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# Schur-positivity of short chords in matchings

Avichai Marmor

## ABSTRACT

We prove that the set of matchings with a fixed number of unmatched vertices is Schur-positive with respect to the set of short chords. Two proofs are presented. The first proof applies a new combinatorial criterion for Schur-positivity, while the second is bijective. The coefficients in the Schur expansion are derived, and interpreted in terms of Bessel polynomials. Then, we present a variant of Knuth equivalence for matchings, and show that every equivalence class corresponds to a Schur function. We proceed to find various refined Schur-positive sets, including the set of matchings with a prescribed crossing number and the set of matchings with a given number of pairs of intersecting chords. Finally, we characterize all the matchings  $m$  such that the set of matchings avoiding  $m$  is Schur-positive.

## 1. INTRODUCTION

Denote the set of nonnegative integers by  $\mathbb{N}$ , and the set of positive integers by  $\mathbb{P}$ . Given  $N \in \mathbb{N}$  and a set  $\mathcal{A}$ , together with a set-valued function  $D : \mathcal{A} \rightarrow 2^{[N-1]}$  which is sometimes called a *statistic*, we define its quasisymmetric generating function

$$\mathcal{Q}_D(\mathcal{A}) = \sum_{a \in \mathcal{A}} F_{N, D(a)},$$

where  $F_{N, S}$  is the *fundamental quasisymmetric function* introduced by Gessel [13], indexed by a subset  $S \subseteq [N-1]$  (see Section 2.1 for more background). We write  $\mathcal{Q}(\mathcal{A})$  instead of  $\mathcal{Q}_D(\mathcal{A})$  when  $D$  is clear from the context. The following long-standing problem was first addressed in [14].

**PROBLEM 1.1** (Gessel and Reutenauer). *Find sets  $\mathcal{A}$  and statistics  $D$  for which  $\mathcal{Q}_D(\mathcal{A})$  is symmetric.*

When  $\mathcal{Q}_D(\mathcal{A})$  is symmetric, we say that  $\mathcal{A}$  is symmetric with respect to the statistic  $D$  (we omit  $D$  when it is clear from the context). If  $\mathcal{A}$  is symmetric, then there is a unique expansion

$$\mathcal{Q}_D(\mathcal{A}) = \sum_{\lambda \vdash N} c_\lambda s_\lambda,$$

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where  $c_\lambda \in \mathbb{Q}$  and  $s_\lambda$  are Schur functions (discussed in Section 2.1). The coefficients  $c_\lambda$  are called the *Schur coefficients* of  $\mathcal{A}$ . If  $c_\lambda \in \mathbb{N}$  for all  $\lambda \vdash N$ , we say that  $\mathcal{A}$  is Schur-positive with respect to  $D$ .

Gessel and Reutenauer also addressed the following problem:

PROBLEM 1.2. *Find Schur-positive sets  $\mathcal{A}$ .*

The study of symmetric functions is partially motivated by the correspondence between symmetric functions of degree  $N$  and class functions on the symmetric group  $S_N$ , which is established through the Frobenius characteristic map described in [26, Chapter 4.7]. A symmetric function is Schur-positive if its corresponding class function is a proper character (see [3] for a detailed explanation). This perspective provides a rich algebraic framework for studying symmetric and Schur-positive sets.

During the last few decades, many Schur-positive subsets of the symmetric group  $S_N$  were constructed with respect to the standard descent function of  $S_N$ , where a permutation  $\pi \in S_N$  has  $i \in \text{Des}(\pi)$  if and only if  $\pi(i) > \pi(i + 1)$ . A fundamental construction is due to Gessel [13], who proved that every set of permutations which is closed under the Knuth equivalence relations is Schur-positive. Furthermore, he showed that every Knuth class  $\mathcal{A} \subset S_N$  (i.e. equivalence class of the Knuth relations) satisfies  $\mathcal{Q}_{\text{Des}}(\mathcal{A}) = s_\lambda$  for some  $\lambda \vdash N$ . This result has many direct consequences. For example, for every  $J \subseteq [N - 1]$ , the set  $D_{N,J}^{-1} = \{\pi^{-1} \mid \text{Des}(\pi) = J\} \subseteq S_N$  is Schur-positive because it is closed under the Knuth relations.

Other examples of Schur-positive sets of permutations include all conjugacy classes, as proven by Gessel and Reutenauer [14], and the set of permutations  $\pi \in S_N$  with a fixed number of inversions (i.e.  $i < j$  such that  $\pi(i) > \pi(j)$ ), a result demonstrated by Adin and Roichman [3, Prop. 9.5].

Some Schur-positive sets that do not consist of permutations were found as well. For instance, Gessel [13] proved that for every partition  $\lambda \vdash N$ , the set  $\text{SYT}(\lambda)$  of standard Young tableaux of shape  $\lambda$  is Schur-positive (see Section 2.2 for details).

In this work, we focus on the set of matchings of a given set of vertices.

DEFINITION 1.3. *A matching  $m$  on a finite set of vertices  $S \subseteq \mathbb{P}$  is an unordered partition of  $S$  into blocks, each block of size 1 or 2. If a block of a matching  $m$  contains only the vertex  $i$ , we say that  $i$  is an unmatched vertex of  $m$  and denote  $(i) \in m$ . If a block contains both  $i$  and  $j$  for  $i < j$ , we say that  $(i, j)$  is a chord of  $m$  and denote  $(i, j) \in m$ . That is, the notation  $(i, j) \in m$  implies that  $i < j$ . When  $(i, j) \in m$ , we also say that  $i$  opens this chord and  $j$  closes it. If a matching  $m$  has no unmatched vertex, then we say that  $m$  is a perfect matching.*

We will focus on matchings on the set  $[N] := \{1, \dots, N\}$  for some  $N \in \mathbb{N}$ . Denote by  $\mathcal{M}_N$  the set of matchings on  $[N]$ , and by  $\mathcal{M}_{N,f}$  the set of matchings on  $[N]$  with  $f$  unmatched vertices.

For example,

$$\mathcal{M}_{4,0} = \{(1, 2), (3, 4)\}, \{(1, 3), (2, 4)\}, \{(1, 4), (2, 3)\}$$

and

$$\mathcal{M}_{3,1} = \{(1, 2), (3)\}, \{(1, 3), (2)\}, \{(1), (2, 3)\}.$$

DEFINITION 1.4. *Let  $m$  be a matching on  $S$ . If  $(i_1, i_3), (i_2, i_4) \in m$  for  $i_1 < i_2 < i_3 < i_4$  then we say that the chords  $(i_1, i_3)$  and  $(i_2, i_4)$  intersect.*

DEFINITION 1.5. *Let  $m \in \mathcal{M}_N$  be a matching on  $[N]$ . If  $(i, i + 1) \in m$ , then it is called a short chord. We denote  $\text{Short}(m) := \{i \in [N - 1] \mid (i, i + 1) \in m\}$ .*

For example, consider the matching  $m = \{(1, 3), (2, 6), (4, 5)\}$  as in Figure 1. In this matching the chords  $(1, 3)$  and  $(2, 6)$  intersect, and  $\text{Short}(m) = \{4\}$  (since  $(4, 5) \in m$ ).

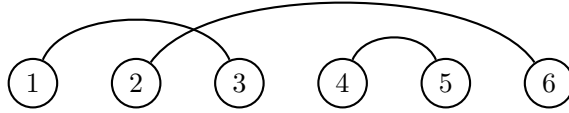


FIGURE 1. The matching  $m = \{(1, 3), (2, 6), (4, 5)\}$ .

Note that  $N \notin \text{Short}(m)$  for every  $m \in \mathcal{M}_N$ . This holds even where  $(1, N) \in m$ .

Enumerative properties of short chords in perfect matchings were studied extensively. The distribution of the number of short chords in perfect matchings appears in entry A079267 of the OEIS [24], and was analyzed for example by Cameron and Killpatrick [8]. McSorley and Feinsilver [23] discovered that short chords of matchings are closely related to Bessel polynomials, as will be discussed in Section 4.3.

The study of short chords of matchings is also motivated by the involutive length and its corresponding poset studied by Adin, Postnikov and Roichman [2]. Related posets derived from the Bruhat order were studied by Richardson and Springer [25], Hultman [17], Deodhar and Srinivasan [10] and others, motivated by the topology of linear algebraic groups (see [17] for details). Further discussion on the algebraic motivations can be found in Section 6.3.

We prove the following:

**THEOREM 1.6.** *Let  $n, f \in \mathbb{N}$  be nonnegative integers, and denote  $N = 2n + f$ . Then the set  $\mathcal{M}_{N,f}$  is Schur-positive with respect to Short. Furthermore, its Schur expansion is given by the following formula:*

$$\mathcal{Q}_{\text{Short}}(\mathcal{M}_{N,f}) = \sum_{k=0}^n |\{m \in \mathcal{M}_{N-2k,f} \mid \text{Short}(m) = \emptyset\}| s_{N-k,k}.$$

It turns out that the Schur coefficients of  $\mathcal{Q}_{\text{Short}}(\mathcal{M}_{N,f})$  may be explicitly interpreted in terms of Bessel polynomials, see Corollary 4.13 below.

We give two proofs of Theorem 1.6. The first proof relies on a new criterion for Schur-positivity of sparse statistics:

**DEFINITION 1.7.** *We say that a set  $J \subseteq [N - 1]$  is sparse if  $\{j, j + 1\} \not\subseteq J$  for every  $1 \leq j \leq N - 2$ .*

*For a set  $\mathcal{A}$  and a function  $D : \mathcal{A} \rightarrow 2^{[N-1]}$ , we say that  $D$  is sparse if  $D(a)$  is sparse for all  $a \in \mathcal{A}$ .*

**THEOREM 1.8.** *Let  $\mathcal{A}$  be a finite set with a statistic  $D : \mathcal{A} \rightarrow 2^{[N-1]}$ , and denote  $n = \lfloor \frac{N}{2} \rfloor$ . Then the following statements are equivalent:*

- *$D$  is sparse, and for every sparse  $J \subseteq [N - 1]$ , the cardinality of the set  $\{a \in \mathcal{A} \mid D(a) \supseteq J\}$  depends on the size of  $J$  only.*
- *$\mathcal{A}$  is symmetric with respect to  $D$ , with a Schur expansion of the form  $\mathcal{Q}_D(\mathcal{A}) = \sum_{k=0}^n c_k s_{N-k,k}$  for some  $c_k \in \mathbb{Z}$ .*

*Furthermore, if these statements hold then  $\mathcal{A}$  is Schur-positive, and its Schur expansion is*

$$\mathcal{Q}_D(\mathcal{A}) = \sum_{k=0}^n |\{a \in \mathcal{A} \mid D(a) = \{1, 3, 5, \dots, 2k - 1\}\}| s_{N-k,k}.$$

This new criterion implies Theorem 1.6, as shown at the beginning of Section 4, and can be applied to many other sets as well. It is also applied in a subsequent paper [1] to prove Schur-positivity of Gallai colorings and transitive colorings of complete graphs.

The second proof of Theorem 1.6 provides a bijection from the set of matchings to a multiset of SYTs, and applies the bijective criterion of Adin and Roichman for Schur-positivity [3, Prop. 9.1], presented in Theorem 2.5.

Following the bijective proof, we present an equivalence relation on matchings, motivated by Knuth equivalence of permutations [19]:

DEFINITION 1.9. *We say that two matchings  $m_1, m_2 \in \mathcal{M}_N$  are Knuth equivalent if one can be obtained from the other by a sequence of elementary Knuth transformations:*

- *Replace the chords  $(i, i + 1), (i + 2)$  with the chords  $(i), (i + 1, i + 2)$  or vice versa (i.e. interchange a short chord with an adjacent unmatched vertex).*
- *Replace the chords  $(i, i + 1), (i + 2, j)$  with the chords  $(i, j), (i + 1, i + 2)$ , or  $(i, i + 1), (j, i + 2)$  with  $(j, i), (i + 1, i + 2)$ , or vice versa (i.e. interchange a short chord with an adjacent endpoint of another chord).*

We also define the core of a given matching:

DEFINITION 1.10 (Definition 4.3 below). *The core of a given matching  $m \in \mathcal{M}_N$ , denoted  $\text{core}(m)$ , is obtained by repeatedly removing short chords from the matching until no short chords remain. The remaining vertices are then re-indexed with natural numbers starting from 1 while preserving their relative order.*

A classical result due to Knuth [19] states that two permutations  $\pi_1, \pi_2 \in S_N$  are Knuth equivalent if and only if  $P(\pi_1) = P(\pi_2)$ , where  $P(\pi)$  denotes the insertion tableau of a permutation  $\pi$  defined by the Robinson-Schensted correspondence (see [26, Section 3] for details). We prove the following analogous result for matchings:

THEOREM 1.11 (Theorem 5.1 below). *Two matchings  $m_1, m_2 \in \mathcal{M}_N$  are Knuth equivalent if and only if  $\text{core}(m_1) = \text{core}(m_2)$ .*

We show that Knuth classes of matchings, similarly to Knuth classes of permutations, have Schur functions as their generating functions:

THEOREM 1.12 (Corollary 5.9 below). *Every set  $\mathcal{M} \subseteq \mathcal{M}_N$  of matchings that is closed under Knuth equivalence is Schur-positive with respect to Short. Moreover, if  $\mathcal{M}$  is a Knuth equivalence class, then its generating function is  $\mathcal{Q}_{\text{Short}}(\mathcal{M}) = s_{N-k,k}$ , where  $N - 2k$  is the number of vertices of  $\text{core}(m)$  for some arbitrary matching  $m \in \mathcal{M}$ .*

Furthermore, utilizing Knuth classes, we explore in Section 5 various refined Schur-positive sets, including:

- The set of  $k$ -crossing matchings (i.e. where  $k$  is the maximal cardinality of a set of pairwise intersecting chords).
- The set of matchings with exactly  $k$  pairs of intersecting chords.

Lastly, we define  $\mathcal{M}_{N,f}(m)$  as the set of matchings in  $\mathcal{M}_{N,f}$  that avoid the pattern  $m$ . In Proposition 5.14, we provide a characterization of the matchings  $m$  for which  $\mathcal{M}_{N,f}(m)$  is Schur-positive for all  $N$  and  $f$ .

The remainder of this work is organized as follows: Section 2 provides necessary background. In Section 3, we prove a necessary and sufficient criterion for sparse Schur-positivity (Theorem 1.8). In Section 4, we focus on matchings. We first derive the Schur-positivity of short chords in matchings (Theorem 1.6) from the above criterion. Then, we present an alternative bijective proof, followed by a description of

the Schur coefficients of matchings in terms of Bessel polynomials. In Section 5, we utilize the bijection to refine the Schur-positivity property of Theorem 1.6. Finally, Section 6 concludes with further remarks and open problems.

## 2. PRELIMINARIES AND NOTATION

DEFINITION 2.1. For nonnegative integers  $i, j$ , we define the interval

$$[i, j] := \begin{cases} \{i, i + 1, \dots, j\} & \text{if } i \leq j, \\ \emptyset & \text{otherwise.} \end{cases}$$

We also define  $[i] = [1, i]$ .

DEFINITION 2.2. Given a set  $\mathcal{A}$ , a function  $D : \mathcal{A} \rightarrow 2^{[N-1]}$  and a set  $J \subseteq [N - 1]$ , we denote  $\mathcal{A}(D = J) := \{a \in \mathcal{A} \mid D(a) = J\}$  and  $\mathcal{A}(D \supseteq J) := \{a \in \mathcal{A} \mid D(a) \supseteq J\}$ .

### 2.1. SYMMETRIC AND QUASISYMMETRIC FUNCTIONS.

DEFINITION 2.3. Given  $N \in \mathbb{N}$ , a partition  $\lambda$  of  $N$  (denoted  $\lambda \vdash N$ ), is a weakly-decreasing sequence  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell)$  of positive integers, such that  $\lambda_1 + \dots + \lambda_\ell = N$ . The values  $(\lambda_1, \dots, \lambda_\ell)$  are called the parts of  $\lambda$ .

A composition  $\alpha$  of  $N$  (denoted  $\alpha \vDash N$ ) is a sequence  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$  of positive integers, such that  $\alpha_1 + \dots + \alpha_\ell = N$ . Every partition is a composition.

Compositions of  $N$  are in bijection with subsets of  $[N - 1] := \{1, \dots, N - 1\}$  by

$$[N - 1] \supseteq \{i_1, i_2, \dots, i_\ell\} \mapsto (i_1, i_2 - i_1, \dots, i_\ell - i_{\ell-1}, N - i_\ell) \vDash N.$$

For example, for  $N = 10$ ,  $\{2, 4, 7, 8\} \mapsto (2, 2, 3, 1, 2)$ . The composition associated to a set  $S$  is denoted  $\alpha_S$ , and the set associated to a composition  $\alpha$  is denoted  $S_\alpha$ .

We say that two compositions  $\alpha, \beta \vDash N$  are *equivalent* (denoted  $\alpha \sim \beta$ ), if  $\beta$  is a rearrangement of the entries of  $\alpha$ .

Denote the ring of symmetric functions by  $\text{Sym}$  and the ring of quasisymmetric functions by  $\text{QSym}$  (see, for example, [16, Section 1] for details). The space of homogeneous symmetric functions of degree  $N$  is denoted by  $\text{Sym}_N$ , and the space of homogeneous quasisymmetric functions of degree  $N$  is denoted by  $\text{QSym}_N$ .

The space  $\text{QSym}_N$  has several important bases. In this work, we focus on the *fundamental basis*, consisting of the functions

$$F_{N,S} := \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_N \\ \forall j \in S: i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_N}, \quad S \subseteq [N - 1].$$

We write  $F_S$  instead of  $F_{N,S}$  when  $N$  is clear from the context.

The space  $\text{Sym}_N$  has several standard bases as well. In this work, we focus on the Schur basis, which consists of the Schur functions  $s_\lambda$ , where  $\lambda$  is a partition of  $N$ . The definition and properties of Schur functions can be found in [26, Section 4.4]. Here, we adopt the combinatorial approach to Schur functions, as described in Theorem 2.4 and Theorem 2.5 below.

2.2. SYMMETRIC AND SCHUR-POSITIVE SETS. As mentioned in Section 1, a set  $\mathcal{A}$  is symmetric with respect to a statistic  $D : \mathcal{A} \rightarrow 2^{[N-1]}$  if the generating function  $\mathcal{Q}_D(\mathcal{A})$  is a symmetric function. Moreover, it is Schur-positive if all Schur coefficients are nonnegative integers.

One of the fundamental constructions of Schur-positive sets, regarding sets of standard Young tableaux (SYT), is due to Gessel [13]. Let  $\text{SYT}(\lambda)$  denote the set of

standard Young tableaux of shape  $\lambda$ . We draw tableaux in English notation, as in Figure 2. The *descent set* of  $T \in \text{SYT}(\lambda)$  is

$$\text{Des}(T) := \{i \in [N - 1] \mid i + 1 \text{ appears in a lower row than } i \text{ in } T\}.$$

For example, the descent set of the SYT in Figure 2 is  $\{2, 4, 7, 8\}$ .

1	2	4	7
3	6		
5	8		
9			

FIGURE 2. A SYT of shape  $\lambda = (4, 2, 2, 1)$ .

The entry in row  $i$  and column  $j$  of a tableau  $T \in \text{SYT}(\lambda)$  is denoted by  $T_{i,j}$ . In addition, we define  $\text{row}_i(T) := \{T_{i,j} \mid 1 \leq j \leq \lambda_i\}$  as the set of entries in the  $i$ -th row of  $T$ . For example, if we consider the SYT shown in Figure 2, then  $T_{3,2} = 8$  and  $\text{row}_3(T) = \{5, 8\}$ .

**THEOREM 2.4** (Gessel [13]). *For every  $\lambda \vdash N$ , the set  $\text{SYT}(\lambda)$  is Schur-positive with respect to  $\text{Des}$ . Moreover,  $\mathcal{Q}(\text{SYT}(\lambda)) = s_\lambda$ .*

In 2015, Adin and Roichman proved the following criterion.

**THEOREM 2.5** ([3, Prop. 9.1]). *A set  $\mathcal{A}$  is symmetric with respect to  $D : \mathcal{A} \rightarrow 2^{[N-1]}$  if and only if*

$$\sum_{a \in \mathcal{A}} \mathbf{t}^{D(a)} = \sum_{\lambda \vdash N} c_\lambda \sum_{T \in \text{SYT}(\lambda)} \mathbf{t}^{\text{Des}(T)}$$

for some values  $c_\lambda$ , where  $\mathbf{t}^J := \prod_{j \in J} t_j$  for  $J \subseteq [N - 1]$ . The coefficients  $c_\lambda$  are the Schur-coefficients of  $\mathcal{A}$ . Moreover,  $\mathcal{A}$  is Schur-positive if and only if  $c_\lambda \in \mathbb{N}$  for all  $\lambda \vdash N$ .

This criterion implies that proving the Schur-positivity of a set is achievable by establishing a statistic-preserving bijection between the set and SYTs of shapes corresponding to a specific multiset.

In this paper, we will also apply a recently formulated criterion for symmetry [21].

**DEFINITION 2.6.** *Let  $\mathcal{A}$  be a finite set with a statistic  $D : \mathcal{A} \rightarrow 2^{[N-1]}$ . The set of elements that respect a given composition  $\alpha \vDash N$ , denoted  $\mathcal{A}_D(\alpha)$ , consists of the elements  $a \in \mathcal{A}$  such that  $D(a) \subseteq S_\alpha$ , where  $S_\alpha$  is the set corresponding to the composition  $\alpha$ . When  $D$  is clear from the context, we may write  $\mathcal{A}(\alpha)$  instead.*

**LEMMA 2.7.** *A set  $\mathcal{A}$  is symmetric if and only if  $|\mathcal{A}(\alpha)| = |\mathcal{A}(\beta)|$  for all  $\alpha \sim \beta \vDash N$ .*

Note that only sets of permutations are considered in [21]. However, Lemma 2.7 applies to other sets as well.

Another useful result about symmetric sets and symmetric functions is due to Bloom and Sagan:

**LEMMA 2.8** (Bloom and Sagan [7, Lemma 2.2]). *For every set  $S \subseteq [N - 1]$ , the function  $F_{N,S}$  is symmetric if and only if  $S = [N - 1]$  or  $S = \emptyset$ .*

### 3. PROOF OF THEOREM 1.8

Denote  $\text{odds}(k) := \{1, 3, 5, \dots, 2k - 1\}$  for a nonnegative integer  $k$ . In particular,  $\text{odds}(0) = \emptyset$ .

Recall Definition 1.7 and Definition 2.2. We divide the assertions of Theorem 1.8 into two propositions:

LEMMA 3.1. *Let  $\mathcal{A}$  be a symmetric set with respect to a statistic  $D : \mathcal{A} \rightarrow 2^{[N-1]}$ , and denote  $n = \lfloor \frac{N}{2} \rfloor$ . Assume that the Schur expansion of  $\mathcal{A}$  is  $\mathcal{Q}_D(\mathcal{A}) = \sum_{k=0}^n c_k s_{N-k,k}$  for some  $c_k \in \mathbb{Z}$ . Then  $D$  is a sparse function, and for every sparse set  $J \subseteq [N - 1]$ , the cardinality of the set  $\mathcal{A}(D \supseteq J)$  depends on the size of  $J$  only.*

LEMMA 3.2. *Let  $\mathcal{A}$  be a finite set with a sparse statistic  $D : \mathcal{A} \rightarrow 2^{[N-1]}$ , and denote  $n = \lfloor \frac{N}{2} \rfloor$ . Assume that for every sparse set  $J \subseteq [N - 1]$ , the cardinality of the set  $\mathcal{A}(D \supseteq J)$  depends on the size of  $J$  only. Then  $\mathcal{A}$  is Schur-positive, and its Schur expansion is*

$$\mathcal{Q}_D(\mathcal{A}) = \sum_{k=0}^n |\mathcal{A}(D = \text{odds}(k))| s_{N-k,k}.$$

We prove Lemma 3.1 in Section 3.1, and then show two proofs for Lemma 3.2. The first proof (presented in Section 3.2) is inductive. The second proof (presented in Section 3.3) is more involved, and it demonstrates the power of column super-standard tableaux, introduced by Hamaker, Pawlowski and Sagan [16], in proving Schur-positivity properties. We believe this approach can be applied in other cases as well.

Before proving these lemmas, let us prove a simple folklore lemma that will be useful for both lemmas:

LEMMA 3.3. *Let  $\mathcal{A}$  be a symmetric set with respect to a statistic  $D : \mathcal{A} \rightarrow 2^{[N-1]}$ . Then  $\mathcal{A}$  is also symmetric with respect to the complementary statistic  $\bar{D} : \mathcal{A} \rightarrow 2^{[N-1]}$  defined by  $a \mapsto [N - 1] \setminus D(a)$ .*

*Proof.* The set  $\mathcal{A}$  is symmetric with respect to  $D$ , so by Theorem 2.5 there exist values  $c_\lambda, \lambda \vdash N$  such that

$$\sum_{a \in \mathcal{A}} \mathbf{t}^{D(a)} = \sum_{\lambda \vdash N} c_\lambda \sum_{T \in \text{SYT}(\lambda)} \mathbf{t}^{\text{Des}(T)},$$

where  $\mathbf{t}^J := \prod_{j \in J} t_j$  for  $J \subseteq [N - 1]$ , and therefore

$$\sum_{a \in \mathcal{A}} \mathbf{t}^{[N-1] \setminus D(a)} = \sum_{\lambda \vdash N} c_\lambda \sum_{T \in \text{SYT}(\lambda)} \mathbf{t}^{[N-1] \setminus \text{Des}(T)} = \sum_{\lambda \vdash N} c_\lambda \sum_{T \in \text{SYT}(\lambda')} \mathbf{t}^{\text{Des}(T)},$$

where  $\lambda'$  is the partition conjugate to  $\lambda$ . □

#### 3.1. PROOF OF LEMMA 3.1.

The set  $\mathcal{A}$  is symmetric, and its Schur expansion is  $\mathcal{Q}_D(\mathcal{A}) = \sum_{k=0}^n c_k s_{N-k,k}$ . Therefore, by Theorem 2.5, we have

$$(1) \quad \sum_{a \in \mathcal{A}} \mathbf{t}^{D(a)} = \sum_{k=0}^n c_k \sum_{T \in \text{SYT}(N-k,k)} \mathbf{t}^{\text{Des}(T)}.$$

First of all, notice that the set  $\text{Des}(T)$  is sparse for all  $T \in \text{SYT}(N - k, k)$ . Therefore, by Equation (1), the set  $D(a)$  is sparse for all  $a \in \mathcal{A}$ .

It remains to show that for every sparse set  $J \subseteq [N - 1]$  of size  $k$ , we have  $|\mathcal{A}(D \supseteq J)| = |\mathcal{A}(D \supseteq \text{odds}(k))|$ . Denote  $\alpha = (2^k, 1^{N-2k}) \vdash N$ . Additionally, let  $i_1 < \dots < i_{N-k-1} \in [N - 1] \setminus J$  denote the elements not in  $J$ . Since  $J$  is sparse, the



differences  $i_{t+1} - i_t$  for  $1 \leq t \leq N - k - 2$ , as well as  $i_1$  and  $N - i_{N-k-1}$ , are either 1 or 2. Those that are equal to 2 correspond to the elements of  $J$ . The composition associated to the set  $[N - 1] \setminus J$  is denoted by  $\beta := (i_1, i_2 - i_1, i_3 - i_2, \dots, i_{N-k-1} - i_{N-k-2}, N - i_{N-k-1}) \vDash N$ . By Lemma 3.3,  $\mathcal{A}$  is symmetric with respect to the complementary statistic  $\bar{D}$ . Notably, the compositions  $\alpha$  and  $\beta$  are equivalent, as they both have  $k$  occurrences of 2 and  $N - 2k$  occurrences of 1. By Lemma 2.7, we obtain that  $|\mathcal{A}_{\bar{D}}(\beta)| = |\mathcal{A}_{\bar{D}}(\alpha)|$ . By Definition 2.6, we obtain that  $\mathcal{A}_{\bar{D}}(\beta) = \mathcal{A}(D \supseteq J)$  and  $\mathcal{A}_{\bar{D}}(\alpha) = \mathcal{A}(D \supseteq \text{odds}(k))$ . Consequently,  $|\mathcal{A}(D \supseteq J)| = |\mathcal{A}(D \supseteq \text{odds}(k))|$ , as required.  $\square$

3.2. FIRST PROOF OF LEMMA 3.2. We start by introducing an observation, which will serve as a crucial step in our inductive argument.

LEMMA 3.4. *Let  $k_1 \geq k_2$  be two nonnegative integers, and let  $J \subseteq [k_1 + k_2 - 1]$  be a set. Then*

$$|\text{SYT}(k_1, k_2)(\text{Des} \supseteq J)| = |\text{SYT}(k_1 + 1, k_2 + 1)(\text{Des} \supseteq (\{1\} \cup (J + 2)))|,$$

where  $J + 2 = \{j + 2 \mid j \in J\}$ .

*Proof.* The proof is bijective: We associate a given tableau  $T \in \text{SYT}(k_1, k_2)$  such that  $\text{Des}(T) \supseteq J$  with the tableau  $T' \in \text{SYT}(k_1 + 1, k_2 + 1)$  defined by  $T'_{1,1} = 1$ ,  $T'_{2,1} = 2$ , and for all  $i \in \{1, 2\}$  and  $j \in [k_i]$ ,  $T'_{i,j+1} = T_{i,j} + 2$  (recall that  $T_{i,j}$  is the entry in row  $i$  and column  $j$  of  $T$ ). Intuitively, we first increase every entry of  $T$  by 2. Then, we insert a new column with the letters 1, 2 on the left of  $T$ , shifting the other columns to the right. Clearly, this process is injective, and  $T'$  satisfies  $T' \in \text{SYT}(k_1 + 1, k_2 + 1)$  and  $\text{Des}(T) \supseteq \{1\} \cup (J + 2)$ . Furthermore, any  $T' \in \text{SYT}(k_1 + 1, k_2 + 1)$  with  $\text{Des}(T') \supseteq \{1\} \cup (J + 2)$  can be obtained through this process.  $\square$

Now we are ready to prove Lemma 3.2.

*First proof of Lemma 3.2.* Let  $\mathcal{A}$  be a finite set with a sparse statistic  $D : \mathcal{A} \rightarrow 2^{[N-1]}$ , and assume that for every sparse set  $J \subseteq [N - 1]$ , the cardinality of the set  $\mathcal{A}(D \supseteq J)$  depends on the size of  $J$  only. By Theorem 2.5, it suffices to show that

$$(2) \quad \sum_{a \in \mathcal{A}} \mathbf{t}^{D(a)} = \sum_{k=0}^n |\mathcal{A}(D = \text{odds}(k))| \sum_{T \in \text{SYT}(N-k, k)} \mathbf{t}^{\text{Des}(T)},$$

where  $n = \lfloor \frac{N}{2} \rfloor$ .

We prove it by induction on  $N \in \mathbb{N}$ :

For  $N \leq 1$ , the statement holds trivially.

Next, we assume that the statement holds for every value smaller than  $N$  and proceed to prove it for  $N$ . Our goal is to establish the equation:

$$(3) \quad |\mathcal{A}(D = J)| = \sum_{k=0}^n c_k |\text{SYT}(N - k, k)(\text{Des} = J)|$$

for every set  $J \subseteq [N - 1]$ , where  $c_k = |\mathcal{A}(D = \text{odds}(k))|$ .

We begin by considering the case  $J = \emptyset$ . In this case, the only SYT  $T$  with  $\text{Des}(T) = \emptyset$  is the unique tableau of one row. Consequently, Equation (3) simplifies to  $|\mathcal{A}(D = \emptyset)| = |\mathcal{A}(D = \text{odds}(0))|$ , which obviously holds.

It now remains to prove Equation (3) for all  $J \neq \emptyset$ . By the inclusion–exclusion principle, it suffices to prove that the following holds for all  $J \neq \emptyset$ :

$$(4) \quad |\mathcal{A}(D \supseteq J)| = \sum_{k=0}^n c_k |\text{SYT}(N - k, k)(\text{Des} \supseteq J)|.$$

The function  $D$  is assumed to be sparse, and the set  $\text{Des}(T)$  is sparse for all  $T \in \text{SYT}(N - k, k)$ . Therefore, if  $J$  is not sparse then Equation (4) holds.

Next, let  $J \neq \emptyset$  be a sparse set, and denote  $|J| = i$ . The set  $\text{odds}(i)$  is sparse too, and has  $i$  elements. Therefore, by the assumptions of the lemma, we obtain that  $|\mathcal{A}(D \supseteq J)| = |\mathcal{A}(D \supseteq \text{odds}(i))|$ . In addition, by Theorem 2.4, the set  $\text{SYT}(N - k, k)$  is Schur-positive and has the Schur expansion  $\mathcal{Q}(\text{SYT}(N - k, k)) = s_{N-k,k}$ . Therefore, we can apply Lemma 3.1 and deduce that

$$|\text{SYT}(N - k, k)(\text{Des} \supseteq J)| = |\text{SYT}(N - k, k)(\text{Des} \supseteq \text{odds}(i))|.$$

Therefore, Equation 4 reduces to

$$(5) \quad |\mathcal{A}(D \supseteq \text{odds}(i))| = \sum_{k=0}^n c_k |\text{SYT}(N - k, k)(\text{Des} \supseteq \text{odds}(i))|,$$

where  $i > 0$ . Thus, the proof will be completed when we establish Equation (5).

Denote  $\mathcal{A}_1 = \{a \in \mathcal{A} \mid 1 \in D(a)\}$ . Furthermore, define a new statistic  $D_1 : \mathcal{A}_1 \rightarrow 2^{[N-3]}$  by  $D_1(a) = (D(a) \setminus \{1\}) - 2$ . Notice that  $\mathcal{A}_1$  satisfies the requirements of Lemma 3.2 with respect to  $D_1$ , where  $N$  is replaced by  $N - 2$ . Therefore, we may assume, by the induction hypothesis, that Lemma 3.2 holds for  $\mathcal{A}_1$ . Thus, we can apply Equation (4) for  $\mathcal{A}_1$ , while replacing  $c_k$  with  $|\mathcal{A}_1(D_1 = \text{odds}(k))| = |\mathcal{A}(D = \text{odds}(k + 1))| = c_{k+1}$ , and obtain

$$(6) \quad |\mathcal{A}_1(D_1 \supseteq J)| = \sum_{k=0}^{n-1} c_{k+1} |\text{SYT}(N - 2 - k, k)(\text{Des} \supseteq J)|.$$

If we substitute  $J = \text{odds}(i - 1)$  into Equation 6, we obtain

$$|\mathcal{A}_1(D_1 \supseteq \text{odds}(i - 1))| = \sum_{k=0}^{n-1} c_{k+1} |\text{SYT}(N - 2 - k, k)(\text{Des} \supseteq \text{odds}(i - 1))|.$$

Notice that  $|\mathcal{A}_1(D_1 \supseteq \text{odds}(i - 1))| = |\mathcal{A}(D \supseteq \text{odds}(i))|$ . Moreover, by Lemma 3.4, the number of tableaux  $T$  of shape  $(N - k - 1, k - 1)$  with  $\text{Des}(T) \supseteq \text{odds}(i - 1)$  is equal to the number of tableaux  $T$  of shape  $(N - k, k)$  with  $\text{Des}(T) \supseteq \text{odds}(i)$ . Therefore, we can reformulate the equation and obtain that

$$|\mathcal{A}(D \supseteq \text{odds}(i))| = \sum_{k=1}^n c_k |\text{SYT}(N - k, k)(\text{Des} \supseteq \text{odds}(i))|.$$

The only difference between this equation and Equation (5) is the omission of the summand corresponding to  $k = 0$ . However, this summand evaluates to zero and does not affect the overall expression. Indeed,  $|\text{SYT}(N, 0)(\text{Des} \supseteq \text{odds}(i))| = 0$ , since the unique SYT of shape  $(N)$  has no descents.  $\square$

3.3. SECOND PROOF OF LEMMA 3.2. As a first step of this proof, we prove that a set satisfying the conditions of Lemma 3.2 is symmetric.

LEMMA 3.5. *Let  $\mathcal{A}$  be a finite set with a sparse statistic  $D : \mathcal{A} \rightarrow 2^{[N-1]}$ , and denote  $n = \lfloor \frac{N}{2} \rfloor$ . Assume that there are constants  $b_k \in \mathbb{N}$  for  $0 \leq k \leq n$ , such that for every sparse set  $J \subseteq [N - 1]$ ,  $|\mathcal{A}(D \supseteq J)| = b_{|J|}$ . Then  $\mathcal{A}$  is symmetric with respect to  $D$ .*

*Proof.* Following Lemma 3.3, it suffices to prove that  $\mathcal{A}$  is symmetric with respect to the complementary statistic  $\bar{D}$ . As we proceed to prove it by Lemma 2.7, let us find  $|\mathcal{A}_{\bar{D}}(\alpha)|$  for all  $\alpha = (\alpha_1, \dots, \alpha_\ell) \vDash N$ .

If a composition  $\alpha$  has  $\max_i(\alpha_i) > 2$ , then there exists  $j \in [N - 2]$  such that  $j, j + 1 \notin S_\alpha$ , where  $S_\alpha$  is the set corresponding to  $\alpha$ . By Definition 2.6, for every element  $a \in \mathcal{A}_{\bar{D}}(\alpha)$  we have  $\bar{D}(a) \subseteq S_\alpha$ , so  $j, j + 1 \notin \bar{D}(a)$ . Consequently,  $j, j + 1 \in$

$D(a)$ , and the set  $D(a)$  is not sparse. However,  $D$  is assumed to be a sparse function, so  $\mathcal{A}_{\bar{D}}(\alpha) = \emptyset$ .

Now assume that  $\max_i(\alpha_i) \leq 2$ , and denote by  $k = |\{i \mid \alpha_i = 2\}|$  the number of occurrences of 2 in  $\alpha$ . Thus, we have  $\ell = N - k$ . Consider the set  $J = [N - 1] \setminus S_\alpha$ . Notably, the set  $J$  is sparse and consists of  $k$  elements. Let  $a \in \mathcal{A}$  be an element. We have  $a \in \mathcal{A}_{\bar{D}}(\alpha)$  if and only if  $\bar{D}(a) \subseteq S_\alpha$ , or equivalently,  $D(a) \supseteq J$ . Thus,  $|\mathcal{A}_{\bar{D}}(\alpha)| = |\mathcal{A}(D \supseteq J)| = b_k$ .

To conclude, if  $\alpha$  contains an element larger than 2 then  $\mathcal{A}_{\bar{D}}(\alpha) = \emptyset$ . Otherwise, the size of  $\mathcal{A}_{\bar{D}}(\alpha)$  depends only on the number of occurrences of 2 in  $\alpha$ . Therefore,  $|\mathcal{A}_{\bar{D}}(\alpha)| = |\mathcal{A}_{\bar{D}}(\beta)|$  for all  $\alpha \sim \beta \vDash N$ . By Lemma 2.7, we obtain that  $\mathcal{A}$  is symmetric.  $\square$

As the next step of the proof, we prove that  $\mathcal{A}$  is Schur-positive and find its Schur coefficients. For this, we define a total order on partitions  $\lambda \vdash N$ :

DEFINITION 3.6. *Let  $\lambda, \mu \vdash N$  be partitions of  $N$ . Let  $\lambda'_i = |\{j \mid \lambda_j \geq i\}|$  denote the length of the  $i$ -th column in the Young diagram of  $\lambda$ . We say that  $\mu$  is larger than  $\lambda$  in the conjugate order, and denote  $\mu' > \lambda'$ , if there exists  $i$  such that  $\mu'_j = \lambda'_j$  for all  $j < i$  and  $\mu'_i > \lambda'_i$ .*

Following Hamaker, Pawlowski and Sagan [16, Section 5], we define the *column superstandard* Young tableau of shape  $\lambda \vdash N$ , obtained by filling the columns of the Young diagram of shape  $\lambda$  one by one. We denote it by  $T_\lambda \in \text{SYT}(\lambda)$ . Formally,  $(T_\lambda)_{i,j} = \lambda'_1 + \dots + \lambda'_{j-1} + i$ . For example, if  $\lambda = (4, 2, 2, 1)$ , then  $T_\lambda$  is the SYT in Figure 3.

1	5	8	9
2	6		
3	7		
4			

FIGURE 3. The SYT  $T_\lambda$  for  $\lambda = (4, 2, 2, 1)$ .

The power of these notions may be reflected by the following statement:

LEMMA 3.7. *Let  $\lambda, \mu \vdash N$  be two partitions. Then:*

- (1)  $\text{Des}(T_\lambda) = [N - 1] \setminus \{\lambda'_1, \lambda'_1 + \lambda'_2, \dots\}$ .
- (2) *If  $\text{Des}(T) = [N - 1] \setminus \{\lambda'_1, \lambda'_1 + \lambda'_2, \dots\}$  for some  $T \in \text{SYT}(\mu)$ , then  $\mu' \geq \lambda'$ . Furthermore, if  $\mu = \lambda$  then  $T = T_\lambda$ .*

*Proof.* The first assertion is obvious, so let us focus on the second assertion.

Let  $T \in \text{SYT}(\mu)$  be a tableau, and assume that  $\text{Des}(T) = \text{Des}(T_\lambda)$  and  $T \neq T_\lambda$ . We aim to show that  $\mu' > \lambda'$ . Let us denote the first column that differs between  $T$  and  $T_\lambda$  by  $i$ . The  $i$ -th column of  $T_\lambda$  contains the entries  $s + 1, \dots, s + \lambda'_i$ , where  $s = \lambda'_1 + \dots + \lambda'_{i-1}$ . To prove  $\mu' > \lambda'$ , it suffices to show that all these entries also appear in the  $i$ -th column of  $T$ .

Assume, by contradiction, that some of these entries appear in other columns of  $T$ . Let  $x$  be the minimal such entry, and let  $j \neq i$  be the column of  $T$  that contains  $x$ . Since  $s + 1$  appears in the  $i$ -th column of  $T$ , we may assume that  $x > s + 1$ . If  $j < i$ , this contradicts the assumption that  $T$  and  $T_\lambda$  agree in the first  $i - 1$  columns. On the other hand, if  $j > i$ , then  $x$  appears in the first row of  $T$ , since it is the minimal entry of the  $j$ -th column of  $T$ . Consequently, we obtain that  $x - 1 \notin \text{Des}(T)$ , while  $x - 1 \in \text{Des}(T_\lambda)$ , which contradicts the assumption that  $\text{Des}(T) = \text{Des}(T_\lambda)$ .  $\square$

Lemma 3.7 associates the Young diagram of shape  $\lambda$  with the set  $\text{Des}(T_\lambda)$ . As we will see later, this association is powerful in analysing the Schur coefficients of symmetric sets.

Standard Young tableaux with at most 2 rows have a slightly stronger property:

LEMMA 3.8. *Let  $\lambda = (N - k_1, k_1)$  and  $\mu = (N - k_2, k_2)$  be two partitions of  $N$  with at most two parts each. Then:*

- (1)  $\text{Des}(T_\lambda) = \text{odds}(k_1)$ .
- (2) *If  $\text{Des}(T) = \text{odds}(k_1)$  for some  $T \in \text{SYT}(\mu)$ , then  $k_1 = k_2$  and  $T = T_\lambda$ .*

*Proof.* The first assertion is obvious, so let us focus on the second assertion.

Let  $T \in \text{SYT}(\mu)$  be a tableau, and suppose that  $\text{Des}(T) = \text{Des}(T_\lambda)$ . Given that both  $T$  and  $T_\lambda$  have two rows each, it suffices to show that  $\text{row}_2(T) = \text{row}_2(T_\lambda)$ . Since  $\text{Des}(T) = \{1, 3, \dots, 2k_1 - 1\}$ , it follows that  $\{1, 3, \dots, 2k_1 - 1\} \subseteq \text{row}_1(T)$  and  $\{2, 4, \dots, 2k_1\} \subseteq \text{row}_2(T)$ . Consequently, we have  $2k_1 + 1 \in \text{row}_1(T)$ . As  $T$  has no descents beyond index  $2k_1 - 1$ , all entries greater than  $2k_1$  must appear in the first row. Therefore, we conclude that  $\text{row}_2(T) = \{2, 4, \dots, 2k_1\}$ .  $\square$

Now we are ready to prove that if a set is symmetric with respect to a sparse statistic then it is Schur-positive:

LEMMA 3.9. *Let  $\mathcal{A}$  be a symmetric set with respect to a sparse statistic  $D : \mathcal{A} \rightarrow 2^{[N-1]}$ , and denote  $n = \lfloor \frac{N}{2} \rfloor$ . Then  $\mathcal{A}$  is Schur-positive, and its Schur expansion is*

$$\mathcal{Q}_D(\mathcal{A}) = \sum_{k=0}^n |\mathcal{A}(D = \text{odds}(k))| s_{N-k,k}.$$

*Proof.* The set  $\mathcal{A}$  is assumed to be symmetric, so by Theorem 2.5, we have

$$(7) \quad \sum_{a \in \mathcal{A}} \mathbf{t}^{D(a)} = \sum_{\lambda \vdash N} c_\lambda \sum_{T \in \text{SYT}(\lambda)} \mathbf{t}^{\text{Des}(T)},$$

where  $c_\lambda$  are the Schur coefficients of  $\mathcal{A}$ . It suffices to show that if there exists  $k$  such that  $\lambda = (N - k, k)$  then  $c_\lambda = |\mathcal{A}(D = \text{odds}(k))|$ , and otherwise  $c_\lambda = 0$ .

If  $\mathcal{A} = \emptyset$ , then  $\mathcal{Q}_D(\mathcal{A}) = 0$ , and the statement holds. Therefore, we may assume that  $\mathcal{A} \neq \emptyset$ , and consequently, there exists  $c_\lambda \neq 0$  for some partition  $\lambda \vdash N$ . Let  $\mu \vdash N$  be a partition with  $c_\mu \neq 0$ , maximal in the conjugate order (i.e. such that  $c_\lambda = 0$  for every  $\lambda \vdash N$  with  $\lambda' > \mu'$ ).

Equation (7) implies that the equation

$$(8) \quad |\mathcal{A}(D = J)| = \sum_{\lambda \vdash N} c_\lambda |\text{SYT}(\lambda)(\text{Des} = J)|$$

holds for all  $J \subseteq [N - 1]$ . Let us find the right-hand side of Equation (8) when substituting  $J = \text{Des}(T_\mu)$ . Let  $T \in \text{SYT}(\lambda)$  be a tableau with  $\text{Des}(T) = \text{Des}(T_\mu)$ . By Lemma 3.7, we have  $T = T_\mu$  or  $\lambda' > \mu'$ . However, if  $\lambda' > \mu'$  then  $c_\lambda = 0$ . Therefore, if  $T \in \text{SYT}(\lambda)$  has  $\text{Des}(T) = \text{Des}(T_\mu)$  and  $c_\lambda \neq 0$ , then  $T = T_\mu$ . Consequently, substituting  $J = \text{Des}(T_\mu)$  into Equation (8), we find that  $|\mathcal{A}(D = \text{Des}(T_\mu))| = c_\mu \neq 0$ .

We may conclude that there exists an element  $a \in \mathcal{A}$  with  $D(a) = \text{Des}(T_\mu)$ . The set  $D(a)$  is sparse, so  $\{1, 2\} \not\subseteq D(a)$ . Lemma 3.7 implies that  $\{1, 2\} \subseteq \text{Des}(T_\mu)$  whenever  $\mu'_1 > 2$ , so we may deduce that  $\mu'_1 \leq 2$ . Therefore,  $c_\lambda = 0$  for every partition  $\lambda$  with more than 2 parts.

Thus, we can reformulate Equation (8) as

$$(9) \quad |\mathcal{A}(D = J)| = \sum_{k=0}^n c_k |\text{SYT}(N - k, k)(\text{Des} = J)|.$$

By Lemma 3.8, since only partitions into at most two parts are involved in Equation (9), substituting  $J = \text{Des}(T_\lambda)$  for  $\lambda = (N - k, k)$  yields  $|\mathcal{A}(D = \text{odds}(k))| = c_k$ , as required.  $\square$

*Second proof of Lemma 3.2.* The lemma follows directly from Lemma 3.5 together with Lemma 3.9.  $\square$

#### 4. SHORT CHORDS OF MATCHINGS

In this section, we analyze the set of matchings  $\mathcal{M}_{N,f}$  with respect to short chords (Recall Definition 1.3 and Definition 1.5). First, we apply Theorem 1.8 to establish Theorem 1.6, which asserts that  $\mathcal{M}_{N,f}$  is Schur-positive. Next, we provide a bijective proof of Theorem 1.6, which will be utilized in Section 5 to refine the Schur-positivity result. Finally, we demonstrate that the Schur expansion of  $\mathcal{Q}_{\text{Short}}(\mathcal{M}_{N,f})$  may be explicitly interpreted in terms of Bessel polynomials.

##### 4.1. FIRST PROOF OF THEOREM 1.6.

Clearly, the function  $\text{Short} : \mathcal{M}_{N,f} \rightarrow 2^{[N-1]}$  is sparse (as defined in Definition 1.7). Let  $J = \{j_1, \dots, j_k\} \subseteq [N - 1]$  be a sparse set, and let us enumerate the elements of  $\mathcal{M}_{N,f}(\text{Short} \supseteq J)$ . In every matching in  $\mathcal{M}_{N,f}(\text{Short} \supseteq J)$ , the vertices  $j_i$  and  $j_i + 1$  are matched for all  $1 \leq i \leq k$ , and the remaining  $N - 2k$  vertices can be matched in any way, subject to the condition that exactly  $f$  vertices remain unmatched. Therefore, we have

$$|\mathcal{M}_{N,f}(\text{Short} \supseteq J)| = |\mathcal{M}_{N-2k,f}|,$$

and thus it depends only on the size of  $J$ .

By applying Theorem 1.8, we conclude that  $\mathcal{M}_{N,f}$  is Schur-positive with respect to Short, with the following Schur expansion:

$$\mathcal{Q}_{\text{Short}}(\mathcal{M}_{N,f}) = \sum_{k=0}^n |\mathcal{M}_{N,f}(\text{Short} = \{1, 3, 5, \dots, 2k - 1\})| s_{N-k,k}.$$

(While Theorem 1.8 sums over  $0 \leq k \leq \lfloor \frac{N}{2} \rfloor$ , this sum only goes up to  $n = \frac{N-f}{2}$ . However, the extra summands equal 0 and do not affect the expression.) Clearly,

$$|\mathcal{M}_{N,f}(\text{Short} = \{1, 3, 5, \dots, 2k - 1\})| = |\mathcal{M}_{N-2k,f}(\text{Short} = \emptyset)|,$$

and we obtain the required Schur expansion.  $\square$

##### 4.2. BIJECTIVE PROOF OF THEOREM 1.6.

Before presenting the bijective proof of Theorem 1.6, we give some definitions and notations for matchings:

**DEFINITION 4.1.** *Given a matching  $m \in \mathcal{M}_N$ , we say that a set  $S \subseteq [N]$  is  $m$ -invariant if for every chord  $(i, j) \in m$  we have  $i \in S$  if and only if  $j \in S$ . Moreover, for a matching  $m$  and an  $m$ -invariant set  $S$  we define the restriction of  $m$  to  $S$ , denoted  $\text{res}_S(m)$ , to be a matching on  $S$  which is obtained by removing all the vertices not in  $S$  from  $m$ .*

For example, consider the matching  $m_1 = \{(1, 2), (3, 5), (4)\} \in \mathcal{M}_{5,1}$  as in Figure 4A. The set  $S_1 = \{1, 2, 4\}$  is  $m_1$ -invariant, and  $\text{res}_{S_1}(m_1) = \{(1, 2), (4)\}$ . On the other hand, the set  $S_2 = \{1, 2, 3\}$  is not  $m_1$ -invariant, so  $\text{res}_{S_2}(m_1)$  is not defined.

**OBSERVATION 4.2.** *Let  $S \subseteq [N]$  be a set of vertices, let  $m_1$  be a matching on  $S$ , and let  $m_2$  be a matching on  $[N] \setminus S$ . Then there exists a unique matching  $m \in \mathcal{M}_N$  such that  $\text{res}_S(m) = m_1$  and  $\text{res}_{[N] \setminus S}(m) = m_2$ .*

Now let us present a bijection

$$F : \mathcal{M}_{N,f} \rightarrow \bigcup_{k=0}^n \mathcal{M}_{N-2k,f}(\text{Short} = \emptyset) \times \text{SYT}(N - k, k),$$

that sends matchings  $m \in \mathcal{M}_{N,f}$  to pairs  $(m_0, T)$ , where  $m_0$  is a short-chord-free matching on  $[N - 2k]$  with  $f$  unmatched vertices and  $T \in \text{SYT}(N - k, k)$  for some  $0 \leq k \leq n$ , such that  $\text{Des } T = \text{Short } m$  for all  $m$ . By Theorem 2.5, the existence of such a bijection implies Theorem 1.6.

4.2.1. *Constructing the bijection.* First, we define the core of a matching:

DEFINITION 4.3. *The reduction process for a given matching  $m$  repeatedly removes any short chords of the matching until there are no short chords left. The remaining vertices are then re-indexed with natural numbers starting from 1 while keeping their relative order, resulting in a matching denoted by  $\text{core}(m)$ .*

*A chord or vertex of  $m$  is called stable if it is not removed during the process, and unstable otherwise. The set of stable vertices of  $m$  is denoted  $\text{Stable}(m)$ .*

Proposition 4.4 below implies that  $\text{core}(m)$  and  $\text{Stable}(m)$  are well-defined and are independent of the order of the steps.

For a matching  $m \in \mathcal{M}_{N,f}$ , we also define  $T(m)$  as the unique SYT consisting of  $N$  cells arranged in two rows, such that

$$\text{row}_2(T) = \{j \mid \text{the chord } (i, j) \in m \text{ is unstable}\}.$$

Recall that when writing  $(i, j) \in m$  we assume that  $i < j$ . Therefore, for every unstable chord  $(i, j) \in m$  with  $i < j$ , we have  $i \in \text{row}_1(T(m))$  and  $j \in \text{row}_2(T(m))$ . We define the bijection by  $F(m) = (\text{core}(m), T(m))$ .

Next, we turn to provide examples of the bijection. Then, in the remaining of the subsection, we will prove that the bijection is well-defined and explore some of its properties. Section 4.2.2 will be devoted to proving that  $F$  is bijective by constructing its inverse function.

As a first example, consider the matching  $m_1 = \{(1, 2), (3, 5), (4)\} \in \mathcal{M}_{5,1}$  as in Figure 4A.

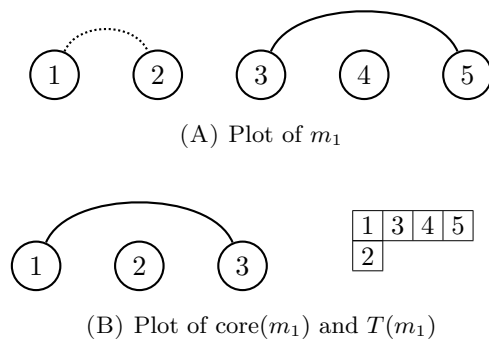


FIGURE 4. Example of the bijection for the matching  $m_1 = \{(1, 2), (3, 5), (4)\} \in \mathcal{M}_{5,1}$

During the reduction process of  $m_1$ , the short chord  $(1, 2)$  is removed. Then, the vertices 3, 4, 5 remain, so  $\text{Stable}(m_1) = \{3, 4, 5\}$ . Next, the stable vertices are renumbered to  $\{1, 2, 3\}$ , so  $\text{core}(m_1) = \{(1, 3), (2)\}$ , as in Figure 4B. In addition, the only unstable chord of  $m_1$  is  $(1, 2)$ , so  $\text{row}_2(T(m_1)) = \{2\}$ . Therefore,  $T(m_1)$  is the tableau presented in Figure 4B.

Next, consider the matching  $m_2 = \{(1, 7), (2, 10), (3, 6), (4, 5), (8, 9)\} \in \mathcal{M}_{10,0}$  as in Figure 5A.

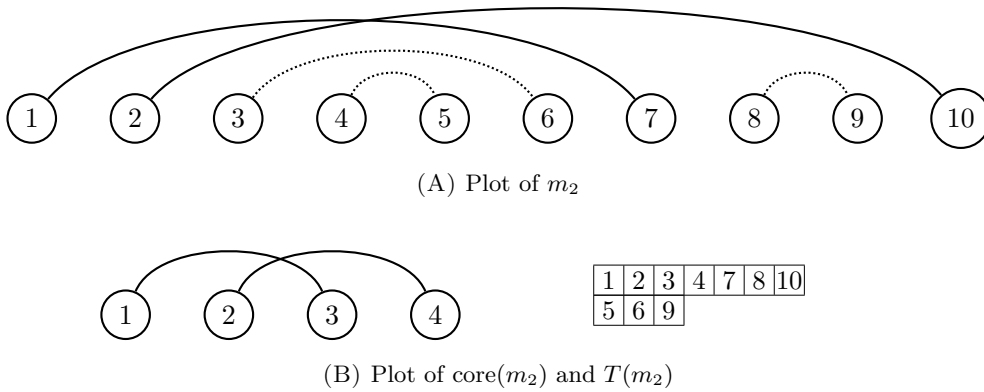


FIGURE 5. Example of the bijection for the matching  $m_2 = \{(1, 7), (2, 10), (3, 6), (4, 5), (8, 9)\} \in \mathcal{M}_{10,0}$

During the reduction process of  $m_2$ , we first remove the short chords  $(4, 5)$  and  $(8, 9)$ . Then, the chord  $(3, 6)$  becomes short and is subsequently removed as well. The remaining vertices are  $1, 2, 7$  and  $10$ , so  $\text{Stable}(m_2) = \{1, 2, 7, 10\}$ . Next, the stable vertices are renumbered to  $[4]$ , so  $\text{core}(m_2) = \{(1, 3), (2, 4)\}$ , as in Figure 5B. In addition, the unstable chords of  $m_2$  are  $(3, 6)$ ,  $(4, 5)$  and  $(8, 9)$ , so  $\text{row}_2(T(m_2)) = \{5, 6, 9\}$ . Therefore,  $T(m_2)$  is the tableau presented in Figure 5B.

Moving on to proving that the bijection  $F$  is well-defined, first let us prove that the core of a matching is well-defined:

PROPOSITION 4.4. *Given a matching, its stable vertices and its core are well-defined.*

*Proof.* Let  $m \in \mathcal{M}_N$  be a matching. Denote the chords that are removed during a given reduction process of  $m$  by  $e_1, \dots, e_k$ , and denote  $e_\ell = (i_\ell, j_\ell)$ . Moreover, denote the chords that are removed during another reduction process by  $e'_1, \dots, e'_{k'}$ , and denote  $e'_\ell = (i'_\ell, j'_\ell)$ . It suffices to prove that  $e_\ell \in \{e'_1, \dots, e'_{k'}\}$  for all  $1 \leq \ell \leq k$  (i.e. every chord that is removed during the first reduction process is removed during the second process as well). Assume by contradiction that  $e_\ell \notin \{e'_1, \dots, e'_{k'}\}$  for some  $\ell$ , and denote by  $\ell_0$  the minimal such  $\ell$ . That is,  $e_1, \dots, e_{\ell_0-1} \in \{e'_1, \dots, e'_{k'}\}$ . Removing the chords  $e_1, \dots, e_k$  from  $m$  constitutes a valid reduction process, so the chord  $e_{\ell_0}$  becomes short before it is removed. That is,  $i \in \{i_1, j_1, \dots, i_{\ell_0-1}, j_{\ell_0-1}\}$  for all  $i_{\ell_0} < i < j_{\ell_0}$ . Therefore, the chord  $e_{\ell_0}$  is a short chord of the core obtained by removing  $e'_1, \dots, e'_{k'}$  of  $m$ , contradicting the requirement that the reduction process continues until there are no short chords remaining. Therefore, we may conclude that if a chord is removed during a reduction process then it is removed during any reduction process, and  $\text{core}(m)$  and  $\text{Stable}(m)$  are well-defined.  $\square$

COROLLARY 4.5. *Let  $N$  and  $f$  be nonnegative integers. Then the function*

$$F : \mathcal{M}_{N,f} \rightarrow \bigcup_{k=0}^n \mathcal{M}_{N-2k,f}(\text{Short} = \emptyset) \times \text{SYT}(N - k, k)$$

*defined by  $F(m) = (\text{core}(m), T(m))$  is well-defined.*

*Proof.* Let  $m \in \mathcal{M}_{N,f}$  be a matching. By Proposition 4.4, we obtain that  $\text{core}(m)$  and  $\text{Stable}(m)$  are well-defined. Therefore, the two-row tableau  $T := T(m)$  consisting of  $N$  cells that is defined by

$$\text{row}_2(T) = \{j \mid \text{the chord } (i, j) \in m \text{ is unstable}\}$$

is also well-defined. In order to prove that  $F$  is well-defined, it remains to show that  $T(m) \in \text{SYT}(N - k, k)$ , where  $k$  is the number of unstable chords of  $m$ .

Obviously, every letter  $i \in [N]$  appears exactly once in  $T(m)$ , and the rows are increasing. Notice that for every entry  $j \in \text{row}_2(T(m))$  there exists  $i < j$  such that  $(i, j) \in m$  is an unstable chord. Therefore, every entry  $j \in \text{row}_2(T(m))$  is associated to an entry  $i \in \text{row}_1(T(m))$  such that  $i < j$ . Moreover, if  $j \neq j' \in \text{row}_2(T(m))$  and  $(i, j), (i', j') \in m$  then  $i \neq i'$ , so the columns of  $T(m)$  are increasing and  $T(m)$  is a standard Young tableau. Finally,  $|\text{row}_2(T(m))| = k$ , implying that  $T(m) \in \text{SYT}(N - k, k)$ .  $\square$

After establishing that  $F$  is a valid function, we turn our attention to exploring some of its properties.

LEMMA 4.6. *The reduction process has the following properties:*

- (1) *If a chord intersects another chord then it is stable.*
- (2) *A chord  $(i, j)$  is stable if and only if there exists a stable vertex in  $\{i+1, \dots, j-1\}$ .*
- (3) *Given a matching  $m$  and  $i < j$ , if the set  $[i, j]$  is  $m$ -invariant (as defined in Definition 4.1) and the restricted matching  $\text{res}_{[i,j]}(m)$  is perfect and non-crossing, then  $\ell \notin \text{Stable}(m)$  for all  $\ell \in [i, j]$ .*

*Proof.*

- (1) Assume that  $(i_1, i_3), (i_2, i_4) \in m$  for  $i_1 < i_2 < i_3 < i_4$ . As long as the chord  $(i_2, i_4)$  is not removed, the chord  $(i_1, i_3)$  does not become short and cannot be removed. On the other hand, as long as the chord  $(i_1, i_3)$  is not removed, the chord  $(i_2, i_4)$  cannot be removed. Therefore, both chords cannot be removed during the reduction process.
- (2) If the chord  $(i, j)$  is unstable, then after removing some unstable vertices it becomes short, implying that every vertex between  $i$  and  $j$  is unstable. On the other hand, if all the vertices between  $i$  and  $j$  are unstable, then they will eventually be removed, making the chord  $(i, j)$  short, so the chord  $(i, j)$  is unstable too.
- (3) Since the set  $[i, j]$  is  $m$ -invariant and the restricted matching is perfect, Definition 4.1 implies that for every  $i \leq i_1 \leq j$  there exists  $i \leq i_2 \leq j$  such that  $i_1 \neq i_2$  and  $(i_1, i_2) \in m$  or  $(i_2, i_1) \in m$ . Notice that the restricted matching  $\text{res}_S(m)$  where  $S = \text{Stable}(m) \cap [i, j]$  (i.e. the matching that consists of the stable chords  $(i_1, i_2) \in m$  with  $i \leq i_1 < i_2 \leq j$ ) is short-chord-free, non-crossing and perfect. The only such a matching is the empty matching  $\emptyset \in \mathcal{M}_0$ , so every  $i \leq i_1 \leq j$  is unstable.  $\square$

As we proceed to apply Theorem 2.5 and establish the Schur-positivity of  $\mathcal{M}_{N,f}$  with respect to short chords, let us prove that  $F$  sends short chords of matchings to descents of SYTs, in the following sense:

PROPOSITION 4.7. *Let  $m \in \mathcal{M}_{N,f}$  be a matching, and denote  $T := T(m)$ . Then  $\text{Des } T = \text{Short } m$ .*

*Proof.* Let  $i \in \text{Short } m$  be an index representing a short chord  $(i, i+1) \in m$ . The chord  $(i, i+1)$  is unstable, so we may deduce from the definition of  $T(m)$  that  $i \in \text{row}_1(T)$  and  $i+1 \in \text{row}_2(T)$ . Therefore,  $i \in \text{Des } T$ .



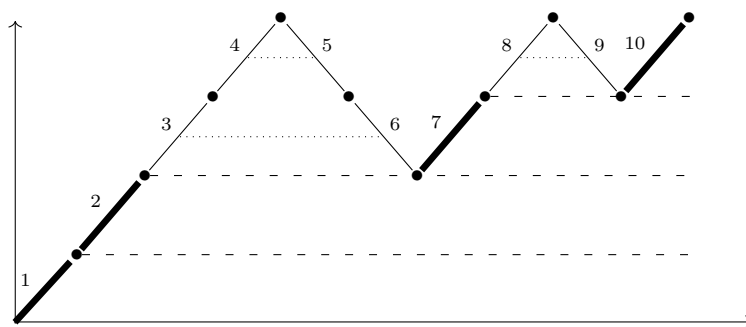


FIGURE 6. A ballot path  $p \in \mathcal{P}_{10,4}$

On the other hand, let  $i \in \text{Des } T$  be a descent of  $T$ . This implies that  $i \in \text{row}_1(T)$  and  $i + 1 \in \text{row}_2(T)$ . Since  $i + 1 \in \text{row}_2(T)$ , we obtain that there exists  $j < i + 1$  such that  $(j, i + 1) \in m$  is an unstable chord. Since the chord  $(j, i + 1)$  is unstable and  $j \leq i \leq i + 1$ , we obtain by Lemma 4.6 (part 2) that  $i$  is an endpoint of an unstable chord of  $m$ . Since  $i \in \text{row}_1(T)$ , we obtain that  $i$  opens an unstable chord  $(i, j')$  of  $m$  for some  $j' > i$ . If  $(i, i + 1) \notin m$  then  $j < i$  and  $j' > i + 1$ , and we obtain that the unstable chord  $(j, i + 1)$  intersects the chord  $(i, j')$ , contradicting Lemma 4.6 (part 1). Therefore, we may conclude that  $(i, i + 1) \in m$  and  $i \in \text{Short } m$ .  $\square$

4.2.2. *Proof of bijection.* In this section we will prove that the transformation  $F$  defined in Section 4.2.1 is indeed a bijection, by constructing its inverse function.

In order to construct the inverse function, we will establish a correspondence between standard Young tableaux of two rows and ballot paths. We define ballot paths as follows:

DEFINITION 4.8. *Let  $N \in \mathbb{N}$  be a nonnegative integer. A ballot path of length  $N$  is a sequence of  $N$  steps, where each step is either  $(1, 1)$  or  $(1, -1)$ . The path starts at the origin  $(0, 0)$ . Namely, each step either moves one unit up and one unit right, or one unit down and one unit right. The path is said to be valid if it never goes below the  $x$ -axis, i.e. the  $y$ -coordinate of a point on the path is always non-negative.*

*The set of ballot paths from  $(0, 0)$  to  $(N, t)$  is denoted  $\mathcal{P}_{N,t}$ . Given a ballot path  $p \in \mathcal{P}_{N,t}$ , denote by  $p_i$  the  $y$ -coordinate of  $p$  after  $i$  steps; in particular,  $p_0 = 0$ . The set  $\text{UP}(p) \subseteq [N]$  ( $\text{DOWN}(p) \subseteq [N]$ ) consists of the indices  $i$  such that  $p_i > p_{i-1}$  (respectively,  $p_i < p_{i-1}$ ). Finally, define the height of the  $i$ -th step of a path  $p$  to be the maximum height of its two endpoints, and denote it by  $\text{height}_p(i) := \max(p_{i-1}, p_i)$ .*

The bijection between SYTs of two rows and ballot paths is direct: Associate  $T \in \text{SYT}(N - k, k)$  with the path  $p(T) \in \mathcal{P}_{N, N-2k}$  such that  $\text{UP}(p(T)) = \text{row}_1(T)$  and  $\text{DOWN}(p(T)) = \text{row}_2(T)$ .

For example, consider the tableau  $T \in \text{SYT}(7, 3)$  described in Figure 5B, with  $\text{row}_2(T) = \{5, 6, 9\}$ . It is associated with the ballot path  $p := p(T) \in \mathcal{P}_{10,4}$  presented in Figure 6. The ballot path  $p$  has, for example,  $2 \in \text{UP}(p)$  because  $p_2 = 2 > p_1 = 1$ . Conversely,  $5 \in \text{DOWN}(p)$  because  $p_5 = 3 < p_4 = 4$ . In addition,  $\text{height}_p(3) = \max(p_2, p_3) = 3$  while  $\text{height}_p(5) = \max(p_4, p_5) = 4$ .

Next, for a given ballot path  $p \in \mathcal{P}_{N,t}$ , we construct a set  $\text{Stable}(p)$  and a perfect matching  $m_{\text{unstable}}(p)$  on  $[N] \setminus \text{Stable}(p)$  as follows: For every vertex  $j \in \text{DOWN}(p)$ , we match it in  $m_{\text{unstable}}(p)$  to the maximal  $i < j$  such that  $\text{height}_p(i) = \text{height}_p(j)$ . Since  $p$  is a valid ballot path, it can be deduced that for every  $j \in \text{DOWN}(p)$ , there

exists a unique  $i < j$  satisfying  $\text{height}_p(i) = \text{height}_p(j)$  such that  $i$  is maximal among all such elements in  $[N]$ . Additionally, it can be inferred from the discrete continuity of the path that this  $i$  is necessarily a part of  $\text{UP}(p)$ . The set  $\text{Stable}(p)$  consists of all  $i \in \text{UP}(p)$  that are not involved in any chord in  $m_{\text{unstable}}(p)$ . Notice that  $i \in \text{Stable}(p)$  if and only if  $i \in \text{UP}(p)$  and  $\text{height}_p(j) > \text{height}_p(i)$  for all  $j > i$ , so  $|\text{Stable}(p)| = t$ .

We are now ready to describe the inverse bijection of  $F$ , denoted

$$\tilde{F} : \bigcup_{k=0}^n \mathcal{M}_{N-2k,f}(\text{Short} = \emptyset) \times \text{SYT}(N - k, k) \rightarrow \mathcal{M}_{N,f}.$$

Given a short-chord-free matching  $m_0 \in \mathcal{M}_{N-2k,f}(\text{Short} = \emptyset)$  and a tableau  $T \in \text{SYT}(N - k, k)$  for some  $k$ , denote  $p = p(T) \in \mathcal{P}_{N,N-2k}$ . We will construct a matching  $m_{\text{stable}}$  on  $\text{Stable}(p)$  and a matching  $m_{\text{unstable}}$  on  $[N] \setminus \text{Stable}(p)$ , and then apply Observation 4.2 to obtain  $\tilde{F}(m_0, T) \in \mathcal{M}_{N,f}$ . We construct these sub-matchings as follows:

- $m_{\text{stable}}$ : Since  $p \in \mathcal{P}_{N,N-2k}$ , we infer that  $|\text{Stable}(p)| = N - 2k$ . Therefore, we may rename the vertices of  $m_0 \in \mathcal{M}_{N-2k,f}$  to  $\text{Stable}(p)$  as follows: There exists a unique bijection  $\varphi : [N - 2k] \rightarrow \text{Stable}(p)$  such that  $i < j$  if and only if  $\varphi(i) < \varphi(j)$  for all  $i, j$ . The matching  $m_{\text{stable}}$  on  $\text{Stable}(p)$  consists of the chords  $(\varphi(i), \varphi(j))$  for all  $(i, j) \in m_0$ .
- $m_{\text{unstable}} = m_{\text{unstable}}(p)$  is the matching described earlier.

For example, consider  $m_0 = \{(1, 3), (2, 4)\} \in \mathcal{M}_{4,0}(\text{Short} = \emptyset)$  and  $T \in \text{SYT}(N - k, k)$  presented in Figure 5B. As mentioned before,  $T$  is associated with the ballot path  $p := p(T) \in \mathcal{P}_{10,4}$  presented in Figure 6. Therefore, we obtain the matching  $m_{\text{unstable}}(p) = \{(3, 6), (4, 5), (8, 9)\}$  (with the pairs of steps that correspond to its chords connected by dotted lines in the figure) and  $\text{Stable}(p) = \{1, 2, 7, 10\}$  (with the steps that correspond to these vertices denoted by bold lines). Applying the order-preserving bijection  $\varphi : [4] \rightarrow \{1, 2, 7, 10\}$  on  $m_0$  yields the matching  $m_{\text{stable}} = \{(1, 7), (2, 10)\}$ . Therefore,

$$\tilde{F}(m_0, T) = \{(1, 7), (2, 10), (3, 6), (4, 5), (8, 9)\}$$

is the matching presented in Figure 5A.

It remains to show that  $\tilde{F}$  is indeed the inverse function of  $F$ . We will do so in two steps.

**Step 1:**  $\tilde{F} \circ F = \text{Id}$ .

LEMMA 4.9. *Let  $m \in \mathcal{M}_{N,f}$  be a matching, and denote  $F(m) = (\text{core}(m), T)$ . Then  $\tilde{F}(\text{core}(m), T) = m$ .*

*Proof.* Denote  $|\text{Stable}(m)| = N - 2k$ , implying that  $\text{core}(m) \in \mathcal{M}_{N-2k,f}$  and  $T \in \text{SYT}(N - k, k)$ . Moreover, denote  $p = p(T) \in \mathcal{P}_{N,N-2k}$ . First, we prove that

$$(10) \quad \text{res}_{[N] \setminus \text{Stable}(m)}(m) = m_{\text{unstable}}(p).$$

We note that the set  $[N] \setminus \text{Stable}(m)$  is  $m$ -invariant, so the left-hand side of Equation (10) is well-defined. Notice that  $|[N] \setminus \text{Stable}(m)| = |[N] \setminus \text{Stable}(p)| = 2k$  and that  $\text{res}_{[N] \setminus \text{Stable}(m)}(m)$  is a perfect matching. Therefore, in order to prove that Equation (10) holds, it suffices to prove that any unstable chord of  $m$  belongs to  $m_{\text{unstable}}(p)$  too. Let  $(i, j) \in m$  be an unstable chord. Thus, we may deduce that  $i \in \text{UP}(p)$  and  $j \in \text{DOWN}(p)$ . Let  $i' \in \text{UP}(p)$  such that  $i < i' < j$ . By Lemma 4.6 (part 2),  $i'$  is an endpoint of an unstable chord  $(i', j') \in m$  with  $i' < j'$ . By Lemma 4.6 (part 1), we obtain that the chord  $(i, j)$  does not intersect  $(i', j')$ , implying that  $i < i' < j' < j$ . On the other hand, every  $j' \in \text{DOWN}(p)$  with  $i < j' < j$  is an endpoint of an unstable chord  $(i', j') \in m$  with  $i < i' < j' < j$ . Therefore, we obtain a bijection from

$\text{UP}(p) \cap [i, j]$  to  $\text{DOWN}(p) \cap [i, j]$  (with  $i$  matched with  $j$ ), so  $\text{height}_p(i) = \text{height}_p(j)$ . Moreover, this bijection has the property that if  $i' \in \text{UP}(p) \cap [i, j]$  is matched with  $j' \in \text{DOWN}(p) \cap [i, j]$  then  $i' < j'$  (i.e. the ascending steps of the path appear before the associated descending steps), so  $\text{height}_p(j) < \text{height}_p(i')$  for all  $i' \in \text{UP}(p) \cap [i + 1, j - 1]$ . We may conclude that  $(i, j) \in m_{\text{unstable}}(p)$  for every unstable chord  $(i, j)$  of  $m$  and prove that Equation (10) holds.

Next, we denote  $m' = \tilde{F}(\text{core}(m), T)$  and prove that  $m = m'$ . From Equation (10) we deduce that the supports of the matchings  $\text{res}_{[N] \setminus \text{Stable}(m)}(m)$  and  $m_{\text{unstable}}(p)$  are identical, and therefore  $\text{Stable}(m) = \text{Stable}(p)$ . Thus, the set  $[N] \setminus \text{Stable}(m)$  is both  $m$ -invariant and  $m'$ -invariant, and restricting each of these matchings to  $[N] \setminus \text{Stable}(m)$  results in  $m_{\text{unstable}}(p)$ . In addition, it can be easily verified from the descriptions of  $F$  and  $\tilde{F}$  that  $\text{res}_{\text{Stable}(m)}(m) = \text{res}_{\text{Stable}(m)}(m')$  is the matching obtained by relabeling the vertices of  $\text{core}(m)$  with the elements of  $\text{Stable}(m)$  in increasing order. By Observation 4.2, we may deduce that  $m = m'$ .  $\square$

**Step 2:**  $F \circ \tilde{F} = \text{Id}$ .

LEMMA 4.10. *Let  $m_0 \in \mathcal{M}_{N-2k, f}$  with  $\text{Short}(m_0) = \emptyset$  and  $T \in \text{SYT}(N - k, k)$  for some  $k$ , and denote  $m = \tilde{F}(m_0, T)$ . Then  $\text{core}(m) = m_0$  and  $T(m) = T$ .*

*Proof.* Denote  $p = p(T) \in \mathcal{P}_{N, N-2k}$  and  $\text{Stable}(p) = \{i_1, \dots, i_{N-2k}\}$  where  $i_1 < \dots < i_{N-2k}$ . We first prove that  $\text{Stable}(m) = \text{Stable}(p)$ . Notice that the matching  $m_{\text{unstable}}(p)$  is non-crossing. This can be viewed visually from Figure 6, where  $m_{\text{unstable}}(p)$  is denoted by horizontal dotted lines that cross the path only in their endpoints. Indeed, let  $j_1 < j_2 < j_3 < j_4$  be four vertices, and assume, by contradiction, that both  $(j_1, j_3)$  and  $(j_2, j_4)$  belong to  $m_{\text{unstable}}(p)$ . By the definition of  $m_{\text{unstable}}(p)$ , the assumption  $(j_1, j_3) \in m_{\text{unstable}}(p)$  implies that  $\text{height}_p(j_1) = \text{height}_p(j_3)$ , and  $\text{height}_p(j) \neq \text{height}_p(j_1)$  for all  $j_1 < j < j_3$ . Since  $\text{height}_p(j_1 + 1) \geq \text{height}_p(j_1)$  and due to the discrete continuity of the path, we obtain that  $\text{height}_p(j) > \text{height}_p(j_1)$  for all  $j_1 < j < j_3$ . Consequently, we obtain  $\text{height}_p(j_2) > \text{height}_p(j_1) = \text{height}_p(j_3)$ . Similarly, the assumption  $(j_2, j_4) \in m_{\text{unstable}}(p)$  implies that  $\text{height}_p(j_3) > \text{height}_p(j_2)$ , in contradiction. Therefore, we may conclude that the matching  $m_{\text{unstable}}(p)$  is non-crossing.

Next, we may infer that for every  $1 \leq \ell < N - 2k$ , the segment  $[i_\ell + 1, i_{\ell+1} - 1]$  is  $m$ -invariant and the restricted matching  $\text{res}_{[i_\ell + 1, i_{\ell+1} - 1]}(m)$  is perfect and non-crossing. By Lemma 4.6 (part 3), we obtain that  $j \notin \text{Stable}(m)$  for all  $i_\ell < j < i_{\ell+1}$ . Similarly, we obtain that if  $j < i_1$  or  $j > i_{N-2k}$  then  $j \notin \text{Stable}(m)$ , and therefore  $\text{Stable}(m) \subseteq \text{Stable}(p)$ . Thus, a valid reduction process of  $m$  may begin with removing every vertex not in  $\text{Stable}(p)$ . We may deduce from the description of  $\tilde{F}$  that  $\text{res}_{\text{Stable}(p)}(m)$  is the matching obtained by relabeling the vertices of  $m_0$  with the elements of  $\text{Stable}(p)$  in increasing order. This matching is short-chord-free, so  $\text{Stable}(m) = \text{Stable}(p)$  and  $\text{core}(m) = m_0$ .

It remains to prove that  $T(m) = T$ , namely that  $p' = p$ , where  $p' := p(T(m))$ . Since  $\text{Stable}(m) = \text{Stable}(p)$ , we may infer from the description of  $\tilde{F}$  that

$$\text{DOWN}(p) = \{j \mid \text{the chord } (i, j) \in m \text{ is unstable}\}.$$

Thus,  $\text{DOWN}(p) = \text{DOWN}(p')$  and therefore  $p = p'$ .  $\square$

Finally, we conclude the bijective proof:

*Bijjective proof of Theorem 1.6.* By Theorem 2.5, it suffices to prove that

$$\sum_{m \in \mathcal{M}_{N, f}} \mathbf{t}^{\text{Short}(m)} = \sum_{k=0}^N |\mathcal{M}_{N-2k, f}(\text{Short} = \emptyset)| \sum_{T \in \text{SYT}(N-k, k)} \mathbf{t}^{\text{Des}(T)},$$

where  $\mathbf{t}^J := \prod_{j \in J} t_j$  for  $J \subseteq [N - 1]$ . Equivalently, it suffices to present a bijection between  $\mathcal{M}_{N,f}$  and  $\bigcup_{k=0}^n \mathcal{M}_{N-2k,f}(\text{Short} = \emptyset) \times \text{SYT}(N - k, k)$ , such that if  $m \mapsto (m_0, T)$  then  $\text{Short}(m) = \text{Des}(T)$ . The transformation  $F$  defined by  $F(m) = (\text{core}(m), T(m))$  is a bijection by Lemma 4.9 combined with Lemma 4.10, and it satisfies  $\text{Short}(m) = \text{Des}(T(m))$  by Proposition 4.7.  $\square$

4.3. ANALYSIS OF THE COEFFICIENTS AND RELATIONS WITH BESSEL POLYNOMIALS. The *Bessel polynomials*  $\theta_n(x)$ , sometimes called the *reverse Bessel polynomials*, are given by the generating function

$$\frac{1}{\sqrt{1-2v}} \exp [x(1 - \sqrt{1-2v})] = \sum_{n=0}^{\infty} \frac{v^n}{n!} \theta_n(x).$$

They also have the explicit formula

$$\theta_n(x) = \sum_{k=0}^n \frac{(2n-k)!}{k!(n-k)!2^{n-k}} x^k.$$

For more information about the Bessel polynomials, the reader is referred to [15].

McSorley and Feinsilver [23, Theorem 3.5] discovered the following identity:

THEOREM 4.11 (McSorley and Feinsilver). *For every  $n \geq 0$ :*

$$\theta_n(x-1) = \sum_{i=0}^n h(P_{2n}, i) x^i,$$

where  $h(P_{2n}, i)$  is the number of perfect matchings on  $2n$  vertices with  $i$  short chords.

An equivalent formulation of this result states that

$$\theta_n(x) = \sum_{i=0}^n h(P_{2n}, i) (x+1)^i,$$

so the sequence  $h(P_{2n}, i)$  can be thought of as the coefficients of the Taylor expansion of  $\theta_n(x)$  around  $x = -1$ .

OBSERVATION 4.12. *For every  $n \geq 0$ :*

$$h(P_{2n}, i) = |\mathcal{M}_{2n-i,i}(\text{Short} = \emptyset)|.$$

*Proof.* We give a bijective proof for the statement. The bijection sends a perfect matching  $m \in \mathcal{M}_{2n,0}$  with  $i$  short chords to a short-chord-free matching on  $2n - i$  vertices with  $i$  unmatched vertices, by replacing every short chord with an unmatched vertex. For example, the perfect matching  $\{(1, 5), (2, 3), (4, 6)\} \in \mathcal{M}_{6,0}$  is sent to  $\{(1, 4), (2), (3, 5)\} \in \mathcal{M}_{5,1}$ . Clearly, this gives a bijection between the two desired sets.  $\square$

Therefore, we can reformulate Theorem 1.6 and obtain:

COROLLARY 4.13. *Let  $n, f \in \mathbb{N}$  be nonnegative integers, and denote  $N = 2n + f$ . Then the Schur expansion of the set  $\mathcal{M}_{N,f}$  with respect to Short is given by the formula*

$$\mathcal{Q}_{\text{Short}}(\mathcal{M}_{N,f}) = \sum_{k=0}^n h(P_{N+f-2k}, f) s_{N-k,k},$$

where  $h(P_{N+f-2k}, f)$  is the coefficient of  $(x+1)^f$  in the Taylor expansion of the Bessel polynomial  $\theta_{n+f-k}(x)$  around  $x = -1$

5. REFINEMENTS OF THE BIJECTION

In this section, we will utilize the bijection  $F$  discussed in Section 4.2 to refine Theorem 1.6 and find many Schur-positive sets of matchings with respect to the set of short chords. Indeed, given non-negative integers  $N$  and  $k$ , for every short-chord-free matching  $m_0 \in \mathcal{M}_{N-2k}(\text{Short} = \emptyset)$ , the set

$$\{m \in \mathcal{M}_N \mid \text{core}(m) = m_0\}$$

is Schur-positive (as we will see later in Corollary 5.9).

5.1. SETS CLOSED UNDER KNUTH EQUIVALENCE. As a first refinement of the Schur-positivity of  $\mathcal{M}_{N,f}$ , we study the Knuth equivalence of matchings presented in Definition 1.9. The power of this notion is reflected by the following theorem:

**THEOREM 5.1.** *Two matchings  $m_1, m_2 \in \mathcal{M}_N$  are Knuth equivalent if and only if  $\text{core}(m_1) = \text{core}(m_2)$ .*

We will prove Theorem 5.1 in two steps:

**Step 1: If two matchings are equivalent then they have the same core.**

**LEMMA 5.2.** *Let  $m_1, m_2 \in \mathcal{M}_N$  be two Knuth equivalent matchings. Then  $\text{core}(m_1) = \text{core}(m_2)$ .*

*Proof.* Since  $m_1$  and  $m_2$  are equivalent, we deduce that  $m_2$  can be obtained from  $m_1$  by a sequence of elementary Knuth transformations. Notably, given a matching  $m \in \mathcal{M}_N$  and a matching  $\varphi(m)$  obtained from  $m$  by applying an elementary Knuth transformation, the matchings  $m$  and  $\varphi(m)$  differ only in the relative position of a certain short chord, and therefore  $\text{core}(\varphi(m)) = \text{core}(m)$ . A direct induction shows that  $\text{core}(m_1) = \text{core}(m_2)$ .  $\square$

**Step 2: If two matchings have the same core then they are equivalent.**

**LEMMA 5.3.** *Let  $m_1, m_2 \in \mathcal{M}_N$  be matchings, and assume that  $\text{core}(m_1) = \text{core}(m_2)$ . Then  $m_1$  and  $m_2$  are Knuth equivalent.*

Before proving Lemma 5.3, we introduce the notion of inserting a short chord into a matching:

**DEFINITION 5.4.** *Let  $m \in \mathcal{M}_N$  be a matching, and let  $1 \leq i \leq N + 1$  be an index. Denote by  $\text{insert}_i(m) \in \mathcal{M}_{N+2}$  the matching obtained by inserting a short chord that matches the vertices  $i$  and  $i + 1$ , while pushing every vertex  $j \geq i$  to position  $j + 2$ . Formally, denote by  $f_i : [N] \rightarrow [N + 2] \setminus \{i, i + 1\}$  the function that is described as follows:*

$$f_i(j) = \begin{cases} j & \text{if } j < i, \\ j + 2 & \text{if } j \geq i. \end{cases}$$

*Then the matching  $\text{insert}_i(m)$  consists of the chords  $(f_i(j_1), f_i(j_2))$  for all  $(j_1, j_2) \in m$  together with  $(i, i + 1)$ , and consists of the unmatched vertices  $(f_i(j))$  for all  $(j) \in m$ .*

For example, if  $m = \{(1, 3), (2, 6), (4, 5)\}$ , then we obtain that  $\text{insert}_3(m) = \{(1, 5), (2, 8), (3, 4), (6, 7)\}$ .

The insertion function is closely related to Knuth equivalence, as demonstrated by the following lemmas:

**LEMMA 5.5.** *Let  $m \in \mathcal{M}_N$  be a matching and let  $1 \leq i, j \leq N + 1$  be indices. Then  $\text{insert}_i(m)$  is Knuth equivalent to  $\text{insert}_j(m)$ .*

*Proof.* We may assume without loss of generality that  $i \leq j$ , and prove the statement by induction on  $j - i$ . The statement is obvious for  $i = j$ .

Assume that  $i < j$ . By Definition 5.4, the matching  $\text{insert}_{i+1}(m)$  is Knuth equivalent to  $\text{insert}_i(m)$ . By the induction hypothesis,  $\text{insert}_{i+1}(m)$  is Knuth equivalent to  $\text{insert}_j(m)$  as well. Therefore,  $\text{insert}_i(m)$  is Knuth equivalent to  $\text{insert}_j(m)$ .  $\square$

LEMMA 5.6. *Let  $m_1, m_2 \in \mathcal{M}_N$  be Knuth equivalent matchings, and let  $1 \leq i \leq N + 1$  be an index. Then  $\text{insert}_i(m_1)$  and  $\text{insert}_i(m_2)$  are Knuth equivalent.*

*Proof.* By Lemma 5.5, we may assume that  $i = N + 1$ . Assume that  $m_2 = \varphi_1 \cdots \varphi_\ell(m_1)$  for some elementary Knuth transformations  $\varphi_1, \dots, \varphi_\ell$ . We prove the statement by induction on  $\ell$ . The statement is obvious for  $\ell = 0$ .

Next, assume that  $\ell > 0$ . By the induction hypothesis, we may assume that  $\text{insert}_{N+1}(m_2)$  is equivalent to  $\text{insert}_{N+1}(\varphi_\ell(m_1))$ . Therefore, it suffices to show that  $\text{insert}_{N+1}(m_1)$  is equivalent to  $\text{insert}_{N+1}(\varphi_\ell(m_1))$ . Assume that  $(j, j + 1), (j + 2, j') \in m_1$  for some  $1 \leq j, j' \leq N$ , and that  $(j, j'), (j + 1, j + 2) \in \varphi_\ell(m_1)$ , as other types of elementary Knuth transformations are handled similarly. Notice that  $\text{insert}_{N+1}(m_1)$  and  $\text{insert}_{N+1}(\varphi_\ell(m_1))$  have all but four chords in common. Specifically,  $\text{insert}_{N+1}(m_1)$  contains the chords  $(j, j + 1)$  and  $(j + 2, j')$ , while  $\text{insert}_{N+1}(\varphi_\ell(m_1))$  contains the chords  $(j, j')$  and  $(j + 1, j + 2)$ . Therefore, there exists an elementary Knuth transformation  $\varphi$ , such that  $\varphi(\text{insert}_{N+1}(m_1)) = \text{insert}_{N+1}(\varphi_\ell(m_1))$ . Thus, the matchings  $\text{insert}_{N+1}(m_1)$  and  $\text{insert}_{N+1}(\varphi_\ell(m_1))$  are equivalent, as required.  $\square$

We can combine Lemma 5.5 with Lemma 5.6 to obtain the following:

LEMMA 5.7. *Let  $m \in \mathcal{M}_n$  be a matching, and let  $i_1, \dots, i_k$  and  $j_1, \dots, j_k$  be two sequences. Then the matchings  $\text{insert}_{i_1} \cdots \text{insert}_{i_k}(m)$  and  $\text{insert}_{j_1} \cdots \text{insert}_{j_k}(m)$  are Knuth equivalent.*

*Proof.* We prove the statement by induction on  $k$ . If  $k = 0$  then the statement is obvious.

Next, we assume that  $k > 0$ , and denote  $m_i = \text{insert}_{i_2} \cdots \text{insert}_{i_k}(m)$  and  $m_j = \text{insert}_{j_2} \cdots \text{insert}_{j_k}(m)$ . We aim to prove that the matchings  $\text{insert}_{i_1}(m_i)$  and  $\text{insert}_{j_1}(m_j)$  are Knuth equivalent. By the induction hypothesis,  $m_i$  and  $m_j$  are equivalent. Therefore, by Lemma 5.6, the matchings  $\text{insert}_{i_1}(m_i)$  and  $\text{insert}_{i_1}(m_j)$  are equivalent too. In addition, by Lemma 5.5, the matchings  $\text{insert}_{i_1}(m_j)$  and  $\text{insert}_{j_1}(m_j)$  are equivalent. Therefore, the matchings  $\text{insert}_{i_1}(m_i)$  and  $\text{insert}_{j_1}(m_j)$  are equivalent.  $\square$

LEMMA 5.8. *Let  $m \in \mathcal{M}_N$  be a matching with  $k$  unstable chords. Then there exist indices  $i_1, \dots, i_k$  such that*

$$m = \text{insert}_{i_1} \cdots \text{insert}_{i_k}(\text{core}(m)).$$

*Proof.* We prove the statement by induction on  $k$ . If  $k = 0$  then  $\text{core}(m) = m$  and the statement is obvious.

Assume that a matching  $m \in \mathcal{M}_N$  has  $k > 0$  unstable chords. Therefore,  $m$  has at least one short chord, denoted  $(i_1, i_1 + 1) \in m$ . Thus, there exists a matching  $m_1 \in \mathcal{M}_{N-2}$  such that  $m = \text{insert}_{i_1}(m_1)$ . Notice that  $\text{core}(m) = \text{core}(m_1)$ , so  $m_1$  has  $k - 1$  unstable chords. Thus, by the induction hypothesis, there exist indices  $i_2, \dots, i_k$  such that  $m_1 = \text{insert}_{i_2} \cdots \text{insert}_{i_k}(\text{core}(m))$  and so  $m = \text{insert}_{i_1} \cdots \text{insert}_{i_k}(\text{core}(m))$ .  $\square$

Now we are ready to prove Lemma 5.3.

*Proof of Lemma 5.3.* Let  $m_1, m_2 \in \mathcal{M}_N$  be matchings with  $m_0 := \text{core}(m_1) = \text{core}(m_2)$ . Thus,  $m_1$  and  $m_2$  have the same number of unstable chords, denoted

$k$ . Following Lemma 5.8, we can write  $m_1 = \text{insert}_{i_1} \cdots \text{insert}_{i_k}(m_0)$  and  $m_2 = \text{insert}_{j_1} \cdots \text{insert}_{j_k}(m_0)$  for some  $i_1, \dots, i_k$  and  $j_1, \dots, j_k$ . The statement now follows directly from Lemma 5.7.  $\square$

Theorem 5.1 implies the following result:

**COROLLARY 5.9.** *If a set  $\mathcal{M} \subseteq \mathcal{M}_N$  is closed under Knuth equivalence then it is Schur-positive with respect to Short. Moreover, if  $\mathcal{M}$  is a Knuth equivalence class then its generating function is*

$$Q(\mathcal{M}) = s_{N-k,k},$$

where  $k$  is the number of unstable chords of some arbitrary matching  $m \in \mathcal{M}$ .

*Proof.* Clearly, a disjoint union of Schur-positive sets is Schur-positive. Therefore, it suffices to establish the statement for equivalence classes. Let  $\mathcal{M} \subseteq \mathcal{M}_N$  be an equivalence class. By Theorem 5.1 there exist  $0 \leq k \leq \frac{N}{2}$ ,  $f \geq 0$  and  $m_0 \in \mathcal{M}_{N-2k,f}(\text{Short} = \emptyset)$ , such that  $\mathcal{M} = \{m \in \mathcal{M}_{N,f} \mid \text{core}(m) = m_0\}$ . Recall the transformation  $F : m \mapsto (\text{core}(m), T(m))$  discussed in Section 4.2, and consider its restriction to  $\mathcal{M}$ . We obtain that the restricted transformation

$$F|_{\mathcal{M}} : \mathcal{M} \rightarrow \{m_0\} \times \text{SYT}(N - k, k)$$

is a statistic-preserving bijection. Applying Theorem 2.5 completes the proof.  $\square$

**5.2. OTHER CONSTRUCTIONS OF SCHUR-POSITIVE SETS.** We may apply Corollary 5.9 to obtain other Schur-positive sets of matchings. For example, we can filter  $\mathcal{M}_N$  by the isomorphism class of the intersection graph:

**DEFINITION 5.10.** *Let  $m$  be a matching. Its intersection graph, denoted  $G(m)$ , is defined to be the undirected simple graph with the chords of  $m$  as its vertices, and with an edge between two vertices if the associated chords of  $m$  intersect.*

It can be easily seen that applying an elementary Knuth transformation on a matching preserves its intersection graph up to graph-isomorphism. Therefore, by Corollary 5.9:

**COROLLARY 5.11.** *For every  $N$  and  $f$ , the set of matchings  $m \in \mathcal{M}_{N,f}$  with a fixed intersection graph up to graph-isomorphism is Schur-positive.*

For example, fix  $N$  and  $f$ . Then for every  $k$ , the set of  $k$ -crossing matchings in  $\mathcal{M}_{N,f}$  (i.e. matchings where  $k$  is the maximal cardinality of a set of pairwise intersecting chords) is Schur-positive. Equivalently, this is the set of matchings whose intersection graph has a maximal clique with  $k$  vertices.

As another example, given a matching  $m \in \mathcal{M}_{N,f}$ , for every  $1 \leq i \leq N$  denote by  $I_i(m)$  the number of chords of  $m$  that intersect the chord that contains  $i$  (if  $i$  is an unmatched vertex define  $I_i(m) = 0$ ). From Corollary 5.11 we obtain that for any fixed multiset  $S$ , the set of matchings  $m \in \mathcal{M}_{N,f}$  such that  $\{I_i(m) \mid i \in [N]\} = S$  (as multisets) is Schur-positive. For example, the following sets are Schur-positive:

- The set of matchings  $m \in \mathcal{M}_{N,f}$  with exactly  $k$  pairs of intersecting chords, i.e. such that  $\frac{1}{2} \sum_i I_i(m) = k$ .
- The set of matchings  $m \in \mathcal{M}_{N,f}$  with exactly  $k$  intersecting chords, i.e. such that  $\frac{1}{2} |\{i \mid I_i(m) > 0\}| = k$ .
- The set of matchings  $m \in \mathcal{M}_{N,f}$  with  $k$  the maximal number of times that a chord intersects other chords, i.e. with  $\max_i I_i(m) = k$ .

5.3. PATTERN AVOIDANCE IN MATCHINGS. Sagan and Woo [27], motivated by Elizalde and Roichman [11], posed the problem of determining which sets  $\Pi$  of permutations satisfy the property that for all  $n$ , the set of permutations in  $S_n$  that avoid every pattern in  $\Pi$  is Schur-positive. This problem has been extensively studied since then [16, 7, 21].

An analogous question may be asked about Schur-positivity of pattern-avoiding matchings as well. Extensive research has been conducted on pattern avoidance in perfect matchings by Simion and Schmidt [28], Jelínek and Mansour [18], Bloom and Elizalde [6], and others, leading to multiple definitions in the literature. Fang, Hamaker, and Troyka [12] explore some of these definitions and provide a comparison. Additionally, various conventions exist for generalizing pattern avoidance to non-perfect matchings [20, 12, 22]. We adopt the definition from McGovern [22], which directly generalizes the definition for perfect matchings from [18] and [9].

DEFINITION 5.12. Let  $m_1 \in \mathcal{M}_{N_1}$  and  $m_2 \in \mathcal{M}_{N_2}$  for some positive integers  $N_1 \leq N_2$ . We say that  $m_2$  contains the pattern  $m_1$ , if there exist indices  $1 \leq i_1 < \dots < i_{N_1} \leq N_2$  such that the following holds:

- For all  $1 \leq j < j' \leq N_1$ ,

$$(j, j') \in m_1 \iff (i_j, i_{j'}) \in m_2.$$

- For all  $1 \leq j \leq N_1$ ,

$$(j) \in m_1 \iff (i_j) \in m_2.$$

Otherwise, we say that  $m_2$  avoids  $m_1$ .

For two sets of matchings  $\mathcal{M}_1 \subseteq \mathcal{M}_{N_1}$  and  $\mathcal{M}_2 \subseteq \mathcal{M}_{N_2}$ , denote by  $\mathcal{M}_2(\mathcal{M}_1)$  the set of matchings in  $\mathcal{M}_2$  that avoid every matching in  $\mathcal{M}_1$ . In addition, denote  $\mathcal{M}_2(m) := \mathcal{M}_2(\{m\})$  for a matching  $m$ .

The following problem is analogous to the problem of Sagan and Woo [27] regarding pattern-avoiding permutations:

PROBLEM 5.13. Determine which sets  $\mathcal{M} \subseteq \mathcal{M}_N$  of matchings satisfy the property that for all  $N', f'$ , the pattern-avoiding set  $\mathcal{M}_{N', f'}(\mathcal{M})$  is Schur-positive with respect to Short.

While a complete solution of Problem 5.13 seems challenging, we are able to solve the problem in the case where  $|\mathcal{M}| = 1$ .

PROPOSITION 5.14. Let  $N$  be a nonnegative integer, and let  $m \in \mathcal{M}_N$  be a matching. Then the pattern-avoiding set  $\mathcal{M}_{N', f'}(m)$  is Schur-positive with respect to Short for all  $N', f'$  if and only if one of the following holds:

- (1)  $\text{Short}(m) = \emptyset$ , or
- (2)  $m = \{(1, 2)\}$ , the unique perfect matching on two vertices.

In the proof of Proposition 5.14, we will apply the following result:

LEMMA 5.15. Let  $m \in \mathcal{M}_N$  be a short-chord-free matching, and let  $N', f'$  be non-negative integers. Then the pattern-avoiding set  $\mathcal{M}_{N', f'}(m)$  is closed under Knuth equivalence.

*Proof.* Let  $m'_1 \in \mathcal{M}_{N', f'}$  be a matching, and let  $m'_2 \in \mathcal{M}_{N', f'}$  be the result of applying an elementary Knuth transformation on  $m'_1$ . Assume that  $m'_1$  contains the pattern  $m$ . It suffices to prove that  $m_2$  contains  $m$  too. By Definition 5.12, there exist indices  $1 \leq i_1 < \dots < i_N \leq N'$  such that  $(j, j') \in m \iff (i_j, i_{j'}) \in m'_1$  for all  $1 \leq j < j' \leq N$  and  $(j) \in m \iff (i_j) \in m'_1$  for all  $1 \leq j \leq N$ . Without loss of generality, we may assume that  $(i, i + 1) \in m'_1$  and  $m'_2$  is obtained from  $m'_1$  by interchanging the chord  $(i, i + 1)$



with the vertex  $i + 2$ . The pattern  $m$  is short-chord-free, so  $i, i + 1 \notin \{i_1, \dots, i_N\}$ . Therefore, by considering the index set  $\{i_1, \dots, i_N\}$  if  $i + 2 \notin \{i_1, \dots, i_N\}$  and the set  $\{i\} \cup \{i_1, \dots, i_N\} \setminus \{i + 2\}$  otherwise, we obtain that the matching  $m'_2$  contains  $m$  as well.  $\square$

Now, let us prove Proposition 5.14:

*Proof of Proposition 5.14.* Denote by  $f$  the number of unmatched vertices of  $m$ , and assume that  $\mathcal{M}_{N',f'}(m)$  is Schur-positive for all  $N', f'$ . In particular, we may take  $N' = N$  and  $f' = f$  and obtain that the set  $\mathcal{M}_{N,f}(m)$  is Schur-positive. Since  $\mathcal{M}_{N,f}(m) = \mathcal{M}_{N,f} \setminus \{m\}$ , we obtain that the generating function  $\mathcal{Q}(\mathcal{M}_{N,f} \setminus \{m\})$  is Schur-positive and, as such, symmetric. Clearly,

$$\mathcal{Q}(\mathcal{M}_{N,f} \setminus \{m\}) = \mathcal{Q}(\mathcal{M}_{N,f}) - \mathcal{Q}(\{m\}).$$

By Theorem 1.6, the function  $\mathcal{Q}(\mathcal{M}_{N,f})$  is symmetric, so the function  $\mathcal{Q}(\{m\}) = F_{\text{Short}(m)}$  is symmetric as well. By Lemma 2.8, we may deduce that  $\text{Short}(m) = \emptyset$  or  $\text{Short}(m) = [N - 1]$ . If  $\text{Short}(m) = \emptyset$  then the statement holds. Otherwise,  $\text{Short}(m) = [N - 1]$ . The statistic  $\text{Short}$  is sparse (recall Definition 1.7), so we may deduce that  $N \leq 2$ . If  $N = 2$  then  $\text{Short}(m) = \{1\}$ , and therefore  $m = \{(1, 2)\}$ . If  $N = 1$  then  $\text{Short}(m) = \emptyset$ .

On the other hand, we assume that  $\text{Short}(m) = \emptyset$  or  $m = \{(1, 2)\}$  and prove that  $\mathcal{M}_{N',f'}(m)$  is Schur-positive for all  $N', f'$ .

- (1) Assume that  $\text{Short}(m) = \emptyset$ , and let  $N', f'$  be nonnegative integers. By Lemma 5.15, the set  $\mathcal{M}_{N',f'}(m)$  is closed under Knuth equivalence. Therefore, by Corollary 5.9, it is Schur-positive.
- (2) Assume that  $m = \{(1, 2)\}$ , and let  $N', f'$  be nonnegative integers. Notice that a matching avoids the pattern  $m$  if and only if it is an empty matching (i.e. with all its vertices unmatched). By Definition 1.9, such a matching is Knuth equivalent only to itself, so  $\mathcal{M}_{N',f'}(m)$  is closed under Knuth equivalence. Therefore by Corollary 5.9,  $\mathcal{M}_{N',f'}(m)$  is Schur-positive.  $\square$

## 6. FURTHER REMARKS AND OPEN PROBLEMS

6.1. CRITERIA FOR SPECIAL CASES OF SCHUR-POSITIVITY. There is a simple well-known criterion for sets under which the generating function  $\mathcal{Q}_D(\mathcal{A})$  is non-negatively spanned by the Schur functions of hook shape  $s_{N-k,1^k}$ :

PROPOSITION 6.1 (folklore). *A set  $\mathcal{A}$  with a statistic  $D : \mathcal{A} \rightarrow 2^{[N-1]}$  has a generating function of the form*

$$\mathcal{Q}_D(\mathcal{A}) = \sum_{k=0}^{N-1} c_k s_{N-k,1^k}, \quad c_k \in \mathbb{N}$$

*if and only if for every  $J \subseteq [N - 1]$ ,  $|\{a \in \mathcal{A} \mid D(a) = J\}|$  depends only on  $|J|$ . In that case,  $c_k = |\{a \in \mathcal{A} \mid D(a) = [k]\}|$ .*

*Proof.* On one hand, assume that there exist  $c_k \in \mathbb{N}$  such that  $|\mathcal{A}(D = J)| = c_{|J|}$  for all  $J \subseteq [N - 1]$ . Notice that every tableau of a hook shape  $T \in \text{SYT}(N - k, 1^k)$  has  $|\text{Des}(T)| = k$ . Conversely, for every set  $J \subseteq [N - 1]$  with  $|J| = k$  there exists a unique tableau  $T \in \text{SYT}(N - k, 1^k)$  such that  $\text{Des}(T) = J$ . Therefore, we obtain the equation

$$\sum_{a \in \mathcal{A}} t^{D(a)} = \sum_{k=0}^{N-1} c_k \sum_{T \in \text{SYT}(N-k,1^k)} t^{\text{Des}(T)},$$

where  $\mathbf{t}^J := \prod_{j \in J} t_j$  for  $J \subseteq [N - 1]$ . By Theorem 2.5, this equation implies the statement.

The other direction of the proposition states that if  $\mathcal{Q}_D(\mathcal{A}) = \sum_{k=0}^{N-1} c_k s_{N-k, 1^k}$ , then  $|\{a \in \mathcal{A} \mid D(a) = J\}| = c_{|J|}$ . Adin and Roichman [3, Corollary 8.1] established that a generating function uniquely determines the descent set distribution, which implies the statement.  $\square$

Similarly to Proposition 6.1, Theorem 1.8 provides a simple criterion under which  $\mathcal{Q}_D(\mathcal{A})$  is non-negatively spanned by the Schur functions of two-row shape  $\{s_{N-k, k} \mid 0 \leq k \leq n\}$ . In the spirit of discovering new methods and expanding our understanding, we pose the problem of finding other such criteria, hopeful that they will contribute to further advancements in the study of Schur-positivity and Schur coefficients.

**PROBLEM 6.2.** *Find other sets of partitions  $\Lambda \subseteq \{\lambda \vdash N\}$  and criteria for a given set  $\mathcal{A}$  under which  $\mathcal{Q}_D(\mathcal{A})$  is non-negatively spanned by the functions  $\{s_\lambda \mid \lambda \in \Lambda\}$ .*

**6.2. SHORT CHORDS OF MATCHINGS.** Theorem 1.6 shows that the set  $\mathcal{M}_{N, f}$  of matchings with a given number of unmatched vertices is Schur-positive with respect to the set of short chords, and describes its Schur expansion. In Section 5, several Schur-positive subsets of  $\mathcal{M}_{N, f}$  are presented too. All these sets are shown to be closed under Knuth equivalence (described in Definition 1.9), so their Schur-positivity can be derived from Corollary 5.9. It is desired to find Schur-positive sets of other types.

**PROBLEM 6.3.** *Find other Schur-positive subsets of  $\mathcal{M}_{N, f}$  with respect to Short. In particular, find Schur-positive subsets that are not closed under Knuth equivalence.*

Schur-positive sets of particular interest are pattern-avoiding sets. As discussed earlier in Section 5.3, Schur-positivity of pattern-avoiding sets was extensively studied in the context of permutations. Recall the problem stated in Section 5.3:

**PROBLEM** (Problem 5.13 above). *Determine which sets  $\mathcal{M} \subseteq \mathcal{M}_N$  of matchings satisfy the property that for all  $N', f'$ , the pattern-avoiding set  $\mathcal{M}_{N', f'}(\mathcal{M})$  is Schur-positive with respect to Short.*

Proposition 5.14 solves this problem for pattern sets of size 1. We do not know a general answer for larger pattern sets.

The connection identified between the Schur expansion of  $\mathcal{M}_{N, f}$  and Bessel polynomials, as revealed in Corollary 4.13, serves as motivation to investigate the Schur expansion of additional sets:

**PROBLEM 6.4** (Sergi Elizalde, personal communication). *Find the Schur expansion of Schur-positive sets of matchings. For instance, find the Schur expansion of the set of  $k$ -crossing matchings in  $\mathcal{M}_{N, f}$ , or the Schur expansion of Schur-positive pattern-avoiding sets.*

**6.3. THE INVOLUTIVE LENGTH.** Here, we discuss one of the primary motivations behind this research. We denote by  $\mathcal{I}_{2n}$  the set of fixed-point-free involutions in the symmetric group  $S_{2n}$ . Furthermore, we define the *simple reflections*  $s_i = (i, i + 1) \in S_{2n}$  for  $1 \leq i < 2n$ , and denote  $w_0 = (1, 2)(3, 4) \cdots (2n - 1, 2n)$ . Adin, Postnikov, and Roichman [2] defined the *involutive length* of an involution  $w \in \mathcal{I}_{2n}$  as

$$\hat{\ell}(w) := \min\{k \mid s_{i_1} \cdots s_{i_k} w (s_{i_1} \cdots s_{i_k})^{-1} = w_0, 1 \leq i_1, \dots, i_k < 2n\}.$$

They also defined the *involutive weak order*  $\leq_{\mathcal{I}}$  on  $\mathcal{I}_{2n}$ , as the reflexive and transitive closure of the relation  $w \prec_{\mathcal{I}} s_i w s_i$  if  $\hat{\ell}(s_i w s_i) = \hat{\ell}(w) + 1$ . The involutive order is motivated by the Bruhat order defined for Coxeter groups [5, Chapter 2].

Since fixed-point-free involutions in  $S_{2n}$  naturally correspond to perfect matchings on  $2n$  vertices, we will adopt the involutive length and the involutive weak order for perfect matchings in  $\mathcal{PM}_{2n} := \mathcal{M}_{2n,0}$  and use the same notation for them.

Consider the natural action of  $S_{2n}$  on the set  $\mathcal{PM}_{2n}$ , and denote  $m_0 = \{(1, 2), (3, 4), \dots, (2n - 1, 2n)\}$ . Notably, the stabilizer of  $m_0$  is isomorphic to the hyperoctahedral group  $H_n = S_2 \wr S_n$ . Furthermore, consider the Schreier graph associated with this action with respect to the simple reflections. For an illustration of this Schreier graph, see Figure 7 which depicts it when  $2n = 4$ .

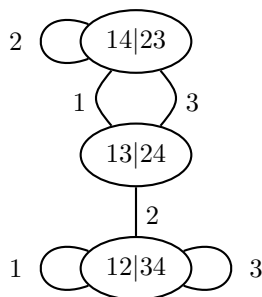


FIGURE 7. Schreier graph of the natural action of  $S_4$  on  $\mathcal{PM}_4$

We can create a layered graph by partitioning the vertices (matchings) into layers based on their distance from  $m_0$ . Specifically, a matching in the  $\ell$ -th layer is at a distance of  $\ell$  from  $m_0$ . Furthermore, Avni [4] proved that this graph is bipartite when ignoring loops. Therefore, a matching in one layer is connected to matchings in the adjacent layers, but not within its own layer or to matchings in non-adjacent layers.

It turns out that this graph corresponds to the involutive length and the involutive weak order. First, the index of the layer of a matching  $m$  is equal to its involutive length  $\hat{\ell}(m)$ . Moreover, for every  $m_1, m_2 \in \mathcal{PM}_{2n}$  we have  $m_1 \leq_{\mathcal{I}} m_2$  if and only if there exists a geodesic path from  $m_0$  to  $m_2$  passing through  $m_1$ .

Based on the structure of the Schreier graph, we define three natural set-valued functions on perfect matchings, denoted by  $\text{Asc}(m)$ ,  $\text{Loop}(m)$ , and  $\text{Des}(m)$ , where  $m$  is a matching in  $\mathcal{PM}_{2n}$ . Given a matching  $m \in \mathcal{PM}_{2n}$  and an index  $1 \leq i \leq 2n - 1$ , we define:

- $i \in \text{Asc}(m)$  if  $s_i \cdot m >_{\mathcal{I}} m$ , where the dot denotes the group action,
- $i \in \text{Des}(m)$  if  $s_i \cdot m <_{\mathcal{I}} m$ , and
- $i \in \text{Loop}(m)$  if  $s_i \cdot m = m$ .

It is noteworthy that  $i \in \text{Asc}(m)$  if and only if  $\hat{\ell}(s_i \cdot m) > \hat{\ell}(m)$ ,  $i \in \text{Des}(m)$  if and only if  $\hat{\ell}(s_i \cdot m) < \hat{\ell}(m)$ , and  $i \in \text{Loop}(m)$  if and only if  $\hat{\ell}(s_i \cdot m) = \hat{\ell}(m)$ .

The algebraic interest of the graph motivates the analysis of these three statistics, specifically focusing on the question of the Schur-positivity of  $\mathcal{PM}_{2n}$  with respect to each of them. As a first step in this direction, it is noteworthy that the definition of  $\text{Asc}$  coincides with the standard ascents of permutations when restricted to  $\mathcal{I}_{2n}$ . Consequently, the Schur-positivity of  $\mathcal{PM}_{2n}$  with respect to  $\text{Asc}$  can be derived directly from classical properties of the Robinson-Schensted correspondence, as noted for example by Gessel and Reutenauer [14, end of Section 7]. Moreover, we have the observation  $\text{Loop}(m) = \text{Short}(m)$  for a perfect matching  $m$ . This observation, when combined with Theorem 1.6, implies that  $\mathcal{PM}_{2n}$  is Schur-positive with respect to  $\text{Loop}$ .

In contrast, the Schur-positivity of  $\mathcal{PM}_{2n}$  with respect to  $\text{Des}$  remains a mystery.

QUESTION 6.5 (Ron Adin and Yuval Roichman, personal communication). *Is  $\mathcal{PM}_{2n}$  Schur-positive with respect to Des?*

Despite extensive analysis of the Des statistic, we are currently unable to provide an answer. However, based on experimental results from simulations, we have discovered that  $\mathcal{PM}_{2n}$  is Schur-positive with respect to Des when  $2n \leq 14$ . Moreover, intriguingly, the statistics Asc and Des are found to be equidistributed for all  $2n \leq 14$ . These findings lead us to propose the following conjecture:

CONJECTURE 6.6 (Ron Adin and Yuval Roichman, personal communication). *For all  $n \in \mathbb{N}$ ,*

$$\sum_{m \in \mathcal{PM}_{2n}} \mathbf{t}^{\text{Asc}(m)} = \sum_{m \in \mathcal{PM}_{2n}} \mathbf{t}^{\text{Des}(m)},$$

where  $\mathbf{t}^J := \prod_{j \in J} t_j$  for  $J \subseteq [2n - 1]$ .

If this conjecture can be proven, it would establish the Schur-positivity of  $\mathcal{PM}_{2n}$  with respect to Des. Furthermore, a bijection on  $\mathcal{PM}_{2n}$  that maps ascents to descents may unveil hidden symmetries within the Schreier graph.

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