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On the Degree of Grothendieck Polynomials

Matt Dreyer, Karola Mešzáros & Avery St. Dizier

ABSTRACT A beautiful degree formula for Grothendieck polynomials was recently given by Pechenik, Speyer, and Weigandt (2021). We provide an alternative proof of their degree formula, utilizing the climbing chain model for Grothendieck polynomials introduced by Lenart, Robinson, and Sottile (2006). Moreover for any term order satisfying $x_1 < x_2 < \cdots < x_n$, we present the leading monomial of each homogeneous component of the Grothendieck polynomial $\mathfrak{G}_w(\mathbf{x})$, confirming a conjecture of Hafner (2022). We conclude with a conjecture for the leading monomial of each homogeneous component of $\mathfrak{G}_w(\mathbf{x})$ in any term order satisfying $x_1 > x_2 > \cdots > x_n$.

1. INTRODUCTION

Schubert polynomials and Grothendieck polynomials are multivariate polynomials associated to permutations in S_n . Schubert and Grothendieck polynomials were introduced by Lascoux and Schützenberger [13, 14] in their study of the *flag variety*, the parameter space of maximal sequences of nested vector spaces in \mathbb{C}^n . The flag variety admits a cell decomposition into *Schubert varieties* indexed by permutations in S_n .

Schubert (resp. Grothendieck) polynomials arise as a set of distinguished representatives for the cohomology (resp. K-theoretic) classes of Schubert varieties in the cohomology ring (resp. K-theory) of the flag variety. Schubert polynomials have a rich and well-studied combinatorial structure, admitting a myriad of formulas such as [1, 2, 6, 7, 12, 15, 25, 5]. Many formulas for Schubert polynomials generalize to Grothendieck polynomials as well.

Schubert and Grothendieck polynomials are geometrically natural choices of class representatives: Knutson and Miller [11] showed them to be the multidegrees and K-polynomials respectively of *matrix Schubert varieties*, determinantal varieties obtained by pulling the Schubert varieties in the flag variety back to $n \times n$ matrix space. See [19, Chapter 15] for a thorough introduction.

In [22], it was noted that Grothendieck polynomials give a combinatorial approach to studying the *regularity* of matrix Schubert varieties, a measure of the complexity of their defining ideals. The regularity is given by the difference between the degrees of the Grothendieck and Schubert polynomial of $w \in S_n$. The Schubert polynomial is homogeneous of degree $\ell(w)$, the number of inversions. The degree of the Grothendieck

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polynomial was not well-understood at the time, and has since seen several developments.

An explicit degree formula for Grothendieck polynomials of Grassmannian permutations was given in [22]. This was extended to vexillary permutations in [23] by Rajchgot, Robichaux, and Weigandt; an alternate proof given by Hafner in [8]. A general explicit degree formula for any Grothendieck polynomial was given by Pechenik, Speyer, and Weigandt in [21], in the form of permutation statistics called the *Rajchgot code* $\text{rajcode}(w)$ and *Rajchgot index* $\text{raj}(w)$:

THEOREM 1.1 ([21, Theorem 1.1]). *For any $w \in S_n$,*

$$\deg \mathfrak{G}_w = \text{raj}(w).$$

Moreover in any term order satisfying $x_1 < x_2 < \dots < x_n$, the leading monomial of the highest-degree homogeneous component $\mathfrak{G}_w^{\text{top}}$ of \mathfrak{G}_w is $\mathbf{x}^{\text{rajcode}(w)}$.

The proof of Theorem 1.1 does not provide an explicit combinatorial representative of the monomial $\mathbf{x}^{\text{rajcode}(w)}$ in the Grothendieck polynomial, but instead gives a complicated recursive algorithm to produce one.

We give a comparable recursive algorithm, furnishing an alternative proof of Theorem 1.1. Our approach extends to yield the leading term of each homogeneous component of any Grothendieck polynomial. Unlike [21] though, we restrict our attention to Grothendieck polynomials in x variables only.

Our main tool is the climbing chain model for Schubert and Grothendieck polynomials, introduced by Lenart, Robinson, and Sottile in [16]. The model expresses the Schubert and Grothendieck polynomials in terms of certain saturated chains in the Bruhat order on permutations. We introduce explicit chains representing the leading monomials of \mathfrak{G}_w and $\mathfrak{G}_w^{\text{top}}$ in term orders with $x_1 < x_2 < \dots < x_n$. We conjecturally do the same for term orders with $x_1 > x_2 > \dots > x_n$ in Section 12.

We also investigate the leading monomials of the other homogeneous components of Grothendieck polynomials, a question first posed by Weigandt [24]. A potential answer was conjectured by Hafner:

CONJECTURE 1.2 ([8, Conjecture 4.1]). *Fix $w \in S_n$ and any term order with $x_1 < x_2 < \dots < x_n$. For $\ell(w) < k \leq \text{raj}(w)$, let $m_k(\mathbf{x})$ be the leading monomial of the degree k homogeneous component of \mathfrak{G}_w . Then*

$$m_k(\mathbf{x}) = x_p m_{k-1}(\mathbf{x})$$

where p is the largest index such that $x_p m_{k-1}(\mathbf{x})$ divides $\mathbf{x}^{\text{rajcode}(w)}$.

We address Hafner’s conjecture by first providing climbing chains for her conjectured monomials (Definition 10.2 and Theorem 10.9). We use these chains to verify Conjecture 1.2 (Theorem 11.3). We formulate an analogue of Conjecture 1.2 for term orders with $x_1 > x_2 > \dots > x_n$ (Conjecture 12.8).

It would be interesting to investigate if there is a commutative algebra connection for the leading monomials of homogeneous components of Grothendieck polynomials – which are the subject of Conjecture 1.2 – that are not in the top degree; the latter is related to the regularity of matrix Schubert varieties. Such an algebraic interpretation is not immediately clear to the present authors. Nonetheless, the leading terms of the homogeneous components of Grothendieck polynomials are of independent interest in the study of the support of the Grothendieck polynomial. The latter has received a lot of recent attention [9, 18, 8, 20, 4].

ROADMAP OF THE PAPER. Section 2 contains general background. Section 3 is a summary of the results of [21]. Section 4 is a presentation of the Lenart, Robinson, and Sottile climbing chain model for Grothendieck polynomials ([16]) via Rothe diagrams. Section 5 details the constructions of four climbing chains. One is studied in Section 6 and connected to the Lehmer code. One is examined in Section 7, and used to give an alternative proof of Theorem 1.1 in Sections 8 and 9. The other two are considered in Section 12, and related to leading terms of Schubert and Grothendieck polynomials when $x_1 > x_2 > \dots > x_n$. In Section 10 and 11, we prove Conjecture 1.2 and conjecture a dual version.

2. BACKGROUND

2.1. CONVENTIONS. For $n \in \mathbb{N}$, we write $[n]$ to mean the set $\{1, 2, \dots, n\}$. For two indices $1 \leq i < j \leq n$ we write t_{ij} for the transposition in the symmetric group S_n swapping i and j . We write s_j for the adjacent transposition $t_{j,j+1}$.

Throughout, we write permutations in one-line notation (as a string) $w = w(1)w(2)\dots w(n)$. We will take permutations as acting on the right – switching positions, not values. For example ws_1 equals w with the numbers $w(1)$ and $w(2)$ swapped.

For $v \in \mathbb{Z}^n$, we write $|v|$ for $v_1 + v_2 + \dots + v_n$. We denote the standard basis vectors of \mathbb{Z}^n by e_1, e_2, \dots, e_n . We use the notation \bar{v} to denote the *vector complement*

$$\bar{v}_k = n - k - v_k \text{ for each } k \in [n].$$

A *term order* on $\mathbb{Z}[x_1, \dots, x_n]$ is a total order $<$ of all (monic) monomials such that

$$x^\alpha \leq x^\beta \text{ implies } x^{\alpha+\gamma} \leq x^{\beta+\gamma} \text{ for all } \gamma \in \mathbb{Z}_{\geq 0}^n.$$

The *leading term* of a polynomial $f \in \mathbb{Z}[x_1, \dots, x_n]$ is the term of the largest monomial under $<$ appearing in f . The *leading monomial* is the leading term divided by its coefficient. For example if $f(x_1, x_2) = x_1^2 + 2x_1x_2 + 3x_2^2$ and $x_1 < x_2$, the leading term of f is $3x_2^2$ and the leading monomial is x_2^2 .

2.2. SCHUBERT AND GROTHENDIECK POLYNOMIALS.

DEFINITION 2.1. Fix any $n \geq 0$. The divided difference operators ∂_j for $j \in [n - 1]$ are operators on the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ defined by

$$\partial_j(f) = \frac{f - (s_j \cdot f)}{x_j - x_{j+1}} = \frac{f(x_1, \dots, x_n) - f(x_1, \dots, x_{j-1}, x_{j+1}, x_j, x_{j+2}, \dots, x_n)}{x_j - x_{j+1}}.$$

The K-divided difference operators $\bar{\partial}_j$ are defined on the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ by $\bar{\partial}_j(f) = \partial_j(f - x_{j+1}f)$.

DEFINITION 2.2. The Schubert polynomial \mathfrak{S}_w of $w \in S_n$ is defined recursively on the weak Bruhat order. Let $w_0 = n(n-1)\dots 21 \in S_n$, the longest permutation in S_n . If $w \neq w_0$ then there is $j \in [n - 1]$ with $w(j) < w(j + 1)$ (called an ascent of w). The polynomial \mathfrak{S}_w is defined by

$$\mathfrak{S}_w = \begin{cases} x_1^{n-1}x_2^{n-2}\dots x_{n-1} & \text{if } w = w_0, \\ \partial_j \mathfrak{S}_{ws_j} & \text{if } w(j) < w(j + 1). \end{cases}$$

DEFINITION 2.3. The Grothendieck polynomial \mathfrak{G}_w of $w \in S_n$ is defined analogously to the Schubert polynomial, with

$$\mathfrak{G}_w = \begin{cases} x_1^{n-1}x_2^{n-2}\dots x_{n-1} & \text{if } w = w_0, \\ \bar{\partial}_j \mathfrak{G}_{ws_j} & \text{if } w(j) < w(j + 1). \end{cases}$$

Recall that $\ell(w) = \#\{(i, j) \mid 1 \leq i < j \leq n \text{ with } w(i) > w(j)\}$ is the number of inversions of $w \in S_n$. It can be seen from the recursive definitions above that \mathfrak{S}_w is homogeneous of degree $\ell(w)$, and equals the lowest-degree nonzero homogeneous component of \mathfrak{G}_w . See [17] for a deeper introduction to Schubert polynomials.

Associated to each permutation $w \in S_n$ is its Rothe diagram $D(w)$.

DEFINITION 2.4. *The Rothe diagram of a permutation $w \in S_n$ is the set*

$$D(w) = \{(i, j) : 1 \leq i, j \leq n, w(i) > j, \text{ and } w^{-1}(j) > i\}.$$

We will make use of the following graphical interpretation of $D(w)$. Start with an $n \times n$ grid of boxes with the usual matrix indexing. Place a dot in box $(i, w(i))$ for each $i \in [n]$. Draw rays emanating south and east of each dot, called *hooks*. Remove any boxes hit by a hook. The remaining boxes lie exactly in the positions $D(w)$.

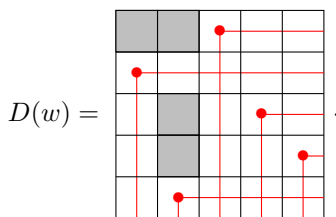
DEFINITION 2.5. *The Lehmer code of $w \in S_n$ is the vector*

$$\text{invcode}(w) = (\text{invcode}(w)_1, \dots, \text{invcode}(w)_n),$$

where $\text{invcode}(w)_i$ equals the number of boxes in row i of $D(w)$. Symbolically,

$$\text{invcode}(w)_i = \#\{j \mid i < j \text{ and } w(j) < w(i)\}.$$

EXAMPLE 2.6. If $w = 31452$, then $D(w) = \{(1, 1), (1, 2), (3, 2), (4, 2)\}$. We draw $D(w)$ by placing east and south hooks at each point $(i, w(i))$ in the 5×5 grid and coloring all boxes not hit by any hook:



The Lehmer code of w is $\text{invcode}(w) = (2, 0, 1, 1, 0)$.

3. DEGREE OF GROTHENDIECK POLYNOMIALS

We recall the results of Pechenik, Speyer, and Weigandt [21] giving the leading monomial of the highest degree component of \mathfrak{G}_w in any term order satisfying $x_1 < x_2 < \dots < x_n$.

DEFINITION 3.1. *An increasing subsequence of a permutation $w \in S_n$ starting from position $q \in [n]$ is a vector $\alpha = (\alpha_1, \dots, \alpha_m)$ such that*

$$q = \alpha_1 < \alpha_2 < \dots < \alpha_m \leq n \quad \text{and} \quad w(\alpha_1) < w(\alpha_2) < \dots < w(\alpha_m).$$

An increasing subsequence of a permutation w starting from position q is called longest if there is no longer such sequence, i.e. m is maximal. We refer to m as the length of α . We denote by $\text{LIS}(w, q)$ the set of all longest increasing subsequences of w starting from position q .

Graphically, increasing subsequences starting from position q of a permutation w are exactly sequences of nested hooks in the Rothe diagram $D(w)$ starting with the hook in row q .

DEFINITION 3.2. *To each $w \in S_n$, associate the vector*

$$\overline{\text{rajcode}}(w) = (\overline{\text{rajcode}}(w)_1, \dots, \overline{\text{rajcode}}(w)_n) \in \mathbb{Z}^n,$$

where for each i , $\overline{\text{rajcode}}(w)_i$ equals the length of any $\alpha \in \text{LIS}(w, i)$ minus one.

DEFINITION 3.3 ([21]). The Rajchgot code of $w \in S_n$ is the vector $\text{rajcode}(w) \in \mathbb{Z}^n$ with

$$\text{rajcode}(w)_k = n - k - \overline{\text{rajcode}}(w)_k \quad \text{for each } k \in [n].$$

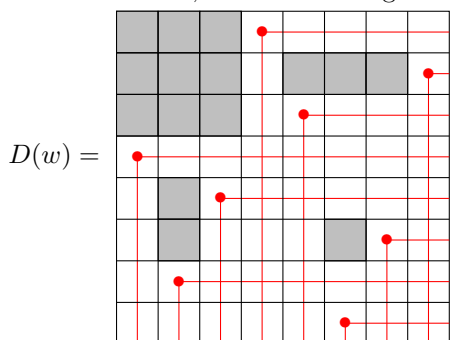
Observe that the entries of $\text{rajcode}(w)$ count the complement of longest increasing sequences: for any fixed $\alpha \in \text{LIS}(w, k)$, $\text{rajcode}(w)_k$ equals the number of elements in $\{k, k + 1, \dots, n\}$ that did not appear in α .

DEFINITION 3.4. The Rajchgot index of $w \in S_n$ is

$$\text{raj}(w) = |\text{rajcode}(w)| = \sum_{k=1}^n \text{rajcode}(w)_k.$$

Similarly, let $\overline{\text{raj}}(w) = |\overline{\text{rajcode}}(w)|$.

EXAMPLE 3.5. Consider $w = 48513726$, with Rothe diagram



The longest increasing sequences in w are

$$\begin{aligned} \text{LIS}(w, 1) &= \{(1, 3, 6), (1, 3, 8)\}, & \text{LIS}(w, 2) &= \{(2)\}, \\ \text{LIS}(w, 3) &= \{(3, 6), (3, 8)\}, & \text{LIS}(w, 4) &= \{(4, 5, 6), (4, 5, 8), (4, 7, 8)\}, \\ \text{LIS}(w, 5) &= \{(5, 6), (5, 8)\}, & \text{LIS}(w, 6) &= \{(6)\}, \\ \text{LIS}(w, 7) &= \{(7, 8)\}, & \text{LIS}(w, 8) &= \{(8)\}. \end{aligned}$$

Observe that these sequences show up graphically as the longest sequences of (row indices of) nested hooks in $D(w)$. One computes

$$\begin{aligned} \overline{\text{rajcode}}(w) &= (2, 0, 1, 2, 1, 0, 1, 0), \\ \overline{\text{raj}}(w) &= 2 + 0 + 1 + 2 + 1 + 0 + 1 + 0 = 7, \\ \text{rajcode}(w) &= (7, 6, 5, 4, 3, 2, 1, 0) - \overline{\text{rajcode}}(w) = (5, 6, 4, 2, 2, 2, 0, 0), \quad \text{and} \\ \text{raj}(w) &= 5 + 6 + 4 + 2 + 2 + 2 + 0 + 0 = 21. \end{aligned}$$

In general, note that $\text{rajcode}(w)_k \geq \text{invcode}(w)_k$ for all $k \in [n]$. This follows since $\text{invcode}(w)_k$ counts the number of inversions of the form (k, \star) in w , and $\text{rajcode}(w)_k$ counts the number of entries in $[k + 1, n]$ excluded in any (fixed) sequence $\alpha \in \text{LIS}(w, k)$.

4. CLIMBING CHAINS AND MARKINGS

We recall climbing chains, a combinatorial model for Schubert and Grothendieck polynomials due to Lenart, Robinson, and Sottile in [16]. We depict their construction in terms of the Rothe diagrams; this diverges from their exposition, but we find this rendering of climbing chains particularly helpful.

The (strong) Bruhat order is the partial order \leq on S_n defined as the transitive closure of the relations

$$w \leq wt_{ij}$$

for any $i < j$ such that $w(i) < w(j)$. We write $w \triangleleft v$ to denote a cover relation in the Bruhat order. The following well-known lemma describes all cover relations of the Bruhat order. See for instance [3, Lemma 2.1.4] for a proof.

LEMMA 4.1. *For $v, w \in S_n$, $w \triangleleft v$ if and only if there is $i, j \in [n]$ with $i < j$ such that:*

- $v = wt_{ij}$,
- $w(i) < w(j)$,
- $\{k \mid i < k < j\} \cap \{k \mid w(i) < w(k) < w(j)\} = \emptyset$.

Lemma 4.1 can be interpreted graphically as follows. For $i < j$, $w \triangleleft wt_{ij}$ if and only if inside $D(w)$:

- The dot at $(i, w(i))$ lies north and west of the dot at $(j, w(j))$,
- No other dot lies in the region bounded by the hooks at $(i, w(i))$ and $(j, w(j))$.

EXAMPLE 4.2. If $w = 31452$ (continuing Example 2.6), then $w \triangleleft wt_{ij}$ exactly when

$$(i, j) \in \{(1, 3), (2, 3), (2, 5), (3, 4)\}.$$

DEFINITION 4.3. *A climbing chain of $w \in S_n$ is a sequence C of pairs*

$$(i_1, j_1), \dots, (i_m, j_m)$$

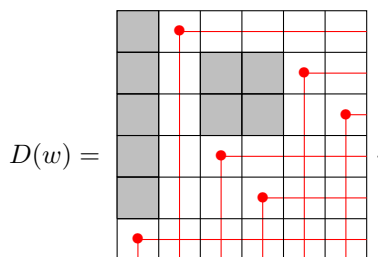
called links, such that

- $i_p < j_p$ for each $p \in [m]$,
- $i_1 \leq i_2 \leq \dots \leq i_m$,
- $wt_{i_1 j_1} \cdots t_{i_m j_m} = w_0$,
- $wt_{i_1 j_1} \cdots t_{i_p j_p} \triangleleft wt_{i_1 j_1} \cdots t_{i_{p+1} j_{p+1}}$ for each $p \in [m]$ (in the Bruhat order on S_n).

We call $\ell(C) = m$ the length of C , and will write C_p to indicate the link (i_p, j_p) for each $p \in [m]$.

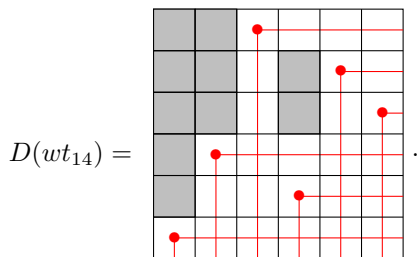
Climbing chains encode special saturated chains in the Bruhat order on S_n from a given permutation w to w_0 .

EXAMPLE 4.4. Let $w = 256341$, so



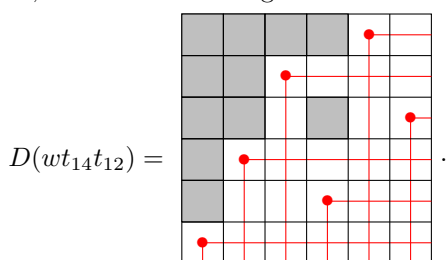
We construct a climbing chain of w . Since $w(1) \neq 6$, the first link of any climbing chain of w will be of the form $(1, j_1)$, where the hook in row j_1 of $D(w)$ sits south and east of the hook in row 1, and no hook sits between them. The available choices are $j_1 \in \{2, 4\}$. We will choose $j_1 = 4$.

Observe that $D(wt_{14}) = D(356241)$ is obtained from $D(w)$ by swapping the column indices of the dots in rows 1 and 4. That is,

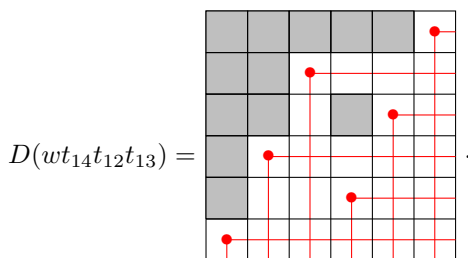


Since $w_{t_{14}}(1) \neq 6$, the next link of the chain must be of the form $(1, j_2)$ where the hook in row j_2 of $D(w_{t_{14}})$ sits south and east of the hook in row 1, and no hook sits between them. The available choices are $j_2 \in \{2, 5\}$. We will choose $j_2 = 2$.

Note $w_{t_{14}t_{12}} = 536241$, and has Rothe diagram



Since $w_{t_{14}t_{12}}(1) \neq 6$, the next link of this chain has to be of the form $(1, j_3)$, and the only available choice is $j_3 = 3$. Then the next permutation is $w_{t_{14}t_{12}t_{13}} = 635241$, with diagram



Finally, $w_{t_{14}t_{12}t_{13}}(1) = 6$. Since $w_{t_{14}t_{12}t_{13}}(2) \neq 5$, the next link of the chain must be of the form $(2, j_4)$. The available choices are $j_4 \in \{3, 5\}$. Continuing in this fashion, one may obtain either of the chains

$$C^{(1)} = ((1, 4), (1, 2), (1, 3), (2, 3), (3, 5), (4, 5)),$$

$$C^{(2)} = ((1, 4), (1, 2), (1, 3), (2, 5), (2, 3), (4, 5)).$$

With different earlier choices, one can obtain any of the chains

$$C^{(3)} = ((1, 2), (1, 3), (2, 3), (3, 4), (3, 5), (4, 5)),$$

$$C^{(4)} = ((1, 2), (1, 3), (2, 4), (2, 3), (3, 5), (4, 5)),$$

$$C^{(5)} = ((1, 2), (1, 3), (2, 4), (2, 5), (2, 3), (4, 5)),$$

$$C^{(6)} = ((1, 4), (1, 5), (1, 2), (1, 3), (2, 3), (4, 5)).$$

These are the six climbing chains of $w = 256341$.

Observe that all climbing chains of a given permutation w have the same length $\ell(w_0) - \ell(w)$.

DEFINITION 4.5. Associated to each climbing chain C of a permutation $w \in S_n$ is a set of special links $M(C)$ called the minimal markings of C . As C is a sequence of distinct pairs (i_p, j_p) , we will abuse notation slightly and write $M(C) \subseteq C$. If the links of C are $C_p = (i_p, j_p)$ for each $p \in [\ell(C)]$, then (taking $i_0 = 0$)

$$M(C) = \{C_p \mid i_{p-1} < i_p \text{ or } i_{p-1} = i_p \text{ and } j_{p-1} < j_p\}.$$

Observe that in the trivial case $w = w_0$, the only climbing chain is the empty sequence. We take the empty sequence to have minimal markings \emptyset .

EXAMPLE 4.6. Continuing Example 4.4, let $w = 256341$. Using overlines to denote markings, the climbing chains of w together with their minimal markings are

$$\begin{aligned} &(\overline{(1, 4)}, (1, 2), \overline{(1, 3)}, \overline{(2, 3)}, \overline{(3, 5)}, \overline{(4, 5)}), \\ &(\overline{(1, 4)}, (1, 2), \overline{(1, 3)}, \overline{(2, 5)}, (2, 3), \overline{(4, 5)}), \\ &(\overline{(1, 2)}, \overline{(1, 3)}, \overline{(2, 3)}, \overline{(3, 4)}, \overline{(3, 5)}, \overline{(4, 5)}), \\ &(\overline{(1, 2)}, \overline{(1, 3)}, \overline{(2, 4)}, (2, 3), \overline{(3, 5)}, \overline{(4, 5)}), \\ &(\overline{(1, 2)}, \overline{(1, 3)}, \overline{(2, 4)}, \overline{(2, 5)}, (2, 3), \overline{(4, 5)}), \\ &(\overline{(1, 4)}, \overline{(1, 5)}, (1, 2), \overline{(1, 3)}, \overline{(2, 3)}, \overline{(4, 5)}). \end{aligned}$$

DEFINITION 4.7. For $w \in S_n$, a marked climbing chain of w is a pair (C, U) where C is a climbing chain of w and U is a superset of the minimal markings of C , that is $M(C) \subseteq U \subseteq C$.

DEFINITION 4.8. The dual weight of a marked climbing chain $\xi = (C, U)$ of $w \in S_n$ is the vector

$$\overline{\text{wt}}(\xi) = (\overline{\text{wt}}(\xi)_1, \dots, \overline{\text{wt}}(\xi)_n)$$

where $\overline{\text{wt}}(\xi)_k$ equals the number of links $(k, \star) \in U$.

DEFINITION 4.9. The weight of a marked climbing chain $\xi = (C, U)$ of $w \in S_n$ is the vector

$$\text{wt}(\xi) = (\text{wt}(\xi)_1, \dots, \text{wt}(\xi)_n)$$

whose k th component equals $n - k - \overline{\text{wt}}(\xi)_k$.

THEOREM 4.10 ([16, Theorem 5.6]). For any $w \in S_n$,

$$\mathfrak{G}_w(x_1, \dots, x_n) = \sum_{\xi} \text{sign}(\xi) \mathbf{x}^{\text{wt}(\xi)} = \sum_{\xi} \text{sign}(\xi) \frac{\mathfrak{G}_{w_0}}{\mathbf{x}^{\overline{\text{wt}}(\xi)}},$$

where the sum is over all marked climbing chains $\xi = (C, U)$ of w , and $\text{sign}(\xi) = (-1)^{\ell(C) - \#U}$.

COROLLARY 4.11 ([16]). For any $w \in S_n$,

$$\mathfrak{G}_w(x_1, \dots, x_n) = \sum_{\xi} \mathbf{x}^{\text{wt}(\xi)} = \sum_{\xi} \frac{\mathfrak{G}_{w_0}}{\mathbf{x}^{\overline{\text{wt}}(\xi)}},$$

where the sum is over all marked climbing chains $\xi = (C, U)$ of w with $U = C$.

EXAMPLE 4.12. Continuing Example 4.6, let $w = 256341$. Then the Schubert and Grothendieck polynomials of w are given by

$$\begin{aligned} \mathfrak{S}_w &= \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^3 x_2 x_3 x_4} + \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^3 x_2^2 x_4} + \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^2 x_2 x_3^2 x_4} + \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^2 x_2^2 x_3 x_4} \\ &\quad + \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^2 x_2^3 x_4} + \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^4 x_2 x_4} \\ \mathfrak{G}_w &= \mathfrak{S}_w \\ &\quad - \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^2 x_2 x_3 x_4} - \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^2 x_2^2 x_4} - \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^3 x_2 x_4} - \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^2 x_2 x_3 x_4} \\ &\quad - \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^2 x_2^2 x_4} - \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^3 x_2 x_4} + \frac{x_1^5 x_2^4 x_3^3 x_4^2 x_5}{x_1^2 x_2 x_4}. \end{aligned}$$

From their definition, climbing chains are clearly recursive. We will make frequent use of the relation between the minimal markings on a climbing chain and those on its truncation by one link. We illustrate this relationship in the following example, and record the observations in the subsequent lemma.

EXAMPLE 4.13. Consider the two climbing chains

$$\begin{aligned} C^{(1)} : w &= 1342 \xrightarrow{(1,2)} 3142 \xrightarrow{(1,3)} 4132 \xrightarrow{(2,3)} 4312 \xrightarrow{(3,4)} 4321 \\ C^{(2)} : v &= 2143 \xrightarrow{(1,4)} 3142 \xrightarrow{(1,3)} 4132 \xrightarrow{(2,3)} 4312 \xrightarrow{(3,4)} 4321 \end{aligned}$$

in S_4 . Truncating the first link in either gives the climbing chain

$$C^{(3)} : u = 3142 \xrightarrow{(1,3)} 4132 \xrightarrow{(2,3)} 4312 \xrightarrow{(3,4)} 4321$$

with minimal markings $M(C_3) = \{(1, 3), (2, 3), (3, 4)\}$. Compare this with

$$\begin{aligned} M(C_1) &= \{(1, 2), (1, 3), (2, 3), (3, 4)\}, \\ M(C_2) &= \{(1, 4), (2, 3), (3, 4)\}. \end{aligned}$$

The first link in the truncation C_3 is always marked in C_3 , but whether it is marked or not in C_1, C_2 is determined by their first link.

LEMMA 4.14. Fix $w \in S_n$ with $w \neq w_0$. Let C be any climbing chain of w . Suppose $C_p = (i_p, j_p)$ for $p \in [\ell(C)]$, and set $w' = wt_{i_1 j_1}$. Define C' to equal C with C_1 removed. Then C' is a climbing chain for w' and

$$M(C) = \begin{cases} (M(C') \setminus \{C_2\}) \cup \{C_1\} & \text{if } \ell(C) > 1 \text{ and } C_2 \notin M(C), \\ M(C') \cup \{C_1\} & \text{otherwise.} \end{cases}$$

Consequently,

$$\overline{\text{wt}}(C, M(C)) = \begin{cases} \overline{\text{wt}}(C', M(C')) & \text{if } \ell(C) > 1 \text{ and } C_2 \notin M(C), \\ \overline{\text{wt}}(C', M(C')) + e_{i_1} & \text{otherwise.} \end{cases}$$

Proof. It is straightforward to see from the definition that C' is a climbing chain for w' . The first equality follows immediately from the local restrictions defining the minimal markings of a climbing chain. The second equality follows from the first by counting markings. \square

5. FOUR SPECIAL CLIMBING CHAINS

We define four climbing chains that represent leading terms of Schubert and Grothendieck polynomials. We give examples clarifying each and explain their differences. This section is intended to be a convenient reference for these four similar definitions.

5.1. GREEDY CHAIN. We show in Theorem 6.1 that the greedy chain defined below yields the monomial $\mathbf{x}^{\text{invcode}(w)}$, the leading term of the Schubert polynomial in any term order with $x_1 < x_2 < \dots < x_n$.

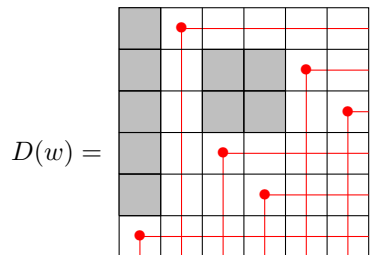
DEFINITION 5.1. To each permutation $w \in S_n$ with $w \neq w_0$, we associate a pair of integers $\text{greedy}(w) = (a, b)$, where

$$a = \min\{k \mid w(k) \neq n + 1 - k\} \quad \text{and} \quad b = w^{-1}(w(a) + 1).$$

DEFINITION 5.2. Define the greedy chain $C^G(w)$ of $w \in S_n$ as follows. If $w = w_0$, then $C^G(w)$ is the empty sequence. Otherwise, define $C^G(w)$ inductively by prepending $\text{greedy}(w)$ to $C^G(wt_{ab})$.

In each iteration, the greedy chain tries to move a hook in $D(w)$ right as little as possible. We illustrate this construction with an example.

EXAMPLE 5.3. Continuing Example 4.6 with $w = 256341$, we compute $C^G(w)$. Recall the Rothe diagram of w is

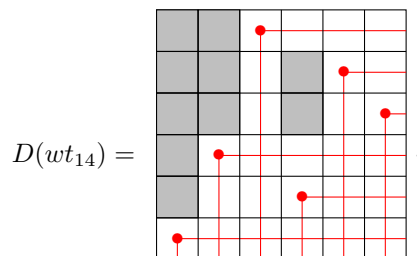


Let $\text{greedy}(w) = (i_1, j_1)$. Since $w(1) \neq 6$, $i_1 = 1$. The definition of $\text{greedy}(w)$ says that

$$j_1 = w^{-1}(w(i_1) + 1),$$

that is j_1 is the row index of the dot in column $w(i_1) + 1$. Graphically, j_1 is the row index of the leftmost dot inside the region south and east of the hook in row i_1 . Thus, $j_1 = 4$.

The Rothe diagram of wt_{14} is



From the Rothe diagram, we observe that $\text{greedy}(wt_{14}) = (1, 5)$. Continuing in this fashion yields

$$C^G(w) = ((1, 4), (1, 5), (1, 2), (1, 3), (2, 3), (4, 5)).$$

This is the chain $C^{(6)}$ in Example 4.4.

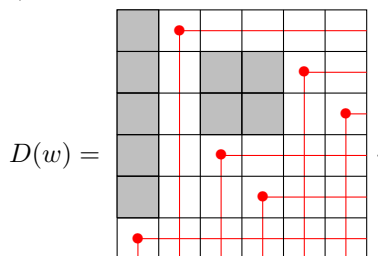
5.2. NESTED CHAIN. We show in Section 9 that the nested chain defined below yields the monomial $\mathbf{x}^{\text{rajcode}(w)}$, the leading monomial of $\mathfrak{G}_w^{\text{top}}$ in any term order with $x_1 < x_2 < \dots < x_n$.

DEFINITION 5.4. To each permutation $w \in S_n$ with $w \neq w_0$, we associate a pair of integers $\text{nested}(w)$ as follows. Set $q = \min\{j \mid w(j) \neq n + 1 - j\}$, and let α be the lexicographically last element of $\text{LIS}(w, q)$. Then $\text{nested}(w) = (\alpha_1, \alpha_2)$.

DEFINITION 5.5. Define the nested chain $C^N(w)$ of $w \in S_n$ as follows. If $w = w_0$, then $C^N(w)$ is the empty sequence. Otherwise, define $C^N(w)$ inductively by prepending $\text{nested}(w)$ to $C^N(wt_{ab})$.

The nested chain acts similar to the greedy chain, trying to move a hook right as little as possible. The difference is that the nested chain only allow moves that belong to maximum-length sequences of nested hooks.

EXAMPLE 5.6. Continuing Example 4.6 with $w = 256341$, we compute $C^N(w)$, $M(C^N(w))$, and $\text{rajcode}(w)$. For reference, the Rothe diagram of w is



We have $\text{LIS}(w, 1) = \{(1, 2, 3), (1, 4, 5)\}$, so $\text{nested}(w) = (1, 4)$. Graphically, this amounts to finding the southmost sequence among all longest sequences of hooks nested under $(1, w(1))$. This is the first link of the nested chain $C^N(w)$.

Next, $wt_{14} = 356241$, so $\text{LIS}(wt_{14}, 1) = \{(1, 2, 3)\}$ and $\text{nested}(wt_{14}) = (1, 2)$. Another iteration: $wt_{14}t_{12} = 536241$, so $\text{LIS}(wt_{14}t_{12}, 1) = \{(1, 3)\}$ and $\text{nested}(wt_{14}t_{12}) = (1, 3)$.

Continuing in this fashion yields

$$C^N(w) = ((1, 4), (1, 2), (1, 3), (2, 5), (2, 3), (4, 5)).$$

This is exactly the chain $C^{(2)}$ from Example 4.4. One also observes

$$M(C^N(w)) = \{(1, 4), (1, 3), (2, 5), (4, 5)\}, \text{ and}$$

$$\overline{\text{rajcode}}(w) = (2, 1, 0, 1, 0, 0) = \overline{\text{wt}}(C^N(w), M(C^N(w))).$$

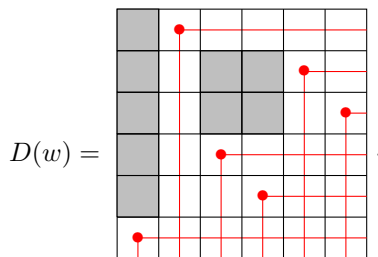
5.3. LEAPING CHAIN. We show in Theorem 12.1 that the leaping chain defined below is “dual” to the greedy chain, yielding the leading term of the Schubert polynomial in any term order with $x_1 > x_2 > \dots > x_n$. Instead of trying to move a hook right as little as possible in each iteration like the greedy chain, the leaping chain tries to move it right as much as possible.

DEFINITION 5.7. To each permutation $w \in S_n$ with $w \neq w_0$, we associate a pair of integers $\text{leap}(w) = (a, b)$, where

$$a = \min\{k \mid w(k) \neq n + 1 - k\} \quad \text{and} \quad b = \min\{k \mid k > a \text{ and } w \leq wt_{ak}\}.$$

DEFINITION 5.8. Define the leaping chain $C^L(w)$ of $w \in S_n$ as follows. If $w = w_0$, then $C^L(w)$ is the empty sequence. Otherwise, define $C^L(w)$ inductively by prepending $\text{leap}(w)$ to $C^L(wt_{ab})$.

EXAMPLE 5.9. Continuing Example 4.6 with $w = 256341$, we compute $C^L(w)$. Recall the Rothe diagram of w is

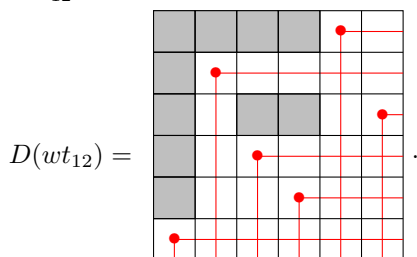


Let $\text{leap}(w) = (i_1, j_1)$. Since $w(1) \neq 6$, $i_1 = 1$. The definition of the leaping pair says that

$$j_1 = \min\{k \mid k > a \text{ and } w \prec wt_{ak}\} = 2.$$

Graphically, j_1 is the row index of the northmost dot inside the region south and east of the hook in row i_1 .

The Rothe diagram of $wt_{12} = 526341$ is



From the Rothe diagram, we observe that $\text{leap}(wt_{12}) = (1, 3)$. Continuing in this fashion yields

$$C^L(w) = ((1, 2), (1, 3), (2, 3), (3, 4), (3, 5), (4, 5)).$$

This is the chain $C^{(3)}$ in Example 4.4.

5.4. STAIRCASE CHAIN. We conjecture in Section 12 that the staircase chain defined below is “dual” to the nested chain, yielding the leading monomial of $\mathfrak{G}_w^{\text{top}}$ in any term order with $x_1 > x_2 > \dots > x_n$.

DEFINITION 5.10. To each permutation $w \in S_n$ we associate a sequence of pairs $\text{stair}(w)$, called the staircase of w , as follows. The staircase of w_0 is undefined. If $w \neq w_0$, let $a = \min\{k \mid w(k) \neq n+1-k\}$ and consider $K = \{k \mid k > a \text{ and } w \prec wt_{ak}\}$. Let K_1 be the set of elements $k \in K$ with

$$\overline{\text{rajcode}}(w)_k = \max(\{\overline{\text{rajcode}}(w)_{k'} \mid k' \in K\}).$$

Iteratively, let K_p be the set of elements $k \in K$ with $k < \min(K_{p-1})$ and

$$\overline{\text{rajcode}}(w)_k = \max \left\{ \overline{\text{rajcode}}(w)_{k'} \mid k' \in K - \bigcup_{q=1}^{p-1} K_q \right\}.$$

Suppose this process results in sets K_1, K_2, \dots, K_q . Then $\text{stair}(w)$ is obtained by ordering the pairs

$$\left\{ (a, k) \mid k \in \bigcup_{p=1}^q K_p \right\}$$

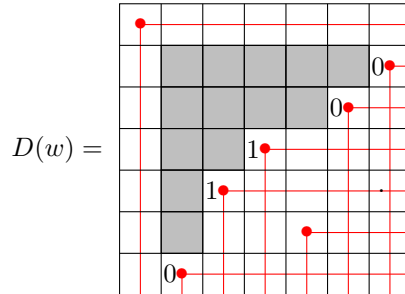
so their second components are decreasing.

DEFINITION 5.11. Define the staircase chain $C^S(w)$ of $w \in S_n$ as follows. If $w = w_0$, then $C^S(w)$ is the empty sequence. Otherwise, let

$$\text{stair}(w) = ((a, b_1), (a, b_2), \dots, (a, b_k)).$$

Define $C^S(w)$ inductively by concatenating $\text{stair}(w)$ and $C^S(wt_{ab_1} \dots t_{ab_k})$.

EXAMPLE 5.12. Consider the permutation $w = 1764352$. Below we draw the Rothe diagram of w and label the hook in each relevant row with the corresponding value of $\text{rajcode}(w)$, the length of the longest hook nesting below itself.



We have $K = \{7, 5, 4, 3, 2\}$, with $K_1 = \{5, 4\}$ and $K_2 = \{3, 2\}$. Thus,

$$\text{stair}(w) = ((1, 5), (1, 4), (1, 3), (1, 2)).$$

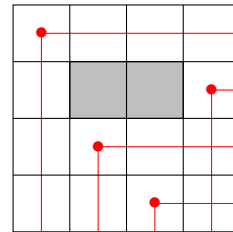
Continuing in this fashion with $wt_{15}t_{14}t_{13}t_{12} = 7643152$, one obtains

$$C^S(w) = ((1, 5), (1, 4), (1, 3), (1, 2), (3, 6), (4, 6), (5, 7), (5, 6)).$$

The definition of the staircase chain is quite different from the other three chains. It does not add a single link at a time, but a whole sequence of links. The following example shows that this feature is unavoidable.

EXAMPLE 5.13. Consider $w = 1423$, which has

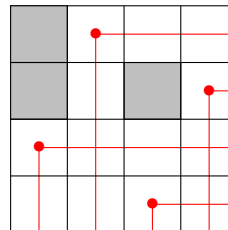
$$\mathfrak{G}_w = (x_1^2 + x_1x_2 + x_2^2) - (x_1x_2^2 + x_1^2x_2), \text{ and } D(w) =$$



The staircase chain is $C^S(w) = ((1, 3), (1, 2), (2, 4), (3, 4))$ with weight $(2, 1)$. It is the only climbing chain of w to achieve this weight.

The first link in $C^S(w)$ is to $v = wt_{13} = 2413$, which has

$$\mathfrak{G}_v = (x_1^2x_2 + x_1x_2^2) - (x_1^2x_2^2), \text{ and } D(v) =$$



The staircase chain of v is $C^S(v) = ((1, 4), (1, 2), (3, 4))$ with weight $(2, 2)$. It is again the only climbing chain of v achieving this weight. However observe that $C^S(v)$ is not the truncation of $C^S(w)$, as was the case for the other three named chains in this section.

LEMMA 5.14. For each $w \in S_n$, each of the constructions $C^G(w)$, $C^N(w)$, $C^L(w)$, and $C^S(w)$ is a climbing chains of w .

Proof. We prove the lemma for $C^N(w)$. The other constructions are all argued analogously. Work by induction on $\ell(w_0) - \ell(w)$, the base case $w = w_0$ being trivial. Let

$C^N(w)_p = (i_p, j_p)$ for each p and set $w' = wt_{i_1 j_1}$. Then $C^N(w')$ is a truncation of $C^N(w)$, and is a climbing chain by induction.

It then remains to verify that $w < w'$ and $i_1 \leq i_2$. The latter is clear from the definition of $C^N(w)$. If the former fails, length considerations imply it fails because $w \not\leq w'$. Then there is q such that $i_1 < q < j_1$ and $w(i_1) < w(q) < w(j_1)$. However, this contradicts the choice of j_1 in $C^N(w)$. \square

6. THE GREEDY CHAIN

We verify the greedy chain $C^G(w)$ (Definition 5.2) realizes the leading term $\mathbf{x}^{\text{invcode}(w)}$ of \mathfrak{S}_w in term orders where $x_1 < \dots < x_n$. We do so by exploiting the connection between climbing chains and cotransition. As explained in Proposition 6.10 of [16], climbing chains of w are in bijection to pipe dreams of w . Specifically fix a reduced pipe dream P of w and replace each elbow tile in P with a cross one-by-one in each row left-to-right, starting with the topmost row and working downwards. With each replacement, record the left-edge exit row numbers $i < j$ of the newly crossed strands. Then the list of pairs (i, j) in the order they were encountered is a climbing chain of w . This bijection is realized inductively by the cotransition formula on Schubert polynomials (see Knutson’s work [10]). We show an example of this bijection in Figure 1.

THEOREM 6.1. *Fix any $w \in S_n$. Set $\xi(w)$ to be the chain $C^G(w)$ with all links marked, that is*

$$\xi(w) = (C^G(w), C^G(w)).$$

Then

$$\text{wt}(\xi(w)) = \text{invcode}(w).$$

Proof. It is straightforward to see that the greedy chain corresponds to the bottom-reduced pipe dream under the cotransition bijection. \square

7. THE NESTED CHAIN

We show that the nested chain $C^N(w)$ (Definition 5.5) yields the monomial $\mathbf{x}^{\text{rajcode}(w)}$ in \mathfrak{S}_w . We prove in Section 9 that this monomial is the leading monomial of $\mathfrak{S}_w^{\text{top}}$ in any term order satisfying $x_1 < x_2 < \dots < x_n$ giving an alternate proof of Theorem 1.1.

We begin with a recursive formula for the vector $\overline{\text{rajcode}}(w)$. In Figure 2, we offer a visual aid for the technical arguments used to prove the recursion.

THEOREM 7.1. *Fix $w \in S_n$ with $w \neq w_0$. Suppose $C^N(w)_p = (i_p, j_p)$ for $p \in [\ell(C^N(w))]$, and set $w' = wt_{i_1 j_1}$. Then*

$$\overline{\text{rajcode}}(w) = \begin{cases} \overline{\text{rajcode}}(w') & \text{if } \ell(C^N(w)) > 1 \text{ and } (i_2, j_2) \notin M(C^N(w)), \\ \overline{\text{rajcode}}(w') + e_{i_1} & \text{otherwise.} \end{cases}$$

Proof. By the definition of climbing chains, i_1 is the earliest position in which w differs from w_0 . Then for $k < i_1$, we have $w(k) = w'(k) = n + 1 - k$. Thus, $\overline{\text{rajcode}}(w)_k = 0 = \overline{\text{rajcode}}(w')_k$ whenever $k < i_1$. Additionally, since w and w' agree after position j_1 , $\overline{\text{rajcode}}(w)_k = \overline{\text{rajcode}}(w')_k$ whenever $k > j_1$.

To show $\overline{\text{rajcode}}(w)_{j_1} = \overline{\text{rajcode}}(w')_{j_1}$, we argue by contradiction.

Clearly $\overline{\text{rajcode}}(w)_{j_1} \leq \overline{\text{rajcode}}(w')_{j_1}$, so necessarily $\overline{\text{rajcode}}(w)_{j_1} < \overline{\text{rajcode}}(w')_{j_1}$. Since i_1 and j_1 occur consecutively in an element of $\text{LIS}(w, i_1)$, it must be that $\overline{\text{rajcode}}(w)_{i_1} = \overline{\text{rajcode}}(w)_{j_1} + 1$. Thus, $\overline{\text{rajcode}}(w)_{i_1} \leq \overline{\text{rajcode}}(w')_{j_1}$.

On the other hand, take $\alpha \in \text{LIS}(w', j_1)$. Replacing $\alpha_1 = j_1$ by i_1 in α yields an increasing sequence β in w that starts from i_1 and has length $\overline{\text{rajcode}}(w')_{j_1} + 1$. Hence,

$\overline{\text{rajcode}}(w)_{i_1} + 1 \geq \overline{\text{rajcode}}(w')_{j_1} + 1$. Thus we have $\overline{\text{rajcode}}(w)_{i_1} = \overline{\text{rajcode}}(w')_{j_1}$. Consequently $\beta \in \text{LIS}(w, i_1)$. However $\beta_2 = \alpha_2 > j_1$, which contradicts the choice of (i_1, j_1) as lexicographically last among $\text{LIS}(w, i_1)$. This contradiction shows $\overline{\text{rajcode}}(w)_{j_1} = \overline{\text{rajcode}}(w')_{j_1}$.

Now, fix any k such that $i_1 < k < j_1$. As $w \triangleleft w'$, it follows that either $w(k) < w(i_1)$ or $w(k) > w(j_1)$. Take $\alpha \in \text{LIS}(w, k)$. If $w(k) > w(j_1)$ then α does not include j_1 , so $\alpha \in \text{LIS}(w', k)$ as well. In this case, $\overline{\text{rajcode}}(w)_k = \overline{\text{rajcode}}(w')_k$.

Otherwise, suppose $w(k) < w(i_1)$. Clearly, $\overline{\text{rajcode}}(w)_k \leq \overline{\text{rajcode}}(w')_k$. To reach a contradiction, suppose $\overline{\text{rajcode}}(w)_k < \overline{\text{rajcode}}(w')_k$. It follows that every $\alpha \in \text{LIS}(w', k)$ includes j_1 . Fix any $\alpha \in \text{LIS}(w', k)$, and suppose $\alpha_p = j_1$. Since $\overline{\text{rajcode}}(w)_{j_1} = \overline{\text{rajcode}}(w')_{j_1}$, we can find an element $\beta \in \text{LIS}(w, j_1) \cap \text{LIS}(w', j_1)$.

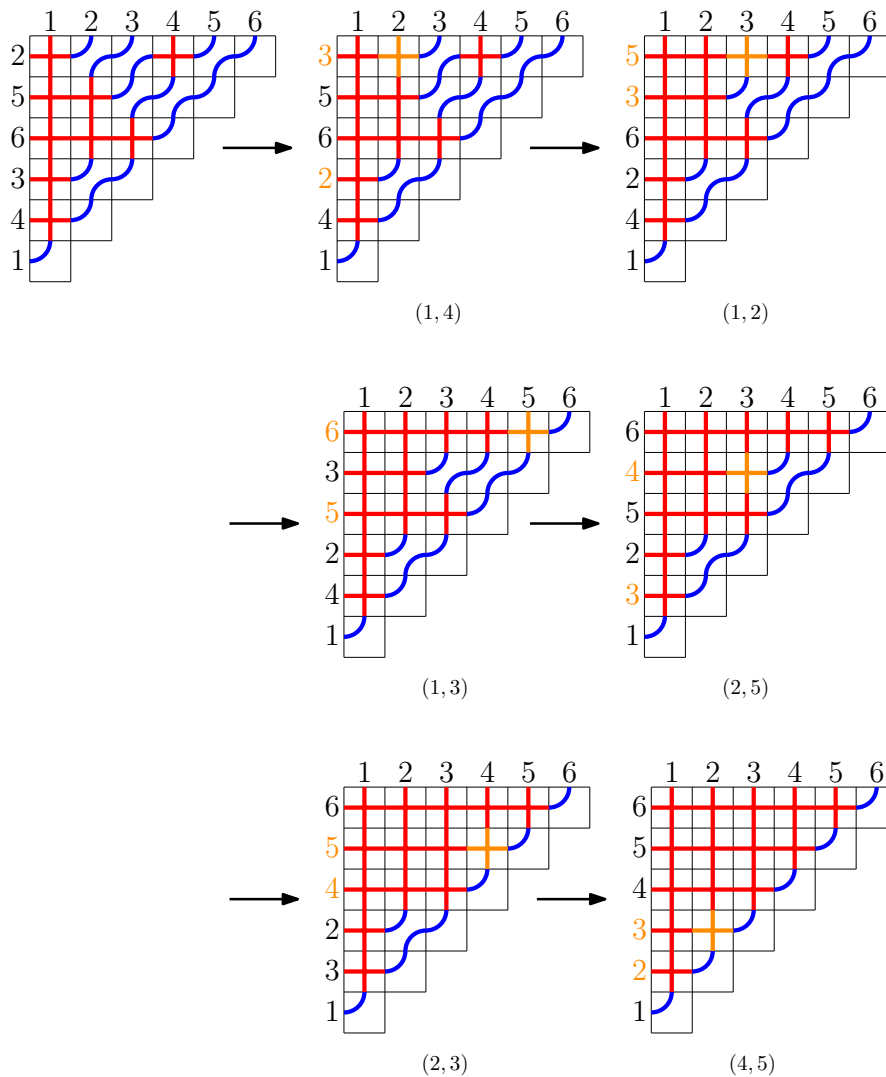


FIGURE 1. The bijection between climbing chains and reduced pipe dreams. The modified cross and strand numbers at each step is shown in orange. Note the result is the nested chain of $w = 256341$ from Example 5.6.

Construct a sequence γ by letting $\gamma_r = \alpha_r$ for $r \in [p]$, and $\gamma_r = \beta_r$ for $r > p$. It follows that γ has the same length as α and that γ is an increasing subsequence of w . Consequently, $\text{rajcode}(w')_k + 1 \leq \text{rajcode}(w)_k + 1$, a contradiction to our assumption that $\text{rajcode}(w)_k < \text{rajcode}(w')_k$. Hence, $\text{rajcode}(w)_k = \text{rajcode}(w')_k$.

We now address the remaining case $k = i_1$. Suppose first that $\ell(C^N(w)) > 1$ and $(i_2, j_2) \notin M(C^N(w))$. It follows that $i_1 = i_2$ and $j_1 > j_2$. Then from the construction of $C^N(w)$, we see $i_1 < j_2 < j_1$ with $w(i_1) < w(j_1) < w(j_2)$.

Clearly $\text{rajcode}(w)_{i_1} \geq \text{rajcode}(w')_{i_1}$. Take $\alpha \in \text{LIS}(w, i_1)$, with $\alpha_2 = j_1$. Let β be α with $\alpha_2 = j_1$ dropped. Then β is an increasing sequence in w' , so $\text{rajcode}(w')_{i_1} \geq \text{rajcode}(w)_{i_1} - 1$. Thus, $\text{rajcode}(w)_{i_1} \geq \text{rajcode}(w')_{i_1} \geq \text{rajcode}(w)_{i_1} - 1$. To reach a contradiction, suppose that $\text{rajcode}(w')_{i_1} = \text{rajcode}(w)_{i_1} - 1$. Then it must be that $\beta \in \text{LIS}(w', i_1)$. The choice of j_2 requires that $\beta_2 \leq j_2$. However, $\beta_2 = \alpha_3 > j_1$. Thus, $j_2 > j_1$, a contradiction. Hence $\text{rajcode}(w)_{i_1} = \text{rajcode}(w')_{i_1}$.

It remains to show that whenever $\ell(C^N(w)) = 1$ or $(i_2, j_2) \in M(C^N(w))$, we have $\text{rajcode}(w)_{i_1} = \text{rajcode}(w')_{i_1} + 1$. When $\ell(C^N(w)) = 1$, we have $w' = w_0$. In this case, it is easy to see that $\text{rajcode}(w)_{i_1} = 1$ as needed. Suppose then that $\ell(C^N(w)) > 1$ and $(i_2, j_2) \in M(C^N(w))$. This implies that either $i_1 < i_2$, or $i_1 = i_2$ with $j_1 < j_2$.

Assume first that $i_1 = i_2$ and $j_1 < j_2$. From the construction of $C^N(w)$, we see $i_1 < j_1 < j_2$ with $w(i_1) < w(j_1) < w(j_2)$. The choice of j_1 asserts that exists $\alpha \in \text{LIS}(w, i_1)$ with $\alpha_2 = j_1$. Hence $\text{rajcode}(w)_{i_1} \leq \text{rajcode}(w')_{i_1} + 1$. We prove the reverse inequality. The choice of j_2 asserts that exists $\beta \in \text{LIS}(w', i_1)$ with $\beta_2 = j_2$. Inserting j_1 into β produces an increasing subsequence of w starting from i_1 of length $\text{rajcode}(w')_{i_1} + 2$, so $\text{rajcode}(w)_{i_1} + 1 \geq \text{rajcode}(w')_{i_1} + 2$. This concludes the case $i_1 = i_2$ and $j_1 < j_2$.

Now, assume $i_1 < i_2$. The construction of $C^N(w)$ then implies that

$$w(j_1) = \max\{w(i_1), w(j_1 + 1), \dots, w(n)\}.$$

Since there must be some $\alpha \in \text{LIS}(w, i_1)$ that includes j_1 , it follows that $\alpha = (i_1, j_1)$ is the only such sequence. Hence $\text{rajcode}(w)_{i_1} = 1$ and $\text{rajcode}(w')_{i_1} = 0$ as needed. \square

THEOREM 7.2. Fix any $w \in S_n$ and let $\xi(w) = (C^N(w), M(C^N(w)))$. Then

$$\overline{\text{wt}}(\xi(w)) = \overline{\text{rajcode}}(w).$$

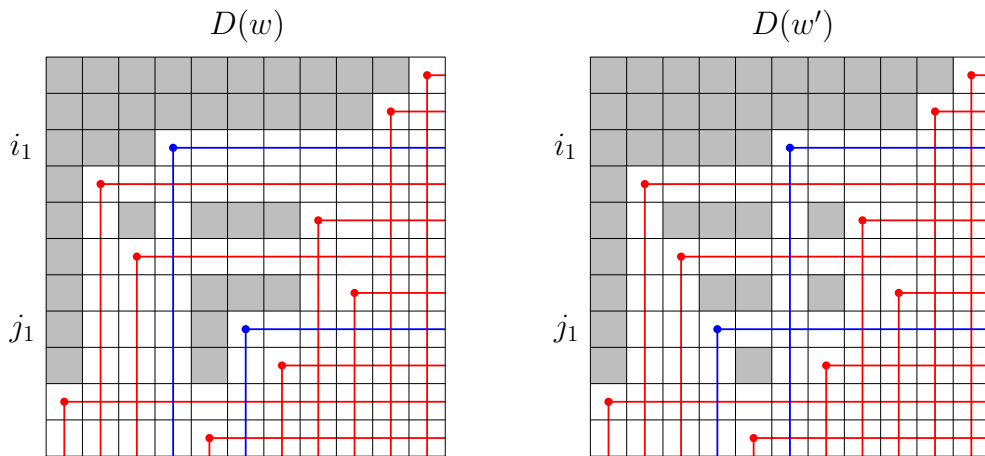


FIGURE 2. A visual aid for the structure of w and w' in the proof of Theorem 7.1.

Proof. We work by induction on $m = \ell(w_0) - \ell(w)$. If $m = 0$, then $w = w_0$. Then $C^N(w_0)$ is empty, and both $\text{rajcode}(w_0)$ and $\overline{\text{wt}}(\xi(w))$ are the zero vector.

Assume the result holds for all $m' < m$. Suppose $\ell(w) = \ell(w_0) - m$. Let $C^N(w)_p = (i_p, j_p)$ for $p \in [\ell(C^N(w))]$ and $w' = wt_{i_1 j_1}$, so $\xi(w') = (C^N(w'), M(C^N(w')))$. By induction, $\overline{\text{wt}}(\xi(w')) = \overline{\text{rajcode}}(w')$. The theorem now follows immediately from Lemma 4.14 and Theorem 7.1. \square

COROLLARY 7.3. For any $w \in S_n$, $\#M(C^N(w)) = \overline{\text{raj}}(w)$.

COROLLARY 7.4. The degree of \mathfrak{G}_w is at least $\text{raj}(w)$.

Proof. By Theorem 7.2, the monomial $\mathbf{x}^{\text{rajcode}(w)}$ has degree $\text{raj}(w)$ and lies in the support of \mathfrak{G}_w . Thus, $\deg \mathfrak{G}_w \geq \text{raj}(w)$. \square

8. THE DEGREE OF \mathfrak{G}_w EQUALS $\text{raj}(w)$

We showed in Corollary 7.4 that $\deg \mathfrak{G}_w \geq \text{raj}(w)$. In Corollary 8.10 we show that $\deg \mathfrak{G}_w = \text{raj}(w)$. This yields an alternate proof of part of Theorem 1.1.

Let C be a climbing chain of $w \in S_n$. Suppose C consists of

$$C : w = w^{(0)} \xrightarrow{C_1=(i_1, j_1)} w^{(1)} \xrightarrow{C_2=(i_2, j_2)} w^{(2)} \xrightarrow{C_3=(i_3, j_3)} \dots \xrightarrow{C_m=(i_m, j_m)} w^{(m)} = w_0.$$

Using Algorithm 1 below, we associate to C two sequences of sets: $\Psi_0, \Psi_1, \dots, \Psi_m$ and $\Omega_0, \Omega_1, \dots, \Omega_m$.

Algorithm 1

```

input  $w \in S_n$ 
input a climbing chain  $C$  of  $w$  with components
 $C : w = w^{(0)} \xrightarrow{C_1=(i_1, j_1)} w^{(1)} \xrightarrow{C_2=(i_2, j_2)} w^{(2)} \xrightarrow{C_3=(i_3, j_3)} \dots \xrightarrow{C_m=(i_m, j_m)} w^{(m)} = w_0$ 

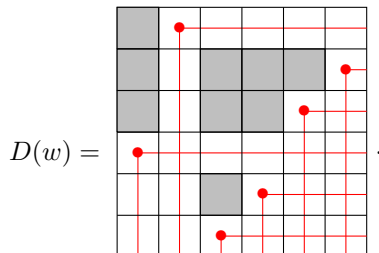
for  $q \in [n - 1]$  do
    compute the lexicographically last element  $\alpha^q = (\alpha_1^q, \dots, \alpha_{k_q}^q)$  of  $\text{LIS}(w, q)$ 
end for
initialize  $\Psi_0 = \bigcup_{q=1}^{n-1} \{(q, \alpha_2^q), \dots, (q, \alpha_{k_q}^q)\}$ 
initialize  $\Omega_0 = \emptyset$ 

for  $k = 1, 2, \dots, m$  do
    if  $C_k \in \Psi_{k-1}$  then
        set  $\Psi_k = \Psi_{k-1} \setminus \{C_k\}$ 
        set  $\Omega_k = \Omega_{k-1} \cup \{C_k\}$ 
    else if  $w^{(k)}(a') > w^{(k)}(b')$  for some  $(a', b') \in \Psi_{k-1}$  then
        set  $\Psi_k = \{(t_{i_k, j_k}(a), b) \mid (a, b) \in \Psi_{k-1}\}$ 
        set  $\Omega_k = \Omega_{k-1}$ 
    else
        set  $\Psi_k = \Psi_{k-1}$ 
        set  $\Omega_k = \Omega_{k-1}$ 
    end if
end for
return  $\Psi_0, \dots, \Psi_m$  and  $\Omega_0, \dots, \Omega_m$ 

```

DEFINITION 8.1. Observe that Algorithm 1 builds Ω_k and Ψ_k by performing exactly one of three available transformations on Ω_{k-1} and Ψ_{k-1} : the “if”, “else if”, and “else” blocks. We will (respectively) name these three operations transfer, adjust, and pass.

EXAMPLE 8.2. Let $w = 265143$. The Rothe diagram of w is



Consider the chain

$$C : 265143 \xrightarrow{(1,3)} 562143 \xrightarrow{(1,2)} 652143 \xrightarrow{(3,6)} 653142 \xrightarrow{(3,5)} 654132 \xrightarrow{(4,5)} 654312 \xrightarrow{(5,6)} 654321.$$

We first initialize Ψ_0 and Ω_0 following Algorithm 1. For each $q \in [5]$, the sets $\text{LIS}(w, q)$ are

$$\begin{aligned} \text{LIS}(w, 1) &= \{(1, 2), (1, 3), (1, 5), (1, 6)\} \\ \text{LIS}(w, 2) &= \{(2)\} \\ \text{LIS}(w, 3) &= \{(3)\} \\ \text{LIS}(w, 4) &= \{(4, 5), (4, 6)\} \\ \text{LIS}(w, 5) &= \{(5)\} \end{aligned}$$

The lexicographically last elements of $\text{LIS}(w, q)$ for $q = 1, 2, 3, 4, 5$ are

$$(1, 6), (2), (3), (4, 6), (5), \quad \text{so } \Psi_0 = \{(1, 6), (4, 6)\}.$$

We also start with $\Omega_0 = \emptyset$.

The algorithm terminates after the six steps shown in Figure 3.

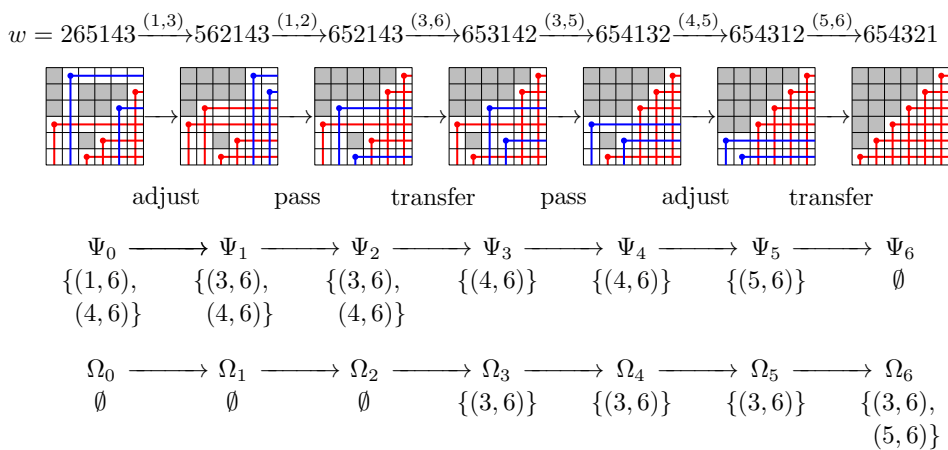


FIGURE 3. Execution of Algorithm 1 on a climbing chain of $w = 265143$.

The following lemma describes a key property of the sets Ψ_k from Algorithm 1.

LEMMA 8.3. Let $w \in S_n$ and C be a climbing chain of w . Assume the notation of Algorithm 1. Fix k with $0 \leq k \leq \ell(C)$. For each $q \in [n - 1]$, let the elements of Ψ_k of the form (q, \star) be $(q, \beta_1^q), \dots, (q, \beta_{l_q}^q)$, labeled so $\beta_1^q < \dots < \beta_{l_q}^q$. Then

$$q < \beta_1^q, \quad \text{and} \quad w^{(k)}(q) < w^{(k)}(\beta_1^q) < \dots < w^{(k)}(\beta_{l_q}^q).$$

Proof. We work by induction on k . When $k = 0$, the lemma follows from the definition of Ψ_0 . Suppose the lemma holds for $k - 1$.

Suppose first that Ψ_k is obtained from Ψ_{k-1} by a transfer operation. Then $\Psi_k = \Psi_{k-1} \setminus \{(i_k, j_k)\}$. Since (i_k, j_k) is a link in C , this means it must be the lexicographically first element of Ψ_{k-1} (by the induction assumption). The statement of the lemma follows easily in this case.

Next, suppose that Ψ_k is obtained from Ψ_{k-1} by an adjust operation (see Figure 4 for a visual representation of this case). It follows $(i_k, j_k) \notin \Psi_{k-1}$, $w^{(k)}(a') > w^{(k)}(b')$ for some $(a', b') \in \Psi_{k-1}$, and

$$\Psi_k = \{(t_{i_k j_k}(a), b) \mid (a, b) \in \Psi_{k-1}\}.$$

Observe that all such (a', b') must satisfy $a' = i_k$, $b' > j_k$, and $w^{(k-1)}(i_k) < w^{(k-1)}(b') < w^{(k-1)}(j_k)$. The adjust operation then guarantees the conditions of the lemma are met.

Lastly, consider the case of a pass operation. This means that $(i_k, j_k) \notin \Psi_{k-1}$ and $w^{(k)}(a) < w^{(k)}(b)$ for all $(a, b) \in \Psi_{k-1}$. The conditions of the lemma are immediate. \square

For later use, we separately record an observation made in the “adjust” case of the proof of Lemma 8.3.

LEMMA 8.4. Let $w \in S_n$ and C be a climbing chain of w . Assume the notation of Algorithm 1. Suppose step k in the execution of Algorithm 1 is an adjust operation caused by $(a', b') \in \Psi_{k-1}$. Then $a' = i_k$ and $b' > j_k$.

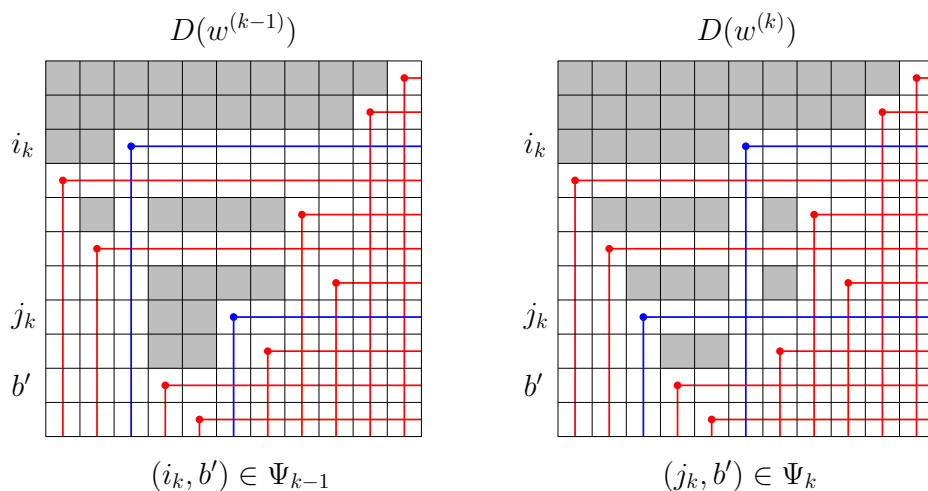


FIGURE 4. Visual representations of the “adjust” case in Lemma 8.3 . The two hooks moved are indicated in blue.

The following lemma records basic properties of Algorithm 1.

LEMMA 8.5. Let $w \in S_n$ and C be a climbing chain of w . Assume the notation of Algorithm 1. Then

- (i) $\emptyset = \Omega_0 \subseteq \dots \subseteq \Omega_m \subseteq \{C_1, \dots, C_m\}$.
- (ii) For each $0 \leq k \leq m$, one has $\Omega_k \subseteq \{C_1, \dots, C_k\}$.
- (iii) If $(i_p, j_p) \in \Omega_k$ for some k , then $\Omega_p = \Omega_{p-1} \cup \{(i_p, j_p)\}$ and $\Psi_p = \Psi_{p-1} \setminus \{(i_p, j_p)\}$.
- (iv) For each $0 \leq k \leq m$, one has $\#\Omega_k + \#\Psi_k = \overline{\text{raj}}(w)$.
- (v) $\Psi_m = \emptyset$ and $\#\Omega_m = \overline{\text{raj}}(w)$.

Proof. We first address (i) and (ii). Initially, $\Omega_0 = \emptyset$. Only the transfer operation causes $\Omega_k \neq \Omega_{k-1}$, specifically by adding the element $C_k = (i_k, j_k)$ to Ω_{k-1} . Thus (i) and (ii) hold. For (iii), note that $(i_p, j_p) \in \Omega_k$ implies step p of Algorithm 1 was a transfer operation. Claim (iv) holds by an argument analogous to that of (ii). The claim that $\Psi_m = \emptyset$ in (v) follows from Lemma 8.3. That $\#\Omega_m = \overline{\text{raj}}(w)$ then follows from (iv). \square

DEFINITION 8.6. Let C be a climbing chain of $w \in S_n$. Suppose the minimal markings $M(C)$ are

$$\{C_{k_1}, C_{k_2}, \dots, C_{k_p}\} \quad \text{with} \quad 1 \leq k_1 < k_2 < \dots < k_p \leq \ell(C).$$

Define the runs of C to be the (disjoint) subsequences of C consisting of links k_q through $k_{q+1} - 1$ for each $q \in [p]$ (taking $k_{p+1} = \ell(C) + 1$).

EXAMPLE 8.7. Continuing Example 8.2, let $w = 265143$ and

$$C = ((1, 3), (1, 2), (3, 6), (3, 5), (4, 5), (5, 6)).$$

Indicating the minimal markings $M(C)$ by overlines and separating runs by “|”, the runs of C are

$$\overline{(1, 3)}, (1, 2) \mid \overline{(3, 6)}, (3, 5) \mid \overline{(4, 5)} \mid \overline{(5, 6)}.$$

LEMMA 8.8. Let $w \in S_n$ and C be a climbing chain of w with length m . Construct the sets $\Omega_0, \dots, \Omega_m$ and Ψ_0, \dots, Ψ_m using Algorithm 1. Then each run of C contains at most one element of Ω_m .

Proof. Consider a run C_p, C_{p+1}, \dots, C_q of C containing at least two elements of Ω_m . The definition of run forces $\{C_p, C_{p+1}, \dots, C_q\} \cap M(C) = \{C_p\}$. It follows that $i_p = i_{p+1} = \dots = i_q$ and $j_p > j_{p+1} > \dots > j_q > i_p$.

First, suppose that $C_a, C_{a+1} \in \Omega_m$ for some $p \leq a < q$. By Lemma 8.5(iii), $C_a, C_{a+1} \in \Psi_{a-1}$. But $i_a = i_{a+1}$ and $j_a > j_{a+1}$, so Lemma 8.3 implies

$$w^{(a-1)}(i_a) < w^{(a-1)}(j_{a+1}) < w^{(a-1)}(j_a).$$

This contradicts the climbing chain condition $w^{(a-1)} \triangleleft w^{(a)} = w^{(a-1)}t_{i_a j_a}$ (see Lemma 4.1).

Now suppose $C_a, C_b \in \Omega_m$ for some a, b with $p \leq a < b - 1$, $b \leq q$ and $b - a$ minimal. Additionally, suppose there are no adjust operations between steps a and b in the execution of Algorithm 1. This implies $\Psi_a = \Psi_{a+1} = \dots = \Psi_{b-1}$. Thus $C_a, C_b \in \Psi_{a-1}$, again contradicting the Bruhat cover condition on climbing chains.

Lastly, suppose there was an adjust operation between steps a and b in the execution of Algorithm 1. Say the first such adjust operation occurs at step k . Recall that $i_a = i_{a+1} = \dots = i_k = \dots = i_b$. Suppose first that there are no elements $(i_k, q) \in \Psi_k$. Then by Lemma 8.4, no further adjust operations occur prior to step b . Thus there are no elements $(i_k, q) \in \Psi_{b-1}$. By Lemma 8.5, this contradicts that $C_b \in \Omega_m$.

Hence we may assume that there is an element of the form $(i_k, q) \in \Psi_k$. Then $(j_k, q) \in \Psi_{k-1}$, so Lemma 8.3 implies $q > j_k$. Hence $q > j_k > j_{k+1} > \dots > j_b$, so $(i_k, q) \notin \{C_k, C_{k+1}, \dots, C_b\}$.

If there is only a single adjust operation between steps a and b , this contradicts that $C_b \in \Omega_m$. If there is more than one adjust operation between steps a and b , the same contradiction is reached by repeating the previous argument for each adjust operation. \square

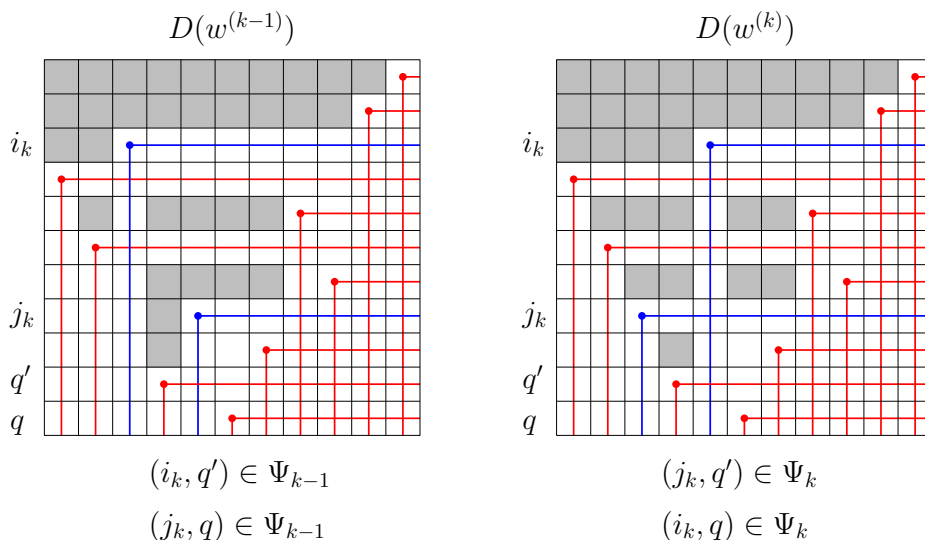


FIGURE 5. Visual aids for the “adjust” case in the proof of Lemma 8.8.

THEOREM 8.9. For any $w \in S_n$ and any climbing chain C of w ,

$$\overline{\text{raj}}(w) \leq \#M(C).$$

Proof. Use Algorithm 1 on C to produce Ω_m . Decompose C into runs. By definition, the number of runs equals $\#M(C)$. Lemma 8.8 shows there is at most one element of Ω_m in each run. Lemma 8.5(v) implies $\#\Omega_m = \overline{\text{raj}}(w)$. Putting this all together,

$$\#M(C) \geq \#\Omega_m = \overline{\text{raj}}(w). \quad \square$$

We are now ready to provide the alternative proof of the degree statement in Theorem 1.1:

COROLLARY 8.10 ([21, Theorem 1.1]). The degree of \mathfrak{G}_w equals $\text{raj}(w)$.

Proof. By Theorem 7.2 and Corollary 7.4, $\text{raj}(w)$ is a lower bound on $\deg \mathfrak{G}_w$ that is attained by the nested chain. Theorem 8.9 implies no climbing chain of w contributes a monomial of degree larger than $\text{raj}(w)$. Thus, $\deg \mathfrak{G}_w = \text{raj}(w)$. \square

9. LEADING TERM OF \mathfrak{G}_w IN TERM ORDERS WITH $x_1 < \dots < x_n$

We complete the alternative proof of Theorem 1.1. In Corollary 8.10 we showed the degree statement; below we show that $\mathbf{x}^{\text{rajcode}(x)}$ is the leading monomial of $\mathfrak{G}_w^{\text{top}}$ in any term order satisfying $x_1 < \dots < x_n$.

LEMMA 9.1. Let C be a climbing chain of $w \in S_n$ with $w \neq w_0$. Suppose $C_p = (i_p, j_p)$ for $p \in [\ell(C)]$, and set $v = wt_{i_1 j_1}$. Then

- (i) $\overline{\text{rajcode}}(w)_p = 0 = \overline{\text{rajcode}}(v)_p$ for $p \in [i_1 - 1]$,
- (ii) $\text{rajcode}(w)_{i_1} \geq \text{rajcode}(v)_{i_1}$,

(iii) $\overline{\text{rajcode}}(w)_p \leq \overline{\text{rajcode}}(v)_p$ for $i_1 + 1 \leq p \leq n$.

Proof. Claim (i) is immediate since i_1 since $w(k) = v(k) = n - k + 1$ for $k \in [i_1]$. Since $v(i_1) > w(i_1)$, any $\alpha \in \text{LIS}(v, i_1)$ is also an increasing sequence in w . This shows (ii).

Consider claim (iii). When $p > j_1$, (iii) follows since w and v agree after position j_1 . Since $v(j_1) < w(j_1)$, (iii) holds when $p = j_1$. It remains to consider $i_1 < p < j_1$. Since $w < v$, either $w(p) < w(i_1)$ or $w(p) > w(j_1)$. In either case, any $\alpha \in \text{LIS}(w, p)$ is also an increasing sequence in v . This proves (iii). \square

DEFINITION 9.2. We call a climbing chain C of $w \in S_n$ heavy if $\#M(C) = \overline{\text{raj}}(w)$.

We choose the name “heavy” since such chains C will have maximal total weight: $|\text{wt}(C)| = \text{raj}(w)$.

LEMMA 9.3. Let C be a climbing chain of $w \in S_n$, and assume the notation of Algorithm 1. Suppose C_k is the last link in C of the form (i_1, \star) . Then

- $\overline{\text{rajcode}}(w)_p = 0 = \overline{\text{rajcode}}(w^{(k)})_p$ for $p \in [i_1 - 1]$,
- $\overline{\text{rajcode}}(w)_{i_1} \geq \overline{\text{rajcode}}(w^{(k)})_{i_1} = 0$,
- $\overline{\text{rajcode}}(w)_p \leq \overline{\text{rajcode}}(w^{(k)})_p$ for $i_1 + 1 \leq p \leq n$.

whenever $i_1 + 1 \leq p \leq n$. Additionally if C is heavy, then the truncation $C' = (C_{k+1}, \dots, C_m)$ is heavy (as a climbing chain of $w^{(k)}$).

Proof. The itemized claims follow from repeated applications of Lemma 9.1. To see the final claim, suppose C' is not heavy. Concatenating (C_1, \dots, C_k) with a heavy chain for $(w^{(k)})$ yields a climbing chain of w with fewer minimal marking than C , implying that C is not heavy. \square

LEMMA 9.4. Let C be a heavy climbing chain of $w \in S_n$. Then

$$\overline{\text{wt}}(C, M(C))_{i_1} \leq \overline{\text{rajcode}}(w)_{i_1}.$$

Proof. Assume the notation of Algorithm 1. Suppose k is the index of the last link in C of the form (i_1, \star) and $C' = (C_{k+1}, \dots, C_m)$. By the choice of k , we have

- $\overline{\text{wt}}(C, M(C))_p = \overline{\text{wt}}(C', M(C'))_p = 0$ for $p < i_1$,
- $\overline{\text{wt}}(C, M(C))_{i_1} > 0$,
- $\overline{\text{wt}}(C', M(C'))_{i_1} = 0$,
- $\overline{\text{wt}}(C, M(C))_p = \overline{\text{wt}}(C', M(C'))_p$ for $p > i_1$.

Observe that the choice of k implies $\overline{\text{rajcode}}(w^{(k)})_p = 0$ for $p \leq i_1$. Applying Lemma 9.3,

$$\begin{aligned} |\overline{\text{wt}}(C, M(C))| &= \sum_{p=1}^n \overline{\text{wt}}(C, M(C))_p \\ &= \overline{\text{wt}}(C, M(C))_{i_1} + \sum_{p=i_1+1}^n \overline{\text{wt}}(C, M(C))_p \\ &= \overline{\text{wt}}(C, M(C))_{i_1} + \sum_{p=i_1+1}^n \overline{\text{wt}}(C', M(C'))_p \\ &= \overline{\text{wt}}(C, M(C))_{i_1} + |\overline{\text{wt}}(C', M(C'))| \\ &= \overline{\text{wt}}(C, M(C))_{i_1} + \overline{\text{raj}}(w^{(k)}) \\ &= \overline{\text{wt}}(C, M(C))_{i_1} + \overline{\text{rajcode}}(w^{(k)})_{i_1+1} + \dots + \overline{\text{rajcode}}(w^{(k)})_n. \end{aligned}$$

On the other hand, C being heavy implies

$$\begin{aligned} |\overline{\text{wt}}(C, M(C))| &= \overline{\text{raj}}(w) \\ &= \overline{\text{rajcode}}(w)_{i_1} + \cdots + \overline{\text{rajcode}}(w)_n. \end{aligned}$$

Hence,

$$(1) \quad \left(\overline{\text{wt}}(C, M(C))_{i_1} - \overline{\text{rajcode}}(w)_{i_1} \right) + \sum_{p=i_1+1}^n \left(\overline{\text{rajcode}}(w^{(k)})_p - \overline{\text{rajcode}}(w)_p \right) = 0.$$

By Lemma 9.1, each term in the summation is nonnegative, so the leftmost term must be nonpositive. Thus,

$$\overline{\text{wt}}(C, M(C))_{i_1} \leq \overline{\text{rajcode}}(w)_{i_1}. \quad \square$$

LEMMA 9.5. *Let C be a heavy climbing chain of $w \in S_n$. Assume the notation of Algorithm 1. Suppose that k is the index of the last link in C of the form (i_1, \star) , and*

$$\overline{\text{wt}}(C, M(C))_{i_1} = \overline{\text{rajcode}}(w)_{i_1}.$$

Then

$$\overline{\text{rajcode}}(w^{(k)})_p = \overline{\text{rajcode}}(w)_p \quad \text{for } p \geq i_1 + 1.$$

Proof. This is an immediate consequence of (1). □

THEOREM 9.6. *For $w \in S_n$ and any term order satisfying $x_1 < x_2 < \cdots < x_n$,*

$$\mathbf{x}^{\overline{\text{rajcode}}(w)} = \min \left\{ \mathbf{x}^{\overline{\text{wt}}(C, M(C))} \mid C \text{ is a heavy climbing chain of } w \right\}.$$

Equivalently,

$$\mathbf{x}^{\overline{\text{rajcode}}(w)} = \max \left\{ \mathbf{x}^{\overline{\text{wt}}(C, M(C))} \mid C \text{ is a heavy climbing chain of } w \right\}.$$

Proof. It is immediate from the definition of wt that the two assertions of the theorem are equivalent. We focus on the first. Since all heavy climbing chains have the same number of minimal markings, the theorem is equivalent to proving

$$\mathbf{x}^{\overline{\text{rajcode}}(w)} = \max \left\{ \mathbf{x}^{\overline{\text{wt}}(C, M(C))} \mid C \text{ is a heavy climbing chain of } w \right\}.$$

in any term order with $x_1 > x_2 > \cdots > x_n$. For each $p \in [n]$, let k_p be the index of the last link in C of the form (p, \star) . The theorem follows by applying Lemmas 9.4 and 9.5 (sequentially) to each of $w^{(k_1)}, w^{(k_2)}, \dots, w^{(k_{n-1})}$. □

Alternate Proof of Theorem 1.1. Combine Theorem 9.6 and Theorem 4.10. □

10. INTERPOLATING CHAINS

We define climbing chains that interpolate between the greedy and nested chains. We show that the monomials predicted by Hafner (Conjecture 1.2) to be the leading monomials of the homogeneous components of the Grothendieck polynomial \mathfrak{G}_w in any term order satisfying $x_1 < \cdots < x_n$ arise from these interpolating chains. In the next section we show that they are indeed the leading monomials (confirming Conjecture 1.2).

DEFINITION 10.1. *Let $w \in S_n$ and C be a climbing chain of w . A link $C_p = (i_p, j_p)$ is called greedy if $\text{greedy}(wt_{i_1 j_1} \cdots t_{i_{p-1} j_{p-1}}) = (i_p, j_p)$, and is called nested if $\text{nested}(wt_{i_1 j_1} \cdots t_{i_{p-1} j_{p-1}}) = (i_p, j_p)$.*

Clearly $C^G(w)$ is the unique climbing chain of w such that every link is greedy, and analogously for $C^N(w)$. Note that a given link can be both greedy and nested. We now define a family of chains that interpolate between $C^G(w)$ and $C^N(w)$.

DEFINITION 10.2. Fix $w \in S_n$ and let m be the length of any climbing chain of w . We define the interpolating chains of w to be the climbing chains $I^0(w), \dots, I^m(w)$ constructed as follows. For $0 \leq k \leq m$, define $I^k(w)$ to be the unique climbing chain of w consisting of $m - k$ greedy links followed by k nested links.

Observe that $C^N(w) = I^m(w)$ and $C^G(w) = I^0(w) = I^1(w)$, since the final link in a chain must be both nested and greedy.

EXAMPLE 10.3. Let $w = 5721463$. The distinct interpolating chains of w together with their underlying permutations are (with nested steps blue and greedy steps red)

$$I^0(w) : 5721463 \xrightarrow{(1,6)} 6721453 \xrightarrow{(1,2)} 7621453 \xrightarrow{(3,7)} 7631452 \xrightarrow{(3,5)} 7641352 \xrightarrow{(3,6)} 7651342 \\ \xrightarrow{(4,7)} 7652341 \xrightarrow{(4,5)} 7653241 \xrightarrow{(4,6)} 7654231 \xrightarrow{(5,6)} 7654321,$$

$$I^4(w) : 5721463 \xrightarrow{(1,6)} 6721453 \xrightarrow{(1,2)} 7621453 \xrightarrow{(3,7)} 7631452 \xrightarrow{(3,5)} 7641352 \xrightarrow{(3,6)} 7651342 \\ \xrightarrow{(4,5)} 7653142 \xrightarrow{(4,6)} 7654132 \xrightarrow{(5,7)} 7654231 \xrightarrow{(5,6)} 7654321,$$

$$I^7(w) : 5721463 \xrightarrow{(1,6)} 6721453 \xrightarrow{(1,2)} 7621453 \xrightarrow{(3,5)} 7641253 \xrightarrow{(3,6)} 7651243 \xrightarrow{(4,5)} 7652143 \\ \xrightarrow{(4,7)} 7653142 \xrightarrow{(4,6)} 7654132 \xrightarrow{(5,7)} 7654231 \xrightarrow{(5,6)} 7654321.$$

The full collection of interpolating chains is

$$I^0(w) = I^1(w) = I^2(w) = I^3(w), \\ I^4(w) = I^5(w) = I^6(w), \\ I^7(w) = I^8(w) = I^9(w).$$

DEFINITION 10.4. To each permutation $w \in S_n$, we associate a sequence of vectors

$$\text{leads}(w) = (L_w(0), \dots, L_w(d)),$$

where $d = \text{raj}(w) - \text{inv}(w)$. Set $L_w(0) = \text{invcode}(w)$. Define $L_w(i)$ for $i \in [d]$ as follows: set $j \in [n]$ to be the largest index such that $L_w(i - 1)_j < \text{rajcode}(w)_j$, and define $L_w(i) = L_w(i - 1) + e_j$.

Recall that $\text{rajcode}(w)_k \geq \text{invcode}(w)_k$ for all $k \in [n]$, so $L_w(d) = \text{rajcode}(w)$.

EXAMPLE 10.5. Continuing Example 10.3, when $w = 5721463$ we compute

$$\text{invcode}(w) = (4, 5, 1, 0, 1, 1, 0), \\ \text{rajcode}(w) = (5, 5, 2, 1, 1, 1, 0).$$

Thus $\text{leads}(w) = (L_w(0), L_w(1), L_w(2), L_w(3))$, where

$$L_w(0) = (4, 5, 1, 0, 1, 1, 0), \\ L_w(1) = (4, 5, 1, 1, 1, 1, 0), \\ L_w(2) = (4, 5, 2, 1, 1, 1, 0), \\ L_w(3) = (5, 5, 2, 1, 1, 1, 0).$$

We now prove that the monomials $\mathbf{x}^{L_w(k)}$ appear with nonzero coefficient in \mathfrak{G}_w . We illustrate the key idea in the following example.

EXAMPLE 10.6. Continuing Examples 10.3 and 10.5, let $w = 5721463$. We mark each interpolating chain $I^k(w)$ by

- completely marking the initial string of greedy links plus the first nested link, and
- minimally marking the remaining links.

Indicating markings with overlines and greedy/nested with red/blue respectively, we obtain the chains and exponents shown in Figure 6.

Marked Interpolating Chain	Weight Exponent
$I^0(w) = (\overline{(1, 6)}, \overline{(1, 2)}, \overline{(3, 7)}, \overline{(3, 5)}, \overline{(3, 6)}, \overline{(4, 7)}, \overline{(4, 5)}, \overline{(4, 6)}, \overline{(5, 6)})$	$L_w(0) = (4, 5, 1, 0, 1, 1, 0)$
$I^1(w) = (\overline{(1, 6)}, \overline{(1, 2)}, \overline{(3, 7)}, \overline{(3, 5)}, \overline{(3, 6)}, \overline{(4, 7)}, \overline{(4, 5)}, \overline{(4, 6)}, \overline{(5, 6)})$	$L_w(0) = (4, 5, 1, 0, 1, 1, 0)$
$I^2(w) = (\overline{(1, 6)}, \overline{(1, 2)}, \overline{(3, 7)}, \overline{(3, 5)}, \overline{(3, 6)}, \overline{(4, 7)}, \overline{(4, 5)}, \overline{(4, 6)}, \overline{(5, 6)})$	$L_w(0) = (4, 5, 1, 0, 1, 1, 0)$
$I^3(w) = (\overline{(1, 6)}, \overline{(1, 2)}, \overline{(3, 7)}, \overline{(3, 5)}, \overline{(3, 6)}, \overline{(4, 7)}, \overline{(4, 5)}, \overline{(4, 6)}, \overline{(5, 6)})$	$L_w(0) = (4, 5, 1, 0, 1, 1, 0)$
$I^4(w) = (\overline{(1, 6)}, \overline{(1, 2)}, \overline{(3, 7)}, \overline{(3, 5)}, \overline{(3, 6)}, \overline{(4, 5)}, \overline{(4, 6)}, \overline{(5, 7)}, \overline{(5, 6)})$	$L_w(1) = (4, 5, 1, 1, 1, 1, 0)$
$I^5(w) = (\overline{(1, 6)}, \overline{(1, 2)}, \overline{(3, 7)}, \overline{(3, 5)}, \overline{(3, 6)}, \overline{(4, 5)}, \overline{(4, 6)}, \overline{(5, 7)}, \overline{(5, 6)})$	$L_w(1) = (4, 5, 1, 1, 1, 1, 0)$
$I^6(w) = (\overline{(1, 6)}, \overline{(1, 2)}, \overline{(3, 7)}, \overline{(3, 5)}, \overline{(3, 6)}, \overline{(4, 5)}, \overline{(4, 6)}, \overline{(5, 7)}, \overline{(5, 6)})$	$L_w(1) = (4, 5, 1, 1, 1, 1, 0)$
$I^7(w) = (\overline{(1, 6)}, \overline{(1, 2)}, \overline{(3, 5)}, \overline{(3, 6)}, \overline{(4, 5)}, \overline{(4, 7)}, \overline{(4, 6)}, \overline{(5, 7)}, \overline{(5, 6)})$	$L_w(2) = (4, 5, 2, 1, 1, 1, 0)$
$I^8(w) = (\overline{(1, 6)}, \overline{(1, 2)}, \overline{(3, 5)}, \overline{(3, 6)}, \overline{(4, 5)}, \overline{(4, 7)}, \overline{(4, 6)}, \overline{(5, 7)}, \overline{(5, 6)})$	$L_w(2) = (4, 5, 2, 1, 1, 1, 0)$
$I^9(w) = (\overline{(1, 6)}, \overline{(1, 2)}, \overline{(3, 5)}, \overline{(3, 6)}, \overline{(4, 5)}, \overline{(4, 7)}, \overline{(4, 6)}, \overline{(5, 7)}, \overline{(5, 6)})$	$L_w(3) = (5, 5, 2, 1, 1, 1, 0)$

FIGURE 6.

LEMMA 10.7. Fix $w \in S_n$. Let $\text{greedy}(w) = (i, j)$ and $v = wt_{ij}$. Then either

$$\overline{\text{rajcode}}(w) = \overline{\text{rajcode}}(v) \quad \text{or} \quad \overline{\text{rajcode}}(w) = \overline{\text{rajcode}}(v) + e_i.$$

Equivalently, either $\text{rajcode}(w) = \text{rajcode}(v)$ or $\text{rajcode}(w) = \text{rajcode}(v) - e_i$.

Proof. By definition, $i = \min\{k \mid w(k) \neq n + 1 - k\}$, and $j = w^{-1}(w(i) + 1)$. Thus v is obtained from w by swapping the numbers $w(i)$ and $w(i) + 1$. It is straightforward to check that $\overline{\text{rajcode}}(w) = \overline{\text{rajcode}}(v)$ unless every sequence $\alpha \in \text{LIS}(w, i)$ has $\alpha_2 = j$. In this case, $\text{rajcode}(w) = \text{rajcode}(v) + e_i$. \square

LEMMA 10.8. Fix $w \in S_n$. Let $\text{greedy}(w) = (i, j)$ and $v = wt_{ij}$. Set $\text{leads}(v)$ to denote the list of vectors $(L_w(0), \dots, L_w(d))$. Then:

- (a) If $\text{rajcode}(w) = \text{rajcode}(v)$, then

$$\text{leads}(w) = (L_w(0) - e_i, \dots, L_w(d) - e_i, \text{rajcode}(w)).$$

- (b) If $\text{rajcode}(w) = \text{rajcode}(v) - e_i$, then

$$\text{leads}(w) = (L_w(0) - e_i, \dots, L_w(d) - e_i)$$

Proof. The lemma follows from Lemma 10.7 and $\text{invcode}(w) = \text{invcode}(v) - e_i$. \square

THEOREM 10.9. Fix $w \in S_n$ and let m be the length of any climbing chain of w . Let

$$\text{leads}(w) = (L_w(0), \dots, L_w(d)).$$

For $0 \leq p \leq m$, let $\xi_w(p) = (I^p(w), U^p(w))$ where $U^0(w) = I^0(w)$, and

$$U^p(w) = M(I^p(w)) \cup \{I^p(w)_1, I^p(w)_2, \dots, I^p(w)_{m-p+1}\} \text{ for } p \in [m].$$

Then

$$\{L_w(0), \dots, L_w(d)\} = \{\text{wt}(\xi_w(0)), \dots, \text{wt}(\xi_w(m))\}.$$

Proof. We work by induction on $\ell(w)$. When $w = w_0$ we have $d = m = 0$ and $\xi_w(0) = (\emptyset, \emptyset)$. Thus

$$\{L_w(0)\} = \{(n - 1, n - 2, \dots, 1, 0)\} = \{\text{wt}(\xi(0))\}.$$

Fix $w \in S_n$ and suppose the theorem holds for all permutations v with $\ell(v) > \ell(w)$. Set $\text{greedy}(w) = (i, j)$ and $v = wt_{i,j}$. By definition, $\text{invcode}(w) = \text{invcode}(v) - e_i$. By Lemma 10.7, either $\text{rajcode}(w) = \text{rajcode}(v)$ or $\text{rajcode}(w) = \text{rajcode}(v) - e_i$.

Suppose first that $\text{rajcode}(w) = \text{rajcode}(v)$. By Lemma 10.8, $\text{leads}(w)$ is obtained by subtracting e_i from each element of $\text{leads}(v)$, then appending $\text{rajcode}(w)$. Thus,

$$\{L_w(0), \dots, L_w(d)\} = \{L_v(0) - e_i, \dots, L_v(d - 1) - e_i, \text{rajcode}(w)\}.$$

On the other hand, prepending and marking (i, j) to all of the interpolating chains $\xi_v(0), \dots, \xi_v(m - 1)$ of v yields $\xi_w(0), \dots, \xi_w(m - 1)$. The final chain $\xi_w(m)$ is simply the nested chain of w with its minimal markings. Thus

$$\{\text{wt}(\xi_w(0)), \dots, \text{wt}(\xi_w(m))\} = \{\text{wt}(\xi_v(0)) - e_i, \dots, \text{wt}(\xi_v(m - 1)) - e_i, \text{rajcode}(w)\}.$$

Applying the induction assumption to v completes the proof.

The proof in the case that $\text{rajcode}(w) = \text{rajcode}(v) - e_i$ is almost identical. It is still true that

$$\{\text{wt}(\xi_w(0)), \dots, \text{wt}(\xi_w(m))\} = \{\text{wt}(\xi_v(0)) - e_i, \dots, \text{wt}(\xi_v(m - 1)) - e_i, \text{rajcode}(w)\}.$$

However, Lemma 10.8 shows one instead obtains

$$\{L_w(0), \dots, L_w(d)\} = \{L_v(0) - e_i, \dots, L_v(d) - e_i\}.$$

In this case, one completes the proof by noting $\text{rajcode}(w) = \text{rajcode}(v) - e_i = \text{wt}(\xi_v(m - 1)) - e_i$. \square

COROLLARY 10.10. *The monomials appearing in Conjecture 1.2 lie in the support of \mathfrak{S}_w .*

11. PROOF OF CONJECTURE 1.2

In Corollary 10.10 we showed that the monomials named in Conjecture 1.2 are in the support of \mathfrak{S}_w . We show that these monomials are actually the leading monomials of the homogeneous components of \mathfrak{S}_w .

LEMMA 11.1. *Let (C, U) be any marked climbing chain of $w \in S_n$ and $C_1 = (i_1, j_1)$. Then*

$$\overline{\text{wt}(C, U)}_{i_1} \leq \overline{\text{invcode}(w)}_{i_1}.$$

If equality holds, then

$$(C_1, C_2, \dots, C_k) = (C_1^G, \dots, C_k^G),$$

where C_k is the last link in C of the form (i_1, \star) .

Proof. From the definition of a climbing chain, we see that all links of the form (i_1, \star) in C are noninversions of w , pairs (a, b) with $a < b$ and $w(a) < w(b)$. There are exactly $n - i_1 - \text{invcode}(w)_{i_1} = \overline{\text{invcode}(w)}_{i_1}$ of these noninversions with $a = i_1$. The second assertion follows from the nested hook interpretation of climbing chain links. \square

LEMMA 11.2. Let (C, U) be a marked climbing chain of $w \in S_n$ with C given by

$$C : w = w^{(0)} \xrightarrow{C_1=(i_1, j_1)} w^{(1)} \xrightarrow{C_2=(i_2, j_2)} w^{(2)} \xrightarrow{C_3=(i_3, j_3)} \dots \xrightarrow{C_m=(i_m, j_m)} w^{(m)} = w_0.$$

Then

$$\overline{\text{wt}}(C, U)_{i_1} \leq \overline{L}_w(m - \#U)_{i_1}.$$

Proof. Suppose C_k is the last link in C of the form (i_1, \star) . By Theorem 8.9,

$$\#U = |\overline{\text{wt}}(C, U)| = \overline{\text{wt}}(C, U)_{i_1} + \sum_{p=i_1+1}^n \overline{\text{wt}}(C, U)_p \geq \overline{\text{wt}}(C, U)_{i_1} + \overline{\text{raj}}(w^{(k)}).$$

On the other hand, Lemma 9.3 implies

$$\overline{\text{raj}}(w) \leq \overline{\text{rajcode}}(w)_{i_1} + \overline{\text{raj}}(w^{(k)}),$$

so that

$$\#U = \overline{\text{raj}}(w) + (\#U - \overline{\text{raj}}(w)) \leq \overline{\text{rajcode}}(w)_{i_1} + \overline{\text{raj}}(w^{(k)}) + (\#U - \overline{\text{raj}}(w)).$$

Hence

$$\overline{\text{wt}}(C, U)_{i_1} \leq \overline{\text{rajcode}}(w)_{i_1} + (\#U - \overline{\text{raj}}(w)).$$

To conclude the proof, we will analyze $\overline{\text{rajcode}}(w)_{i_1} + (\#U - \overline{\text{raj}}(w))$. Let $d = m - \#U$. See Figure 7 for a visual representation of the following argument. Recall that $L_w(d)$ is obtained from $\text{invcode}(w)$ by iteratively increasing components d times, moving right-to-left so entries stay weakly below the corresponding entries of $\text{rajcode}(w)$. Then $\overline{L}_w(d)$ is obtained from $\text{invcode}(w)$ by iteratively decreasing components d times, moving right-to-left so entries stay weakly above the corresponding entries of $\text{rajcode}(w)$.

Equivalently, $\overline{L}_w(d)$ is obtained from $\text{rajcode}(w)$ by iteratively increasing components $m - \overline{\text{raj}}(w) - d$ times, moving left-to-right so entries stay weakly below the corresponding entries of $\text{invcode}(w)$. Note that

$$m - \overline{\text{raj}}(w) - d = \#U - \overline{\text{raj}}(w).$$

If $\overline{\text{rajcode}}(w)_{i_1} + (\#U - \overline{\text{raj}}(w)) \geq \text{invcode}(w)_{i_1}$, then $\overline{L}_w(d)_{i_1} = \text{invcode}(w)_{i_1}$ and Lemma 11.1 completes the proof.

Otherwise, $\overline{\text{rajcode}}(w)_{i_1} + (\#U - \overline{\text{raj}}(w)) < \text{invcode}(w)_{i_1}$, so

$$\overline{L}_w(m - \#U)_{i_1} = \overline{\text{rajcode}}(w)_{i_1} + (\#U - \overline{\text{raj}}(w)) \geq \overline{\text{wt}}(C, U)_{i_1}$$

as shown above, so we are done. □

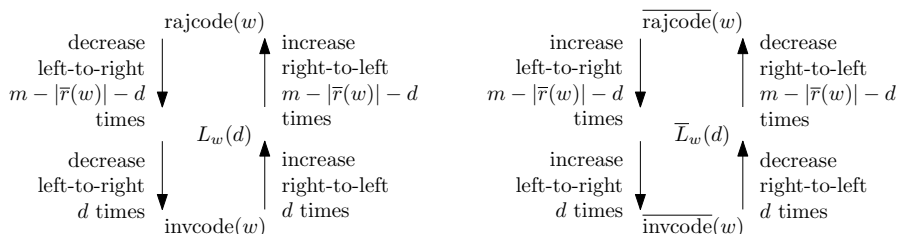


FIGURE 7.

THEOREM 11.3. *Fix $w \in S_n$, and let \mathcal{Z}_w denote the set of marked climbing chains of w . For each d with $1 \leq d \leq \text{raj}(w) - \text{inv}(w)$ and any term order satisfying $x_1 < x_2 < \dots < x_n$,*

$$\mathbf{x}^{\bar{L}_w(d)} = \min \left\{ \mathbf{x}^{\overline{\text{wt}}(C,U)} \mid (C,U) \in \mathcal{Z}_w \text{ with } \#U = \ell(C) - d \right\}.$$

Equivalently,

$$\mathbf{x}^{L_w(d)} = \max \left\{ \mathbf{x}^{\text{wt}(C,U)} \mid (C,U) \in \mathcal{Z}_w \text{ with } \#U = \ell(C) - d \right\}.$$

Proof. It is immediate from the definition of wt that the two assertions of the theorem are equivalent. We focus on the first. Since we are considering marked climbing chains with a fixed number of marks, the theorem is equivalent to proving

$$\mathbf{x}^{\bar{L}_w(d)} = \max \left\{ \mathbf{x}^{\overline{\text{wt}}(C,M(C))} \mid (C,U) \in \mathcal{Z}_w \text{ with } \#U = \ell(C) - d \right\}.$$

in any term order with $x_1 > x_2 > \dots > x_n$.

Assume the notation of Algorithm 1. By definition, $\bar{L}_w(d)_p = \overline{\text{wt}}(C,U)_p$ for $p < i_1$. Lemma 11.2 shows that $\bar{L}_w(d)_{i_1} \geq \overline{\text{wt}}(C,U)_{i_1}$. If the inequality is strict, then there is nothing to prove.

Suppose first that $\bar{L}_w(d)_{i_1} = \overline{\text{wt}}(C,U)_{i_1} < \overline{\text{invcode}}(w)_{i_1}$. Then for $p > i_1$, $\bar{L}_w(d)_p = \overline{\text{rajcode}}(w)_p$. Let C_k be the last link in C of the form (i_1, \star) . Let $C' = (C_{k+1}, \dots, C_m)$ and $U' = U \cap C'$.

By Theorem 8.9,

$$\sum_{p=i_1+1}^n \overline{\text{wt}}(C,U)_p = |\overline{\text{wt}}(C',U')| \geq \overline{\text{raj}}(w^{(k)}).$$

For $p > i_1$, Lemma 9.3 shows $\overline{\text{rajcode}}(w^{(k)})_p \geq \overline{\text{rajcode}}(w)_p$. Then

$$(2) \quad \sum_{p=i_1+1}^n \overline{\text{wt}}(C,U)_p \geq \overline{\text{raj}}(w^{(k)}) \geq \sum_{p=i_1+1}^n \overline{\text{rajcode}}(w)_p = \sum_{p=i_1+1}^n \bar{L}_w(d)_p.$$

Since $|\bar{L}_w(d)| = |\overline{\text{wt}}(C,U)|$ and $\bar{L}_w(d)_p = \overline{\text{wt}}(C,U)_p$ for $p \leq i_1$, it follows that equality holds throughout in (2). Thus C' is heavy for $w^{(k)}$, so the theorem now follows from Theorem 9.6.

It remains to consider the case that $\bar{L}_w(d)_{i_1} = \overline{\text{wt}}(C,U)_{i_1} = \overline{\text{invcode}}(w)_{i_1}$. In this case, we may pass to $w^{(k)}$ and repeat the above arguments and iterate. The iteration will terminate at worst with w_0 , for which the theorem is trivial. \square

12. ON LEADING MONOMIALS IN TERM ORDERS SATISFYING $x_1 > \dots > x_n$

We consider the leading monomials of the homogeneous components of \mathfrak{S}_w in any term order satisfying $x_1 > x_2 > \dots > x_n$. The lowest degree component, the Schubert polynomial \mathfrak{S}_w , is well-known to have leading term coming from the top-reduced pipe dream of the permutation [1]. We verify the leaping chain $C^L(w)$ (Definition 5.8) realizes this monomial. We prove the staircase chain $C^S(w)$ (Definition 5.11) is heavy, and conjecture it yields the leading monomial of $\mathfrak{S}_w^{\text{top}}$ in any term order satisfying $x_1 > x_2 > \dots > x_n$. We conclude with an analogue of Conjecture 1.2.

12.1. LEAPING CHAIN.

THEOREM 12.1. *In any term order with $x_1 > x_2 > \dots > x_n$, the leading monomial of the Schubert polynomial \mathfrak{S}_w is $\mathbf{x}^{\text{wt}(C^L(w),C^L(w))}$.*

Proof. It is straightforward to check that the leaping chain corresponds to the top-reduced pipe dream under the cotransition bijection. \square

12.2. STAIRCASE CHAIN.

THEOREM 12.2. For any $w \in S_n$, $C^S(w)$ is a heavy climbing chain.

Proof. Recall that $C^S(w)$ is the concatenation of $\text{stair}(w)$ and another staircase chain $C^S(w')$. Observe that the sequence $\text{stair}(w)$ will contribute only its first element to $M(C^S(w))$ by definition. If we can show that $\overline{\text{raj}}(w) = \overline{\text{raj}}(w') + 1$, then we are done by induction.

We show that $\overline{\text{raj}}(w) = \overline{\text{raj}}(w') + 1$ by tracking how $\overline{\text{raj}}(\cdot)$ changes as the transpositions corresponding to the links in $\text{stair}(w)$ are applied iteratively to w . Recall the sets K_1, \dots, K_q used to define $\text{stair}(w)$.

To see the following claims, refer to the visual aid in Figure 8. Each link from K_1 other than the last such link does not change the previous value $\overline{\text{raj}}(\cdot)$. The last link from K_1 decreases the previous value $\overline{\text{raj}}(\cdot)$ by 1. For links from each subsequent set K_p ,

- the first such link increases $\overline{\text{raj}}(\cdot)$ by 1;
- intermediate such links do not change $\overline{\text{raj}}(\cdot)$;
- the final such link decreases $\overline{\text{raj}}(\cdot)$ by 1.

Any singleton K_p acts as both the first and final link, leaving $\overline{\text{raj}}(\cdot)$ unchanged. Thus,

$$\overline{\text{raj}}(w) = \overline{\text{raj}}(w') + 1. \quad \square$$

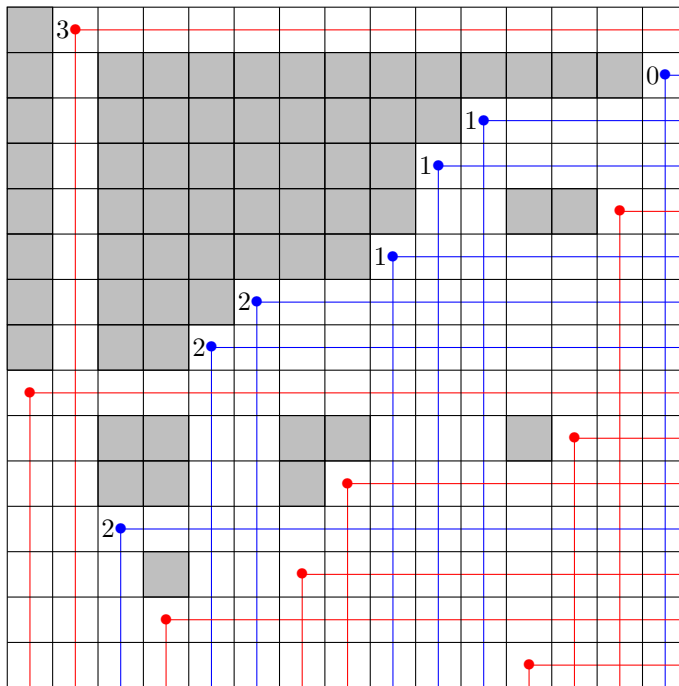


FIGURE 8. A visual aid for the proof of Theorem 12.2. The sets used to define $\text{stair}(w)$ here are $K = \{12, 8, 7, 6, 4, 3, 2\}$ (indicated in blue), $K_1 = \{12, 8, 7\}$, $K_2 = \{6, 4, 3\}$, and $K_3 = \{2\}$.

DEFINITION 12.3. Fix $w \in S_n$. Define the highest nesting length $h(w)$ as follows. If $w = w_0$, set $h(w) = 0$. Otherwise, set $(q_0, q_1) = \text{leap}(w)$. Iteratively define

$$q_k = \min\{p \mid p > q_{k-1} \text{ and } wt_{q_0 q_1} \cdots t_{q_0 q_{k-1}} \triangleleft wt_{q_0 q_1} \cdots t_{q_0 q_{k-1}} t_{q_0 p}\}$$

until the set on the right side is empty. Let the $h(w) = \#\{q_1, q_2, \dots\}$ be the number of steps taken.

Graphically, the quantity $h(w)$ counts the sequence of highest nested hooks south and east of the dot in row q_0 of $D(w)$.

EXAMPLE 12.4. For the $w = 1465273$, $h(w) = 3$ counts the hooks indicated in blue in Figure 9.

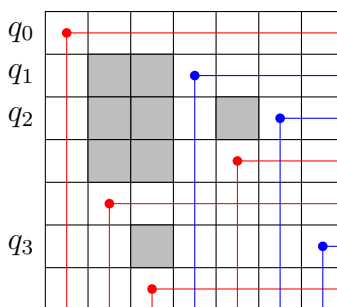


FIGURE 9.

LEMMA 12.5. For any $w \in S_n$ and marked climbing chain (C, U) of w ,

$$\overline{\text{wt}}(C, U)_{i_1} \geq h(w).$$

Proof. If $w = w_0$, the lemma reduces to $0 \geq 0$. Otherwise, let $q = \min\{j \mid w(j) \neq n + 1 - j\}$. We work by induction. Let $C_p = (i_p, j_p)$ for all p , so $q = i_1$. Set $w' = wt_{i_1 j_1}$ and (C', U') be the corresponding truncation of (C, U) . Suppose $\text{leap}(w) = (i_1, b)$. If $j_1 \neq b$, then $j_1 > b$, so $h(w') = h(w)$. In this case, induction implies

$$\overline{\text{wt}}(C, U)_{i_1} \geq \overline{\text{wt}}(C', U')_{i_1} \geq h(w') = h(w).$$

Hence we may suppose that $j_1 = b$. If $w'(i_1) = n - i_1 + 1$, then $\overline{\text{wt}}(C, U)_{i_1} = 1 = h(w)$. Otherwise, $i_1 = i_2$ and the definition of $\text{leap}(w)$ forces $j_2 > j_1$. Then $(i_2, j_2) \in U$, so by induction,

$$\overline{\text{wt}}(C, U)_{i_1} = \overline{\text{wt}}(C', U')_{i_1} + 1 \geq h(w') + 1 = h(w). \quad \square$$

LEMMA 12.6. Let C be a heavy climbing chain of $w \in S_n$. Then

$$\overline{\text{wt}}(C, M(C))_{i_1} \geq \overline{\text{wt}}(C^S(w), M(C^S(w)))_{i_1}.$$

Proof. By Lemma 12.5, it is enough to show that

$$\overline{\text{wt}}(C^S(w), C^S(w))_{i_1} = h(w).$$

The chain $C^S(w)$ is constructed iteratively by adding chunks of the form $\text{stair}(v)$ for $v \in S_n$. Within a chunk $\text{stair}(v)$, the first components are all the same and the second components are decreasing. The last element (a, b) in $\text{stair}(v)$ will exactly be the highest hook nested under the hook in row a of $D(v)$. The first link of the next chunk will either have first component $a' > a$, or first component a and second component $b' > b$. Thus, the links in $M(C^S(w))$ are exactly the first elements in each chunk $\text{stair}(v)$ used in constructing $C^S(w)$. The last links of the chunks are exactly the row indices of the hooks in $D(w)$ counted by $h(w)$. \square

CONJECTURE 12.7. Fix $w \in S_n$ and pick any term order satisfying $x_1 > x_2 > \dots > x_n$. Let $\xi = (C^S(w), M(C^S(w)))$. Then

$$\mathbf{x}^{\overline{\text{wt}}(\xi)} = \min \left\{ \mathbf{x}^{\overline{\text{wt}}(C, M(C))} \mid C \text{ is a heavy climbing chain of } w \right\}.$$

Equivalently,

$$\mathbf{x}^{\text{wt}(\xi)} = \max \left\{ \mathbf{x}^{\text{wt}(C, M(C))} \mid C \text{ is a heavy climbing chain of } w \right\}.$$

By Theorem 12.1, the leading monomial of \mathfrak{S}_w in any term order with $x_1 > x_2 > \dots > x_n$ is witnessed by $C^L(w)$. The following conjecture is an analogue of Conjecture 1.2/Theorem 9.6. We have tested it for all permutations in S_8 .

CONJECTURE 12.8. Fix $w \in S_n$ and any term order with $x_1 > x_2 > \dots > x_n$. For $\ell(w) < k \leq \text{raj}(w)$, let $m_k(\mathbf{x})$ be the leading monomial of the degree k homogeneous component of \mathfrak{S}_w . Then

$$m_k(\mathbf{x}) = x_p m_{k-1}(\mathbf{x})$$

where p is the smallest index such that $x_p m_{k-1}(\mathbf{x})$ divides $\mathbf{x}^{\text{wt}(C^S(w), M(C^S(w)))}$.

EXAMPLE 12.9. Mirroring Example 10.5, let $w = 5721463$. Then

$$\begin{aligned} \text{wt}(C^L(w), C^L(w)) &= (5, 4, 2, 1, 0, 0, 0), \\ \text{wt}(C^S(w), M(C^S(w))) &= (5, 5, 2, 1, 1, 1, 0). \end{aligned}$$

Then the conjectured leading monomials of the homogeneous components of \mathfrak{S}_w have exponents

$$\begin{aligned} (5, 4, 2, 1, 0, 0, 0), \\ (5, \mathbf{5}, 2, 1, 0, 0, 0), \\ (5, 5, 2, 1, \mathbf{1}, 0, 0), \\ (5, 5, 2, 1, 1, \mathbf{1}, 0). \end{aligned}$$

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REFERENCES

- [1] Nantel Bergeron and Sara Billey, *RC-graphs and Schubert polynomials*, Experiment. Math. **2** (1993), no. 4, 257–269.
- [2] Sara C. Billey, William Jockusch, and Richard P. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), no. 4, 345–374.
- [3] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.
- [4] Federico Castillo, Yairon Cid-Ruiz, Fatemeh Mohammadi, and Jonathan Montaña, *K-polynomials of multiplicity-free varieties*, 2022, <https://arxiv.org/abs/2212.13091>.
- [5] Sergey Fomin, Curtis Greene, Victor Reiner, and Mark Shimozono, *Balanced labellings and Schubert polynomials*, European J. Combin. **18** (1997), no. 4, 373–389.
- [6] Sergey Fomin and Anatol N. Kirillov, *The Yang–Baxter equation, symmetric functions, and Schubert polynomials*, Discrete Math. **153** (1996), no. 1-3, 123–143, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993).
- [7] Sergey Fomin and Richard P. Stanley, *Schubert polynomials and the nilCoxeter algebra*, Adv. in Math. **103** (1994), no. 2, 196–207.
- [8] Elena S. Hafner, *Vexillary Grothendieck polynomials via bumpless pipe dreams*, 2022, <https://arxiv.org/abs/2201.12432>.
- [9] June Huh, Jacob P. Matherne, Karola Mészáros, and Avery St. Dizier, *Logarithmic concavity of Schur and related polynomials*, Trans. Amer. Math. Soc. **375** (2022), no. 6, 4411–4427.

- [10] Allen Knutson, *Schubert polynomials, pipe dreams, equivariant classes, and a co-transition formula*, in Facets of algebraic geometry. Vol. II, London Math. Soc. Lecture Note Ser., vol. 473, Cambridge Univ. Press, Cambridge, 2022, pp. 63–83.
- [11] Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, Ann. of Math. (2) **161** (2005), no. 3, 1245–1318.
- [12] Thomas Lam, Seung Jin Lee, and Mark Shimozono, *Back stable Schubert calculus*, Compos. Math. **157** (2021), no. 5, 883–962.
- [13] Alain Lascoux and Marcel-Paul Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 13, 447–450.
- [14] ———, *Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 11, 629–633.
- [15] Cristian Lenart, *A unified approach to combinatorial formulas for Schubert polynomials*, J. Algebraic Combin. **20** (2004), no. 3, 263–299.
- [16] Cristian Lenart, Shawn Robinson, and Frank Sottile, *Grothendieck polynomials via permutation patterns and chains in the Bruhat order*, Amer. J. Math. **128** (2006), no. 4, 805–848.
- [17] Laurent Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS Texts and Monographs, vol. 6, American Mathematical Society; Société Mathématique de France, 2001, Translated from the 1998 French original by John R. Swallow, Cours Spécialisés, 3. [Specialized Courses].
- [18] Karola Mészáros, Linus Setiabrata, and Avery St. Dizier, *On the support of Grothendieck polynomials*, 2022, <https://arxiv.org/abs/2201.09452>.
- [19] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005.
- [20] Oliver Pechenik and Matthew Satriano, *Proof of a conjectured Möbius inversion formula for Grothendieck polynomials*, 2022, <https://arxiv.org/abs/2202.02897>.
- [21] Oliver Pechenik, David E. Speyer, and Anna Weigandt, *Regularity of matrix Schubert varieties*, Sém. Lothar. Combin. **86B** (2022), article no. 47 (12 pages).
- [22] Jenna Rajchgot, Yi Ren, Colleen Robichaux, Avery St. Dizier, and Anna Weigandt, *Degrees of symmetric Grothendieck polynomials and Castelnuovo-Mumford regularity*, Proc. Amer. Math. Soc. **149** (2021), no. 4, 1405–1416.
- [23] Jenna Rajchgot, Colleen Robichaux, and Anna Weigandt, *Castelnuovo-Mumford regularity of ladder determinantal varieties and patches of Grassmannian Schubert varieties*, J. Algebra **617** (2023), 160–191.
- [24] Anna Weigandt, *The Castelnuovo-Mumford regularity of matrix Schubert varieties*, Presented at Cascade Lectures in Combinatorics, 2021.
- [25] Anna Weigandt and Alexander Yong, *The prism tableau model for Schubert polynomials*, J. Combin. Theory Ser. A **154** (2018), 551–582.

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