




ALGEBRAIC COMBINATORICS

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Volume 7, issue 4 (2024), p. 1183-1195.

<https://doi.org/10.5802/alco.360>

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*Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing*
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486





Some restriction coefficients for the trivial and sign representations

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ABSTRACT We use character polynomials to obtain a positive combinatorial interpretation of the multiplicity of the sign representation in irreducible polynomial representations of $GL_n(\mathbb{C})$ indexed by two-column and hook partitions. Our method also yields a positive combinatorial interpretation for the multiplicity of the trivial representation of S_n in an irreducible polynomial representation indexed by a hook partition.

1. INTRODUCTION

The representation theory of symmetric groups and general linear groups lies at the heart of algebraic combinatorics. The irreducible polynomial representations of the general linear group $GL_n(\mathbb{C})$ are given by Weyl modules $W_\lambda(\mathbb{C}^n)$, for partitions λ with at most n parts, while the irreducible representations of the symmetric group S_n are given by Specht modules V_μ , for partitions μ of n .

Decomposing into irreducibles the restriction of $W_\lambda(\mathbb{C}^n)$ to the subgroup of permutation matrices, which is isomorphic to S_n , we have

$$\operatorname{Res}_{S_n}^{GL_n(\mathbb{C})} W_\lambda(\mathbb{C}^n) = \bigoplus_{\mu} V_\mu^{\oplus r_{\lambda\mu}},$$

where the multiplicities $r_{\lambda\mu}$ are called the *restriction coefficients*. A combinatorial description of these coefficients is a natural and, as yet, unresolved question, which we will henceforth call *the restriction problem*.

Littlewood [8] proved that (see also [18, Exercise 7.74])

$$(1) \quad s_\mu[1 + h_1 + h_2 + \cdots] = \sum_{\lambda} r_{\lambda\mu} s_\lambda,$$

where $s_\mu[1 + h_1 + h_2 + \cdots]$ is the plethystic substitution of the sum of the complete homogeneous symmetric functions h_i into the Schur function $s_\mu(x_1, x_2, \dots)$. Concretely, it is the substitution of the monomials occurring in $1 + h_1 + h_2 + \cdots$ into the variables x_1, x_2, \dots of s_μ . The reader is referred to [9, 10, 18] for a complete treatment of symmetric functions and plethysm. In [12], we interpreted Littlewood's identity in terms of an induction functor from representations of S_n to polynomial representations of $GL_n(\mathbb{C})$.

Manuscript received 5th December 2022, accepted 31st January 2024.

KEYWORDS. plethysm, character polynomials, moments, restriction problem, sign-reversing involution.

ACKNOWLEDGEMENTS. Shraddha Srivastava was supported by DST INSPIRE faculty fellowship DST/INSPIRE/04/2021/00026.

In recent years, many new approaches to the restriction problem have been developed. Assaf–Speyer [2] and Orellana–Zabrocki [15] independently defined the unique basis for the ring of symmetric functions such that the restriction coefficients appear in the linear expansion of Schur functions with respect to this basis. In [13], Orellana–Zabrocki–Saliola–Schilling related the restriction problem to the subalgebra of uniform block permutations within the partition algebra.

The spaces $\text{Sym}^k(\mathbb{C}^n) \otimes \wedge^l(\mathbb{C}^n)$ can be thought of as spaces of symmetric functions in *superspace* (see also, [1, 3, 4, 7]). These spaces were studied by Solomon [17] in a more general setting to prove a formula of Shephard and Todd. Orellana–Zabrocki [14] gave a combinatorial interpretation for the multiplicity of any Specht module in the restriction of $\text{Sym}(\mathbb{C}^n) \otimes \wedge(\mathbb{C}^n)$ to S_n .

The elements of $\text{Sym}^k(\mathbb{C}^n) \otimes \wedge^l(\mathbb{C}^n)$ can be thought of as differential l -forms on \mathbb{C}^n whose coefficients are polynomials of degree k . The hook Weyl module $W_{(k,1^l)}(\mathbb{C}^n)$ is the cokernel of the exterior derivative applied to $\text{Sym}^k(\mathbb{C}^n) \otimes \wedge^l(\mathbb{C}^n)$ (see [19, Ch. 2, Ex. 2]).

Heaton–Sriwongsa–Willenbring [6] proved that every irreducible representation of S_n occurs in at least one of a certain specified family of Weyl modules parameterized by two-row partitions.

Character polynomials encode the character values of certain families of S_n representations across n (see Section 2). The application of character polynomials to the restriction problem was suggested by Garsia and Goupil in [5]. In [11], we computed character polynomials for the family $(W_\lambda(\mathbb{C}^n))$ as S_n -representations across all n . Using these character polynomials, we interpreted restriction coefficients for the trivial and sign representations as signed sums of vector-partitions. This allowed us to derive necessary and sufficient conditions for the occurrence of the trivial representation of S_n in $W_\lambda(\mathbb{C}^n)$ when λ is a partition with two rows, or two columns, or is hook-shaped.

In this paper, we introduce the notion of signed moment of a character polynomial at each nonnegative integer n (see Definition 2.2). In Theorems 3.2, 3.8, 3.17 we find the generating functions from which we deduce the following main results of this paper. An expository account of some of these ideas can be found in [16].

THEOREM (A). *For $n \geq 2$, let $(k, \ell)'$ be the conjugate of the partition (k, ℓ) , where $0 < \ell \leq k \leq n$. Then the sign representation of S_n occurs in $W_{(k, \ell)'}(\mathbb{C}^n)$ if and only if $(k, \ell) \in \{(n-1, 0), (n, 0), (n-1, 1), (n, 1)\}$. In all these cases it occurs with multiplicity one.*

THEOREM (B). *For all $a, b \geq 0$ with $b+1 \leq n$, the multiplicity of the sign representation of S_n in $W_{(a+1, 1^b)}(\mathbb{C}^n)$ is the number of superpartitions (λ, μ) (see Definition 3.9) such that*

- (1) $\lambda = (\lambda_1 \geq \dots \geq \lambda_b \geq 0)$ and $\mu = (\mu_1 > \dots > \mu_{n-b} \geq 0)$,
- (2) $\lambda_1 + \dots + \lambda_b + \mu_1 + \dots + \mu_{n-b} = a + 1$, and $\mu_1 > \lambda_1$.

THEOREM (C). *For all $a, b \geq 0$ with $b+1 \leq n$, the multiplicity of trivial representation of S_n in $W_{(a+1, 1^b)}(\mathbb{C}^n)$ is the number of superpartitions (λ, μ) of partitions such that*

- (1) $\lambda = (\lambda_1 \geq \dots \geq \lambda_{n-b} \geq 0)$ and $\mu = (\mu_1 > \dots > \mu_b \geq 0)$,
- (2) $\lambda_1 + \dots + \lambda_{n-b} + \mu_1 + \dots + \mu_b = a + 1$ and $\mu_1 < \lambda_1 - 1$.

Owing to Littlewood’s formula (1), each of these results reveals a term in the expansion of $s_\nu[1 + h_1 + h_2 + \dots]$ in the basis of Schur functions, where ν is either of the partitions (1^n) or (n) (Corollaries 3.7, 3.16, and 3.20).

2. CHARACTER POLYNOMIALS AND THEIR MOMENTS

For $i \geq 1$, define the class function X_i on S_n as

$$X_i(\sigma) = \text{number of cycles of length } i \text{ in } \sigma.$$

A polynomial in the ring $\mathbb{C}[X_1, X_2, \dots]$ is called a *character polynomial*. For each $n \geq 1$, let V_n be a representation of S_n . The sequence (V_n) is *eventually polynomial* if there exists a polynomial $p \in \mathbb{C}[X_1, X_2, \dots]$ and an integer $N \geq 0$ such that, for each $n \geq N$ and each $\sigma \in S_n$

$$p(X_1(\sigma), X_2(\sigma), \dots) = \chi_{V_n}(\sigma),$$

where χ_{V_n} is the character of the representation V_n .

The polynomial p is unique since if $f(X_1(\sigma), X_2(\sigma), \dots) = 0$ for a polynomial f and for all permutations σ , then it is zero for all sequences of nonnegative integer values, and must thus be identically zero. The polynomial p is the character polynomial associated to the sequence (V_n) .

EXAMPLE 2.1. Let $V_n = \text{Sym}^2(\mathbb{C}^n)$. $\text{Sym}^2(\mathbb{C}^n)$ has a basis indexed by multisets (collections of possibly repeated elements) of size 2 drawn from $[n] = \{1, \dots, n\}$. We calculate the character polynomial of (V_n) by counting the number of fixed points of the action of an arbitrary permutation $\sigma \in S_n$ on this basis.

A permutation σ acts on a multiset $\{i, j\}$ as

$$\sigma \cdot \{i, j\} = \{\sigma(i), \sigma(j)\}.$$

A multiset is fixed by σ if $\{i, j\} = \{\sigma(i), \sigma(j)\}$. Thus either $i = \sigma(i)$ and $j = \sigma(j)$, or $i = \sigma(j)$ and $j = \sigma(i)$. There are $X_1(\sigma) + \binom{X_1(\sigma)}{2}$ ways of picking i, j from the $X_1(\sigma)$ fixed points of σ for the former case and $X_2(\sigma)$ ways of picking i, j to be elements of a 2-cycle of σ for the latter. Thus

$$p = X_1 + \binom{X_1}{2} + X_2.$$

DEFINITION 2.2. Given a polynomial $p \in \mathbb{C}[X_1, X_2, \dots]$, the moment of p is defined to be

$$\langle p \rangle_n = \frac{1}{n!} \sum_{\sigma \in S_n} p(X_1(\sigma), X_2(\sigma), \dots).$$

The signed moment of p is defined to be

$$\{p\}_n = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) p(X_1(\sigma), X_2(\sigma), \dots),$$

where $\text{sgn}(\sigma)$ is the value of the sign character on σ .

REMARK 2.3. Let p be the character polynomial of the eventually polynomial sequence of representations (V_n) . Clearly $\langle p \rangle_n$ and $\{p\}_n$ are respectively the multiplicities of the trivial and sign representation of S_n in V_n . Moments were introduced and their generating functions were studied in [11]. Here we extend that theory to signed moments.

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0)$, the *length* of the partition $l(\lambda) = \ell$ is the number of nonzero parts of the partition. Let a_i be the number of parts of λ of size i . Then λ is expressed in the *exponential notation* as

$$\lambda = 1^{a_1} 2^{a_2} \dots$$

Define

$$z_\lambda = \frac{|\lambda|!}{\prod_i i^{a_i} a_i!},$$

the cardinality of the centralizer of a permutation of cycle-type λ . Here $|\lambda| = \sum_i ia_i$ is the size of the partition λ . Let Par denote the set of all partitions and let Par_n denote the subset of partitions of n for any nonnegative integer n .

DEFINITION 2.4. For a partition $\alpha = 1^{a_1}2^{a_2}\dots$, define the following elements in $\mathbb{C}[X_1, X_2, \dots]$:

$$\binom{X}{\alpha} = \prod_i \binom{X_i}{a_i}$$

and

$$\left(\binom{X}{\alpha}\right) = \prod_i \left(\binom{X_i}{a_i}\right),$$

where $\left(\binom{X_i}{a_i}\right) := \binom{X_i+a_i-1}{a_i}$.

The sets $\{\binom{X}{\alpha} \mid \alpha \in \text{Par}\}$ and $\{\left(\binom{X}{\alpha}\right) \mid \alpha \in \text{Par}\}$ are bases of $\mathbb{C}[X_1, X_2, \dots]$. The former is called the *binomial basis* of $\mathbb{C}[X_1, X_2, \dots]$.

THEOREM 2.5. For a partition α we have

$$\left\{ \binom{X}{\alpha} \right\}_n = \begin{cases} \frac{\text{sgn}(\alpha)}{z_\alpha} & \text{if } n \in \{|\alpha|, |\alpha| + 1\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\text{sgn}(\alpha)$ is the value of the sign representation on a permutation of type α .

Proof. The signed moment $\left\{ \binom{X}{\alpha} \right\} = \frac{1}{n!} \sum_{w \in S_n} \text{sgn}(w) \binom{\beta(w)}{\alpha}$, where $\beta(w)$ is the cycle-type of the permutation w . Gathering together permutations in each conjugacy class we have

$$\begin{aligned} \sum_{n \geq 0} \left\{ \binom{X}{\alpha} \right\}_n v^n &= \sum_{n \geq 0} \sum_{\beta \vdash n = 1^{b_1} 2^{b_2} \dots} \frac{1}{z_\beta} (-1)^{b_2+b_4+\dots} \binom{\beta}{\alpha} v^n \\ &= \sum_{b_i \geq a_i} \prod_i \frac{(-1)^{(i-1)b_i}}{i^{b_i} b_i!} \frac{b_i!}{a_i! (b_i - a_i)!} v^{ib_i} \\ &= \prod_i \frac{(-1)^{(i-1)a_i} v^{ia_i}}{i^{a_i} a_i!} \sum_{c_i \geq 0} \frac{(-1)^{(i-1)c_i} v^{ic_i}}{i^{c_i} c_i!} \\ &= \frac{\text{sgn}(\alpha) v^{|\alpha|}}{z_\alpha} \sum_{n \geq 0} \frac{v^n}{n!} \sum_{w \in S_n} \text{sgn}(w) \\ &= \frac{\text{sgn}(\alpha) v^{|\alpha|}}{z_\alpha} (1 + v), \end{aligned}$$

where the final equality is due to the signed sum of permutations being nonzero precisely when $n = 0, 1$.

The coefficient of v^n is thus $\frac{\text{sgn}(\alpha)}{z_\alpha}$ if $n = |\alpha|$ or $n = |\alpha| + 1$, and is 0 otherwise. \square

Theorem 2.5 allows us thus to make general conclusions about the presence of the sign representation.

COROLLARY 2.6. Let (V_n) be an eventually polynomial sequence of representations of S_n . Then the sign representation does not occur in V_n for sufficiently large n . In particular, the sign representation does not occur in the restriction of $W_\lambda(\mathbb{C}^n)$ to S_n for any $n > |\lambda| + 1$.

Proof. The first statement is a straightforward consequence of Theorem 2.5. From [11, Section 2.3], we know that the expansion of the character polynomial of $W_\lambda(\mathbb{C}^n)$ in the binomial basis involves partitions at most as large as $|\lambda|$. \square

3. COMBINATORIAL INTERPRETATIONS FOR SOME RESTRICTION COEFFICIENTS

This section contains the main results of this paper. Here we give combinatorial interpretations for the multiplicity of the sign representation of S_n in $W_\lambda(\mathbb{C}^n)$ when λ is a two-column or hook partition. The proof of the latter result is easily modified to yield the multiplicity of the trivial representation of S_n when λ is a hook partition. We recall the following character polynomials from [11].

THEOREM 3.1. *Let $\text{Sym}^k(\mathbb{C}^n)$ and $\wedge^\ell(\mathbb{C}^n)$ be the k -th symmetric and ℓ -th exterior powers of \mathbb{C}^n respectively. Then*

- (1) *The sequence of representations $(\text{Sym}^k(\mathbb{C}^n))$ is eventually polynomial, and the character polynomial is*

$$H_k = \sum_{\alpha \vdash k} \binom{X}{\alpha}.$$

- (2) *The sequence of representations $(\wedge^\ell(\mathbb{C}^n))$ is eventually polynomial, and the character polynomial is*

$$E_\ell = \sum_{\alpha \vdash k} (-1)^{|\alpha| - l(\alpha)} \binom{X}{\alpha}.$$

3.1. MULTIPLICITY OF SIGN REPRESENTATION WHEN λ HAS TWO COLUMNS. Let $(k, \ell)'$ be the conjugate of the partition (k, ℓ) , where $0 < \ell \leq k \leq n$. The character polynomial associated to $(W_{(k, \ell)'})$ is defined using the dual Jacobi-Trudi identity (see [18, Corollary 7.16.2])

$$S_{(k, \ell)'} = E_k E_\ell - E_{k+1} E_{\ell-1}.$$

The signed moment of $S_{(k, \ell)'}$ is thus the difference of signed moments, for which we have the following generating functions.

THEOREM 3.2. *For integers $k, \ell \geq 0$, the signed moment of $E_k E_\ell$ is*

$$\sum_{k, \ell, n} \{E_k E_\ell\}_n u^k v^\ell z^n = \frac{(1+z)(1+uvz)}{(1-uz)(1-vz)}.$$

Proof. By Theorem 3.1, the signed moment $\{E_k E_\ell\}_n$ is

$$\{E_k E_\ell\}_n = \sum_{\alpha \vdash n} \frac{1}{z_\alpha} (-1)^{|\alpha| - l(\alpha)} \sum_{\beta \vdash k, \gamma \vdash \ell} (-1)^{|\beta| - l(\beta)} (-1)^{|\gamma| - l(\gamma)} \binom{\alpha}{\beta} \binom{\alpha}{\gamma}.$$

Which is the coefficient of $[u^k v^\ell z^n]$ in

$$\prod_{i \geq 1} \sum_{a_i, b_i, c_i \geq 0} \frac{1}{i^{a_i} a_i!} (-1)^{(i-1)a_i + (i-1)b_i + (i-1)c_i} \binom{a_i}{b_i} \binom{a_i}{c_i} u^{ib_i} v^{ic_i} z^{ia_i}.$$

The inner sum is $\exp\left(\frac{-(-z)^i(1-(-u)^i)(1-(-v)^i)}{i}\right)$. Thus

$$\begin{aligned} \{E_k E_\ell\}_n &= [u^k v^\ell z^n] \prod_{i \geq 1} \exp\left(\frac{-(-z)^i(1-(-u)^i)(1-(-v)^i)}{i}\right) \\ &= [u^k v^\ell z^n] \exp\left(\sum_{i \geq 1} \frac{(uz)^i + (vz)^i}{i}\right) \exp\left(\sum_{i \geq 1} \frac{-(-z)^i - (-uvz)^i}{i}\right) \\ &= [u^k v^\ell z^n] \frac{(1+z)(1+uvz)}{(1-uz)(1-vz)}, \end{aligned}$$

as required. □

The following corollary of Theorem 3.2 expresses the signed moment of $S_{(k,\ell)}$ as the difference of the cardinality of two sets.

DEFINITION 3.3. Fix a vector $\mathbf{v} = (a_1, \dots, a_n)$ with nonnegative integer coordinates. A vector-partition of \mathbf{v} is a set $\{\mathbf{v}_1, \dots, \mathbf{v}_l\}$ of non-zero vectors with nonnegative integer coordinates such that $\sum_{i=1}^l \mathbf{v}_i = \mathbf{v}$.

COROLLARY 3.4. Let $q_n^*(k, \ell)$ denote the number of vector-partitions of (k, ℓ) into n parts, where each part is in $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$, and where $(0, 0)$ and $(1, 1)$ occur at most once, while $(1, 0)$ and $(0, 1)$ may occur multiple times. Then the signed moment of $S_{(k,\ell)}$ for $k \geq \ell$ is given by

$$\{S_{(k,\ell)}\}_n = \begin{cases} q_n^*(k, \ell) - q_n^*(k + 1, \ell - 1) & \text{if } \ell > 0, \\ q_n^*(k, \ell) & \text{otherwise.} \end{cases}$$

LEMMA 3.5. For integers $k \geq \ell$ we have

$$q_n^*(k, \ell) = \begin{cases} 1 & \text{if } \ell > 0 \text{ and } k + \ell \in \{n - 1, n + 1\}, \\ 2 & \text{if } \ell > 0 \text{ and } k + \ell = n, \\ 1 & \text{if } \ell = 0 \text{ and } k \in \{n - 1, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\ell > 0$, and $k + \ell = n - 1$, then the only vector-partition is

$$(k, \ell) = (0, 0) + k(1, 0) + \ell(0, 1)$$

If $\ell > 0$ and $k + \ell = n$, then there are two vector-partitions

$$\begin{aligned} (k, \ell) &= (0, 0) + (1, 1) + (k - 1)(1, 0) + (\ell - 1)(0, 1) \\ &= k(1, 0) + \ell(0, 1). \end{aligned}$$

If $\ell > 0$ and $k + \ell = n + 1$, then the only vector-partition is

$$(k, \ell) = (1, 1) + (k - 1)(1, 0) + (\ell - 1)(0, 1).$$

If $\ell = 0$, and $k = n - 1$ then the only vector-partition is

$$(k, 0) = (0, 0) + k(1, 0).$$

If $\ell = 0$ and $k = n$, then the only vector-partition is

$$(k, 0) = k(1, 0),$$

completing the proof. □

THEOREM 3.6. For each $n \geq 2$, the sign representation of S_n occurs in $W_{(k,\ell)}(\mathbb{C}^n)$ if and only if $(k, \ell) \in \{(n - 1, 0), (n, 0), (n - 1, 1), (n, 1)\}$. In all these cases it occurs with multiplicity one.

Proof. From Corollary 3.4, the multiplicity of the sign representation in $W_{(k,\ell)}(\mathbb{C}^n)$ is $q_n^*(k, \ell) - q_n^*(k + 1, \ell - 1)$.

If $\ell = 0$, the second term is 0. The first term, $q_n^*(k, 0)$ is 1 if $k \in \{n - 1, n\}$ and 0 otherwise, since $(0, 0)$ may occur at most once, and the other parts must all be $(1, 0)$.

For $\ell \geq 2$, each vector-partition of $(k + 1, \ell - 1)$ that contributes to $q_n^*(k + 1, \ell - 1)$ must contain at least one $(1, 0)$ part. The map replacing one $(1, 0)$ by $(0, 1)$ is a bijection from vector-partitions of $(k + 1, \ell - 1)$ and (k, ℓ) . Thus in this case the multiplicity is zero.

When $\ell = 1$, both $q_n^*(k, \ell)$ and $q_n^*(k + 1, \ell - 1)$ are zero unless either $k = n$ or $k = n - 1$. When $k = n - 1$, we have $q_n^*(k, \ell) = 2$ and $q_n^*(k + 1, \ell - 1) = 1$ from Lemma 3.5. When $k = n$, Lemma 3.5 again verifies that the multiplicity of the sign representation is 1. □

By Littlewood's formula (Equation (1)) we have the following corollary.

COROLLARY 3.7. *For a partition $\lambda = (k, \ell)'$, the coefficient of s_λ in the expansion of the plethysm $s_{(1^n)}[1 + h_1 + h_2 + \dots]$ is*

$$r_{\lambda\mu} = \begin{cases} 1 & (k, \ell) \in \{(n-1, 0), (n, 0), (n-1, 1), (n, 1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

3.2. MULTIPLICITY OF THE SIGN REPRESENTATION IN THE RESTRICTION OF HOOKS. For all $a, b \geq 0$, let $\lambda = (a|b)$ denote the hook partition $(a+1, 1^b)$. We now consider the multiplicity of the sign representation of S_n in $W_\lambda(\mathbb{C}^n)$ for $n \geq b+1$. The Pieri identity (see [18, Theorem 7.15.7]) expresses the Schur function $s_{(a|b)}$ as the alternating sum

$$s_{(a|b)} = \sum_{j=0}^b (-1)^j h_{a+1+j} e_{b-j}.$$

This relation also holds for the corresponding character polynomials, so that

$$(2) \quad S_{(a|b)} = \sum_{j=0}^b (-1)^j H_{a+1+j} E_{b-j}.$$

Thus the signed moment of $S_{(a|b)}$ is an alternating sum of signed moments, for which we have the following generating function.

THEOREM 3.8. *For $k, \ell \geq 0$, the signed moment of $H_k E_\ell$ is*

$$(3) \quad \sum_{k, \ell, n} \{H_k E_\ell\}_n u^k v^\ell z^n = \frac{\prod_{j \geq 0} (1 + u^j z)}{\prod_{j \geq 0} (1 - u^j v z)}.$$

Proof. From Theorem 3.1 and the definition of the signed moment,

$$\{H_k E_\ell\}_n = \sum_{\alpha \vdash n} \frac{1}{z_\alpha} (-1)^{|\alpha| - l(\alpha)} \sum_{\beta \vdash k, \gamma \vdash \ell} (-1)^{|\gamma| - l(\gamma)} \binom{\alpha}{\beta} \binom{\alpha}{\gamma}.$$

This is the coefficient of $u^k v^\ell z^n$ in $\prod_{i \geq 1} \exp\left(\frac{-(-z)^i (1 - (-v)^i)}{1 - u^i}\right)$. Recalling that $\log x = \sum_{i \geq 1} \frac{x^i}{i}$ gives us the desired expression. \square

A combinatorial interpretation of the signed moment $\{H_{a+1+j} E_{b-j}\}_n$ is given in terms of superpartitions.

DEFINITION 3.9. *A superpartition is a pair (λ, μ) where λ is a nonincreasing sequence of nonnegative integers and μ is a nonincreasing sequence of distinct nonnegative integers. We define $|\lambda|$ and $|\mu|$ to be the sum of the parts of λ and μ respectively.*

REMARK 3.10. Superpartitions play a role in the theory of symmetric functions in superspace. See [1, 3, 4, 7].

PROPOSITION 3.11. *For all $a, b \geq 0$ and $j \leq b$, $\{H_{a+1+j} E_{b-j}\}_n$ is the number of superpartitions (λ, μ) such that*

- (1) $\lambda = (\lambda_1, \dots, \lambda_{b-j})$, where $\lambda_1 \geq \dots \geq \lambda_{b-j} \geq 0$,
- (2) $\mu = (\mu_1, \dots, \mu_{n-b+j})$, with $\mu_1 > \dots > \mu_{n-b+j} \geq 0$,
- (3) $|\lambda| + |\mu| = a + 1 + j$.

Proof. The terms in the denominator of the right hand side of (3) contribute to the sequence λ . There are $b-j$ of these terms (since terms with v in them come only from the denominator and there must be $b-j$ such terms). The terms from the numerator contribute to the sequence μ . Each of these terms contain a distinct power of u , and there are $n - b + j$ of them. \square

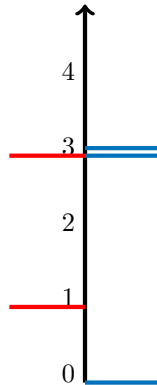


FIGURE 1. The sequence $\lambda = (3, 3, 0)$ and $\mu = (3, 1)$.

We will think of each sequence (λ, μ) marked on a single axis, with the parts of λ marked in blue and the parts of μ marked in red, as in Figure 1. We define a sign-reversing involution on the alternating sum in (2) to obtain a positive combinatorial formula for the multiplicity of the sign representation in $W_{(a|b)}(\mathbb{C}^n)$.

THEOREM 3.12. *For all $a, b \geq 0$, then $\{S_{(a|b)}\}_n$ is the number of superpartitions (λ, μ) such that*

- (1) $\lambda = (\lambda_1, \dots, \lambda_b)$, where $\lambda_1 \geq \dots \geq \lambda_b \geq 0$,
- (2) $\mu = (\mu_1, \dots, \mu_{n-b})$, with $\mu_1 > \dots > \mu_{n-b} \geq 0$,
- (3) $|\lambda| + |\mu| = a + 1$,
- (4) $\mu_1 > \lambda_1$.

Equivalently,

$$\{S_{(a|b)}\}_n = \sum_{\rho \in P(a,n)} \binom{r_\rho}{n-b-1},$$

where $P(a, n)$ denotes the set of partitions of $a + n$ with n parts, and for a partition $\rho \in P(a, n)$, r_ρ is the number of removable cells of ρ (i.e. cells whose removal from the Young diagram of ρ leaves behind a valid partition) that are not in its first row.

Proof. Let $\mathcal{C}_n(a, b)$ denote the set of superpartitions (λ, μ) , where

- (1) $\lambda = (\lambda_1, \dots, \lambda_b)$, where $\lambda_1 \geq \dots \geq \lambda_b \geq 0$,
- (2) $\mu = (\mu_1, \dots, \mu_{n-b})$, where $\mu_1 > \dots > \mu_{n-b} \geq 0$,
- (3) $|\lambda| + |\mu| = a$.

Then evidently, $|\mathcal{C}_n(a, b)|$ is the coefficient of $u^a v^b z^n$ in $\frac{\prod_{j \geq 0} (1 + u^j z)}{\prod_{j \geq 0} (1 - u^j v z)}$, and hence, by Theorem 3.12 $\{H_a E_b\}_n$. Note that

$$(4) \quad \{S_{(a|b)}\}_n = \sum_{i=0}^b (-1)^i \{H_{a+1+i} E_{b-i}\}_n = \sum_{i=0}^b (-1)^i |\mathcal{C}_n(a+1+i, b-i)|.$$

Partition $\mathcal{C}_n(a, b)$ into two parts: $\mathcal{B}_n(a, b)$, which consists of superpartitions $(\lambda, \mu) \in \mathcal{C}_n(a, b)$ such that the largest among the parts of λ and μ occurs in λ , and $\mathcal{R}_n(a, b)$, which consists of superpartitions $(\lambda, \mu) \in \mathcal{C}_n(a, b)$ such that the largest among the parts of λ and μ occurs only in μ . Observe that

$$\mathcal{C}_n(a, b) = \mathcal{B}_n(a, b) \sqcup \mathcal{R}_n(a, b).$$



FIGURE 2. A demonstration of the involution between the monomials $(u^1z)(u^3z)(u^0vz)(u^3vz)^2$ on the left and $(u^1z)(u^3z)(u^4z)(u^0vz)(u^3vz)$ on the right

Define a bijection

$$\omega : \mathcal{B}_n(a, b) \rightarrow \mathcal{R}_n(a + 1, b - 1)$$

by setting $\omega(\lambda, \mu)$ to be the superpartition obtained by moving a largest part of λ to a μ after increasing it by 1. Then $\omega^{-1}(\lambda, \mu)$ is the superpartition obtained by moving the largest part of μ to λ after reducing it by 1 (see Figure 2). We may rewrite (4) as

$$\begin{aligned} \{S_{(a|b)}\}_n &= +|\mathcal{B}_n(a + 1, b)| + |\mathcal{R}_n(a + 1, b)| \\ &\quad - |\mathcal{B}_n(a + 2, b - 1)| - |\mathcal{R}_n(a + 2, b - 1)| \\ &\quad + |\mathcal{B}_n(a + 3, b - 1)| - |\mathcal{R}_n(a + 3, b - 1)| \\ &\quad \vdots \\ &\quad (-1)^b |\mathcal{B}_n(a + b + 1, 0)| + (-1)^b |\mathcal{R}_n(a + b + 1, 0)|. \end{aligned}$$

By virtue of the bijection ω , we can cancel out the blue term in each row of the above expression with the red term in the row below it. Noting further, that $\mathcal{B}_n(a + b + 1, 0) = 0$, we have

$$\{S_{(a|b)}\}_n = |\mathcal{R}_n(a + 1, b)|,$$

which is precisely the number claimed in the first part of the theorem.

Given $(\lambda, \mu) \in \mathcal{R}_n(a + 1, b)$, form a partition ρ by subtracting one from the largest part of the partition $\lambda \cup \mu$ formed by arranging the parts of λ and μ together in weakly decreasing order. Also, the parts of μ , excepting the first, form a set of $n - b - 1$ distinct parts of ρ , including a possible 0-part. Thus they correspond to a choice of $n - b - 1$ removable cells of ρ that do not lie in its first row, except when ρ has a part equal to 0, in which case we can also choose a zero part of μ , giving the expression in the second part of the statement of the theorem. \square

EXAMPLE 3.13. To compute $\{S_{(4,1,1)}\}_5$ we take $a = 3$, $b = 2$, and $n = 5$. According to Theorem 3.12, we must enumerate integers

$$\lambda_1 \geq \lambda_2 \geq 0, \quad \mu_1 > \mu_2 > \mu_3 \geq 0, \quad \mu_1 > \lambda_1$$

such that

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 = 4.$$

The only possibilities for (λ, μ) are $((1, 0), (2, 1, 0))$ and $((0, 0), (3, 1, 0))$. Therefore $\{S_{(4,1,1)}\}_5 = 2$.

In the second formulation, the sum is over partitions ρ of 3 with 5 non-negative parts. For $\rho = (3, 0, 0, 0, 0)$, $r_\rho = 1$; for $\rho = (2, 1, 0, 0, 0)$, $r_\rho = 2$, and for $\rho = (1, 1, 1, 0, 0, 0)$, $r_\rho = 2$. On the other hand $n - b - 1 = 2$. Thus we get

$$\{S_{(4,1,1)}\}_5 = \binom{1}{2} + \binom{2}{2} + \binom{2}{2} = 2.$$

EXAMPLE 3.14. To compute $\{S_{(3,1)}\}_3$, take $a = 2$, $b = 1$, and $n = 3$. We must count the number of possibilities for

$$\lambda_1 \geq 0, \quad \mu_1 > \mu_2 \geq 0, \quad \mu_1 > \lambda_1, \quad \lambda_1 + \mu_1 + \mu_2 = 3.$$

The possibilities for (λ, μ) are $((0), (2, 1))$, $((0), (3, 0))$, $((1), (2, 0))$, so that $\{S_{(3,1)}\}_3 = 3$.

In the second formulation, $P(2, 3) = (2, 0, 0), (1, 1, 0)$. We have $r_{(2,0,0)} = 1$, while $r_{(1,1,0)} = 2$, and $n - b - 1 = 1$, so

$$\{S_{(3,1)}\}_3 = \binom{1}{1} + \binom{2}{1} = 3.$$

COROLLARY 3.15. For all $a, b \geq 0$,

$$\{S_{(a|b)}\}_n > 0 \text{ if and only if } b < n \text{ and } \binom{n-b}{2} \leq a + 1.$$

Proof. Suppose $b < n$, $\binom{n-b}{2} \leq a + 1$, take $\lambda_i = 0$ for all $i = 1, \dots, b$, and take $\mu_1 = a + 1 - \binom{n-b-1}{2}$, and $\mu_i = n - b - i$ for $1 < i \leq n - b$. This satisfies the conditions (1) through (4) of Theorem 3.12, so that $\{S_{(a|b)}\}_n > 0$.

Conversely, if $\lambda_1, \dots, \lambda_b, \mu_1, \dots, \mu_{n-b}$ exist satisfying the conditions (1) to (4), then surely $b < n$, and also $\mu_1 + \dots + \mu_{n-b} \leq a + 1$. Since $\mu_1 > \dots > \mu_{n-b} \geq 0$, $\mu_1 + \dots + \mu_{n-b} \geq \binom{n-b}{2}$. It follows that $\binom{n-b}{2} \leq a + 1$. \square

By Littlewood’s formula (Equation (1)) we have the following corollary.

COROLLARY 3.16. For all $a, b \geq 0$ with $b + 1 \leq n$, the coefficient of $s_{(a|b)}$ in the expansion of the plethysm $s_{(1^n)}[1 + h_1 + h_2 + \dots]$ is the number of superpartitions of partitions (λ, μ) that satisfy the conditions of Theorem 3.12.

3.3. MULTIPLICITY OF THE TRIVIAL REPRESENTATION IN THE RESTRICTION OF HOOKS. By the Pieri formula (2) the moment of $S_{(a|b)}$ is an alternating sum of the moments of the form $H_r E_s$. The moment of $H_r E_s$ is a special case of Corollary 4.2 of [11], which we reproduce here.

THEOREM 3.17. For integers $k, \ell \geq 0$, the moment of $H_k E_\ell$ is

$$\sum_{k, \ell, n} \langle H_k E_\ell \rangle_n u^k v^\ell z^n = \frac{\prod_{j \geq 0} (1 + u^j v z)}{\prod_{j \geq 0} (1 - u^j z)}.$$

Proof. The moment $\langle H_k E_\ell \rangle_n$ is the coefficient of $u^k v^\ell z^n$ in $\prod_{i \geq 1} \exp\left(\frac{z^i (1 - (-v)^i)}{1 - u^i}\right)$. We obtain the generating function from simplifying this exponent as in the proof of Theorem 3.8. \square

THEOREM 3.18. For all $a, b \geq 0$, then $\langle S_{(a|b)} \rangle_n$ is the number of superpartitions (λ, μ) of partitions such that

- (1) $\lambda = (\lambda_1, \dots, \lambda_{n-b})$, where $\lambda_1 \geq \dots \geq \lambda_{n-b} \geq 0$,
- (2) $\mu = (\mu_1, \dots, \mu_b)$, with $\mu_1 > \dots > \mu_b \geq 0$,



FIGURE 3. A demonstration of the involution between the monomials $(v^1z)(v^3z)(uv^0z)(uv^3z)^2$ on the left and $(v^1z)(uv^0z)(uv^3z)(v^3z)(v^4z)$ on the right.

- (3) $|\lambda| + |\mu| = a + 1,$
- (4) $\mu_1 < \lambda_1 - 1.$

Equivalently,

$$\langle S_{(a|b)} \rangle_n = \sum_{\rho \in P(a+1, n)} \binom{r_\rho}{b},$$

where $P(a + 1, n)$ denotes the set of partitions of $a + 1 + n$ with n non-negative parts, and for a partition $\rho \in P(a + 1, n)$, r_ρ is the number of removable cells of ρ that are in a row ρ_i with $\rho_1 - \rho_i \geq 2$.

Proof. Let $\mathcal{B}_n(a, b)$ denote the set of superpartitions (λ, μ) , where

- (1) $\lambda = (\lambda_1, \dots, \lambda_{n-b})$, where $\lambda_1 \geq \dots \geq \lambda_{n-b} \geq 0,$
- (2) $\mu = (\mu_1, \dots, \mu_b)$, where $\mu_1 > \dots > \mu_b \geq 0,$
- (3) $|\lambda| + |\mu| = a.$

Then evidently, $|\mathcal{B}_n(a, b)|$ is the coefficient of $u^a v^b z^n$ in $\frac{\prod_{j \geq 0} (1 + u^j v z)}{\prod_{j \geq 0} (1 - u^j z)}$, and hence, by Theorem 3.12, equals $\langle H_a E_b \rangle_n$. Note that

$$(5) \quad \langle S_{(a|b)} \rangle_n = \sum_{i=0}^b (-1)^i \langle H_{a+1+i} E_{b-i} \rangle_n = \sum_{i=0}^b (-1)^i |\mathcal{B}_n(a + 1 + i, b - i)|.$$

The proof proceeds along the lines of that to Theorem 3.12. Observe that the roles of λ and μ are reversed in this case. Recall that the earlier involution proceeded as in Figure 2 by either increasing a largest blue part (i.e. a part in λ) by one and changing its colour to red (i.e. appending this part to μ) or reducing a largest red part (i.e. a part in μ) and changing its colour to blue (i.e. appending it to λ). In this case, the involution attempts to do the reverse, as illustrated in Figure 3.

This presents an obstacle however, when the largest red part (i.e. a part in μ) is one less than the largest blue part (i.e. a part in λ). In this case, the involution attempts to push down a blue part and change its colour to red, but the resulting superpartition is not a valid one since the parts of μ must be distinct. Thus in this case, we prescribe that the involution ignore the largest part of the superpartition and proceed to the next largest part, as demonstrated in Figure 4.



FIGURE 4. The involution when the largest blue part cannot be pushed down, since a red part already occupies the level below it. Instead, the largest red part is raised up to a blue part.

Partition $\mathcal{B}_n(a, b)$ into two parts: $\mathcal{L}_n(a, b)$, which consists of superpartitions $(\lambda, \mu) \in \mathcal{B}_n(a, b)$ such that the largest among the parts of λ and μ occurs in λ and the largest part in μ is at least 2 less than the largest part in λ and $\mathcal{U}_n(a, b)$, which consists of all other superpartitions $(\lambda, \mu) \in \mathcal{B}_n(a, b)$. Observe that

$$\mathcal{B}_n(a, b) = \mathcal{L}_n(a, b) \sqcup \mathcal{U}_n(a, b).$$

Then the multiplicity of the trivial representation of S_n in $S_{(a|b)}$ is given by

$$\begin{aligned} \langle S_{(a|b)} \rangle_n &= + |\mathcal{L}_n(a + 1, b)| + |\mathcal{U}_n(a + 1, b)| \\ &\quad - |\mathcal{L}_n(a + 2, b - 1)| - |\mathcal{U}_n(a + 2, b - 1)| \\ &\quad + |\mathcal{L}_n(a + 3, b - 1)| - |\mathcal{U}_n(a + 3, b - 1)| \\ &\quad \vdots \\ &\quad (-1)^b |\mathcal{L}_n(a + b + 1, 0)| + (-1)^b |\mathcal{U}_n(a + b + 1, 0)|. \end{aligned}$$

By virtue of the involution, we can cancel out the blue term in each row of the above expression with the red term in the row above it. Noting further, that $\mathcal{U}_n(a + b + 1, 0) = 0$, we have

$$\langle S_{(a|b)} \rangle_n = |\mathcal{L}_n(a + 1, b)|,$$

which is precisely the number claimed in the first part of the theorem.

The cardinality of $\mathcal{L}_n(a + 1, b)$ is the same as that of $\mathcal{U}_n(a, b + 1)$. Given $(\lambda, \mu) \in \mathcal{U}_n(a, b + 1)$, form a partition ρ by arranging the parts of λ and μ together in weakly decreasing order. Also, the parts of μ , form a set of $b + 1$ distinct parts of ρ , including a possible 0-part. Thus they correspond to a choice of $b + 1$ removable cells of ρ , including either the first largest part of ρ , or excluding it but including then the second largest part of ρ (in this case though the largest part is blue, it cannot be pushed down due to a red part already occupying the lower position). \square

This interpretation allows us to easily recover Theorem 4.8.3 of [6]

COROLLARY 3.19. For all $a, b \geq 0$ with $b + 1 \leq n$,

$$\langle S_{(a|b)} \rangle_n > 0 \text{ if and only if } b < n \text{ and } \binom{b + 1}{2} \leq a.$$

COROLLARY 3.20. For all $a, b \geq 0$ with $b + 1 \leq n$, the coefficient of $s_{(a|b)}$ in the expansion of the plethysm $s_{(n)}[1 + h_1 + h_2 + \dots]$ is the number of superpartitions (λ, μ) that satisfy the conditions of Theorem 3.18.

Acknowledgements. We thank the anonymous referees for several helpful suggestions which have helped improve the final version of this paper.

REFERENCES

- [1] L. Alarie-Vezina, O. Blondeau-Fournier, P. Desrosiers, L. Lapointe, and P. Mathieu, *Symmetric functions in superspace: a compendium of results and open problems (including a SageMath worksheet)*, 2019, <https://arxiv.org/abs/1903.07777>.
- [2] Sami H. Assaf and David E. Speyer, *Specht modules decompose as alternating sums of restrictions of Schur modules*, Proc. Amer. Math. Soc. **148** (2020), no. 3, 1015–1029.
- [3] P. Desrosiers, L. Lapointe, and P. Mathieu, *Classical symmetric functions in superspace*, J. Algebraic Combin. **24** (2006), no. 2, 209–238.
- [4] S. Fishel, L. Lapointe, and Maria E. Pinto, *Hopf algebra structure of symmetric and quasisymmetric functions in superspace*, J. Combin. Theory Ser. A **166** (2019), 144–170.
- [5] A. M. Garsia and A. Goupil, *Character polynomials, their q -analogs and the Kronecker product*, Electron. J. Combin. **16** (2009), no. 2, article no. 19 (40 pages).
- [6] A. Heaton, S. Sriwongsa, and J. F. Willenbring, *Branching from the general linear group to the symmetric group and the principal embedding*, Algebr. Combin. **4** (2021), no. 2, 189–200.
- [7] J. Kim and B. Rhoades, *Lefschetz Theory for Exterior Algebras and Fermionic Diagonal Coinvariants*, Int. Math. Res. Not. IMRN **2022** (2020), no. 4, 2906–2933.
- [8] D. E. Littlewood, *Products and plethysms of characters with orthogonal, symplectic and symmetric groups*, Canadian J. Math. **10** (1958), 17–32.
- [9] N. A. Loehr and J. B. Remmel, *A computational and combinatorial exposé of plethystic calculus*, J. Algebraic Combin. **33** (2011), no. 2, 163–198.
- [10] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015.
- [11] Sridhar P. Narayanan, Digjoy Paul, Amritanshu Prasad, and Shraddha Srivastava, *Character polynomials and the restriction problem*, Algebr. Comb. **4** (2021), no. 4, 703–722.
- [12] ———, *Polynomial induction and the restriction problem*, Indian J. Pure Appl. Math. **52** (2021), no. 3, 643–651.
- [13] R. Orellana, F. Saliola, A. Schilling, and M. Zabrocki, *Plethysm and the algebra of uniform block permutations*, Algebr. Combin. **5** (2022), no. 5, 1165–1203.
- [14] Rosa Orellana and Mike Zabrocki, *A combinatorial model for the decomposition of multivariate polynomial rings as S_n -modules*, Electron. J. Combin. **27** (2020), no. 3, article no. 3.24 (18 pages).
- [15] ———, *Symmetric group characters as symmetric functions*, Adv. Math. **390** (2021), article no. 107943 (34 pages).
- [16] A. Prasad, *The Frobenius characteristic of character polynomials*, J. Indian Inst. Sci. **102** (2022), no. 3, 947–959.
- [17] L. Solomon, *Invariants of finite reflection groups*, Nagoya Math. J. **22** (1963), 57–64.
- [18] R. P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.
- [19] Jerzy Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, vol. 149, Cambridge University Press, Cambridge, 2003.

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