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On the growth of the Jacobians in $\mathbb{Z}_p^l$-voltage covers of graphs

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Abstract We investigate the growth of the p-part of the Jacobians in voltage covers of finite connected graphs, where the voltage group is isomorphic to $\mathbb{Z}_p^l$ for some $l \geqslant 2$, and we study analogues of a conjecture of Greenberg on the growth of class numbers in multiple $\mathbb{Z}_p$-extensions of number fields. Moreover we prove an Iwasawa main conjecture in this setting, and we study the variation of (generalised) Iwasawa invariants as one runs over the $\mathbb{Z}_p^l$-covers of a fixed finite graph X. We discuss many examples; in particular, we construct examples with non-trivial Iwasawa invariants.

1. Introduction

Let p be a prime, and let K_∞ be a $\mathbb{Z}_p$-extension of a number field K. Then for each $m \in \mathbb{N}$ there exists a unique subextension K_m/K of degree p^m. Let h_m be the class number of K_m, and let $v_p(h_m) \geqslant 0$ be the exponent of the largest power of p which divides h_m, respectively. Then Iwasawa (see [9]) proved that we have an asymptotic formula

$$v_p(h_m) = \mu(K_\infty/K) \cdot p^m + \lambda(K_\infty/K) \cdot m + \nu(K_\infty/K) \tag{1}$$

for each sufficiently large m, where $\mu(K_\infty/K) \in \mathbb{Z}_{\geqslant 0}$, $\lambda(K_\infty/K) \in \mathbb{Z}_{\geqslant 0}$ and $\nu(K_\infty/K) \in \mathbb{Z}$ denote the Iwasawa invariants of the $\mathbb{Z}_p$-extension K_∞/K. More generally, let K_∞ be a $\mathbb{Z}_p^l$-extension of a number field K, with intermediate fields K_m (i.e. K_m is the unique intermediate field such that $\mathrm{Gal}(K_m/K) \cong (\mathbb{Z}/p^m\mathbb{Z})^l$, $m \in \mathbb{N}$). Again, we denote by h_m the class number of K_m. Then Greenberg conjectured (see [3, Section 7]) that there exists a polynomial $P(X,Y) \in \mathbb{Q}[X,Y]$ of total degree at most l and of degree at most 1 in Y such that

$$v_p(h_m) = P(p^m, m)$$

for each sufficiently large m. In this paper, we will refer to this conjecture as *Greenberg's conjecture*. Cuoco and Monsky have generalised Iwasawa's formula to $\mathbb{Z}_p^l$-extensions, but their formula contains an error term $O(p^{n(d-1)})$ and so they were not able to prove Greenberg's conjecture (in fact, it seems that this conjecture might not even be true in general, see [3]).

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In the Iwasawa theory of graphs one starts with a finite connected graph X instead of a number field K and one considers a sequence of Galois covers

$$X \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow X_4 \leftarrow \cdots$$

such that $\mathrm{Gal}(X_m/X) \cong (\mathbb{Z}/p^m\mathbb{Z})^l$. Then each X_m is again a finite graph and all the X_m are connected. Instead of the p-part of the class number of the field K_m one considers the q-part of the number of spanning trees of X_m (where q is a prime not necessarily distinct from p). Vallières and McGown-Vallières proved in a sequence of papers (see [20], [15], [16]) that (under mild technical hypothesis) the number of spanning trees in a $\mathbb{Z}_p$-cover (i.e. $l = 1$) grows asymptotically as in Iwasawa's formula. In a subsequent work Lei and Vallières considered the case $q \neq p$ (see [14]). The proofs of the above results are purely analytic and make use of Ihara L-functions. On the other hand, Gonet proved an Iwasawa-like formula in the case $p = q$ via a more algebraic approach, which is closer to Iwasawa's module theoretical approach (see [7]).

In this paper we study analogous growth patterns in the theory of voltage covers of graphs for $l \geqslant 2$ and $q = p$. Like Gonet's one our approach is purely algebraic and uses the machinery developed by Cuoco and Monsky, together with the fact that the number of spanning trees is given by the size of the Jacobian of a finite connected graph (see Section 2 for precise definitions). We are able to prove Greenberg's conjecture under the assumption that all the involved graphs are connected (see Theorem 4.3). This result had also been obtained before under some mild additional assumptions by Vallières and DuBose, generalising the approach of McGown-Vallières, using Ihara L-functions [4, Theorem A].

A central aspect of classical Iwasawa theory is the interplay between algebraic objects such as certain Galois modules on the one hand and analytic objects such as L-functions (e.g. the Hasse-Weil L-function $L(E, s)$ for an elliptic curve E or the Dedekind zeta function $\zeta_K(s)$ for a number field K) on the other hand. In many cases one can define a p-adic function interpolating special values of these complex analytic functions. Given a finite connected graph X and a finite Galois cover Y/X we consider the complex analytic Ihara L-function $L_X(\chi, s)$ for every character $\chi \in \widehat{\mathrm{Gal}(Y/X)}$. This L-function has a factor $(1 - s^2)^{\chi(X)}$, where $\chi(X)$ is the Euler characteristic of X. As this factor is independent of the cover Y/X and just produces trivial zeros at $s = 1$, we will ignore it and view $Q_X(\chi, s) = L_X(\chi, s)/(1 - s^2)^{\chi(X)}$ as the algebraic part of the Ihara L-function.

Let now X be a finite connected graph (we allow multiple edges and loops), and suppose that X has n vertices. Let X_m be the m-th intermediate graph in a $\mathbb{Z}_p^l$-cover X_∞ of X, i.e. $\mathrm{Gal}(X_m/X) \cong (\mathbb{Z}/p^m\mathbb{Z})^l$. Then there exists an element

$$\Delta_\infty \in \mathrm{Mat}_{n,n}(\mathbb{Z}_p[[\mathrm{Gal}(X_\infty/X)]])$$

such that $\chi(\det(\Delta_\infty))$ interpolates the algebraic part of $L_X(\chi, s)$ at $s = 1$ in the sense that

$$\chi(\det(\Delta_\infty))^{-1} = Q_X(\chi, 1).$$

In this setting we obtain the following

Theorem (Iwasawa main conjecture, Theorem 5.3). *Let $J(X_\infty)$ be the Jacobian of X_∞ and let $R = \mathbb{Z}_p[T_1, \ldots, T_l]$ and $\Lambda = \mathbb{Z}_p[\![T_1, \ldots, T_l]\!]$. Then $J(X_\infty) \otimes_R \Lambda$ is a finitely generated torsion Λ-module, and*

$$\mathrm{Char}_\Lambda(J(X_\infty) \otimes_R \Lambda) = (\det(\Delta_\infty)).$$

Here $\mathrm{Char}_\Lambda(M)$ denotes the characteristic ideal of a finitely generated and torsion Λ-module M (for details we refer to Section 2.1). This should be understood as an analogous statement to the classical Iwasawa main conjecture along $\mathbb{Q}_p(\mu_{p^\infty})$ that relates the characteristic ideals of the projective limits of p-class groups to power series interpolating special values of Dirichlet L-functions.

The last central topic which we want to address in the present paper is the behaviour of so-called *(generalised) Iwasawa invariants* if we vary the $\mathbb{Z}_p^l$-cover of a given finite connected graph X. Let $P(X,Y)$ be the polynomial appearing in Greenberg's conjecture. Then

$$P(p^m, m) = m_0 p^{ml} + l_0 m p^{m(l-1)} + O(p^{m(l-1)}).$$

Following Cuoco and Monsky (see [3]) we call m_0 and l_0 the (generalised) Iwasawa invariants of $J(X_\infty) \otimes \Lambda$, and we write $m_0(X_\infty)$ and $l_0(X_\infty)$ for these parameters. In Section 6 we define a topology on the set $\mathcal{E}^l(X)$ of voltage $\mathbb{Z}_p^l$-covers of X and we prove the following results.

Theorem (Theorems 6.4 and 6.5). *We have the following two results.*

(1) Assume that $l = 1$ and let $X_\infty \in \mathcal{E}^1(X)$. Then there exists a (sufficiently small) neighbourhood U of X_∞ such that the following statements hold:

(a) For each $\tilde{X}_\infty \in U$, we have

$$m_0(\tilde{X}_\infty) \leqslant m_0(X_\infty).$$

(b) $l_0(\tilde{X}_\infty) = l_0(X_\infty)$ for each $\tilde{X}_\infty \in U$ for which $m_0(\tilde{X}_\infty) = m_0(X_\infty)$.

(2) Fix an element $X_\infty \in \mathcal{E}^l(X)$, $l \geqslant 2$. Then there exist an integer $k \in \mathbb{N}$ and a neighbourhood U of X_∞ such that the following two statements hold for each $\tilde{X}_\infty$ in U.

(a) $m_0(\tilde{X}_\infty) \leqslant m_0(X_\infty)$, and

(b) $l_0(\tilde{X}_\infty) \leqslant k$ holds if $m_0(\tilde{X}_\infty) = m_0(X_\infty)$.

Let us briefly describe the structure of the article. Section 2 is preliminary in nature – here we introduce the basic notation and the setup. In Section 3 we relate the Jacobians of the Galois covers X_m/X, $m \in \mathbb{N}$, to certain quotients of an Iwasawa module. This enables us to describe the growth of these Jacobians by using results of Cuoco and Monsky, and we already obtain a weak result in the direction of Greenberg's conjecture (see Lemma 3.4). In Section 4, we prove Greenberg's conjecture by studying the voltage p-Laplacian of the Galois cover and using a suitable matrix representation in order to describe the growth of the Jacobians in terms of a certain power series. In Section 5 we prove the Iwasawa main conjecture and in Section 6 we prove Theorems 6.4 and 6.5 which have been stated above. The proofs are completely algebraic, and they are based on work of Fukuda (see [5]) and the first named author (see [11] and [12]). In the final two sections, we compute many numerical examples by applying the results from Sections 4 and 5. In particular, we construct examples of Galois covers X_∞/X of graphs with non-trivial m_0- and l_0-invariant. To construct these examples we need to be able to compare the Jacobians in a $\mathbb{Z}_p^l$-cover X_∞/X with the Jacobian of a $\mathbb{Z}_p$-cover Y/X contained in X_∞. In the setting of Iwasawa theory of class groups or elliptic curves results of this form are often referred to as control theorems. We will prove the control theorem needed in our context in Section 7.

In the classical Iwasawa theory of $\mathbb{Z}_p$-extensions (i.e. $l = 1$) of number fields, Iwasawa himself was the first who constructed examples with a non-trivial m_0-invariant (see [10]). On the other hand, to the authors' knowledge no example of a $\mathbb{Z}_p^l$-extension (for $l \geqslant 2$) of a number field K is known where one can show that the l_0-invariant is greater than zero (however, there exist such examples in the setting of Selmer groups of elliptic curves, see [13]). Therefore the construction of an example with non-trivial l_0-invariant in the final section might be of some interest.

2. Notation and definitions

2.1. Iwasawa modules. Let Γ be a group which is topologically isomorphic to $\mathbb{Z}_p^l$. Then the completed group ring $\mathbb{Z}_p[\![\Gamma]\!]$ is defined as the projective limit

$$\mathbb{Z}_p[\![\Gamma]\!] = \varprojlim_U \mathbb{Z}_p[\Gamma/U],$$

where $U \subseteq \Gamma$ runs over the open normal subgroups. The ring $\mathbb{Z}_p[\![\Gamma]\!]$ can be identified (non-canonically) with the ring $\mathbb{Z}_p[\![T_1, \ldots, T_l]\!]$ of formal power series in l variables, as follows: choose a set of topological generators $\gamma_1, \ldots, \gamma_l$ of Γ, and consider the bijective homomorphism between the group rings which is induced by mapping γ_i to $T_i + 1$, respectively.

In the following we always denote by $\Lambda_l = \mathbb{Z}_p[\![T_1, \ldots, T_l]\!]$ the Iwasawa algebra in l variables. If l is clear from the context, then we abbreviate Λ_l to Λ.

Now we describe the structure theory of *Iwasawa modules*. In this article, any finitely generated Λ-module A will be called an Iwasawa module. An Iwasawa module is called *pseudo-null* if it is annihilated by two relatively prime elements of the unique factorisation domain Λ. A *pseudo-isomorphism* of finitely generated Λ-modules A and B is a Λ-module homomorphism $\varphi\colon A \longrightarrow B$ such that the kernel and the cokernel of φ are pseudo-null.

Let now A be any Iwasawa module. Then by the general structure theory (see [18, Section 5.1]) there exist an *elementary* Λ-module

$$E_A = \Lambda^r \oplus \bigoplus_{i=1}^{s} \Lambda/(p^{e_i}) \oplus \bigoplus_{j=1}^{t} \Lambda/(g_j)$$

and a pseudo-isomorphism $\varphi\colon A \longrightarrow E_A$. Here $g_1, \ldots, g_t$ are elements of Λ which are coprime with p. If $r = 0$, then the *characteristic power series* F_A attached to A is the element

$$F_A = p^{\sum_{i=1}^{s} e_i} \cdot \prod_{j=1}^{t} g_j,$$

and the *characteristic ideal* $\mathrm{Char}(E_A)$ of E_A (and A) is the principal ideal (F_A). If $r \geqslant 1$, i.e. if A is not Λ-torsion, then we define $F_A = 0$ and $\mathrm{Char}(A) = (0)$. The characteristic power series is determined up to units of Λ by the Iwasawa module A, and therefore the characteristic ideal is well-defined.

Now suppose that A is Λ-torsion. Then we define the *(generalised) Iwasawa invariants* of A as follows. To this purpose, we fix an isomorphism between $\mathbb{Z}_p[\![\Gamma]\!]$ and Λ, as above, and we consider the characteristic power series F_A attached to A. Let

$$m_0(A) = \sum_{i=1}^{s} e_i,$$

i.e. $p^{m_0(A)}$ is the exact power of p which divides F_A. Moreover, we write

$$F_A = p^{m_0(A)} \cdot G_A,$$

i.e. G_A is coprime with p, and we consider the coset $\overline{G_A}$ of G_A in the quotient algebra $\overline{\Lambda} = \Lambda/(p)$. Then we define

$$l_0(A) = \sum_{\mathcal{P}} v_{\mathcal{P}}(\overline{G_A}),$$

where the sum runs over all the prime ideals of $\overline{\Lambda}$ of the form $\mathcal{P} = (\overline{\sigma - 1})$ for some element $\sigma \in \Gamma \setminus \Gamma^p$, and where $v_{\mathcal{P}}$ denotes the $\mathcal{P}$-adic valuation. Note that this sum is finite since $\overline{\Lambda}$ is again a unique factorisation domain.

In the case of $l = 1$, the structure theory of Iwasawa modules is better understood. In this case, an Iwasawa module is pseudo-null if and only if it is *finite*. Moreover, it follows from the Weierstraß Preparation Theorem for $\Lambda = \mathbb{Z}_p[\![T]\!]$ (see [21, Theorem 7.3]) that the characteristic power series F_A of a torsion Λ-module A is associated to a *polynomial* of the form

$$p^{m_0(A)} \cdot G_A,$$

where G_A is *distinguished*. This means that G_A is monic, and that each coefficient of G_A, apart from the leading one, is divisible by p. Then we define the *classical Iwasawa invariants* of A as

$$\mu(A) = m_0(A), \quad \lambda(A) = \deg(G_A) = l_0(G_A).$$

If $A = \varprojlim A_n$ is the classical Iwasawa module of ideal class groups in the intermediate layers of a $\mathbb{Z}_p$-extension K_∞/K, then the Iwasawa invariants of A are the constants from the asymptotic growth formula (1) mentioned in the introduction (see [9] or [21, Chapter 13] for more details).

We conclude the current section by introducing some more notation. For any $i \in \{1, \ldots, l\}$ and every $m \in \mathbb{N}$, we let

$$\omega_m(T_i) = (T_i + 1)^{p^m} - 1.$$

Moreover, for every $0 \leqslant j \leqslant m$, we introduce

$$\nu_{m,j}(T_i) = \omega_m(T_i)/\omega_j(T_i).$$

Recall that $\Lambda \cong \mathbb{Z}_p[[\Gamma]]$, where Γ is topologically isomorphic to $\mathbb{Z}_p^l$. In fact, the polynomials $\omega_m(T_i)$ correspond to elements in $R = \mathbb{Z}_p[\Gamma] \subseteq \Lambda$. We denote by I_m the ideal of Λ generated by all the $\omega_m(T_i)$, and we denote the quotient Λ/I_m by R_m, $m \in \mathbb{N}$. Note that $R_m \cong \mathbb{Z}_p[\Gamma/\Gamma^{p^m}]$.

2.2. Graphs. We briefly define the notions from graph theory which will be needed; for a more detailed introduction see for example the paper [4] of DuBose and Vallières.

Let Y be a graph, and let $\mathbf{E}_Y$ and E_Y be the sets of *directed* and *undirected edges* of Y. If $e \in \mathbf{E}_Y$ has origin v and target w then we denote by e^{-1} the directed edge with origin w and target v. In this article we denote the set of directed edges between two vertices v and w of a graph Y by $E(v,w) \subseteq \mathbf{E}_Y$, i.e. each $e \in E(v,w)$ has origin v and target w. Sometimes we say that $e \in E(v,w)$ is an edge *from v to w*. Note that $E(v,w)$ may contain more than one element (i.e. we allow *multigraphs*). For any graph Y we make the following definitions.

Definition 2.1. *Let Y be a (not necessarily finite) graph. Then we denote by $V(Y)$ the set of vertices of Y. We let*

$$\deg(v) = \sum_{w \in V(Y), w \neq v} |E(v,w)| + 2|\{\text{undirected loops from } v \text{ to } v\}|$$

be the degree *of a vertex v and we define* $\mathrm{mult}(v,w) = |E(v,w)|$ *to be the number of edges with origin v and target w (i.e.* $\mathrm{mult}(v,w) = 0$ *if v and w are not adjacent).*

In this article we always assume that $\deg(v)$ *is finite for each $v \in V(Y)$ (such graphs are called locally finite).*

We define

$$\mathrm{Div}(Y) = \left\{ \sum_{v \in V(Y)} a_v v \;\middle|\; a_v \in \mathbb{Z}_p, a_v = 0 \text{ for all but finitely many } v \right\}$$

$$\mathrm{Div}^0(Y) = \left\{ d = \sum_{v \in V(Y)} a_v v \in \mathrm{Div}(Y) \;\middle|\; \sum_{v \in V(Y)} a_v = 0 \right\}$$

$$\mathrm{Pr}(Y) = \mathcal{L}(\mathrm{Div}(Y)),$$

where $\mathcal{L}$ is the Laplacian operator which is defined as follows: For a vertex $v \in V(Y)$ we define $\mathcal{D}(v) = \deg(v)v$ and $\mathcal{A}(v) = \sum_{w\sim v} \mathrm{mult}(v,w)w$, where $w \sim v$ means that the vertices w and v are adjacent. We now define

$$\mathcal{L}(v) = \mathcal{D}(v) - \mathcal{A}(v).$$

We further define the two quotient groups

$$\mathrm{Pic}_p(Y) = \mathrm{Div}(Y)/\mathrm{Pr}(Y),$$
$$J_p(Y) = \mathrm{Div}^0(Y)/\mathrm{Pr}(Y).$$

The subscript p is used to emphasize that we are considering $\mathbb{Z}_p$-modules here. Let $\mathrm{Jac}(Y)$ be the *Jacobian* of Y. Then $J_p(Y) = \mathrm{Jac}(Y) \otimes \mathbb{Z}_p$. Note that $J_p(Y)$ is the p-Sylow subgroup of $\mathrm{Jac}(Y)$ if $\mathrm{Jac}(Y)$ is finite. As the prime p is fixed once and for all we will write $J(Y)$ instead of $J_p(Y)$ from now on. If Y is finite and connected, then it is well-known that $J(Y)$ is a finite abelian group (see [2, Proposition 2.37]). Furthermore, the group $\mathrm{Pr}(Y)$ is generated by the elements

$$p_v = \mathcal{L}(v) = \deg(v)v - \sum_{w\sim v, w\neq v} \mathrm{mult}(v,w)w - 2|\{\text{undirected loops from } v \text{ to } v\}|v,$$

$v \in V(Y)$.

Let now X be a finite graph with n vertices. We specialise to the situation of voltage covers X_m (see also [6]). Let $\gamma\colon E_X \longrightarrow \mathbf{E}_X$ be a section of the natural map $\mathbf{E}_X \longrightarrow E_X$. Let $S = \gamma(E_X)$. The set $S \subseteq \mathbf{E}_X$ corresponds to a choice of orientation.

Let Γ be a multiplicative group which is isomorphic to $\mathbb{Z}_p^l$. We fix a *voltage assignment*, i.e. a map

$$\alpha\colon S \longrightarrow \Gamma.$$

Using the natural projection maps $\Gamma \longrightarrow \Gamma/\Gamma^{p^m}$, $m \in \mathbb{N}$, α induces well-defined assignments

$$\alpha_m\colon S \longrightarrow \Gamma/\Gamma^{p^m}.$$

Since the target groups of α in our applications usually will be Galois groups, we write the target groups of α and of the α_m multiplicatively. Note that for each element $e \in \mathbf{E}_X$ either $e \in S$ or $e^{-1} \in S$. If $e \notin S$, we define $\alpha(e) = \alpha(e^{-1})^{-1}$ and analogously for α_m. Thus, we can interpret α and α_m as maps defined on $\mathbf{E}_X$.

Definition 2.2. *We define the derived graph $X_m := X(\Gamma/\Gamma^{p^m}, S, \alpha_m)$ on the vertices $(v, g \pmod{\Gamma^{p^m}})$, where $g \in \Gamma$ and $g \pmod{\Gamma^{p^m}}$ denotes the equivalence class of g in Γ/Γ^{p^m}. The set $\mathbf{E}_{X_m}$ of derived edges of X_m is defined as follows: We draw an edge between $(v, g \pmod{\Gamma^{p^m}})$ and $(v', g' \pmod{\Gamma^{p^m}})$ if there is an edge e from v to v' in $\mathbf{E}_X$ such that $g' \pmod{\Gamma^{p^m}} = g \cdot \alpha_m(e) \pmod{\Gamma^{p^m}}$.*

We abbreviate the vertex $(v, 1)$ of the graph X_m to v.

We have a natural action of Γ/Γ^{p^m} on X_m given by

$$(g' \pmod{\Gamma^{p^m}}) \circ (x, g \pmod{\Gamma^{p^m}}) = (x, g\cdot g' \pmod{\Gamma^{p^m}}),$$

where x is a vertex of X. By using the natural projection $\Gamma \longrightarrow \Gamma/\Gamma^{p^m}$, we also have a natural action of Γ on X_m. This induces a well-defined action of Γ on $J(X_m)$. We also obtain an action of the two canonical group rings

$$R = \mathbb{Z}_p[\Gamma], \quad \mathbb{Z}_p[[\Gamma]].$$

The completed group ring is isomorphic non-canonically to the Iwasawa algebra $\Lambda = \Lambda_l = \mathbb{Z}_p[[T_1, \ldots, T_l]]$, as has been explained in the previous subsection.

Definition 2.3. *We let X_∞ be a $\mathbb{Z}_p^l$-voltage cover of X. Then we define*

$$J_\Lambda := J(X_\infty) \otimes_R \Lambda.$$

Note that J_Λ is a finitely generated Λ-module, i.e. it is an Iwasawa module in the language of Section 2.1 (this will follow from Lemma 5.2 and the results in Section 3). Therefore it makes sense to speak of the (generalised) Iwasawa invariants of the graph X_∞.

In this article, we always assume that all the graphs X_m are connected. We will provide a sufficient criterion for this property below (see Lemma 2.5). Thus, X_m/X is a Galois cover with group $\Gamma/\Gamma^{p^m} \cong (\mathbb{Z}/p^m\mathbb{Z})^l$. Then as a $\mathbb{Z}_p[\mathrm{Gal}(X_m/X)]$-module $\mathrm{Pr}(X_m)$ is generated by the elements

$$\begin{aligned} p_{v,0} &= -\sum_{e\in \mathbf{E}_v(X_m)} (t(e)-v) \\ &= -\sum_{e\in \mathbf{E}_v(X), t(e)\neq v} (\alpha_m(e)t(e)-v) - \sum_{e\in \mathbf{E}_v(X), t(e)=v} (\alpha_m(e)+\alpha_m(e)^{-1}-2)v, \end{aligned}$$

where $v\in V(X)$, $\mathbf{E}_v(Y)$ denotes the set of directed edges with origin v in a graph Y, and where $t(e)$ is the target of an edge e.

If e is a loop with trivial voltage assignment then the term $\alpha_m(e)+\alpha_m(e)^{-1}-2$ vanishes in the above sum. The coefficient of $v=(v,1)$ in $p_{v,0}$ (over $\mathbb{Z}_p$) is given by

$$\deg(v)v - 2|\{e \mid t(e)=v, \alpha_m(e)=1\}|.$$

Now suppose that X is a finite graph, and let $n=|V(X)|$. Fixing an indexing of the vertices of X, say $V(X)=\{v_1,\dots,v_n\}$, we will also write $p_{i,0}=p_{v_i,0}$ for these elements. We define a $\mathbb{Z}_p$-linear map

$$\mathcal{L}_\alpha \colon \mathrm{Div}(X_\infty) \longrightarrow \mathrm{Pr}(X_\infty), \quad (v_i,1)\mapsto p_{i,0}.$$

It is easy to see from the definitions and the action of $\mathrm{Gal}(X_\infty/X)$ on $\mathrm{Div}(X_\infty)$ that this map is unique and well-defined. It is called the *voltage p-Laplacian*, see also [6].

Definition 2.4. *We say that a graph X contains a non-trivial cycle of length $n>1$ if there exist a sequence of vertices $v_0, v_1, \dots v_{n-1}, v_n, v_0$ and a sequence of directed edges $e_0,\dots,e_n$ such that e_i is an edge from v_i to v_{i+1} for $0\leqslant i\leqslant n-1$ and e_n is an edge from v_n to v_0. We assume furthermore that $e_{i+1}\neq {e_i}^{-1}$ for $0\leqslant i\leqslant n-1$ and $e_n\neq e_0^{-1}$.*

For any such cycle C we write β_C for the group element $\prod_{i=0}^n \alpha(e_i)$.

Lemma 2.5. *X_m is connected for all m if and only if we can find cycles $C_1,\dots,C_l$ such that $\{\beta_{C_1},\dots,\beta_{C_l}\}$ is a set of topological generators of Γ.*

Proof. Recall that we write $V(X_m)$ for the set of vertices of X_m. We say that two vertices (v,γ) and (v',γ') of X_m are connected if there is a path connecting (v,γ) and (v',γ'). It is easy to check that this defines an equivalence relation on $V(X_m)$.

Assume first that there are cycles $C_1,\dots,C_l$ satisfying the above condition. In what follows, we abbreviate β_{C_i} to β_i for each $1\leqslant i\leqslant l$. Let $v\in V(X)$. We will first show that $(v,1)$ is connected to (v,γ) for every $\gamma\in\Gamma/\Gamma^{p^m}\smallsetminus\{1\}$. As $\beta_1,\dots,\beta_l$ are topological generators of Γ, it suffices to consider $\gamma=\beta_i^{a_i}$ for $1\leqslant i\leqslant l$ and $a_i\in\mathbb{Z}/p^m\mathbb{Z}$. Let v_0 be a vertex lying on C_i. Then (v_0,γ') is connected to $(v_0,\gamma'\cdot\beta_i^{a_i})$ for every choice of γ'. As X is connected we can find a path $e_0,e_1,\dots,e_k$ from v to v_0 and we see that $(v,1)$ is connected to $(v_0,\prod_{j=1}^k\alpha_m(e_j))$ and $(v_0,\tilde\gamma)$ is connected to $(v,\tilde\gamma\prod_{j=1}^k\alpha_m(e_j^{-1}))$ for every choice of $\tilde\gamma$. We obtain the following relations

$$(v,1)\sim\left(v_0,\prod_{j=1}^k\alpha_m(e_j)\right)\sim\left(v_0,\prod_{j=1}^k\alpha_m(e_j)\beta_i^{a_i}\right)\sim(v,\beta_i^{a_i})$$

proving our first claim.

For every $v' \in V(X)$ we can find an element $\gamma \in \Gamma/\Gamma^{p^m}$ such that $(v,1)$ is connected to (v',γ) – using that X is connected. Using the first part of the proof we see that $(v,1) \sim (v',\gamma) \sim (v',\gamma')$ for every choice of γ'. Therefore X_m is indeed connected.

Assume now that all the X_m are connected. In particular X_1 is connected. Let $v \in V(X)$ and $\gamma \in \Gamma/\Gamma^p \setminus \{1\}$ be arbitrary. Then $(v,1)$ is connected to (v,γ). Thus, there is a path $e_1,\dots,e_k$ from v to v such that $\gamma = \prod_{j=1}^k \alpha_1(e_j)$. Without loss of generality we can assume that $e_i \neq e_{i+1}^{-1}$ for $1 \leqslant i \leqslant k-1$. Let v_i be the target of e_i and set $v = v_0$. Let k_v be the minimal index such that $e_{k_v} \neq e_{k-k_v+1}^{-1}$. Such an index exists, as $\gamma \neq 1$. Then $e_{k_v},\dots,e_{k-k_v+1}$ is a path from v_{k_v-1} to itself and it is indeed a cycle. Furthermore, $\gamma = \prod_{j=k_v}^{k-k_v+1} \alpha_1(e_j)$. □

In the rest of the paper we will always assume that all the graphs X_m are connected.

3. The module theoretic perspective in the multidimensional case

In this section we generalise the module-theoretic point of view from [6] to the case of arbitrary l: we will explain the structure of the Iwasawa module $\varprojlim_m J(X_m)$ and its relation to the asymptotic growth of $J(X_m)$.

Let $R = \mathbb{Z}_p[\Gamma]$, let $\Lambda = \mathbb{Z}_p[[T_1,\dots,T_l]]$ be as before, and let I_m and $R_m = \Lambda/I_m$ be defined as in Section 2.1. We write Div_Λ, Div^0_Λ, Pr_Λ and Pic_Λ for the corresponding Λ-modules $\mathrm{Div}(X_\infty) \otimes \Lambda$, $\mathrm{Div}^0(X_\infty) \otimes \Lambda$, $\mathrm{Pr}(X_\infty) \otimes \Lambda$ and $\mathrm{Pic}(X_\infty) \otimes \Lambda$. In what follows we write $|V(X)| = n$. It is easy to see that Div_Λ is Λ-free of rank n.

Remark 3.1. The ring Λ is actually flat over R. The argument boils down to showing that $I \otimes \Lambda = I\Lambda$ for every ideal $I \subseteq R$. This argument involves several diagram chases and the fact that projective limits and tensor products commute under certain conditions. As the proof is not very enlightening and the computation of the necessary Tor modules is rather easy in the cases of interest to us, we decided not to include here a proof of the fact that Λ is flat over R.

Lemma 3.2. *The Λ-module Pr_Λ is free of rank n. Furthermore, the natural map $\mathrm{Pr}_\Lambda \longrightarrow \mathrm{Div}_\Lambda$ is injective. Thus, we obtain a short exact sequence*

$$0 \longrightarrow \mathrm{Pr}_\Lambda \longrightarrow \mathrm{Div}_\Lambda \longrightarrow \mathrm{Pic}_\Lambda \longrightarrow 0.$$

Proof. Let V be the submodule of Div_Λ generated by the $p_{i,0}$, i.e. V is the image of Pr_Λ in Div_Λ. As $\mathrm{Div}(X_m)$ is generated by the v_i as an R_m-module, there is clearly a surjective map

$$\mathrm{Div}_\Lambda/I_m \longrightarrow \mathrm{Div}(X_m).$$

Assume that there are elements $F_i \in \Lambda$, not all zero, such that $\sum F_i \cdot p_{i,0} = 0$ in Div_Λ. Without loss of generality we can assume that $F_1 \neq 0$. Then there is a surjection $\Lambda/F_1 \oplus \Lambda^{n-1} \twoheadrightarrow V$. Taking quotients by $(I_m\Lambda)^n$ we see that the $\mathbb{Z}_p$-rank of $\mathrm{Pr}(X_m)$ is at most $(n-1)p^{ml} + E(m)$ for some error term $E(m) = O(p^{m(l-1)})$. It follows that

$$np^{ml} - 1 = \mathbb{Z}_p\text{-rank}(\mathrm{Pr}(X_m)) \leqslant (n-1)p^{ml} + E(m), \quad E(m) = O(p^{m(l-1)}),$$

yielding a contradiction. Thus, V is Λ-free as well and $\mathrm{Pr}_\Lambda \longrightarrow \mathrm{Div}_\Lambda$ is injective.

The exactness of the sequence in the statement of the lemma now follows from tensoring

$$0 \longrightarrow \mathrm{Pr}(X_\infty) \longrightarrow \mathrm{Div}(X_\infty) \longrightarrow \mathrm{Pic}(X_\infty) \longrightarrow 0$$

with Λ. Note that the sequence stays left exact after tensoring with Λ as Pr_Λ embeds into Div_Λ. □

Let $M_\Lambda \supseteq \mathrm{Pr}_\Lambda$ be the submodule of Div_Λ generated by the elements $p_{i,0}$, the elements $v_i - v_j$ for $1 \leqslant j < i \leqslant n$ and $(T_1, \dots, T_l)\mathrm{Div}_\Lambda$. The following theorem generalises results of [6, Section 5].

Theorem 3.3. *We have $J(X_m) \cong M_\Lambda/(I_m \cdot \mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda)$.*

Proof. Let $\pi_m \colon \mathrm{Div}(X_\infty) \longrightarrow \mathrm{Div}(X_m)$ be the canonical map of voltage covers. Note that we can see this as an R-module homomorphism. We start by the following series of observations:

(a) π_m is surjective and induces surjective maps $\mathrm{Div}^0(X_\infty) \longrightarrow \mathrm{Div}^0(X_m)$ and $\mathrm{Pr}(X_\infty) \longrightarrow \mathrm{Pr}(X_m)$.
(b) The kernel of π_m is given by $I_m\mathrm{Div}_\Lambda \cap \mathrm{Div}(X_\infty)$.
(c) We have $I_m\mathrm{Div}_\Lambda + \mathrm{Div}^0(X_\infty) = M_\Lambda$.
(d) $I_m\mathrm{Div}_\Lambda + \mathrm{Pr}(X_\infty) = \mathrm{Pr}_\Lambda + I_m\mathrm{Div}_\Lambda$.

Let us first see that this series of observations suffices to prove the theorem. By points (a) and (b) we have an isomorphism

$$\mathrm{Div}^0(X_m) \cong \mathrm{Div}^0(X_\infty)/(I_m\mathrm{Div}_\Lambda \cap \mathrm{Div}^0(X_\infty)),$$

where the right hand term is isomorphic to

$$(\mathrm{Div}^0(X_\infty) + I_m\mathrm{Div}_\Lambda)/I_m\mathrm{Div}_\Lambda.$$

Combining this with (c) and (d), we have

$$\begin{aligned} J(X_m) &= \mathrm{Div}^0(X_m)/\mathrm{Pr}(X_m) \\ &= M_\Lambda/(\mathrm{Pr}_\Lambda + I_m\mathrm{Div}_\Lambda). \end{aligned}$$

It remains to check the observations (a)-(d). For observation (a) note that v_i, $1 \leqslant i \leqslant n$, is a set of generators for X_m (seen as an R-module). To see that π_m remains surjective when restricted to $\mathrm{Div}^0(X_\infty)$, it suffices to note that the elements $v_j - v_i$ and $(v_j, \tau_k) - (v_j, 1)$ for a set of topological generators $\tau_1, \dots, \tau_l$ of $\mathrm{Gal}(X_\infty/X)$ generate $\mathrm{Div}^0(X_m)$ as well as $\mathrm{Div}^0(X_\infty)$ as R-modules. Using a similar argument for the generators $p_{i,0}$ instead shows the corresponding claim for $\mathrm{Pr}(X_\infty)$ and $\mathrm{Pr}(X_m)$.

For point (b) note that Div_Λ is a free Λ-module of rank n with generators $(v_j, 1)$ for $1 \leqslant j \leqslant n$. Furthermore, $\mathrm{Div}(X_m)$ is a free R_m-module with the same generators. As $\mathrm{Div}(X_\infty)$ contains the R-module generated by $(v_j, 1)$, we see that $\mathrm{Div}(X_\infty)$ lies densely in Div_Λ. We obtain the following isomorphisms.

$$\begin{aligned} \mathrm{Div}(X_m) &\cong \bigoplus_{i=1}^n R_m v_i \cong \bigoplus_{i=1}^n \Lambda/I_m v_i \\ &\cong \mathrm{Div}_\Lambda/I_m \\ &\cong \mathrm{Div}(X_\infty)/(\mathrm{Div}(X_\infty) \cap I_m\mathrm{Div}_\Lambda), \end{aligned}$$

which proves claim (b).

Now we prove (c). By construction $\mathrm{Div}^0(X_\infty) \subseteq M_\Lambda$, and as our chosen generators of M_Λ lie in $\mathrm{Div}^0(X_\infty)$, we see that $\mathrm{Div}^0(X_\infty)$ lies densely in M_Λ. Note that $I_m\mathrm{Div}_\Lambda$ is a neighbourhood of 0 in M_Λ. Let $y \in M_\Lambda$ be an arbitrary element, then there exists $z \in \mathrm{Div}^0(X_\infty) \cap (y + I_m\mathrm{Div}_\Lambda)$. Thus,

$$\begin{aligned} y + I_m\mathrm{Div}_\Lambda &\subseteq z + I_m\mathrm{Div}_\Lambda \\ &\subseteq \mathrm{Div}^0(X_\infty) + I_m\mathrm{Div}_\Lambda. \end{aligned}$$

Varying y in M_Λ gives the claim.

Claim (d) can be proved similarly to point (c) by using the generators $p_{i,0}$. Clearly, $I_m\mathrm{Div}_\Lambda + \mathrm{Pr}(X_\infty) \subseteq \mathrm{Pr}_\Lambda + I_m\mathrm{Div}_\Lambda$. By definition, $\mathrm{Pr}(X_\infty)$ lies densely in Pr_Λ and

$I_m \mathrm{Div}_\Lambda \cap \mathrm{Pr}_\Lambda$ is a neighbourhood of $0 \in \mathrm{Pr}_\Lambda$ in the subspace topology. Let now $y \in \mathrm{Pr}_\Lambda$. Then there exists $z \in \mathrm{Pr}(X_\infty) \cap (y + (I_m \mathrm{Div}_\Lambda \cap \mathrm{Pr}_\Lambda))$. Thus,

$$y + I_m \mathrm{Div}_\Lambda \subseteq \mathrm{Pr}(X_\infty) + I_m \mathrm{Div}_\Lambda.$$

Varying $y \in \mathrm{Pr}_\Lambda$ gives the claim. □

Lemma 3.4. *Let L be a finitely generated Λ-torsion module and let $N \subseteq L$ be such that $L_m := I_m L \subseteq N$ for each $m \in \mathbb{N}$. Assume that N/L_m is finite for all m. Then there are non-negative integers m_0 and l_0 such that*

$$v_p(|N/L_m|) = m_0 p^{ml} + l_0 m p^{m(l-1)} + O(p^{m(l-1)}). \tag{2}$$

Proof. Since N/L_m is finite for each m, the quotient L_0/L_m is finite for all m. In particular, L/L_m and L/L_0 have the same $\mathbb{Z}_p$-rank.

To compute $v_p(|N/L_m|)$ note that we have

$$|N/L_m| = |N/L_0| \cdot |L_0/L_m|.$$

So it remains to compute $|L_0/L_m|$. Since L/L_m and L/L_0 have the same $\mathbb{Z}_p$-rank and therefore the kernel of the natural surjective map $L/L_m \longrightarrow L/L_0$ is contained in the p-torsion of L/L_m, we deduce

$$|L_0/L_m| = \frac{|(L/L_m)[p^\infty]|}{|(L/L_0)[p^\infty]|}.$$

The desired result follows directly from [3, Theorem 3.4] (note that in our case we even have $\mathbb{Z}_p$-rank$(L/L_m) = O(1)$ instead of $O(p^{m(l-2)})$). □

This lemma can be applied, according to Theorem 3.3, to the following Λ-modules: consider $N = M_\Lambda/\mathrm{Pr}_\Lambda$ and $L = \mathrm{Div}_\Lambda/\mathrm{Pr}_\Lambda$. By definition $v_i - v_j \in M_\Lambda$ for all $1 \leqslant i \leqslant j \leqslant n$. Hence, $\mathrm{Div}_\Lambda/M_\Lambda$ is cyclic as Λ-module. For example, we can write $\mathrm{Div}_\Lambda/M_\Lambda = \Lambda v_1$. Furthermore, $\mathrm{Div}_\Lambda/M_\Lambda$ is annihilated by $(T_1, \dots, T_l)$. Thus, $L/N \cong \mathrm{Div}_\Lambda/M_\Lambda = \Lambda v_1 = \mathbb{Z}_p v_1 \cong \mathbb{Z}_p$. Furthermore,

$$L_m = I_m \cdot L$$

is contained in N for each $m \in \mathbb{N}$, and

$$N/L_m \cong J(X_m)$$

is finite for each m by Theorem 3.3. As $L/N \cong \mathbb{Z}_p$ it follows that L/L_m has $\mathbb{Z}_p$-rank 1 for each m. In particular, L has to be Λ-torsion. As N is a submodule of L it has to be torsion as well.

In this case, we obtain the following arithmetic description of the numbers m_0 and l_0 from (2):

Corollary 3.5. *If $l \geqslant 2$, then the parameters m_0 and l_0 from* (2) *are the generalised Iwasawa invariants of L. Since $L/N \cong \mathbb{Z}_p$ is pseudo-null over Λ, these are also the generalised Iwasawa invariants of N.*

If $l = 1$, then we obtain the Iwasawa invariants of N. In particular, m_0 equals the μ-invariant of L, while $l_0 = \lambda(N) = \lambda(L) - 1$ in this case.

Proof. If $l \geqslant 2$ this follows directly from the above proof. In the case $l = 1$ it is a straight forward computation over one-dimensional Iwasawa algebras (note that $\varprojlim_m N/I_m \cong N$, since $\bigcap_m I_m = \{0\}$). □

4. Computation of the order of $J(X_m)$

Let X be a finite graph, and let $\alpha\colon S \longrightarrow \Gamma$ be as in Section 2. Recall the definition of the $\mathbb{Z}_p$-linear map $\mathcal{L}_\alpha$ from Section 2. We can extend naturally the linear map $\mathcal{L}_\alpha$ to a Λ-linear map. Recall that $\mathrm{Div}_\Lambda \cong \Lambda^n$ with basis $\{v_1, \dots, v_n\}$. We would like to give a matrix-representation of $\mathcal{L}_\alpha$ with respect to this basis. Let D be the degree matrix of the base graph X and define the voltage assignment matrix $A_\alpha = (\alpha_{i,j})_{1\leqslant i,j\leqslant n}$ as follows:

$$\alpha_{i,j} = \begin{cases} \sum_{e\in E(v_i,v_j)} \alpha(e) & \text{if } i \neq j, \\ \sum_{e\in E(v_i,v_i)} \alpha(e) + \alpha(e)^{-1} & \text{if } i = j. \end{cases}$$

Then $\alpha_{i,j} \in \Lambda$ for all i, j. It is easy to verify that $\mathcal{L}_\alpha$ is represented by the transpose Δ_∞^t of the matrix $\Delta_\infty = D - A_\alpha$ (here we use the v_i as a basis). Recall that

$$p_{v_i,0} = -\sum_{e\in E_{v_i}(X_\infty)} (\alpha(e)t(e) - v_i) = \deg(v_i)v_i - \sum_{j=1}^{n} \alpha_{i,j}v_j,$$

which is represented by the i-th row of $D - A_\alpha$ and not by the i-th column. Note that this ambiguity does not occur if we understand $\mathcal{L}_\alpha$ only as a $\mathbb{Z}_p$-linear map. In this case a $\mathbb{Z}_p$-basis of $\mathrm{Div}(X_m)$ is $\{(v_i, g) \mid v_i \in V(X), g \in (\mathbb{Z}/p^m\mathbb{Z})^l\}$, $\mathcal{L}_\alpha$ is represented by a symmetric matrix and we do not have to consider the transpose.

In what follows, we fix some $m \in \mathbb{N}$. Recall the definition of $R_m = \Lambda/I_m$ from Section 2.1. If we define Δ_m^t to be the representing matrix for the map

$$\mathcal{L}_{\alpha_m}\colon \mathrm{Div}(X_m) \longrightarrow \mathrm{Div}(X_m)$$

(again in the R_m-basis v_i), we obtain that Δ_m is the image of Δ_∞ in $\mathrm{Mat}_{n,n}(R_m)$.

Let Ω be the group of all p-power roots of unity in some fixed algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. We say that two elements in Ω^l are equivalent if they are Galois conjugate to each other. Let $\zeta = (\zeta_1, \dots, \zeta_l) \in \Omega^l$. We denote by $\mathbb{Z}_p[\zeta]$ the ring $\mathbb{Z}_p[\zeta_1, \dots, \zeta_l]$ obtained by adjoining all the components ζ_i to $\mathbb{Z}_p$. Following [3] we define a map

$$\Lambda/I_m \longrightarrow \bigoplus \mathbb{Z}_p[\zeta], \quad F(T_1, \dots, T_l) \mapsto (F(\zeta_1 - 1, \dots, \zeta_l - 1))_\zeta,$$

where the sum on the right hand side runs over a set of representatives $\zeta = (\zeta_1, \dots, \zeta_l)$ for the equivalence classes in $\Omega^l[p^m]$. Using that Div_Λ is Λ-free of rank n, we can extend the above map linearly and define

$$\phi_m\colon \mathrm{Div}_\Lambda/I_m \longrightarrow (\bigoplus \mathbb{Z}_p[\zeta])^n.$$

Let $\tilde{\Lambda}_m$ be the image of ϕ_m. Note that by [3, Theorem 2.2], ϕ_m has a finite cokernel and trivial kernel. Recall that $\mathrm{Pr}_\Lambda = \mathcal{L}_\alpha(\mathrm{Div}_\Lambda)$ and that Δ_∞^t is a matrix representing the map $\mathcal{L}_\alpha$.

It is immediate that $\mathcal{L}_\alpha(I_m\mathrm{Div}_\Lambda) \subseteq I_m\mathrm{Div}_\Lambda$. So $\mathcal{L}_\alpha$ induces well-defined maps on Div_Λ/I_m. By abuse of notation, we denote these maps by $\mathcal{L}_\alpha$ again, and we define a map $\tilde{\mathcal{L}_\alpha}$ on $\tilde{\Lambda}_m$ such that

$$\phi_m \circ \mathcal{L}_\alpha = \tilde{\mathcal{L}_\alpha} \circ \phi_m$$

on Div_Λ/I_m. Note that $\tilde{\mathcal{L}_\alpha}$ is a linear map of $\mathbb{Z}_p$-modules. Recall that ϕ_m has finite cokernel. So we can extend $\tilde{\mathcal{L}_\alpha}$ to a linear map of $\tilde{\Lambda}_m \otimes \mathbb{Q}_p = (\oplus_{z=1}^k \mathbb{Q}_p[\zeta_z])^n$, where $\zeta_1, \dots, \zeta_k$ represent the different equivalence classes in $\Omega^l[p^m]$ modulo Galois equivalence.

For every matrix A with coefficients $a_{i,j} \in \Lambda$ and each $\zeta \in \Omega^l$ we denote by $A(\zeta - 1)$ the matrix with coefficients $a_{i,j}(\zeta - 1) \in \mathbb{Z}_p[\zeta]$ (i.e. we replace T_i by the i-th component of $\zeta - 1$).

Lemma 4.1. *We can choose a basis of $\tilde{\Lambda}_m \otimes \mathbb{Q}_p$ such that $\tilde{\mathcal{L}_\alpha}$ is represented by a $p^{ml}n \times p^{ml}n$-block matrix*

$$A_m = \begin{pmatrix} \Delta_\infty^t(\zeta_1 - 1) & 0 & \dots & \dots & \dots & 0 \\ 0 & \Delta_\infty^t(\zeta_1 - 1) & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta_\infty^t(\zeta_2 - 1) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & \Delta_\infty^t(\zeta_k - 1) \end{pmatrix},$$

where $\zeta_1, \dots, \zeta_k$ represent the different equivalence classes in $\Omega^l[p^m]$ modulo Galois equivalence and each block $\Delta_\infty^t(\zeta_i - 1)$ occurs $\varphi(\mathrm{ord}(\zeta_i))$ times, where φ denotes Euler's totient function.

Proof. Let F be an element in Div_Λ. Note that $\phi_m \circ \mathcal{L}_\alpha(F)$ is a vector with components $\Delta_\infty^t(\zeta_i - 1)F(\zeta_i - 1)$.

We fix an index j $(1 \leqslant j \leqslant k)$ and let $F_j v_i \in \Lambda v_i$ be such that $\phi_m(F_j v_i)$ vanishes at all $\zeta_z - 1$ for $z \neq j$ and is an element $a \in \mathbb{Z}_p \smallsetminus \{0\}$ at $\zeta_j - 1$. Let G be an arbitrary element in Λ. Then we have

$$\tilde{\mathcal{L}_\alpha}(\phi_m(GF_j v_i)) = \phi_m(\mathcal{L}_\alpha(GF_j v_{0,i})) = \phi_m(GF_j \Delta_\infty^t v_i).$$

Recall that $G(\zeta - 1)F_j(\zeta - 1) = 0$ for all ζ that do not lie in the equivalence class of ζ_j. Thus

$$G(\zeta - 1)F_j(\zeta - 1)\Delta_\infty^t(\zeta - 1) = 0$$

for all these ζ. In particular, $\phi_m(GF_j\Delta_\infty^t v_i)$ has only one non-trivial component, namely at $\zeta_j - 1$. Note that $\phi_m((\Lambda/I_m)F_j v_i)$ has finite index in $\mathbb{Z}_p[\zeta_j]$. Using the fact that $\bigoplus_{z=1}^k \mathbb{Z}_p[\zeta_z]$ is $\mathbb{Z}_p$-free, we have shown that if we start with an element in $\tilde{\Lambda}_m$ that is non-trivial at exactly one $\zeta_j - 1$, then the same holds for its image under $\tilde{\mathcal{L}_\alpha}$. From this we can deduce that $\tilde{\mathcal{L}_\alpha}(\mathbb{Q}_p[\zeta_j])^n \subseteq (\mathbb{Q}_p[\zeta_j])^n$.

It remains to compute the corresponding matrix. Let $G_1, \dots, G_{\varphi(\mathrm{ord}(\zeta_j))-1}$ be polynomials such that $G_w(\zeta_j - 1) = \zeta_j^w$ and let $G_0 = 1$. Note that $\phi_m(G_w F_j v_i)$ for $0 \leqslant w \leqslant \varphi(\mathrm{ord}(\zeta_j)) - 1$ spans a sublattice of finite index in $\mathbb{Z}_p[\zeta_j]$. Using the above computations (with G_w for G) we see that

$$\tilde{\mathcal{L}_\alpha}(\zeta_j^w a\phi_m(v_i)) = \sum_{u=1}^n \Delta_\infty^t(\zeta_j - 1)\zeta_j^w a\phi_m(v_u).$$

Therefore, we see that the block $\Delta_\infty^t(\zeta_j - 1)$ has to occur $\varphi(\mathrm{ord}(\zeta_j))$ times in the matrix representing $\tilde{\mathcal{L}_\alpha}$ on $\tilde{\Lambda}_m \otimes \mathbb{Q}_p$. ☐

Lemma 4.2. *We have the following identity*

$$v_p(|J(X_m)|) = \sum_{\zeta \neq 1} v_p(\det(\Delta_\infty^t(\zeta - 1))) + v_p(|J(X_0)|),$$

where the sum runs over all elements in $\Omega^l[p^m]$ that are different from $(1, 1, \dots, 1)$.

Proof. Recall that $N/I_m L$ is finite for all m and that $L/I_m L$ has $\mathbb{Z}_p$-rank one for each m. By definition of L we see that

$$L/I_m L = \mathrm{Div}_\Lambda/(I_m + \mathrm{Pr}_\Lambda) = \mathrm{Div}(X_m)/\mathrm{Pr}(X_m) = \mathrm{Div}(X_m)/\Delta_m^t \mathrm{Div}(X_m).$$

Therefore the kernel of the linear map given by multiplication with Δ_m^t has $\mathbb{Z}_p$-rank 1. Since ϕ_m is an injection, it follows that the matrix A_m introduced in Lemma 4.1 has rank $p^{ml}n - 1$ for all m. In particular, the matrix $\Delta_\infty^t(0)$ has determinant zero, but all the matrices $\Delta_\infty^t(\zeta_i - 1)$ have non-zero determinant.

Fix a basis of $\tilde{\Lambda}_m$ and let A'_m be the matrix describing $\tilde{\mathcal{L}_\alpha}$ in this basis. Then A_m and A'_m are conjugate. Moreover, $\mathrm{Div}(X_m)$ is a free $\mathbb{Z}_p$-module, and

$$\mathrm{Div}(X_m)/\Delta_m^t \mathrm{Div}(X_m) \cong \tilde{\Lambda}_m/A'_m\tilde{\Lambda}_m.$$

We are interested in $(\mathrm{Div}(X_m)/\Delta_m^t\mathrm{Div}(X_m))[p^\infty]$. Let B_m and B'_m be the matrices representing

$$\mathcal{L}_\alpha\colon (\mathrm{Div}_\Lambda/I_m \otimes \mathbb{Q}_p)/\ker(A_m) \longrightarrow (\mathrm{Div}_\Lambda/I_m \otimes \mathbb{Q}_p)/\ker(A_m)$$

and

$$\tilde{\mathcal{L}_\alpha}\colon \tilde{\Lambda}_m/\ker(A'_m) \longrightarrow \tilde{\Lambda}_m/\ker(A'_m)$$

in our chosen basis. Then B_m and B'_m are conjugate and we obtain

$$v_p(|(\mathrm{Div}(X_m)/\Delta_m^t\mathrm{Div}(X_m))[p^\infty]|) = v_p(\det(B'_m)) = v_p(\det(B_m)).$$

We let $\zeta_k = 1$. Then B_m has the form

$$\begin{pmatrix} \Delta_\infty^t(\zeta_1-1) & 0 & \dots & \dots & \dots & 0 \\ 0 & \Delta_\infty^t(\zeta_1-1) & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta_\infty^t(\zeta_2-1) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & B_0 \end{pmatrix},$$

where $v_p(\det(B_0)) = v_p(|J(X_0)|)$. Using the structure of the matrix B_m, the claim follows. □

In particular, together with [17, Theorem 5.6] this proves Greenberg's conjecture, i.e. we obtain the following

Theorem 4.3. *Greenberg's conjecture holds true for the growth of $v_p(|J(X_m)|)$, i.e. there exists a polynomial $Q(X,Y) \in \mathbb{Z}_p[X,Y]$ of total degree l and degree at most one in the second variable such that*

$$v_p(|J(X_m)|) = Q(p^m, m)$$

for all sufficiently large $m \in \mathbb{N}$.

Remark 4.4. Our method gives the same polynomial as the method using Ihara L-functions from [4, Theorem 6.2]. This follows directly from the construction of the polynomial $P(X,Y)$ occurring in the statement of *loc. cit.* The difference in the methods of proof lies in the fact that *loc. cit.* relates this polynomial to Ihara L-functions which in turn describe the number of spanning trees, whereas the proof in the present paper is based on the fact that the matrix Δ_∞ describes the Jacobian and that the cardinality of the Jacobian is precisely the number of spanning trees.

5. An Iwasawa main conjecture

Let as before X be a finite connected graph. Let X_∞ be a $\mathbb{Z}_p^l$-voltage cover of X. For every finite graph we define the zeta function and the Ihara L-functions as in [4]. Let Δ_∞ be defined as in Section 4. Let ψ be a character of $\mathbb{Z}_p^l$ with finite image and let A_ψ be the matrix defined in [4]. From the definitions in [4] we see that

$$P_\psi(u) := \det(I - A_\psi u + (D-I)u^2) = \frac{(1-u^2)^{\chi(X)}}{L_X(u,\psi)}.$$

In this section we assume that no vertex has degree 1. If ψ is not the trivial character then $P_\psi(1) \neq 0$ (see for example [16, Section 5]). Using the definition of A_ψ, it follows that

$$P_\psi(1) = \psi(\det(\Delta_\infty)).$$

If we interpret the function $\lim_{u\to 1}\frac{(1-u^2)^{\chi(X)}}{L_X(u,\psi)}$ as the algebraic part of the Ihara L-function, then $\det(\Delta_\infty)\in\Lambda$ can be seen as a p-adic L-series interpolating special values of the algebraic part of $L_X(u,\psi)$.

To understand the algebraic side we have to analyse the Λ-module structure of Pic_Λ and N in more detail. Recall that Pic_Λ was defined as $\mathrm{Pic}(X_\infty)\otimes\Lambda$.

Lemma 5.1.

$$M_\Lambda\cong \mathrm{Div}^0_\Lambda.$$

Proof. Recall that $\mathrm{Div}^0(X_\infty)+I_m\mathrm{Div}_\Lambda=M_\Lambda$ for all m (by observation (c) in the proof of Theorem 3.3). It follows that the image of Div^0_Λ in Div_Λ is equal to M_Λ. Note that $\mathrm{Div}(X_\infty)/\mathrm{Div}^0(X_\infty)$ is annihilated by $I_0=(T_1,T_2,\dots,T_l)$. It follows that

$$I_0^{-1}\mathrm{Tor}_1^R(\mathbb{Z}_p,\Lambda)=\mathrm{Tor}_1^R(\mathbb{Z}_p\otimes I_0^{-1}R,\Lambda\otimes RI_0^{-1})=0.$$

Therefore $\mathrm{Tor}_1^R(\mathbb{Z}_p,\Lambda)$ is torsion as its annihilator intersects with I_0 non-trivially. But

$$\mathrm{Div}^0(X_m)[I_0]=(\prod_{i=1}^{l}\nu_{m,0}(T_i))\mathrm{Div}(X_m)\cap\mathrm{Div}^0(X_m).$$

It follows that Div^0_Λ does not contain non-trivial I_0-torsion. In particular, the image of $\mathrm{Tor}_1^R(\mathrm{Div}(X_\infty)/\mathrm{Div}^0(X_\infty),\Lambda)$ in Div^0_Λ is trivial and we obtain a natural injection

$$\mathrm{Div}^0_\Lambda\longrightarrow\mathrm{Div}_\Lambda.$$

Thus, $\mathrm{Div}^0_\Lambda\cong M_\Lambda$. □

Lemma 5.2. *Let X_m be the level m subcover of X_∞/X. Then we have*

$$J_\Lambda:=J(X_\infty)\otimes\Lambda=\varprojlim_m J(X_m)\cong\varprojlim_m N/(I_mL)\cong N.$$

In particular, J_Λ is a finitely generated Λ-module and we have a short exact sequence

$$0\longrightarrow\mathrm{Pr}_\Lambda\longrightarrow\mathrm{Div}^0_\Lambda\longrightarrow J_\Lambda\longrightarrow 0.$$

Proof. By definition $J(X_\infty)=\mathrm{Div}^0(X_\infty)/\mathrm{Pr}(X_\infty)$ as R-modules. Consider the exact sequence

$$0\longrightarrow\mathrm{Pr}(X_\infty)\longrightarrow\mathrm{Div}^0(X_\infty)\longrightarrow J(X_\infty)\longrightarrow 0.\tag{3}$$

Upon tensoring (3) with Λ we obtain, by Lemma 5.1, an exact sequence

$$\mathrm{Pr}_\Lambda\longrightarrow M_\Lambda\longrightarrow J_\Lambda\longrightarrow 0.$$

Recall that $\mathrm{Pr}_\Lambda=\mathcal{L}_\alpha(\mathrm{Div}_\Lambda)\subseteq M_\Lambda$ injects into Div_Λ by Lemma 3.2. In particular, the natural map $\mathrm{Pr}_\Lambda\longrightarrow M_\Lambda$ is injective and we obtain an isomorphism

$$M_\Lambda/\mathrm{Pr}_\Lambda\cong J_\Lambda.\tag{4}$$

Taking projective limits of the exact sequence

$$0\longrightarrow\mathrm{Pr}(X_m)\longrightarrow\mathrm{Div}^0(X_m)\longrightarrow J(X_m)\longrightarrow 0$$

gives us a sequence

$$0\longrightarrow\varprojlim_m\mathrm{Pr}(X_m)\longrightarrow\varprojlim_m\mathrm{Div}^0(X_m)\longrightarrow\varprojlim_m J(X_m).$$

By Theorem 3.3 $\varprojlim_m J(X_m)\cong N$. Note that $\mathrm{Pr}(X_m)$ is generated by the $p_{i,0}$ and that these generators do not have a relation over Λ. It follows that $\varprojlim_m\mathrm{Pr}(X_m)\cong\mathrm{Pr}_\Lambda$. The module $\mathrm{Div}^0(X_m)$ is generated by $I_0\mathrm{Div}^0(X_m)$, v_i-v_j for $1\leqslant i<j\leqslant n$ and the elements $p_{i,0}$. It follows that $\varprojlim_m\mathrm{Div}^0(X_m)$ is generated by the same elements

over Λ and we obtain that $\varprojlim_m \mathrm{Div}^0(X_m) \cong M_\Lambda$. Summarising, we obtain an exact sequence

$$0 \longrightarrow \mathrm{Pr}_\Lambda \longrightarrow M_\Lambda \longrightarrow \varprojlim_m J(X_m) \longrightarrow 0.$$

Together with (4) the claim follows. □

Fixing this notation, we can prove an *Iwasawa main conjecture* for graphs:

THEOREM 5.3. *Let $l \geqslant 2$. Then*

$$\mathrm{Char}(J_\Lambda) = (\det(\Delta_\infty)).$$

Proof. Recall that Δ_∞^t is the matrix representing the Λ-linear map

$$\mathcal{L}_\alpha \colon \mathrm{Div}_\Lambda \longrightarrow \mathrm{Div}_\Lambda,$$

such that $\mathcal{L}_\alpha(\mathrm{Div}_\Lambda) = \mathrm{Pr}_\Lambda$. By Lemma 3.2 we have a short exact sequence

$$0 \longrightarrow \mathrm{Pr}_\Lambda \longrightarrow \mathrm{Div}_\Lambda \longrightarrow (\mathrm{Div}(X_\infty)/\mathrm{Pr}(X_\infty)) \otimes \Lambda \longrightarrow 0.$$

By Lemma 5.2 we have a second exact sequence

$$0 \longrightarrow \mathrm{Pr}_\Lambda \longrightarrow \mathrm{Div}_\Lambda^0 \longrightarrow J_\Lambda \longrightarrow 0.$$

There is a natural injection

$$\psi \colon J_\Lambda \longrightarrow (\mathrm{Div}(X_\infty)/\mathrm{Pr}(X_\infty)) \otimes \Lambda,$$

whose cokernel has $\mathbb{Z}_p$-rank one. In particular

$$\mathrm{Char}((\mathrm{Div}(X_\infty)/\mathrm{Pr}(X_\infty)) \otimes \Lambda) = \mathrm{Char}(J_\Lambda),$$

since $l \geqslant 2$ and therefore each finitely generated $\mathbb{Z}_p$-module is pseudo-null over Λ. □

REMARK 5.4. Note that this proof crucially depends on the fact that $l \geqslant 2$. In the case $l = 1$, we see that $\mathrm{Char}(\mathrm{coker}\psi) = (T)$. Thus, in this case

$$(T) \cdot \mathrm{Char}(J_\Lambda) = (\det(\Delta_\infty)).$$

6. FUKUDA'S THEOREM AND LOCAL BOUNDEDNESS RESULTS

We start from the Λ-module isomorphisms

$$J(X_m) \cong N/L_m$$

from Theorem 3.3 (recall that X_0 has to be connected for this theorem to hold). Let $\mathfrak{m} = (p, T_1, \dots, T_l)$ be the maximal ideal of the Iwasawa algebra Λ. For any ideal $I \subseteq \mathfrak{m}$ of Λ and any finitely generated torsion Λ-module A, we define

$$\mathrm{rank}_I(A) = v_p(|A/(I \cdot A)|),$$

whenever this is finite.

The following key result is proven as [11, Theorem 2.5].

THEOREM 6.1. *Suppose that there exists an ideal $I \subseteq \Lambda$ such that*

$$\mathrm{rank}_I(J(X_m)) = \mathrm{rank}_I(J(X_{m+1}))$$

for some $m \in \mathbb{N}$. Then

$$\mathrm{rank}_I(J(X_k)) = \mathrm{rank}_I(J(X_m))$$

for every $k \geqslant m$, and in fact $\mathrm{rank}_I(N)$ *is finite and equal to* $\mathrm{rank}_I(J(X_m))$.

Proof. Since $N/(I \cdot N + L_m)$ and $N/(I \cdot N + L_{m+1})$ have the same cardinality, we may conclude that

$$I \cdot N + L_m = I \cdot N + L_{m+1}.$$

Letting $Z := (I \cdot N + L_m)/(I \cdot N)$ and recalling the definition of L_m, it follows that

$$Z \subseteq \mathfrak{m} \cdot Z.$$

Since Z is a compact Λ-module, Nakayama's Lemma implies that $Z = 0$, i.e. L_m is contained in $I \cdot N$. The assertion of the theorem follows immediately. □

Lemma 6.2. *Let $N = M_\Lambda/\mathrm{Pr}_\Lambda \cong J_\Lambda$ (by (4)). Then $N \subseteq L = \mathrm{Div}_\Lambda/\mathrm{Pr}_\Lambda$. The two Λ-modules N and L do not contain any non-zero pseudo-null submodules.*

Proof. Note that Div_Λ and Pr_Λ are both free Λ-modules of rank n (see also Lemma 3.2). Suppose that $x \in \mathrm{Div}_\Lambda$ is an element such that $x + \mathrm{Pr}_\Lambda$ generates a pseudo-null submodule in L. Choose two coprime elements a and c such that $ax, cx \in \mathrm{Pr}_\Lambda$. Fix a basis $b_1, \ldots, b_n$ of Pr_Λ and choose coefficients

$$ax = \sum_{i=1}^{n} a_i b_i, \quad cx = \sum_{i=1}^{n} c_i b_i.$$

Note that the coefficients a_i and c_i are unique. It follows that

$$ca_i = ac_i \text{ for } 1 \leqslant i \leqslant n.$$

As $(c, a) = 1$, it follows that $a \mid a_i$ for $1 \leqslant i \leqslant n$.

Therefore we can write

$$ax = \sum_{i=1}^{n} a \cdot \frac{a_i}{a} \cdot b_i = a \cdot \sum_{i=1}^{n} \frac{a_i}{a} \cdot b_i,$$

and thus

$$a \cdot \left(x - \sum_{i=1}^{n} \frac{a_i}{a} \cdot b_i \right) = 0.$$

Since Div_Λ is a free Λ-module, this is possible only if already

$$x = \sum_{i=1}^{n} \frac{a_i}{a} \cdot b_i$$

is contained in Pr_Λ. Therefore L (and N) do not contain any non-trivial pseudo-null submodule. □

The property proved above has many important consequences. For example, since L does not contain any non-trivial pseudo-null submodule, we obtain, starting from a pseudo-isomorphism, an *injection* $\psi \colon L \longrightarrow E$ into some elementary Λ-module E such that the cokernel of ψ is pseudo-null.

Now we introduce a topology on the set of $\mathbb{Z}_p^l$-covers of our fixed finite graph X. Let $\Gamma \cong \mathbb{Z}_p^l$ be as in Section 2. The voltage graphs X_∞ and $\tilde{X}_\infty$ assigned to two voltage covers

$$\alpha, \tilde{\alpha} \colon S \longrightarrow \Gamma$$

will be considered as 'close' if the induced maps

$$\alpha_m, \tilde{\alpha}_m \colon S \longrightarrow \Gamma/\Gamma^{p^m}$$

coincide for a large integer m. More formally, let $\mathcal{E}^l(X)$ denote the set of voltage Γ-covers of X. For any voltage cover $X_\infty \in \mathcal{E}^l(X)$ and each $m \in \mathbb{N}$, we denote by $\mathcal{E}(X_\infty, m)$ the set of voltage covers $\tilde{X}_\infty$ which satisfy the above condition (i.e.

$\alpha_m = \tilde{\alpha}_m$). It follows from our definition of the voltage cover graphs X_m (see Definition 2.2) that

$$\tilde{X}_k = X_k$$

for each $k \leqslant m$ if $\tilde{X}_\infty \in \mathcal{E}(X_\infty, m)$.

The sets $\mathcal{E}(X_\infty, m)$, with $X_\infty \in \mathcal{E}^l(X)$ and $m \in \mathbb{N}$, form the basis of a topology on $\mathcal{E}^l(X)$. This topology is motivated by what we call *Greenberg's topology* on the set $\mathcal{E}^l(K)$ of $\mathbb{Z}_p^l$-extensions of a number field K (the latter topology has been introduced in this classical setting in [8]).

Lemma 6.3. *The set $\mathcal{E}^l(X)$ is compact with respect to Greenberg's topology.*

Proof. Fix a finite graph X and a voltage cover

$$\alpha\colon S \longrightarrow \Gamma.$$

For every $m \in \mathbb{N}$, the set $\mathcal{E}^{l,m}(S)$ of maps

$$\alpha_m\colon S \longrightarrow \Gamma/\Gamma^{p^m}$$

is finite. Therefore $\mathcal{E}^{l,m}(S)$ is compact (with regard to the discrete topology) for each m. Since $\mathcal{E}^l(X) = \varprojlim_m \mathcal{E}^{l,m}(S)$, it follows that $\mathcal{E}^l(X)$ is also compact. □

In the following, we want to compare the (generalised) Iwasawa invariants of voltage covers $X_\infty, \tilde{X}_\infty \in \mathcal{E}^l(X)$ which are close with respect to Greenberg's topology. We start with the case $l \geqslant 2$.

Theorem 6.4. *Fix an element $X_\infty \in \mathcal{E}^l(X)$, $l \geqslant 2$. Then there exist integers $i, k \in \mathbb{N}$ such that with $U := \mathcal{E}(X_\infty, i)$ the following two statements hold for each $\tilde{X}_\infty$ in U.*
(a) $m_0(\tilde{X}_\infty) \leqslant m_0(X_\infty)$, and
(b) $l_0(\tilde{X}_\infty) \leqslant k$ holds if $m_0(\tilde{X}_\infty) = m_0(X_\infty)$.

Proof. Recall that we want to study the generalised Iwasawa invariants of the Λ-module $N = M_\Lambda/\mathrm{Pr}_\Lambda$ (see also Corollary 3.5). Let $f_N \in \Lambda$ be the characteristic power series of N. Recall that $\Lambda = \mathbb{Z}_p[[T_1, \ldots, T_l]]$. We say that f_N is *in Weierstraß normal form with respect to T_l* if f_N is associated with a product $p^{m_0(N)} \cdot g_N$ for some element g_N of the form

$$g_N = T_l^k + h_{k-1}T_l^{k-1} + \cdots + h_1 \cdot T_l + h_0, \tag{5}$$

where $k \geqslant 1$ and $h_0, \ldots, h_{k-1} \in (p, T_1, \ldots, T_{l-1}) \subseteq \mathbb{Z}_p[[T_1, \ldots, T_{l-1}]]$. It follows from [1, Lemma 1] that given any set $\gamma_1, \ldots, \gamma_l$ of topological generators of

$$\mathrm{Gal}(X_\infty/X) \cong \mathbb{Z}_p^l,$$

we can change this set of generators to a set of generators $\gamma_1', \ldots, \gamma_{l-1}', \gamma_l$ (i.e. without changing the last element) such that with respect to these new generators, f_N is in Weierstraß normal form with respect to the last variable T_l.

Now let E_N be an elementary Λ-module attached to N, and fix a pseudo-isomorphism $\varphi : E_N \longrightarrow N$. Then the cokernel $B := N/\varphi(E_N)$ is a pseudo-null Λ-module, and it follows from [12, Lemma 4.4] that the topological generators $\gamma_1, \ldots, \gamma_l$ of $\mathrm{Gal}(X_\infty/X) \cong \mathbb{Z}_p^l$ can be chosen such that

- $\gamma_l = T_l - 1$ remains unchanged, f_N is still in Weierstraß normal form with respect to T_l, and the degree k in the representation (5) of g_N does not change, and
- $B/(T_1, \ldots, T_{l-2}, T_{l-1} - p^x)$ is finite for all but finitely many $x \in \mathbb{N}$.

Fix such an integer x. It follows from [12, Lemma 4.5] that

$$\text{rank}_s(N) := v_p(|N/(T_1, \ldots, T_{l-2}, T_{l-1} - p^x, \nu_{2s,s}(T_l)|)$$

is finite for each sufficiently large $s \in \mathbb{N}$. Moreover, it follows from the short exact sequence

$$0 \longrightarrow E_N \xrightarrow{\varphi} N \longrightarrow B \longrightarrow 0$$

that

$$\text{rank}_s(N) \leqslant \text{rank}_s(E_N) + \text{rank}_s(B)$$

(in particular, all the terms in this inequality are finite; cf. also the proof of [12, Theorem 3.2]). Note that

$$\text{rank}_s(B) = v_p(|B/(T_1, \ldots, T_{l-2}, T_{l-1} - p^x)|) =: C$$

for each sufficiently large $s \in \mathbb{N}$ (it will suffice if $s > C$), i.e. $\text{rank}_s(B)$ is bounded in s.

From now on we will consider s large enough to ensure that
(a) $\text{rank}_s(N)$ is finite,
(b) $s > C$, and
(c) $p^s(p-1) > s \cdot k + C$, where k is as in (5).
Then it follows from the proof of [12, Theorem 4.6] (cf. in particular equation (4.4)) that

$$\text{rank}_s(E_N) = m_0(X_\infty) \cdot (p^{2s} - p^s) + s \cdot k.$$

Since $J(X_m) \cong N/L_m$ and therefore $\text{rank}_s(J(X_m)) \leqslant \text{rank}_s(N)$ for each $m \in \mathbb{N}$, it follows that we can choose m sufficiently large such that

$$\text{rank}_s(J(X_m)) = \text{rank}_s(J(X_k)) = \text{rank}_s(N)$$

for each $k \geqslant m$ (cf. also the proof of Theorem 6.1). Fix such an integer m, and consider $U := \mathcal{E}^l(X_\infty, m+1)$. Note that the integer m depends on s; this is not a problem since s has been chosen and fixed above.

Let $\tilde{X}_\infty \in U$ be arbitrary. We denote the corresponding Λ-module $N(\tilde{X}_\infty)$ by $\tilde{N}$. Since $\tilde{X}_\infty \in U$, we have

$$\text{rank}_s(J(\tilde{X}_m)) = \text{rank}_s(J(\tilde{X}_{m+1})) = \text{rank}_s(N).$$

Therefore Theorem 6.1 implies that

$$\text{rank}_s(\tilde{N}) = \text{rank}_s(N).$$

Moreover, it follows from [12, Theorem 3.2] that

$$\text{rank}_s(E_{\tilde{N}}) \leqslant \text{rank}_s(\tilde{N}),$$

where we denote by $E_{\tilde{N}}$ an elementary Λ-module attached to $\tilde{N}$. Summarising, we have shown that

$$\text{rank}_s(E_{\tilde{N}}) \leqslant m_0(X_\infty) \cdot (p^{2s} - p^s) + s \cdot k + C.$$

Since $\text{rank}_s(E_{\tilde{N}}) \geqslant m_0(\tilde{N}) \cdot (p^{2s} - p^s)$, it follows from our choice of s (in particular, cf. property (c)) that

$$m_0(\tilde{X}_\infty) = m_0(\tilde{N}) \leqslant m_0(X_\infty).$$

As $\tilde{X}_\infty \in U$ had been chosen arbitrarily, this proves assertion (a) of the theorem.

Now suppose that $m_0(\tilde{X}_\infty) = m_0(X_\infty)$ for some fixed $\tilde{X}_\infty \in U$, and write the characteristic power series of $\tilde{N}$ as

$$f_{\tilde{N}} = p^{m_0(\tilde{N})} \cdot g_{\tilde{N}},$$

with $p \nmid \tilde{g}_N$. It follows from the proof of [12, Theorem 4.6] that either $g_{\tilde{N}}$ is in Weierstraß normal form with respect to T_l, say,

$$g_{\tilde{N}} = T_l^{\tilde{k}} + \tilde{h}_{\tilde{k}-1} T_l^{\tilde{k}-1} + \cdots + \tilde{h}_0$$

for suitable $\tilde{h}_0, \ldots, \tilde{h}_{\tilde{k}-1} \in \mathbb{Z}_p[[T_1, \ldots, T_{l-1}]]$, in which case we may conclude (from the same proof) that $\tilde{k} \leqslant k$, or

$$\operatorname{rank}_s(E_{\tilde{N}}) \geqslant m_0(\tilde{N}) \cdot (p^{2s} - p^s) + p^s(p-1).$$

In view of our choice of s, the latter alternative is not possible. Since moreover

$$l_0(\tilde{N}) \leqslant \tilde{k}$$

by [12, Lemma 4.1], assertion (b) from the theorem follows. □

In the case $l = 1$ of $\mathbb{Z}_p$-covers of X, we can actually prove a more precise statement.

Theorem 6.5. *Assume that $l = 1$ and let $X_\infty \in \mathcal{E}^1(X)$. Then there exists an integer $m \in \mathbb{N}$ such that the following statements hold:*

(a) For each $\tilde{X}_\infty \in U := \mathcal{E}^1(X_\infty, m)$, we have

$$\mu(\tilde{X}_\infty) \leqslant \mu(X_\infty).$$

(b) $\lambda(\tilde{X}_\infty) = \lambda(X_\infty)$ for each $\tilde{X}_\infty \in U$ which satisfies $\mu(\tilde{X}_\infty) = \mu(X_\infty)$.

Proof. For statement (a), we could use the previous theorem, since

$$\mu(N) = \mu(\varprojlim_m J(X_m))$$

holds also in the case $l = 1$ (see also Corollary 3.5). However, it is not hard to give an argument which will prove both (a) and (b).

Fix a pseudo-isomorphism $\varphi : N \longrightarrow E_N$, where E_N is an elementary Λ-module. Then $N^\circ = \ker(\varphi)$ is trivial, i.e. φ is an injection. Indeed, N° is actually equal to the maximal finite Λ-submodule of N; therefore the claim follows from Lemma 6.2. We stress that when compared to the proof of Theorem 6.4, we have interchanged the roles of the set of definition and the image set in the definition of φ.

Moreover, the quotient $N/\nu_{2s,s}(T)$ is finite for each sufficiently large s since N is a torsion Λ-module and the polynomials $\nu_{r+1,r}(T)$ are pairwise coprime (recall that $\nu_{2s,s}(T) = \nu_{2s,2s-1}(T) \cdots \nu_{s+1,s}(T)$).

Letting

$$\operatorname{rank}_s(N) := v_p(|N/\nu_{2s,s}(T)|)$$

(and similarly for other torsion Λ-modules), we have

$$\begin{aligned}\operatorname{rank}_s(N) &= \operatorname{rank}_s(E_N) + \operatorname{rank}_s(N^\circ) \\ &= \operatorname{rank}_s(E_N) + v_p(|N^\circ|),\end{aligned}$$

where the last equality holds for each sufficiently large s (see [11, proof of Theorem 3.10] for the first equation). Since N° is trivial, we actually may conclude that

$$\operatorname{rank}_s(N) = \operatorname{rank}_s(E_N)$$

for each $s \in \mathbb{N}$.

Now let s be large enough such that

(a) $\operatorname{rank}_s(N)$ is finite, and

(b) $p^s(p-1) > s \cdot \lambda(N)$.

As in the proof of Theorem 6.4 we can use the stabilisation property from Theorem 6.1 in order to define a neighbourhood $U = \mathcal{E}^1(X_\infty, m)$ of X_∞ such that

$$\operatorname{rank}_s(\tilde{X}_\infty) = \operatorname{rank}_s(X_\infty)$$

for each $\tilde{X}_\infty \in U$. By the above, since the maximal pseudo-null submodule of $\tilde{N}$ is also trivial, we have

$$\begin{aligned}\operatorname{rank}_s(E_{\tilde{N}}) &= \operatorname{rank}_s(E_N) \\ &= \mu(N) \cdot (p^{2s} - p^s) + \lambda \cdot s \qquad (6)\end{aligned}$$

for each $\tilde{X}_\infty \in U$, where $E_{\tilde{N}}$ is an elementary Λ-module attached to $\tilde{N}$, and where the last equation holds in view of the property (b) above (see the proof of [11, Theorem 3.10]).

Since $\mathrm{rank}_s(E_{\tilde{N}}) \geqslant \mu(\tilde{N}) \cdot (p^{2s} - p^s)$, the choice of s implies that $\mu(\tilde{N}) \leqslant \mu(N)$. This proves (a).

Now suppose that $\mu(\tilde{N}) = \mu(N)$. If $\lambda(\tilde{N}) \geqslant p^s(p-1)$, then

$$\mathrm{rank}_s(E_{\tilde{N}}) \geqslant \mu(\tilde{N}) \cdot (p^{2s} - p^s) + p^s(p-1)$$

by the proof of [11, Theorem 3.10]. Since this is not possible in view of our choice of s, it follows that $\lambda(\tilde{N}) < p^s(p-1)$. Then the proof of [11, Theorem 3.10] implies that

$$\mathrm{rank}_s(E_{\tilde{N}}) = \mu(\tilde{N}) \cdot (p^{2s} - p^s) + s \cdot \lambda(\tilde{N}).$$

Since $\mathrm{rank}_s(E_{\tilde{N}}) = \mathrm{rank}_s(E_N)$ by the above, we may conclude that $\lambda(\tilde{N}) = \lambda(N)$. □

Corollary 6.6. *The m_0-invariant is bounded globally on $\mathcal{E}^l(X)$, i.e. there exists a constant C such that*

$$m_0(X_\infty) \leqslant C$$

for each $X_\infty \in \mathcal{E}^l(X)$.

Moreover, if the m_0-invariant is constant on $\mathcal{E}^l(X)$, then the l_0-invariant is bounded globally on $\mathcal{E}^l(X)$.

Proof. This follows by combining Theorem 6.4 with Lemma 6.3. □

Again, we prove a more precise statement in the one-dimensional case.

Corollary 6.7. *Let $l = 1$, and let $X_\infty \in \mathcal{E}^l(X)$. We fix a neighbourhood U of X_∞ as in Theorem 6.5. Then there exists a potentially smaller neighbourhood $U' \subseteq U$ of X_∞ with the following property:*

Either $\mu(\tilde{X}_\infty) = \mu(X_\infty)$ for each $\tilde{X}_\infty \in U'$, or λ is unbounded on U'.

Proof. We will use the notation from the proof of Theorem 6.5. Suppose that s has been chosen such that $p^s(p-1) > s \cdot \lambda(N)$, and that $\mu(\tilde{N}) < \mu(N)$. For any $\tilde{X}_\infty \in U$ with $\mu(\tilde{N}) < \mu(N)$, we have that either $\lambda(\tilde{N}) \geqslant p^s(p-1)$, or the equation

$$\mathrm{rank}_s(E_{\tilde{N}}) = \mathrm{rank}_s(E_N)$$

implies that

$$\begin{aligned}\lambda(\tilde{N}) &\geqslant \lambda(N) + \Big\lfloor \frac{p^{2s} - p^s}{s} \Big\rfloor \cdot (\mu(N) - \mu(\tilde{N})) \\ &\geqslant \Big\lfloor \frac{p^{2s} - p^s}{s} \Big\rfloor.\end{aligned}$$

Here $\lfloor a \rfloor$ means the largest integer which is smaller than or equal to a.

Now choose an integer $s' > s$. Then we obtain a possibly smaller neighbourhood of X_∞ such that

$$\mathrm{rank}_{s'}(E_{\tilde{N}}) = \mathrm{rank}_{s'}(E_N)$$

for each $\tilde{X}_\infty$ which is contained in this smaller neighbourhood. In this neighbourhood, we will have that

$$\lambda(\tilde{N}) \geqslant \min(p^{s'}(p-1), \Big\lfloor \frac{p^{2s'} - p^{s'}}{s'} \Big\rfloor).$$

Letting s' tend to infinity, we may conclude that the λ-invariant is unbounded on a neighbourhood of X_∞. □

7. A weak control theorem

Let $m \in \mathbb{N}$. Recall that we write $\mathrm{Pic}(X_m) = \mathrm{Div}(X_m)/\mathrm{Pr}(X_m)$, $J_\Lambda = J(X_\infty) \otimes \Lambda$ and $\mathrm{Pic}_\Lambda = \mathrm{Pic}(X_\infty) \otimes \Lambda$. Note that $\mathrm{Pic}(X_m)$ is an infinite abelian group of $\mathbb{Z}_p$-rank 1 for each m.

We have seen in Lemma 5.2 that $J_\Lambda \cong \varprojlim_m J(X_m)$. Analogously to Lemma 5.2 one can show that $\mathrm{Pic}_\Lambda \cong \varprojlim_m \mathrm{Pic}(X_m)$. Recall from Lemma 3.2 that we have a short exact sequence

$$0 \longrightarrow \mathrm{Pr}_\Lambda \longrightarrow \mathrm{Div}_\Lambda \longrightarrow \mathrm{Pic}_\Lambda \longrightarrow 0.$$

Moreover, we have seen in the proof of Lemma 5.1 that the natural map

$$\mathrm{Div}_\Lambda^0 \longrightarrow \mathrm{Div}_\Lambda$$

is injective. Thus, we obtain a natural injection

$$J_\Lambda \longrightarrow \mathrm{Pic}_\Lambda.$$

Finally, we recall the notion of the ideals $I_m = (\omega_m(T_1), \ldots, \omega_m(T_l)) \subseteq \Lambda$ from Sections 2 and 3.

Lemma 7.1. *The natural maps*

$$r_m \colon \mathrm{Pic}_\Lambda/I_m\mathrm{Pic}_\Lambda \longrightarrow \mathrm{Pic}(X_m)$$

are isomorphisms.

Proof. Consider the tautological commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pr}_\Lambda & \longrightarrow & \mathrm{Div}_\Lambda & \longrightarrow & \mathrm{Pic}_\Lambda & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_m} & & \downarrow{\scriptstyle g_m} & & \downarrow{\scriptstyle r_m} & & \\ 0 & \longrightarrow & \mathrm{Pr}(X_m) & \longrightarrow & \mathrm{Div}(X_m) & \longrightarrow & \mathrm{Pic}(X_m) & \longrightarrow & 0 \end{array}$$

It follows from the proof of Theorem 3.3 that h_m and g_m are surjective. Hence, r_m is surjective. Applying the snake lemma we obtain a short exact sequence

$$0 \longrightarrow \ker(h_m) \longrightarrow \ker(g_m) \longrightarrow \ker(r_m) \longrightarrow 0.$$

Recall that Div_Λ is Λ-free in the generators v_i. Likewise $\mathrm{Div}(X_m)$ is Λ/I_m-free in the same generators. It follows that $\ker(g_m) = I_m\mathrm{Div}_\Lambda$ and the above sequence becomes

$$0 \longrightarrow \mathrm{Pr}_\Lambda \cap I_m\mathrm{Div}_\Lambda \longrightarrow I_m\mathrm{Div}_\Lambda \longrightarrow \ker(r_m) \longrightarrow 0.$$

Thus, $\ker(r_m)$ is the image of $I_m\mathrm{Div}_\Lambda$ in Pic_Λ, and therefore $\ker(r_m) \cong I_m\mathrm{Pic}_\Lambda$. □

Lemma 7.2. *The natural maps*

$$r'_m \colon J_\Lambda/I_m \longrightarrow J(X_m)$$

are surjective. The p-rank of their kernels is bounded by l.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_\Lambda & \longrightarrow & \mathrm{Pic}_\Lambda & \longrightarrow & \Lambda/I_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle r'_m} & & \downarrow{\scriptstyle r_m} & & \downarrow & & \\ 0 & \longrightarrow & J(X_m) & \longrightarrow & \mathrm{Pic}(X_m) & \longrightarrow & \Lambda/I_0 & \longrightarrow & 0 \end{array}$$

The right vertical map is an isomorphism. Thus, r'_m is surjective and

$$\ker(r'_m) = \ker(r_m) = (I_m\mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda)/\mathrm{Pr}_\Lambda.$$

Let x be a generator of $\mathrm{Pic}_\Lambda/J_\Lambda$. Then $\ker(r'_m)/I_mJ_\Lambda$ is generated by

$$\{\omega_m(T_1)x, \ldots, \omega_m(T_l)x\}.$$

□

We are not only interested in the projection to finite level, but also in the projection to $\mathbb{Z}_p$-subcovers. Let Y_∞ be a $\mathbb{Z}_p$-subcover of a given $\mathbb{Z}_p^l$-cover X_∞. Let Λ and $\Lambda(Y_\infty)$ be the corresponding Iwasawa algebras. Without loss of generality we can assume that the kernel of the induced map $\Lambda \longrightarrow \Lambda(Y_\infty)$ on the Iwasawa algebras is $(T_2, \dots, T_l)$. Let $I'_m \subseteq \Lambda$ be the ideal generated by $\omega_m(T_2), \dots, \omega_m(T_l)$. We write $J_\Lambda(Y_\infty)$ and $J_\Lambda(X_\infty)$ etc. to make the corresponding graphs clear.

Lemma 7.3. *The natural map*

$$t\colon \mathrm{Pic}_\Lambda(X_\infty)/I'_0 \longrightarrow \mathrm{Pic}_\Lambda(Y_\infty)$$

is an isomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pr}_\Lambda(X_\infty) & \longrightarrow & \mathrm{Div}_\Lambda(X_\infty) & \longrightarrow & \mathrm{Pic}_\Lambda(X_\infty) & \longrightarrow & 0 \\
 & & \downarrow h & & \downarrow g & & \downarrow t & & \\
0 & \longrightarrow & \mathrm{Pr}_\Lambda(Y_\infty) & \longrightarrow & \mathrm{Div}_\Lambda(Y_\infty) & \longrightarrow & \mathrm{Pic}_\Lambda(Y_\infty) & \longrightarrow & 0
\end{array}$$

Again all three vertical maps are surjective and the kernel of g is $I'_0\mathrm{Div}_\Lambda(X_\infty)$. The remaining part of the proof is the same as for Lemma 7.1. □

Using the same ideas as in the proof of Lemma 7.2 we obtain

Lemma 7.4. *The natural map*

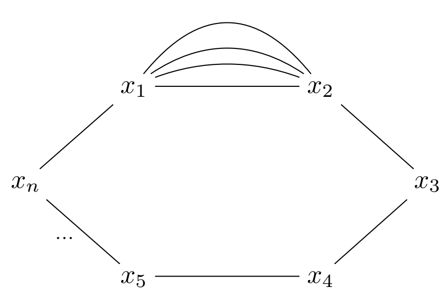
$$t'\colon J_\Lambda(X_\infty)/I'_0 \longrightarrow J_\Lambda(Y_\infty)$$

is surjective and the kernel has p-rank at most $l-1$.

8. Examples

In this section we explicitly compute the (generalised) Iwasawa invariants of certain concrete examples. The starting point is the main conjecture (see Theorem 5.3). In view of this result, we can compute the characteristic power series of a $\mathbb{Z}_p^l$-cover (if $l = 1$, then we have to keep in mind the Remark 5.4).

In this section, we consider a base graph of the following form:



In other words, we let $X = X_0$ be a graph with n vertices $x_1, \dots, x_n$, which are connected in a cycle, with one multiple edge, say, between x_1 and x_2. In this section we denote the number of edges between x_1 and x_2 by k, and we let $e_1, \dots, e_k$ be these edges. We give each edge an orientation as follows: For $1 \leqslant i \leqslant n-1$ any edge between x_i and x_{i+1} starts at x_i and ends at x_{i+1}. The edge between x_n and x_1 starts at x_n and ends at x_1. Let S be the set of these oriented edges. Let

$$\alpha\colon S \longrightarrow \Gamma \cong \mathbb{Z}_p^l$$

be a voltage assignment such that $\alpha(e)=1$ for each edge e different from $e_1,\dots,e_k$. As usual we write the group Γ multiplicatively. We have to study the $n\times n$-matrix

$$\Delta_\infty = D - A_\alpha = \begin{pmatrix} a & x & 0 & 0 & 0 & -1 \\ y & a & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Here we denote by 1 the trivial element of the group Γ, and $x,y\in\mathbb{Z}_p[\Gamma]$ are the group ring elements corresponding to the voltage assignment of the multi-edge. More precisely,

$$x=-\alpha(e_1)-\cdots-\alpha(e_k),\quad y=-\alpha(e_1)^{-1}-\cdots-\alpha(e_k)^{-1},$$

and $a=k+1$.

We want to compute the determinant of Δ_∞, which is almost tridiagonal. To this purpose, we use an idea from the proof of [22, Theorem 1]: Let ρ be the permutation matrix which has entries

$$\rho_{i,j}=\begin{cases}1 & i=n, j=1,\\ 1 & j=i+1,\\ 0 & \text{otherwise.}\end{cases}$$

Then the determinant of ρ is equal to $(-1)^{n-1}$, and the matrix product $\Delta_\infty\cdot\rho$ is of the form

$$\Delta_\infty\cdot\rho = \left(\begin{array}{cc|cccccc} -1 & a & x & 0 & \dots & \dots & \dots & 0 \\ 0 & y & a & -1 & 0 & \dots & \dots & 0 \\ \hline 0 & 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & -1 & 2 & -1 & & \vdots \\ \vdots & & \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & & & & -1 & 2 & -1 \\ -1 & 0 & \vdots & & & & -1 & 2 \\ 2 & -1 & 0 & \cdots & & \cdots & 0 & -1 \end{array}\right),$$

which can be viewed as a block matrix $\begin{pmatrix} A & B \\ C & D\end{pmatrix}$.

Now we apply *Schur's determinant formula* (see [19, Theorem 4.1]):

$$\det\left(\begin{pmatrix} A & B \\ C & D\end{pmatrix}\right)=\det(D)\cdot\det(A-B\cdot D^{-1}\cdot C).$$

It is easy to see that

$$D^{-1}=\begin{pmatrix} -1 & -2 & -3 & -4 & \cdots & -(n-2) \\ 0 & -1 & -2 & \cdots & & -(n-3) \\ \vdots & & & & & -(n-4) \\ & & & & & \vdots \\ 0 & \cdots & & & & -1 \end{pmatrix}=(m_{ij}),$$

where

$$m_{ij}=\begin{cases}0 & \text{if } i>j,\\ -j+i-1 & \text{if } i\leqslant j.\end{cases}$$

It follows from a straight-forward computation that

$$A - BD^{-1}C = \begin{pmatrix} -1 + (n-1)x & a - (n-2)x \\ 2 - n + (n-1)a & y + (n-3) - (n-2)a \end{pmatrix}.$$

Computing the determinants, we may conclude that

$$\det(\Delta_\infty) = -[((n-1)y - 1) \cdot x - y + (-a^2 + 2a - 1) \cdot n + a^2 - 4a + 3].$$

Here we have used that $\det(\rho) \cdot \det(D) = (-1)^{n-1} \cdot (-1)^{n-2} = -1$.

In our examples, we first concentrate on the case $l = 1$.

Example 8.1. The easiest case is $a = 2$ (i.e., no multiple edge). In this case $y = x^{-1}$, and the above formula simplifies to

$$\det(\Delta_\infty) = 2 + x + x^{-1}.$$

In particular, the determinant does not depend on n in this special case. Letting $\tau = T + 1$ be a topological generator of the Galois group of X_∞ over X, we have $x = -\tau = -(T+1)$ and we obtain the power series

$$(T+1)^{-1} \cdot (2(T+1) - (T+1)^2 - 1) = (T+1)^{-1} \cdot (-T^2).$$

Note that $T + 1$ is a unit in Λ. It follows from Remark 5.4 that the characteristic power series of J_Λ is equal to T. Thus, in this case we obtain $\lambda(J_\Lambda) = 1$ independently of $n = |V(X_0)|$.

Example 8.2. In our second example we consider the case $a = 3$, i.e. we have two edges between x_1 and x_2. If the voltage assignments of these two edges are τ and 1, respectively, then a similar computation yields that the power series $\det(\Delta_\infty)$ is associated with

$$n \cdot T^2.$$

Therefore $\lambda(J_\Lambda) = 1$ for all primes p, and $\mu(J_\Lambda) = 0$ except for the prime divisors of n. In particular, we see that this family of examples includes voltage covers with *arbitrarily large μ-invariant* (for fixed prime p, let $n = p^r$ for some large integer r).

If the voltage assignments are τ and τ, then we obtain that the characteristic power series of J_Λ is associated with

$$2T,$$

i.e. it does not depend on the number n of vertices of X_0.

Finally, suppose that $\alpha(e_1) = \tau$ and $\alpha(e_2) = \tau^2$. Then the power series $\det(\Delta_\infty)$ is associated with

$$T^2 \cdot (T^2 + (4+n)T + 4 + n).$$

Therefore $\mu(J_\Lambda) = 0$ for all primes p, and

$$\lambda(J_\Lambda) = \begin{cases} 3 & \text{if } p \mid (n+4); \\ 1 & \text{if } p \nmid (n+4). \end{cases}$$

Example 8.3. Now suppose that a is arbitrary, and that each $\alpha(e_i)$ is equal to τ, $i = 1, \dots, a-1$. Then we obtain that

$$\det(\Delta_\infty) = -(T+1)^{-1}(a-1)T^2,$$

i.e. the characteristic polynomial of J_Λ is associated with $(a-1)T$ in this case; in particular this polynomial does not depend on the number of vertices of X_0.

Example 8.4. Now suppose that a is arbitrary and that there exists some integer $b \leqslant a - 1$ such that

$$\alpha(e_1) = \dots = \alpha(e_b) = \tau, \quad \alpha(e_{b+1}) = \dots = \alpha(e_{a-1}) = 1.$$

Then we obtain that $\det(\Delta_\infty)$ is associated to

$$T^2((ab-b^2-b)\cdot n - ab + b^2 + 2b).$$

In particular, this determinant is independent of the number n of vertices of X_0 if and only if $b=0$ or $a=b+1$. This fact is in accordance with our previous examples. Note that we did not consider examples with $b=0$ since in that case the hypothesis from Lemma 2.5 is not satisfied, which ensures that all the X_m will be connected.

Example 8.5. In our last example we consider a $\mathbb{Z}_p^2$-cover of X_0. Let σ and τ denote topological generators of $\mathrm{Gal}(X_\infty/X) \cong \mathbb{Z}_p^2$. Consider the case

$$a = 3, \quad \alpha(e_1) = \tau, \quad \alpha(e_2) = \sigma.$$

Write $\tau = T+1$ and $\sigma = S+1$. Then we obtain that the characteristic power series of J_Λ (which is associated with $\det(\Delta_\infty)$ in this example, since $l>1$, i.e. Remark 5.4 does not apply) gives

$$-\frac{1}{(S+1)(T+1)}\cdot\left(n(T-S)^2 + 2ST + ST^2 + S^2T\right).$$

We see that $m_0(J_\Lambda)=0$. We will show in the next section that $l_0(J_\Lambda)$, however, is positive.

9. A non-trivial l_0-invariant

The main goal of this section is to provide an example of a $\mathbb{Z}_p^l$-cover X_∞/X, $l>1$, such that $l_0(J_\Lambda(X_\infty))>0$. For any $\mathbb{Z}_p^l$-cover X_∞/X, we denote by $\mathcal{E}^{\subseteq X_\infty}(X)$ the set of $\mathbb{Z}_p$-covers of X which are contained in X_∞. We make use of the following

Theorem 9.1. *Let X_∞/X be a $\mathbb{Z}_p^2$-cover, and suppose that X_∞ contains two $\mathbb{Z}_p$-subcovers Y_1 and Y_2 of X such that*

$$\mu(Y_1)=0, \quad \mu(Y_2)>0.$$

Then λ is unbounded on $\mathcal{E}^{\subseteq X_\infty}(X)$, and $l_0(J_\Lambda(X_\infty))>0$.

The proof of this theorem will be given at the end of the section.

Example 9.2. Let X_∞/X be the $\mathbb{Z}_p^2$-cover from Example 8.5, with $n=3$ and $a=3$. Then X_∞ contains the two $\mathbb{Z}_p$-voltage covers Y_1 and Y_2 which are given as follows: for Y_1, we have

$$\alpha(e_1)=\tau, \quad \alpha(e_2)=1$$

(this corresponds to the choice $\mathrm{Gal}(X_\infty/Y_1)=\langle\sigma\rangle$), and for Y_2 we have

$$\alpha(e_1)=\alpha(e_2)=\tau$$

(this corresponds to $\mathrm{Gal}(X_\infty/Y_2)=\langle\sigma\tau^{-1}\rangle$).

We write $\Lambda = \mathbb{Z}_p[\![S,T]\!]$, and we do not abbreviate $J_\Lambda(X_\infty) := J(X_\infty)\otimes\Lambda$ to J_Λ for clarity. Moreover, we let $J_\Lambda(Y_i)$ be the Λ_1-module $J(Y_i)\otimes\Lambda_1$, where

$$\Lambda_1 = \mathbb{Z}_p[\![T]\!] \cong \mathbb{Z}_p[\![\mathrm{Gal}(Y_i/X)]\!],$$

respectively. Then we know from Example 8.2 that the characteristic power series of $J_\Lambda(Y_1)$ is associated with $3T$, and the characteristic power series of $J_\Lambda(Y_2)$ is associated with $2T$. In particular, if we consider $p=3$, then $\mu(Y_1)>0$ and $\mu(Y_2)=0$, i.e. the hypotheses from Theorem 9.1 are satisfied. This can be seen also by looking at the characteristic power series $F=F(S,T)$ of $J_\Lambda(X_\infty)$ (see Example 8.5): modulo $S=\sigma-1$ we obtain

$$nT^2,$$

which is clearly divisible by $p = 3$ since $n = 3$ by assumption. On the other hand, if we let $S = T$ in F, then we obtain a polynomial which is associated with

$$2T^2 + 2T^3 = T^2(T+2) \sim T^2$$

for $p \neq 2$; therefore $\mu(Y_2) = 0$. One can even see that $l_0(F) > 0$ directly here, since $F \in (p, S)$ for the choice $n = p$.

The following lemma will be the crucial ingredient in the proof of Theorem 9.1.

Lemma 9.3. *Let X_∞/X be a $\mathbb{Z}_p^2$-cover. If $\mu(Y) = 0$ for some $\mathbb{Z}_p$-cover $Y \in \mathcal{E}^{\subseteq X_\infty}(X)$ of X, then $\mu(\tilde{Y}) = 0$ for all but finitely many $\tilde{Y} \in \mathcal{E}^{\subseteq X_\infty}(X)$.*

Proof. Fix $Y \in \mathcal{E}^{\subseteq X_\infty}(X)$ such that $\mu(Y) = 0$. In what follows we write $\Lambda = \mathbb{Z}_p[\![S,T]\!]$, and we do not abbreviate $J_\Lambda(X_\infty) := J(X_\infty) \otimes \Lambda$ to J_Λ for clarity. Moreover, we let $J_\Lambda(Y)$ be the Λ_1-module $J(Y) \otimes \Lambda_1$, where $\Lambda_1 = \mathbb{Z}_p[\![\mathrm{Gal}(Y/X)]\!]$.

Suppose that the topological generators σ and τ of $\mathrm{Gal}(X_\infty/X) \cong \mathbb{Z}_p^2$ have been chosen such that $Y = X_\infty^{\langle\sigma\rangle}$, and let $S = \sigma - 1$. Then Λ_1 can be identified with $\mathbb{Z}_p[\![T]\!] \cong \Lambda/S$, and Lemma 7.4 implies that the kernel of the canonical map

$$t' \colon J_\Lambda(X_\infty)/S \longrightarrow J_\Lambda(Y)$$

has p-rank at most 1. Therefore the finitely generated Λ_1-module $J_\Lambda(X_\infty)/S$ is Λ_1-torsion and has μ-invariant 0. This means that we can choose an annihilator $f \in \Lambda_1$ of $J_\Lambda(X_\infty)/S$ which is not divisible by p.

Now consider the Λ-module $J_\Lambda(X_\infty)$, and suppose that $m_0(X_\infty) > 0$. Since the Fitting ideal $\mathrm{Fitt}_\Lambda(J_\Lambda(X_\infty))$ is contained in the annihilator ideal of $J_\Lambda(X_\infty)$, this assumption would imply that

$$\mathrm{Fitt}_\Lambda(J_\Lambda(X_\infty)) \subseteq (p).$$

But it follows from the general properties of Fitting ideals (see also [13, Proposition 2.1]) that
(i) $\pi(\mathrm{Fitt}_\Lambda(J_\Lambda(X_\infty)) = \mathrm{Fitt}_{\Lambda_1}(J_\Lambda(X_\infty)/S)$ and
(ii) $\mathrm{Ann}_{\Lambda_1}(J_\Lambda(X_\infty)/S)^r \subseteq \mathrm{Fitt}_{\Lambda_1}(J_\Lambda(X_\infty)/S)$ for all sufficiently large $r \in \mathbb{N}$,
where

$$\pi \colon \Lambda \longrightarrow \Lambda_1$$

denotes the surjection induced by mapping S to 0, and where

$$\mathrm{Ann}_{\Lambda_1}(J_\Lambda(X_\infty/S)) \subseteq \Lambda_1$$

denotes the annihilator ideal.

Therefore each annihilator of $J_\Lambda(X_\infty)/S$ would be divisible by p, which would yield a contradiction to the existence of the annihilator f above.

We have shown that $m_0(X_\infty) = 0$. Now let $\tilde{Y} \in \mathcal{E}^{\subseteq X_\infty}(X)$ be arbitrary, and choose topological generators $\tilde{\sigma}$ and $\tilde{\tau}$ of $\mathrm{Gal}(X_\infty/X)$ such that $\tilde{Y} = X_\infty^{\langle\tilde{\sigma}\rangle}$. Then $\mathbb{Z}_p[\![\mathrm{Gal}(\tilde{Y}/X)]\!] \cong \mathbb{Z}_p[\![\tilde{T}]\!] =: \tilde{\Lambda}_1$, where $\tilde{T} = \tilde{\tau} - 1$. Since $m_0(X_\infty) = 0$, we can choose an annihilator $F \in \Lambda$ which is not divisible by p. In fact, F can be chosen such that F is not contained in the ideal $(p, \tilde{S})$, $\tilde{Y} = X_\infty^{\langle\tilde{S}+1\rangle}$, for all but finitely many choices of $\tilde{Y}$. Indeed, the characteristic power series $F_{J_\Lambda(X_\infty)}$ of $J_\Lambda(X_\infty)$ is contained in finitely many of the ideals $(p, \tilde{S})$ (note that each such ideal contributes to the l_0-invariant of X_∞). Suppose now that $\tilde{Y}$ has been chosen such that $F_{J_\Lambda(X_\infty)}$ is not contained in the corresponding ideal $(p, \tilde{S})$. Since $J_\Lambda(X_\infty)$ does not contain any non-trivial pseudo-null submodules by Lemma 6.2, it follows from the general structure of Λ-modules that $F_{J_\Lambda(X_\infty)}$ annihilates $J_\Lambda(X_\infty)$ (see also [13, Proposition 2.1(3)]). Therefore we can choose $F = F_{J_\Lambda(X_\infty)}$ above; this proves our claim.

Again, a suitable power F^r of F (the exponent of this power depends on the number of generators of $J_\Lambda(X_\infty)$ over Λ) is contained in the Fitting ideal $\mathrm{Fitt}_\Lambda(J_\Lambda(X_\infty))$, and

$$\tilde{\pi}(F^r) \in \mathrm{Fitt}_{\Lambda_1}(J_\Lambda(X_\infty)/\tilde{S}) \subseteq \mathrm{Ann}_{\tilde{\Lambda}_1}(J_\Lambda(X_\infty)/\tilde{S}),$$

where now $\tilde{\pi}\colon \Lambda \longrightarrow \tilde{\Lambda}_1$ is induced by mapping $\tilde{S}$ to zero.

If $\tilde{\pi}(F^r) = \tilde{\pi}(F)^r$ is not divisible by p for our particular choice of $\tilde{S}$ (by the above, this can be achieved for all but finitely many $\tilde{Y} \in \mathcal{E}^{\subseteq X_\infty}(X)$), then it follows that $\mu(J_\Lambda(X_\infty)/\tilde{S}) = 0$, i.e. $J_\Lambda(X_\infty)/\tilde{S}$ is a finitely generated $\mathbb{Z}_p$-module. By Lemma 7.4, the canonical map

$$J_\Lambda(X_\infty)/\tilde{S} \longrightarrow J_\Lambda(\tilde{Y})$$

is surjective. This shows that $J_\Lambda(\tilde{Y})$ is also finitely generated over $\mathbb{Z}_p$, and therefore $\mu(J_\Lambda(\tilde{Y})) = 0$. □

Proof of Theorem 9.1. Since $\mu(Y_1) = 0$, it follows from Lemma 9.3 that $\mu(\tilde{Y}) = 0$ for all but finitely many $\tilde{Y} \in \mathcal{E}^{\subseteq X_\infty}(X)$. Now fix Y_2 such that $\mu(Y_2) > 0$, and let $U' = \mathcal{E}(Y_2, n)$ be a neighbourhood as in Corollary 6.7. Then $\mu(\tilde{Y}) = 0$ for all but finitely many $\tilde{Y} \in U'$. The first assertion of Theorem 9.1 now follows from Corollary 6.7.

Moreover, it follows from the proof of Lemma 9.3 that $m_0(X_\infty) = 0$, and that $\mu(Y_2) > 0$ is possible only if the characteristic power series $F_{J_\Lambda(X_\infty)} \in \Lambda$ is contained in the ideal $(p, \tilde{S})$, where $\tilde{S} = \tilde{\sigma} - 1$ for some topological generator $\tilde{\sigma}$ of $\mathrm{Gal}(X_\infty/Y_2)$. This implies that $l_0(J_\Lambda(X_\infty)) > 0$, by the definitions. □

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