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Abstract The purpose of this paper is to extend the scope of Ehrhart theory to periodic graphs. We give sufficient conditions for the growth sequences of periodic graphs to be a quasi-polynomial and to satisfy the reciprocity laws. Furthermore, we apply our theory to determine the growth series in several new examples.

1. Introduction

In this paper, a graph Γ means a directed graph that may have loops and multiple edges. We define an *n-dimensional periodic graph* (Γ, L) as a graph Γ on which a free abelian group L of rank n acts freely and its quotient graph Γ/L is finite (see Definition 2.2). For a vertex x_0 of Γ, the *growth sequence* $(s_{\Gamma,x_0,i})_{i\geqslant 0}$ (resp. *cumulative growth sequence* $(b_{\Gamma,x_0,i})_{i\geqslant 0}$) is defined as the number of vertices of Γ whose distance from x_0 is i (resp. at most i). The purpose of this paper is to discuss phenomena similar to Ehrhart theory that appear in the growth sequences of periodic graphs.

Periodic graphs naturally appear in crystallography, and their growth sequences have been studied intensively in this field. In crystallography, the growth sequence is also referred to as the *coordination sequence*. For a periodic graph Γ corresponding to a crystal, $s_{\Gamma,x_0,1}$ is nothing but the usual coordination number of the atom x_0. The coordination sequence is utilized in several crystal database entries (cf. [4]), and it can be useful, for instance, in distinguishing between two allotropes that cannot be distinguished by the coordination number.

In [13], Grosse-Kunstleve, Brunner and Sloane conjectured that the growth sequences of periodic graphs are of quasi-polynomial type, i.e., there exist an integer M and a quasi-polynomial $f_s : \mathbb{Z} \to \mathbb{Z}$ such that $s_{\Gamma,x_0,i} = f_s(i)$ holds for all $i \geqslant M$ (see Definition 2.16). In [19], the second author, Sakamoto, Mase, and Nakagawa prove this conjecture to be true for any periodic graphs (Theorem 2.17). Although it was proved to be of quasi-polynomial type, determining the quasi-polynomial in practice is still difficult. Thus, the following natural question arises.

Question 1.1. *Find an effective algorithm to determine the explicit formulae of the growth sequences.*

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Keywords. Ehrhart theory, periodic graphs, tiling problems, generating functions, coordination sequences, growth sequences.

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So far, various computational methods have been established for several special classes of periodic graphs. In [7], Conway and Sloane give growth sequences of the contact graphs of some lattices from the viewpoint of Ehrhart theory (cf. [3, 2]). In [12], Goodman-Strauss and Sloane propose "the coloring book approach" to obtain the growth sequence for some periodic tilings. In [21, 22], Shutov and Maleev obtained the growth sequences for tilings satisfying certain conditions that contain the 20 2-uniform tilings. However, as far as we know, no algorithm that can be applied to general periodic graphs has been proposed, even in dimension two.

The difficulty of Question 1.1 is due to the difficulty of determining M that appears in the definition of "quasi-polynomial type" above. Indeed, if this M can be determined and a quasi-period of f_s is known, then the explicit formula of $(s_{\Gamma,x_0,i})_i$ can be determined by its first few terms. In this paper, in the context of Ehrhart theory, we focus on the category of graphs whose growth sequences are honest quasi-polynomials (i.e. quasi-polynomials on $i > 0$). We note in advance that for these graphs, the difficulty of Question 1.1 is avoided.

In Ehrhart theory, for a rational polytope $Q \subset \mathbb{R}^N$, it is proved that the function

$$h_Q : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}; \quad i \mapsto \#\big(iQ \cap \mathbb{Z}^N\big)$$

is a quasi-polynomial on $i \geqslant 0$. As we will discuss in Subsection 3.2, for a rational polytope $Q \subset \mathbb{R}^N$ with $0 \in Q$, we can construct a periodic graph $(\Gamma_Q, \mathbb{Z}^N)$ such that its cumulative growth sequence $b_{\Gamma_Q,0,i}$ coincides with $h_Q(i)$. Therefore, we can say that the study of the growth sequences of periodic graphs essentially contains the Ehrhart theory of rational polytopes Q satisfying $0 \in Q$. Since the cumulative growth sequence $(b_{\Gamma_Q,0,i})_i$ is a quasi-polynomial on $i \geqslant 0$, the following natural question arises.

Question 1.2. *Find a reasonable class $\mathcal{P}$ of pairs (Γ, x_0) that consist of a periodic graph Γ and one of its vertices x_0 such that*

- *$\mathcal{P}$ contains the class $\{(\Gamma_Q, 0) \mid Q$ is a rational polytope with $0 \in Q\}$, and*
- *for any $(\Gamma, x_0) \in \mathcal{P}$, the sequence $(b_{\Gamma,x_0,i})_i$ is an honest quasi-polynomial (i.e. a quasi-polynomial on $i \geqslant 0$).*

The word "reasonable" here means that $\mathcal{P}$ should be a class that can be described in terms of graph theory. Note that, unlike the case of Ehrhart theory, the growth sequences of periodic graphs in general are not necessarily quasi-polynomials, and they may have finite exceptional terms (see Example 2.18). However, it has been observed that for some highly symmetric periodic graphs, they are often honest quasi-polynomials. The intention of this question is to describe the properties of such good periodic graphs.

Another important topic of Ehrhart theory is the reciprocity law. When we think of the function h_Q as a quasi-polynomial and substitute a negative value for it, we have $h_Q(-i) = (-1)^{\dim Q}\#\big(i \cdot \mathrm{relint}(Q) \cap \mathbb{Z}^N\big)$ for $i > 0$. In the growth sequences of some n-dimensional periodic graphs, it has been observed that they sometimes satisfy the equations

$$(\diamondsuit) \qquad f_b(-i) = (-1)^n f_b(i-1), \qquad f_s(-i) = (-1)^{n+1} f_s(i),$$

where f_b and f_s are the corresponding quasi-polynomials to the sequences $(b_{\Gamma,x_0,i})_i$ and $(s_{\Gamma,x_0,i})_i$ (see [7, 21, 26]). These equations in $(\diamondsuit)$ are consistent with the reciprocity laws of reflexive polytopes. Thus, the following natural question arises.

Question 1.3. *Find a reasonable class $\mathcal{P}'$ of pairs (Γ, x_0) such that*

- *$\mathcal{P}'$ contains the class $\{(\Gamma_Q, 0) \mid Q$ is a reflexive polytope$\}$, and*
- *for any $(\Gamma, x_0) \in \mathcal{P}'$, its growth sequence satisfies the reciprocity laws $(\diamondsuit)$.*

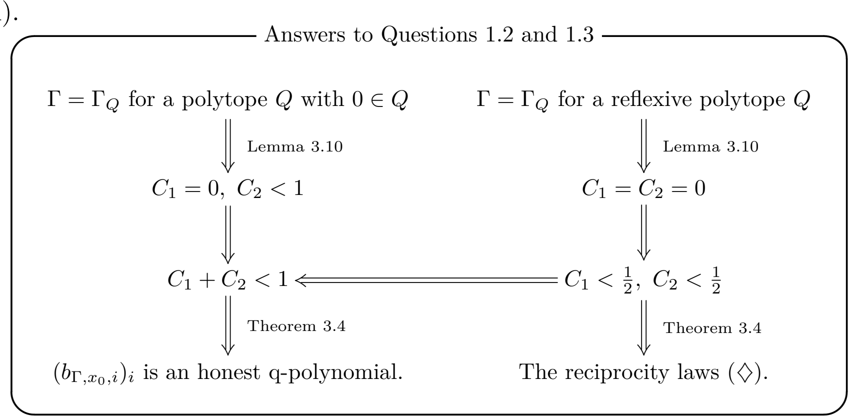
The purpose of this paper is to give answers to Questions 1.2 and 1.3. First, we introduce invariants $C_1(\Gamma, \Phi, x_0) \in \mathbb{R}_{\geqslant 0}$ and $C_2(\Gamma, \Phi, x_0) \in \mathbb{R}_{\geqslant 0}$ for (Γ, x_0) and a periodic realization $\Phi\colon \Gamma \to L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ (i.e. a map preserving an L-action). These invariants measure the deviation from the polytope approximation of d_Γ, where d_Γ is the distance function of the graph Γ. Using these invariants, we can give sufficient conditions for the cumulative growth sequence to be a quasi-polynomial and to satisfy the reciprocity laws $(\diamondsuit)$.

Theorem 1.4 (= Theorem 3.4). *Let (Γ, L) be a strongly connected n-dimensional periodic graph, and let x_0 be a vertex of Γ. Then, the following assertions hold.*

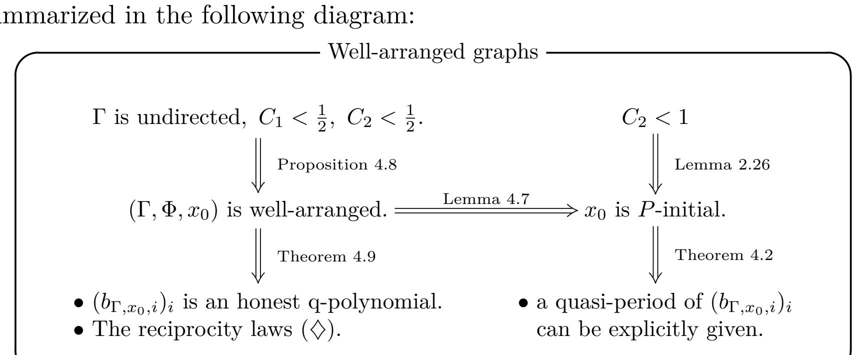
(1) *If a periodic realization Φ of (Γ, L) satisfies $C_1(\Gamma, \Phi, x_0) + C_2(\Gamma, \Phi, x_0) < 1$, then the cumulative growth sequence $(b_{\Gamma, x_0, i})_i$ is a quasi-polynomial on $i \geqslant 0$.*

(2) *If a periodic realization Φ of (Γ, L) satisfies both $C_1(\Gamma, \Phi, x_0) < \frac{1}{2}$ and $C_2(\Gamma, \Phi, x_0) < \frac{1}{2}$, then the reciprocity laws $(\diamondsuit)$ are satisfied.*

Note that if $\Gamma = \Gamma_Q$ is a periodic graph obtained by a polytope Q with $0 \in Q$, then it follows that $C_1(\Gamma, \Phi, x_0) = 0$ and $C_2(\Gamma, \Phi, x_0) < 1$. Furthermore, if $\Gamma = \Gamma_Q$ is a periodic graph obtained by a reflexive polytope Q, then it follows that $C_1(\Gamma, \Phi, x_0) = C_2(\Gamma, \Phi, x_0) = 0$. Therefore, Theorem 1.4 is an answer to Questions 1.2 and 1.3. We also note that Theorem 1.4 is a generalization of a result by Conway and Sloane [7], where they only consider the contact graphs of lattices (see Remark 3.6 for more detail).



For undirected periodic graphs, we can give a larger class that satisfies the reciprocity laws $(\diamondsuit)$. In Subsection 4.2, we define a class of undirected periodic graphs called "well-arranged", and we prove that their growth sequences satisfy the reciprocity laws $(\diamondsuit)$ (Theorem 4.9). We also introduce the notion of "P-initial" for a vertex x_0, and we see that a quasi-period of the growth sequence can be explicitly given in this case (Theorem 4.2). The relationship with the invariants C_1 and C_2 can be summarized in the following diagram:



When (Γ, Φ, x_0) is well-arranged, the growth sequence $(s_{\Gamma,x_0,i})_i$ is a quasi-polynomial on $i \geqslant 1$, and its quasi-period can be explicitly given. Therefore, it is possible to determine the explicit formula of $(s_{\Gamma,x_0,i})_i$ by computing the first few terms of it. By this method, we compute the growth series in several new examples in Section 5. As far as we know, this is the first time that the growth sequences have been computed for nontrivial 3-dimensional periodic graphs. However, this method is only applicable to well-arranged graphs, and answers to Question 1.1 for general periodic graphs remain as future work.

The paper is organized as follows: in Section 2, we first summarize the notations of graphs, and we define periodic graphs and their growth sequences. We then define an important concept, the *growth polytope*, and use it to define the invariants C_1 and C_2. In Section 3, we give sufficient conditions for the cumulative growth sequence to be a quasi-polynomial type and to satisfy the reciprocity laws (Theorem 3.4). Furthermore, in Subsection 3.2, we see that Theorem 3.4 can be seen as a generalization of the usual Ehrhart theory for polytopes Q with $0 \in Q$ and for reflexive polytopes Q. In Section 4, we treat a periodic graph (Γ, L) with a P-initial vertex x_0. In this case, we can calculate the invariant C_2 (Proposition 4.4) and a quasi-period of the growth sequence of Γ with the start point x_0 (Theorem 4.2). Furthermore, we also introduce a class of periodic graphs called "well-arranged", and we prove that their growth sequences satisfy the reciprocity laws (Theorem 4.9). In Section 5, for some specific periodic graphs, we will see whether they are well-arranged and discuss their growth series. Furthermore, as an application of Theorem 4.9, we determine the growth series in several new examples (Subsections 5.2 and 5.3). In Appendix A, we summarize the properties of the growth polytope necessary for the definition of the invariants C_1 and C_2. In Section B, we discuss a variant of Ehrhart theory (Theorem B.4), which is necessary for the proof of Theorem 3.4. The difference from the usual Ehrhart theory is that the center of the dilation need not be the origin, and the dilation factor may be shifted by a constant.

2. Notation and Preliminaries

2.1. Notation. For a set X, $\#X$ denotes the cardinality of X, and 2^X denotes the power set of X.

For a finite subset $S \subset \mathbb{Z}_{>0}$, $\mathrm{LCM}(S)$ denotes the least common multiple of the elements of S.

For a polytope $P \subset \mathbb{R}^N$, $\mathrm{Facet}(P)$ denotes the set of facets of P, $\mathrm{Face}(P)$ denotes the set of faces of P, and $V(P)$ denotes the set of vertices of P. Note that both P itself and the empty set $\varnothing$ are considered as faces of P.

For a subset $C \subset \mathbb{R}^N$, $\mathrm{int}(C)$ denotes the interior of C, and $\mathrm{relint}(C)$ denotes the relative interior of C.

For a polytope $P \subset \mathbb{R}^N$ of dimension d, a *triangulation* T of P means a finite collection of d-simplices with the following two conditions:

- $P = \bigcup_{\Delta \in T} \Delta$.
- For any $\Delta_1, \Delta_2 \in T$, $\Delta_1 \cap \Delta_2$ is a face of Δ_1 and Δ_2.

In this paper, monoids always mean commutative monoids. We refer the reader to [6] and [19] for the terminology of monoid and its module theory.

Let M be a set equipped with a binary operation $*$. For $u \in M$ and subsets $X, Y \subset M$, we define subsets $u * X, X * Y \subset M$ by

$$u * X := \{u * x \mid x \in X\}, \qquad X * Y := \{x * y \mid (x, y) \in X \times Y\}.$$

2.2. Graphs and walks. In this paper, a *graph* means a directed weighted graph which may have loops and multiple edges. A graph $\Gamma = (V_\Gamma, E_\Gamma, s_\Gamma, t_\Gamma, w_\Gamma)$ consists of the set V_Γ of vertices, the set E_Γ of edges, the source function $s_\Gamma : E_\Gamma \to V_\Gamma$, the target function $t_\Gamma : E_\Gamma \to V_\Gamma$, and the (integer) weight function $w_\Gamma : E_\Gamma \to \mathbb{Z}_{>0}$. We often abbreviate s_Γ, t_Γ and w_Γ to s, t and w when no confusion can arise.

Definition 2.1. *Let $\Gamma = (V_\Gamma, E_\Gamma, s, t, w)$ be a graph.*

(1) *Γ is called to be unweighted if $w(e) = 1$ holds for every $e \in E_\Gamma$. Γ is called to be undirected when there exists an involution $E_\Gamma \to E_\Gamma$; $e \mapsto e'$ such that $s(e) = t(e')$, $t(e) = s(e')$ and $w(e) = w(e')$. Γ is called to be locally finite when for all $x \in V_\Gamma$, there are only finitely many edges e satisfying $s(e) = x$ and only finitely many edges e satisfying $t(e) = x$.*

(2) *A walk p in Γ is a sequence $e_1 e_2 \cdots e_\ell$ of edges e_i of Γ satisfying $t(e_i) = s(e_{i+1})$ for each $i = 1, \dots, \ell - 1$. We define*

$$s(p) \coloneqq s(e_1), \quad t(p) \coloneqq t(e_\ell), \quad w(p) \coloneqq \sum_{i=1}^{\ell} w(e_i), \quad \mathrm{length}(p) \coloneqq \ell.$$

Note that we have $w(p) = \mathrm{length}(p)$ if Γ is unweighted.

We say that "p is a walk from x to y" when $x = s(p)$ and $y = t(p)$. We also define the support $\mathrm{supp}(p) \subset V_\Gamma$ of p by

$$\mathrm{supp}(p) \coloneqq \{s(e_1), t(e_1), t(e_2), \dots, t(e_\ell)\} \subset V_\Gamma.$$

By convention, each vertex $v \in V_\Gamma$ is also considered as a walk of length 0. This is called the trivial walk at v and denoted by $\varnothing_v$: i.e., we define

$$s(\varnothing_v) \coloneqq v, \quad t(\varnothing_v) \coloneqq v, \quad w(\varnothing_v) \coloneqq 0, \quad \mathrm{length}(\varnothing_v) \coloneqq 0, \quad \mathrm{supp}(\varnothing_v) \coloneqq \{v\}.$$

(3) *A path in Γ is a walk $e_1 \cdots e_\ell$ such that $s(e_1), t(e_1), t(e_2), \dots, t(e_\ell)$ are distinct. A walk of length 0 is considered as a path.*

(4) *A cycle in Γ is a walk $e_1 \cdots e_\ell$ with $s(e_1) = t(e_\ell)$ such that $t(e_1), t(e_2), \dots, t(e_\ell)$ are distinct. A walk of length 0 is not considered as a cycle. Cyc_Γ denotes the set of cycles in Γ.*

(5) *For $x, y \in V_\Gamma$, $d_\Gamma(x, y) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ denotes the smallest weight $w(p)$ of any walk p from x to y. By convention, we define $d_\Gamma(x, y) = \infty$ when there is no walk from x to y. A graph Γ is said to be strongly connected when we have $d_\Gamma(x, y) < \infty$ for all $x, y \in V_\Gamma$. When Γ is undirected, we have $d_\Gamma(x, y) = d_\Gamma(y, x)$ for all $x, y \in V_\Gamma$.*

(6) *$C_1(\Gamma, \mathbb{Z})$ denotes the group of 1-chains on Γ with coefficients in $\mathbb{Z}$, i.e., $C_1(\Gamma, \mathbb{Z})$ is the free abelian group generated by E_Γ. For a walk $p = e_1 \cdots e_\ell$ in Γ, let $\langle p \rangle$ denote the 1-chain $\sum_{i=1}^{\ell} e_i \in C_1(\Gamma, \mathbb{Z})$. $H_1(\Gamma, \mathbb{Z}) \subset C_1(\Gamma, \mathbb{Z})$ denotes the 1-st homology group, i.e., $H_1(\Gamma, \mathbb{Z})$ is a subgroup generated by $\langle p \rangle$ for $p \in \mathrm{Cyc}_\Gamma$. We refer the reader to* [25] *for more detail.*

2.3. Periodic graphs.

Definition 2.2. *Let n be a positive integer. An n-dimensional periodic graph (Γ, L) is a graph Γ and a free abelian group $L \simeq \mathbb{Z}^n$ of rank n with the following two conditions:*

- *L freely acts on both V_Γ and E_Γ, and their quotients V_Γ/L and E_Γ/L are finite sets.*
- *This action preserves the edge relations, i.e., for any $u \in L$ and $e \in E_\Gamma$, we have $s_\Gamma(u(e)) = u(s_\Gamma(e))$, $t_\Gamma(u(e)) = u(t_\Gamma(e))$ and $w_\Gamma(u(e)) = w_\Gamma(e)$.*

Then, L is called the period lattice of Γ. Note that Γ automatically becomes a locally finite graph.

If (Γ, L) is an n-dimensional periodic graph, then the *quotient graph* $\Gamma/L = (V_{\Gamma/L}, E_{\Gamma/L}, s_{\Gamma/L}, t_{\Gamma/L}, w_{\Gamma/L})$ is defined by $V_{\Gamma/L} \coloneqq V_\Gamma/L$, $E_{\Gamma/L} \coloneqq E_\Gamma/L$, and the functions $s_{\Gamma/L} : E_\Gamma/L \to V_\Gamma/L$, $t_{\Gamma/L} : E_\Gamma/L \to V_\Gamma/L$, and $w_{\Gamma/L} : E_\Gamma/L \to \mathbb{Z}_{>0}$ induced from s_Γ, t_Γ, and w_Γ. Note that the functions $s_{\Gamma/L}$, $t_{\Gamma/L}$ and $w_{\Gamma/L}$ are well-defined due to the second condition in Definition 2.2.

Definition 2.3. *Let (Γ, L) be an n-dimensional periodic graph.*

(1) *Since L is an abelian group, we use the additive notation: for $u \in L$, $x \in V_\Gamma$, $e \in E_\Gamma$ and a walk $p = e_1 \cdots e_\ell$, $u+x$, $u+e$ and $u+p$ denote their translations by u.*
(2) *For any $x \in V_\Gamma$ and $e \in E_\Gamma$, let $\overline{x} \in V_{\Gamma/L}$ and $\overline{e} \in E_{\Gamma/L}$ denote their images in $V_{\Gamma/L} = V_\Gamma/L$ and $E_{\Gamma/L} = E_\Gamma/L$. For a walk $p = e_1 \cdots e_\ell$ in Γ, let $\overline{p} := \overline{e_1} \cdots \overline{e_\ell}$ denote its image in Γ/L.*
(3) *When $x, y \in V_\Gamma$ satisfy $\overline{x} = \overline{y}$, there exists an element $u \in L$ such that $u + x = y$. Since the action is free, such $u \in L$ uniquely exists and is denoted by $y - x$.*
(4) *For a walk p in Γ with $\overline{s(p)} = \overline{t(p)}$, we define*

$$\mathrm{vec}(p) := t(p) - s(p) \in L.$$

Definition 2.4. *Let (Γ, L) be an n-dimensional periodic graph. We define $L_\mathbb{R} := L \otimes_\mathbb{Z} \mathbb{R}$.*

(1) *A periodic realization $\Phi\colon V_\Gamma \to L_\mathbb{R}$ is a map satisfying $\Phi(u + x) = u + \Phi(x)$ for any $u \in L$ and $x \in V_\Gamma$. When we fix an injective periodic realization of $\Phi\colon V_\Gamma \to L_\mathbb{R}$, we sometimes identify V_Γ with the subset of $L_\mathbb{R}$.*
(2) *Let Φ be a periodic realization of (Γ, L). For an edge e and a walk p in Γ, we define*

$$\mathrm{vec}_\Phi(e) := \Phi(t(e)) - \Phi(s(e)) \in L_\mathbb{R}, \quad \mathrm{vec}_\Phi(p) := \Phi(t(p)) - \Phi(s(p)) \in L_\mathbb{R}.$$

It is easy to see that the value $\mathrm{vec}_\Phi(e) \in L_\mathbb{R}$ depends only on the class $\overline{e} \in E_{\Gamma/L}$, and therefore, the map

$$\mu_\Phi : E_{\Gamma/L} \to L_\mathbb{R}; \quad \overline{e} \mapsto \mathrm{vec}_\Phi(e)$$

is well-defined. It can be extended to a homomorphism

$$\mu_\Phi : C_1(\Gamma/L, \mathbb{Z}) \to L_\mathbb{R}; \quad \sum a_i \overline{e_i} \mapsto \sum a_i \mu_\Phi(\overline{e_i}).$$

By construction, it satisfies $\mu_\Phi(\langle \overline{p} \rangle) = \mathrm{vec}_\Phi(p)$ for any walk p in Γ.

Remark 2.5.

(1) An injective periodic realization of (Γ, L) always exists. To see this fact, we take any injective map $V_\Gamma/L \to L_\mathbb{R}/L$. It is possible because $\#(V_\Gamma/L) < \infty$. Then, any injective map $V_\Gamma/L \to L_\mathbb{R}/L$ lifts to an injective periodic realization $V_\Gamma \to L_\mathbb{R}$.
(2) Even if a periodic realization Φ is not injective, the map $V_\Gamma \to L_\mathbb{R} \times V_\Gamma/L$; $x \mapsto \big(\Phi(x), \overline{x}\big)$ is always injective.
(3) In Definition 2.4(2), we have $\mathrm{vec}_\Phi(p) = \mathrm{vec}(p)$ for any p with $\overline{s(p)} = \overline{t(p)}$.

Example 2.6 (cf. [26], [19, Figure 3]). The *Wakatsuki graph* is an undirected unweighted graph $\Gamma = (V_\Gamma, E_\Gamma, s_\Gamma, t_\Gamma)$ defined by

$$V_\Gamma = \{v_0, v_1, v_2\} \times \mathbb{Z}^2, \quad E_\Gamma = \{e_0, e_1, \dots, e_9\} \times \mathbb{Z}^2,$$

and

$$\begin{gathered}
s_\Gamma(e_0,(x,y)) = s_\Gamma(e_1,(x,y)) = s_\Gamma(e_2,(x,y)) = s_\Gamma(e_3,(x,y)) = (v_0,(x,y)),\\
s_\Gamma(e_4,(x,y)) = s_\Gamma(e_5,(x,y)) = s_\Gamma(e_6,(x,y)) = s_\Gamma(e_7,(x,y)) = (v_1,(x,y)),\\
s_\Gamma(e_8,(x,y)) = s_\Gamma(e_9,(x,y)) = (v_2,(x,y)),\\
t_\Gamma(e_0,(x,y)) = (v_1,(x,y)), \quad t_\Gamma(e_1,(x,y)) = (v_1,(x-1,y)),\\
t_\Gamma(e_2,(x,y)) = (v_1,(x-1,y-1)), \quad t_\Gamma(e_3,(x,y)) = (v_2,(x,y)),\\
t_\Gamma(e_4,(x,y)) = (v_0,(x,y)), \quad t_\Gamma(e_5,(x,y)) = (v_0,(x+1,y)),\\
t_\Gamma(e_6,(x,y)) = (v_0,(x+1,y+1)), \quad t_\Gamma(e_7,(x,y)) = (v_2,(x,y)),\\
t_\Gamma(e_8,(x,y)) = (v_0,(x,y)), \quad t_\Gamma(e_9,(x,y)) = (v_1,(x,y))
\end{gathered}$$

for any $(x,y)\in\mathbb{Z}^2$. Then, Γ admits an action of $L=\mathbb{Z}^2$ by

$$\begin{aligned}
(a,b)+(v_i,(x,y)) &\coloneqq (v_i,(x+a,y+b)),\\
(a,b)+(e_j,(x,y)) &\coloneqq (e_j,(x+a,y+b))
\end{aligned}$$

for each $(a,b),(x,y)\in\mathbb{Z}^2$, $0\leqslant i\leqslant 2$ and $0\leqslant j\leqslant 9$. By this action, (Γ,L) becomes a 2-dimensional periodic graph. A periodic realization Φ of (Γ,L) is defined by

$$\Phi(v_0,(x,y)) = (x,y),\ \Phi(v_1,(x,y)) = \left(x+\frac{1}{2}, y+\frac{1}{2}\right),\ \Phi(v_2,(x,y)) = \left(x+\frac{1}{2}, y\right)$$

Let $v_i' \coloneqq (v_i,(0,0))$ for each $0\leqslant i\leqslant 2$, and let $e_j' \coloneqq (e_j,(0,0))$ for each $0\leqslant j\leqslant 9$. Then, the realization and the quotient graph Γ/L can be illustrated as in Figures 1, 2 and 3.

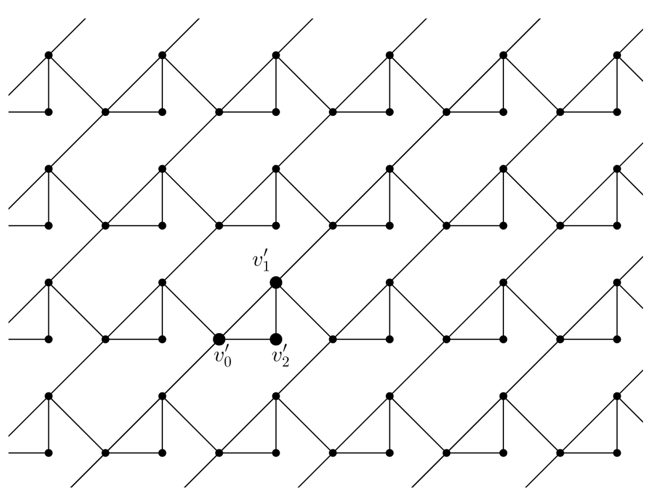


Figure 1. The Wakatsuki graph Γ.

Lemma 2.7. *Let (Γ,L) be an n-dimensional periodic graph. Let μ_Φ be the homomorphism defined in Definition 2.4(2). The restriction map $\mu_\Phi|_{H_1(\Gamma/L,\mathbb{Z})} : H_1(\Gamma/L,\mathbb{Z})\to L_\mathbb{R}$ is independent of the choice of the periodic realization Φ. Furthermore, its image is contained in L.*

Proof. Since $H_1(\Gamma/L,\mathbb{Z})$ is generated by $\langle q\rangle$ for $q\in \mathrm{Cyc}_{\Gamma/L}$, it is sufficient to show that $\mu_\Phi(\langle q\rangle)\in L$ and that the value $\mu_\Phi(\langle q\rangle)$ is independent of the choice of Φ. Take any walk p in Γ such that $\overline{p}=q$. Since $\overline{t(p)}=\overline{s(p)}$, we have

$$\mu_\Phi(\langle q\rangle) = \mathrm{vec}_\Phi(p) = \Phi(t(p)) - \Phi(s(p)) = t(p)-s(p)\in L,$$

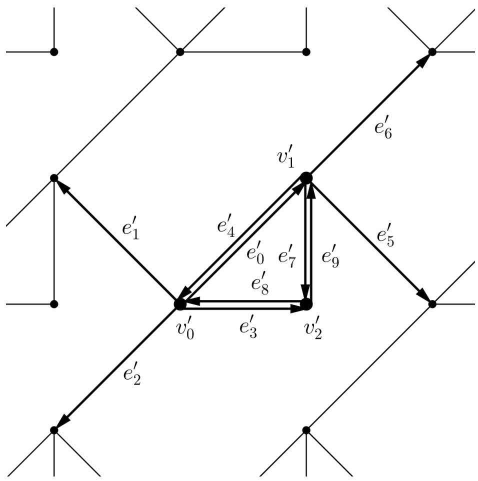


Figure 2. $e'_0, \ldots, e'_9$.

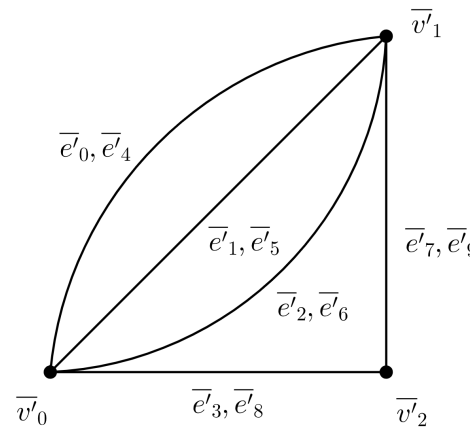


Figure 3. Γ/L.

and this is independent of the choice of Φ. ☐

Definition 2.8. *Let (Γ, L) be an n-dimensional periodic graph. We denote by $\mu\colon H_1(\Gamma/L, \mathbb{Z}) \to L$ the restriction of μ_Φ in Definition 2.4(2) to $H_1(\Gamma/L, \mathbb{Z})$. Note that this restriction map is well-defined by Lemma 2.7.*

Remark 2.9. The homomorphism μ in Definition 2.8 coincides with μ defined in [25, Section 6.1].

Example 2.10. In the Wakatsuki graph (see Example 2.6), the walks $\overline{e'_5}\,\overline{e'_0}$ and $\overline{e'_6}\,\overline{e'_3}\,\overline{e'_9}$ in Γ/L are examples of cycles, and we have

$$\mu\left(\left\langle \overline{e'_5}\,\overline{e'_0} \right\rangle\right) = (1,0), \quad \mu\left(\left\langle \overline{e'_6}\,\overline{e'_3}\,\overline{e'_9} \right\rangle\right) = (1,1).$$

We finish this subsection with some observations on the decomposition and the composition of walks.

Definition 2.11. *Let (Γ, L) be an n-dimensional periodic graph. Let q_0 be a path in Γ/L, and let $q_1, \ldots, q_\ell \in \mathrm{Cyc}_{\Gamma/L}$ be cycles. The sequence $(q_0, q_1, \ldots, q_\ell)$ is called walkable if there exists a walk q' in Γ/L such that $\langle q' \rangle = \sum_{i=0}^{\ell} \langle q_i \rangle$.*

Lemma 2.12. *Let (Γ, L) be an n-dimensional periodic graph.*

(1) *For a walk q' in Γ/L, there exists a walkable sequence $(q_0, q_1, \ldots, q_\ell)$ such that $\langle q' \rangle = \sum_{i=0}^{\ell} \langle q_i \rangle$.*
(2) *Let q_0 be a path in Γ/L, and let $q_1, \ldots, q_\ell \in \mathrm{Cyc}_{\Gamma/L}$ be cycles. Then, $(q_0, q_1, \ldots, q_\ell)$ is walkable if and only if there exists a permutation $\sigma : \{1, 2, \ldots, \ell\} \to \{1, 2, \ldots, \ell\}$ such that*

$$\left(\mathrm{supp}(q_0) \cup \bigcup_{1 \leqslant i \leqslant k} \mathrm{supp}(q_{\sigma(i)})\right) \cap \mathrm{supp}(q_{\sigma(k+1)}) \neq \varnothing$$

holds for any $0 \leqslant k \leqslant \ell - 1$.

Proof. For any walk q' in Γ/L, if q' is not a path, then there exist a walk q'' and a cycle q_1 in Γ/L such that $\langle q' \rangle = \langle q'' \rangle + \langle q_1 \rangle$. Therefore, the assertion (1) follows by the induction on the length of q'. The assertion (2) also follows from the induction. ☐

Remark 2.13.

(1) If q' in Lemma 2.12(1) satisfies $s(q') = t(q')$, then q_0 must be a trivial path (i.e., $\mathrm{length}(q_0) = 0$).
(2) If a walk q' in Γ/L and a vertex $x_0 \in V_\Gamma$ satisfy $s(q') = \overline{x_0}$, then there exists the unique walk p in Γ satisfying $\overline{p} = q'$ and $s(p) = x_0$ (we call such p the *lift* of q' with initial point x_0). Therefore, for a walkable sequence $(q_0, q_1, \dots, q_\ell)$ and a vertex $x_0 \in V_\Gamma$ satisfying $s(q_0) = \overline{x_0}$, there exists a walk p in Γ such that $s(p) = x_0$ and $\langle \overline{p} \rangle = \sum_{i=0}^{\ell} \langle q_i \rangle$. Conversely, for a walk p in Γ, applying Lemma 2.12(1) to $\overline{p}$, there exists a walkable sequence $(q_0, q_1, \dots, q_\ell)$ satisfying $\langle \overline{p} \rangle = \sum_{i=0}^{\ell} \langle q_i \rangle$.
(3) For a walk p in Γ and a walkable sequence $(q_0, q_1, \dots, q_\ell)$ satisfying $\langle \overline{p} \rangle = \sum_{i=0}^{\ell} \langle q_i \rangle$, it follows that

$$w(p) = \sum_{i=0}^{\ell} w(q_i), \qquad \mathrm{length}(p) = \sum_{i=0}^{\ell} \mathrm{length}(q_i).$$

Furthermore, if we fix a periodic realization Φ, we also have

$$\mathrm{vec}_\Phi(p) = \sum_{i=0}^{\ell} \mu_\Phi(\langle q_i \rangle).$$

Example 2.14. In the Wakatsuki graph (see Example 2.6), we consider a walk p as in Figure 4. Then, the image of p in the quotient graph Γ/L is given by

$$\overline{p} = \overline{e'_3}\,\overline{e'_9}\,\overline{e'_5}\,\overline{e'_0}\,\overline{e'_6}\,\overline{e'_3}\,\overline{e'_9}\,\overline{e'_6}\,\overline{e'_1}\,\overline{e'_6}\,\overline{e'_3}\,\overline{e'_9},$$

Then, we have two decompositions

$$\langle \overline{p} \rangle = \left\langle \overline{e'_1} \right\rangle + \left\langle \overline{e'_5}\,\overline{e'_0} \right\rangle + 3 \left\langle \overline{e'_6}\,\overline{e'_3}\,\overline{e'_9} \right\rangle,$$
$$\langle \overline{p} \rangle = \left\langle \overline{e'_3}\,\overline{e'_9} \right\rangle + \left\langle \overline{e'_5}\,\overline{e'_0} \right\rangle + \left\langle \overline{e'_6}\,\overline{e'_1} \right\rangle + 2 \left\langle \overline{e'_6}\,\overline{e'_3}\,\overline{e'_9} \right\rangle.$$

Here, $\overline{e'_1}$ and $\overline{e'_3}\,\overline{e'_9}$ are paths in Γ/L, and $\overline{e'_5}\,\overline{e'_0}$, $\overline{e'_6}\,\overline{e'_3}\,\overline{e'_9}$, and $\overline{e'_6}\,\overline{e'_1}$ are cycles in Γ/L.

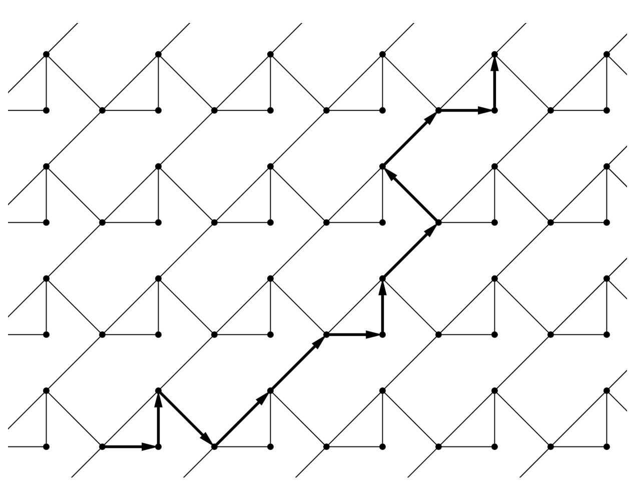


Figure 4. The walk p.

2.4. Growth sequences of periodic graphs. Let Γ be a locally finite graph, and let $x_0 \in V_\Gamma$. For $i \in \mathbb{Z}_{\geqslant 0}$, we define subsets $B_{\Gamma,x_0,i}, S_{\Gamma,x_0,i} \subset V_\Gamma$ by

$$B_{\Gamma,x_0,i} \coloneqq \{y \in V_\Gamma \mid d_\Gamma(x_0,y) \leqslant i\}, \quad S_{\Gamma,x_0,i} \coloneqq \{y \in V_\Gamma \mid d_\Gamma(x_0,y) = i\}.$$

Let $b_{\Gamma,x_0,i} \coloneqq \# B_{\Gamma,x_0,i}$ and $s_{\Gamma,x_0,i} \coloneqq \# S_{\Gamma,x_0,i}$ denote their cardinalities. The sequence $(s_{\Gamma,x_0,i})_i$ is called the *growth sequence* of Γ with the start point x_0. The sequence $(b_{\Gamma,x_0,i})_i$ is called the *cumulative growth sequence*.

The *growth series* $G_{\Gamma,x_0}(t)$ is the generating function

$$G_{\Gamma,x_0}(t) \coloneqq \sum_{i \geqslant 0} s_{\Gamma,x_0,i} t^i$$

of the growth sequence $(s_{\Gamma,x_0,i})_i$.

Remark 2.15. In crystallography, the growth sequence is called a *coordination sequence* (see [19]).

Definition 2.16 (cf. [24, Chapter 0]).

(1) *A function $f : \mathbb{Z} \to \mathbb{C}$ is called a quasi-polynomial if there exist a positive integer N and polynomials $Q_0, \dots, Q_{N-1} \in \mathbb{C}[x]$ such that $f(n) = Q_i(n)$ holds for all $n \in \mathbb{Z}$ and $i \in \{0, \dots, N-1\}$ with $n \equiv i \pmod N$. The polynomials $Q_0, \dots, Q_{N-1}$ are called the constituents of f.*

(2) *A function $g : \mathbb{Z} \to \mathbb{C}$ is called to be of quasi-polynomial type if there exists a non-negative integer $M \in \mathbb{Z}_{\geqslant 0}$ and a quasi-polynomial f such that $g(n) = f(n)$ holds for all $n > M$. The positive integer N is called a quasi-period of g when f is of the form in (1). Note that the notion of quasi-period is not unique. The minimum quasi-period is called the period of g. We say that the function g is a quasi-polynomial on $n \geqslant m$ if $g(n) = f(n)$ holds for $n \geqslant m$.*

The growth sequences of periodic graphs are known to be of quasi-polynomial type (Theorem 2.17).

Theorem 2.17 ([19, Theorem 2.2]). *Let (Γ, L) be a periodic graph, and let $x_0 \in V_\Gamma$. Then, the functions $b : i \mapsto b_{\Gamma,x_0,i}$ and $s : i \mapsto s_{\Gamma,x_0,i}$ are of quasi-polynomial type. In particular, its growth series is a rational function.*

In [19], Theorem 2.17 is proved for unweighted periodic graphs, and the same proof also works for weighted periodic graphs.

Example 2.18. One can show that the growth sequence of the Wakatsuki graph (see Example 2.6) with the start point v'_0 is given by $s_{\Gamma,v'_0,0} = 1$ and

$$s_{\Gamma,v'_0,n} = \begin{cases} \frac{9}{2}n - 1 & (n \equiv 0 \mod 2) \\ \frac{9}{2}n - \frac{1}{2} & (n \equiv 1 \mod 2) \end{cases}$$

for $n \geqslant 1$. The growth sequence is exactly the same when the start point is v'_1.

When the start point is v'_2, the growth sequence is given by $s_{\Gamma,v'_2,0} = 1$, $s_{\Gamma,v'_2,1} = 2$, $s_{\Gamma,v'_2,2} = 4$ and

$$s_{\Gamma,v'_2,n} = \begin{cases} 3n & (n \equiv 0 \mod 2) \\ 6n - 6 & (n \equiv 1 \mod 2) \end{cases}$$

for $n \geqslant 3$.

2.5. Growth polytope (periodic graphs → polytopes). In this subsection, we define the *growth polytope* $P_\Gamma \subset L_\mathbb{R}$ for a periodic graph (Γ, L). The concept of a growth polytope has been defined and studied in various contexts [15, 16, 27, 10, 20, 11, 1]. This is helpful in understanding the asymptotic behavior of the growth sequence via convex geometry (see Theorem A.2).

Definition 2.19. *Let (Γ, L) be an n-dimensional periodic graph.*

(1) *We define the normalization map $\nu : \mathrm{Cyc}_{\Gamma/L} \to L_\mathbb{R} := L \otimes_\mathbb{Z} \mathbb{R}$ by*

$$\nu : \mathrm{Cyc}_{\Gamma/L} \to L_\mathbb{R}; \quad p \mapsto \frac{\mu(\langle p \rangle)}{w(p)}.$$

We define the growth polytope

$$P_\Gamma := \mathrm{conv}\big(\mathrm{Im}(\nu) \cup \{0\}\big) \subset L_\mathbb{R}$$

as the convex hull of the set $\mathrm{Im}(\nu) \cup \{0\} \subset L_\mathbb{R}$. Note that $\mathrm{Cyc}_{\Gamma/L}$ is a finite set. Furthermore, we have $\mathrm{Im}(\nu) \subset L_\mathbb{Q}$ since $w(p) \in \mathbb{Z}_{>0}$ and $\mu(\langle p \rangle) \in L$ (cf. Definition 2.8). Therefore, P_Γ is a rational polytope (i.e., P_Γ is a polytope whose vertices are on $L_\mathbb{Q} := L \otimes_\mathbb{Z} \mathbb{Q}$). When Γ is strongly connected, we have $0 \in \mathrm{int}(P_\Gamma)$ by Lemma A.1.

(2) *For a polytope $Q \subset L_\mathbb{R}$ and $y \in L_\mathbb{R}$, we define*

$$d_Q(y) := \inf\{t \in \mathbb{R}_{\geqslant 0} \mid y \in tQ\} \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}.$$

When $0 \in \mathrm{int}(Q)$, we have $d_Q(y) < \infty$ for any $y \in L_\mathbb{R}$.

(3) *For a periodic realization $\Phi\colon V_\Gamma \to L_\mathbb{R}$, we define*

$$d_{P_\Gamma,\Phi}(x, y) := d_{P_\Gamma}\big(\Phi(y) - \Phi(x)\big)$$

for $x, y \in V_\Gamma$.

Remark 2.20. In this paper, we assume that the weight function w takes integer values (see Subsection 2.2). This assumption is used for P_Γ to be a rational polytope.

Next, we define the notation "a vertex is P-initial" as follows. As far as we know, this concept was first considered by Shutov and Maleev in [21].

Definition 2.21. *Let (Γ, L) be a periodic graph. A vertex $y \in V_\Gamma$ is called P-initial if the following condition holds:*

- *For any vertex $u \in V(P_\Gamma) \smallsetminus \{0\}$, there exists a cycle $p_u \in \mathrm{Cyc}_{\Gamma/L}$ such that $\nu(p_u) = u$ and $\overline{y} \in \mathrm{supp}(p_u)$.*

Example 2.22. For the Wakatsuki graph (see Example 2.6), $\mathrm{Im}(\nu)$ can be illustrated as in Figure 5. Here, the numbers written beside each point are the possible lengths of the cycles that give that point. For example, the cycle $q_1 := \overline{e'_5}\ \overline{e'_0}$ gives a point $\frac{\mu(\langle q_1 \rangle)}{w(q_1)} = \frac{1}{2}(1, 0)$. The cycle $q_2 := \overline{e'_6}\ \overline{e'_3}\ \overline{e'_9}$ gives a point $\frac{\mu(\langle q_2 \rangle)}{w(q_2)} = \frac{1}{3}(1, 1)$. In this case, the growth polytope P_Γ is a hexagon.

Furthermore, we can see that v'_0 and v'_1 are P-initial, but v'_2 is not. The six paths from v'_0 in Figure 6 give the six vertices of P_Γ, which shows that v'_0 is P-initial. On the other hand, there are no paths from v'_2 to $v'_2 + (0, 1)$ of length two. This causes v'_2 not to be P-initial.

Lemma 2.23. *Let (Γ, L) be a strongly connected periodic graph, and let $x_0 \in V_\Gamma$. Then we have*

$$d_{P_\Gamma}(y - x_0) \leqslant d_\Gamma(x_0, y)$$

for any $y \in V_\Gamma$ satisfying $\overline{y} = \overline{x_0}$.

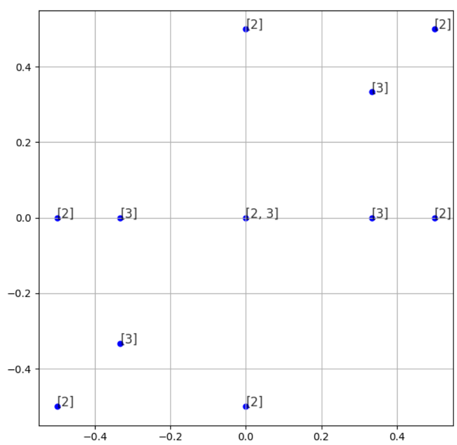


Figure 5. $\mathrm{Im}(\nu)$.

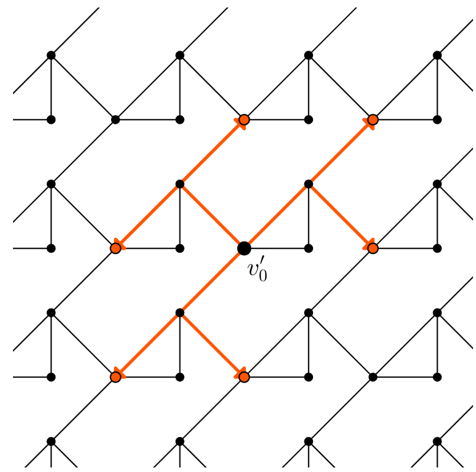


Figure 6. Paths giving $V(P_\Gamma)$.

Proof. Let p be a walk in Γ from x_0 to y satisfying $w(p) = d_\Gamma(x_0, y)$. By Lemma 2.12, $\overline{p}$ decomposes to a walkable sequence $(q_0, q_1, \dots, q_\ell)$ with $q_0 = \varnothing_{\overline{x_0}}$ such that $\langle \overline{p} \rangle = \sum_{i=1}^{\ell} \langle q_i \rangle$. Then, we have

$$y - x_0 = \sum_{i=1}^{\ell} \mu(\langle q_i \rangle) \in \sum_{i=1}^{\ell} w(q_i) \cdot P_\Gamma = w(p) \cdot P_\Gamma,$$

which proves the desired inequality. □

We define $C_1(\Gamma, \Phi, x_0)$ and $C_2(\Gamma, \Phi, x_0)$ as invariants that measure the difference between d_Γ and $d_{P_\Gamma, \Phi}$.

Definition 2.24. *Let (Γ, L) be a strongly connected periodic graph. Let $\Phi : V_\Gamma \to L_\mathbb{R}$ be a periodic realization, and let $x_0 \in V_\Gamma$. Then, we define*

$$C_1(\Gamma, \Phi, x_0) \coloneqq \sup_{y \in V_\Gamma} \big(d_{P_\Gamma, \Phi}(x_0, y) - d_\Gamma(x_0, y) \big),$$

$$C_2(\Gamma, \Phi, x_0) \coloneqq \sup_{y \in V_\Gamma} \big(d_\Gamma(x_0, y) - d_{P_\Gamma, \Phi}(x_0, y) \big).$$

By Theorem A.2*, we have $C_1(\Gamma, \Phi, x_0) < \infty$ and $C_2(\Gamma, \Phi, x_0) < \infty$.*

Remark 2.25.

(1) By the proof of Theorem A.2, we have

$$C_1(\Gamma, \Phi, x_0) = \max_{y \in B'_{c-1}} \big(d_{P_\Gamma, \Phi}(x_0, y) - d_\Gamma(x_0, y) \big),$$

where $c \coloneqq \#(V_\Gamma / L)$ and

$$B'_{c-1} \coloneqq \{ y \in V_\Gamma \mid \text{there exists a walk } p \text{ from } x_0 \text{ to } y \text{ with length}(p) \leqslant c - 1 \}.$$

(2) It is not so easy to calculate $C_2(\Gamma, \Phi, x_0)$ in general. In Proposition 4.4, we will discuss a way of the calculation of $C_2(\Gamma, \Phi, x_0)$ when x_0 is P-initial.

Lemma 2.26. *Let (Γ, L) be a strongly connected periodic graph. Let $\Phi : V_\Gamma \to L_\mathbb{R}$ be a periodic realization, and let $x_0 \in V_\Gamma$.*

(1) *We have $C_1(\Gamma, \Phi, x_0) \geqslant 0$ and $C_2(\Gamma, \Phi, x_0) \geqslant 0$.*
(2) *Suppose $\#(V_\Gamma / L) = 1$. Then, any $y \in V_\Gamma$ is P-initial. Furthermore, we have $C_1(\Gamma, \Phi, x_0) = 0$.*
(3) *If $C_2(\Gamma, \Phi, x_0) < 1$, then x_0 is P-initial.*

Proof. (1) can be easily shown by setting $y = x_0$ in the definition of $C_1(\Gamma, \Phi, x_0)$ and $C_2(\Gamma, \Phi, x_0)$. The first assertion of (2) immediately follows from the definition of being P-initial. The second assertion of (2) follows from Lemma 2.23.

We prove (3). Suppose $C_2(\Gamma, \Phi, x_0) < 1$. Let $u \in V(P_\Gamma)$. We take $d \in \mathbb{Z}_{>0}$ such that $du \in L$. Then, by Lemma 2.23, we have

$$d_\Gamma(x_0, du + x_0) \geqslant d_{P_\Gamma}(du) = d \in \mathbb{Z}_{>0}.$$

Since $C_2(\Gamma, \Phi, x_0) < 1$, we have $d_\Gamma(x_0, du + x_0) = d$. Here, we have used the fact that the weight function w_Γ is defined to be integral. Therefore, there exists a walk p in Γ from x_0 to $du + x_0$ such that $w(p) = d$. By Lemma 2.12(1), $\overline{p}$ decomposes to a walkable sequence $(q_0, q_1, \ldots, q_\ell)$ with $q_0 = \varnothing_{\overline{x_0}}$ such that $\langle \overline{p} \rangle = \sum_{i=1}^{\ell} \langle q_i \rangle$. Then, we have

$$\frac{\sum_{i=1}^{\ell} \mu(\langle q_i \rangle)}{\sum_{i=1}^{\ell} w(q_i)} = \frac{\mu(\langle \overline{p} \rangle)}{d} = u \in V(P_\Gamma).$$

Here, $\frac{\sum_{i=1}^{\ell} \mu(\langle q_i \rangle)}{\sum_{i=1}^{\ell} w(q_i)}$ is a convex combination of $\frac{\mu(\langle q_1 \rangle)}{w(q_1)}, \ldots, \frac{\mu(\langle q_\ell \rangle)}{w(q_\ell)} \in P_\Gamma$. Since $\frac{\sum_{i=1}^{\ell} \mu(\langle q_i \rangle)}{\sum_{i=1}^{\ell} w(q_i)} = u$ is a vertex of P_Γ, we conclude that $\frac{\mu(\langle q_i \rangle)}{w(q_i)} = u$ for any $1 \leqslant i \leqslant \ell$. By Lemma 2.12(2), $\overline{x_0} \in \operatorname{supp}(q_i)$ holds for some $1 \leqslant i \leqslant \ell$. Therefore, we can conclude that x_0 is P-initial. □

3. Ehrhart theory on periodic graphs

In Subsection 3.1, we treat a class of periodic graphs for which Ehrhart theory can be applied. More precisely, in Theorem 3.4(1)(2), we see that the cumulative growth sequence $(b_{\Gamma,x_0,i})_i$ is a quasi-polynomial on $i \geqslant 0$ if a periodic realization Φ satisfies $C_1(\Gamma, \Phi, x_0) + C_2(\Gamma, \Phi, x_0) < 1$. Furthermore, in Theorem 3.4(3), we see that the growth series has the same reciprocity law as the Ehrhart series of reflexive polytopes if a periodic realization Φ satisfies both $C_1(\Gamma, \Phi, x_0) < \frac{1}{2}$ and $C_2(\Gamma, \Phi, x_0) < \frac{1}{2}$. Theorem 3.4 can be seen as a generalization of a result of Conway and Sloane [7], where they treat the contact graphs of lattices (see Remark 3.6). In the proof of Theorem 3.4, we essentially use a variant of Ehrhart theory that is proved in Appendix B.

In Subsection 3.2, we construct periodic graphs (Γ_Q, L) from rational polytopes Q. By this construction, Theorem 3.4 can be seen as a generalization of the Ehrhart theory for polytopes Q with $0 \in Q$.

3.1. Ehrhart graphs.

Definition 3.1. *Let (Γ, L) be a strongly connected periodic graph, and let $\Phi : V_\Gamma \to L_\mathbb{R}$ be a periodic realization. Let $x_0 \in V_\Gamma$ and let $\alpha \in \mathbb{R}$. The triple (Γ, Φ, x_0) is called to be α-Ehrhart if we have*

$$B_{\Gamma,x_0,i} = \{y \in V_\Gamma \mid d_{P_\Gamma,\Phi}(x_0, y) \leqslant i + \alpha\}$$

for all $i \in \mathbb{Z}_{\geqslant 0}$.

This condition is equivalent to the condition that

$$d_{P_\Gamma,\Phi}(x_0, y) - \alpha \leqslant d_\Gamma(x_0, y) < d_{P_\Gamma,\Phi}(x_0, y) + 1 - \alpha$$

holds for all $y \in V_\Gamma$.

Definition 3.2. *Let (Γ, L) be a strongly connected n-dimensional periodic graph. Let $\Phi : V_\Gamma \to L_\mathbb{R}$ be a periodic realization, and let $x_0 \in V_\Gamma$. We say that Φ is symmetric with respect to x_0 if $\#(\Phi^{-1}(y)) = \#(\Phi^{-1}(y'))$ holds for all $y, y' \in L_\mathbb{R}$ satisfying $y' + y = 2\Phi(x_0)$.*

Remark 3.3. When $\#(V_\Gamma/L) = 1$, there exists an essentially unique periodic realization Φ (unique up to translation), and this Φ is symmetric with respect to any vertex $x_0 \in V_\Gamma$.

Theorem 3.4. *Let (Γ, L) be a strongly connected n-dimensional periodic graph. Let $\Phi\colon V_\Gamma \to L_\mathbb{R}$ be a periodic realization, and let $x_0 \in V_\Gamma$. Let $s_i := s_{\Gamma,x_0,i}$ and $b_i := b_{\Gamma,x_0,i}$ be the growth sequence and the cumulative growth sequence with the start point x_0. Let $G_s(t) := \sum_{i\geqslant 0} s_i t^i$ and $G_b(t) := \sum_{i\geqslant 0} b_i t^i$ be their generating functions. Set $C_1 := C_1(\Gamma, \Phi, x_0)$ and $C_2 := C_2(\Gamma, \Phi, x_0)$.*

(1) *Suppose that the triple (Γ, Φ, x_0) is α-Ehrhart for some $\alpha \in [0, 1)$. Then, the function $i \mapsto b_i$ is a quasi-polynomial on $i \geqslant 0$.*
(2) *Suppose $C_1 + C_2 < 1$. Then, (Γ, Φ, x_0) is α-Ehrhart for any $\alpha \in [C_1, 1 - C_2)$.*
(3) *Suppose both $C_1 < \frac{1}{2}$ and $C_2 < \frac{1}{2}$. Suppose one of the following conditions holds:*
 (i) *Γ is undirected, or*
 (ii) *Φ is symmetric with respect to x_0.*
 Then, we have
$$G_b(1/t) = (-1)^{n+1} t G_b(t), \qquad G_s(1/t) = (-1)^n G_s(t).$$
 In particular, we have
$$f_b(-i) = (-1)^n f_b(i-1)$$
 for any $i \in \mathbb{Z}$, and
$$f_s(-i) = (-1)^{n+1} f_s(i)$$
 for any $i \in \mathbb{Z} \setminus \{0\}$, where f_b and f_s are the quasi-polynomials corresponding to the sequences $(b_i)_i$ and $(s_i)_i$.

Proof. As in Appendix B, for a rational polytope $P \subset L_\mathbb{R}$, $v \in L_\mathbb{R}$, and $\beta \in \mathbb{R}$, we define a function $h_{P,v,\beta} : \mathbb{Z} \to \mathbb{Z}$ by
$$h_{P,v,\beta}(i) := \#\big((v + (i+\beta)P) \cap L\big),$$
and its generating function $H_{P,v,\beta}(t) = \sum_{i\in\mathbb{Z}} h_{P,v,\beta}(i) t^i$. We also define $\overset{\circ}{h}_{P,v,\beta}$ and $\overset{\circ}{H}_{P,v,\beta}$ similarly. Note that we have $\operatorname{relint}(P_\Gamma) = \operatorname{int}(P_\Gamma)$ by the assumption that Γ is strongly connected (see Lemma A.1).

Let $c := \#(V_\Gamma/L)$. Take $y_1, \dots, y_c \in V_\Gamma$ such that $\{\overline{y}_1, \dots, \overline{y}_c\} = V_\Gamma/L$. Then, we have $V_\Gamma = \bigsqcup_{j=1}^c (y_j + L)$, and hence,
$$\begin{aligned} B_{\Gamma,x_0,i} &= \{y \in V_\Gamma \mid d_{P_\Gamma,\Phi}(x_0, y) \leqslant i + \alpha\} \\ &= \Phi^{-1}\big(\Phi(x_0) + (i+\alpha)P_\Gamma\big) \\ &= \bigsqcup_{j=1}^c \big(\Phi^{-1}\big(\Phi(x_0) + (i+\alpha)P_\Gamma\big) \cap (y_j + L)\big) \\ &= \bigsqcup_{j=1}^c \big(y_j + \big\{m \in L \;\big|\; \Phi(y_j) + m \in \Phi(x_0) + (i+\alpha)P_\Gamma\big\}\big) \\ &= \bigsqcup_{j=1}^c \big(y_j + \big(\Phi(x_0) - \Phi(y_j) + (i+\alpha)P_\Gamma\big) \cap L\big). \end{aligned}$$

Hence, we have
$$b_i = \sum_{j=1}^c \#\big(\big(\Phi(x_0) - \Phi(y_j) + (i+\alpha)P_\Gamma\big) \cap L\big) = \sum_{j=1}^c h_{P_\Gamma, \Phi(x_0)-\Phi(y_j), \alpha}(i).$$

Therefore, (1) follows from Theorem B.4(1).

(2) follows from the definitions of C_1 and C_2.

We prove (3). When the condition (ii) is satisfied, we have

$$(\heartsuit)\qquad \#\big(\Phi^{-1}(\Phi(x_0)+a\cdot\mathrm{int}(P_\Gamma))\big)=\#\big(\Phi^{-1}(\Phi(x_0)+a\cdot\mathrm{int}(-P_\Gamma))\big)$$

for any $a\in\mathbb{R}_{\geqslant 0}$. When the condition (i) is satisfied, we have $-P_\Gamma=P_\Gamma$, and the same assertion ($\heartsuit$) holds.

By (2), (Γ,Φ,x_0) is $\frac{1}{2}$-Ehrhart. Furthermore, (Γ,Φ,x_0) is α-Ehrhart for $\alpha=\frac{1}{2}-\epsilon$ for sufficiently small $\epsilon>0$. Therefore, we have both

$$B_{\Gamma,x_0,i}=\Phi^{-1}\left(\Phi(x_0)+\left(i+\frac{1}{2}\right)P_\Gamma\right),$$
$$B_{\Gamma,x_0,i}=\Phi^{-1}\left(\Phi(x_0)+\left(i+\frac{1}{2}\right)\mathrm{int}(P_\Gamma)\right)$$

for any $i\in\mathbb{Z}$. Therefore, we have

$$\begin{aligned}
G_b(t^{-1})&=\sum_{j=1}^{c}H_{P_\Gamma,\Phi(x_0)-\Phi(y_j),\frac{1}{2}}(t^{-1})\\
&=(-1)^{n+1}\sum_{j=1}^{c}\mathring{H}_{P_\Gamma,-(\Phi(x_0)-\Phi(y_j)),-\frac{1}{2}}(t)\\
&=(-1)^{n+1}\sum_{j=1}^{c}\mathring{H}_{-P_\Gamma,\Phi(x_0)-\Phi(y_j),-\frac{1}{2}}(t)\\
&=(-1)^{n+1}t\sum_{j=1}^{c}\mathring{H}_{-P_\Gamma,\Phi(x_0)-\Phi(y_j),\frac{1}{2}}(t)\\
&=(-1)^{n+1}t\sum_{j=1}^{c}\mathring{H}_{P_\Gamma,\Phi(x_0)-\Phi(y_j),\frac{1}{2}}(t)\\
&=(-1)^{n+1}tG_b(t).
\end{aligned}$$

Here, the second equality follows from Theorem B.4(3), the third follows from Lemma B.3, and the fifth follows from ($\heartsuit$). Since $G_s(t)=(1-t)G_b(t)$, we have

$$\begin{aligned}
G_s(t^{-1})&=(1-t^{-1})G_b(t^{-1})=(-1)^{n+1}(1-t^{-1})tG_b(t)\\
&=(-1)^{n+1}(1-t^{-1})t(1-t)^{-1}G_s(t)=(-1)^nG_s(t).
\end{aligned}$$

Since f_b is a quasi-polynomial, we have

$$\sum_{i\in\mathbb{Z}_{<0}}f_b(i)t^i=-\sum_{i\in\mathbb{Z}_{\geqslant 0}}f_b(i)t^i$$

as rational functions (cf. [5, Exercise 4.7]). Therefore, we have

$$\begin{aligned}
\sum_{i\in\mathbb{Z}_{>0}}f_b(-i)t^{-i}&=-\sum_{i\in\mathbb{Z}_{\geqslant 0}}f_b(i)t^i\\
&=-G_b(t)\\
&=(-1)^nt^{-1}G_b(t^{-1})\\
&=(-1)^nt^{-1}\sum_{i\in\mathbb{Z}_{\geqslant 0}}f_b(i)t^{-i}\\
&=(-1)^n\sum_{i\in\mathbb{Z}_{>0}}f_b(i-1)t^{-i}
\end{aligned}$$

Here, the second equality follows because the function $i \mapsto b_i$ is a quasi-polynomial on $i \geqslant 0$. By comparing the coefficients, we can conclude that $f_b(-i) = (-1)^n f_b(i-1)$ for any $i \in \mathbb{Z}_{>0}$. The statement for $f_s(-i)$ follows from the same argument. □

Remark 3.5. The reciprocity laws appearing in Theorem 3.4(3) are the same as the reciprocity laws of the Ehrhart series of reflexive polytopes (cf. [5, Section 4.4]). We will see in Remark 3.12 that Theorem 3.4(3) can be seen as a generalization of the reciprocity laws of the Ehrhart series of reflexive polytopes.

Remark 3.6. In this remark, we explain that Theorem 3.4 is a generalization of results of Conway and Sloane in [7], where only the case $\#(V_\Gamma/L) = 1$ is treated.

In [7], Conway and Sloane study the growth sequence of the contact graph Γ of an n-dimensional lattice L in $\mathbb{R}^n$ that is spanned by its minimal vectors. More precisely, they considered the graphs obtained in the following way:

- $L \subset \mathbb{R}^n$ is a lattice of rank n. Let F be the set of all $v \in L \setminus \{0\}$ such that its Euclidean norm $||v||$ is the smallest among $L \setminus \{0\}$. Suppose that L is spanned by F.
- Define a periodic graph (Γ, L) by
  - $V_\Gamma := L$, $E_\Gamma := L \times F$, and
  - for $e = (x, v) \in E_\Gamma$, we set

$$s(e) := x, \qquad t(e) := x + v, \qquad w(e) = 1.$$

Note that $\#(V_\Gamma/L) = 1$ in this case.

Conway and Sloane define the "contact polytope" $\mathcal{P}$ of L as the convex hull of F, and they study the growth sequence of Γ using Ehrhart theory on $\mathcal{P}$. Note that $\mathcal{P}$ coincides with the growth polytope P_Γ in our notation. Since $\#(V_\Gamma/L) = 1$, we have $C_1 = 0$.

Conway and Sloane also introduce the notations "well-placed", "well-rounded" and "well-coordinated" according to the property of L. The condition "well-placed" coincides with the condition that the $\mathcal{P}$ is a reflexive polytope. The condition "well-rounded" coincides with the condition "$C_2 < 1$". L is called "well-coordinated" if L is well-placed and well-rounded. They prove the following assertions ([7, Theorems 2.5 and 2.9]):

- If L is well-rounded, the growth sequence $(b_i)_i$ of Γ is a polynomial on $i \geqslant 0$.
- If L is well-coordinated, the growth sequence satisfies the reciprocity laws in Theorem 3.4(3).

Therefore, Theorem 3.4 can be seen as the generalization of these results to the case where $\#(V_\Gamma/L) > 1$.

3.2. Polytopes → periodic graphs.

In this subsection, we define a periodic graph Γ_Q from a rational polytope Q, and we see that the study of the growth sequences of periodic graphs can be essentially seen as a generalization of the Ehrhart theory of rational polytopes Q satisfying $0 \in Q$.

First, we define a periodic graph Γ_Q for a rational polytope Q (possibly $0 \notin Q$).

Definition 3.7. *Let $Q \subset \mathbb{R}^N$ be a d-dimensional rational polytope. Let a be the minimum positive integer such that aQ is a lattice polytope. We define a graph Γ_Q as follows:*

- $V_{\Gamma_Q} := \mathbb{Z}^N$.
- $E_{\Gamma_Q} := \mathbb{Z}^N \times F_Q$, *where* $F_Q := \left\{(i, m) \in \mathbb{Z}_{>0} \times \mathbb{Z}^N \;\middle|\; i < a(d+1),\ m \in iQ\right\}$.
- *For* $e = (x, (i, m)) \in E_{\Gamma_Q}$, *we define*

$$s(e) := x, \qquad t(e) := x + m, \qquad w(e) := i.$$

Then, (Γ_Q, L) *for* $L \coloneqq \mathbb{Z}^N$ *becomes an* N*-dimensional periodic graph. Since* $\#\left(V_{\Gamma_Q}/L\right) = 1$*, there exists the unique realization* $\Phi : V_{\Gamma_Q} \to L_{\mathbb{R}}$ *such that* $\Phi(0) = 0$*. We set* $C_i \coloneqq C_i(\Gamma_Q, \Phi, 0)$ *for* $i \in \{1, 2\}$*.*

Remark 3.8. Let i be a positive integer satisfying $i < a(d+1)$, and let $x, y \in V_{\Gamma_Q} = \mathbb{Z}^N$ be any two vertices. Then, the graph Γ_Q is defined so that the following two conditions are equivalent:

- There exists an edge from x to y of weight i.
- $y - x \in iQ$.

Note that without the boundedness condition "$i < a(d+1)$" in the definition of F_Q, we could have $\#\left(E_{\Gamma_Q}/L\right) = \#F_Q = \infty$, and therefore, Γ_Q could not be a periodic graph. This specific value "$a(d+1)$" will be used in the proof of Lemma 3.10(1) when applying Lemma 3.11.

Example 3.9. Let $N = d = 2$, and let $Q = \mathrm{conv}\big(\{(0,0), (0,1/2), (1/2,0)\}\big)$. In this case, we have $a = 2$ and

$$\begin{aligned} F_Q = &\{(i,(0,0)) \mid 1 \leqslant i \leqslant 5\} \cup \{(i,(1,0)) \mid 2 \leqslant i \leqslant 5\} \cup \{(i,(0,1)) \mid 2 \leqslant i \leqslant 5\} \\ &\cup \{(i,(1,1)) \mid i = 4,5\} \cup \{(i,(2,0)) \mid i = 4,5\} \cup \{(i,(0,2)) \mid i = 4,5\}. \end{aligned}$$

For example, for all $x \in V_{\Gamma_Q} = \mathbb{Z}^2$, there are four distinct edges from x to $x + (1,0)$ of weights $2, 3, 4$ and 5.

Lemma 3.10. *Let* $x \in \mathbb{Z}^N$*.*

(1) *For any* $i \in \mathbb{Z}_{\geqslant 0}$*, the condition* $x \in \bigcup_{0 \leqslant j \leqslant i} jQ$ *is equivalent to the condition* $d_{\Gamma_Q}(0, x) \leqslant i$*. In particular, the cumulative growth sequence* $b_{\Gamma_Q,0,i}$ *coincides with*
$$\#\left(\left(\bigcup_{0 \leqslant j \leqslant i} jQ\right) \cap \mathbb{Z}^N\right).$$
(2) *The growth polytope* P_{Γ_Q} *coincides with* $\mathrm{conv}(Q \cup \{0\})$*.*
(3) *When* $0 \in Q$*, we have* $b_{\Gamma_Q,0,i} = \#\big(iQ \cap \mathbb{Z}^N\big)$*.*
(4) *If* $0 \in Q$ *and* $d_Q(x) < \infty$*, we have* $d_{\Gamma_Q}(0, x) = \lceil d_Q(x) \rceil$*.*
(5) *The strong connectedness of* Γ_Q *is equivalent to the condition that* $0 \in \mathrm{int}(Q)$*.*
(6) *If* $0 \in \mathrm{int}(Q)$*, we have*
$$C_1 = 0, \qquad C_2 \in \{0\} \cup \left[\frac{1}{2}, 1\right).$$
(7) *Suppose* $0 \in \mathrm{int}(Q)$*. Then,* $C_2 = 0$ *holds if and only if*
$$(i+1)\,\mathrm{int}(Q) \cap \mathbb{Z}^N = iQ \cap \mathbb{Z}^N$$
holds for all $i \in \mathbb{Z}_{\geqslant 0}$*.*

Proof. We prove (1). Let $i \in \mathbb{Z}_{\geqslant 0}$. First, we suppose that $d_{\Gamma_Q}(0, x) \leqslant i$. Then, there exists a path $p = e_1 \cdots e_\ell$ of Γ_Q from 0 to x with $w(p) \leqslant i$. By the definition of Γ_Q (cf. Remark 3.8), we have $t(e_i) - s(e_i) \in w(e_i) \cdot Q$ for all $1 \leqslant i \leqslant \ell$. Hence, we have

$$x = \sum_{i=1}^{\ell} \big(t(e_i) - s(e_i)\big) \in \left(\sum_{i=1}^{\ell} w(e_i)\right) \cdot Q = w(p) \cdot Q \subset \bigcup_{0 \leqslant j \leqslant i} jQ.$$

Next, we suppose that $x \in jQ$ for some $0 \leqslant j \leqslant i$. Set $b \coloneqq \max\left\{0, \left\lfloor \frac{j - a(d+1)}{a} \right\rfloor + 1\right\}$. Then, by Lemma 3.11 below, we have

$$x \in jQ \cap \mathbb{Z}^N \subset \left((j - ba)Q \cap \mathbb{Z}^N\right) + b\left(aQ \cap \mathbb{Z}^N\right).$$

Therefore, there exist $m_0 \in (j-ba)Q \cap \mathbb{Z}^N$ and $m_1, \dots, m_b \in aQ \cap \mathbb{Z}^N$ such that $x = \sum_{k=0}^{b} m_k$. For each $0 \leqslant \ell \leqslant b$, we set $x_\ell \coloneqq \sum_{k=0}^{\ell} m_k$. If $j - ba \neq 0$, there exists an edge from 0 to x_0 of weight $j - ba$ by the definition of Γ_Q (cf. Remark 3.8). Here, we have used the fact $j - ba < a(d+1)$ which follows from the choice of b. Therefore, we have $d_{\Gamma_Q}(0, x_0) \leqslant j - ba$ (this is correct even if $j - ba = 0$). Similarly, we have $d_{\Gamma_Q}(x_\ell, x_{\ell+1}) \leqslant a$ for all $0 \leqslant \ell \leqslant b-1$. Hence, we have

$$d_{\Gamma_Q}(0, x) \leqslant d_{\Gamma_Q}(0, x_0) + \sum_{\ell=0}^{b-1} d_{\Gamma_Q}(x_\ell, x_{\ell+1}) \leqslant j - ba + ba = j \leqslant i,$$

which completes the proof of (1).

We prove (2). By the definition of the growth polytope P_{Γ_Q}, we have

$$\begin{aligned} P_{\Gamma_Q} &= \operatorname{conv}\left(\left\{ \frac{t(e) - s(e)}{w(e)} \;\middle|\; e \in E_{\Gamma_Q} \right\} \cup \{0\} \right) \\ &= \operatorname{conv}\left(\left\{ \frac{m}{i} \;\middle|\; (i, m) \in F_Q \right\} \cup \{0\} \right). \end{aligned}$$

By the definition of F_Q, we have

$$\operatorname{conv}\left(\left\{ \frac{m}{i} \;\middle|\; (i, m) \in F_Q \right\} \right) = Q,$$

which completes the proof of (2).

(3) and (4) follow from (1) since we have $\bigcup_{0 \leqslant j \leqslant i} jQ = iQ$ when $0 \in Q$. (5) also follows from (1). (7) follows from (4).

We shall see (6) below. The assertion $C_1 = 0$ follows from Lemma 2.26(2) (or directly from (4)). By (4), we have

$$C_2 = \sup_{x \in \mathbb{Z}^N} \big(\lceil d_Q(x) \rceil - d_Q(x) \big) = \max_{x \in \mathbb{Z}^N} \big(\lceil d_Q(x) \rceil - d_Q(x) \big) < 1.$$

If $\lceil d_Q(x) \rceil - d_Q(x) \in \left[0, \frac{1}{2}\right)$, we have $\lceil d_Q(2x) \rceil - d_Q(2x) = 2(\lceil d_Q(x) \rceil - d_Q(x))$. Therefore, we can conclude $C_2 \notin \left(0, \frac{1}{2}\right)$. ☐

Lemma 3.11 (cf. [8, Theorem 2.2.12]). *For $k \in \mathbb{R}_{\geqslant a(d+1)}$, we have*

$$kQ \cap \mathbb{Z}^N \subset \big((k-a)Q \cap \mathbb{Z}^N \big) + \big(aQ \cap \mathbb{Z}^N \big).$$

Proof. By taking a triangulation of Q, we may assume that Q is a simplex. Let $v_0, \dots, v_d \in \mathbb{Q}^N$ be its vertices. By the choice of a, we may write $v_i = m_i / a$ for some $m_i \in \mathbb{Z}^N$. Let $m \in kQ \cap \mathbb{Z}^N$. Then, we may uniquely write

$$m = \sum_{i=0}^{d} \alpha_i v_i = \sum_{i=0}^{d} \frac{\alpha_i}{a} \cdot m_i$$

for some $\alpha_i \in \mathbb{R}_{\geqslant 0}$ satisfying $\sum_{i=0}^{d} \alpha_i = k$. Since $\sum_{i=0}^{d} \alpha_i = k \geqslant a(d+1)$, there exists $i \in \{0, 1, \dots, d\}$ such that $\frac{\alpha_i}{a} \geqslant 1$. Then, for such i, we have

$$m = (m - m_i) + m_i \in \big((k-a)Q \cap \mathbb{Z}^N \big) + \big(aQ \cap \mathbb{Z}^N \big).$$

We complete the proof. ☐

Remark 3.12. For a lattice polytope Q, it is known that Q is a reflexive polytope if and only if the condition

$$(i+1)\operatorname{int}(Q) \cap \mathbb{Z}^N = iQ \cap \mathbb{Z}^N$$

holds for all $i \in \mathbb{Z}_{\geqslant 0}$ (cf. [5, Section 4.4]). Therefore, Theorem 3.4(3) can be seen as a generalization of the reciprocity laws of the Ehrhart series of reflexive polytopes.

4. P-initial vertex and well-arranged graphs

4.1. The P-initial case. In this subsection, we treat a periodic graph (Γ, L) and a P-initial vertex $x_0 \in V_\Gamma$. In this case, we can calculate the invariant C_2 (Proposition 4.4) and a quasi-period of the growth sequence of Γ with the start point x_0 (Theorem 4.2).

The following lemma will be used in Theorem 4.2.

Lemma 4.1. *Let (Γ, L) be a periodic graph, and let $x_0 \in V_\Gamma$. Suppose that x_0 is P-initial. For each $v \in V(P_\Gamma) \setminus \{0\}$, we pick a cycle $q_v \in \nu^{-1}(v)$ such that $\overline{x_0} \in \operatorname{supp}(q_v)$. We define*

$$B \coloneqq \{(i, y) \in \mathbb{Z}_{\geqslant 0} \times V_\Gamma \mid d_\Gamma(x_0, y) \leqslant i\} \subset \mathbb{Z}_{\geqslant 0} \times V_\Gamma.$$

We define a subset $M' \subset \mathbb{Z}_{\geqslant 0} \times L$ by

$$M' \coloneqq \big\{ \big(w(q_v), \mu(\langle q_v \rangle)\big) \;\big|\; v \in V(P_\Gamma) \setminus \{0\} \big\}.$$

Let $M \subset \mathbb{Z}_{\geqslant 0} \times L$ be the submonoid generated by M' and $(1, 0)$.

Then, B is a finitely generated M-module.

Proof. First, we prove that B is an M-module (i.e. $M + B \subset B$). Take $(i, y) \in B$ and $v \in V(P_\Gamma) \setminus \{0\}$. Then, by the definition of B, there exists a walk p in Γ from x_0 to y satisfying $w(p) \leqslant i$. By the choice of q_v, we have $\overline{x_0} \in \operatorname{supp}(q_v)$. In particular, we have $\operatorname{supp}(\overline{p}) \cap \operatorname{supp}(q_v) \neq \varnothing$. Therefore, there exists a path p' in Γ from x_0 such that

$$\langle \overline{p'} \rangle = \langle q_v \rangle + \langle \overline{p} \rangle.$$

Then, we have

$$t(p') = \mu(\langle q_v \rangle) + t(p) = \mu(\langle q_v \rangle) + y.$$

Furthermore, we have

$$w(p') = w(q_v) + w(p) \leqslant w(q_v) + i.$$

They show that $\big(w(q_v) + i, \mu(\langle q_v \rangle) + y\big) \in B$. Hence, we can conclude that $M' + B \subset B$. Since it is clear that $(1, 0) + B \subset B$, we conclude that $M + B \subset B$.

Next, we prove that the M-module B is generated by some finite subset $B' \subset B$. For each $q \in \operatorname{Cyc}_{\Gamma/L}$, we take a positive integer d_q with the following condition:

- Let $\operatorname{Facet}'(P_\Gamma)$ be the set of $\sigma \in \operatorname{Facet}(P_\Gamma)$ satisfying $0 \notin \sigma$. First, for each $\sigma \in \operatorname{Facet}'(P_\Gamma)$, we fix a triangulation T_σ of σ such that $V(\Delta) \subset V(\sigma)$ holds for all $\Delta \in T_\sigma$.
- For each $q \in \operatorname{Cyc}_{\Gamma/L}$, we take $\sigma \in \operatorname{Facet}'(P_\Gamma)$ and $\Delta \in T_\sigma$ such that $\nu(q) \in \mathbb{R}_{\geqslant 0}\Delta$. Then, we take a positive integer d_q such that

  $$d_q \cdot \mu(\langle q \rangle) = \sum_{v \in V(\Delta)} b_v \cdot \mu(\langle q_v \rangle)$$

  holds for some $b_v \in \mathbb{Z}_{\geqslant 0}$.

We note that for $q \in \operatorname{Cyc}_{\Gamma/L}$, Δ and b_v's above, we have

$$d_q \cdot w(q) \geqslant \sum_{v \in V(\Delta)} b_v \cdot w(q_v)$$

since $\frac{\mu(\langle q \rangle)}{w(q)} = \nu(q) \in [0, 1] \cdot \Delta$. Therefore, we have

$$d_q \cdot \big(w(q), \mu(\langle q \rangle)\big) \in M.$$

We shall show that the M-module B is generated by

$$B' \coloneqq \left\{ (i, y) \in B \;\middle|\; i \leqslant W \cdot \big(\#(V_\Gamma/L)\big)^2 + \sum_{q \in \operatorname{Cyc}_{\Gamma/L}} (d_q - 1) \cdot w(q) \right\},$$

where $W := \max_{e\in E_\Gamma} w(e)$.

Take $(i,y)\in B$. Then, by the definition of B, there exists a walk p in Γ from x_0 to y satisfying $w(p)\leqslant i$. By decomposing $\overline{p}$ (Lemma 2.12(1)), there exists a walkable sequence $(q_0,q_1,\dots,q_\ell)$ such that $\langle\overline{p}\rangle=\sum_{i=0}^{\ell}\langle q_i\rangle$. By Lemma 2.12(2), by rearranging the indices of $q_1,\dots,q_\ell$, we may assume the following condition for each $0\leqslant j\leqslant \ell-1$:

- $\left(\bigcup_{0\leqslant i\leqslant j}\operatorname{supp}(q_i)\right)\cap\operatorname{supp}(q_{j+1})\neq\varnothing$.

Furthermore, we may also assume the following condition for each $0\leqslant j\leqslant\ell-1$:

- If $\bigcup_{0\leqslant i\leqslant j}\operatorname{supp}(q_i)\neq\operatorname{supp}(\overline{p})$, then $\operatorname{supp}(q_{j+1})\not\subset\bigcup_{0\leqslant i\leqslant j}\operatorname{supp}(q_i)$.

In particular, for $\ell' := \#(\operatorname{supp}(\overline{p}))-\#(\operatorname{supp}(q_0))\leqslant\#(V_\Gamma/L)-\operatorname{length}(q_0)-1$, it follows that

- $(q_0,q_1,\dots,q_{\ell'})$ is a walkable sequence, and
- $\bigcup_{0\leqslant i\leqslant\ell'}\operatorname{supp}(q_i)=\operatorname{supp}(\overline{p})$.

For each $q\in\mathrm{Cyc}_{\Gamma/L}$, we define $\alpha_q\in\mathbb{Z}_{\geqslant 0}$ by

$$\alpha_q := \#\{\ell'+1\leqslant i\leqslant\ell \mid q_i=q\}.$$

Let $\beta_q\in\mathbb{Z}_{\geqslant 0}$ be the integer satisfying $0\leqslant\beta_q<d_q$ and $\beta_q\equiv\alpha_q \pmod{d_q}$. We set $\ell'' := \ell'+\sum_{q\in\mathrm{Cyc}_{\Gamma/L}}\beta_q$. Then, by rearranging the indices of $q_{\ell'+1},\dots,q_\ell$, we may assume the following condition

$$\#\{\ell'+1\leqslant i\leqslant\ell''\mid q_i=q\}=\beta_q.$$

Since $\operatorname{supp}(q_i)\subset\operatorname{supp}(\overline{p})$ for any $1\leqslant i\leqslant\ell$, the sequence $(q_0,q_1,\dots,q_{\ell''})$ is also a walkable sequence. Furthermore, since

$$\begin{aligned}\sum_{i=0}^{\ell'}\operatorname{length}(q_i)&=\operatorname{length}(q_0)+\sum_{i=1}^{\ell'}\operatorname{length}(q_i)\\&\leqslant\operatorname{length}(q_0)+\ell'\cdot\#(V_\Gamma/L)\\&\leqslant\operatorname{length}(q_0)+(\#(V_\Gamma/L)-\operatorname{length}(q_0)-1)\cdot\#(V_\Gamma/L)\\&\leqslant(\#(V_\Gamma/L))^2,\end{aligned}$$

we have

$$\sum_{i=0}^{\ell'}w(q_i)\leqslant W\cdot(\#(V_\Gamma/L))^2.$$

We also have

$$\sum_{i=\ell'+1}^{\ell''}w(q_i)=\sum_{q\in\mathrm{Cyc}_{\Gamma/L}}\beta_q\cdot w(q)\leqslant\sum_{q\in\mathrm{Cyc}_{\Gamma/L}}(d_q-1)\cdot w(q)$$

Since $(q_0,q_1,\dots,q_{\ell''})$ is a walkable sequence, there exists a path p' in Γ from x_0 such that $\langle\overline{p'}\rangle=\sum_{i=0}^{\ell''}\langle q_i\rangle$. Then, we have

$$\begin{aligned}w(p')&=\sum_{i=0}^{\ell'}w(q_i)+\sum_{i=\ell'+1}^{\ell''}w(q_i)\\&\leqslant W\cdot\left(\#(V_\Gamma/L)\right)^2+\sum_{q\in\mathrm{Cyc}_{\Gamma/L}}(d_q-1)\cdot w(q),\end{aligned}$$

and hence, $(w(p'), t(p')) \in B'$. Since

$$i - w(p') \geqslant w(p) - w(p') = \sum_{i=\ell''+1}^{\ell} w(q_i) = \sum_{q \in \mathrm{Cyc}_{\Gamma/L}} (\alpha_q - \beta_q) \cdot w(q),$$

$$y - t(p') = t(p) - t(p') = \sum_{i=\ell''+1}^{\ell} \mu(\langle q_i \rangle) = \sum_{q \in \mathrm{Cyc}_{\Gamma/L}} (\alpha_q - \beta_q) \cdot \mu(\langle q \rangle),$$

and $d_q \mid (\alpha_q - \beta_q)$ for each $q \in \mathrm{Cyc}_{\Gamma/L}$, we have

$$(i, y) - \big(w(p'), t(p')\big) = \big(i - w(p), 0\big) + \sum_{q \in \mathrm{Cyc}_{\Gamma/L}} \frac{\alpha_q - \beta_q}{d_q} \cdot d_q \cdot \big(w(q), \mu(\langle q \rangle)\big) \in M.$$

Therefore, we have $B = M + B'$. Since B' is a finite set, we can conclude that B is a finitely generated M-module. □

Theorem 4.2. *Let (Γ, L) be a periodic graph, and let $x_0 \in V_\Gamma$. Suppose that x_0 is P-initial. For each $v \in V(P_\Gamma) \setminus \{0\}$, we pick a cycle $q_v \in \nu^{-1}(v)$ such that $\overline{x_0} \in \mathrm{supp}(q_v)$. Then,*

$$\mathrm{LCM}\{w(q_v) \mid v \in V(P_\Gamma) \setminus \{0\}\}$$

is a quasi-period of the growth sequence $(s_{\Gamma,x_0,d})_d$. More precisely, the growth series $G_{\Gamma,x_0}(t)$ is of the form

$$G_{\Gamma,x_0}(t) = \frac{Q(t)}{\prod_{v \in V(P_\Gamma) \setminus \{0\}} (1 - t^{w(q_v)})}$$

with some polynomial $Q(t)$.

In particular, if the graph Γ is unweighted, then $\mathrm{LCM}\{1, 2, \dots, \#(V_\Gamma/L)\}$ is a quasi-period of the growth sequence.

Proof. We keep the notation B, M' and M in Lemma 4.1. For $i \in \mathbb{Z}_{\geqslant 0}$, we define $B_i \coloneqq \{y \in V_\Gamma \mid (i, y) \in B\}$. In this notation, we have $b_{\Gamma,x_0,d} = \# B_d$.

By Lemma 4.1, B is a finitely generated M-module. Furthermore, the monoid M is generated by the finite set $M' \cup \{(1,0)\}$, and the degree of each element of $M' \cup \{(1,0)\}$ divides $\mathrm{LCM}\{w(q_v) \mid v \in V(P_\Gamma) \setminus \{0\}\}$. Therefore, the assertion follows from Theorem 4.3. □

The following theorem is well-known.

Theorem 4.3 (cf. [6, Theorem 6.38]). *Let N be a monoid. Let $M' \subset \mathbb{Z}_{>0} \times N$ be a finite subset, and let $M \subset \mathbb{Z}_{\geqslant 0} \times N$ be the submonoid generated by M'. Let $X \subset \mathbb{Z}_{\geqslant 0} \times N$ be a finitely generated M-submodule. Then, the function*

$$h : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}; \quad i \mapsto \#\{x \in N \mid (i, x) \in X\}$$

is of quasi-polynomial type.

More precisely, its generating function $\sum_{i \geqslant 0} h(i) t^i$ is of the form

$$\frac{Q(t)}{\prod_{a \in M'} (1 - t^{\deg a})}$$

with some polynomial $Q(t)$. Here, $\deg : \mathbb{Z}_{\geqslant 0} \times N \to \mathbb{Z}_{\geqslant 0}$ denotes the first projection.

Proposition 4.4. *Let (Γ, L) be a strongly connected periodic graph. Let $\Phi \colon V_\Gamma \to L_\mathbb{R}$ be a periodic realization, and let $x_0 \in V_\Gamma$. Suppose that x_0 is P-initial. We define a bounded set $Q \subset L_\mathbb{R}$ as follows:*

- *For each $v \in V(P_\Gamma)$, we pick a cycle $q_v \in \nu^{-1}(v)$ such that $\overline{x_0} \in \mathrm{supp}(q_v)$, and we define $d_v \coloneqq w(q_v)$.*

- *For each $\sigma \in \mathrm{Facet}(P_\Gamma)$, we fix a triangulation T_σ of σ such that $V(\Delta) \subset V(\sigma)$ holds for any $\Delta \in T_\sigma$.*
- *We define a bounded set $Q \subset L_\mathbb{R}$ as follows:*

$$Q \coloneqq \bigcup_{\substack{\sigma \in \mathrm{Facet}(P_\Gamma),\\ \Delta \in T_\sigma}} \left(\sum_{v \in V(\Delta)} [0,1) d_v v \right) \subset L_\mathbb{R}.$$

Then, we have

$$C_2(\Gamma, \Phi, x_0) = \max\bigl\{ d_\Gamma(x_0, y) - d_{P_\Gamma, \Phi}(x_0, y) \bigm| y \in V_\Gamma,\ \Phi(y) - \Phi(x_0) \in Q \bigr\}.$$

Proof. By the definition of $C_2(\Gamma, \Phi, x_0)$, we have

$$C_2(\Gamma, \Phi, x_0) \geqslant \max\bigl\{ d_\Gamma(x_0, y) - d_{P_\Gamma, \Phi}(x_0, y) \bigm| y \in V_\Gamma,\ \Phi(y) - \Phi(x_0) \in Q \bigr\}.$$

The opposite inequality follows from the same argument as the proof of Theorem A.2 by making the following modifications:

- Replacing d_v and q_v in the proof of Theorem A.2 with d_v and q_v in the statement of Proposition 4.4.
- Replacing C_2' in the proof of Theorem A.2 with

$$\max\bigl\{ d_\Gamma(x_0, y) - d_{P_\Gamma, \Phi}(x_0, y) \bigm| y \in V_\Gamma,\ \Phi(y) - \Phi(x_0) \in Q \bigr\}.$$

Then, by the choice of q_v, we have $\overline{x_0} \in \mathrm{supp}(q_v)$. In particular, for any walk p in Γ from x_0, we have $\mathrm{supp}(\overline{p}) \cap \mathrm{supp}(q_v) \neq \varnothing$. Therefore, we can see that the same argument as the proof of Theorem A.2 works. □

4.2. Well-arranged periodic graphs. In this subsection, we introduce a class of periodic graphs called "well-arranged", and we prove that their growth sequences satisfy the same reciprocity laws as in Theorem 3.4(3).

Definition 4.5. *Let (Γ, L) be a strongly connected n-dimensional periodic undirected graph. Let $\Phi : V_\Gamma \to L_\mathbb{R}$ be a periodic realization, and let $x_0 \in V_\Gamma$. We say that the triple (Γ, Φ, x_0) is well-arranged if the following condition holds: there exist*

- *an integer $d_v \in \mathbb{Z}_{>0}$ for each $v \in V(P_\Gamma)$, and*
- *a triangulation T_σ of σ for each $\sigma \in \mathrm{Facet}(P_\Gamma)$*

with the following conditions:

1. *$V(\Delta) \subset V(\sigma)$ holds for any $\Delta \in T_\sigma$.*
2. *$d_v v \in L$.*
3. *For any $\sigma \in \mathrm{Facet}(P_\Gamma)$, $\Delta \in T_\sigma$ and any subset $V' \subset V(\Delta)$, we have*

$$d_\Gamma(x_0, y) + d_\Gamma(y, z) = \sum_{v \in V'} d_v$$

for $z \coloneqq \left(\sum_{v \in V'} d_v v\right) + x_0$ and for any $y \in V_\Gamma$ such that $\Phi(y) - \Phi(x_0) \in \sum_{v \in V'} [0, d_v) v$.

Remark 4.6. Applying Definition 4.5(3) to the case $y = x_0$, we have $d_\Gamma(x_0, z) = \sum_{v \in V'} d_v$ in Definition 4.5(3). Therefore, the equation in (3) says "for any y, there exists a shortest walk from x_0 to z factors through y".

Lemma 4.7. *If (Γ, Φ, x_0) is well-arranged, then x_0 is P-initial.*

Proof. Let $v \in V(P_\Gamma)$. Take $\sigma \in \mathrm{Facet}(P_\Gamma)$ and $\Delta \in T_\sigma$ satisfying $v \in V(\Delta)$. Then, by Definition 4.5(3) for $V' = \{v\}$, we have $d_\Gamma(x_0, x_0 + d_v v) = d_v$. Therefore, there exists a walk p in Γ from x_0 such that $w(p) = d_v$ and $t(p) = d_v v + s(p)$. By Lemma 2.12, $\overline{p}$ decomposes to a walkable sequence $(q_0, q_1, \ldots, q_\ell)$ with $q_0 = \varnothing_{\overline{x_0}}$ such that $\langle \overline{p} \rangle = \sum_{i=1}^{\ell} \langle q_i \rangle$. Note that we have $\frac{\mu(\langle \overline{p} \rangle)}{w(p)} = v \in V(P_\Gamma)$ and $\frac{\mu(\langle q_i \rangle)}{w(q_i)} = \nu(q_i) \in P_\Gamma$. Therefore, we have $\nu(q_i) = v$ for each $1 \leqslant i \leqslant \ell$. Since $\overline{x_0} \in \mathrm{supp}(q_i)$ for some i, we conclude that x_0 is P-initial. □

Proposition 4.8 below shows that the graphs treated in Theorem 3.4(3)(i) are well-arranged. Therefore, Theorem 4.9 below can be seen as the generalization of Theorem 3.4(3) for undirected periodic graphs.

Proposition 4.8. *Let (Γ, L) be a strongly connected n-dimensional periodic undirected graph. Let $\Phi : V_\Gamma \to L_\mathbb{R}$ be a periodic realization, and let $x_0 \in V_\Gamma$. Suppose that $C_1(\Gamma, \Phi, x_0) < \frac{1}{2}$ and $C_2(\Gamma, \Phi, x_0) < \frac{1}{2}$. Then (Γ, Φ, x_0) is well-arranged.*

Proof. Since Γ is undirected, we have $d_\Gamma(y_1, y_2) = d_\Gamma(y_2, y_1)$ for any $y_1, y_2 \in V_\Gamma$. By Theorem 3.4(2), we have both

$$(\diamondsuit) \qquad d_\Gamma(y_1,y_2) = \left\lceil d_{P_\Gamma,\Phi}(y_1,y_2) - \frac{1}{2} \right\rceil, \quad d_\Gamma(y_1,y_2) = \left\lfloor d_{P_\Gamma,\Phi}(y_1,y_2) + \frac{1}{2} \right\rfloor$$

for any $y_1, y_2 \in V_\Gamma$ satisfying $\overline{y_1} = \overline{x_0}$.

For each $v \in V(P_\Gamma)$, take $d_v \in \mathbb{Z}_{>0}$ satisfying Definition 4.5(2). For each $\sigma \in \mathrm{Facet}(P_\Gamma)$, take a triangulation T_σ of σ satisfying Definition 4.5(1). We prove that Definition 4.5(3) is satisfied for any choice of such d_v's and T_σ's.

Let $\sigma \in \mathrm{Facet}(P_\Gamma)$, $\Delta \in T_\sigma$, and $V' \subset V(\Delta)$. Take $y \in V_\Gamma$ such that $\Phi(y) - \Phi(x_0) \in \sum_{v \in V'} [0, d_v) v$. Then, by the condition on y, we have

$$d_{P_\Gamma,\Phi}(x_0, y) + d_{P_\Gamma,\Phi}(y, z) = \sum_{v \in V'} d_v \in \mathbb{Z}_{\geqslant 0}.$$

Then, the desired equality $d_\Gamma(x_0, y) + d_\Gamma(y, z) = \sum_{v \in V'} d_v$ follows from $(\diamondsuit)$. □

Theorem 4.9. *Let (Γ, L) be a strongly connected n-dimensional periodic undirected graph. Let $\Phi : V_\Gamma \to L_\mathbb{R}$ be a periodic realization, and let $x_0 \in V_\Gamma$. Suppose that (Γ, Φ, x_0) is well-arranged. Then, the following assertions hold.*

(1) *The function $i \mapsto s_{\Gamma,x_0,i}$ obtained by the growth sequence is a quasi-polynomial on $i \geqslant 1$. The function $i \mapsto b_{\Gamma,x_0,i}$ is a quasi-polynomial on $i \geqslant 0$.*
(2) *More precisely, the generating function $G_s(t) \coloneqq \sum_{i \geqslant 0} s_{\Gamma,x_0,i} t^i$ of the growth sequence $(s_{\Gamma,x_0,i})_i$ can be expressed as*

$$G_s(t) = \frac{Q_1(t)}{Q_2(t)}$$

with polynomials Q_1 and Q_2 satisfying the two conditions:
  - $Q_2(t) = \mathrm{LCM}\left\{ \prod_{v \in V(\Delta)} (1 - t^{d_v}) \;\middle|\; \sigma \in \mathrm{Facet}(P),\ \Delta \in T_\sigma \right\}$.
  - $\deg Q_1 \leqslant \deg Q_2$.

(3) *The same reciprocity as in Theorem 3.4(3) is satisfied.*

Proof. The assertion (1) follows from (2) since the generating function corresponding to the function $i \mapsto b_{\Gamma,x_0,i}$ is equal to $\frac{Q_1(t)}{(1-t)Q_2(t)}$. For the reciprocity in Theorem 3.4(3), it is sufficient to show the formula $G_s(1/t) = (-1)^n G_s(t)$ because this formula implies the other three formulas. For a subset $E \subset L_\mathbb{R}$, we define

$$G_E(t) \coloneqq \sum_{y \in V_\Gamma \cap \Phi^{-1}(E)} t^{d_\Gamma(x_0, y)}.$$

Using this notation, $G_s(t)$ in Theorem 3.4(3) can be expressed as $G_s(t) = G_{L_\mathbb{R}}(t)$.

Take d_v's for $v \in V(P_\Gamma)$ and a triangulation T_σ of σ for each $\sigma \in \mathrm{Facet}(P_\Gamma)$ as in Definition 4.5. We set

$$T \coloneqq \{ \Delta' \mid \sigma \in \mathrm{Facet}(P_\Gamma), \Delta \in T_\sigma, \Delta' \in \mathrm{Face}(\Delta) \}$$

For $\Delta' \in T$, we define subsets $D^+_{\Delta'}, D^-_{\Delta'} \subset L_{\mathbb{R}}$ by

$$D^+_{\Delta'} \coloneqq \sum_{v \in V(\Delta')} [0,\infty)v, \qquad D^-_{\Delta'} \coloneqq \sum_{v \in V(\Delta')} (-\infty,0)v.$$

Then, we have both

$$(\spadesuit) \qquad G_{L_{\mathbb{R}}}(t) = G_{\{0\}}(t) + \sum_{\Delta' \in T} G_{D^-_{\Delta'}}(t) = 1 + \sum_{\Delta' \in T} G_{D^-_{\Delta'}}(t).$$

$$\begin{aligned} G_{L_{\mathbb{R}}}(t) &= (-1)^n \left(G_{\{0\}}(t) + \sum_{\Delta' \in T} (-1)^{\dim \Delta'} G_{D^+_{\Delta'}}(t) \right) \\ &= (-1)^n \left(1 + \sum_{\Delta' \in T} (-1)^{\dim \Delta'} G_{D^+_{\Delta'}}(t) \right). \end{aligned}$$

For $\Delta' \in T$, we define

$$\diamondsuit^+_{\Delta'} \coloneqq \sum_{v \in V(\Delta')} [0,d_v)v, \qquad \diamondsuit^-_{\Delta'} \coloneqq \sum_{v \in V(\Delta')} [-d_v,0)v.$$

Then the following assertions hold.

Claim 4.10. (1) *For any $y \in V_\Gamma \cap \diamondsuit^+_{\Delta'}$ and for $a_v \in \mathbb{Z}_{\geqslant 0}$, we have*

$$d_\Gamma\left(x_0, y + \sum_{v \in V(\Delta')} a_v d_v v\right) = d_\Gamma(x_0,y) + \sum_{v \in V(\Delta')} a_v d_v.$$

(2) *For any $y \in V_\Gamma \cap \diamondsuit^-_{\Delta'}$ and for $a_v \in \mathbb{Z}_{\geqslant 0}$, we have*

$$d_\Gamma\left(x_0, y - \sum_{v \in V(\Delta')} a_v d_v v\right) = d_\Gamma(x_0,y) + \sum_{v \in V(\Delta')} a_v d_v.$$

(3) *For any $y \in V_\Gamma \cap \diamondsuit^+_{\Delta'}$, we have*

$$d_\Gamma(x_0,y) + d_\Gamma\left(x_0, y - \sum_{v \in V(\Delta')} d_v v\right) = \sum_{v \in V(\Delta')} d_v.$$

(4) *We have*

$$G_{D^+_{\Delta'}}(t) = \frac{G_{\diamondsuit^+_{\Delta'}}(t)}{\prod_{v \in V(\Delta')}(1-t^{d_v})}, \quad G_{D^-_{\Delta'}}(t) = \frac{G_{\diamondsuit^-_{\Delta'}}(t)}{\prod_{v \in V(\Delta')}(1-t^{d_v})}.$$

(5) *We have $t^{d_{\Delta'}} G_{\diamondsuit^-_{\Delta'}}(1/t) = G_{\diamondsuit^+_{\Delta'}}(1/t)$, where $d_{\Delta'} \coloneqq \sum_{v \in V(\Delta')} d_v$.*

(6) *We have $G_{D^-_{\Delta'}}(1/t) = (-1)^{\dim \Delta' + 1} G_{D^+_{\Delta'}}(t)$.*

(7) *We have $\deg G_{\diamondsuit^+_{\Delta'}}(t) \leqslant \sum_{v \in V(\Delta')} d_v$.*

Proof. We prove (1). We set

$$\begin{aligned} z &\coloneqq x_0 + \sum_{v \in V(\Delta')} d_v v, \\ y' &\coloneqq y + \sum_{v \in V(\Delta')} a_v d_v v, \\ z' &\coloneqq z + \sum_{v \in V(\Delta')} a_v d_v v. \end{aligned}$$

For each $v \in V(P_\Gamma)$, by the proof of Lemma 4.7, we can take a closed walk q_v in Γ/L such that

$$w(q_v) = d_v, \quad \mu(\langle q_v \rangle) = d_v v, \quad x_0 \in \operatorname{supp}(q_v).$$

Let p be a walk in Γ from x_0 to y with $w(p) = d_\Gamma(x_0, y)$. Since $\operatorname{supp} \overline{p} \cap \operatorname{supp}(q_v) \neq \varnothing$, there exists a walk p' in Γ from x_0 to y' such that $w(p') = w(p) + \sum_{v \in V(\Delta')} a_v d_v$. Therefore, we have

$$\text{(i)} \qquad d_\Gamma(x_0, y') \leqslant d_\Gamma(x_0, y) + \sum_{v \in V(\Delta')} a_v d_v.$$

On the other hand, we have

$$d_\Gamma(x_0, z') \leqslant d_\Gamma(x_0, y') + d_\Gamma(y', z').$$

Here, we have

$$d_\Gamma(x_0, z') = \sum_{v \in V(\Delta')} (a_v + 1) d_v$$

by Lemma 2.23 and the inequality (i) for $y = x_0$. Furthermore, we have

$$d_\Gamma(y', z') = d_\Gamma(y, z) = -d_\Gamma(x_0, y) + \sum_{v \in V(\Delta')} d_v$$

by Definition 4.5(3). Therefore, we get the opposite inequality of (i). We complete the proof of the assertion (1). The assertion (2) is proved by the similar way.

By translation, we have

$$d_\Gamma\left(x_0, y - \sum_{v \in V(\Delta')} d_v v\right) = d_\Gamma(z, y).$$

Therefore, the assertion (3) follows from Definition 4.5(3).

The assertion (4) follows from (1) and (2). The assertion (5) follows from (3). The assertion (6) follows from (4) and (5).

By the inequality (i), we have $d_\Gamma(x_0, y) \leqslant \sum_{v \in V(\Delta')} d_v$ for any $y \in V_\Gamma \cap \diamondsuit^+_{\Delta'}$. Therefore, we conclude the assertion (7). $\square$

By (♠) and Claim 4.10(4)(7), we conclude that $G_{L_\mathbb{R}}(t)$ is a rational function of the form $\frac{Q_1(t)}{Q_2(t)}$ satisfying the two conditions in (2). By (♠) and Claim 4.10(6), we have the reciprocity $G_{L_\mathbb{R}}(1/t) = (-1)^n G_{L_\mathbb{R}}(t)$. We complete the proof. $\square$

Remark 4.11. When a periodic graph is well-arranged for some realization, we can conclude that the growth sequence $(s_{\Gamma,x_0,i})_i$ is a quasi-polynomial on $d \geqslant 1$ (Theorem 4.9(1)), and we can find a quasi-period (Theorem 4.9(2)). Therefore, we can determine the growth sequence from its first few terms. In Section 5, by using this method, we determine the growth series for several new examples.

5. Examples

In this section, for some specific periodic graphs, we will see whether they are well-arranged or not. Furthermore, we determine the growth series in several new examples by the method explained in Remark 4.11.

- In Subsection 5.1, we will examine seven tilings in [12]. For these examples, the growth sequences have already been determined by Goodman-Strauss and Sloane in [12]. However, we expect that this subsection will help the reader to understand the concepts “well-arranged” and “P-initial”.

- In Subsection 5.2, we treat another tiling called the $(3^6; 3^2.4.3.4; 3^2.4.3.4)$ 3-uniform tiling. We check that the corresponding periodic graph is well-arranged under a suitable realization. Furthermore, we determine its growth series. As far as we know, this is the first time that its growth series have been determined with a proof.
- In Subsection 5.3, we treat 3-dimensional periodic graphs obtained by carbon allotropes. We confirm that 22 of them are well-arranged, and we determine their growth series. As far as we know, this is the first time that the growth sequences have been computed for nontrivial 3-dimensional periodic graphs.

Remark 5.1. It is not difficult to implement the following calculations and verifications in a computer program.

(1) Calculate the first few terms of the growth sequence $(s_{\Gamma,x_0,i})_i$ (breadth-first search algorithm).
(2) Check that x_0 is P-initial or not.
(3) Calculate $C_1(\Gamma, \Phi, x_0)$ (Remark 2.25(1)).
(4) Calculate $C_2(\Gamma, \Phi, x_0)$ when x_0 is P-initial (Proposition 4.4).
(5) Check that (Γ, Φ, x_0) is well-arranged or not for given d_v's and triangulations T_σ's.
(6) Determine the growth series for well-arranged periodic graphs (Remark 4.11).

We prepare implementations of the algorithms in Python to compute or check (1)–(6) above for unweighted periodic graphs. For details, see the source code:

`https://github.com/yokozuna57/Ehrhart_on_PG`

5.1. 2-dimensional periodic graphs from [12]. In this subsection, we examine seven specific periodic tilings from [12] illustrated in Figures 7-13. Let (Γ, L) be the corresponding unweighted undirected periodic graphs, and let Φ be the periodic realizations shown in the figures. The parallelogram drawn with red lines represents a fundamental region of the periodic graph (Γ, L). In [12], Goodman-Strauss and Sloane determine their growth sequences. We list their generating functions in the item "Growth series" of Table 1. With the help of a computer program (cf. Remark 5.1), we can check whether these tilings (and their starting points) are P-initial and whether they are well-arranged as in Table 1. In the table, "PI" stands for P-initial, "WA" for well-arranged, and "RL" for reciprocity law.

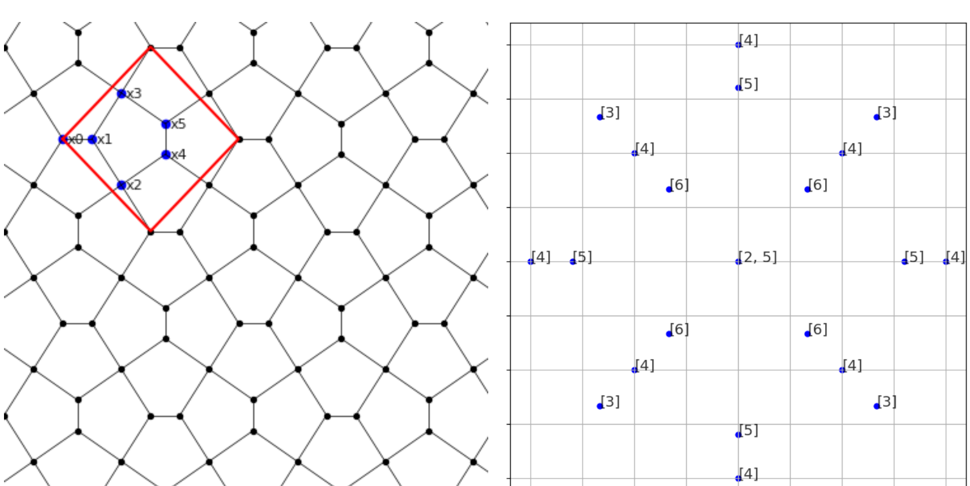


Figure 7. The Cairo tiling and its $\mathrm{Im}(\nu)$.

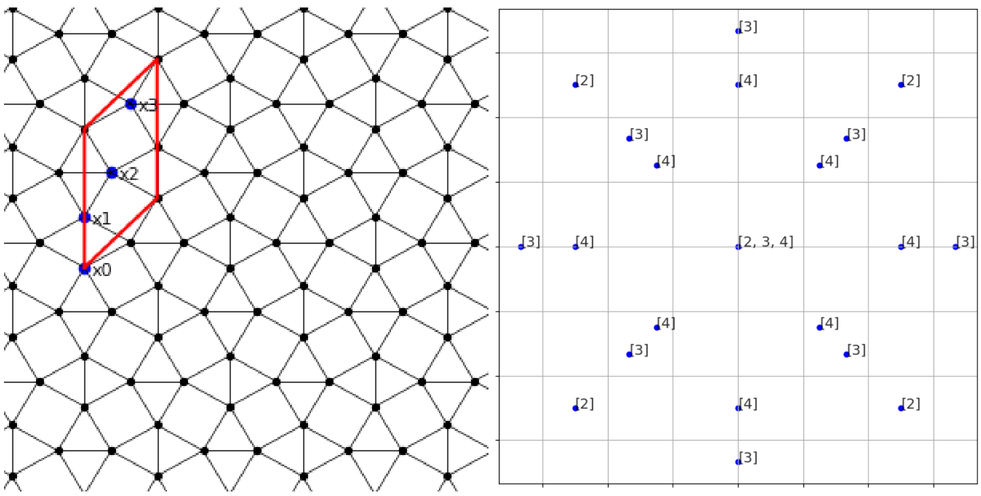


Figure 8. The $3^2.4.3.4$ uniform tiling and its $\mathrm{Im}(\nu)$.

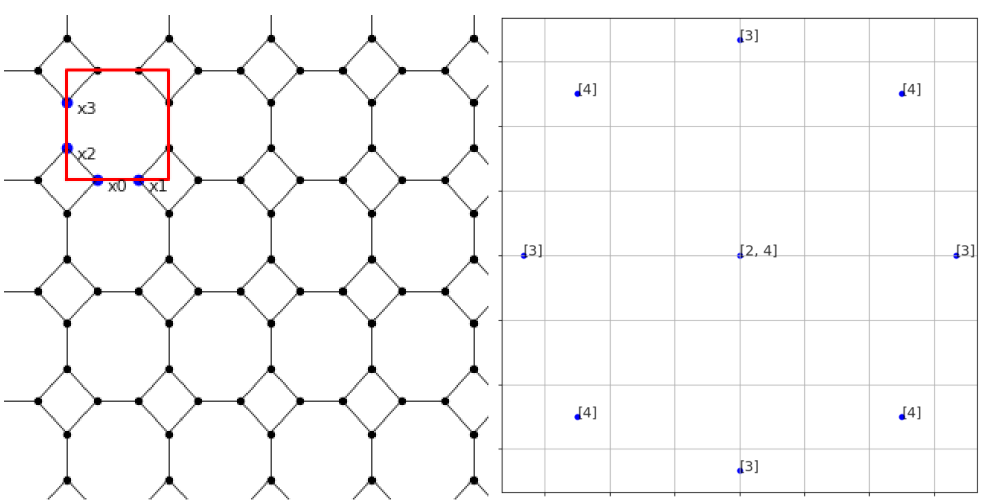


Figure 9. The 4.8^2 uniform tiling and its $\mathrm{Im}(\nu)$.

5.2. The $(3^6; 3^2.4.3.4; 3^2.4.3.4)$ 3-uniform tiling. The $(3^6; 3^2.4.3.4; 3^2.4.3.4)$ 3-uniform tiling is a two-dimensional tiling illustrated in Figure 14. Let (Γ, L) be the corresponding unweighted undirected periodic graph, and let Φ be the periodic realization shown in the figure. The parallelogram drawn with red lines represents a fundamental region of the periodic graph (Γ, L). The vertices $x_1, x_2, x_3, x_5, x_{10}$ and x_{11} are symmetric with respect to $\mathrm{Aut}\,\Gamma$. The vertices x_4, x_6, x_7, x_8, x_9 and x_{12} are also symmetric.

With the help of a computer program, we can see that all vertices are P-initial, and we have

$$
\begin{aligned}
C_1(\Gamma, \Phi, x_0) &= C_2(\Gamma, \Phi, x_0) = 0.267...,\\
C_1(\Gamma, \Phi, x_1) &= C_2(\Gamma, \Phi, x_1) = 0.535...,\\
C_1(\Gamma, \Phi, x_4) &= C_2(\Gamma, \Phi, x_4) = 0.422....
\end{aligned}
$$

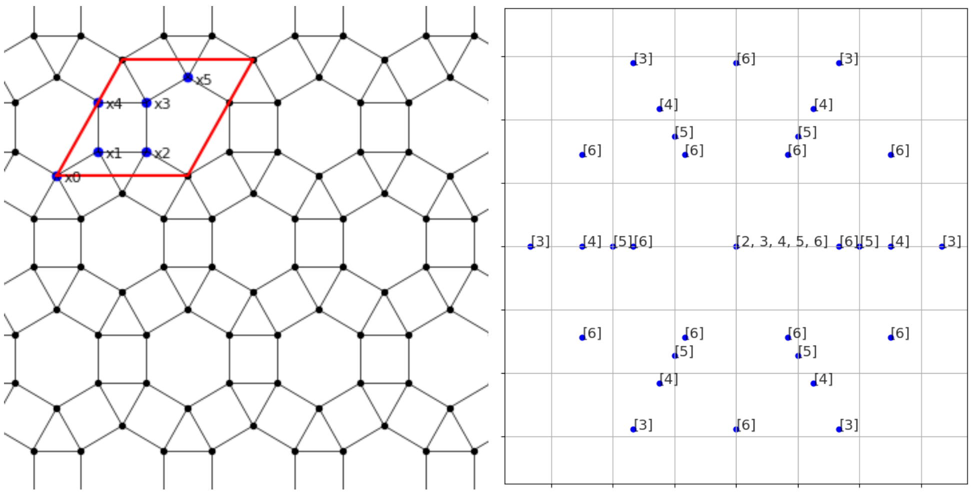


Figure 10. The 3.4.6.4 uniform tiling and its $\mathrm{Im}(\nu)$.

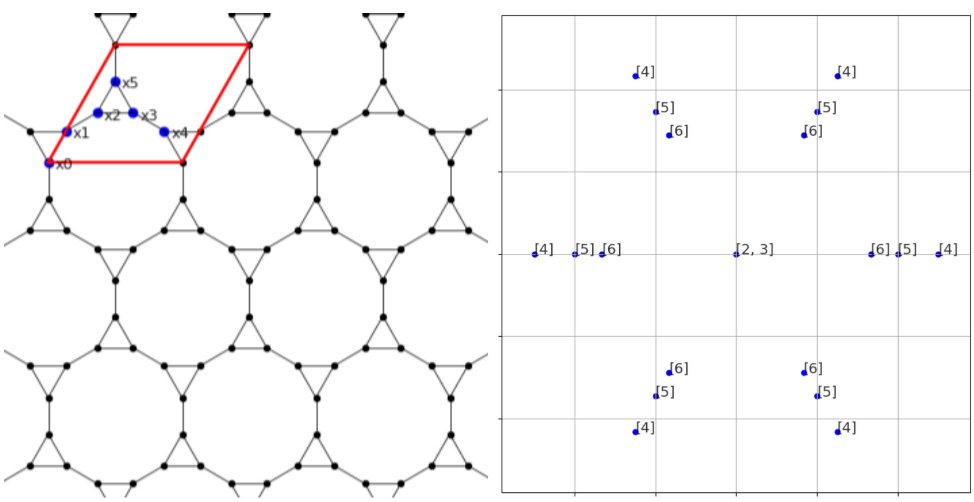


Figure 11. The 3.12^2 uniform tiling and its $\mathrm{Im}(\nu)$.

Therefore, (Γ, Φ, x_0) and (Γ, Φ, x_4) are well-arranged by Proposition 4.8. Note that we cannot apply Proposition 4.8 to x_1 because $C_1(\Gamma, \Phi, x_1) + C_2(\Gamma, \Phi, x_1) \geqslant 1$. Furthermore, we could not check whether (Γ, Φ, x_1) is well-arranged or not by our computer program (indeed, we checked the desired condition (3) in Definition 4.5 for one triangulation T_σ and d_v satisfying (1) and (2), but the result was negative). Fortunately, by changing the periodic realization as Figure 15, we can confirm that (Γ, Φ', x_1) is well-arranged.

Since we have confirmed that (Γ, Φ, x_0), (Γ, Φ', x_1), and (Γ, Φ, x_4) are well-arranged, according to the method explained in Remark 4.11, we can determine their growth series as in Table 2.

In what follows, we shall give the calculation of G_{Γ, x_0} in detail. By Theorem 4.9, it follows that $G_{\Gamma, x_0}(t)$ is of the form

$$G_{\Gamma, x_0}(t) = \frac{Q(t)}{(1 - t^4)(1 - t^7)}$$

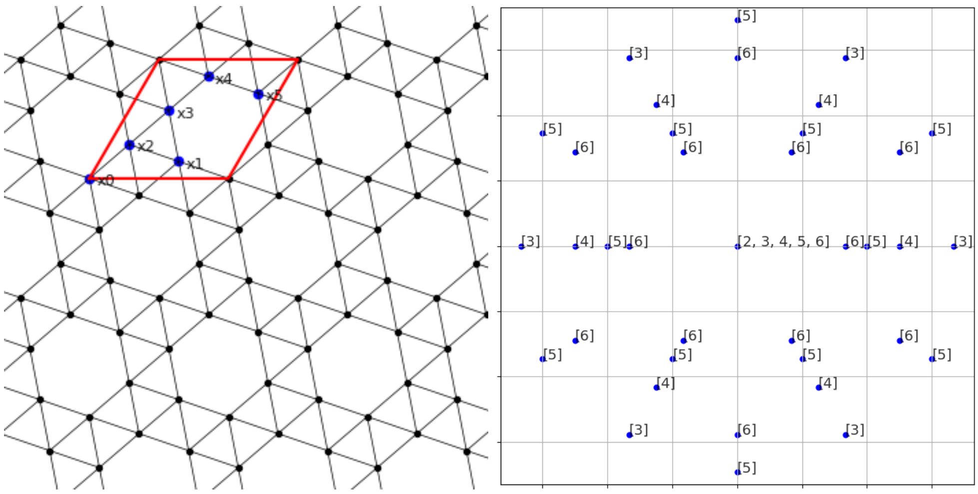


FIGURE 12. The $3^4.6$ uniform tiling and its $\mathrm{Im}(\nu)$.

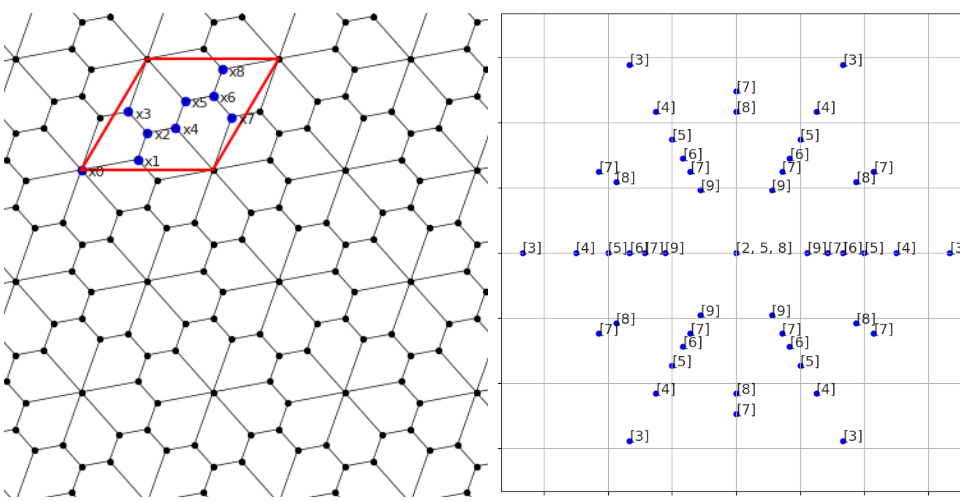


FIGURE 13. The snub-632 3-uniform tiling and its $\mathrm{Im}(\nu)$.

with some polynomial $Q(t)$ with $\deg Q \leqslant 11$. In particular, we can conclude that the growth sequence $(s_{\Gamma,x_0,i})_{i\geqslant 0}$ satisfies the linear recurrence relation corresponding to $(1-t^4)(1-t^7)$ for $i \geqslant 1$.

With the help of a computer program (breadth-first search algorithm), the first 12 terms $(s_{\Gamma,x_0,i})_{0\leqslant i\leqslant 11}$ can be computed: 1, 6, 12, 12, 24, 30, 36, 36, 42, 54, 54, 60. Then the sequence $(s_{\Gamma,x_0,i})_{i\geqslant 0}$ is completely determined by the linear recurrence relation, and its generating function can be calculated as

$$\begin{aligned} G_{\Gamma,x_0}(t) &= \frac{\big(\text{The terms of } (1-t^4)(1-t^7)\sum_{i=0}^{11} s_{\Gamma,x_0,i}t^i \text{ of degree 11 or less}\big)}{(1-t^4)(1-t^7)} \\ &= \frac{1+6t+12t^2+12t^3+23t^4+24t^5+24t^6+23t^7+12t^8+12t^9+6t^{10}+t^{11}}{(1-t^4)(1-t^7)}. \end{aligned}$$

Here, we can see that $G_{\Gamma,x_0}(t)$ actually satisfies the reciprocity law $G_{\Gamma,x_0}(1/t) = G_{\Gamma,x_0}(t)$.

Tiling / start pt.	$\#(V_{\Gamma/L})$	PI	WA	RL	C_1	C_2	Growth series
Fig. 7 / x_2, x_3	6	YES	YES	YES	1/3	1/3	$\frac{1+2t+t^2}{(1-t)^2}$
Fig. 7 / otherwise	6	NO	NO	NO	2/3	$\geqslant 1$	$\frac{1+2t+5t^2+4t^3+2t^4+3t^5-t^7}{(1-t)(1-t^4)}$
Fig. 8 / all	4	YES	YES	YES	0.36...	0.36...	$\frac{1+4t+6t^2+4t^3+t^4}{(1-t)(1-t^3)}$
Fig. 9 / all	4	YES	YES	YES	0.58...	0.58...	$\frac{1+2t+2t^2+2t^3+t^4}{(1-t)(1-t^3)}$
Fig. 10 / all	6	YES	YES	YES	0.46...	0.46...	$\frac{1+2t+t^2}{(1-t)^2}$
Fig. 11 / all	6	NO	NO	NO	0.38...	$\geqslant 1$	$\frac{1+3t+4t^2+6t^3+6t^4+6t^5+6t^6+3t^7+3t^8-2t^{10}}{(1-t^4)^2}$
Fig. 12 / all	6	YES	YES	YES	5/7	5/7	$\frac{1+4t+4t^2+6t^3+4t^4+4t^5+t^6}{(1-t)(1-t^5)}$
Fig. 13 / x_0	9	YES	NO	NO	3/7	1	$\frac{1+6t+12t^2+10t^3+12t^4+12t^5+t^6}{(1-t^3)^2}$
Fig. 13 / x_2, x_6	9	NO	NO	NO	3/7	$\geqslant 1$	$\frac{1+3t+6t^2+13t^3+15t^4+6t^5+4t^6+9t^7-3t^{10}}{(1-t^3)^2}$
Fig. 13 / otherwise	9	NO	NO	NO	6/7	$\geqslant 1$	$\frac{1+3t+9t^2+13t^3+12t^4+9t^5+8t^6+4t^7-t^8-2t^9-2t^{10}}{(1-t^3)^2}$

Table 1. Growth series of the seven tilings in [12].

5.3. 3-dimensional periodic graphs. In this subsection, we treat 3-dimensional periodic graphs obtained by some carbon allotropes. *Samara Carbon Allotrope Database* [14] currently lists 525 carbon allotropes. In this paper, we examine only the carbon allotropes that satisfy the following conditions:

- The corresponding graph Γ is a uniform graph (i.e. all vertices of Γ are symmetric with respect to $\mathrm{Aut}(\Gamma)$).
- Each vertex of Γ has order 4.

There are 49 such carbon allotropes in SACADA database: #1, 8, 10, 11, 12, 13, 20, 21, 29, 30, 31, 33, 35, 37, 39, 51, 52, 56, 57, 58, 59, 60, 65, 66, 67, 68, 69, 70, 71, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 613, 628, 629 in SACADA database. Of these graphs, 22 graphs can be verified to be well-arranged by a computer program: #1, 10, 21, 37, 39, 52, 56, 60, 65, 67, 74, 75, 76, 77, 80, 81, 82, 86, 87, 88,

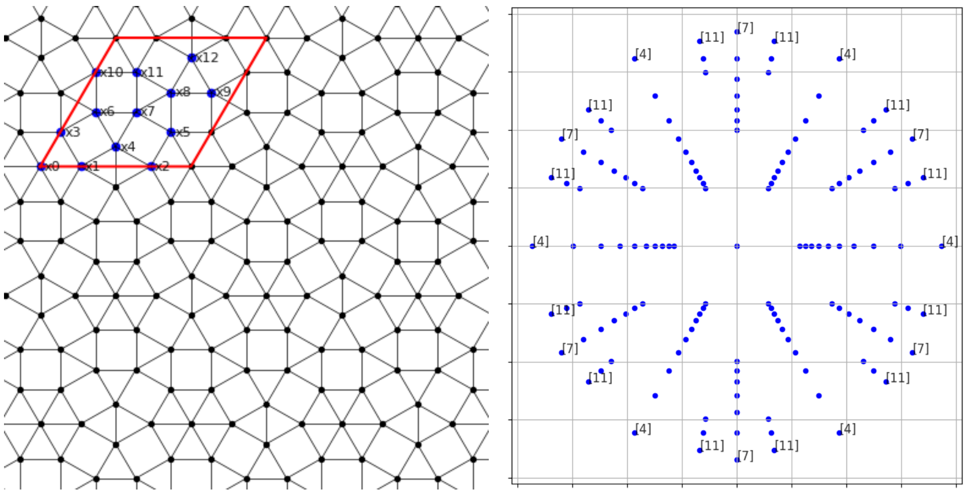


FIGURE 14. The $(3^6; 3^2.4.3.4; 3^2.4.3.4)$ 3-uniform tiling and its $\mathrm{Im}(\nu)$.

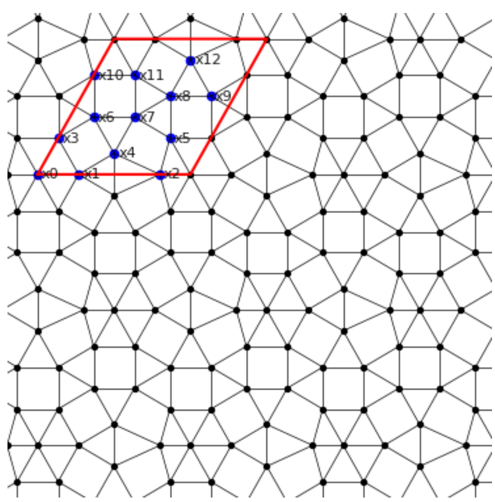


FIGURE 15. New realization Φ' obtained by moving the vertex x_2 to the right.

Start pt.	Growth series
x_0	$\frac{1+6t+12t^2+12t^3+23t^4+24t^5+24t^6+23t^7+12t^8+12t^9+6t^{10}+t^{11}}{(1-t^4)(1-t^7)}$
$x_1, x_2, x_3, x_5, x_{10}, x_{11}$	$\frac{1+4t+5t^2+7t^3+5t^4+7t^5+5t^6+4t^7+t^8}{(1-t)(1-t^7)}$
$x_4, x_6, x_7, x_8, x_9, x_{12}$	$\frac{1+5t+12t^2+17t^3+22t^4+21t^5+21t^6+22t^7+17t^8+12t^9+5t^{10}+t^{11}}{(1-t^4)(1-t^7)}$

TABLE 2. Growth series of the $(3^6; 3^2.4.3.4; 3^2.4.3.4)$ 3-uniform tiling.

89, 613 in SACADA database. Using the method explained in Remark 4.11, we can determine the growth series of these 22 periodic graphs as in Table 3. We also display the graphs Γ and the growth polytopes P_Γ for #1 and #60 in Figures 16 and 17.

SACADA#	Growth series
#1	$\frac{1+2t+4t^2+2t^3+t^4}{(1-t)^2(1-t^2)}$
#10	$\frac{1+t+t^2+t^3}{(1-t)^3}$
#21	$\frac{1+2t+4t^2+2t^3+t^4}{(1-t)^2(1-t^2)}$
#37	$\frac{1+2t+5t^2+5t^3+5t^4+2t^5+t^6}{(1-t)^2(1-t^4)}$
#39	$\frac{1+3t+5t^2+9t^3+12t^4+9t^5+5t^6+3t^7+t^8}{(1-t)(1-t^3)(1-t^4)}$
#52	$\frac{1+3t+5t^2+8t^3+10t^4+8t^5+5t^6+3t^7+t^8}{(1-t)(1-t^3)(1-t^4)}$
#56	$\frac{1+2t+5t^2+6t^3+5t^4+2t^5+t^6}{(1-t)^2(1-t^4)}$
#60	$\frac{1+3t+7t^2+11t^3+11t^4+7t^5+3t^6+t^7}{(1-t)(1-t^3)^2}$
#65	$\frac{1+2t+2t^2+3t^3+3t^4+2t^5+2t^6+t^7}{(1-t)^2(1-t^5)}$
#67	$\frac{1+3t+6t^2+9t^3+9t^4+6t^5+3t^6+t^7}{(1-t)(1-t^3)^2}$
#74	$\frac{1+3t+6t^2+10t^3+14t^4+18t^5+18t^6+14t^7+10t^8+6t^9+3t^{10}+t^{11}}{(1-t)(1-t^5)^2}$
#75	$\frac{1+3t+6t^2+9t^3+9t^4+6t^5+3t^6+t^7}{(1-t)(1-t^3)^2}$
#76	$\frac{1+2t+4t^2+4t^3+6t^4+4t^5+4t^6+2t^7+t^8}{(1-t)^2(1-t^6)}$
#77	$\frac{1+2t+2t^2+3t^3+3t^4+2t^5+2t^6+t^7}{(1-t)^2(1-t^5)}$
#80	$\frac{1+3t+6t^2+10t^3+12t^4+12t^5+10t^6+6t^7+3t^8+t^9}{(1-t)(1-t^3)(1-t^5)}$
#81	$\frac{1+3t+7t^2+12t^3+14t^4+15t^5+15t^6+14t^7+12t^8+7t^9+3t^{10}+t^{11}}{(1-t)(1-t^3)(1-t^7)}$
#82	$\frac{1+2t+3t^2+5t^3+5t^4+3t^5+2t^6+t^7}{(1-t)^2(1-t^5)}$
#86	$\frac{1+3t+5t^2+7t^3+9t^4+12t^5+15t^6+16t^7+15t^8+12t^9+9t^{10}+7t^{11}+5t^{12}+3t^{13}+t^{14}}{(1-t)(1-t^6)(1-t^7)}$
#87	$\frac{1+3t+5t^2+8t^3+11t^4+11t^5+8t^6+5t^7+3t^8+t^9}{(1-t)(1-t^3)(1-t^5)}$
#88	$\frac{1+3t+7t^2+11t^3+15t^4+20t^5+20t^6+15t^7+11t^8+7t^9+3t^{10}+t^{11}}{(1-t)(1-t^5)^2}$
#89	$\frac{1+2t+4t^2+3t^3+4t^4+4t^5+6t^6+4t^7+4t^8+3t^9+4t^{10}+2t^{11}+t^{12}}{(1-t)^2(1-t^{10})}$
#613	$\frac{1+3t+6t^2+9t^3+9t^4+6t^5+3t^6+t^7}{1-t-2t^3+2t^4+t^6-t^7}$

Table 3. Growth series of the 22 carbon allotropes.

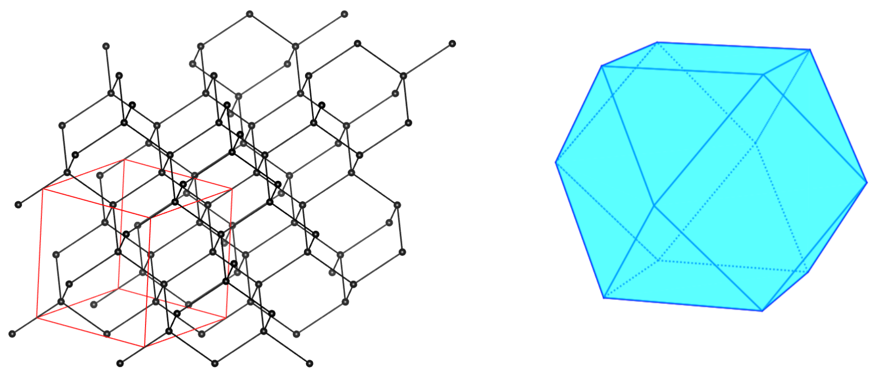


FIGURE 16. The diamond crystal structure (SACADA #1) and its P_Γ.

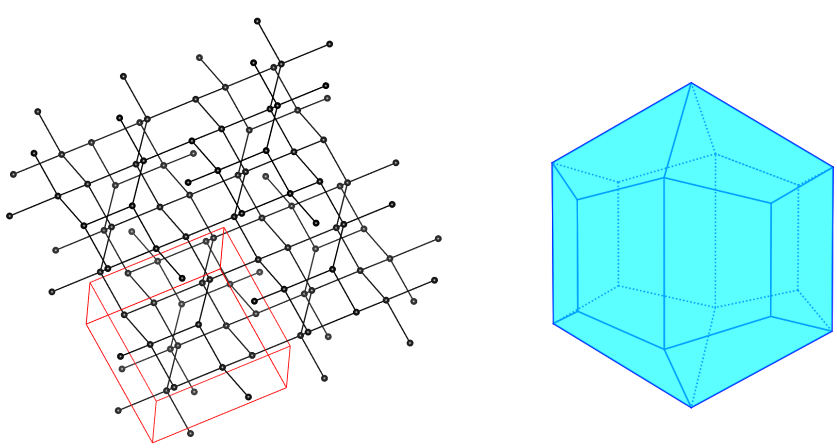


FIGURE 17. A carbon allotrope SACADA #60 and its P_Γ.

Appendix A. Properties of the growth polytope

In this section, we explain some properties of the growth polytope defined in Subsection 2.5.

LEMMA A.1 ([11, Proposition 21]). *If (Γ, L) is a strongly connected n-dimensional periodic graph, then we have $0 \in \operatorname{int}(P_\Gamma)$. In particular, we have $d_{P_\Gamma}(y) < \infty$ for any $y \in L_\mathbb{R}$.*

Proof. It is sufficient to prove the following claim:

- For any $u \in L$, there exists $m \in \mathbb{Z}_{>0}$ such that $u \in mP_\Gamma$.

We fix $x_0 \in V_\Gamma$. Since the graph Γ is strongly connected by assumption, there exists a walk p in Γ from x_0 to $u + x_0$. Then, by applying Lemma 2.12(1) to $\overline{p}$ (cf. Remark 2.13(1)), there exists a sequence $q_1, \ldots, q_\ell \in \mathrm{Cyc}_{\Gamma/L}$ satisfying $\langle \overline{p} \rangle = \sum_{i=1}^{\ell} \langle q_i \rangle$.

Since $\frac{\mu(\langle q\rangle)}{w(q)} \in P_\Gamma$ holds for any $q \in \mathrm{Cyc}_{\Gamma/L}$, we have

$$u = \mu(\langle \overline{p}\rangle) = \sum_{i=1}^{\ell} \mu(\langle q_i\rangle) \in \left(\sum_{i=1}^{\ell} w(q_i)\right) P_\Gamma,$$

which completes the proof. □

The following theorem shows that the growth polytope P_Γ can describe the asymptotic behavior of the growth sequence. In the theorem, we fix a periodic realization Φ, and C_1' and C_2' depend on the choice of Φ. We should emphasize that the growth polytope P_Γ itself does not depend on the choice of the realization.

In [23, Theorem 1], Shutov and Maleev explain that the following theorem is proved in the papers [27] and [20] (written in Russian). We also emphasize that Kotani and Sunada in [16], Fritz in [11], and Akiyama, Caalim, Imai and Kaneko in [1] have similar results. We will give a proof since the proof is referred to the proof of Proposition 4.4.

Theorem A.2. *Let (Γ, L) be a strongly connected n-dimensional periodic graph. Let $\Phi\colon V_\Gamma \to L_\mathbb{R}$ be a periodic realization, and let $x_0 \in V_\Gamma$. Then there exist $C_1', C_2' \in \mathbb{R}_{\geqslant 0}$ such that for any $y \in V_\Gamma$, we have*

$$d_{P_\Gamma,\Phi}(x_0, y) - C_1' \leqslant d_\Gamma(x_0, y) \leqslant d_{P_\Gamma,\Phi}(x_0, y) + C_2'.$$

Proof. Set $c \coloneqq \#(V_\Gamma/L)$ and

$$B_{c-1}' \coloneqq \{y \in V_\Gamma \mid \text{there exists a walk } p \text{ from } x_0 \text{ to } y \text{ with } \mathrm{length}(p) \leqslant c-1\}.$$

We define $C_1' \in \mathbb{R}_{\geqslant 0}$ by

$$C_1' \coloneqq \max_{y \in B_{c-1}'} \big(d_{P_\Gamma,\Phi}(x_0, y) - d_\Gamma(x_0, y)\big).$$

Note that C_1' exists by Lemma A.1 and the fact that B_{c-1}' is a finite set.

Let $y \in V_\Gamma$. We prove $d_{P_\Gamma,\Phi}(x_0, y) - C_1' \leqslant d_\Gamma(x_0, y)$. Let p be a walk in Γ from x_0 to y such that $w(p) = d \coloneqq d_\Gamma(x_0, y)$. By applying Lemma 2.12(1) to $\overline{p}$, there exists a walkable sequence $(q_0, q_1, \dots, q_\ell)$ such that $\langle \overline{p}\rangle = \sum_{i=0}^{\ell} \langle q_i\rangle$. Then, the following three conditions hold.

- $\mathrm{length}(q_0) \leqslant c-1$.
- $\Phi(y) - \Phi(x_0) = \mu_\Phi(\langle \overline{p}\rangle) = \sum_{i=0}^{\ell} \mu_\Phi(\langle q_i\rangle)$.
- $d = w(p) = \sum_{i=0}^{\ell} w(q_i)$.

Here, the first condition follows from the fact that q_0 is a path. Since we have $\mu(\langle q\rangle) \in w(q) \cdot P_\Gamma$ for each $q \in \mathrm{Cyc}_{\Gamma/L}$, we have

$$\sum_{i=1}^{\ell} \mu(\langle q_i\rangle) \in \left(\sum_{i=1}^{\ell} w(q_i)\right) P_\Gamma.$$

Let p_0 be the unique lift of q_0 with initial point x_0. Then, by the choice of C_1', we have

$$d_{P_\Gamma,\Phi}(x_0, t(p_0)) - d_\Gamma\big(x_0, t(p_0)\big) \leqslant C_1',$$

and hence,

$$\mu_\Phi(\langle q_0\rangle) = \mathrm{vec}_\Phi(p_0) = \Phi(t(p_0)) - \Phi(x_0) \in \big(C_1' + w(p_0)\big) P_\Gamma.$$

Therefore, we have

$$\begin{aligned}\Phi(y)-\Phi(x_0) &= \sum_{i=0}^{\ell}\mu_\Phi(\langle q_i\rangle)\\ &\in \left(C_1'+w(q_0)+\sum_{i=1}^{\ell}w(q_i)\right)P_\Gamma\\ &= (C_1'+d)P_\Gamma,\end{aligned}$$

and hence, we have $d_{P_\Gamma,\Phi}(x_0,y)\leqslant C_1'+d$.

Next, we define $C_2'\in\mathbb{R}_{\geqslant 0}$ as follows.

- First, we define $d_v \coloneqq \min_{q\in\nu^{-1}(v)} w(q)$ for each $v\in V(P_\Gamma)$.
- For $y\in V_\Gamma$, we define $d'(x_0,y)$ as the smallest weight $w(p)$ of a walk p from x_0 to y satisfying $\operatorname{supp}(\overline{p})=V_\Gamma/L$. We have $d'(x_0,y)<\infty$ since Γ is assumed to be strongly connected.
- For each $\sigma\in\operatorname{Facet}(P_\Gamma)$, we fix a triangulation T_σ of σ such that $V(\Delta)\subset V(\sigma)$ holds for any $\Delta\in T_\sigma$.
- We define a bounded set $Q\subset L_\mathbb{R}$ as follows:

$$Q\coloneqq \bigcup_{\substack{\sigma\in\operatorname{Facet}(P_\Gamma),\\ \Delta\in T_\sigma}}\left(\sum_{v\in V(\Delta)}[0,1)d_v v\right)\subset L_\mathbb{R}.$$

- Then, we set

$$C_2'\coloneqq \max\bigl\{d'(x_0,y)-d_{P_\Gamma,\Phi}(x_0,y)\;\big|\; y\in V_\Gamma,\ \Phi(y)-\Phi(x_0)\in Q\bigr\}.$$

Such C_2' exists since the set

$$\bigl\{y\in V_\Gamma \;\big|\; \Phi(y)-\Phi(x_0)\in Q\bigr\}$$

is a finite set.

Let $y\in V_\Gamma$. We prove $d_\Gamma(x_0,y)\leqslant d_{P_\Gamma,\Phi}(x_0,y)+C_2'$. By Lemma A.1, there exist $\sigma\in\operatorname{Facet}(P_\Gamma)$ and $\Delta\in T_\sigma$ such that $\Phi(y)-\Phi(x_0)\in\mathbb{R}_{\geqslant 0}\Delta$. Then we can uniquely write

$$\Phi(y)-\Phi(x_0)=\sum_{v\in V(\Delta)}b_v d_v v$$

with $b_v\in\mathbb{R}_{\geqslant 0}$. Then, we have

$$d_{P_\Gamma,\Phi}(x_0,y)=\sum_{v\in V(\Delta)}b_v d_v.$$

We define $y'\coloneqq -\bigl(\sum_{v\in V(\Delta)}\lfloor b_v\rfloor d_v v\bigr)+y$. Here, we have $d_v v\in L$ by the choice of d_v. Then, y' satisfies

$$\begin{aligned}\Phi(y')-\Phi(x_0) &= \Phi(y)-\Phi(x_0)-\sum_{v\in V(\Delta)}\lfloor b_v\rfloor d_v v\\ &= \sum_{v\in V(\Delta)}(b_v-\lfloor b_v\rfloor)d_v v\\ &\in \sum_{v\in V(\Delta)}[0,1)d_v v\subset Q.\end{aligned}$$

By the choice of C_2', there exists a walk q in Γ from x_0 to y' satisfying

$$\operatorname{supp}(\overline{q})=V_{\Gamma/L},\qquad w(q)\leqslant C_2'+d_{P_\Gamma,\Phi}(x_0,y').$$

For each $v \in V(\Delta)$, we can take $q_v \in \mathrm{Cyc}_{\Gamma/L}$ such that

$$w(q_v) = d_v, \qquad \mu(\langle q_v \rangle) = d_v v.$$

Since we have $\mathrm{supp}(\overline{q}) = V_{\Gamma/L}$, and q_v's are closed walks, there exists a walk r' in Γ/L such that

$$\langle r' \rangle = \langle \overline{q} \rangle + \sum_{v \in V(\Delta)} \lfloor b_v \rfloor \langle q_v \rangle.$$

Let r be the unique lift of r' with initial point x_0. Then, we have

$$t(r) = \left(\sum_{v \in V(\Delta)} \lfloor b_v \rfloor d_v v \right) + t(q) = \left(\sum_{v \in V(\Delta)} \lfloor b_v \rfloor d_v v \right) + y' = y.$$

Furthermore, its weight satisfies

$$\begin{aligned} w(r) &= w(q) + \sum_{v \in V(\Delta)} \lfloor b_v \rfloor w(q_v) \\ &\leqslant C_2' + d_{P_\Gamma, \Phi}(x_0, y') + \sum_{v \in V(\Delta)} \lfloor b_v \rfloor d_v \\ &= C_2' + \sum_{v \in V(\Delta)} (b_v - \lfloor b_v \rfloor) d_v + \sum_{v \in V(\Delta)} \lfloor b_v \rfloor d_v \\ &= C_2' + d_{P_\Gamma, \Phi}(x_0, y). \end{aligned}$$

Therefore, we have $d_\Gamma(x_0, y) \leqslant C_2' + d_{P_\Gamma, \Phi}(x_0, y)$. □

Example A.3. For the Wakatsuki graph with the injective periodic realization Φ in Example 2.6 and the start point $x_0 = v_2'$, we shall compute the values C_1' and C_2'. We identify V_Γ with the subset of $L_\mathbb{R}$.

First, C_1' is given by

$$C_1' \coloneqq \max_{y \in B'_{c-1}} \left(d_{P_\Gamma, \Phi}(x_0, y) - d_\Gamma(x_0, y) \right) = 1.$$

Note that we have $c = \#(V_\Gamma / L) = 3$ in this case, and B_2' consists of seven vertices $x_0, x_1, \ldots, x_6$ shown in Figure 18. The values $d_{P_\Gamma, \Phi}(x_0, x_i)$ and $d_\Gamma(x_0, x_i)$ for $i = 0, 1, \ldots, 6$ are as in Table 4.

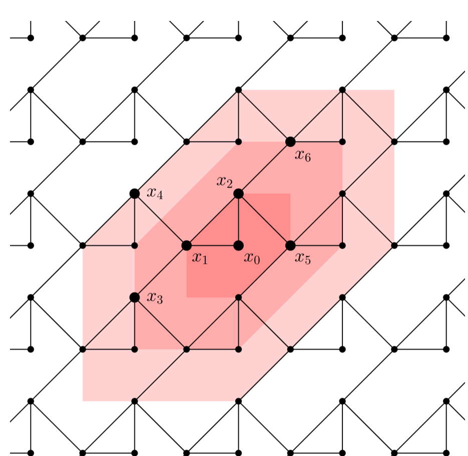


Figure 18. $x_0, x_1, \ldots, x_6$, and $x_0 + iP_\Gamma$ for $i = 1, 2, 3$.

i	0	1	2	3	4	5	6
$d_{P_\Gamma,\Phi}(x_0,x_i)$	0	1	1	2	3	1	2
$d_\Gamma(x_0,x_i)$	0	1	1	2	2	2	2

Table 4. $d_{P_\Gamma,\Phi}(x_0,x_i)$ and $d_\Gamma(x_0,x_i)$ for $i=0,1,\dots,6$.

Next, C_2' is given by

$$C_2' = \max_{y\in \mathrm{int}(Q)\cap V_\Gamma} \bigl(d'(x_0,y) - d_{P_\Gamma,\Phi}(x_0,y)\bigr) = 3,$$

where Q is the hexagram-like figure illustrated as in Figure 19. Note that $\mathrm{int}(Q)\cap V_\Gamma$ consists of nine points $x_0,x_1,\dots,x_8$. The values $d'(x_0,x_i)$ and $d_{P_\Gamma,\Phi}(x_0,x_i)$ for $i = 0,1,\dots,8$ are as in Table 5.

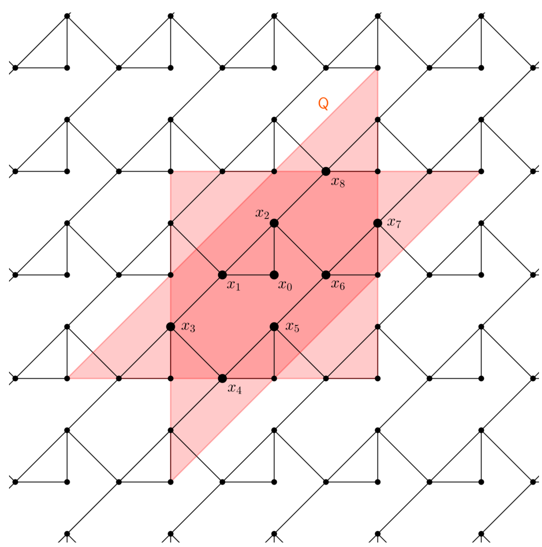


Figure 19. Q and $x_0,x_1,\dots,x_8$.

i	0	1	2	3	4	5	6	7	8
$d'(x_0,x_i)$	3	2	2	2	3	3	2	3	2
$d_{P_\Gamma,\Phi}(x_0,x_i)$	0	1	1	2	2	1	1	2	2

Table 5. $d'(x_0,x_i)$ and $d_{P_\Gamma,\Phi}(x_0,x_i)$ for $i=0,1,\dots,8$.

The following corollary is proved by Shutov and Maleev [23], and this is an immediate consequence of Theorem A.2. For a lattice $L\simeq \mathbb{Z}^n$ and a polytope $P\subset L_\mathbb{R}$, we can define the volume $\mathrm{vol}_L(P)$ of P as follows. We fix a group isomorphism $i : L \xrightarrow{\simeq} \mathbb{Z}^n$. Then, i is extended to an isomorphism $i_\mathbb{R} : L_\mathbb{R} \xrightarrow{\simeq} \mathbb{R}^n$, and we define $\mathrm{vol}_L(P)$ as the volume of $i_\mathbb{R}(P)\subset \mathbb{R}^n$. Note that the value $\mathrm{vol}_L(P)$ is independent of the choice of i.

Corollary A.4 ([23, Theorem 2]). *Let (Γ, L) be a strongly connected n-dimensional periodic graph, and let $x_0\in V_\Gamma$. Let f_b and f_s be the quasi-polynomials corresponding to the functions $b : i\mapsto b_{\Gamma,x_0,i}$ and $s : i\mapsto s_{\Gamma,x_0,i}$ (Theorem 2.17). Then the following assertions hold.*

(1) *Each constituent of* f_b *is a polynomial of degree* n*, and its leading coefficient is* $\#(V_\Gamma/L) \cdot \mathrm{Vol}_L(P_\Gamma)$.
(2) *Let* $Q_0, \dots, Q_{N-1}$ *be the constituents of* f_s*. Then, for each* $i = 0, \dots, N-1$*, we have* $\deg Q_i \leqslant n-1$*. Furthermore, let* a_i *denote the coefficient of* Q_i *of degree* $n-1$*. Then, we have* $\frac{1}{N}\sum_{i=0}^{N-1} a_i = n \cdot \#(V_\Gamma/L) \cdot \mathrm{Vol}_L(P_\Gamma)$.

Proof. We fix an injective periodic realization Φ satisfying $\Phi(x_0) = 0$. By Φ, we identify V_Γ with the subset of $L_\mathbb{R}$. We take C_1' and C_2' to satisfy Theorem A.2. Then, for each $d \in \mathbb{Z}_{\geqslant 0}$, we have

$$\#((d - C_2')P_\Gamma \cap V_\Gamma) \leqslant b_{\Gamma,x_0,d} \leqslant \#((d + C_1')P_\Gamma \cap V_\Gamma).$$

Therefore, (1) follows from the following fact (cf. [5, Lemma 3.19]):

$$\#(V_\Gamma/L) \cdot \mathrm{Vol}_L(P_\Gamma) = \lim_{d\to\infty} \frac{1}{d^n} \cdot \#\big(dP_\Gamma \cap V_\Gamma\big).$$

(2) follows from (1). □

Remark A.5. In crystallography, the invariant $\frac{1}{N}\sum_{i=0}^{N-1} a_i$ in Corollary A.4(2) is called the *topological density* (cf. [13]).

Example A.6. For the Wakatsuki graph (see Example 2.6), the invariant $\frac{1}{N}\sum_{i=0}^{N-1} a_i$ in Corollary A.4(2) for the start point $x_0 = v_0'$ is given by $\frac{1}{2}\left(\frac{9}{2} + \frac{9}{2}\right) = \frac{9}{2}$ (see Example 2.18). The same invariant for the start point $x_0 = v_2'$ is also given by $\frac{1}{2}(3+6) = \frac{9}{2}$. They are actually equal to $2 \cdot \#(V_\Gamma/L) \cdot \mathrm{Vol}_L(P_\Gamma) = 2 \cdot 3 \cdot \frac{3}{4} = \frac{9}{2}$ as proved in Corollary A.4(2).

Appendix B. Ehrhart theory

In this section, we discuss a variant of Ehrhart theory (Theorem B.4), which is necessary for the proof of Theorem 3.4. The difference from the usual Ehrhart theory is that the center v of the dilation need not be the origin, and the dilation factor may be shifted by a constant α.

Definition B.1. *Let* $P \subset \mathbb{R}^N$ *be a rational polytope, and let* $v \in \mathbb{R}^N$ *and* $\alpha \in \mathbb{R}$.

(1) *We define a function* $h_{P,v,\alpha} : \mathbb{Z} \to \mathbb{Z}$ *as follows*

$$h_{P,v,\alpha}(d) \coloneqq \#\left((v + (d+\alpha)P) \cap \mathbb{Z}^N\right).$$

Here, we define $tP = \varnothing$ *for* $t < 0$.

(2) *Let* $\mathrm{relint}(P)$ *denote the relative interior of* P*. Then, we also define a function* $\mathring{h}_{P,v,\alpha} : \mathbb{Z} \to \mathbb{Z}$ *as follows*

$$\mathring{h}_{P,v,\alpha}(d) \coloneqq \#\left((v + (d+\alpha) \cdot \mathrm{relint}(P)) \cap \mathbb{Z}^N\right).$$

Here, we define $t \cdot \mathrm{relint}(P) = \varnothing$ *for* $t \leqslant 0$.

(3) *Let*

$$H_{P,v,\alpha}(t) \coloneqq \sum_{i\in\mathbb{Z}} h_{P,v,\alpha}(i)t^i, \qquad \mathring{H}_{P,v,\alpha}(t) \coloneqq \sum_{i\in\mathbb{Z}} \mathring{h}_{P,v,\alpha}(i)t^i$$

denote their generating functions.

Remark B.2.

(1) In the definition above, we have defined $0 \cdot P = \{0\}$ but $0 \cdot \mathrm{relint}(P) = \varnothing$. This definition is necessary for the equations (♠) in the proof of Theorem B.4.
(2) The usual Ehrhart theory (cf. [5]) treats the case where $v = 0$ and $\alpha = 0$. McMullen (in [17]) and de Vries and Yoshinaga (in [9, Section 3]) discuss $h_{P,v,\alpha}$ when $\alpha = 0$.

Lemma B.3. *We have* $h_{P,v,\alpha} = h_{-P,-v,\alpha}$ *and* $\mathring{h}_{P,v,\alpha} = \mathring{h}_{-P,-v,\alpha}$.

Proof. The first assertion follows from

$$\begin{aligned}\#\left((v+(d+\alpha)P)\cap\mathbb{Z}^N\right) &= \#\left((-(v+(d+\alpha)P))\cap\mathbb{Z}^N\right)\\ &= \#\left((-v+(d+\alpha)(-P)\cap\mathbb{Z}^N\right).\end{aligned}$$

The second assertion can be proved in the same way. □

Theorem B.4. *Let* $P \subset \mathbb{R}^N$ *be a rational polytope of dimension* M*, and let* $v \in \mathbb{R}^N$ *and* $\alpha \in \mathbb{R}$*. Then, the following assertions hold:*

(1) $h_{P,v,\alpha}$ *is a quasi-polynomial on* $d \geqslant -\alpha$*, and* $h_{P,v,\alpha}(d) = 0$ *holds for* $d < -\alpha$.
(2) $\mathring{h}_{P,v,\alpha}$ *is a quasi-polynomial on* $d > -\alpha$*, and* $\mathring{h}_{P,v,\alpha}(d) = 0$ *holds for* $d \leqslant -\alpha$.
(3) $H_{P,v,\alpha}(1/t) = (-1)^{M+1}\mathring{H}_{P,-v,-\alpha}(t)$.
(4) *Let* $f_{P,v,\alpha}$ *be the corresponding quasi-polynomial to the function* $h_{P,v,\alpha}$ *on* $d \geqslant -\alpha$*. Then, we have* $f_{P,v,\alpha}(-i) = (-1)^M\mathring{h}_{P,-v,-\alpha}(i)$ *for any* $i \in \mathbb{Z}_{>\alpha}$.

Proof. For a subset $S \subset \mathbb{R}^{N+1}$, let

$$\sigma_S(\mathbf{z}) = \sigma_S(z_1,\ldots,z_{N+1}) \coloneqq \sum_{\mathbf{m}\in S\cap\mathbb{Z}^{N+1}} \mathbf{z}^{\mathbf{m}}$$

denote the integer-point transform of S (see [5, Section 3.3]). This is a formal sum of Laurent monomials $\mathbf{z}^{\mathbf{m}} = z_1^{m_1}\cdots z_{N+1}^{m_{N+1}}$ for $\mathbf{m} = (m_1,\ldots,m_{N+1}) \in S\cap\mathbb{Z}^{N+1}$.

Let $K \coloneqq \mathbb{R}_{\geqslant 0}\left(\{1\}\times P\right) \subset \mathbb{R}^{N+1}$ be the cone over P. Then, we have

$$(\spadesuit)\qquad \begin{aligned} H_{P,v,\alpha}(t) &= \sigma_{(-\alpha,v)+K}(t,1,\ldots,1),\\ \mathring{H}_{P,-v,-\alpha}(t) &= \sigma_{(\alpha,-v)+\mathrm{relint}(K)}(t,1,\ldots,1).\end{aligned}$$

Furthermore, we have

$$\sigma_{(-\alpha,v)+K}(z_1^{-1},\ldots,z_{N+1}^{-1}) = (-1)^{M+1}\sigma_{(\alpha,-v)+\mathrm{relint}(K)}(z_1,\ldots,z_{N+1})$$

by [5, Exercises 4.5 and 4.6]. Therefore, we get the desired equality in (3).

The second assertions of (1) and (2) are obvious from the definition of $h_{P,v,\alpha}$ and $\mathring{h}_{P,v,\alpha}$.

By taking a triangulation of P, the first assertions in (1) and (2) can be reduced to the same assertions for simplices P. Assume that P is a simplex. Let $v_1,\ldots,v_{M+1}$ be the vertices of P. Since P is rational polytope, we may write $v_i = u_i/a_i$ for some $u_i \in \mathbb{Z}^N$ and $a_i \in \mathbb{Z}_{>0}$. We set $D, D^\circ \subset \mathbb{R}^{N+1}$ by

$$\begin{aligned} D &\coloneqq \left\{ \sum_{i=1}^{M+1}\alpha_i(a_i,u_i) \;\middle|\; \alpha_1,\ldots,\alpha_{M+1}\in[0,1)\right\},\\ D^\circ &\coloneqq \left\{ \sum_{i=1}^{M+1}\alpha_i(a_i,u_i) \;\middle|\; \alpha_1,\ldots,\alpha_{M+1}\in(0,1]\right\}.\end{aligned}$$

Then, by [5, Theorem 3.5], we have

$$\begin{aligned} \sigma_{(-\alpha,v)+K}(z_1,\mathbf{z}) &= \frac{\sigma_{(-\alpha,v)+D}(z_1,\mathbf{z})}{\prod_{i=1}^{M+1}(1-z_1^{a_i}\mathbf{z}^{u_i})},\\ \sigma_{(-\alpha,v)+\mathrm{relint}(K)}(z_1,\mathbf{z}) &= \frac{\sigma_{(-\alpha,v)+D^\circ}(z_1,\mathbf{z})}{\prod_{i=1}^{M+1}(1-z_1^{a_i}\mathbf{z}^{u_i})},\end{aligned}$$

where $\mathbf{z} = (z_2, \ldots, z_{N+1})$. Therefore, we have

$$H_{P,v,\alpha}(t) = \sigma_{(-\alpha,v)+K}(t,1,\ldots,1) = \frac{\sigma_{(-\alpha,v)+D}(t,1,\ldots,1)}{\prod_{i=1}^{M+1}(1-t^{a_i})},$$
$$\mathring{H}_{P,v,\alpha}(t) = \sigma_{(-\alpha,v)+\mathrm{relint}(K)}(t,1,\ldots,1) = \frac{\sigma_{(-\alpha,v)+D^\circ}(t,1,\ldots,1)}{\prod_{i=1}^{M+1}(1-t^{a_i})}.$$

Here, we have

$$\deg \sigma_{(-\alpha,v)+D}(t,1,\ldots,1) < -\alpha + \sum_{i=1}^{M+1} a_i,$$
$$\deg \sigma_{(-\alpha,v)+D^\circ}(t,1,\ldots,1) \leqslant -\alpha + \sum_{i=1}^{M+1} a_i.$$

Therefore, we can conclude that $h_{P,v,\alpha}$ is a quasi-polynomial on $d \geqslant -\alpha$, and that $\mathring{h}_{P,v,\alpha}$ is a quasi-polynomial on $d > -\alpha$. We complete the proof of (1) and (2).

Finally, we prove (4). Since $f_{P,v,\alpha}$ is a quasi-polynomial, we have

$$\sum_{i \in \mathbb{Z}_{<-\alpha}} f_{P,v,\alpha}(i)t^i = -\sum_{i \in \mathbb{Z}_{\geqslant -\alpha}} f_{P,v,\alpha}(i)t^i$$

as rational functions (cf. [5, Exercise 4.7]). Therefore, we have

$$\begin{aligned}\sum_{i \in \mathbb{Z}_{>\alpha}} f_{P,v,\alpha}(-i)t^{-i} &= \sum_{i \in \mathbb{Z}_{<-\alpha}} f_{P,v,\alpha}(i)t^i \\ &= -\sum_{i \in \mathbb{Z}_{\geqslant -\alpha}} f_{P,v,\alpha}(i)t^i \\ &= -H_{P,v,\alpha}(t) \\ &= (-1)^M \mathring{H}_{P,-v,-\alpha}(t^{-1}) \\ &= (-1)^M \sum_{i \in \mathbb{Z}} \mathring{h}_{P,-v,-\alpha}(i)t^{-i}.\end{aligned}$$

Here, the third equality follows from (1), and the fourth follows from (3). By comparing the coefficients, we conclude that $f_{P,v,\alpha}(-i) = (-1)^M \mathring{h}_{P,-v,-\alpha}(i)$ for $i \in \mathbb{Z}_{>\alpha}$. □

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