ALGEBRAIC COMBINATORICS

Young-Hoon Kiem & Donggun Lee

Geometry of the twin manifolds of regular semisimple Hessenberg varieties and unicellular LLT polynomials

Volume 7, issue 3 (2024), p. 861-885.

https://doi.org/10.5802/alco.349

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org   www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 7, issue 3 (2024), p. 861–885
https://doi.org/10.5802/alco.349

Geometry of the twin manifolds of regular semisimple Hessenberg varieties and unicellular LLT polynomials

Young-Hoon Kiem & Donggun Lee

Abstract Recently, Masuda-Sato and Precup-Sommers independently proved an LLT version of the Shareshian-Wachs conjecture, which says that the Frobenius characteristics of the cohomology of the twin manifolds of regular semisimple Hessenberg varieties are unicellular LLT polynomials. The purpose of this paper is to study the geometry of twin manifolds and we prove that they are related by explicit blowups and fiber bundle maps. Upon taking their cohomology, we obtain a direct proof of the modular law which establishes the LLT Shareshian-Wachs conjecture.

1. Introduction

LLT polynomials are symmetric functions that serve as q-deformations of the product of Schur functions, introduced by Lascoux, Leclerc, and Thibon [18] in their study of quantum affine algebras. A specific class of these polynomials known as unicellular LLT polynomials (Definition 2.2) was explored in [9] using Dyck paths, or Hessenberg functions, in parallel with the chromatic quasisymmetric functions. The purpose of this paper is to investigate on the geometry of the twin manifolds of regular semisimple Hessenberg varieties (Propositions 4.6 and 4.7) and provide a direct geometric proof of the fact that unicellular LLT polynomials are the Frobenius characteristics of representations of symmetric groups S_n on the cohomology of the twin manifolds (Theorem 5.4).

Hessenberg varieties are subvarieties of flag varieties with interesting properties in geometric, representation theoretic and combinatorial aspects (cf. [11, 2]). One of their notable features is the S_n-action on their cohomology [27], where the induced graded S_n-representations are equivalent to the purely combinatorially defined symmetric functions known as the chromatic quasisymmetric functions [24, 23] of the associated indifference graphs. This equivalence (cf. (5.5)), known as the Shareshian-Wachs conjecture [23], proved in [8, 14], translates the longstanding conjecture by Stanley and Stembridge [25] on e-positivity of the chromatic (quasi)symmetric functions into a positivity problem on the S_n-representations on the cohomology of Hessenberg varieties.

Manuscript received 14th July 2023, revised 29th November 2023, accepted 12th December 2023.

Keywords. Hessenberg varieties, twin manifolds, unicellular LLT polynomials, Shareshian-Wachs conjecture, modular law.

Acknowledgements. YHK was partially supported by Korea NRF grant 2021R1F1A1046556. DL was supported by the Institute for Basic Science (IBS-R032-D1).

A natural question arises whether there exist geometric objects that encode unicellular LLT polynomials through their cohomology, as in the Shareshian-Wachs conjecture. Recently, an answer was found by Masuda-Sato and Precup-Sommers in [21, 22] where they proved that unicellular LLT polynomials are the Frobenius characteristics of the cohomology of the *twin manifolds* of regular semisimple Hessenberg varieties.

The unitary group $U(n)$ is acted on by its maximal torus $T = U(1)^n$ by left and right multiplications. So we have the quotient maps

$$Y := T\backslash U(n) \xleftarrow{p_1} U(n) \xrightarrow{p_2} U(n)/T \cong \mathrm{Fl}(n) =: X$$

where X denotes the flag variety $\mathrm{Fl}(n)$ and Y denotes the isospectral manifold of Hermitian matrices with a fixed spectrum (cf. §3.1). The twin manifold of a Hessenberg variety $X_h \subset \mathrm{Fl}(n) = X$ is now defined in [6] as the submanifold

$$Y_h := p_1(p_2^{-1}(X_h))$$

of the isospectral manifold Y. These twin manifolds Y_h, which are the spaces of staircase Hermitian matrices with a fixed given spectrum, are interesting compact orientable smooth real algebraic varieties. They generalize the space of tridiagonal matrices of a given spectrum [26, 7, 10] and we have natural isomorphisms

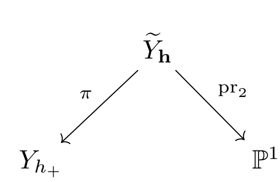
$$H^*_T(Y_h) \cong H^*_{T\times T}(p_1^{-1}(X_h)) \cong H^*_T(X_h) \tag{1.1}$$

which induce an S_n-action on the cohomology $H^*(Y_h)$ from that on $H^*(X_h)$ in [27]. The LLT analogue of the Shareshian-Wachs conjecture (*LLT-SW conjecture*, for short) tells us that the Frobenius characteristics of $H^*(Y_h)$ are the unicellular LLT polynomials (cf. Theorem 5.4). The known proofs in [21, 22] are rather indirect and use only the Hessenberg varieties without looking into the geometry of twin manifolds themselves. See Remark 5.5 for more details. Therefore, it seems natural to ask for a direct approach through the geometry of twin manifolds.

The *modular law* (cf. Definition 2.4), introduced in [3, 13] for chromatic quasisymmetric functions and in [5, 19] for unicellular LLT polynomials, is a significant relation involving specific triples of these functions. It serves as a symmetric function analogue to the well known deletion-contraction relation of chromatic polynomials. In fact, Abreu and Nigro proved in [3] that together with an initial condition (for the case of $h(i) = n$ for all i) and the multiplicativity (cf. (2.5)), the modular law completely determines the chromatic quasisymmetric functions and unicellular LLT polynomials.

In [16], the authors investigated on the geometry of Hessenberg varieties X_h and proved that the Hessenberg varieties X_{h_-}, X_h and X_{h_+} for a modular triple $\mathbf{h} = (h_-, h, h_+)$ (cf. Definition 2.3) are related by explicit blowups and projective bundle maps. By applying the blowup formula and projective bundle formula, we then immediately obtain the modular law for the cohomology of X_h, which provides us with an elementary proof of the Shareshian-Wachs conjecture.

In this paper, we investigate on the geometry of the twin manifolds Y_h. The key for our comparison of twin manifolds is the *roof manifold* $\widetilde{Y}_{\mathbf{h}}$ defined in Definition 4.1. For a modular triple $\mathbf{h} = (h_-, h, h_+)$ (cf. Definition 2.3), we construct maps



where pr_2 is a smooth fibration over the complex projective line $\mathbb{P}^1$ with fiber Y_h (Proposition 4.6) and π is the blowup along the submanifold Y_{h_-} of complex codimension 2 (Proposition 4.7). We define an S_n-action on the cohomology $H^*(\widetilde{Y}_{\mathbf{h}})$ and show that the induced maps on cohomology by π and pr_2 are S_n-equivariant. We thus obtain S_n-equivariant isomorphisms

$$H^*(Y_h) \oplus H^{*-2}(Y_h) \cong H^*(\widetilde{Y}_{\mathbf{h}}) \cong H^*(Y_{h_+}) \oplus H^{*-2}(Y_{h_-}).$$

Upon taking the Frobenius characteristic, we have the modular law for $H^*(Y_h)$ and hence the LLT-SW conjecture

$$\sum_{k \geqslant 0} \mathrm{ch}(H^{2k}(Y_h)) q^k = \mathrm{LLT}_h(q)$$

where $\mathrm{LLT}_h(q)$ denotes the unicellular LLT polynomial associated to h.

The layout of this paper is as follows. In §2, we review the definition of unicelullar LLT polynomials and their characterization by the modular law. In §3, we review the results in [6] on twin manifolds including the S_n-action defined on their cohomology. In §4, we study the geometry of twin manifolds of triples and in §5, we establish the modular law for S_n-representations on their cohomology.

All cohomology groups in this paper have rational coefficients. By $\mathbb{P}^r$, we denote the complex projective space of one dimensional subspaces in $\mathbb{C}^{r+1}$.

2. Unicellular LLT polynomials

LLT polynomials are symmetric functions introduced by Lascoux, Leclerc, and Thibon [18] as q-deformations of the product of Schur functions in their study of quantum affine algebras. In the case of unicellular LLT polynomials, which form a subfamily of these symmetric functions, a more convenient model is presented in [9] using Hessenberg functions. This model represents unicellular LLT polynomials as symmetric functions that encode colorings of graphs, which may not be proper.

In this section, we recall the definition of unicellular LLT polynomials in terms of Hessenberg functions from [9] and the characterization by the modular law from [4].

2.1. Definitions.
Unicellular LLT polynomials can be defined as follows.

Definition 2.1. *Let $[n] := \{1, \cdots, n\}$ for an integer $n \geqslant 1$.*

1. *A* Hessenberg function *is a nondecreasing function $h : [n] \to [n]$ satisfying $h(i) \geqslant i$ for all i*
2. *The* indifference graph Γ_h *associated to h is the graph whose set of vertices is $V(\Gamma_h) = [n]$ and whose set of edges is*
$$\mathrm{E}(\Gamma_h) = \{(i,j) \in [n] \times [n] : i < j \leqslant h(i)\}.$$

Every unicellular LLT polynomial can be written as a symmetric function which encodes vertex-colorings of the indifference graph Γ_h, similar to the definition of the chromatic quasisymmetric function [24, 23].

A map $\gamma : \mathrm{V}(\Gamma_h) \to \mathbb{N}$ is said to be a (vertex-)coloring of Γ_h, and it is said to be *proper* if and only if $\gamma(i) \neq \gamma(j)$ whenever $(i,j) \in \mathrm{E}(\Gamma_h)$, where $\mathbb{N}$ is the set of colors indexed by positive integers.

Definition 2.2. [9, 24, 23] *Let $h : [n] \to [n]$ be a Hessenberg function. Let Λ be the ring of symmetric functions in variables $x_1, x_2, \cdots$.*

1. *The* unicellular LLT polynomial *associated to h is*

$$\mathrm{LLT}_h(q) := \sum_{\gamma : \mathrm{V}(\Gamma_h) \to \mathbb{N}} q^{\mathsf{asc}_h(\gamma)} x_{\gamma(1)} \cdots x_{\gamma(n)} \quad \in \Lambda[q] \tag{2.1}$$

where γ runs over all colorings which are not necessarily proper and

$$\mathsf{asc}_h(\gamma) := |\{(i,j) \in \mathrm{E}(\Gamma_h) : \gamma(i) < \gamma(j)\}|.$$

(2) *The* chromatic quasisymmetric function *associated to h is*

$$\mathsf{csf}_h(q) := \sum_{\substack{\gamma:\mathrm{V}(\Gamma_h)\to\mathbb{N} \\ proper}} q^{\mathsf{asc}_h(\gamma)} x_{\gamma(1)} \cdots x_{\gamma(n)} \quad \in \Lambda[q] \tag{2.2}$$

where γ runs over all proper *colorings.*

2.2. Modular law and characterization of $\mathrm{LLT}_{(-)}$. The symmetric functions LLT_h and csf_h satisfy a linear relation called the *modular law*.

Definition 2.3 (Modular triple). *Let $h_-, h, h_+ : [n] \to [n]$ be Hessenberg functions. The triple (h_-, h, h_+) is called a* modular triple *if it satisfies one of the following.*

(1) *If $h(j) = h(j+1)$ and $h^{-1}(j) = \{j_0\}$ for some $1 \leqslant j_0 < j < n$, then h_- and h_+ are defined by*

$$h_-(i) = \begin{cases} j-1 & \text{for } i = j_0 \\ h(i) & \text{otherwise} \end{cases} \quad \text{and} \quad h_+(i) = \begin{cases} j+1 & \text{for } i = j_0 \\ h(i) & \text{otherwise.} \end{cases}$$

(2) *If $h(j) + 1 = h(j+1) \neq j+1$ and $h^{-1}(j) = \varnothing$ for some $1 \leqslant j < n$, then h_- and h_+ are defined by*

$$h_-(i) = \begin{cases} h(j) & \text{for } i = j+1 \\ h(i) & \text{otherwise} \end{cases} \quad \text{and} \quad h_+(i) = \begin{cases} h(j)+1 & \text{for } i = j \\ h(i) & \text{otherwise.} \end{cases}$$

The two conditions (1) and (2) in Definition 2.3 are actually dual to each other. See Remark 4.2.

Definition 2.4 (Modular law). *Let F be a function from the set of Hessenberg functions to $\Lambda[q]$. We say that F satisfies the* modular law *if*

$$(1+q)F(h) = F(h_+) + qF(h_-) \tag{2.3}$$

for every modular triple (h_-, h, h_+).

Note that unicellular LLT polynomials and chromatic quasisymmetric functions can be viewed as functions $\mathrm{LLT}_{(-)}$ and $\mathsf{csf}_{(-)}$ from the set of Hessenberg functions to $\Lambda[q]$.

Proposition 2.5. [3, 5, 19] *Unicellular LLT polynomials and chromatic quasisymmetric functions satisfy the modular law.*

The modular law, analogous to the deletion-contraction property in chromatic polynomials, plays a crucial role in determining these symmetric functions recursively.

For $m \geqslant 1$, let

$$[m]_q := \frac{1-q^m}{1-q} = 1 + q + \cdots + q^{m-1}, \quad [m]_q! := \prod_{i=1}^{m} [i]_q!.$$

We set $[0]_q! := 1$. Moreover, let $e_m := \sum_{1 \leqslant i_1 < \cdots < i_m} x_{i_1} \cdots x_{i_m} \in \Lambda$ be the m-th elementary symmetric function.

Theorem 2.6. [3, 4] $\mathrm{LLT}_{(-)}$ *(resp. $\mathsf{csf}_{(-)}$) is the unique function F from the set of Hessenberg functions to $\Lambda[q]$ satisfying the following.*

(1) *For $h : [n] \to [n]$ with $h(i) = n$ for all i, $\mathcal{K}_n := F(h)$ satisfies*

$$\mathcal{K}_n = \sum_{i=1}^{n} (q-1)^{i-1} \frac{[n-1]_q!}{[n-i]_q!} e_i \mathcal{K}_{n-i}, \qquad \mathcal{K}_0 := 1 \tag{2.4}$$

(resp. $\mathcal{K}_n = [n]_q! e_n$).

(2) *It is multiplicative: when $h(j) = j$ for $1 \leqslant j < n$,*

$$F(h) = F(h')F(h'') \tag{2.5}$$

where $h' : [j] \to [j]$ and $h'' : [n-j] \to [n-j]$ are Hessenberg functions defined by

$$h'(i) = i \quad \text{for} \quad i \in [j] \quad \text{and} \quad h''(i) = h(i+j) - j \quad \text{for} \quad i \in [n-j].$$

(3) *It satisfies the modular law* (2.3).

Remark 2.7. One fundamental technique in representation theory and combinatorics is to construct a geometric object corresponding to an object of interest. Hard combinatorics problems are often translated into well known geometry problems and solved subsequently, as demonstrated by the recent spectacular works of June Huh.

The Shareshian-Wachs conjecture formulated in [23] and proved in [8, 14] tells us that the chromatic quasisymmetric function (2.2) is the ω-dual of the Frobenius characteristic

$$\mathcal{F}(h) := \sum_{k \geqslant 0} \mathrm{ch}(H^{2k}(X_h)) q^k \quad \in \quad \Lambda[q]$$

of regular semisimple Hessenberg varieties X_h in §3.2 below. Here ω is an involution of Λ interchanging each Schur function with its transpose. The first two conditions (1) and (2) in Theorem 2.6 for csf are easy to check for $\omega\mathcal{F}(h)$ and hence the Shareshian-Wachs conjecture follows immediately from the modular law (2.3) for $\mathcal{F}(h)$. In [16], we investigated on the geometry of generalized Hessenberg varieties and constructed canonical S_n-equivariant isomorphisms

$$H^{2k}(X_h) \oplus H^{2k-2}(X_h) \cong H^{2k}(X_{h_+}) \oplus H^{2k-2}(X_{h_-}).$$

Upon taking the Frobenius characteristic, we obtain the modular law (2.3) and hence the Shareshian-Wachs conjecture

$$\mathcal{F}(h) = \omega\, \mathsf{csf}_h(q).$$

In the remainder of this paper, we will prove that the three conditions in Theorem 2.6 for unicellular LLT are satisfied for the Frobenius characteristics of representations of S_n on the cohomology of the twin manifolds Y_h of regular semisimple Hessenberg varieties X_h, by finding geometric relations among the twin manifolds that give rise to the modular law (2.3) upon taking cohomology. This will give us a direct proof of the LLT-SW conjecture (cf. Theorem 5.4) without relying on the Shareshian-Wachs conjecture.

3. Twin manifolds and their cohomology

In this section, we collect necessary facts about the twin manifolds Y_h of regular semisimple Hessenberg varieties X_h of type A from [6].

Let $h : [n] \to [n]$ be a Hessenberg function (Definition 2.1) where $[n] = \{1, \cdots, n\}$. Let

$$x = \mathrm{diag}(\lambda_1, \cdots, \lambda_n), \quad \lambda_1 > \lambda_2 > \cdots > \lambda_n \in \mathbb{R}$$

be a fixed regular semisimple diagonal matrix. Let $T \cong U(1)^n$ denote the group of diagonal unitary matrices.

3.1. Isospectral manifolds. Let $\mathcal{H}$ denote the real vector space of $n\times n$ Hermitian matrices. Let $Y = Y(x) \subset \mathcal{H}$ be the set of $n \times n$ Hermitian matrices whose characteristic polynomial is $\prod_{i=1}^{n}(t-\lambda_i)$. In other words, Y is the set of $n \times n$ Hermitian matrices with fixed (unordered) spectrum $\{\lambda_i\}$. As the diffeomorphism type of Y is independent of x by [6, Theorem 3.5], we will suppress x to simplify the notation.

By the spectral decomposition theorem in linear algebra, any matrix in Y is of the form $g^{-1}xg$, with $g \in U(n)$ and the map

$$U(n) \longrightarrow Y \subset \mathcal{H}, \quad g \mapsto g^{-1}xg$$

induces a diffeomorphism

$$T\backslash U(n) \cong Y, \quad Tg \mapsto g^{-1}xg. \tag{3.1}$$

In particular, Y is a compact smooth orientable manifold of real dimension $n^2 - n$.

Let $X := \mathrm{Fl}(n)$ denote the variety of flags of $\mathbb{C}$-linear subspaces

$$(V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n), \quad \dim V_i = i$$

which is a smooth projective variety of real dimension $n^2 - n$. As the columns of a unitary matrix define a flag in $\mathbb{C}^n$, we have

$$X = \mathrm{Fl}(n) \cong U(n)/T.$$

The isospectral manifold Y and the flag variety X fit into the following diagram

$$Y \cong T\backslash U(n) \xleftarrow{p_1} U(n) \xrightarrow{p_2} U(n)/T \cong \mathrm{Fl}(n) = X \tag{3.2}$$

where p_1 is the left quotient and p_2 is the right quotient.

3.2. Hessenberg varieties and their twins. For a Hessenberg function $h : [n] \to [n]$, the *Hessenberg variety* associated to h is defined in [11] as

$$X_h := \{(V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \in X : xV_i \subset V_{h(i)} \text{ for all } i\}.$$

Under our assumptions, by [11], the Hessenberg variety X_h is a smooth projective variety of real dimension

$$2\sum_{i=1}^{n}(h(i)-i). \tag{3.3}$$

See [2] for a recent survey on Hessenberg varieties.

The *twin manifold* Y_h of X_h is a submanifold of Y defined in [6] by

$$Y_h := p_1(Z_h), \quad Z_h := p_2^{-1}(X_h) \tag{3.4}$$

where p_1 and p_2 are the quotient maps in (3.2).

By [6, Theorem 3.5], Y_h is a compact real smooth manifold of dimension (3.3) whose diffeomorphism type is independent of the choice of x. Using (3.1), it is straightforward to check that Y_h is precisely, the locus of staircase matrices

$$Y_h = \{y \in Y \ : \ y_{ij} = 0 \text{ if either } i > h(j) \text{ or } j > h(i)\} \tag{3.5}$$

where $y_{ij} = \overline{y_{ji}}$ denotes the entry of the Hermitian matrix y at the i-th row and j-th column.

Since the real dimension of Y is $n^2 - n = 2\sum_{i=1}^{n}(n-i)$ and Y_h is defined by the vanishing of $\sum_{i=1}^{n}(n-h(i))$ complex valued functions by (3.5), the expected dimension

$$2\sum_{i=1}^{n}(n-i) - 2\sum_{i=1}^{n}(n-h(i))$$

of Y_h coincides with the actual dimension (3.3). In particular, Y_h is the transversal vanishing locus of

$$f_{ij} : Y \longrightarrow \mathbb{C}, \quad y \mapsto y_{ij} \tag{3.6}$$

where (i,j) runs over the pairs with $h(i) < j$, or equivalently $h(j) < i$.

By (3.1), we have a right action of T on Y by

$$Y \times T \longrightarrow Y, \quad y \cdot t = t^{-1} y t. \tag{3.7}$$

If we let $t = \mathrm{diag}(t_1, \cdots, t_n) \in T$, then

$$f_{ij}(y \cdot t) = t_i^{-1} t_j f_{ij}(y). \tag{3.8}$$

In particular, we have an induced T-action on Y_h for every Hessenberg function h, whose fixed point set is exactly

$$Y_h^T = \{\mathrm{diag}(\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(n)}) \,|\, \sigma \in S_n\} \cong S_n. \tag{3.9}$$

Note that our notation is different from that in [6], where Hessenberg varieties and their twin manifolds are denoted by Y_h and X_h respectively.

3.3. Goresky-Kottwitz-MacPherson theory. When a manifold admits a nice torus action, we can compute its equivariant cohomology from the data of 0 and 1-dimensional orbits.

Definition 3.1. *(See* [6, Definition 5.1]*.) A compact orientable manifold M with a smooth action of a compact torus $T = U(1)^n$ is called a* GKM manifold *if it satisfies the following conditions.*

(1) *M is equivariantly formal.*
(2) *The set M^T of T-fixed points is finite.*
(3) *The weights of the induced representation of T on the tangent space of M at each $y \in M^T$ are pairwise non-collinear: if*

$$\mathbb{T}_M|_y \cong \bigoplus_{i=1}^m \mathbb{C}\alpha_i, \quad \alpha_i \in \mathrm{Hom}(T, U(1)) \cong \mathbb{Z}^n,$$

then α_i and α_j are non-collinear as vectors whenever $i \neq j$.
(4) *Every 2-dimensional T-invariant closed submanifold which is the union of T-orbits of dimension at most one, has a T-fixed point.*

By (2)–(4) in Definition 3.1, the *1-skeleton* of M, which is by definition the union of 0 or 1-dimensional T-orbits, is the union of M^T and T-invariant 2-spheres. The induced T-action on each T-invariant 2-sphere $S^2 \cong \mathbb{P}^1$ is of the form

$$T \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad (t, [z_0 : z_1]) \mapsto [z_0 : \alpha(t) z_1]$$

for some $\alpha \in \mathrm{Hom}(T, U(1)) \cong \mathbb{Z}^n$. In particular, each T-invariant 2-sphere connects precisely two T-fixed points, with the associated weight α determined uniquely up to sign.

The GKM theory [12, 17] tells us that the T-equivariant cohomology $H_T^*(M)$ of a GKM manifold M is determined by the combinatorial data of its 1-skeleton as a subring of the T-equivariant cohomology $H_T^*(M^T)$ of its T-fixed point set. Indeed, by torus localization, $H_T^*(M)$ is embedded into

$$H_T^*(M^T) \cong \bigoplus_{y \in M^T} \mathbb{Q}[t_1, \cdots, t_n]$$

by the pullback homomorphism induced by the inclusion $M^T \subset M$.

Theorem 3.2. [6, Theorem 5.2] *The image of* $H_T^*(M)$ *in* $H_T^*(M^T)$ *is*

$$H_T^*(M) \cong \{(f_y)_{y\in M^T} : f_{y_e} \equiv f_{y'_e} \text{ modulo } \alpha_e\}$$

as an algebra over $H_T^* := H_T^*(\mathrm{pt}) \cong \mathbb{Q}[t_1, \cdots, t_n]$*, where* $f_y \in H_T^*(\{y\}) \cong \mathbb{Q}[t_1, \cdots, t_n]$*, e runs over the set of T-invariant 2-spheres in M,* $\{y_e, y'_e\}$ *is the set of T-fixed points in e and* α_e *is the T-weight associated to e.*

3.4. Equivariant cohomology of Y_h. For the T-action on Y_h by (3.7), we may use the GKM theory to investigate the cohomology of Y_h.

Theorem 3.3. Y_h *is a GKM manifold by the following.*

(1) *The cohomology groups of* Y_h *vanish in odd degrees. In particular,* Y_h *is equivariantly formal.*
(2) *The set of T-fixed points is* $Y_h^T \cong S_n$.
(3) *Two T-fixed points* $\sigma, \tau \in S_n$ *are connected by a T-invariant 2-sphere in* Y_h *if and only if* $\tau = \sigma \cdot (i,j)$ *for some* $i < j \leqslant h(i)$*, where* (i,j) *denotes the transposition interchanging i and j.*
(4) *The tangent space* $\mathbb{T}_{Y_h}|_y$ *of* Y_h *at* $y \in Y_h^T$ *is isomorphic to*

$$\mathbb{T}_{Y_h}|_y \cong \bigoplus_{i<j\leqslant h(i)} \mathbb{C}\epsilon_{ij}, \quad \epsilon_{ij} = \epsilon_i - \epsilon_j$$

as a T-representation, where $\epsilon_i \in \mathrm{Hom}(T, U(1))$ *denotes the character of T sending* $(t_1, \cdots, t_n) \in T$ *to* t_i.

Proof. For (1), see [6, Theorem 3.5 and Remark 3.7]. (3.9) gives (2). For (3), see below the proof of [6, Proposition 3.9]. For (4), see [6, Proposition 3.9] and its proof in page 16687. □

By Theorem 3.2, we thus have the following description of $H_T^*(Y_h)$, via the restriction map

$$\mathsf{res} : H_T^*(Y_h) \hookrightarrow H_T^*(Y_h^T) \cong \bigoplus_{v\in S_n} \mathbb{Q}[t_1, \cdots, t_n] \tag{3.10}$$

induced by the inclusion $Y_h^T \subset Y_h$.

Proposition 3.4. [6, Proposition 5.3]

$$H_T^*(Y_h) \cong \left\{(p_v)_{v\in S_n} : p_v \equiv p_{v\cdot(i,j)} \text{ modulo } t_i - t_j \quad \forall i < j \leqslant h(i)\right\} \tag{3.11}$$

as algebras over $H_T^* = \mathbb{Q}[t_1, \cdots, t_n]$*, where* $p_v \in \mathbb{Q}[t_1, \cdots, t_n]$.

Recall that the T-equivariant cohomology of the Hessenberg variety X_h admits a similar description in [27] (see also [1, §8])

$$H_T^*(X_h) \cong \left\{(p_v)_{v\in S_n} : p_v \equiv p_{v\cdot(i,j)} \text{ modulo } t_{v(i)} - t_{v(j)} \quad \forall i < j \leqslant h(i)\right\}. \tag{3.12}$$

Comparing (3.11) with (3.12), we find that the ring isomorphism

$$\xi : \prod_{v\in S_n} \mathbb{Q}[t_1, \cdots, t_n] \longrightarrow \prod_{v\in S_n} \mathbb{Q}[t_1, \cdots, t_n], \quad (p_v)_{v\in S_n} \mapsto (vp_v)_{v\in S_n} \tag{3.13}$$

restricts to an isomorphism

$$\xi : H_T^*(Y_h) \longrightarrow H_T^*(X_h) \tag{3.14}$$

of subrings [6, p.16689] where vp_v denotes the action of v on p_v by (3.16). Indeed, if $p_v \equiv p_{v\cdot(i,j)}$ modulo $t_i - t_j$, then

$$vp_v - v\cdot(i,j)p_{v\cdot(i,j)} = v(p_v - p_{v\cdot(i,j)}) + v(p_{v\cdot(i,j)} - (i,j)p_{v\cdot(i,j)}) \equiv 0$$

modulo $v(t_i - t_j) = t_{v(i)} - t_{v(j)}$. Note that for monomials $f = t_1^{a_1} \cdots t_n^{a_n}$,

$$f - (i,j)f = (t_i^{a_i} t_j^{a_j} - t_i^{a_j} t_j^{a_i}) \prod_{k \neq i,j} t_k^{a_k} \equiv 0 \ \text{ modulo } \ t_i - t_j.$$

Remark 3.5. The T-weights on the tangent spaces of the fixed points are well defined only up to sign. We use the sign choices in Theorem 3.3 (4). (See [6, (10) and p.16687].)

Remark 3.6. One can check that the isomorphism ξ is in fact the natural one given by

$$H_T^*(Y_h) \cong H_{T\times T}^*(Z_h) \cong H_T^*(X_h)$$

where $Z_h = p_2^{-1}(X_h) \subset U(n)$ in (3.4) is acted on by $T \times T$ by left and right multiplications so that $Y_h = T \backslash Z_h$ and $X_h = Z_h/T$. If we identify S_n with the group of permutation matrices in $U(n)$ so that $TS_n = S_nT$, one can easily check that (3.13) is given by the natural isomorphism

$$H_T^*(Y_h^T) \cong H_{T\times T}^*(TS_n) = H_{T\times T}^*(S_nT) \cong H_T^*(X_h^T),$$

where $Y_h^T = T\backslash TS_n$ and $X_h^T = S_nT/T$.

3.5. An S_n-action on $H^*(Y_h)$. In [27], Tymoczko defined an action of S_n on the cohomology of the Hessenberg variety X_h whose Frobenius characteristic turns out to coincide with the chromatic quasisymmetric function (2.2) by [8, 14]. Similarly, there is a natural action of S_n on the cohomology of Y_h.

Let us first recall the dot action on $H^*(X_h)$. Consider the S_n-action on $H_T^*(X_h^T) \cong \bigoplus_{v\in S_n} \mathbb{Q}[t_1, \cdots, t_n]$ defined by

$$(\mu, (p_v)_{v\in S_n}) \mapsto (\mu p_{\mu^{-1}v})_{v\in S_n}, \quad \text{for } \ \mu \in S_n \tag{3.15}$$

where

$$(\mu p)(t_1, \cdots, t_n) := p(t_{\mu(1)}, \cdots t_{\mu(n)}) \tag{3.16}$$

for $p \in \mathbb{Q}[t_1, \cdots, t_n]$. From (3.12), it is straightforward to check that this action preserves the subring $H_T^*(X_h) \subset H_T^*(X_h^T)$. Hence we have an induced action of S_n on $H_T^*(X_h)$ which in turn defines an S_n-action on

$$H^*(X_h) \cong H_T^*(X_h)/\mathfrak{m}H_T^*(X_h),$$

as it preserves the submodule generated by $\mathfrak{m} := (t_1, \cdots, t_n)$. This is called *Tymoczko's dot action.*

Definition 3.7. [6] *By the isomorphism ξ in* (3.14)*, the dot action on $H_T^*(X_h)$ pulls back to an action of S_n on $H_T^*(Y_h)$ defined by*

$$(\mu, (p_v)_{v\in S_n}) \mapsto (p_{\mu^{-1}v})_{v\in S_n} \quad \textit{for } \ \mu \in S_n. \tag{3.17}$$

As this preserves $\mathfrak{m}H_T^*(Y_h)$, (3.17) *defines an action of S_n on*

$$H^*(Y_h) \cong H_T^*(Y_h)/\mathfrak{m}H_T^*(Y_h). \tag{3.18}$$

This is called the dagger action *in* [21].

Using this dagger action, we have the following.

Definition 3.8. *For a graded S_n-module $V = \bigoplus_{k\geqslant 0} V_k$, we define*

$$\mathrm{ch}_q(V) = \sum_{k\geqslant 0} \mathrm{ch}(V_k) q^k \in \Lambda[q]$$

where ch *is the Frobenius characteristic from the ring of representations of symmetric groups S_n onto the ring Λ of symmetric functions* [20, §I.7]. *For a Hessenberg function $h : [n] \to [n]$, we define*

$$\mathcal{P}(h) := \sum_{k\geqslant 0} \mathrm{ch}(H^{2k}(Y_h))q^k = \mathrm{ch}_q(H^{2*}(Y_h)) \in \Lambda[q].$$

Example 3.9. Suppose $h(i) = n$ for all $1 \leqslant i \leqslant n$ so that Y_h is the isospectral manifold $Y \cong T\backslash U(n)$ in (3.1) and X_h is the flag variety

$$X = \mathrm{Fl}(n) \cong U(n)/T.$$

We identify S_n with the group of permutation matrices in $U(n)$. In this case, we have a left (resp. right) action of S_n on X (resp. Y) by left (resp. right) multiplication of permutation matrices.

Let $\mathcal{V}_i$ denote the rank i tautological vector bundle on the flag variety X. As the cohomology ring of X is generated by the line bundles $\mathcal{V}_i/\mathcal{V}_{i-1}$ and the S_n action preserves the line bundles, we find that the action of S_n on $H^*(X)$ is trivial (cf. [27, Proposition 4.2], [16, Example 2.12]). As the action of S_n on the equivariant line bundles $\mathcal{V}_i/\mathcal{V}_{i-1}$ permutes the equivariant weights, we find that the T-equivariant cohomology of X is

$$H^*_T(X) \cong H^*(X) \otimes \mathbb{Q}[t_1, \cdots, t_n]$$

where S_n acts trivially on $H^*(X)$ and by (3.16) on $H^*_T = \mathbb{Q}[t_1, \cdots, t_n]$.

By Remark 3.6 and (3.13), we find that

$$H^*(Y) \otimes \mathbb{Q}[t_1, \cdots, t_n] \cong H^*_T(Y) \cong H^*_T(X) \cong H^*(X) \otimes \mathbb{Q}[t_1, \cdots, t_n] \tag{3.19}$$

is S_n-equivariant where S_n acts trivially on the left $\mathbb{Q}[t_1, \cdots, t_n]$ and by (3.16) on the right $\mathbb{Q}[t_1, \cdots, t_n]$.

Applying the Frobenius characteristic to (3.19), we obtain

$$\mathcal{K}_n := \mathrm{ch}_q(H^{2*}(Y)) = (1-q)^n [n]_q! \mathrm{ch}(\mathbb{Q}[t_1, \cdots, t_n]) \ \in \Lambda[q] \tag{3.20}$$

because $\mathrm{ch}_q(H^*(X)) = [n]_q!$. From this, it follows that $\mathcal{K}_n$ is uniquely determined by the inductive formula (2.4). Indeed, $f_n := \mathrm{ch}_q(\mathbb{Q}[t_1, \cdots, t_n])$ in (3.20) satisfies

$$f_n = \frac{1}{1-q^n}\left(e_1 f_{n-1} - e_2 f_{n-2} + \cdots + (-1)^{n-1} e_n f_0\right), \qquad f_0 := 1$$

by Lemma 3.11 (2) below.

Example 3.9 immediately implies the following.

Proposition 3.10. *The function $h \mapsto \mathcal{P}(h)$ satisfies Theorem* 2.6 *(1).*

The remaining two conditions in Theorem 2.6 will be proved in §3.6 and in §5.

Lemma 3.11. *Let $f_n := \mathrm{ch}(\mathbb{Q}[t_1, \cdots, t_n])$ in* (3.20) *with $f_0 = 1$. Then $\{f_n\}_{n\geqslant 0}$ satisfies the following:*

(1) $f_n = \sum_{i=0}^{n} q^{n-i} h_i f_{n-i}$ *and*
(2) $q^n f_n = \sum_{i=0}^{n} (-1)^i e_i f_{n-i}$

where $h_m = \sum_{i_1 \leqslant \cdots \leqslant i_m} x_{i_1} \cdots x_{i_m} \in \Lambda$ is the m-th complete homogeneous symmetric function with $h_0 = 1$.

Proof. (1) Let W_n (resp. W'_n) be the submodule in $\mathbb{Q}[t_1, \cdots, t_n]$ in (3.20) spanned by monomials generated by less than (resp. precisely) n variables. Then, we have

$$\mathbb{Q}[t_1, \cdots, t_n] = \bigoplus_{k\geqslant 0} (t_1 \cdots t_n)^k W_n = W_n \oplus W'_n$$

Hence $f_n = \frac{1}{1-q^n}\mathrm{ch}_q(W_n) = \mathrm{ch}_q(W_n) + \mathrm{ch}_q(W'_n)$. In particular,

$$\mathrm{ch}_q(W_n) = (1-q^n)f_n \quad \text{and} \quad \mathrm{ch}_q(W'_n) = q^n f_n. \tag{3.21}$$

Moreover, W_n admits a decomposition by the number of generating variables

$$W_n = \bigoplus_{i=1}^{n} \mathrm{Ind}_{S_i\times S_{n-i}}^{S_n} W'_{n-i}$$

where $\mathrm{Ind}_{S_i\times S_{n-i}}^{S_n} W'_{n-i}$ is the induced representation of W'_{n-i} as an $S_i \times S_{n-i}$ representation with a trivial S_i-action. Since ch is multiplicative with respect to the multiplication on representations of symmetric groups given by taking the induced representation of tensor products ([20, I, (7.3)]), we have

$$(1-q^n)f_n = \mathrm{ch}_q(W_n) = \sum_{i=1}^{n} h_i \mathrm{ch}_q(W'_{n-i}) = \sum_{i=1}^{n} q^{n-i} h_i f_{n-i}$$

by (3.21). This proves (1).

(2) We use (1) and a well-known identity ([20, I, (2.6)])

$$h_n = (-1)^{n-1}e_n + \sum_{j=1}^{n-1}(-1)^j h_{n-j}e_j \quad \text{for } n \geqslant 2 \quad \text{and} \quad h_1 = e_1, \tag{3.22}$$

together with induction on n.

When $n=0$, (2) trivially holds since $q^0 f_0 = 1 = e_0 f_0$. Assume that $n \geqslant 1$ and (2) holds for every $m < n$, so that

$$q^m f_m = f_m + \sum_{j=1}^{m}(-1)^j e_j f_{m-j}. \tag{3.23}$$

Then, by the assertion (1), (3.23) and (3.22), we have

$$\begin{aligned}
(1-q^n)f_n &= \sum_{i=1}^{n}(q^{n-i}f_{n-i})h_i = \sum_{i=1}^{n} h_i f_{n-i} + \sum_{i=1}^{n}\sum_{j=1}^{n-i}(-1)^j h_i e_j f_{n-i-j} \\
&= \sum_{i=1}^{n}(-1)^{i-1}e_i f_{n-i} + \sum_{i=2}^{n}\sum_{j=1}^{i-1}(-1)^{j-1}h_{i-j}e_j f_{n-i} + \sum_{i=1}^{n}\sum_{j=1}^{n-i}(-1)^j h_i e_j f_{n-i-j} \\
&= \sum_{i=1}^{n}(-1)^{i-1}e_i f_{n-i}
\end{aligned}$$

where the last equality holds by

$$\sum_{i=2}^{n}\sum_{j=1}^{i-1}(-) = \sum_{k,j\geqslant 1, k+j\leqslant n}(-) = \sum_{k=1}^{n}\sum_{j=1}^{n-k}(-)$$

with $i = j+k$. This proves the assertion (2). □

A by-product of Lemma 3.11 is an elementary proof of the following.

Corollary 3.12. [21, Lemma 5.0.1 (2)] *Let $\omega : \Lambda \cong \Lambda$ be the involution of the ring Λ which interchanges e_n and h_n. The we have*

$$f_n(q^{-1}) = (-q)^n(\omega f_n)(q).$$

Proof. Let $f'_n := \omega f_n$ for $n \geqslant 0$. We use induction on n. The assertion trivially holds for $n = 0$. Assume that $n \geqslant 1$ and $f_m(q^{-1}) = (-q)^m f'_m(q)$ for $m < n$. By applying ω to Lemma 3.11 (2), we have

$$\begin{aligned}(1-q^n)\omega f_n(q) = (1-q^n)f'_n(q) &= \sum_{i=1}^{n}(-1)^{i-1}h_i f'_{n-i}(q)\\ &= (-1)^{n-1}\sum_{i=1}^{n} q^{-(n-i)}h_i f_{n-i}(q^{-1})\\ &= (-1)^{n-1}(1-q^{-n})f_n(q^{-1})\\ &= (-q)^{-n}(1-q^n)f_n(q^{-1})\end{aligned}$$

where the third and the fourth equalities hold by the induction hypothesis and Lemma 3.11 (1) respectively. Therefore the corollary holds for all n. □

Corollary 3.12 is a key ingredient in the proof of the palindromicity of $\mathcal{P}(h)$ up to the involution ω by Masuda and Sato in [21] which says that

$$(3.24)\qquad \begin{aligned}&H^*(Y_h) \cong H^{\dim_{\mathbb{R}} Y_h - 2*}(Y_h)\otimes \mathrm{sgn} \quad \text{or equivalently,}\\ &\mathrm{ch}_q(H^{2*}(Y_h)) = \omega\,\mathrm{ch}_q(H^{\dim_{\mathbb{R}} Y_h - 2*}(Y_h))\end{aligned}$$

where sgn denotes the sign representation. In fact, this palindromicity (3.24) follows from Corollary 3.12 and the modular law (Theorem 5.3) as follows: First one can immediately check (3.24) for the isospectral manifold Y in (3.1) using Corollary 3.12 and (3.20). Next the modular law enables us to deduce (3.24) for Y_h from the palindromicity for Y.

Remark 3.13. Complete homogeneous symmetric functions h_n are used only in Lemma 3.11 and the proof of Corollary 3.12. We hope these not to be confused with Hessenberg functions.

3.6. Connectedness and multiplicativity. In this subsection, we give a criterion for connectedness of twin manifolds, and check the multiplicative property of the function $\mathcal{P}$ in Definition 3.8.

Lemma 3.14. *For a Hessenberg function $h : [n] \to [n]$ with $n \geqslant 2$, the twin manifold Y_h is connected if and only if $h(i) > i$ for all $1 \leqslant i < n$. If $h(j) = j$ for some $j < n$, then Y_h is the disjoint union of $\binom{n}{j}$ copies of $Y_{h'} \times Y_{h''}$ for h' and h'' defined as in Theorem* 2.6 *(2).*

Proof. This is given by the explicit diffeomorphism

$$(3.25)\qquad Y_h \cong \bigsqcup_{I\subset[n],\ |I|=j} Y_{h',I}\times Y_{h'',I^c},\qquad \begin{pmatrix}A' & O\\ O & A''\end{pmatrix} \leftrightarrow (A', A'')$$

where $Y_{h',I} \cong Y_{h'}$ denotes the twin manifold with spectrum $\{\lambda_i \,|\, i \in I\}$ and $Y_{h'',I^c} \cong Y_{h''}$ is defined similarly. □

Proposition 3.15. *The function $h \mapsto \mathcal{P}(h)$ satisfies Theorem* 2.6 *(2).*

Proof. Under the identifications (3.5) and (3.9), the isomorphism (3.25) restricts to an isomorphism on T-fixed point sets

$$Y_h^T \cong S_n = \bigsqcup_{I\subset[n],\ |I|=j} (S_I\times S_{I^c})w_I \cong \bigsqcup_{I\subset[n],\ |I|=j} Y_{h',I}^T\times Y_{h'',I^c}^T$$

where w_I is an element of S_n sending $[j]$ and $[j]^c$ to I and I^c respectively such that $w_I|_{[j]}$ and $w_I|_{[j]^c}$ are increasing. This induces an isomorphism

$$\bigoplus_{I\subset[n],\ |I|=j} H^*_T(Y^T_{h',I}) \otimes_{H^*_T} H^*_T(Y^T_{h'',I^c}) \xrightarrow{\cong} H^*_T(Y^T_h)$$

sending $(p_{v_I})_{v_I\in S_I} \otimes (p_{v_{I^c}})_{v_{I^c}\in S_{I^c}}$ to $(p_v)_{v\in S_n}$ with $p_v = p_{v_I}p_{v_{I^c}}$ if $v = v_I v_{I^c} w_I$ and $p_v = 0$ otherwise for each I. This is equivariant under the induced actions of $S_I \times S_{I^c}$ and S_n from (3.17) for each component. Furthermore, one can easily see that the above isomorphism restricts to an isomorphism

$$\bigoplus_{I\subset[n],\ |I|=j} H^*_T(Y_{h',I}) \otimes_{H^*_T} H^*_T(Y_{h'',I^c}) \xrightarrow{\cong} H^*_T(Y_h)$$

which exhibits $H^*_T(Y_h)$ as the induced representation of the $S_j \times S_{n-j}$-representation $H^*_T(Y_{h'}) \otimes_{H^*_T} H^*_T(Y_{h''})$. Taking quotients by the submodules generated by $\mathfrak{m}$, $H^*(Y_h)$ is the induced representation of the $S_j \times S_{n-j}$-representation $H^*(Y_{h'}) \otimes H^*(Y_{h''})$. Since the Frobenius characteristic ch is multiplicative with respect to tensor products (cf. [20, I, (7.3)]), we have

$$\mathcal{P}(h) = \mathcal{P}(h')\mathcal{P}(h'')$$

as desired. □

By comparing the twin manifolds geometrically, we will prove below that the last condition (3) in Theorem 2.6 is also satisfied for $\mathcal{P}(h)$ (cf. Theorem 5.3 below) and hence give a direct proof of the LLT-SW conjecture (cf. Theorem 5.4).

4. Geometry of twin manifolds

In this section, we compare the twin manifolds associated to a modular triple $\mathbf{h} = (h_-, h, h_+)$ of Hessenberg functions in Definition 2.3. More precisely, we construct a manifold $\widetilde{Y}_{\mathbf{h}}$, called the *roof manifold* of $\mathbf{h}$, together with maps

$$\begin{array}{ccc} & \widetilde{Y}_{\mathbf{h}} & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_2} \\ Y_{h_+} & & \mathbb{P}^1 \end{array}$$

where pr_2 is a smooth fibration with fiber Y_h and π is the *blowup* along the submanifold Y_{h_-} of complex codimension 2. The modular law

$$H^{2k}(Y_h) \oplus H^{2k-2}(Y_h) \cong H^{2k}(Y_{h_+}) \oplus H^{2k-2}(Y_{h_-}). \tag{4.1}$$

for $\mathcal{P}(h)$ then follows immediately from the *blowup formula* for π and the spectral sequence for pr_2.

4.1. Roof of a triple.

In this subsection, we define the roof $\widetilde{Y}_{\mathbf{h}}$ of a triple $\mathbf{h} = (h_-, h, h_+)$.

For $I = [a, b] = \{a, a+1, \cdots, b\} \subset [n]$, let

$$\iota_I : U(b-a+1) \longrightarrow U(n)$$

be the embedding

$$A \ \mapsto \begin{pmatrix} I_{a-1} & & \\ & A & \\ & & I_{n-b} \end{pmatrix}$$

where I_k denotes the $k \times k$ identity matrix for each k. When $a = b$, let $\iota_a := \iota_I$. For disjoint $I = [a, b]$ and $J = [c, d] \subset [n]$, we denote by

$$\iota_I \times \iota_J : U(b-a+1) \times U(d-c+1) \longrightarrow U(n)$$

the map $(A,B)\mapsto \iota_I(A)\iota_J(B)$ given by the matrix multiplication.

Definition 4.1. *Given a Hessenberg function $h:[n]\to[n]$, consider a triple $\mathbf{h}=(h_-,h,h_+)$ of Hessenberg functions defined by either of the following.*

(1) *When $h(j)=h(j+1)$ and $h^{-1}(j)=\{j_0\}$ for some $1\leqslant j_0<j<n$, let $1\leqslant r\leqslant n-j$ be any integer such that*

$$h(j)=\cdots=h(j+r)\quad\text{and}\quad h^{-1}(\{j+1,\cdots,j+r-1\})=\varnothing.$$

Let $h_-,h_+:[n]\to[n]$ be defined by

$$h_-(i)=\begin{cases} j-1 & \text{for } i=j_0\\ h(i) & \text{otherwise}\end{cases}\quad\text{and}\quad h_+(i)=\begin{cases} j+r & \text{for } i=j_0\\ h(i) & \text{otherwise.}\end{cases}$$

In this case, we define

$$\widetilde{Y}_{\mathbf{h}}:=Y_h\times_{U(1)\times U(r)}U(r+1)\quad\text{and}$$
$$E_{\mathbf{h}}:=Y_{h_-}\times_{U(1)\times U(r)}U(r+1)$$

to be the quotients of $Y_h\times U(r+1)$ and $Y_{h_-}\times U(r+1)$ by the free actions of $U(1)\times U(r)$ given by

$$(Tg,A).B=(Tg\iota(B),\iota'(B)^{-1}A)\tag{4.2}$$

for $A\in U(r+1)$ and $B\in U(1)\times U(r)$, via

$$\iota=\iota_j\times\iota_{[j+1,j+r]}:U(1)\times U(r)\hookrightarrow U(n)\quad\text{and}$$
$$\iota'=\iota_1\times\iota_{[2,r+1]}:U(1)\times U(r)\hookrightarrow U(r+1).$$

(2) *When $h(j)+1=h(j+1)\neq j+1$ and $h^{-1}(j)=\varnothing$ for some $j<n$, let $1\leqslant r\leqslant j$ be any integer such that*

$$h(j-r+1)=\cdots=h(j)\quad\text{and}\quad h^{-1}(\{j-r+1,\cdots,j\})=\varnothing.$$

Let $h_-,h_+:[n]\to[n]$ be defined by

$$h_-(i)=\begin{cases} h(j) & \text{for } i=j+1\\ h(i) & \text{otherwise}\end{cases}\quad\text{and}\quad h_+(i)=\begin{cases} h(j)+1 & \text{for } j-r<i\leqslant j\\ h(i) & \text{otherwise.}\end{cases}$$

In this case, similarly we define

$$\widetilde{Y}_{\mathbf{h}}:=Y_h\times_{U(r)\times U(1)}U(r+1)\quad\text{and}$$
$$E_{\mathbf{h}}:=Y_{h_-}\times_{U(r)\times U(1)}U(r+1)$$

given by the actions (4.2) *for $B\in U(r)\times U(1)$, via*

$$\iota=\iota_{[j-r+1,j]}\times\iota_{j+1}:U(r)\times U(1)\hookrightarrow U(n)\quad\text{and}$$
$$\iota'=\iota_{[1,r]}\times\iota_{\{r+1\}}:U(r)\times U(1)\hookrightarrow U(r+1).$$

We call a triple $\mathbf{h}=(h_-,h,h_+)$ in (1) and (2) a triple of type (1) *and* (2) *respectively.*

Remark 4.2. Note that when $r=1$, $\mathbf{h}$ is a modular triple in Definition 2.3. Also note that triples of type (1) and (2) with the same r are dual to each other via the involution map $h\mapsto h^t$ on the set of Hessenberg functions, where for a Hessenberg function $h:[n]\to[n]$, its transpose $h^t:[n]\to[n]$ is defined by

$$h^t(i)=n-i_{\max},\quad i_{\max}=\max\{j\in[n]:h(j)<n+1-i\}.$$

One can easily see that h is a triple of type (1) (resp. (2)) if and only if h^t is a triple of type (2) (resp. (1)).

Example 4.3. Let $h = (2,5,5,5,6,6)$ where we write $h = (h(1), \cdots, h(n))$. Then $\mathbf{h} = (h_-, h, h_+)$ with

$$h_- = (1,5,5,5,6,6) \quad \text{and} \quad h_+ = (4,5,5,5,6,6)$$

is a triple of type (1) with $(j_0, j, r) = (1,2,2)$. Similarly, $\mathbf{h} = (h_-, h, h_+)$ with

$$h_- = (2,5,5,5,5,6) \quad \text{and} \quad h_+ = (2,5,6,6,6,6)$$

is a triple of type (2) with $(j, r) = (4,2)$.

4.2. Maps from the roof. In this subsection, we show that for a triple $\mathbf{h} = (h_-, h, h_+)$ in Definition 4.1,

(*) $\widetilde{Y}_{\mathbf{h}}$ is the blowup of Y_{h_+} along Y_{h_-} with exceptional divisor $E_{\mathbf{h}}$ and
(**) $\widetilde{Y}_{\mathbf{h}}$ is a fiber bundle over $\mathbb{P}^r$ with fiber Y_h.

These are our *geometric relations* among the twin manifolds that will give us the modular law (4.1).

To prove (*) and (**), we have to construct maps among the manifolds $\widetilde{Y}_{\mathbf{h}}$, $E_{\mathbf{h}}$, Y_{h_+}, Y_{h_-} and $\mathbb{P}^r$. As

$$(U(1) \times U(r)) \backslash U(r+1) \cong \mathbb{P}^r \cong (U(r) \times U(1)) \backslash U(r+1), \tag{4.3}$$

the second projection

$$Y_h \times U(r+1) \longrightarrow U(r+1), \quad (\text{resp. } Y_{h_-} \times U(r+1) \longrightarrow U(r+1))$$

induces the fiber bundle

$$\mathrm{pr}_2 : \widetilde{Y}_{\mathbf{h}} \longrightarrow \mathbb{P}^r, \quad (\text{resp. } E_{\mathbf{h}} \longrightarrow \mathbb{P}^r) \tag{4.4}$$

whose fibers are Y_h (resp. Y_{h_-}). The obvious inclusion $Y_{h_-} \subset Y_h$ induces the inclusion

$$\jmath : E_{\mathbf{h}} \hookrightarrow \widetilde{Y}_{\mathbf{h}} \tag{4.5}$$

of fiber bundles over $\mathbb{P}^r$. Moreover, we have the map

$$\pi : \widetilde{Y}_{\mathbf{h}} \longrightarrow Y_{h_+}, \quad [Tg, A] \mapsto Tg\iota(A) \tag{4.6}$$

for $\iota = \iota_{[j,j+r]}$ (resp. $\iota = \iota_{[j-r+1,j+1]}$) if $\mathbf{h}$ is of type (1) (resp. type (2)). Similarly, we have the map

$$\pi_- : E_{\mathbf{h}} \longrightarrow Y_{h_-}, \quad [Tg, A] \mapsto Tg\iota(A). \tag{4.7}$$

These are well defined, since Y_{h_+} and Y_{h_-} are invariant under the right multiplication of $\iota(U(r+1))$ by (3.5).

Using the notation of (3.6), let $f_{\mathbf{h}} : Y_{h_+} \to \mathbb{C}^{r+1}$ be a map defined by

$$f_{\mathbf{h}}(y) = \begin{cases} (f_{j_0,j}(y), \ \cdots, \ f_{j_0,j+r}(y)) & \text{for type (1)} \\ (f_{h(j+1),j-r+1}(y), \ \cdots, \ f_{h(j+1),j+1}(y)) & \text{for type (2)} \end{cases}$$

so that

$$Y_{h_-} = \{y \in Y_{h_+} : f_{\mathbf{h}}(y) = 0\} \tag{4.8}$$

is the transversal vanishing locus of $f_{\mathbf{h}}$ by (3.5) and (3.6). This induces a map

$$Y_{h_+} - Y_{h_-} \longrightarrow \mathbb{P}^r, \qquad y \mapsto [f_{\mathbf{h}}(y)]. \tag{4.9}$$

Example 4.4. Let $\mathbf{h}$ be as in Example 4.3 (1). Then, we have

$$Y_h = \{y \in Y_{h_+} : f_{13}(y) = f_{14}(y) = 0\} \quad \text{and}$$
$$Y_{h_-} = \{y \in Y_{h_+} : f_{12}(y) = f_{13}(y) = f_{14}(y) = 0\}.$$

The map (4.9) is given by

$$Y_{h_+} - Y_{h_-} \longrightarrow \mathbb{P}^2, \quad y \mapsto [f_{12}(y) : f_{13}(y) : f_{14}(y)].$$

The following propositions illustrate the geometry of Y_h for a triple $\mathbf{h}$ which is very similar to that of a triple of Hessenberg varieties X_h studied in [16, §3.3], via blowups and projective bundles.

Proposition 4.5. *Let* $\mathbf{h} = (h_-, h, h_+)$ *be a triple in Definition* 4.1*. Then,* (4.4) *and* (4.7) *give us a diffeomorphism*

$$(\pi_-, \mathrm{pr}_2) : E_{\mathbf{h}} \xrightarrow{\cong} Y_{h_-} \times \mathbb{P}^r. \tag{4.10}$$

Proof. Note that Y_{h_-} is invariant under the action of $\iota(U(r+1))$. Hence the map $(Tg, [A]) \mapsto [Tg\iota(A)^{-1}, A]$ is the well defined inverse. □

Proposition 4.6. *The map*

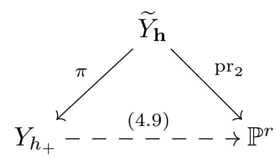
$$(\pi, \mathrm{pr}_2) : \widetilde{Y}_{\mathbf{h}} \longrightarrow Y_{h_+} \times \mathbb{P}^r \tag{4.11}$$

induced by (4.4) *and* (4.6) *is an embedding of* $\widetilde{Y}_{\mathbf{h}}$ *onto the submanifold defined by*

$$\widetilde{Y}_{\mathbf{h}} \cong \{(y, [v]) \in Y_{h_+} \times \mathbb{P}^r \ : \ f_{\mathbf{h}}(y) \in \mathbb{C}v\}. \tag{4.12}$$

In particular, (4.9) *fits into the diagram*

(4.13)



where pr_2 *is a smooth fibration with fiber* Y_h*.* (4.9)

Proof. Since Y_{h_+} is invariant under the action of $\iota(U(r+1))$ for ι defined in Definition 4.1, the same argument in the proof in Proposition 4.5 proves that $Y_{h_+} \times \mathbb{P}^r$ is isomorphic to the quotients $Y_{h_+} \times_{U(1)\times U(r)} U(r+1)$ or $Y_{h_+} \times_{U(r)\times U(1)} U(r+1)$ defined in the same manner as in Definition 4.1 (1) and (2) respectively. Under this isomorphism, (4.11) is induced from the canonical inclusion $Y_h \subset Y_{h_+}$, in particular it is an embedding.

To see (4.12), note that a point $(y, a) = (Tg, [A]) \in Y_{h_+} \times \mathbb{P}^r$ with $A \in U(r+1)$ lies in the image of (4.11) if and only if $Tg\iota(A)^{-1} \in Y_h$, or equivalently, $\iota(A)(g^{-1}xg)\iota(A)^{-1}$ is contained in (3.5). The latter is equivalent to that the last (resp. first) r coordinates of the j_0-th (resp. $h(j+1)$-th) column vector

$$\begin{array}{rl} A(f_{j,j_0}(y),\ \cdots,\ f_{j+r,j_0}(y))^t & = A\overline{f_{\mathbf{h}}(y)}^t \\ (\text{resp. } \ A(f_{j-r+1,h(j+1)}(y),\ \cdots,\ f_{j+1,h(j+1)})^t) & \end{array} \tag{4.14}$$

of $\iota(A)(g^{-1}xg)\iota(A)^{-1}$ vanish if $\mathbf{h}$ is of type (1) (resp. type (2)), by (3.5) and (3.6). Here $f_{i,k}(y)$ denotes the (i,k)-th component of the matrix $g^{-1}xg$ by (3.1) and (3.6). Also note that the right multiplication of $\iota(A)^{-1}$ in $\iota(A)(g^{-1}xg)\iota(A)^{-1}$ does not change the j_0-th (resp. $h(j+1)$-th) column vector since we assume $j_0 < j$ (resp. $h(j+1) > j+1$) in Definition 4.1.

Under the isomorphisms (4.3) which send the $a = [A]$ to the class represented by the first (resp. last) row vectors of A, this is equivalent to that the vector $f_{\mathbf{h}}(y)$ is parallel to a vector v representing $a = [v]$, or equivalently, $f_{\mathbf{h}}(y) \in \mathbb{C}v$. The last assertion is immediate. □

Proposition 4.7.

(1) *The left square in the commutative diagram*

$$\begin{array}{ccccc} E_{\mathbf{h}} & \overset{\jmath}{\hookrightarrow} & \widetilde{Y}_{\mathbf{h}} & \hookrightarrow & Y_{h_+} \times \mathbb{P}^r \\ {\scriptstyle \pi_-}\downarrow & & {\scriptstyle \pi}\downarrow & \swarrow & \\ Y_{h_-} & \hookrightarrow & Y_{h_+} & & \end{array}$$

is Cartesian.

(2) π *is a diffeomorphism over* $Y_{h_+} - Y_{h_-}$ *with the inverse map*

$$Y_{h_+} - Y_{h_-} \longrightarrow \widetilde{Y}_{\mathbf{h}} \subset Y_{h_+} \times \mathbb{P}^r, \qquad y \mapsto (y, [f_{\mathbf{h}}(y)]). \tag{4.15}$$

(3) π *is the trivial* $\mathbb{P}^r$*-bundle over* Y_{h_-} *via* (4.10).

(4) *The normal bundle of* $\jmath$ *is isomorphic to the pullback of the tautological complex line bundle* $\mathcal{O}_{\mathbb{P}^r}(-1)$ *via* (4.10)*, as a real vector bundle.*

Proof. (1) and (3) follow from (4.8), which implies the vanishing of (4.14) for every $A \in U(r+1)$, so that $\pi^{-1}(Y_{h_-}) = Y_{h_-} \times \mathbb{P}^r \cong E_{\mathbf{h}}$. Furthermore, (2) follows from Proposition 4.6. (4) follows from the fact that $E_{\mathbf{h}}$ is the vanishing locus in $\widetilde{Y}_{\mathbf{h}}$ of the map

$$(f_{j_0,j}, \mathrm{id}) : \widetilde{Y}_{\mathbf{h}} = Y_h \times_{U(1)\times U(r)} U(r+1) \longrightarrow \mathbb{C} \times_{U(1)\times U(r)} U(r+1)$$

when $\mathbf{h}$ is of type (1) and

$$(f_{h(j+1),j+1}, \mathrm{id}) : \widetilde{Y}_{\mathbf{h}} = Y_h \times_{U(r)\times U(1)} U(r+1) \longrightarrow \mathbb{C} \times_{U(r)\times U(1)} U(r+1)$$

when $\mathbf{h}$ is of type (2) respectively, and the fact that $E_{\mathbf{h}} \subset \widetilde{Y}_{\mathbf{h}}$ is submanifold of real codimension two. The complex line bundles $\mathbb{C} \times_{U(1)\times U(r)} U(r+1)$ and $\mathbb{C} \times_{U(r)\times U(1)} U(r+1)$ are the tautological complex line bundle $\mathcal{O}_{\mathbb{P}^r}(-1)$ over $\mathbb{P}^r$ by (3.8), via the isomorphisms (4.3). □

4.3. Cohomology of the roof.

In this subsection, we compare the cohomology and T-equivariant cohomology of the twin manifolds Y_{h_-}, Y_h and Y_{h_+} associated to a triple $\mathbf{h} = (h_-, h, h_+)$ by Propositions 4.6 and 4.7.

First observe that we have natural T-actions on $\widetilde{Y}_{\mathbf{h}}$ and $E_{\mathbf{h}}$ as follows.

Definition 4.8. *Let* $\mathbf{h} = (h_-, h, h_+)$ *be a triple in Definition* 4.1*. Define (right)* T*-actions on* $\widetilde{Y}_{\mathbf{h}}$ *and* $E_{\mathbf{h}}$ *by*

$$[Tg, A].t = [Tgt, (t')^{-1}At'] \tag{4.16}$$

for $t = \mathrm{diag}(t_1, \cdots, t_n) \in T$ *and* $t' \in U(r+1)$ *given by*

$$t' = \begin{cases} \mathrm{diag}(t_j, \cdots, t_{j+r}) & \textit{if } \mathbf{h} \textit{ is of type (1),} \\ \mathrm{diag}(t_{j-r+1}, \cdots, t_{j+1}) & \textit{if } \mathbf{h} \textit{ is of type (2).} \end{cases}$$

This induces a natural T*-action on* $\mathbb{P}^r$ *via* (4.3)*, which coincides with the componentwise multiplication of* t'*.*

It is straightforward to see that all the morphisms in Propositions 4.5, 4.6 and 4.7 are T-equivariant.

By Proposition 4.6, pr_2 is a fiber bundle with fiber Y_h. By Proposition 3.4 and (3.18), the odd degree parts of $H^*(Y_h)$ and $H^*_T(Y_h)$ vanish. Hence the spectral sequence for pr_2 degenerates and we have isomorphisms

$$H^*(\widetilde{Y}_{\mathbf{h}}) \cong H^*(Y_h) \otimes H^*(\mathbb{P}^r) \quad \text{and} \quad H^*(E_{\mathbf{h}}) \cong H^*(Y_{h_-}) \otimes H^*(\mathbb{P}^r). \tag{4.17}$$

Letting $\gamma = c_1(\mathcal{O}_{\mathbb{P}^r}(-1))$, we have a ring isomorphism $H^*(\mathbb{P}^r) \cong \mathbb{Q}[\gamma]/(\gamma^{r+1})$ and the second isomorphism in (4.17) is in fact the inverse of

$$\sum_i \beta_i \otimes \gamma^i \mapsto \sum_i \gamma^i \cup \pi_-^* \beta_i, \quad \beta_i \in H^{*-2i}(Y_{h_-}) \tag{4.18}$$

by Proposition 4.5 above. Here we are abusing the notation by denoting the pullback of γ to $E_{\mathbf{h}}$ by γ to simplify the notation.

Similarly as in (4.17), we have

$$H_T^*(\widetilde{Y}_{\mathbf{h}}) \cong H_T^*(Y_h) \otimes H^*(\mathbb{P}^r) \quad \text{and} \quad H_T^*(E_{\mathbf{h}}) \cong H_T^*(Y_{h_-}) \otimes H^*(\mathbb{P}^r) \tag{4.19}$$

for the T-equivariant cohomology.

By Proposition 4.7, we have the following *blowup formula.*

Proposition 4.9. *Let $\mathbf{h} = (h_-, h, h_+)$ be as above. Then the map*

$$H^*(Y_{h_+}) \oplus \bigoplus_{i=1}^r H^{*-2i}(Y_{h_-}) \xrightarrow{\cong} H^*(\widetilde{Y}_{\mathbf{h}}) \tag{4.20}$$

sending $(\alpha, \beta_1, \cdots, \beta_r)$ to $\pi^*\alpha + \sum_{i=1}^r \jmath_*(e(N_\jmath)^{i-1} \cup \pi_-^*\beta_i)$ is an isomorphism, where $e(N_\jmath)$ denotes the Euler class of the normal bundle $N_\jmath$ of the canonical inclusion $\jmath : E_{\mathbf{h}} \hookrightarrow \widetilde{Y}_{\mathbf{h}}$ and $\jmath_*$ denotes the Gysin homomorphism induced by $\jmath$. Similarly, the map

$$H_T^*(Y_{h_+}) \oplus \bigoplus_{i=1}^r H_T^{*-2i}(Y_{h_-}) \xrightarrow{\cong} H_T^*(\widetilde{Y}_{\mathbf{h}}) \tag{4.21}$$

sending $(\alpha, \beta_1, \cdots, \beta_r)$ to $\pi^*\alpha + \sum_{i=1}^r \jmath_*(e^T(N_\jmath)^{i-1} \cup \pi_-^*\beta_i)$ is an H_T^*-module isomorphism, where $e^T(N_\jmath)$ denotes the T-equivariant Euler class of $N_\jmath$.

Proof. We will show that $\jmath^* : H^*(\widetilde{Y}_{\mathbf{h}}) \to H^*(E_{\mathbf{h}})$ induces an isomorphism

$$\operatorname{Coker}(\pi^*) \xrightarrow{\cong} \operatorname{Coker}\left(\pi_-^*\right) \cong \bigoplus_{i=1}^r H^{*-2i}(Y_{h_-}) \otimes H^{2i}(\mathbb{P}^r) \tag{4.22}$$

using Propositions 4.5 and 4.7. Then a splitting of a short exact sequence

$$0 \longrightarrow H^*(Y_{h_+}) \xrightarrow{\pi^*} H^*(\widetilde{Y}_{\mathbf{h}}) \longrightarrow \bigoplus_{i=1}^r H^{*-2i}(Y_{h_-}) \longrightarrow 0$$

is given by the map

$$\bigoplus_{i=1}^r H^{*-2i}(Y_{h_-}) \longrightarrow H^*(\widetilde{Y}_{\mathbf{h}}), \quad (\beta_1, \cdots, \beta_r) \mapsto \sum_{i=1}^r \jmath_*\left(\gamma^{i-1} \cup \pi_-^*\beta_i\right)$$

since $\jmath^*\jmath_*\left(\gamma^{i-1} \cup \pi_-^*\beta_i\right) = \gamma^i \cup \pi_-^*\beta_i$ by Proposition 4.7 (4) and this corresponds to β_i via (4.18).

In the rest of the proof, we show that (4.22) is an isomorphism. By (4.17), $H^{2k+1}(\widetilde{Y}_{\mathbf{h}}) = H^{2k+1}(E_{\mathbf{h}}) = 0$ for all k. Similarly, we have $H_T^{2k+1}(\widetilde{Y}_{\mathbf{h}}) = H_T^{2k+1}(E_{\mathbf{h}}) = 0$ for all k. From (1) and (2) in Proposition 4.7, we have a commutative diagram of

exact sequences
(4.23)

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_c^{2k}(Y_{h_+} - Y_{h_-}) & \longrightarrow & H^{2k}(Y_{h_+}) & \longrightarrow & H^{2k}(Y_{h_-}) & \longrightarrow & H_c^{2k+1}(Y_{h_+} - Y_{h_-}) \longrightarrow 0 \\
 & & \downarrow \cong & & \downarrow \pi^* & & \downarrow \pi_-^* & & \downarrow \cong \\
0 & \longrightarrow & H_c^{2k}(\widetilde{Y}_{\mathbf{h}} - E_{\mathbf{h}}) & \longrightarrow & H^{2k}(\widetilde{Y}_{\mathbf{h}}) & \xrightarrow{\jmath^*} & H^{2k}(E_{\mathbf{h}}) & \longrightarrow & H_c^{2k+1}(\widetilde{Y}_{\mathbf{h}} - E_{\mathbf{h}}) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Coker}(\pi^*) & \xrightarrow{\cong} & \mathrm{Coker}(\pi_-^*) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

without two 0's at the top, for each k, where the two rows are parts of the long exact sequences of cohomology with compact supports. By (4.10) and the five lemma, π_-^* and π^* are injective respectively. Then one can check that the horizontal arrow at the bottom is an isomorphism by a diagram chase. In particular, we have

$$\mathrm{Coker}(\pi^*) \cong \mathrm{Coker}(\pi_-^*) \cong \bigoplus_{i=1}^{r} H^{2k-2i}(Y_{h_-}) \otimes H^{2i}(\mathbb{P}^r)$$

in (4.22), where the second isomorphism follows by Proposition 4.5.

By the same argument with the equivariant cohomology instead of the ordinary cohomology, the second assertion also follows, since all maps used in the argument above, including π_-, π and $\jmath$, are T-equivariant. □

4.4. S_n-representations on the cohomology of the roof. In this subsection, we prove the following.

Proposition 4.10. *There are S_n-actions on $H_T^*(\widetilde{Y}_{\mathbf{h}})$ and $H^*(\widetilde{Y}_{\mathbf{h}})$ so that* (4.17)*,* (4.19)*,* (4.20) *and* (4.21) *are all isomorphisms of S_n-representations.*

Proof. The statement for the ordinary cohomology follows from that for the equivariant cohomology by (3.18). By using the S_n-actions on $H_T^*(Y_h)$, $H_T^*(Y_{h_-})$ and $H_T^*(Y_{h_+})$, we can define two actions of S_n on $H_T^*(\widetilde{Y}_{\mathbf{h}})$ by the isomorphisms (4.19) and (4.21). We have to show that the two actions of S_n coincide.

Note that the set of T-fixed points in $\widetilde{Y}_{\mathbf{h}}$ is

$$\widetilde{Y}_{\mathbf{h}}^T \cong Y_{h_+}^T \times (\mathbb{P}^r)^T \cong S_n \times \{\sigma_i\}_{0 \leqslant i \leqslant r} \tag{4.24}$$

where σ_i denote the i-th coordinate points. Indeed, by (4.10) and (4.11), we have

$$Y_{h_-}^T \times (\mathbb{P}^r)^T = E_{\mathbf{h}}^T \ \subset\ \widetilde{Y}_h^T \ \subset\ (Y_{h_+} \times \mathbb{P}^r)^T = Y_{h_+}^T \times (\mathbb{P}^r)^T$$

where all the inclusions are equalities since $Y_{h_-}^T = Y_{h_+}^T \cong S_n$.

By (4.17) and (4.19), the odd degree part of $H^*(\widetilde{Y}_{\mathbf{h}})$ vanishes and hence the roof $\widetilde{Y}_{\mathbf{h}}$ is equivariantly formal. In particular, the restriction map

$$\mathsf{res} : H_T^*(\widetilde{Y}_{\mathbf{h}}) \longrightarrow H_T^*(\widetilde{Y}_{\mathbf{h}}^T) \tag{4.25}$$

by the inclusion $\widetilde{Y}_{\mathbf{h}}^T \subset \widetilde{Y}_{\mathbf{h}}$ is injective.

Consider a natural S_n-action on $H_T^*(\widetilde{Y}_{\mathbf{h}}^T)$ defined by

$$(\mu, (p_{v,\sigma})_{(v,\sigma)\in \widetilde{Y}_{\mathbf{h}}^T}) \mapsto (p_{\mu^{-1}v,\sigma})_{(v,\sigma)\in \widetilde{Y}_{\mathbf{h}}^T} \tag{4.26}$$

for $\mu \in S_n$ and $p_{(v,\sigma)} \in H_T^* \cong \mathbb{Q}[t_1, \cdots, t_n]$. Proposition 4.10 then follows if we show that the isomorphisms (4.19) and (4.21) composed with (4.25) is S_n-equivariant with respect to the action (4.26). Therefore the proposition follows from Lemmas 4.11 and 4.14 below. □

Lemma 4.11. *The composition of* (4.21) *and* (4.25) *is S_n-equivariant.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} \widetilde{Y}_{\mathbf{h}}^T & \hookrightarrow & E_{\mathbf{h}} & \overset{\jmath}{\hookrightarrow} & \widetilde{Y}_{\mathbf{h}} \\ \downarrow{\scriptstyle \pi^T} & & \downarrow{\scriptstyle \pi_-} & & \downarrow{\scriptstyle \pi} \\ Y_{h_+}^T & \hookrightarrow & Y_{h_-} & \hookrightarrow & Y_{h_+} \end{array}$$

where π^T denotes the restriction of π to the fixed point set. It suffices to show that the embedding of each component of the left hand side of (4.21) into $H_T^*(\widetilde{Y}_{\mathbf{h}}^T)$ is S_n-equivariant with respect to the actions (3.17) and (4.26). Equivalently, it suffices to show that the compositions

$$\mathsf{res} \circ \pi^* = (\pi^T)^* \circ \mathsf{res} : H_T^*(Y_{h_+}) \longrightarrow H_T^*(\widetilde{Y}_{\mathbf{h}}^T)$$
$$\mathsf{res} \circ \jmath_* \circ e^T(N_\jmath)^k \circ \pi_-^* = e^T(N_\jmath)^{k+1}|_{\widetilde{Y}_{\mathbf{h}}^T} \circ (\pi^T)^* \circ \mathsf{res} : H_T^*(Y_{h_-}) \longrightarrow H_T^*(\widetilde{Y}_{\mathbf{h}}^T)$$

are S_n-equivariant for $0 \leqslant k < r$.

Since res in (4.25) and $(\pi^T)^*$ are S_n-equivariant as $\pi^T(v,\sigma) = v$ under the isomorphism (4.24), it is enough to show that the T-equivariant Euler class map $e^T(N_\jmath)|_{\widetilde{Y}_{\mathbf{h}}^T} : H_T^*(\widetilde{Y}_{\mathbf{h}}^T) \to H_T^{*+2}(\widetilde{Y}_{\mathbf{h}}^T)$ is S_n-equivariant. Indeed, by Proposition 4.7 (2), each $N_\jmath|_{(v,\sigma_i)} = \mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^r}(-1)|_{\sigma_i})$ with $v \in S_n$ and $0 \leqslant i \leqslant r$ does not depend on v and hence $e^T(N_\jmath)|_{\widetilde{Y}_{\mathbf{h}}^T}$ is S_n-equivariant. □

Remark 4.12. More precisely, the T-equivariant Euler class of $\mathcal{O}_{\mathbb{P}^r}(-1)|_{\sigma_i}$ above is $t_{j_0} - t_{j+i}$ for type (1) and $t_{j-r+1+i} - t_{h(j+1)}$ for type (2) respectively, up to sign, by the proof of Proposition 4.7 (4) and (3.8).

Example 4.13 ($r = 1$). Let $\mathbf{h} = (h_-, h, h_+)$ be a modular triple of type (1). In particular, $r = 1$. Then, the inclusions

$$Y_h \cong \mathrm{pr}_2^{-1}(0) \overset{\jmath}{\hookrightarrow} \widetilde{Y}_{\mathbf{h}} \overset{\imath}{\hookleftarrow} \mathrm{pr}_2^{-1}(\infty)$$

induce the short exact sequence

$$H_T^{2k-2}(\mathrm{pr}_2^{-1}(0)) \xrightarrow{\jmath_*} H_T^{2k}(\widetilde{Y}_{\mathbf{h}}) \xrightarrow{\imath^*} H_T^{2k}(\mathrm{pr}_2^{-1}(\mathbb{P}^1 \setminus \{0\}) \cong H_T^{2k}(\mathrm{pr}_2^{-1}(\infty)) \longrightarrow 0$$

where $\jmath_*$ is indeed injective, since $\jmath^*\jmath_*$ is the multiplication by $e^T(N_\jmath) = \pm(t_{j+1} - t_j)$ which is not a zero divisor. This gives the decomposition

$$H_T^*(\widetilde{Y}_{\mathbf{h}}) \cong H_T^{*-2}(\mathrm{pr}_2^{-1}(0)) \oplus H_T^*(\mathrm{pr}_2^{-1}(\infty)). \tag{4.27}$$

Under the identification

$$\phi : Y_h \cong \mathrm{pr}_2^{-1}(0) \xrightarrow{\cong} \mathrm{pr}_2^{-1}(\infty), \quad Tg \mapsto Tg \cdot (j, j+1)$$

the isomorphism (4.27) now reads as

$$H_T^*(\widetilde{Y}_{\mathbf{h}}) \cong H_T^{*-2}(Y_h) \oplus H_T^*(Y_h),$$

which preserves the submodule generated by $\mathfrak{m}$. One can also immediately check that it is S_n-equivariant since $e^T(N_\jmath)$ and $\phi^* \circ \imath^*$ are. Hence, we have

$$H^*(\widetilde{Y}_{\mathbf{h}}) \cong H^{*-2}(Y_h) \oplus H^*(Y_h)$$

which is S_n-equivariant.

The arguments used in the above example easily extend to a more general setting in the following lemma.

Lemma 4.14. *The composition*

$$\bigoplus_{i=0}^{r} H_T^{*-2i}(Y_h) \cong H_T^*(Y_h) \otimes H^*(\mathbb{P}^r) \cong H_T^*(\widetilde{Y}_{\mathbf{h}}) \hookrightarrow H_T^*(\widetilde{Y}_{\mathbf{h}}^T)$$

of (4.19) *and* (4.25) *is S_n-equivariant where S_n acts trivially on $H^*(\mathbb{P}^r)$. Furthermore, the above isomorphism preserves the submodule generated by $\mathfrak{m}$.*

Proof. Suppose $\mathbf{h}$ is of type (1). Let $H_k \cong \mathbb{P}^k$ be the coordinate plane in $\mathbb{P}^r$ spanned by the coordinate points $\sigma_0, \cdots, \sigma_k$. This induces a T-equivariant filtration

$$Y_h \cong \widetilde{Y}_0 \subset \cdots \subset \widetilde{Y}_r = \widetilde{Y}_{\mathbf{h}}$$

of $\widetilde{Y}_{\mathbf{h}}$, where $\widetilde{Y}_k$ are given by

$$\widetilde{Y}_k := \mathrm{pr}_2^{-1}(H_k) \cong \{(y,[v]) \in Y_{h_+} \times H_k : f_{\mathbf{h}}(y) \in \mathbb{C}v\}.$$

Equivalently, $\widetilde{Y}_k$ is the (smooth) intersection of $\widetilde{Y}_{\mathbf{h}}$ and $Y_{h_+} \times H_k$ in $Y_{h_+} \times \mathbb{P}^r$. Let $\jmath_k : \widetilde{Y}_{k-1} \subset \widetilde{Y}_k$ denote the inclusion. Associated to this filtration, there is a Gysin sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_T^{*-2}(\widetilde{Y}_{k-1}) & \xrightarrow{(\jmath_k)_*} & H_T^*(\widetilde{Y}_k) & \longrightarrow & H_T^*(\widetilde{Y}_k - \widetilde{Y}_{k-1}) & \longrightarrow & 0 \\ & & & & & \searrow^{\rho_k} & \downarrow \cong & & \\ & & & & & & H_T^*(\mathrm{pr}_2^{-1}(\sigma_k)) & & \end{array} \tag{4.28}$$

for each $1 \leqslant k \leqslant r$, which is split. Indeed, $\jmath_k^* \circ (\jmath_k)_*$ is equal to the multiplication by $\alpha_k := e^T(N_{H_{k-1}/H_k}) = (\pm(t_{j+k} - t_{j+i}))_{(v,\sigma_i) \in \widetilde{Y}_{k-1}^T}$ which is not a zero divisor.

The short exact sequence (4.28) gives us the isomorphism

$$H_T^*(\widetilde{Y}_{\mathbf{h}}) \cong \bigoplus_{i=0}^{r} H_T^{*-2k}(\mathrm{pr}_2^{-1}(\sigma_k))$$

as graded vector spaces. Furthermore, there is an explicit isomorphism

$$Y_h \cong \widetilde{Y}_0 \xrightarrow{\cong} \mathrm{pr}_2^{-1}(\sigma_k) = \{y \in Y_{h_+} : f_{\mathbf{h}}(y) \in \mathbb{C}e_k\} \tag{4.29}$$

which sends Tg to $Tg \cdot (j, j+k)$ where $(j, j+k)$ denotes the permutation matrix in $U(n)$ associated to the transposition $(j, j+k) \in S_n$. Hence, it remains to show that $(\jmath_k)_*$ and ρ_k are S_n-equivariant. To see this, observe that (4.28) fits into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_T^{*-2}(\widetilde{Y}_{k-1}) & \xrightarrow{(\jmath_k)_*} & H_T^*(\widetilde{Y}_k) & \xrightarrow{\rho_k} & H_T^*(\mathrm{pr}_2^{-1}(\sigma_k)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_k \circ \mathsf{res}} & & \downarrow{\scriptstyle \mathsf{res}} & & \downarrow{\scriptstyle \mathsf{res}} & & \\ 0 & \longrightarrow & H_T^*(\widetilde{Y}_{k-1}^T) & \longrightarrow & H_T^*(\widetilde{Y}_k^T) & \xrightarrow{\rho_k^T} & H_T^*(\mathrm{pr}_2^{-1}(\sigma_k)^T) & \longrightarrow & 0 \end{array}$$

of short exact sequences, where the vertical maps are all injective and the bottom row is induced by $\widetilde{Y}_k^T = \widetilde{Y}_{k-1}^T \sqcup \mathrm{pr}_2^{-1}(\sigma_k)^T$. Therefore, $(\jmath_k)_*$ is S_n-equivariant since α_k is S_n-equivariant and $\widetilde{Y}_{k-1}^T$ is S_n-invariant in $\widetilde{Y}_k^T$.

Moreover, ρ_k is S_n-equivariant since ρ_k^T is the projection map induced by the inclusion $S_n \times \{\sigma_k\} \hookrightarrow S_n \times \{\sigma_i\}_{i \leqslant k}$, which is S_n-equivariant. On the other hand, the isomorphism (4.29) is T-equivariant with respect to the usual T-action on Y_h

composed with the interchange of t_j and t_{j+k}. This completes the proof for $\mathbf{h}$ of type (1).

For $\mathbf{h}$ of type (2), consider the coordinate planes $H_k \cong \mathbb{P}^k$ in $\mathbb{P}^r$ spanned by the coordinate points $\sigma_r, \cdots, \sigma_{r-k+1}$ and the induced T-equivariant filtration $Y_h \cong \widetilde{Y}_0 \subset \cdots \subset \widetilde{Y}_r = \widetilde{Y}_{\mathbf{h}}$ of $\widetilde{Y}_{\mathbf{h}}$ given by $\widetilde{Y}_k = \mathrm{pr}_2^{-1}(H_k)$. Then the proof is parallel to that for $\mathbf{h}$ of type (1) and we omit the detail. □

5. Unicellular LLT polynomials and twin manifolds

Let $\mathbf{h} = (h_-, h, h_+)$ be a triple of Hessenberg functions in Definition 4.1. Let $\widetilde{Y}_{\mathbf{h}}$ denote the roof manifold which is a Y_h-fiber bundle over $\mathbb{P}^r$ by Proposition 4.6 and also the blowup of Y_{h_+} along Y_{h_-} by Proposition 4.7.

In this section, we will apply the geometry of the twin manifolds associated to $\mathbf{h}$ to compare the cohomology of the twin manifolds. In particular, we will establish the modular law (2.3) for a modular triple $\mathbf{h}$ (when $r = 1$).

5.1. The modular law. By Proposition 4.10, we can consider the Frobenius characteristic of the cohomology of the roof manifold $\widetilde{Y}_{\mathbf{h}}$.

Definition 5.1. *Following Definition* 3.8*, we let*

$$\mathcal{P}(\mathbf{h}) := \sum_{k\geqslant 0} \mathrm{ch}(H^{2k}(\widetilde{Y}_{\mathbf{h}}))q^k \in \Lambda[q].$$

For every triple $\mathbf{h} = (h_-, h, h_+)$ in Definition 4.1, the following is immediate from (4.17), (4.20) and Proposition 4.10.

Proposition 5.2.

$$\mathcal{P}(\mathbf{h}) = \mathcal{P}(h_+) + q[r]_q\mathcal{P}(h_-). \tag{5.1}$$

$$\mathcal{P}(\mathbf{h}) = [r+1]_q\mathcal{P}(h). \tag{5.2}$$

Combining (5.1) and (5.2), we have the following.

Theorem 5.3. *Let* $\mathbf{h} = (h_-, h, h_+)$ *be a triple in Definition* 4.1*. Then,*

$$[r+1]_q\mathcal{P}(h) = \mathcal{P}(h_+) + q[r]_q\mathcal{P}(h_-).$$

In particular, the modular law

$$[2]_q\mathcal{P}(h) = \mathcal{P}(h_+) + q\mathcal{P}(h_-) \tag{5.3}$$

holds for a modular triple $\mathbf{h}$ *(when* $r = 1$*).*

Moreover, we have canonical S_n-equivariant isomorphisms

$$H^*(Y_h) \oplus H^{*-2}(Y_h) \cong H^*(\widetilde{Y}_{\mathbf{h}}) \cong H^*(Y_{h_+}) \oplus H^{*-2}(Y_{h_-}) \tag{5.4}$$

for a modular triple $\mathbf{h}$.

5.2. Unicellular LLT and twins. The chromatic quasisymmetric functions and the representations of S_n on the cohomology of Hessenberg varieties are related by the Shareshian-Wachs conjecture [23], proved in [8, 14],

$$\mathcal{F}(h) := \sum_{k\geqslant 0} \mathrm{ch}(H^{2k}(X_h))q^k = \omega\, \mathsf{csf}_h(q) \tag{5.5}$$

where the S_n-action on $H^*(X_h)$ is given by the dot action (3.15) and ω denotes the involution of Λ which interchanges each Schur function with its transpose.

An analogous connection between unicellular LLT polynomials and the representations of S_n on the cohomology of twin manifolds was discovered by Masuda-Sato and Precup-Sommers.

Theorem 5.4. [21, Proposition 3.2.1] [22, Corollary 7.9 (2)]

$$\mathcal{P}(h) = \mathrm{LLT}_h(q)$$

for every Hessenberg function h.

Proof. By the characterization of LLT_h in Theorem 2.6, it suffices to show that $\mathcal{P}$ satisfies the three conditions (1), (2) and (3). (1) was proved in Proposition 3.10 and (2) was proved in Proposition 3.15. Finally, Theorem 5.3 proves (3). □

Remark 5.5. Masuda-Sato's proof of Theorem 5.4 in [21] makes an essential use of the Shareshian-Wachs conjecture while our proof is based on the geometry of twin manifolds, independent of the SW conjecture. Their proof is a direct consequence of the following three major ingredients:

(i) the Shareshian-Wachs conjecture (5.5),

(ii) the Carlsson-Mellit relation (cf. [9, Proposition 3.5])

$$(q-1)^n \mathsf{csf}_h(q) = \mathrm{LLT}_h[(q-1)X; q]$$

where $X = x_1 + x_2 + \cdots$ with variables x_i of Λ and $[\ ]$ denotes the plethystic substitution, and

(iii) the parallel plethystic relation (cf. [21, Proposition 3.0.2])

$$(1-q)^n \mathcal{F}(h) = \mathcal{P}(h)\,[(1-q)X; q]$$

obtained by applying the formula in [15, Proposition 3.3.1] to the isomorphism (3.14).

Pictorially, it can be summarized in the diagram

$$\begin{array}{ccc} \text{Hessenberg varieties} & \overset{\text{(i)}}{\text{———}} & \text{chromatic quasisymmetric functions} \\ \Big|\,\text{(iii)} & & \Big|\,\text{(ii)} \\ \text{twin manifolds} & \xleftrightarrow{\text{Theorem 5.4}} & \text{unicellular LLT polynomials.} \end{array} \tag{5.6}$$

Note that our proof of Theorem 5.4 is direct without relying on (i), (ii) or (iii).

On the other hand, in [22], Precup-Sommers studied a relation between the coefficients of $\mathcal{P}(h)$ in the Schur basis expansion, which are polynomials in q, and the cohomology of *nilpotent* Hessenberg varieties (cf. [22, §1]). Based on this, they proved the modular law (5.3), by establishing the modular law for the nilpotent Hessenberg varieties. However, their work does not address the geometry of twin manifolds.

Remark 5.6. Our proof of Theorem 5.4 together with (ii) and (iii) in (5.6) provide us with a new proof of the Shareshian-Wachs conjecture (5.5).

Acknowledgements. We thank Anton Ayzenberg, Jaehyun Hong, Antonio Nigro and Takashi Sato for enlightening discussions and comments.

References

[1] Hiraku Abe, Megumi Harada, Tatsuya Horiguchi, and Mikiya Masuda, *The cohomology rings of regular nilpotent Hessenberg varieties in Lie type A*, Int. Math. Res. Not. IMRN (2019), no. 17, 5316–5388, https://doi.org/10.1093/imrn/rnx275.

[2] Hiraku Abe and Tatsuya Horiguchi, *A survey of recent developments on Hessenberg varieties*, in Schubert calculus and its applications in combinatorics and representation theory, Springer Proc. Math. Stat., vol. 332, Springer, Singapore, 2020, pp. 251–279, https://doi.org/10.1007/978-981-15-7451-1_10.

[3] Alex Abreu and Antonio Nigro, *Chromatic symmetric functions from the modular law*, J. Combin. Theory Ser. A **180** (2021), article no. 105407 (30 pages).

[4] ———, *A symmetric function of increasing forests*, Forum Math. Sigma **9** (2021), article no. e35 (21 pages).

[5] Per Alexandersson, *LLT polynomials, elementary symmetric functions and melting lollipops*, J. Algebraic Combin. **53** (2021), no. 2, 299–325, https://doi.org/10.1007/s10801-019-00929-z.

[6] Anton Ayzenberg and Victor Buchstaber, *Manifolds of isospectral matrices and Hessenberg varieties*, Int. Math. Res. Not. IMRN (2021), no. 21, 16671–16692, https://doi.org/10.1093/imrn/rnz388.

[7] A. M. Bloch, H. Flaschka, and T. Ratiu, *A convexity theorem for isospectral manifolds of Jacobi matrices in a compact Lie algebra*, Duke Math. J. **61** (1990), no. 1, 41–65, https://doi.org/10.1215/S0012-7094-90-06103-4.

[8] Patrick Brosnan and Timothy Y. Chow, *Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties*, Adv. Math. **329** (2018), 955–1001, https://doi.org/10.1016/j.aim.2018.02.020.

[9] Erik Carlsson and Anton Mellit, *A proof of the shuffle conjecture*, J. Amer. Math. Soc. **31** (2018), no. 3, 661–697, https://doi.org/10.1090/jams/893.

[10] Michael W. Davis and Tadeusz Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451, https://doi.org/10.1215/S0012-7094-91-06217-4.

[11] Filippo De Mari, Claudio Procesi, and Mark A. Shayman, *Hessenberg varieties*, Trans. Amer. Math. Soc. **332** (1992), no. 2, 529–534, https://doi.org/10.2307/2154181.

[12] Mark Goresky, Robert Kottwitz, and Robert MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25–83, https://doi.org/10.1007/s002220050197.

[13] Mathieu Guay-Paquet, *A modular relation for the chromatic symmetric functions of (3+1)-free posets*, 2013, https://arxiv.org/abs/1306.2400.

[14] ———, *A second proof of the Shareshian–Wachs conjecture, by way of a new Hopf algebra*, 2016, https://arxiv.org/abs/1601.05498.

[15] Mark Haiman, *Combinatorics, symmetric functions, and Hilbert schemes*, in Current developments in mathematics, 2002, Int. Press, Somerville, MA, 2003, pp. 39–111.

[16] Young-Hoon Kiem and Donggun Lee, *Birational geometry of generalized Hessenberg varieties and the generalized Shareshian-Wachs conjecture*, J. Combin. Theory Ser. A **206** (2024), article no. 105884 (45 pages).

[17] Shintarô Kuroki, *Introduction to GKM theory*, Trends in Mathematics – New Series, vol. 11, Proceedings of ASARC Workshop, no. 2, KAIST, Information Center for Mathematical Sciences, 2009, pp. 113–129, https://mathsci.kaist.ac.kr/~31871/ASARC_workshop.pdf.

[18] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon, *Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties*, J. Math. Phys. **38** (1997), no. 2, 1041–1068, https://doi.org/10.1063/1.531807.

[19] Seung Jin Lee, *Linear relations on LLT polynomials and their k-Schur positivity for* $k = 2$, J. Algebraic Combin. **53** (2021), no. 4, 973–990, https://doi.org/10.1007/s10801-020-00950-7.

[20] Ian G. Macdonald, *Symmetric functions and Hall polynomials*, Second ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015.

[21] Mikiya Masuda and Takashi Sato, *Unicellular LLT polynomials and twins of regular semisimple Hessenberg varieties*, Int. Math. Res. Not. IMRN (2024), no. 2, 964–996, https://doi.org/10.1093/imrn/rnac359.

[22] Martha Precup and Eric Sommers, *Perverse sheaves, nilpotent Hessenberg varieties, and the modular law*, 2022, https://arxiv.org/abs/2201.13346.

[23] John Shareshian and Michelle L. Wachs, *Chromatic quasisymmetric functions*, Adv. Math. **295** (2016), 497–551, https://doi.org/10.1016/j.aim.2015.12.018.

[24] Richard P. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, Adv. Math. **111** (1995), no. 1, 166–194, https://doi.org/10.1006/aima.1995.1020.

[25] Richard P. Stanley and John R. Stembridge, *On immanants of Jacobi-Trudi matrices and permutations with restricted position*, J. Combin. Theory Ser. A **62** (1993), no. 2, 261–279, https://doi.org/10.1016/0097-3165(93)90048-D.

[26] Carlos Tomei, *The topology of isospectral manifolds of tridiagonal matrices*, Duke Math. J. **51** (1984), no. 4, 981–996, https://doi.org/10.1215/S0012-7094-84-05144-5.

[27] Julianna S. Tymoczko, *Permutation actions on equivariant cohomology of flag varieties*, in Toric topology, Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 365–384, https://doi.org/10.1090/conm/460/09030.

Young-Hoon Kiem, School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Korea
E-mail : kiem@kias.re.kr

Donggun Lee, Center for Complex Geometry, Institute for Basic Science, 55 Expo-ro, Yuseong-gu, Daejeon 34126, Korea
E-mail : dglee@ibs.re.kr