (u - 1)c.$$

This follows by repeatedly applying the inequality $w_H(f(uv)-f((u-1)v)-f(v)) \leqslant c$ if $u > 1$; the case $u = 1$ is trivial.

For $x = kp + r$ with $k \in \mathbb{Z}$ and $0 \leqslant r \leqslant p - 1$, we have

$$w_H(f(x) - xf(1)) = w_H(f(kp + r) - rf(1));$$

here we have used that $pf(1) = 0$. We rewrite the latter as

$$w_H(f(kp + r) - f(kp) - f(r) + f(kp) + f(r) - rf(1)).$$

We have $w_H(f(kp+r)-f(kp)-f(r)) \leqslant c$; $w_H(f(kp)) \leqslant (p-1)c$ using our observation with $u = p, v = k$; and also $w_H(f(r) - rf(1)) \leqslant (p - 2)c$ (in the case $r > 0$). In total, this gives $w_H(f(x) - xf(1)) \leqslant 2(p - 1)c$, so we can take $C = 2(p - 1)c$.

The remainder of this paper is organized as follows. In Section 2 we prove an auxiliary result of independent interest: maps from a finite abelian group into a torsion-free group that are almost a homomorphism, are in fact almost zero. Then, in Section 3, we apply this auxiliary result to the component functions of a c-quasihomomorphism $\mathbb{Z} \to \mathbb{Q}^n$ to prove the Main Theorem.

2. Almost homomorphisms are almost zero

Let A be a finite abelian group and let H be a torsion-free abelian group. The only homomorphism $A \to H$ is the zero map. The following proposition says that maps that are, in a suitable sense, close to being homomorphisms, are in fact also close to the zero map.

Proposition 2.1. *Let a be a positive integer, A an abelian group of order a, H a torsion-free abelian group, $q \in [0, 1]$, and $f : A \to H$ a map. Suppose that the* zero set

$$Z(f) := \{b \in A \mid f(b) = 0\}$$

has cardinality at most qa. Then the problem set

$$P(f) := \{(b, c) \in A \times A \mid f(b + c) \neq f(b) + f(c)\}$$

has cardinality at least $\frac{(1-q)^2}{4}a^2 + \frac{(1-q)}{2}a$.

The contraposition of this statement says that if $P(f)$ is a small fraction of a^2, so that f can be thought of as an (additive) "almost homomorphism" $A \to H$, then q must be close to 1 so that f is essentially zero.

Proof. Since H is torsion-free, it embeds into the $\mathbb{Q}$-vector space $V := \mathbb{Q} \otimes_{\mathbb{Z}} H$. By basic linear algebra, there exists a $\mathbb{Q}$-linear function $\xi : V \to \mathbb{Q}$ such that $\xi(f(b)) \neq 0$ for all $b \notin Z(f)$, so that $Z(\xi \circ f) = Z(f)$. Since $P(\xi \circ f) \subseteq P(f)$, it suffices to prove the proposition for $\xi \circ f$ instead of f. In other words, we may assume from the beginning that $H = \mathbb{Q}$.

Set

$$B := \{b \in A \mid f(b) > 0\}.$$

Let $\lambda_1 > \lambda_2 > \ldots > \lambda_k > 0$ be the distinct values in $f(B)$, and for each $i = 1, \ldots, k$ set

$$B_i := \{b \in B \mid f(b) = \lambda_i\} \text{ and } n_i := |B_i|;$$

as well as $n := n_1 + \cdots + n_k = |B|$.

Now for each $c \in B_1$ and each $b \in B$ we have

$$f(b) + f(c) = f(b) + \lambda_1 > \lambda_1$$

so that the left-hand side is not in $f(B)$ and in particular not equal to $f(b+c)$. We have thus found $n_1(n_1 + \cdots + n_k)$ pairs $(b, c) \in P(f)$ with $c \in B_1$.

Next, suppose (b, c) is a pair with $c \in B_2$, $b \in B$, and $(b, c) \notin P(f)$. Then

$$f(b+c) = f(b) + f(c) > f(c) = \lambda_2$$

and hence $b+c \in B_1$. But given c, there are at most n_1 values of b with $b+c \in B_1$. (Note that here we have used that A is a group.) Hence we have at least $n_2(n_2 + \cdots + n_k)$ pairs $(b, c) \in P(f)$ with $c \in B_2$.

Similarly, we find at least $n_i(n_i + \cdots + n_k)$ pairs $(b, c) \in P(f)$ with $c \in B_i$. In total, we have therefore found at least

$$\sum_{i=1}^{k} n_i(n_i + \cdots + n_k) \geqslant \frac{n(n+1)}{2} \tag{3}$$

pairs in $P(f)$; see Figure 1.

Let $B' := \{b' \in A \mid f(b') < 0\}$ and $n' := |B'|$. Repeating the same argument above with B' and n', we find at least $n'(n'+1)/2$ further pairs in $P(f)$, disjoint from those found above. Since $|Z(f)| \leqslant qa$, we have $n + n' \geqslant a(1-q)$. Therefore

$$|P(f)| \geqslant \frac{n(n+1)}{2} + \frac{n'(n'+1)}{2} = \frac{n^2 + n'^2}{2} + \frac{n+n'}{2} \geqslant \left(\frac{n+n'}{2}\right)^2 + \frac{n+n'}{2},$$

where the second inequality is the Cauchy-Schwarz inequality

$$\left(n^2 + n'^2\right)\left(\frac{1}{2^2} + \frac{1}{2^2}\right) \geqslant \left(\frac{n}{2} + \frac{n'}{2}\right)^2.$$

Since $n + n' \geqslant a(1-q)$, we conclude that

$$|P(f)| \geqslant \left(\frac{a-qa}{2}\right)\left(\frac{a-qa}{2} + 1\right). \qquad \square$$

Remark 2.2. The lower bound in Proposition 2.1 is sharp. Let $a = 2k+1 \in \mathbb{Z}$, consider $A := \mathbb{Z}/a\mathbb{Z}$ and define $f : A \to \mathbb{Z}$ as $f(x) :=$the representative of $x + a\mathbb{Z}$ in $\{-k, \ldots, 0, \ldots, k\}$. We take $q = \frac{Z(f)}{a} = \frac{1}{2k+1}$. Then $f(x+y) = f(x) + f(y)$ if and only if the right-hand side is still inside the interval $\{-k, \ldots, k\}$, and a straightforward count shows that this is the case for $3k^2 + 3k + 1$ pairs $(x, y) \in A^2$. Hence $P(f)$ has size $k(k+1)$, which equals $\frac{(1-q)^2}{4}a^2 + \frac{(1-q)}{2}a$.

A similar construction for $a = 2k$ yields a problem set of size $\frac{a^2}{4} = k^2$, which equals the ceiling of the lower bound $\frac{a^2}{4} - \frac{1}{4}$.

Below, we will use the following strengthening of Proposition 2.1:

Proposition 2.3. *Let a, A, H, q and f be as in Proposition 2.1. Furthermore, let $p \in [0, \frac{1-q}{2})$ and let $S \subseteq A$ be a subset of cardinality at most pa. Then the set*

$$P_S(f) := \{(b,c) \in A \times A \mid f(b+c) \neq f(b) + f(c) \text{ and } b+c \notin S\}.$$

has cardinality at least $\frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a$.

Proof. Keep the notation from the proof of Proposition 2.1. Recall $n = |B|$ and $n' = |B'|$. Note that for a fixed b, there can be at most pa choices of c with $b + c \in S$. We then find at least $n_i(n_i + \cdots + n_k - pa)$ pairs $(b,c) \in P_S(f)$ with $b \in B_i$. Letting $k' \leqslant k$ be the largest index for which the second factor $(n_{k'} + \cdots + n_k - pa)$ is nonnegative, as in the proof of Proposition 2.1, we find that B contributes at least

$$\begin{aligned} \sum_{i=1}^{k'} n_i(n_i + \cdots + n_k - pa) &= \sum_{i=1}^{k'} n_i(n - n_1 - \cdots - n_{i-1} - pa) \\ &\geqslant (n-pa)(n-pa+1)/2 \end{aligned} \tag{4}$$

to $P_S(f)$; see Figure 1. Similarly, B' contributes at least $(n'-pa)(n'-pa+1)/2$, and these contributions are disjoint. The desired inequality follows as in the proof of Proposition 2.1 but with n, n' replaced by $n - pa, n' - pa$. □

The key ingredient for the proof of Theorem 1.6 is the following corollary of Proposition 2.3. Here, and in the rest of the paper, we write $[a]$ for the set $\{1, 2, \ldots, a\}$.

Corollary 2.4. *Let $p, q \in [0,1]$ such that $p < \frac{1-q}{2}$. Let $f : [2a] \to \mathbb{Q}$ such that:*

(1) *$|Z_a(f)| \leqslant qa$, where $Z_a(f) := \{x \in [a] \mid f(x) = 0\}$ is the zero set of $f\big|_{[a]}$.*
(2) *$|NP(f)| \leqslant pa$, where $NP(f) := \{x \in [a] \mid f(x+a) \neq f(x)\}$ is the nonperiodicity set.*

Then

$$|P(f)| \geqslant \frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a,$$

where

$$P(f) = \{(x,y) \in [a] \times [a] \mid f(x+y) \neq f(x) + f(y)\}.$$

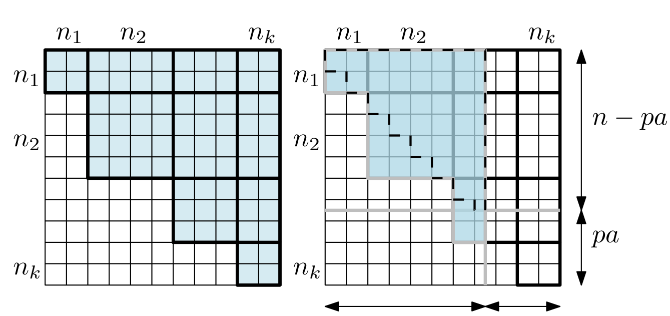

Figure 1. On the left, a graphical proof of the inequality (3): the left-hand side is the number of small squares in the shaded region, the right-hand side is the number of squares on or above the main diagonal. On the right, a proof of the inequality (4): the two expressions on top represent the area of the shaded region, while the bottom expression represents the area enclosed by the dashed line.

Proof. Let $\tilde{f}$ be the restriction of f to the interval $[a]$, and identify $\mathbb{Z}/a\mathbb{Z}$ with $[a]$ with the group operation $\star$ defined by $x \star y := x + y \pmod{a}$.

Let $S = NP(f)$, and apply Proposition 2.3 to $\tilde{f}$. We find that

$$P_S(\tilde{f}) = \{(b,c) \in \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z} \mid \tilde{f}(b \star c) \neq \tilde{f}(b) + \tilde{f}(c) \text{ and } b \star c \notin S\}$$

has cardinality at least $\frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a$. Since $b \star c \notin S$ implies that $\tilde{f}(b \star c) = f(b+c)$, this set is contained in the problem set $P(f)$. □

3. Proof of the main theorem

The main goal of this section is to prove Theorem 1.6. We start with some definitions.

Definition 3.1. *Let $1 < a$, and $f : [2a] \to \mathbb{Q}$. We define the following* problem sets *of f:*

$$P(f) := \{(x,y) \in [a] \times [a] \mid f(x+y) \neq f(x) + f(y)\},$$

and

$$P_1(f) := \{x \in [a] \mid f(x+1) \neq f(x) + f(1)\},$$

and

$$P_a(f) := \{x \in [a] \mid f(x+a) \neq f(x) + f(a)\}.$$

Furthermore, we recall that $Z_a(f)$ denotes the zero set of $f\big|_{[a]}$:

$$Z_a(f) := \{x \in [a] \mid f(x) = 0\}.$$

The following proposition says that $P_1(f), P_a(f), P(f)$ cannot be simultaneously small.

Proposition 3.2. *Let $p, q \in (0,1)$ such that $1 - q - 2p > 0$, $a \in \mathbb{N}$ with $1 < a$, and let $f : [2a] \to \mathbb{Q}$ such that $f(a) \neq af(1)$. Then at least one of the following holds:*

(i) $|P_1(f)| > qa$,
(ii) $|P_a(f)| > pa$,
(iii) $|P(f)| \geqslant F(p,q)a^2$,

where

$$F(p,q) := \frac{(1-q-2p)^2}{4}.$$

Proof. Without loss of generality we can assume $f(a) = 0$ and hence $f(1) \neq 0$. Indeed, suppose we have shown the statement for every $\tilde{f}$ with $\tilde{f}(a) = 0$. Then for any $f : [2a] \to \mathbb{Q}$ with $f(a) \neq af(1)$, we take $\tilde{f} : [2a] \to \mathbb{Q}$ to be $\tilde{f}(x) = af(x) - xf(a)$. Now we observe that $\tilde{f}(a) = 0 \neq a\tilde{f}(1)$, and that $P(f) = P(\tilde{f})$, $P_1(f) = P_1(\tilde{f})$, $P_a(f) = P_a(\tilde{f})$.

To prove the proposition we will assume that ((i)) and ((ii)) are false, and prove that then ((iii)) must hold. Write $Z_a(f) = \{x_1, \ldots, x_m\}$, where $x_1 < \cdots < x_m$. Note that for $1 \leqslant i < m$, one of the elements $x_i, x_i + 1, \ldots, x_{i+1} - 1$ needs to be in $P_1(f)$ since $f(x_{i+1}) \neq f(x_i) + (x_{i+1} - x_i)f(1)$. Likewise, at least one of the elements $1, 2, \ldots, x_1 - 1$ needs to be in $P_1(f)$. Thus we have

$$|Z_a(f)| \leqslant |P_1(f)| \leqslant qa,$$

and by assumption we have $|NP(f)| = |P_a(f)| \leqslant pa$. Now we can apply Corollary 2.4 to conclude. □

We now prove Theorem 1.6.

Proof of the Main Theorem. Consider a c-quasihomomorphism $f = (f_1, \dots, f_n) : \mathbb{Z} \to \mathbb{Q}^n$. Our goal is to show that for every $a \in \mathbb{Z}$ we have $w_H(f(a) - af(1)) \leqslant C$ for some constant C depending only on c. We start with the case $a > 0$.

Write $I_a \coloneqq \{i \in [a] \mid f_i(a) \neq af_i(1)\}$, and note that $|I_a| = w_H(f(a) - af(1))$. We will show that $|I_a| \leqslant C'$ for some constant C' depending on c only. To this end, fix small parameters $p, q \in (0,1)$ (to be optimized over later) and write $f_i^a \coloneqq f_i\big|_{[2a]}$ for the restriction of f_i to $[2a]$. By Proposition 3.2, for every $i \in I_a$, we have

(i) $|P_1(f_i^a)| > qa$, or
(ii) $|P_a(f_i^a)| > pa$, or
(iii) $|P(f_i^a)| \geqslant F(p,q)a^2$.

Let m_0 be the number of coordinates $i \in I_a$ such that (iii) holds. We define m_1 and m_2 analogously, for (i) and (ii) respectively.

By counting the number of triples $(i, x, y) \in [n] \times [a] \times [a]$ such that $f_i(x+y) - f_i(x) - f_i(y) \neq 0$ in two ways, we see that

$$\sum_{x=1}^{a} \sum_{y=1}^{a} w_H\left(f(x+y) - f(x) - f(y)\right) = \sum_{i=1}^{n} |P(f_i^a)| \geqslant \sum_{i \in I_a} |P(f_i^a)|.$$

Because f is a c-quasihomomorphism, the very left-hand side is at most a^2c. On the other hand, the very right-hand side is at least $m_0F(p,q)a^2$, so

$$a^2c \geqslant \sum_{x=1}^{a} \sum_{y=1}^{a} w_H\left(f(x+y) - f(x) - f(y)\right) \geqslant \sum_{i \in I_a} |P(f_i^a)| \geqslant m_0F(p,q)a^2.$$

So we obtain $m_0 \leqslant \frac{c}{F(p,q)}$. Similarly we find

$$ac \geqslant \sum_{x=1}^{a} w_H\left(f(x+1) - f(x) - f(1)\right) = \sum_{i=1}^{n} |P_1(f_i^a)| \geqslant \sum_{i \in I_a} |P_1(f_i^a)| > m_1qa,$$

so that $m_1 < \frac{c}{q}$. Finally,

$$ac \geqslant \sum_{x=1}^{a} w_H\left(f(x+a) - f(x) - f(a)\right) = \sum_{i=1}^{n} |P_a(f_i^a)| \geqslant \sum_{i \in I_a} |P_a(f_i^a)| > m_2pa.$$

So $m_2 < \frac{c}{p}$. But now $|I_a| \leqslant m_0 + m_1 + m_2 < c(\frac{1}{F(p,q)} + \frac{1}{q} + \frac{1}{p}) =: C'$.

The case $a = 0$ is easy: we have

$$w_H(f(0)) = w_H(f(0) - f(0) - f(0)) \leqslant c.$$

Finally, let us consider the case $a < 0$. Then

$$\begin{aligned} w_H(f(a) - af(1)) \leqslant & w_H(f(a) + f(-a) - f(0)) + w_H(f(0)) \\ & + w_H(f(-a) - (-a)f(1)) \leqslant 2c + C' =: C. \end{aligned}$$

This completes the proof of the qualitative part of the Main Theorem. To obtain the explicit bound $28c$, we minimize the function

$$2 + \frac{1}{q} + \frac{1}{p} + \frac{1}{F(p,q)} = 2 + \frac{1}{q} + \frac{1}{p} + \frac{4}{(1-q-2p)^2}.$$

This function is strictly convex for $(p,q) \in \mathbb{R}^2_{>0}$, so it has at most one minimum in the positive orthant. We find this by setting the partial derivatives to zero and solving for p, q. The minimum is ≈ 27.6817 and attained at $(p,q) \approx (0.1167, 0.16500)$. □

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