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Saturation for Flagged Skew Littlewood–Richardson coefficients

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Abstract We define and study a generalization of the Littlewood–Richardson (LR) coefficients, which we call the flagged skew LR coefficients. These subsume several previously studied extensions of the LR coefficients. We establish the saturation property for these coefficients, generalizing results of Knutson-Tao and Kushwaha-Raghavan-Viswanath.

1. Introduction

The Littlewood–Richardson (LR) coefficients are among the most celebrated numbers in algebraic combinatorics. They are the multiplicities in the decomposition into irreducibles of tensor products of irreducible polynomial representations of general linear groups. As such, they are the structure constants for the multiplication of Schur polynomials (which form a basis for the ring of symmetric polynomials). These coefficients naturally occur in several other contexts. For example:

- They determine the branching of irreducible representations of symmetric groups on restriction to Young subgroups.
- They are the structure constants for the multiplication of Schubert cohomology classes of Grassmanians.

Several generalizations of these coefficients can be found in the literature. Our broader goal is to investigate, for some of these generalizations, the analogue of the saturation theorem of Knutson and Tao for the LR coefficients. In this paper, we consider a simultaneous generalization of Zelevinsky's skew LR coefficients in [17] and the flagged LR coefficients of Kushwaha et al in [8]. Our main result is that these *flagged skew LR coefficients* exhibit the saturation property (Theorem 2.3).

In order to do this, we first lift the main *key-positivity* result of Reiner and Shimozono in [13] to crystals (Theorem 3.11), using a recent result of Assaf, Dranowski and Gonzalez in [1]. We then generalize the hive-like model in [8] for flagged LR coefficients to their skew versions (§4.4). Our model gives a simple bijective proof of a well known identity satisfied by skew Schur functions in the context of the Hall inner product (Remark 4.5). We believe that this is a new bijective proof of this identity. As a final step, we prove that every flagged skew hive polytope is isomorphic to some flagged hive polytope (§5). It turns out crucially that scaling the parameters of the former polytope results in a scaling of the parameters of the latter polytope. We can then

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invoke the saturation result from [8] for flagged hive polytopes, to finish the proof of our main result. Note that, in contrast to the case of hive polytopes, we do not know if every skew Gelfand–Tsetlin (GT) polytope is isomorphic to some GT polytope.

This paper is organized as follows. In §2 we recall some notations and state our main results. In §3, after recalling the notion of crystals we prove that the set of flagged skew tableaux forms a disjoint union of Demazure crystals. This result is applied in §4 to obtain a polytopal model for the flagged skew LR coefficients. In §5, we prove the main result of this paper, namely that the saturation property holds for flagged skew LR coefficients. In Appendix A we give a precise description of the connected components of the set of flagged skew tableaux.

2. Preliminaries and Statements of Main Theorems

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a weakly decreasing sequence of non-negative integers with finitely many non-zero terms (or parts). The *length* of the partition λ is defined to be the largest integer i such that λ_i is non-zero and we denote it by $l(\lambda)$. The *weight* of λ is the sum of its parts and we denote it by $|\lambda|$. We denote by $\mathcal{P}[n]$ the set of all partitions of length at most n. The *Young diagram* of the partition λ is the left and top justified collection of boxes such that row i contains λ_i boxes. We denote it again by λ. For Young diagrams λ and μ such that $\lambda \supset \mu$ (i.e., $\lambda_i \geqslant \mu_i\ \forall i$), the *skew diagram* λ/μ is obtained by removing the boxes of μ from those of λ.

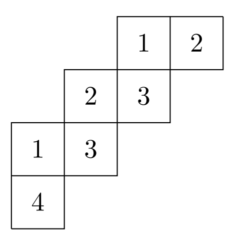


Figure 1. A skew tableau of shape $(4,3,2,1)/(2,1)$, weight $(2,2,2,1)$ and reverse-row reading word 2132314.

Given partitions $\lambda \supset \mu$, a *semi-standard skew tableau* of shape λ/μ is a filling of the skew diagram λ/μ that is weakly increasing along the rows (from left to right) and strictly increasing along the columns (from top to bottom). A semi-standard tableau of shape λ is just a semi-standard skew tableau of shape λ/empty. We denote by $\mathrm{Tab}(\lambda/\mu)$ the set of all semi-standard skew tableaux of shape λ/μ and $\mathrm{Tab}(\lambda/\mu, n)$ the subset of $\mathrm{Tab}(\lambda/\mu)$ where the fillings are all $\leqslant n$. It will be convenient to let $\mathrm{Tab}(\lambda/\mu)$ be the empty set if λ, μ are partitions with $\lambda \not\supset \mu$. The *weight* of a tableau $T \in \mathrm{Tab}(\lambda/\mu, n)$ is defined as $wt(T) = (t_1, t_2, \ldots, t_n)$, where t_i is the number of times i occurs in T. A standard (skew) tableau T is a semi-standard (skew) tableau of the same shape in which $1, 2, \ldots, k$ appears exactly once, where k is the number of boxes in T. For $T \in \mathrm{Tab}(\lambda/\mu)$, we write b_T to denote the *reverse-row reading word of* T which is the word obtained by reading T right to left and from top to bottom.

Fix n a positive integer. A flag $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n)$ is a weakly increasing sequence of positive integers such that[(1)] $\Phi_n = n$. For $\lambda, \mu \in \mathcal{P}[n]$ and flag $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n)$, $\mathrm{Tab}(\lambda/\mu, \Phi)$ is the set of all elements T in $\mathrm{Tab}(\lambda/\mu)$ such that the entries in row i of

[(1)] In the literature, a flag need not have $\Phi_n = n$, but for our purposes it is sufficient to consider only such flags.

T are at most Φ_i for $1 \leqslant i \leqslant n$. Following Reiner and Shimozono [13], we define the *flagged skew Schur polynomial*

$$s_{\lambda/\mu}(X_\Phi) = \sum_T \mathbf{x}^{wt(T)}$$

where T varies over $\mathrm{Tab}(\lambda/\mu, \Phi)$ and for $t = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_+^n$, $\mathbf{x}^t$ denotes the monomial $x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}$. When $\Phi = (n, n, \ldots, n)$, these reduce to the skew Schur polynomials $s_{\lambda/\mu}(x_1, x_2, \ldots, x_n)$. When μ is the empty partition, they become the flagged Schur polynomials $s_\lambda(X_\Phi)$, which coincide with key polynomials corresponding to 312-avoiding permutations [12, Theorem 14.1]. The flagged skew Schur polynomials $s_{\lambda/\mu}(X_\Phi)$ also have a representation-theoretic interpretation as characters of certain Borel modules called *flagged Schur modules* [15].

For $1 \leqslant i \leqslant n-1$, define the Demazure operator T_i on the ring of polynomials in the variables $x_1, x_2, \ldots, x_n$ as follows:

$$(T_i f)(x_1, x_2, \ldots, x_n) = \frac{x_i\, f(x_1, x_2, \ldots, x_n) - x_{i+1}\, f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)}{x_i - x_{i+1}}$$

For $w \in \mathfrak{S}_n$ (the symmetric group), we define

$$T_w = T_{i_1} T_{i_2} \cdots T_{i_k}$$

where $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression for w. This is well-defined because the T_i's satisfy the braid relations.

For $\alpha \in \mathbb{Z}_+^n$, let $\alpha^\dagger$ be the partition obtained by sorting the parts of α in descending order and let $w \in \mathfrak{S}_n$ be any permutation such that $w\alpha^\dagger = \alpha$ (here, the action of w is the usual left action of $\mathfrak{S}_n$ on n-tuples). We recall that the *key polynomial* κ_α is defined to be $\kappa_\alpha = T_w(\mathbf{x}^{\alpha^\dagger})$, and that this is independent of the choice of w. A polynomial f is said to be *key-positive* if it is a sum of key polynomials. If f is key-positive, then $\mathbf{x}^\lambda f$ and $T_w(f)$ are also key-positive, for all $\lambda \in \mathcal{P}[n]$ and all $w \in \mathfrak{S}_\mathfrak{n}$. The former follows from a theorem of Joseph [5, §2.11] and the latter from the fact that a composition of Demazure operators is itself a Demazure operator. Reiner and Shimozono [13, Theorem 20] showed that flagged skew Schur polynomials $s_{\lambda/\mu}(X_\Phi)$ are key-positive.

Now, if w_0 denotes the longest element of $\mathfrak{S}_n$, we have

$$T_{w_0}(\kappa_\alpha) = s_{\alpha^\dagger}(x_1, x_2, \ldots, x_n), \tag{1}$$

the Schur polynomial indexed by $\alpha^\dagger$. More generally, (since key polynomials form a $\mathbb{Z}$-basis of the polynomial ring in n variables [13, Corollary 7]) given any polynomial $f = f(x_1, x_2, \ldots, x_n)$, we have that $T_{w_0}(f)$ is a symmetric polynomial, which can therefore be expanded in the basis of Schur polynomials. If further f is key-positive, then Eq. (1) shows that $T_{w_0}(f)$ is Schur-positive, i.e., a sum of Schur polynomials. This leads us to the main objects of our study.

Definition 2.1. *For* $\lambda, \mu, \gamma \in \mathcal{P}[n]$ *and flag* $\Phi = (\Phi_1, \ldots, \Phi_n)$*, let*

$$T_{w_0}(\mathbf{x}^\lambda\, s_{\mu/\gamma}(X_\Phi)) = \sum_{\nu \in \mathcal{P}[n]} c^{\nu}_{\lambda,\, \mu/\gamma}(\Phi)\ \ s_\nu(x_1, x_2, \ldots, x_n) \tag{2}$$

We call the coefficients $c^{\nu}_{\lambda,\, \mu/\gamma}(\Phi)$ *as the* ***flagged skew LR coefficients****.*

By the preceding remarks, it follows that the LHS of Eq. (2) is Schur positive, and thus the flagged skew LR coefficients are non-negative integers. It is clear by definition that these coefficients are zero if $\mu \not\supset \gamma$. It will follow from Theorem 2.2 below that they are also zero if $\nu \not\supset \lambda$.

These coefficients subsume many other extensions of the LR coefficients. When $\Phi = (n, n, \ldots, n)$, these become Zelevinsky's extension [17] of the LR coefficients

$c^{\nu}_{\lambda,\,\mu/\gamma}$ defined by $s_\lambda(\mathbf{x})\, s_{\mu/\gamma}(\mathbf{x}) = \sum_{\nu\in\mathcal{P}[n]} c^{\nu}_{\lambda,\,\mu/\gamma}\, s_\nu(\mathbf{x})$. These in turn reduce to the usual LR coefficients when we further take $\gamma = (0, 0, \ldots, 0)$.

On the other hand, if we take $\gamma = (0, 0, \ldots, 0)$ but let Φ remain arbitrary, we get the *w-refined LR coefficients* of [8] for *312-avoiding permutations* w.

If we set $\lambda = (0, 0, \ldots, 0)$, we have $c^{\nu}_{\lambda,\,\mu/\gamma}(\Phi) = \sum_\alpha c^{\mu/\gamma,\Phi}_{\alpha}$ where the sum runs over all compositions α that are obtained by permuting the parts of ν. The coefficients on the right are the ones which appear in the flagged LR expansion of [13, §7].

Our first result provides two combinatorial models for flagged skew LR coefficients (see §3, 4 for undefined terms) that generalize those for LR coefficients:

Theorem 2.2. *Let $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n)$ be a flag and $\lambda, \mu, \nu, \gamma \in \mathcal{P}[n]$. The flagged skew LR coefficient $c^{\nu}_{\lambda,\,\mu/\gamma}(\Phi)$ equals:*

1. *the cardinality of the set of all λ-dominant tableaux in* $\mathrm{Tab}(\mu/\gamma, \Phi)$ *of weight $\nu - \lambda$.*
2. *the number of the integral points of the flagged skew "hive polytope"* $\mathrm{SHive}(\lambda, \mu, \gamma, \nu, \Phi)$.

Our proof of Theorem 2.2 hinges on understanding the crystal structure on the set of flagged skew tableaux $\mathrm{Tab}(\mu/\gamma, \Phi)$. We show in particular that this set is a disjoint union of Demazure crystals; this lifts the key-positivity result of Reiner and Shimozono [13, Theorem 20] from the level of characters to that of crystals.

The main theorem of this paper is the following *saturation property* of the flagged skew LR coefficients:

Theorem 2.3. *Let Φ be a flag and $\lambda, \mu, \nu, \gamma \in \mathcal{P}[n]$. Then,*

$$c^{k\nu}_{k\lambda,\,k\mu/k\gamma}(\Phi) > 0 \text{ for some } k \geqslant 1 \implies c^{\nu}_{\lambda,\,\mu/\gamma}(\Phi) > 0$$

We remark that, as in the classical LR case, the converse statement holds. In fact, scaling $\lambda, \mu, \gamma, \nu$ by k dilates the polytope $\mathrm{SHive}(\lambda, \mu, \gamma, \nu, \Phi)$ by the factor k. Thus, $c^{\nu}_{\lambda,\,\mu/\gamma}(\Phi) > 0$ implies that $c^{k\nu}_{k\lambda,\,k\mu/k\gamma}(\Phi) > 0$ for every $k \geqslant 1$, by Theorem 2.2.

To prove Theorem 2.3, we construct an affine linear isomorphism between $\mathrm{SHive}(\lambda, \mu, \gamma, \nu)$ and a certain hive polytope, which preserves flags and integral points. The theorem then follows from the corresponding one for hive polytopes [8, Theorem 1.4].

3. Flagged skew tableaux and crystals

The purpose of this section is to prove that the subset $\mathrm{Tab}(\mu/\gamma, \Phi)$ of the type A_{n-1} crystal $\mathrm{Tab}(\mu/\gamma, n)$ is a disjoint union of Demazure crystals. This is the key step in proving the first part of Theorem 2.2, as we will see at the end of this section.

A *crystal*[(2)] *of type* A_{n-1} consists of an underlying finite and non-empty set $\mathcal{B}$ together with an auxiliary element $0 \notin \mathcal{B}$ and maps

$$wt : \mathcal{B} \to \mathbb{Z}^n \qquad \text{and} \qquad e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \;\; \text{for } i \in \{1, 2, \ldots, n-1\}$$

subject to the following axioms:

1. For $x, y \in \mathcal{B}$, we have $e_i(x) = y$ if and only if $f_i(y) = x$. In this case, we further have
$$wt(y) = wt(x) + (\epsilon_i - \epsilon_{i+1})$$
where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are the standard basis vectors in $\mathbb{Z}^n$.

(2) This is called a seminormal crystal in [3, §2.2].

(2) For $x \in \mathcal{B}$ and $i \in \{1, 2, \ldots, n-1\}$, we have

$$\phi_i(x) = wt(x) \cdot (\epsilon_i - \epsilon_{i+1}) + \varepsilon_i(x)$$

where $\varepsilon_i(x)$ (respectively $\phi_i(x)$) is the maximum number of times e_i (respectively f_i) can be applied to x without making it 0.

The maps e_i and f_i are called the *raising* and *lowering* operators respectively. By a slight abuse of notation, a crystal is often denoted by its underlying set $\mathcal{B}$.

Example: The *standard type A_{n-1} crystal* consists of the underlying set $\mathbb{B} = \{1, 2, \ldots, n\}$ and maps

$$wt(i) = \epsilon_i \quad \text{and} \quad f_k(l) = \begin{cases} k+1 & \text{if} \quad l = k \\ 0 & \text{otherwise} \end{cases}$$

This crystal can be depicted by the following "crystal graph":

$$\boxed{1} \xrightarrow{\ 1\ } \boxed{2} \xrightarrow{\ 2\ } \cdots \xrightarrow{\ n-1\ } \boxed{n}$$

The *crystal graph* associated to a type A_{n-1} crystal is an edge-coloured directed graph (the colours being $1, 2, \ldots, n-1$) whose vertex set is the underlying set $\mathcal{B}$ of the crystal. An edge with colour k originates from $x \in \mathcal{B}$ and terminates at $y \in \mathcal{B}$ if and only if $f_k(x) = y$. We say a crystal is *connected* if its crystal graph is connected (viewed as an undirected graph).

A subset $\mathcal{B}'$ of a crystal $\mathcal{B}$ which is a union of connected components of $\mathcal{B}$ inherits a crystal structure from that of $\mathcal{B}$. In this case, we call $\mathcal{B}'$ a *full subcrystal* of $\mathcal{B}$.

3.1. Tensor Product of Crystals. If $\mathcal{B}$ and $\mathcal{C}$ are type A_{n-1} crystals, there is a notion of the tensor product crystal $\mathcal{B} \otimes \mathcal{C}$. Its underlying set is $\{x \otimes y : x \in \mathcal{B}, y \in \mathcal{C}\}$ (where $x \otimes y$ is just a symbol). We define $wt(x \otimes y)$ to be $wt(x) + wt(y)$. The raising and lowering operators are defined as follows:

$$e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y & \text{if} \quad \varepsilon_i(y) \leqslant \phi_i(x) \\ x \otimes e_i(y) & \text{if} \quad \varepsilon_i(y) > \phi_i(x) \end{cases} \tag{3}$$

and

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \text{if} \quad \varepsilon_i(y) < \phi_i(x) \\ x \otimes f_i(y) & \text{if} \quad \varepsilon_i(y) \geqslant \phi_i(x) \end{cases} \tag{4}$$

It is understood that $x \otimes 0 = 0 \otimes y = 0$.

Tensor products are associative [3, §2.3]. As noted in [3, Remark 1.1], the convention used there for tensor products differs from the widely-used convention that we adopt in this paper.

We define the *character* of a crystal $\mathcal{B}$ by $ch(\mathcal{B}) := \sum_{u \in \mathcal{B}} \mathbf{x}^{wt(u)}$. From the definition of the tensor product crystal, it is elementary to observe that its character equals the product of the characters of its factors.

The k-fold tensor product $\mathbb{B}^{\otimes k}$ of the standard (type A_{n-1}) crystal is called the *crystal of words* (of length k). Elements of the underlying set of $\mathbb{B}^{\otimes k}$ are called *words*. A word ζ is *dominant* if $e_i \zeta = 0$ for all i.

Remark 3.1. Let $\lambda, \mu \in \mathcal{P}[n]$ such that $|\lambda| - |\mu| = k$. Then $\mathrm{Tab}(\lambda/\mu, n)$ is given the structure of a type A_{n-1} crystal by the following embedding into $\mathbb{B}^{\otimes k}$ (where b_T is the reverse-row reading word of T, defined in §2):

$$T \mapsto b_T = w_1 w_2 \cdots w_k \mapsto w_1 \otimes w_2 \otimes \cdots \otimes w_k$$

It is well-known that the image under this embedding is a full subcrystal of $\mathbb{B}^{\otimes k}$. See for example [3, Theorem 8.8].

3.2. Demazure Crystals. Let $\lambda \in \mathcal{P}[n]$ and w a permutation in the symmetric group $\mathfrak{S}_n$. The *Demazure crystal* $\mathcal{B}_w(\lambda)$ is defined by:

$$\mathcal{B}_w(\lambda) := \{f_{i_1}^{k_1} f_{i_2}^{k_2} \cdots f_{i_p}^{k_p} T_\lambda^0 : k_j \geqslant 0\} \smallsetminus \{0\} \subset \operatorname{Tab}(\lambda, n) \tag{5}$$

where T_λ^0 is the unique *dominant* tableau of shape λ (i.e., with shape and weight both equal to λ) and $s_{i_1} s_{i_2} \cdots s_{i_p}$ is any reduced expression of w.

Remark 3.2. (1) It is a well-known fact that the definition of $\mathcal{B}_w(\lambda)$ is independent of the choice of reduced expression [3, Theorem 13.5].

(2) For any $w \in \mathfrak{S}_n$ and $\lambda \in \mathcal{P}[n]$, T_λ^0 is the unique element in $\mathcal{B}_w(\lambda)$ such that $e_i(T_\lambda^0) = 0$ for all i. In Eq. (5) above, we could replace T_λ^0 with any other dominant word $b_\lambda \in \mathbb{B}^{\otimes|\lambda|}$ of weight λ. We thereby obtain a subset of $\mathbb{B}^{\otimes|\lambda|}$:

$$\mathcal{B}_w(b_\lambda) := \{f_{i_1}^{k_1} f_{i_2}^{k_2} \cdots f_{i_p}^{k_p} b_\lambda : k_j \geqslant 0\} \smallsetminus \{0\}$$

which is isomorphic to $\mathcal{B}_w(\lambda)$ as crystals, i.e., there is a weight preserving bijection between these sets which intertwines the crystal raising and lowering operators (where defined). We also refer to $\mathcal{B}_w(b_\lambda)$ as a Demazure crystal in what follows, and write (by abuse of notation) $\mathcal{B}_w(\lambda)$ in place of $\mathcal{B}_w(b_\lambda)$.

The following proposition is the refined Demazure character formula in [6].

Proposition 3.3. *Let $\lambda \in \mathcal{P}[n]$ and $w \in \mathfrak{S}_n$. Then, $ch(\mathcal{B}_w(\lambda)) = \kappa_{w\lambda}$.*

Examples:

- $\mathcal{B}_{w_0}(\lambda) = \operatorname{Tab}(\lambda, n)$. This crystal is denoted as $B(\lambda)$ in the literature.
- $\mathcal{B}_{s_{k-1}\cdots s_2 s_1}((1)) = \operatorname{Tab}((1), k) \subset \operatorname{Tab}((1), n)$ for $1 \leqslant k \leqslant n$. We will denote this Demazure crystal by $\mathbb{B}_k$.

A subset $\mathcal{S}$ of a crystal $\mathcal{C}$ is said to have the *string property* if, given $i \in \{1, 2, \ldots, n-1\}$ and $x \in \mathcal{S}$ such that $e_i(x) \neq 0$, we have:

(1) $e_i(x) \in \mathcal{S}$,
(2) $f_i(x) \neq 0$ implies $f_i(x) \in \mathcal{S}$.

Proposition 3.4. [6, Proposition 3.3.5] *For any $\lambda \in \mathcal{P}[n]$ and $w \in \mathfrak{S}_n$, the Demazure subcrystal $\mathcal{B}_w(\lambda)$ of $\operatorname{Tab}(\lambda, n)$ has the string property.*

Remark 3.5. The converse of Proposition 3.4 is not true in general (see [1, §4]).

A characterization of when a tensor product of Demazure crystals decomposes into Demazure crystals was given in [7]. Following this, a different characterization was obtained in [1] as follows:

Theorem 3.6. [1, Theorem 1.2] *For $\lambda, \mu \in \mathcal{P}[n]$ and $w, w' \in \mathfrak{S}_n$, the subset $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_{w'}(\mu)$ of $\operatorname{Tab}(\lambda, n) \otimes \operatorname{Tab}(\mu, n)$ is a disjoint union of Demazure crystals if and only if it has the string property.*

Proposition 3.7. *Let X_1 and X_2 be subsets of crystals C_1 and C_2 respectively. Assume that for each $i \in \{1, 2, \ldots, n-1\}$ there exists $x_i^\dagger \in X_2$ such that $e_i(x_i^\dagger) = 0$ (i.e., for each i, X_2 contains a head of some "i-string"). Then, if $X_1 \otimes X_2$ has the string property, so does X_1.*

Proof. Let $i \in \{1, 2, \ldots, n-1\}$ be arbitrary. Suppose $u \in X_1$ such that $e_i(u) \neq 0$. Then $e_i(u \otimes x_i^\dagger) = e_i(u) \otimes x_i^\dagger \neq 0$ since $\varepsilon_i(x_i^\dagger) \leqslant \phi_i(u)$. Therefore by the string property of $X_1 \otimes X_2$ we have $e_i(u) \otimes x_i^\dagger \in X_1 \otimes X_2$, which implies that $e_i(u) \in X_1$.

Additionally, if $f_i(u) \neq 0$ then $f_i(u \otimes x_i^\dagger) = f_i(u) \otimes x_i^\dagger \neq 0$ since $\varepsilon_i(x_i^\dagger) < \phi_i(u)$. Therefore by the string property of $X_1 \otimes X_2$ we have $f_i(u) \otimes x_i^\dagger \in X_1 \otimes X_2$, which implies that $f_i(u) \in X_1$. ☐

Remark 3.8. Subsets of crystals that exhibit the string property are referred to as *extremal* in [1]. The above proposition is a strengthening of [1, Proposition 8.1]. For example, let $X_1 = \{112\} \subset \mathbb{B}^{\otimes 3}$ and $X_2 = \{12, 13\} \subset \mathbb{B}^{\otimes 2}$, where $\mathbb{B}$ is the standard type A_2 crystal. Then $e_1(13) = 0, e_2(12) = 0$ but $e_1(12) = 11 \notin X_2$. Thus, X_2 satisfies the hypotheses of Proposition 3.7, but not those of [1, Proposition 8.1]. Also, $X_1 \otimes X_2 = \{11212, 11213\}$ is a disjoint union of Demazure crystals since $11212, 11213$ are both dominant words. Proposition 3.7 allows us to conclude that X_1 must also have the string property (which is straightforward to see in this case anyhow).

Corollary 3.9. *Let $D_1, D_2, \ldots, D_k$ be Demazure crystals. If $D_1 \otimes D_2 \otimes \cdots \otimes D_k$ has the string property, then it is a disjoint union of Demazure crystals.*

Proof. We prove the assertion by induction on k. The case $k = 1$ is vacuous. Suppose $k > 1$. Then it follows from Proposition 3.7 that $D_1 \otimes D_2 \otimes \cdots \otimes D_{k-1}$ has the string property since D_k has $x_i^\dagger$ such that $e_i(x_i^\dagger) = 0\ \forall i$. Therefore by induction hypothesis $D_1 \otimes D_2 \otimes \cdots \otimes D_{k-1}$ is a disjoint union of Demazure crystals (say $= \bigsqcup \tilde{D}$). Now,

$$D_1 \otimes D_2 \otimes \cdots \otimes D_k = \bigsqcup (\tilde{D} \otimes D_k)$$

Observe that if $i \in \{1, 2, \ldots, n-1\}$ and $x \otimes y \in \tilde{D} \otimes D_k$ then $e_i(x \otimes y) \in (\tilde{D} \otimes D_k) \cup \{0\}$ because of Proposition 3.4.

Let $x \otimes y \in \tilde{D} \otimes D_k$ such that $e_i(x \otimes y) \neq 0$ and $f_i(x \otimes y) \neq 0$. It follows that $f_i(x \otimes y) \in \tilde{D} \otimes D_k$ because $e_i(f_i(x \otimes y)) = x \otimes y$ and the fact that the decomposition $D_1 \otimes D_2 \otimes \cdots \otimes D_{k-1} = \bigsqcup \tilde{D}$ is disconnected. Therefore $\tilde{D} \otimes D_k$ has the string property. By Theorem 3.6, it now follows that $\tilde{D} \otimes D_k$ is a disjoint union of Demazure crystals. □

Let Φ be a flag and $\rho \in \mathbb{Z}_+^n$ be a composition of $k \in \mathbb{Z}_+$ (i.e., $\rho_1 + \rho_2 + \cdots + \rho_n = k$). Define the subset of $\mathbb{B}^{\otimes k}$:

$$\mathbb{B}_\Phi^\rho := \mathbb{B}_{\Phi_1}^{\otimes \rho_1} \otimes \mathbb{B}_{\Phi_2}^{\otimes \rho_2} \otimes \cdots \otimes \mathbb{B}_{\Phi_n}^{\otimes \rho_n}$$

Lemma 3.10. *$\mathbb{B}_\Phi^\rho$ is a disjoint union of Demazure crystals.*

Proof. By Corollary 3.9, it is sufficient to show that $\mathbb{B}_\Phi^\rho$ has the string property.

Let $u = u_1 \otimes u_2 \otimes \cdots \otimes u_k \in \mathbb{B}_\Phi^\rho$. Suppose that $e_i(u) \neq 0$. Then $e_i(u) \in \mathbb{B}_\Phi^\rho$, because of Proposition 3.4 and the fact that $\mathbb{B}_{\Phi_i}$ is a Demazure crystal for each i.

Now suppose furthermore that $f_i(u) \neq 0$. Let t be the index where e_i acts in u. i.e.,

$$u = u_1 \otimes \cdots \otimes u_{t-1} \otimes (i+1) \otimes u_{t+1} \otimes \cdots \otimes u_k$$
$$e_i(u) = u_1 \otimes \cdots \otimes u_{t-1} \otimes i \otimes u_{t+1} \otimes \cdots \otimes u_k$$

Define $u' = u_1 \otimes \cdots \otimes u_{t-1}$ and $u'' = (i+1) \otimes u_{t+1} \otimes \cdots \otimes u_k$. Note that when $t = 1$ we have u' to be the empty word and $u'' = u$. Then by (3) it follows that $\varepsilon_i(u'') > \phi_i(u')$. Therefore, (4) implies $f_i(u) = u' \otimes f_i(u'')$. The action of f_i on u'' amounts to changing a u_{t_0} ($t_0 > t$) which is an i to an $i+1$. If u_t comes from the factor $\mathbb{B}_{\Phi_m}$ in $\mathbb{B}_\Phi^\rho$, then $\Phi_m \geqslant i+1$ and since Φ is weakly increasing, it follows that $f_i(u) \in \mathbb{B}_\Phi^\rho$. □

Theorem 3.11. *Let Φ be a flag and $\mu, \gamma \in \mathcal{P}[n]$ such that $\mu \subset \gamma$. Then* $\mathrm{Tab}(\mu/\gamma, \Phi)$ *is a disjoint union of Demazure crystals.*

Proof. Define the composition ρ by $\rho_i = \mu_i - \gamma_i, \quad 1 \leqslant i \leqslant n$. Let $k = |\rho| = \rho_1 + \cdots + \rho_n$. We then have

$$\mathrm{Tab}(\mu/\gamma, \Phi) = \mathrm{Tab}(\mu/\gamma, n) \cap \mathbb{B}_\Phi^\rho$$

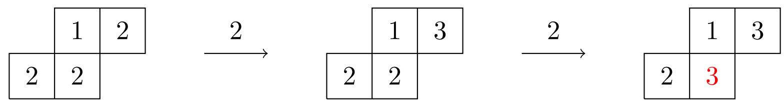


Figure 2. $\mathrm{Tab}(\mu/\gamma, \Phi)$ need not have the string property if Φ is not weakly increasing.

Since $\mathrm{Tab}(\mu/\gamma, n)$ is a full subcrystal of $\mathbb{B}^{\otimes k}$, the theorem follows from Lemma 3.10. ☐

Remark 3.12. The above theorem at the level of characters appears as Theorem 20 in [13]. But, in fact, as explained in the appendix below, a stronger version of it, giving the explicit decomposition of $\mathrm{Tab}(\mu/\gamma, \Phi)$ into a disjoint union of Demazure crystals, is implicit in that paper.

Remark 3.13. It is important that we assume Φ to be weakly increasing. For example, let $\mu = (3, 2)$, $\gamma = (1, 0)$ and $\Phi = (3, 2)$. Then $\mathrm{Tab}(\mu/\gamma, \Phi)$ is not a union of Demazure crystals. It does not even have the string property since it does not contain the third tableau in Figure 2.

Consider the tensor product $\mathcal{C}(\lambda; \mu, \gamma, \Phi) := \mathrm{Tab}(\lambda, \Phi_0) \otimes \mathrm{Tab}(\mu/\gamma, \Phi)$, where Φ_0 denotes the *standard flag* $(1, 2, 3, \ldots, n)$. The Demazure crystal $\mathrm{Tab}(\lambda, \Phi_0)$ is the singleton set containing T^0_λ.

Proposition 3.14. *$\mathcal{C}(\lambda; \mu, \gamma, \Phi)$ is a disjoint union of Demazure crystals.*

Proof. By Theorem 3.11, we have $\mathrm{Tab}(\mu/\gamma, \Phi) = \bigsqcup_p D_p$ where each D_p is a Demazure crystal. Then

$$\mathcal{C}(\lambda; \mu, \gamma, \Phi) = \mathrm{Tab}(\lambda, \Phi_0) \otimes (\bigsqcup D_p) = \bigsqcup \mathrm{Tab}(\lambda, \Phi_0) \otimes D_p$$

But $\mathrm{Tab}(\lambda, \Phi_0) \otimes D_p$ is a disjoint union of Demazure crystals by Joseph's theorem [5, §2.11]. This fact also follows from Theorem 3.6 because $\mathrm{Tab}(\lambda, \Phi_0) \otimes D$ has the string property as we show below:

For $u \otimes v \in \mathrm{Tab}(\lambda, \Phi_0) \otimes D$, $e_i(u \otimes v) \neq 0$ only if $e_i(u \otimes v) = u \otimes e_i(v)$. This implies that $\varepsilon_i(v) > \phi_i(u)$. By the tensor product rule we therefore have $f_i(u \otimes v) = u \otimes f_i(v)$. By assumptions, $e_i(v) \neq 0$ and $f_i(v) \neq 0$. Since D is a Demazure crystal, by Proposition 3.4 it follows that $f_i(v) \in D$ and hence $f_i(u \otimes v) \in \mathrm{Tab}(\lambda, \Phi_0) \otimes D$. ☐

Corollary 3.15. *$\mathbf{x}^\lambda\, s_{\mu/\gamma}(\Phi) = ch(\mathcal{C}(\lambda; \mu, \gamma, \Phi))$ is key-positive.*

This corollary generalizes [13, Theorem 20] to which it reduces when λ is empty.

A skew tableau $T \in \mathrm{Tab}(\mu/\gamma, n)$ is λ-*dominant* if the concatenated word $b_{T^0_\lambda} * b_T$ is a dominant word. We now prove the first part of Theorem 2.2.

Theorem 3.16. *$c^\nu_{\lambda,\,\mu/\gamma}(\Phi)$ is the cardinality of the set $\mathrm{Tab}^\nu_\lambda(\mu/\gamma, \Phi)$ of all λ-dominant tableaux in $\mathrm{Tab}(\mu/\gamma, \Phi)$ of weight $\nu - \lambda$.*

Proof. By Proposition 3.14 (and Remark 3.2), for all $\nu \in \mathcal{P}[n]$ there exists a multisubset $\mathcal{W}(\nu) \subseteq \mathfrak{S}_n$ such that

$$\mathcal{C}(\lambda; \mu, \gamma, \Phi) = \bigsqcup_{\nu \in \mathcal{P}[n]} \bigsqcup_{w \in \mathcal{W}(\nu)} \mathcal{B}_w(\nu) \tag{6}$$

Taking characters, we obtain

$$\mathbf{x}^\lambda\, s_{\mu/\gamma}(\Phi) = \sum_\nu \sum_{w \in \mathcal{W}(\nu)} \kappa_{w\nu}$$

Applying T_{w_0} and using Eq. (1), we get $T_{w_0}(\mathbf{x}^\lambda s_{\mu/\gamma}(\Phi)) = \sum_\nu |\mathcal{W}(\nu)| s_\nu(\mathbf{x})$. Thus

$$c^\nu_{\lambda,\,\mu/\gamma}(\Phi) = |\mathcal{W}(\nu)|$$

Now, by definition of λ-dominance, the number of elements $\zeta \in \mathcal{C}(\lambda;\mu,\gamma,\Phi)$ of weight ν satisfying $e_i\zeta = 0$ for all i is precisely $|\operatorname{Tab}^\nu_\lambda(\mu/\gamma,\Phi)|$. On the other hand, in the RHS of Eq. (6), each $\mathcal{B}_w(\nu)$ has a unique element ξ such that $e_i\xi = 0$ for all i; this element has weight ν. Thus, the number of elements ζ as above is also equal to $|\mathcal{W}(\nu)|$. Putting all these together establishes Theorem 3.16. □

4. A hive model for flagged skew LR coefficients

In this section we define the skew hive polytope and some associated notions. We then prove the second part of Theorem 2.2.

4.1. Skew GT patterns. Given $m, n \geqslant 1$, a skew GT pattern is an array of real numbers $\{x_{ij} : 0 \leqslant i \leqslant m,\ \ 1 \leqslant j \leqslant n\}$ satisfying the following inequalities:

$$NE_{ij} = x_{ij} - x_{(i-1)j} \geqslant 0 \qquad 1 \leqslant i \leqslant m;\quad 1 \leqslant j \leqslant n$$
$$SE_{ij} = x_{(i-1)j} - x_{i(j+1)} \geqslant 0 \qquad 1 \leqslant i \leqslant m;\quad 1 \leqslant j \leqslant n-1$$

The above inequalities simply mean that the consecutive rows "interlace". Following standard convention, we arrange the rows in the shape of a parallelogram as follows (shown for $m = n = 4$):

$$\begin{array}{cccccccc}
 & & & & x_{01} & x_{02} & x_{03} & x_{04} \\
 & & & x_{11} & x_{12} & x_{13} & x_{14} & \\
 & & x_{21} & x_{22} & x_{23} & x_{24} & & \\
 & x_{31} & x_{32} & x_{33} & x_{34} & & & \\
x_{41} & x_{42} & x_{43} & x_{44} & & & &
\end{array}$$

For $\mu, \gamma \in \mathcal{P}[n]$, such that $\gamma \subset \mu$, the skew GT polytope $\operatorname{GT}(\mu/\gamma, m, n)$ is the set of all skew GT patterns (x_{ij}) with $0 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ satisfying $x_{0j} = \gamma_j$, $x_{mj} = \mu_j$ for all j. Define,

$$\operatorname{GT}_{\mathbb{Z}}(\mu/\gamma, m, n) := \{X = (x_{ij}) \in \operatorname{GT}(\mu/\gamma, m, n) : x_{ij} \in \mathbb{Z}\}$$

In the sequel, we will only have occasion to consider the case when $m = n$. We write $\operatorname{GT}(\mu/\gamma, n)$ for $\operatorname{GT}(\mu/\gamma, n, n)$ and $\operatorname{GT}_{\mathbb{Z}}(\mu/\gamma, n)$ for $\operatorname{GT}_{\mathbb{Z}}(\mu/\gamma, n, n)$.

Consider the map

$$\Upsilon : \operatorname{GT}_{\mathbb{Z}}(\mu/\gamma, n) \to \operatorname{Tab}(\mu/\gamma, n)$$

where $X = (x_{ij}) \in \operatorname{GT}_{\mathbb{Z}}(\mu/\gamma, n)$ maps to the unique tableau $\Upsilon(X)$ in $\operatorname{Tab}(\mu/\gamma)$ such that the number of i that appears in the j^{th} row of $\Upsilon(X)$ is $x_{ij} - x_{(i-1)j}$. The following statement is well known—see for instance [11, §3] (whose pattern drawing convention differs from ours by a vertical flip).

Lemma 4.1. *The map Υ is a bijection.*

4.2. Flagged skew GT patterns. We keep the notation of the previous subsection, but in addition assume that we are given a flag $\Phi = (\Phi_1, \dots, \Phi_n)$, i.e., $0 < \Phi_1 \leqslant \cdots \leqslant \Phi_n = n$. Define the set of flagged skew GT patterns:

(7) $\quad \operatorname{GT}(\mu/\gamma, \Phi) = \{(x_{ij}) \in \operatorname{GT}(\mu/\gamma, n) : x_{n,j} = x_{n-1,j} = \cdots = x_{\Phi_j,j} \quad \forall 1 \leqslant j \leqslant n\}$

and let $\operatorname{GT}_{\mathbb{Z}}(\mu/\gamma, \Phi)$ denote the set of integer points in this polytope. We have:

Lemma 4.2. *The map Υ restricts to a bijection between $\operatorname{GT}_{\mathbb{Z}}(\mu/\gamma, \Phi)$ and $\operatorname{Tab}(\mu/\gamma, \Phi)$.*

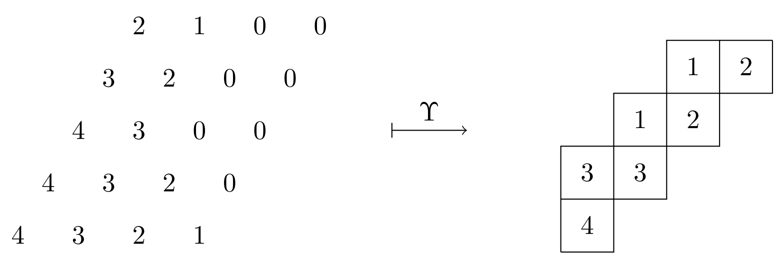


Figure 3. The skew GT pattern on the left maps to the skew tableau on the right under the map Υ.

Proof. Let $X = (x_{ij}) \in \mathrm{GT}_{\mathbb{Z}}(\mu/\gamma, \Phi)$. If $\Phi = (n, n, \ldots, n)$ then it is easy to see $\Upsilon(X) \in \mathrm{Tab}(\mu/\gamma, \Phi)$. Otherwise let k be the maximum such that $\Phi_k \neq n$. Then the number of $i\,(\Phi_j < i \leqslant n)$ that appear in the j^{th} row $(1 \leqslant j \leqslant k)$ of $\Upsilon(X)$ is $x_{ij} - x_{(i-1)j} = 0$. Since $\forall j > k$, $\Phi_j = n$ so for those j in j^{th} row of $\Upsilon(X)$ all entries are $\leqslant \Phi_j (= n)$. Thus $\Upsilon(X) \in \mathrm{Tab}(\mu/\gamma, \Phi)$. Also, if $T \in \mathrm{Tab}(\mu/\gamma, \Phi)$ then $(i,j)^{th}$ entry of $\Upsilon^{-1}(T)$ is the number of entries $\leqslant i$ that appear in the j^{th} row of T. So $\Upsilon^{-1}(T) \in \mathrm{GT}_{\mathbb{Z}}(\mu/\gamma, \Phi)$. □

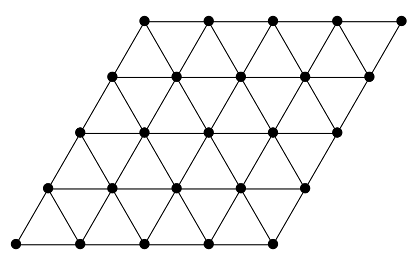
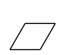
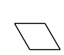


Figure 4. The 4-hive parallelogram

In the next two subsections, we give a hive model for the flagged skew LR coefficients.

4.3. Skew hives. The $(n+1) \times (n+1)$ array of nodes as in Figure 4 is called the *n-hive parallelogram*. Observe that the *small rhombi*—those with unit side length 1—in the n-hive parallelogram are oriented in three different ways:

Northeast (NE):  Southeast (SE):  Vertical: 

Let $\bar{\lambda} := (0, \lambda_1, \lambda_1 + \lambda_2, \ldots, |\lambda|)$ and $\partial\lambda := (\lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \ldots, \lambda_n - \lambda_{n-1})$ for $\lambda \in \mathcal{P}[n]$. Let $\lambda, \mu, \gamma, \nu \in \mathcal{P}[n]$ be such that $\gamma \subset \mu$, $\lambda \subset \nu$ and $|\lambda| + |\mu| = |\nu| + |\gamma|$. We define the *skew hive polytope* $\mathrm{SHive}(\lambda, \mu, \gamma, \nu)$ as the set of all $\mathbb{R}$-labellings of the nodes of the n-hive parallelogram such that:

(1) The boundary labels are as follows (see Figure 5(a)):
 left (top to bottom) given by $\bar{\lambda}$
 bottom (left to right) given by $|\lambda| + \bar{\mu}$
 top (left to right) given by $\bar{\gamma}$
 right (top to bottom) given by $|\gamma| + \bar{\nu}$

(2) The *contents* of all the small rhombi are non-negative. We recall that the content of a small rhombus is the sum of the labels on its obtuse-angled nodes minus the sum of the labels on its acute-angled nodes. A rhombus that has zero content is said to be *flat*.

We denote the set of integer points in $\mathrm{SHive}(\lambda,\mu,\gamma,\nu)$ by $\mathrm{SHive}_{\mathbb{Z}}(\lambda,\mu,\gamma,\nu)$. An el-

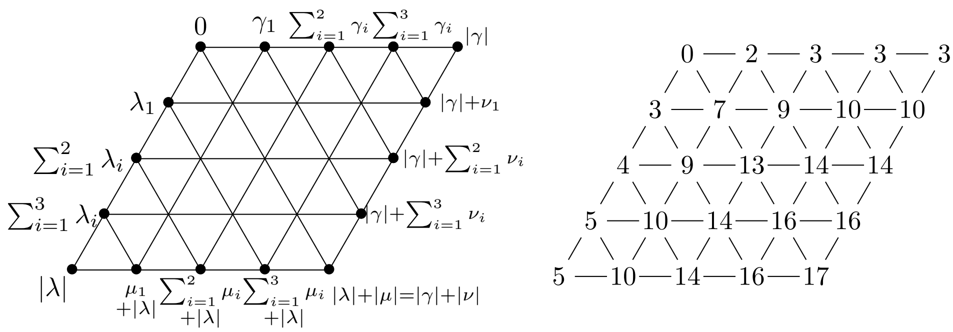


FIGURE 5. (a) The boundary labels of skew hives in $\mathrm{SHive}(\lambda,\mu,\gamma,\nu)$. (b) A skew hive in $\mathrm{SHive}_{\mathbb{Z}}(\lambda,\mu,\gamma,\nu)$ where $\lambda=(3,1,1,0)$; $\gamma=(2,1,0,0)$; $\mu=(5,4,2,1)$; $\nu=(7,4,2,1)$.

ement of $\mathrm{SHive}(\lambda,\mu,\gamma,\nu)$ is called a skew hive with boundary (λ,μ,γ,ν). The rows of a skew hive $h\in\mathrm{SHive}(\lambda,\mu,\gamma,\nu)$ (from top to bottom) give a sequence of vectors $h_0,h_1,\dots,h_n\in\mathbb{R}^{n+1}$. Consider the parallelogram array ∂h with $(n+1)$ rows (and n nodes in each row) whose rows (from top to bottom) are $\partial h_0,\partial h_1,\dots,\partial h_n$. It follows directly (adapting the arguments of [8]) that the positivity of the contents of Northeast and Southeast rhombi in h correspond to positivity (in the skew GT setting of §4.1) of NE_{ij} and SE_{ij} of ∂h respectively. Thus ∂h is a skew GT pattern with top row γ and bottom row μ. It is elementary to check that the map $\partial:\mathrm{SHive}(\lambda,\mu,\gamma,\nu)\longrightarrow\mathrm{GT}(\mu/\gamma,n)$ is linear, injective and maps integer points to integer points.

THEOREM 4.3. *For $\lambda,\mu,\gamma,\nu\in\mathcal{P}[n]$ such that $\gamma\subset\mu$, $\lambda\subset\nu$ and $|\lambda|+|\mu|=|\nu|+|\gamma|$, the map $\Upsilon\circ\partial$ is a bijection between $\mathrm{SHive}_{\mathbb{Z}}(\lambda,\mu,\gamma,\nu)$ and $\mathrm{Tab}_{\lambda}^{\nu}(\mu/\gamma,n)=\{T\in\mathrm{Tab}(\mu/\gamma,n): b_{T_\lambda^0}*b_T$ is a dominant word of weight $\nu\}$.*

Proof. We follow closely the proof of [9, Proposition 4]. Let $h\in\mathrm{SHive}_{\mathbb{Z}}(\lambda,\mu,\gamma,\nu)$. By the preceding discussion and Lemma 4.1, it is clear that $T=\Upsilon(\partial h)\in\mathrm{Tab}(\mu/\gamma,n)$. Now we will show that $T\in\mathrm{Tab}_{\lambda}^{\nu}(\mu/\gamma,n)$. Let $h=(h_0,h_1,\dots,h_n)$, where $h_i=(h_{i0},h_{i1},\dots,h_{in})$ for all $0\leqslant i\leqslant n$. Then $\partial h=(x_{ij})$ such that $x_{ij}=h_{ij}-h_{i(j-1)}$ for all $0\leqslant i\leqslant n, 1\leqslant j\leqslant n$. So the number of times i appears in row j of $T=x_{ij}-x_{(i-1)j}$. Now we have to prove that $b_{T_\lambda^0}*b_T$ is a dominant word of weight ν. Let $b_T=b_{T_1}*b_{T_2}*\cdots*b_{T_n}$ where b_{T_k} is the reverse reading word of the k-th row of T. Also, let N_{ik} be the number of times i appears in the word $b_{T_0}*b_{T_1}*\cdots*b_{T_k}$ (with $b_{T_0}=b_{T_\lambda^0}$). Then by definition, $b_{T_\lambda^0}*b_T$ is dominant if and only if $N_{ik}\geqslant N_{(i+1)(k+1)}$ for all $1\leqslant i\leqslant n,\ 0\leqslant k\leqslant n$. We get

$$N_{ik}=\lambda_i+(x_{i1}-x_{(i-1)1})+\cdots+(x_{ik}-x_{(i-1)k})=h_{ik}-h_{(i-1)k}\text{ (in terms of }h)$$

So $N_{ik}-N_{(i+1)(k+1)}=h_{ik}+h_{i(k+1)}-h_{(i-1)k}-h_{(i+1)(k+1)}\geqslant 0$ by the corresponding vertical rhombus inequality in h. Now the number of times i occurs in $b_{T_\lambda^0}*b_T$ is

$$\lambda_i+(x_{i1}-x_{(i-1)1})+\cdots+(x_{in}-x_{(i-1)n})=h_{in}-h_{(i-1)n}=\nu_i$$

So $b_{T^0_\lambda} * b_T$ is a dominant word of weight ν. Thus $T \in \mathrm{Tab}^\nu_\lambda(\mu/\gamma, n)$.

Now, we will show $\Upsilon \circ \partial$ is injective. It suffices to show that ∂ is injective. Let $h = (h_0, h_1, \dots, h_n)$, $h' = (h'_0, h'_1, \dots, h'_n) \in \mathrm{SHive}_{\mathbb{Z}}(\lambda, \mu, \gamma, \nu)$ where $h_i, h'_i \in \mathbb{Z}^{n+1}$ and $\partial h = \partial h'$. Also, let $h_i = (h_{i0}, h_{i1}, \dots, h_{in})$ and $h'_i = (h'_{i0}, h'_{i1}, \dots, h'_{in})$. We will show that $h_{ik} = h'_{ik}$ for all i by induction on k. Clearly, $h_{00} = h'_{00} = 0$ and $h_{i0} = \sum_{j=1}^{i} \lambda_j = h'_{i0}$. So for $k = 0$ we get $h_{ik} = h'_{ik}$ for all i. Let $k \geqslant 1$ and $h_{i(k-1)} = h'_{i(k-1)}$ for all i. Since $\partial h_i = \partial h'_i$. So $h_{ik} - h_{i(k-1)} = h'_{ik} - h'_{i(k-1)}$. Thus $h_{ik} = h'_{ik}$ for all i. Hence $h = h'$. Therefore ∂ is injective.

Next we will prove $\Upsilon \circ \partial$ is surjective. Let $S \in \mathrm{Tab}^\nu_\lambda(\mu/\gamma, n)$ and $\Upsilon^{-1}(S) = (s_{ij}) \in \mathrm{GT}_{\mathbb{Z}}(\mu/\gamma, n)$. Then, for any $j, 1 \leqslant j \leqslant n$, we have $s_{0j} = \gamma_j$ and $s_{ij} = \gamma_j + n(i, j)$, where $n(i, j)$ denotes the number of elements not exceeding i in j^{th} row of S. Define $\lambda_0 = 0$, $h_i = (h_{i0}, h_{i1}, \dots, h_{in})$ where $h_{i0} = \sum_{k=0}^{i} \lambda_k$ and

$$h_{ij} = h_{i0} + \sum_{k=1}^{j} s_{ik} \quad (0 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n).$$

Consider the n-hive parallelogram h whose rows (from top to bottom) are $h_0, h_1, \dots, h_n$. We claim that $h \in \mathrm{SHive}_{\mathbb{Z}}(\lambda, \mu, \gamma, \nu)$. The Northeast and Southeast rhombi inequalities of h hold since $S \in \mathrm{Tab}(\mu/\gamma, n)$ and the vertical rhombi inequalities of h hold because $b_{T^0_\lambda} * b_S$ is dominant. The boundary labels of h can be easily seen to be $\bar{\lambda}$ (left edge, top to bottom), $|\lambda| + \bar{\mu}$ (bottom edge, left to right), $\bar{\gamma}$ (top edge, left to right) and $|\gamma| + \bar{\nu}$ (right edge, top to bottom) respectively. So $h \in \mathrm{SHive}_{\mathbb{Z}}(\lambda, \mu, \gamma, \nu)$ and $(\Upsilon \circ \partial)^{-1}(S) = h$. Thus $\Upsilon \circ \partial$ is surjective. □

4.4. Flagged skew hives. Given a flag Φ, consider the *face* of the skew hive polytope defined by $\mathrm{SHive}(\lambda, \mu, \gamma, \nu, \Phi) := \partial^{-1}(\mathrm{GT}(\mu/\gamma, \Phi))$. We call this the *flagged skew hive polytope*. The set of integer points in this polytope is denoted by $\mathrm{SHive}_{\mathbb{Z}}(\lambda, \mu, \gamma, \nu, \Phi)$. It is elementary to observe that $\mathrm{SHive}_{\mathbb{Z}}(\lambda, \mu, \gamma, \nu, \Phi) = \partial^{-1}(\mathrm{GT}_{\mathbb{Z}}(\mu/\gamma, \Phi))$. Lemma 4.2 and Theorem 4.3 together give us the second part of Theorem 2.2:

Theorem 4.4. *The map $\Upsilon \circ \partial$ restricts to a bijection between* $\mathrm{SHive}_{\mathbb{Z}}(\lambda, \mu, \gamma, \nu, \Phi)$ *and* $\mathrm{Tab}^\nu_\lambda(\mu/\gamma, \Phi)$. *Then* $c^\nu_{\lambda,\,\mu/\gamma}(\Phi) = |\,\mathrm{SHive}_{\mathbb{Z}}(\lambda, \mu, \gamma, \nu, \Phi)| = |\,\mathrm{Tab}^\nu_\lambda(\mu/\gamma, \Phi)|$.

The NE rhombi of the n-hive parallelogram are labelled R_{ij} with $1 \leqslant i, j \leqslant n$ as shown in the example in Figure 6. To the flag Φ, we associate the set $I = \bigcup_{i=1}^{n} \{(\Phi_i + 1, i), \dots, (n, i)\}$. Then $R(\Phi) = \bigcup_{(i,j) \in I} \{R_{ij}\}$ forms a bottom-left-justified region of NE rhombi (shown in purple in Figure 6). It is easy to see from Eq. (7) that $\mathrm{SHive}_{\mathbb{Z}}(\lambda, \mu, \gamma, \nu, \Phi)$ is obtained from $\mathrm{SHive}_{\mathbb{Z}}(\lambda, \mu, \gamma, \nu)$ by imposing the condition that all rhombi in $R(\Phi)$ are flat. For example, the hive in Figure 5(b) is such that every NE rhombus in $R(\Phi)$ is flat where $\Phi = (2,2,3,4)$.

Remark 4.5. Recall the following relations in the space of symmetric functions

$$\langle s_\lambda s_{\mu/\gamma}, s_\nu \rangle = \langle s_{\mu/\gamma}, s_{\nu/\lambda} \rangle = \langle s_{\nu/\lambda}, s_{\mu/\gamma} \rangle = \langle s_\gamma s_{\nu/\lambda}, s_\mu \rangle$$

where, the inner product is the Hall inner product in the ring of symmetric polynomials. The above theorem gives a bijective proof for this identity namely, flip each skew hive (i.e., transpose) in $\mathrm{SHive}_{\mathbb{Z}}(\lambda, \mu, \gamma, \nu)$ to get a unique skew hive in $\mathrm{SHive}_{\mathbb{Z}}(\gamma, \nu, \lambda, \mu)$.

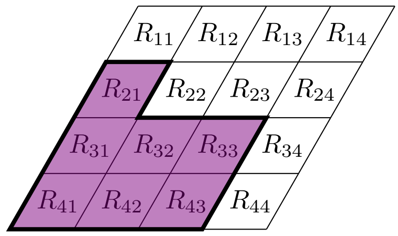


FIGURE 6. Labelling of NE oriented rhombi in 4-hive parallelogram and the shaded region is a typical configuration of $R(\Phi)$ (for $\Phi = (1,2,2,4)$)

5. Flagged Skew LR coefficients are w-refined LR coefficients

In this section we prove that any skew hive polytope is affinely isomorphic to some *hive polytope* (albeit in twice as many ambient dimensions). Moreover, this isomorphism maps the flagged skew hive polytope to a hive Kogan face corresponding to some 312-avoiding permutation [8, §2.4] and preserves the integral points.

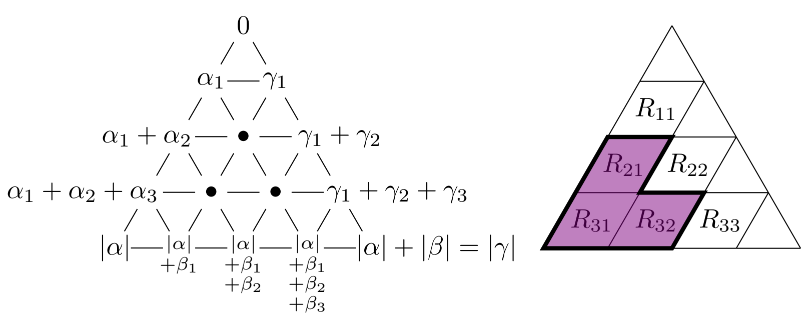


FIGURE 7. (a) A 4-hive with boundary (α,β,γ) (b) Labelling of NE oriented rhombi. The shaded region is a typical configuration of $R(\Phi)$, shown here for $\Phi = (2,3,4,4)$

Given partitions $\alpha,\beta,\gamma \in \mathcal{P}[n]$ such that $|\alpha| + |\beta| = |\gamma|$, the hive polytope $\mathrm{Hive}(\alpha,\beta,\gamma)$ is the set of all $\mathbb{R}$-labellings of an $(n+1)$-triangular array of nodes such that

(1) the boundary labels are given by $(0,\alpha_1,\alpha_1+\alpha_2,\ldots,|\alpha|,|\alpha|+\beta_1,|\alpha|+\beta_1+\beta_2,\ldots,|\alpha|+|\beta|,\gamma_1+\cdots+\gamma_{n-1},\ldots,\gamma_1+\gamma_2,\gamma_1)$, reading the nodes anticlockwise beginning from the topmost node as in Figure 7.
(2) the contents of all the small rhombi are non-negative.

The horizontal strings of nodes ("rows") of the triangular array are termed the zeroth row, first row, second row, etc starting from the top. Consider the labelling of the NE oriented small rhombi as shown in the example in Figure 7. Given a flag Φ, consider the set of NE rhombi $R(\Phi) = \{R_{ij} \,|\, n > i \geqslant \Phi_j\}$. We define the face $\mathrm{Hive}(\alpha,\beta,\gamma,\Phi)$ of the polytope $\mathrm{Hive}(\alpha,\beta,\gamma)$ as the collection of those hives in which all the rhombi in $R(\Phi)$ are flat. As we vary Φ, these run over the *hive Kogan faces* corresponding to 312-avoiding permutations, in the terminology of [8, §2.4]. Refer [8] for more details.

LEMMA 5.1. *Let $\alpha,\beta,\gamma \in \mathbb{Z}_+^n$ be weakly decreasing sequences such that either α or β is a constant sequence. Then $\mathrm{Hive}(\alpha,\beta,\gamma)$ is either empty or a singleton set. The latter is true if and only if $\alpha+\beta=\gamma$.*

Proof. Supposing first that $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is constant, i.e., $\beta_i = b$ (say) for all i. If $\mathrm{Hive}(\alpha, \beta, \gamma)$ is non-empty, let h be an element. Define $\kappa = \gamma - \alpha \in \mathbb{R}^n$. The labels along the left and right edges of h are $\sum_{i=1}^{k} \alpha_i$ and $\sum_{i=1}^{k} \gamma_i$ for $0 \leqslant k \leqslant n$. Now consider the following types of trapezia formed by two overlapping unit rhombi, one NE and the other SE.

$$\begin{array}{ccccccc} & & d & — & e & & \\ & / & & \backslash\ / & & \backslash & \\ c & — & & b & & — & a \end{array}$$

From the two rhombus inequalities in this picture, we conclude that if $a-b = b-c$, then each is also equal to $e-d$. Now $\beta_i = b$ for all i implies that the successive differences of labels on the bottom (i.e., n^{th}) row of h are all equal to b. The observation above means that the successive differences of labels on the $(n-1)^{th}$ row are also all b. We proceed by induction, moving up the hive triangle, to conclude that the successive differences of labels along every row of h is equal to b. In particular, all labels of h are uniquely determined from those on the left boundary alone. An additional compatibility condition arises from summing the differences in each row - this gives $\sum_{i=1}^{k} \kappa_i = kb$ for each k, or equivalently that $\kappa = \beta$ as claimed.

The proof for the case that α is constant is similar. One considers instead the trapezia of the form:

$$\begin{array}{ccccc} & & a & & \\ & / & & \backslash & \\ & b & — & e & \\ / & & \backslash\ / & & \\ c & — & d & & \end{array}$$

□

Let $\lambda, \mu, \nu, \gamma \in \mathcal{P}[n]$ be such that $\gamma \subset \mu$, $\lambda \subset \nu$ and $|\lambda| + |\mu| = |\nu| + |\gamma|$. Define $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \in \mathcal{P}[2n]$ by $\tilde{\lambda} = (\nu_1, \dots, \nu_1, \lambda_1, \dots, \lambda_n)$, $\tilde{\mu} = (\mu_1, \mu_2, \dots, \mu_n, 0, 0, \dots, 0)$ and $\tilde{\nu} = (\nu_1 + \gamma_1, \dots, \nu_1 + \gamma_n, \nu_1, \dots, \nu_n)$. Then we define the following map

$$\psi : \mathrm{SHive}(\lambda, \mu, \gamma, \nu) \to \mathrm{Hive}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$$

For $h \in \mathrm{SHive}(\lambda, \mu, \gamma, \nu)$ we describe $\psi(h)$ as follows (see Figure 8 for a representative example):

(1) $\psi(h)$ is a labelling of the $(2n+1)$- triangular array, with boundary labels coinciding with those of hives in $\mathrm{Hive}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$.
(2) The bottom-left-justified $(n+1) \times (n+1)$ parallelogram in $\psi(h)$ (white, with a red border in Figure 8) is labelled by $h + n \cdot \nu_1$, i.e., the labels of the nodes of h are translated by the constant $n \cdot \nu_1$.
(3) The labels of the top n rows of $\psi(h)$ (highlighted in yellow in Figure 8) are determined by the boundary conditions of $\mathrm{Hive}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$ and the choice of parallelogram labels in (2) above. This follows from Lemma 5.1, using the fact that the first n components of $\tilde{\lambda}$ are equal.
(4) Again, using (2) and the fact that the last n components of $\tilde{\mu}$ are equal, Lemma 5.1 implies that the labels of the bottom-right-justified $(n+1)$-triangular subarray in $\psi(h)$ (highlighted in blue in Figure 8) are determined by the boundary conditions of $\mathrm{Hive}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$.

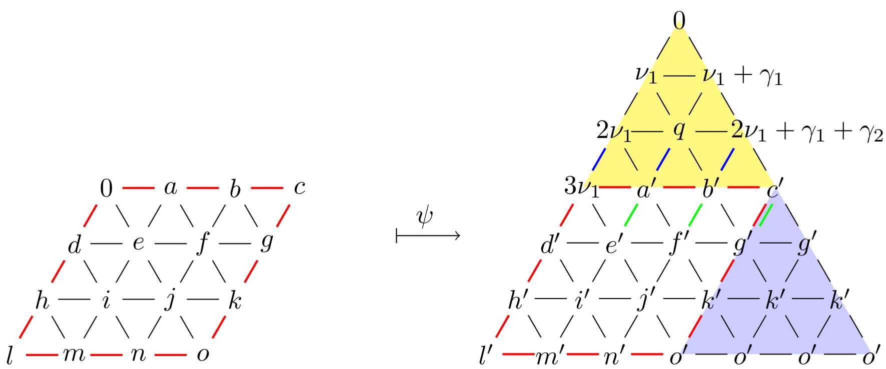


Figure 8. Here, $q = 2\nu_1 + \gamma_1$ and for x a label in h we write x' to denote $x + 3\nu_1$

.

Proposition 5.2. *ψ is well-defined and an affine linear isomorphism between* $\mathrm{SHive}(\lambda, \mu, \gamma, \nu)$ *and* $\mathrm{Hive}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$. *Moreover, the isomorphism preserves integral points and maps* $\mathrm{SHive}(\lambda, \mu, \gamma, \nu, \Phi)$ *onto* $\mathrm{Hive}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}, \tilde{\Phi})$, *where* $\tilde{\Phi} = (\Phi_1 + n, \dots, \Phi_n + n, 2n, \dots, 2n) \in \mathbb{Z}_+^{2n}$.

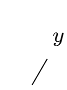
Proof. To verify the well-definedness of the map ψ, it suffices to check the rhombus inequalities in $\psi(h)$ for the following $2n$ rhombi:

- The n SE rhombi each of which straddles the regions described in (2) and (4).
- The n vertical rhombi each of which straddles the regions described in (2) and (3).

The first n rhombi inequalities hold because of the fact that the entries of h increase along the rows (this follows from the Southeast rhombi inequalities and the fact that $\mu_t \geqslant 0 \ \forall t$). The second n rhombi inequalities hold because the edge labels(3) of the Northeast edges (green edges in Figure 8) originating from the $(n+2)^{th}$ row is bounded above by ν_1, which is equal to the edge label of the Northeast edges originating from the $(n+1)^{th}$ row (blue edges in Figure 8). This follows from the Northeast rhombi inequalities and the boundary condition on h. For example, in Figure 8 we have $e' - a' \leqslant f' - b' \leqslant g' - c' = \nu_1$. Therefore the map ψ is well defined.

It is clear from the definition of the map that it is injective, affine linear and sends integral skew hives to integral hives. We now establish surjectivity. Given a hive in $\mathrm{Hive}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$, consider its triangular subarrays marked in yellow and blue in Figure 8. These are themselves hives, and Lemma 5.1 implies that these hives are uniquely determined (since the corresponding hive polytopes are non-empty). In particular, the labels on the bottom row of the yellow triangle are $\bar{\gamma} + n \cdot \nu_1$ where $\bar{\gamma}$ denotes the vector of partial sums of γ as defined in §4.3. Likewise, the labels on the left edge of the blue triangle must be $|\gamma| + \bar{\nu} + n \cdot \nu_1$. These are also the edges of the white parallelogram in Figure 8. The other two edges of the parallelogram have edge labels $\bar{\lambda} + n \cdot \nu_1$ and $|\lambda| + \bar{\mu} + n \cdot \nu_1$. This proves the surjectivity of ψ.

Finally, since the map ψ does not change rhombus contents, it does not alter any flatness conditions within the white parallelogram. Thus if the left and bottom justified region $R(\Phi)$ is flat in h, then it remains flat in $\psi(h)$; however since $\psi(h)$

(3)The edge label of the Northeast edge 

is defined to be $x - y$.

is a triangular hive in twice as many ambient dimensions, this region would now correspond to $R(\tilde{\Phi})$ in $\psi(h)$, where $\tilde{\Phi} = (\Phi_1 + n, \dots, \Phi_n + n, 2n, \dots, 2n) \in \mathbb{Z}_+^{2n}$. ☐

We remark that $\mathrm{Hive}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}, \tilde{\Phi})$ coincides with $K^{\mathrm{Hive}}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}, w(\tilde{\Phi}))$, the hive Kogan face of [8], where $w(\tilde{\Phi}) \in \mathfrak{S}_{2n}$ is the unique 312-avoiding permutation corresponding to the flag $\tilde{\Phi}$ [12, §14], [8, §2.4]. Now, as a consequence of the proposition, we have:

Theorem 5.3. *The flagged skew LR coefficients coincide with certain w-refined (where, w is 312-avoiding) LR coefficients of* [8]. *More precisely,* $c^{\nu}_{\lambda,\,\mu/\gamma}(\Phi) = c^{\tilde{\nu}}_{\tilde{\lambda},\,\tilde{\mu}}(w(\tilde{\Phi}))$ *where $w(\tilde{\Phi})$ is the unique 312-avoiding permutation corresponding to the flag $\tilde{\Phi}$.*

It is elementary to check that $\widetilde{k\lambda} = k\tilde{\lambda}$ and $\widetilde{k\nu} = k\tilde{\nu}$. This implies, $c^{k\nu}_{k\lambda,\,k\mu/k\gamma}(\Phi) = c^{k\tilde{\nu}}_{k\tilde{\lambda},\,k\tilde{\mu}}(w(\tilde{\Phi}))$. But Theorem 1.4 of [8] establishes the saturation property of the w-refined LR coefficients when w is 312-avoiding. Together with the preceding remarks, this proves Theorem 2.3 which asserts that the saturation property holds for the flagged LR coefficients.

We remark here that Theorem 2.3 can also be proved by working directly with the skew hive polytope, rather than with its isomorphic hive polytope. This involves mimicking all the arguments of [2, 8] for skew hives. While we have chosen a shorter approach in this paper, this alternate approach naturally suggests numerous other refinements of the LR coefficients with the saturation property. These will be considered in a future publication.

Appendix A. Decomposition of flagged skew tableaux into Demazure crystals

In this section, we make Theorem 3.11 effective, describing algorithmically the Demazure crystals which occur in the decomposition. The arguments below are implicit in the character level proof of [13], and so we content ourselves with sketching their broad contours.

We start with a brief discussion of the Burge correspondence [4]. We use the standard notation $[n] = \{1, 2, \dots, n\}$. Consider a matrix $M = (m_{ij})$ of size $r \times n$ with non-negative integer entries. We associate a biword to M as follows

$$w_M = \begin{bmatrix} i_t & \cdots & i_2 & i_1 \\ j_t & \cdots & j_2 & j_1 \end{bmatrix}$$

such that for any pair (i, j) that indexes an entry m_{ij} of M, there are m_{ij} columns equal to $\begin{bmatrix} i \\ j \end{bmatrix}$ in w_M, and the columns of w_M are ordered as follows:

(1) $i_t \geqslant \cdots \geqslant i_2 \geqslant i_1 \geqslant 1$.
(2) $i_{k+1} > i_k$ whenever $j_{k+1} > j_k$.

In other words, form the biword w_M by reading the entries m_{ij} of M from left to right within each row starting with the bottom row and proceeding upwards, recording each $\begin{bmatrix} i \\ j \end{bmatrix}$ with multiplicity m_{ij}. We will often denote the row and column indices of the biword by $\mathbf{i} = i_1 i_2 \cdots i_t$ and $\mathbf{j} = j_1 j_2 \cdots j_t$. Additionally, given a flag Φ, if $i_k \leqslant \Phi_{j_k}$ for all k (in particular, the matrix M is block upper-triangular) then $\mathbf{i}$ is said to be $(\mathbf{j}, \Phi)$-compatible (see [13]).

Theorem A.1. [4, Appendix A, Proposition 2] *The Burge correspondence gives a bijection between the set of all $r \times n$ matrices with non-negative integer entries $\mathrm{Mat}_{r\times n}(\mathbb{Z}_+)$ and the set of pairs (P, Q) of semi-standard tableaux with the same* *shape where entries of P are in $[n]$ and entries of Q are in $[r]$. We use the notation $(w_A \to \varnothing) = (P,Q)$ if A corresponds to (P,Q).*

Theorem A.2. [4, Appendix A, Symmetry Theorem (b)] *If $M \in \mathrm{Mat}_{r\times n}(\mathbb{Z}_+)$ corresponds to (P,Q) then its transpose M^t corresponds to (Q,P).*

The reverse filling of the skew shape λ/μ, denoted $\mathrm{RF}(\lambda/\mu)$, is defined to be the filling of the boxes of the shape λ/μ by $1,2,3,\ldots,|\lambda/\mu|$, sequentially from right to left within each row, starting with the top row and proceeding downwards.

We say a standard (skew) tableau Q with $|\mathrm{shape}(Q)| = |\lambda/\mu|$ is *λ/μ-compatible* if it satisfies the following:

1. If $i+1, i$ are adjacent in a row of $\mathrm{RF}(\lambda/\mu)$ then $i+1$ appears weakly north and strictly east of i in Q.
2. If i occurs directly above j in a column of $\mathrm{RF}(\lambda/\mu)$ then j appears weakly west and strictly south of i in Q.

Remark A.3. The set of all λ/μ-compatible standard tableaux of shape ν is in one-to-one correspondence with the set of LR tableaux of shape λ/μ and weight ν [4, Chapter 5, Proposition 4].

Fix a λ/μ-compatible standard tableau Q. Given a composition $\alpha = (\alpha_1,\alpha_2,\ldots)$, let $\mathbf{b}(\alpha)$ be the word $\cdots b^{(2)}b^{(1)}$ in which $b^{(j)}$ consists of a string of α_j copies of j. Also, consider the semi-standard tableau R whose standardization[(4)] is Q and weight is the composition $\rho = \lambda - \mu$. If $\mathbf{i} = i_1 i_2 \cdots i_t$ then by $\mathrm{rev}(\mathbf{i})$ we mean the word $i_t \cdots i_2 i_1$.

Define $\mathcal{A}(Q,\lambda/\mu,\Phi) = \{T \in \mathrm{Tab}(\lambda/\mu,\Phi) : (\begin{bmatrix}\mathbf{b}(\rho)\\ \mathrm{rev}(b_T)\end{bmatrix} \to \varnothing) = (-,R)\}$. Then $\mathrm{Tab}(\lambda/\mu,\Phi) = \bigsqcup \mathcal{A}(Q,\lambda/\mu,\Phi)$ where the union is over all λ/μ-compatible standard tableaux Q [13]. We will show that $\mathcal{A}(Q,\lambda/\mu,\Phi)$ is isomorphic to some Demazure crystal as crystals, i.e., there is a weight-preserving bijection between these sets which intertwines the crystal raising and lowering operators (where defined).

For a composition α, $\mathrm{key}(\alpha)$ is the semi-standard tableau of shape $\alpha^\dagger$ whose first α_k columns contain the letter k for all k. One can see that $\mathrm{key}(\alpha)$ is the unique tableau of shape $\alpha^\dagger$ and weight α. We define $\mathcal{W}(\alpha,\Phi)$ as the set of all words $\mathbf{u} = \cdots u^2 u^1$, where each u^i is a maximal row word of length α_i together with the properties that each letter in u^i can be at most Φ_i and $(\begin{bmatrix}\mathbf{b}(\alpha)\\ \mathbf{u}\end{bmatrix} \to \varnothing) = (-,\mathrm{key}(\alpha))$.

Recall the standard flag $\Phi_0 = (1,2,3,\ldots,n)$. We have:

Theorem A.4. [10, Proposition 5.6] *Let $\alpha = w\alpha^\dagger$. Then the set $\mathcal{W}(\alpha,\Phi_0)$ has a one-to-one correspondence with the set $\mathcal{B}_w(\alpha^\dagger)$ via $\mathbf{u} \mapsto P(\mathbf{u})$ where $P(\mathbf{u})$ is the unique tableau that is Knuth equivalent to $\mathbf{u}$.*

Let $\mathbf{a} = a_1 a_2 \cdots a_t$ be a word. Then by an *ascent* of the word $\mathbf{a}$ we mean a positive integer $1 \leqslant k \leqslant t-1$ such that $a_k < a_{k+1}$. We recursively define the *essential subword* $\mathrm{ess}_L(\mathbf{a})$ of $\mathbf{a}$ with respect to a positive integer L or ∞ to be the following indexed subword of $\mathbf{a}$:

1. $\mathrm{ess}_L(\mathbf{a})$ is the empty word if $\mathbf{a}$ is empty.
2. $\mathrm{ess}_L(\mathbf{a}) = \mathrm{ess}_{a_t}(a_1 a_2 \cdots a_{t-1})\, a_t$ if $a_t < L$.
3. $\mathrm{ess}_L(\mathbf{a}) = \mathrm{ess}_L(a_1 a_2 \cdots a_{t-1})$ if $a_t \geqslant L$.

Define the essential subword of $\mathbf{a}$ as $\mathrm{ess}(\mathbf{a}) = \mathrm{ess}_\infty(\mathbf{a})$.

[(4)] The standardization of a tableau T (denoted by $\mathrm{std}(T)$) is the tableau obtained by changing the 1's in T from left to right to $1,2,\ldots,\alpha_1$, then the 2's to $\alpha_1+1, \alpha_1+2, \ldots, \alpha_1+\alpha_2$ etc, where $\alpha = wt(T)$.

Lemma A.5. *Let Φ be a flag and $\mathbf{i} = i_1 i_2 \cdots i_t$ be a word. If $\mathbf{a} = a_1 a_2 \cdots a_t$, $\mathbf{b} = b_1 b_2 \cdots b_t$ are words in $[n]$ having the same essential subword and ascents, then $\mathbf{i}$ is $(\mathbf{a}, \Phi)$-compatible if and only if $\mathbf{i}$ is $(\mathbf{b}, \Phi)$-compatible.*

Proof. The proof is similar to that of Lemma 8 of [13]. □

For a semi-standard tableau T, let $T|_{<L}$ denote the subtableau of T consisting of the entries of T which are less than L, and let $K\_(T)$ denote the left key tableau of T. For more details on computing left and right keys, see [13], [16] and [14].

Lemma A.6. *Let $L \geqslant 1$. Suppose that*

(1) *P, P' and Q are semi-standard tableaux of the same shape such that $wt(Q) = (1, 1, \ldots, 1)$ and $K\_(P)|_{<L} = K\_(P')|_{<L}$.*

(2) *$\mathbf{a} = a_1 a_2 \cdots a_t$, $\mathbf{a}' = a'_1 a'_2 \cdots a'_t$ are two words with $(\begin{bmatrix} \mathbf{b}(\mathbf{1}_t) \\ \mathrm{rev}(\mathbf{a}) \end{bmatrix} \to \varnothing) = (P, Q)$ and $(\begin{bmatrix} \mathbf{b}(\mathbf{1}_t) \\ \mathrm{rev}(\mathbf{a}') \end{bmatrix} \to \varnothing) = (P', Q)$, where $\mathbf{1}_t$ is the composition $(1, 1, \ldots, 1)$ of length t.*

Then $\mathrm{ess}_L(\mathbf{a}) = \mathrm{ess}_L(\mathbf{a}')$ and $\mathbf{a}, \mathbf{a}'$ have the same ascents.

Proof. The proof is similar to that of Lemma 9 of [13]. □

Now we have the following proposition:

Proposition A.7. *There is a bijection Ω between the sets $\mathcal{A}(Q, \lambda/\mu, \Phi)$ and $\mathcal{W}(\beta(R), \Phi)$ such that if $T \mapsto \Omega(T)$ then $\mathrm{rev}(b_T)$ and $\Omega(T)$ are Knuth equivalent. Here $\beta(R)$ denote the weight of the left key tableau $K\_(R)$ of R.*

Proof. Let $T \in \mathcal{A}(Q, \lambda/\mu, \Phi)$ and $M(T)$ be the matrix corresponding to $\begin{bmatrix} \mathbf{b}(\rho) \\ \mathrm{rev}(b_T) \end{bmatrix}$. Suppose that $\begin{bmatrix} \mathrm{rev}(\mathbf{i}) \\ \mathrm{rev}(\mathbf{a}) \end{bmatrix}$ is the biword for $M(T)^t$. Thus $\mathbf{i}$ is $(\mathbf{a}, \Phi)$-compatible because $T \in \mathrm{Tab}(\lambda/\mu, \Phi)$.

Let $\mathrm{rect}(T)$ denote the rectification of the skew tableau T. Then $(\begin{bmatrix} \mathbf{b}(\rho) \\ \mathrm{rev}(b_T) \end{bmatrix} \to \varnothing) = (\mathrm{rect}(T), R) \implies (\begin{bmatrix} \mathrm{rev}(\mathbf{i}) \\ \mathrm{rev}(\mathbf{a}) \end{bmatrix} \to \varnothing) = (R, \mathrm{rect}(T))$ (by Theorem A.2). Consider the unique word $\mathbf{a}'$ such that $(\begin{bmatrix} \mathrm{rev}(\mathbf{i}) \\ \mathrm{rev}(\mathbf{a}') \end{bmatrix} \to \varnothing) = (K\_(R), \mathrm{rect}(T))$. Then by Lemma A.5 and Lemma A.6, $\mathbf{i}$ is $(\mathbf{a}', \Phi)$-compatible. Let $\begin{bmatrix} \mathrm{rev}(\mathbf{j}) \\ \mathrm{rev}(\mathbf{v}) \end{bmatrix}$ be the biword associated to the matrix A such that A^t corresponds to $\begin{bmatrix} \mathrm{rev}(\mathbf{i}) \\ \mathrm{rev}(\mathbf{a}') \end{bmatrix}$. Hence $(\begin{bmatrix} \mathrm{rev}(\mathbf{j}) \\ \mathrm{rev}(\mathbf{v}) \end{bmatrix} \to \varnothing) = (\mathrm{rect}(T), K\_(R))$ (by Theorem A.2). So by Corollary 12 of [13], we have $\mathrm{rev}(\mathbf{v}) \in \mathcal{W}(\beta(R), \Phi)$. We define $\Omega(T) = \mathrm{rev}(\mathbf{v})$. Then Ω is a bijection and $\mathrm{rev}(b_T)$ and $\Omega(T)$ are Knuth equivalent. □

Theorem A.8. [13, Theorem 21] *For a flag Φ and a composition β, either $\mathcal{W}(\beta, \Phi)$ is empty or there is a bijection ζ between the sets $\mathcal{W}(\beta, \Phi)$ and $\mathcal{W}(\hat{\beta}, \Phi_0)$ for some composition $\hat{\beta}$ with $\beta^\dagger = \hat{\beta}^\dagger$ such that if $\mathbf{u} \mapsto \zeta(\mathbf{u})$ then $\mathbf{u}$ and $\zeta(\mathbf{u})$ are Knuth equivalent.*

Now the following proposition tells us that $\mathrm{Tab}(\lambda/\mu, \Phi)$ is a disjoint union of Demazure crystals.

Proposition A.9. *The rectification map* rect : $\mathcal{A}(Q,\lambda/\mu,\Phi) \to \mathcal{B}_\tau(\widehat{\beta(R)}^\dagger)$ *is a weight-preserving bijection which intertwines the crystal raising and lowering operators. Here* τ *is any permutation such that* $\tau.\widehat{\beta(R)}^\dagger = \widehat{\beta(R)}$.

Proof. Now $\mathrm{rev}(b_T)$, $\Omega(T)$, $\zeta(\Omega(T))$, $P(\zeta(\Omega(T)))$ are all Knuth equivalent. So $\mathrm{rect}(T) = P(\zeta(\Omega(T))) \in \mathcal{B}_\tau(\widehat{\beta(R)}^\dagger)$. So the map is well-defined. Clearly, the rectification map is a weight-preserving bijection. The commutativity of the rectification map with the crystal raising and lowering operators comes from properties of Knuth equivalence. ☐

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