ALGEBRAIC COMBINATORICS

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Existence of elementwise invariant vectors in representations of symmetric groups

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ABSTRACT We determine when a permutation with cycle type μ admits a non-zero invariant vector in the irreducible representation V_λ of the symmetric group. We find that a majority of pairs (λ, μ) have this property, with only a few simple exceptions.

1. INTRODUCTION

Let $(\rho_\lambda, V_\lambda)$ denote the irreducible representation of S_n associated with a partition λ of n. Let $w_\mu \in S_n$ denote a permutation with cycle type μ for a partition μ of n. A vector $v \in V_\lambda$ is said to be an invariant vector for $w \in S_n$ if $\rho_\lambda(w)v = v$.

MAIN THEOREM. *The only pairs of partitions (λ, μ) of a given integer n such that w_μ does not admit a nonzero invariant vector in V_λ are the following:*

  (1) $\lambda = (1^n)$, μ *is any partition of n for which w_μ is odd,*
  (2) $\lambda = (n-1, 1)$, $\mu = (n)$, $n \geqslant 2$,
  (3) $\lambda = (2, 1^{n-2})$, $\mu = (n)$, $n \geqslant 3$ *is odd,*
  (4) $\lambda = (2^2, 1^{n-4})$, $\mu = (n-2, 2)$, $n \geqslant 5$ *is odd,*
  (5) $\lambda = (2, 2)$, $\mu = (3, 1)$,
  (6) $\lambda = (2^3)$, $\mu = (3, 2, 1)$,
  (7) $\lambda = (2^4)$, $\mu = (5, 3)$,
  (8) $\lambda = (4, 4)$, $\mu = (5, 3)$,
  (9) $\lambda = (2^5)$, $\mu = (5, 3, 2)$.

In the special case where $\mu = (n)$, the following characterisation, restated in terms of invariant vectors, was obtained by Joshua Swanson [5] (see also [7]), proving a conjecture of Sheila Sundaram [3, Remark 4.8].

THEOREM 1.1. *For every positive integer n and partition λ of n, $w_{(n)}$ admits a nonzero invariant vector in V_λ except in the following cases:*

  (1) $\lambda = (n-1, 1)$,
  (2) $\lambda = (1^n)$ *when n is even,*
  (3) $\lambda = (2, 1^{n-2})$ *when $n \geqslant 3$ is odd.*

We use the above result and the Littlewood–Richardson rule to obtain our main theorem.

For each partition μ of n, let C_μ denote the cyclic group generated by w_μ. The permutation w_μ admits a non-zero invariant vector in V_λ if and only if the trivial representation of C_μ occurs in the restriction of V_λ to C_μ. By Frobenius reciprocity, this is equivalent to V_λ occurring in the induced representation $\mathrm{Ind}_{C_\mu}^{S_n} 1$.

A closely related problem was considered by Sheila Sundaram who proved the following result [4, Theorem 5.1].

THEOREM 1.2. *Let μ be a partition of an integer $n \neq 4, 8$. Let Z_μ denote the centralizer of w_μ in S_n. Then V_λ occurs in $\mathrm{Ind}_{Z_\mu}^{S_n} 1$ for every partition λ of n if and only if μ has at least two parts, and all its parts are odd and distinct.*

A conjugacy class in a group G whose permutation representation for the conjugation action contains every irreducible representation of G is called a *global class*. Thus her theorem characterizes the global classes of symmetric groups.

Since C_μ is a subgroup of Z_μ, $\mathrm{Ind}_{Z_\mu}^{S_n} 1$ is a subrepresentation of $\mathrm{Ind}_{C_\mu}^{S_n} 1$. Hence if V_λ occurs in $\mathrm{Ind}_{Z_\mu}^{S_n} 1$, it also occurs in $\mathrm{Ind}_{C_\mu}^{S_n} 1$. If the parts of μ are distinct and pairwise relatively prime, then $Z_\mu = C_\mu$. In this case our result is consistent with hers. Our proof strategy is also quite similar to hers.

Dipendra Prasad and Ravi Raghunathan [2] proposed a partial order on automorphic representations called immersion. Adapted to finite groups, it may be defined as follows.

DEFINITION 1.3. *Given representations (ρ, V) and (σ, W) of G, say that V is immersed in W, denoted $V \preccurlyeq W$, if for every $g \in G$ and every $\lambda \in \mathbf{C}$, the multiplicity of λ as an eigenvalue of $\rho(g)$ does not exceed the multiplicity of λ as an eigenvalue of $\sigma(g)$.*

In particular, if V is a subrepresentation of W, then $V \preccurlyeq W$. In the context of immersion, our main theorem implies the following result.

THEOREM 1.4. *For every positive integer n, $V_{(n)} \preccurlyeq V_\lambda$ unless λ is one of the partitions of n that occur in the statement of the main theorem. Similarly $V_{(1^n)} \preccurlyeq V_\lambda$ unless the conjugate of λ is one of the partitions that occurs in the statement of the main theorem.*

Proof. The first assertion is a direct consequence of our main theorem. The second follows from the first using the observation that if $\gamma : G \to \mathbf{C}^*$ is a multiplicative character, then $V \preccurlyeq W$ if and only if $V \otimes \gamma \preccurlyeq W \otimes \gamma$. Taking γ to be the sign character of S_n, we get $V_{(n)} \preccurlyeq W_\lambda$ if and only if $V_{(1^n)} \preccurlyeq W_{\lambda'}$, where λ' is the partition conjugate to λ. $\qquad\square$

SUPPORTING CODE. Some steps in the proof involve direct calculations using the Sage Mathematical Software system [6]. Code for carrying out these calculations can be downloaded from: `https://www.imsc.res.in/~amri/invariant_vectors/`

2. PROOF OF THE MAIN THEOREM

Following [1, Section I.2], let Λ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, \dots$ with integer coefficients. For each partition λ, let s_λ denote the Schur function corresponding to λ. Given $f, g \in \Lambda$, we say that $f \geqslant g$ if $f - g$ is a non-negative integer combination of Schur functions.

For each representation V of S_n, let $\mathrm{ch}\, V$ denote the Frobenius characteristic of V, which is a homogeneous symmetric function of degree n [1, Section I.7]. Since we have $\mathrm{ch}\, V_\lambda = s_\lambda$, V_λ occurs in $\mathrm{Ind}_{C_\mu}^{S_n} 1$ if and only if

$$(1) \qquad\qquad \mathrm{ch}\, \mathrm{Ind}_{C_\mu}^{S_n} 1 \geqslant s_\lambda.$$

Define

$$f_\mu = \operatorname{ch} \operatorname{Ind}_{C_\mu}^{S_n} 1.$$

Let $S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_k}$ be the Young subgroup corresponding to μ. Let D_μ denote the subgroup of S_μ generated by the cycles of w_μ. Thus D_μ is a product of cyclic groups of orders $\mu_1, \mu_2, \ldots, \mu_k$. Using induction in stages,

$$f_\mu = \operatorname{ch} \operatorname{Ind}_{S_\mu}^{S_n} \operatorname{Ind}_{D_\mu}^{S_\mu} \operatorname{Ind}_{C_\mu}^{D_\mu} 1.$$

Therefore

$$(2) \qquad f_\mu \geqslant \operatorname{ch} \operatorname{Ind}_{S_\mu}^{S_n} \operatorname{Ind}_{D_\mu}^{S_\mu} 1 = \prod_{i=1}^{k} f_{(\mu_i)}.$$

Swanson's theorem (Theorem 1.1) tells us that $f_{(n)} \geqslant s_\lambda$ for most partitions λ of n. We will use this fact, together with the inequality (2), to establish (1) in most cases. This will be carried out by using the Littlewood–Richardson rule. Recall that Littlewood–Richardson coefficients are defined by

$$s_\alpha s_\beta = \sum_\lambda c_{\alpha\beta}^\lambda s_\lambda.$$

The Littlewood–Richardson rule [1, Section I.9] asserts that $c_{\alpha\beta}^\lambda$ is the number of LR-tableaux of shape λ/α and weight β. Recall that an LR-tableau is a semistandard skew-tableau whose reverse row reading word is a lattice permutation.

LEMMA 2.1. *For every partition λ of $p+q$, and every partition α of p that is contained in λ, there exists a partition β of q such that $s_\alpha s_\beta \geqslant s_\lambda$.*

Proof. Let $T_{\lambda\alpha}$ denote the skew-tableau obtained by putting i in the ith cell (from the top) of each column of λ/α. Let β be the weight of $T_{\lambda\alpha}$. For example, if $\lambda = (5, 4, 4, 1)$ and $\alpha = (3, 2, 1)$ then

$$T_{\lambda\alpha} = \;\vcenter{\hbox{}}\;,$$

and β is $(5, 2, 1)$. Since every $i + 1$ occurs below an i in the same column, $T_{\lambda\alpha}$ is an LR-tableau. The Littlewood–Richardson rule implies that $s_\alpha s_\beta \geqslant s_\lambda$. $\qquad\square$

REMARK 2.2. Given $\alpha \subset \lambda$ (i.e., α is contained in λ), the construction of β in the proof of the above lemma has the following property: if $\gamma \vdash q$ also satisfies $s_\alpha s_\gamma \geqslant s_\lambda$, then $\beta \geqslant \gamma$ in the dominance order. This is because, in a skew tableau of shape λ/α, the maximum number of cells with entry i cannot exceed the number of columns of length at least i.

LEMMA 2.3. *Given integers $p \geqslant 2$, $q \geqslant 1$, and a partition $\lambda \vdash (p + q)$ different from $(1^{(p+q)})$, there exists a partition $\beta \vdash q$ such that $f_{(q)} \geqslant s_\beta$ and $\beta \subset \lambda$.*

Proof. We need to show that there exists $\beta \vdash q$ with $\beta \subset \lambda$ that is not among the exceptions listed in Theorem 1.1. We consider the following cases:

Case 1: $\lambda \supset (q-1, 1)$. Since $p \geqslant 2$, the skew shape $\lambda/(q-1, 1)$ has at least two cells. If at least one of these cells lies in the first row of λ, then choose $\beta = (q)$. If at least one of these cells lies in the first column of λ, then choose $\beta = (q-2, 1, 1)$ If neither of the above happens, then $\lambda/(q - 1, 1)$ has at least two cells in its second row. In this case $q - 1 \geqslant 3$. Choose $\beta = (q - 2, 2)$. The possible placements of the cells of $\lambda/(q - 1, 1)$ are shown in Figure 1. In all these cases, Theorem 1.1 implies that $f_{(q)} \geqslant s_\beta$.
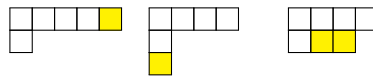
FIGURE 1. Possible placements of two cells of $\lambda/(q-1,1)$

Case 2: $\lambda \supset (1^q)$ and q is even. Since $\lambda \neq (1^{p+q})$, the skew-shape $\lambda/(1^q)$ must contain at least one cell in the first row. Take $\beta = (2, 1^{q-2})$. By Theorem 1.1, $f_{(q)} \geqslant s_\beta$, since q is even.

Case 3: $\lambda \supset (2, 1^{q-2})$ and q is odd. If $\lambda/(2, 1^{q-2})$ has a cell in its first row, take $\beta = (3, 1^{q-3})$. If $\lambda/(2, 1^{q-2})$ has a cell in its first column, take $\beta = (1^q)$. By Theorem 1.1, $f_{(q)} \geqslant s_\beta$, since q is odd. Otherwise the second column of $\lambda/(2, 1^{q-2})$ must have at least two cells. In this case $q \geqslant 4$. Take $\beta = (2, 2, 1^{q-4})$. The possible placements of
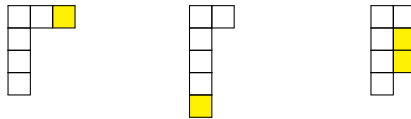

FIGURE 2. Possible placements of cells of $\lambda/(2, 1^{q-2})$.

the cells of $\lambda/(2, 1^{q-2})$ are shown in Figure 2.

All remaining λ: Take β to be any partition of q that is contained in λ. Since λ does not contain any of the exceptions of Theorem 1.1, $f_{(q)} \geqslant s_\beta$. □

DEFINITION 2.4 (Persistent partition). *A partition μ of n is said to be* persistent *if every irreducible representation of S_n, with the possible exception of the sign representation, admits a non-zero w_μ-invariant vector.*

When μ is persistent,

$$f_\mu \geqslant s_\alpha \text{ for every } \alpha \vdash n, \ \alpha \neq (1^n).$$

The main step in proving the main theorem is to show that μ is persistent for every integer partition μ except $\mu = (n)$ for all n and $\mu = (n-2, 2)$ for n odd and sufficiently large. We accomplish this in Lemmas 2.5–2.8. We first consider the case where μ has two parts.

LEMMA 2.5. *If $\mu = (p, q)$ where $p \geqslant q \geqslant 4$, then μ is persistent.*

Proof. We will assume that $p \geqslant 6$. For $p = 4$ and 5, direct computation (using Sage [6]) establishes the lemma. By (2), to show that $f_{(p,q)} \geqslant s_\lambda$ it suffices to find partitions α and β of p and q respectively, such that $f_{(p)} \geqslant s_\alpha$, $f_{(q)} \geqslant s_\beta$ and $s_\alpha s_\beta \geqslant s_\lambda$.

By Lemma 2.3, choose $\beta \subset \lambda$ such that $f_{(q)} \geqslant s_\beta$. Using Lemma 2.1 (with the roles of α and β reversed) we may choose $\alpha \vdash p$ such that $s_\alpha s_\beta \geqslant s_\lambda$. If $f_{(p)} \geqslant s_\alpha$ we are done. Otherwise, by Theorem 1.1, one of the following cases must occur:

Case 1: $\alpha = (p-1, 1)$.

- Suppose $\lambda_2 \geqslant 2$. Replace α by $(p-2, 2)$ and choose β as in the proof of Lemma 2.1. If $f_{(q)} \geqslant s_\beta$ we are done, otherwise by Theorem 1.1, one of the following must occur.

– $\beta = (q-1,1)$. Since each column of $T_{\lambda\alpha}$ in the proof of Lemma 2.1 is filled with integers $1, 2, \cdots$ in increasing order, λ/α has $q-2$ columns with one cell and one column with two cells. Either the first column, the third column, or the $(p-1)$st column can have two cells.

If the first column of λ/α has two cells, then we may change β to $(q-2,1,1)$ by constructing a skew-tableau as shown in the example below.

Note that the reverse reading word will be a lattice permutation because $\lambda \supset (p-1,1)$, so there will be at least one cell in the first row of λ/α.

If the third column of λ/α has two cells, then we may change β to $(q-2,2)$ as shown in the example below.

Note that the reverse reading word will be a lattice permutation because at least one 1 occurs in the first row (since $\lambda \supset (p-1,1)$) and at least one 1 occurs in the second row of this tableau (in the third column).

If the $(p-1)$st column of λ/α has two cells, we may change α to $(p-3,3)$ (note that $p \geqslant 6$, so α is a partition). We may then take $\beta = (q-2,2)$ as shown in the example below.


– $\beta = (2, 1^{q-2})$ with q odd. This would mean that λ/α has two columns, having 1 and $q-1$ cells respectively. But since $\lambda \supset (p-1,1)$ the $(p-1)$st cell in the first row lies in a column of length one. Since $q \geqslant 4$, the other column of λ/α has to be the first one. In particular, λ/α is a vertical strip, so we can replace beta with (1^q).

– $\beta = (1^q)$ with q even. In this case, λ/α has only one column. But λ/α contains the cell $(1, p-1)$ whose column cannot contain any other cells. It follows that this case cannot occur.

• Suppose $\lambda_2 = 1$ and $\lambda_1 \geqslant p$. Replace α by (p) to make $f_{(p)} \geqslant s_\alpha$, and choose β using the proof of Lemma 2.1. By Theorem 1.1 $f_{(q)} \geqslant s_\beta$ unless one of the following cases occurs.

– $\beta = (q-1,1)$. In this case, λ/α has $q-1$ columns, of which one column has exactly two cells (and the others have only one cell). The column with two cells has to be the first column of λ/α, since λ is a hook. Incrementing its entries by 1 allows us to replace β by $(q-2,1,1)$.

– $\beta = (2, 1^{q-2})$ with q odd. Then λ/α has two columns, having 1 and $q-1$ cells respectively. Since λ is a hook, the column with one cell must lie in the first row and the column with $q-1$ cells has to be the first column. Incrementing the entries of the first column by 1 allows us to replace β by (1^q).

– $\beta = (1^q)$ with q even. All the cells of λ/α must lie in the first column. We may replace α by $(p-2,1,1)$ and β by $(2, 1^{q-2})$ as shown in the

following example.
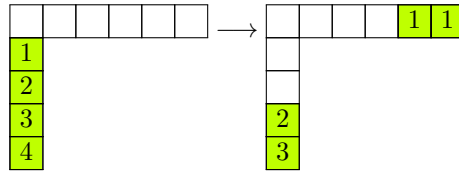
- Suppose $\lambda_2 = 1$ and $\lambda_1 = p - 1$. In other words, $\lambda = (p-1, 1^{q+1})$. Replace α by $(p-2, 1, 1)$. Replace β by $(3, 1^{q-3})$ if $q > 4$ as shown in the example below.
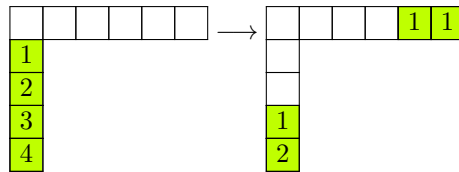
In case $q = 4$, replace β by $(2, 1, 1)$

Case 2: $\alpha = (2, 1^{p-2})$ *with p odd.*

- Suppose $\lambda_2 \geqslant 2$. Change α to $(2, 2, 1^{p-4})$ so that $f_{(p)} \geqslant s_\alpha$. Choose β as in the proof of Lemma 2.1. If $f_{(q)} \geqslant s_\beta$ we are done. Otherwise, by Theorem 1.1, one of the following cases must occur.
  - $\beta = (q-1, 1)$. In this case, λ/α has $q - 1$ columns, with one column having two cells. The column with two cells has to be one of the first three columns.
   If the first or second column of λ/α has two cells, increment the entries of those two cells by 1 to replace β by $(q-2, 1, 1)$. Since $q \geqslant 4$, λ/α has at least one cell in the first row, so the resulting skew-tableau is an LR-tableau.
    Suppose the third column of λ/α has two cells. Since $\lambda \supset (2, 1^{p-2})$, the first column of λ/α must have exactly one cell. Changing the entry of this cell from 1 to 3 gives us an LR-tableau of weight $(q-2, 1, 1)$.
  - $\beta = (2, 1^{q-2})$ with q odd. In this case, λ/α has exactly two columns, having 1 and $q - 1$ cells respectively. Again, since $\lambda \supset (2, 1^{p-2})$, λ/α has at least one cell in the first column.
   If the first column has $q - 1$ cells, then λ/α has to be a vertical strip, and we can take $\beta = (1^q)$.
    If the first column has one cell and the second has $q - 1$ cells, then the other column, which has at least three cells must be the second column. Replace α by $(2^3, 1^{p-6})$ and take $\beta = (2, 2, 1^{q-4})$ as shown in the following example.
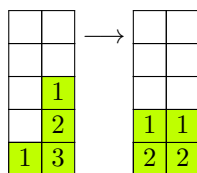

  - $\beta = (1^q)$ with q even. In this case, λ/α is a single column, which has to be the first column since $q \geqslant 4$. Replace α by (1^p) and β by $(2, 1^{q-2})$ as

shown in the following example.
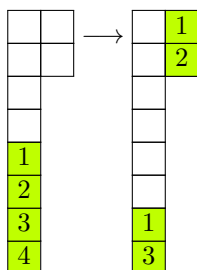
- Suppose $\lambda_2 = 1$ (so that λ is a hook) and $\lambda_1 \geqslant 3$. Replace α by $(3, 1^{p-3})$ and choose β according to Lemma 2.1. If $f_{(q)} \geqslant s_\beta$ we are done. Otherwise by Theorem 1.1 one of the following must occur.
  - $\beta = (q-1, 1)$. In this case, λ has $q-1$ columns, one having two cells and the other having only one cell. Since λ is a hook, only the first column of λ/α can have two cells. Incrementing the entries in these cells by one will allow us to change β to $(q-2, 1, 1)$.
  - $\beta = (2, 1^{q-2})$ with q odd. In this case, λ/α has two columns. Since $\lambda \supset (2, 1^{p-2})$, λ/α has cells in the first column. Since λ is a hook, the fourth column must have exactly one cell. Incrementing the entries of the first column by 1 will allow us to replace β by (1^q).
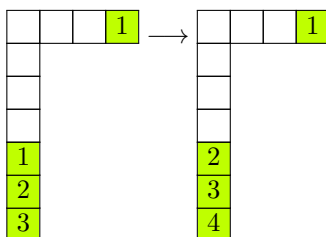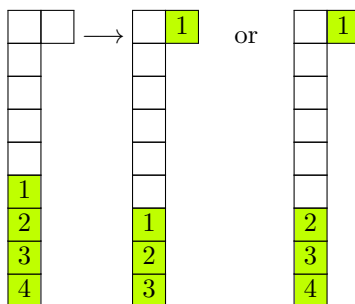

  - $\beta = (1^q)$, with q even. In this case, λ/α is contained in the first column. We can replace α by (1^p) (since p is odd) and β by $(2, 1^{q-2})$ as shown in the example below.
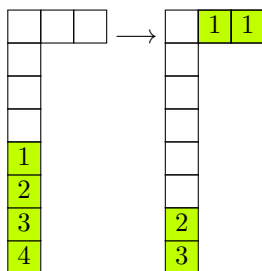
- Suppose $\lambda_2 = 1$ and $\lambda_1 = 2$. In this case λ must be $(2, 1^{p+q-2})$. So we may replace α by (1^p) and β by $(2, 1^{q-2})$ or (1^q) depending on the parity of q.


Case 3: $\alpha = (1^p)$ *with p even.* Since we have assumed that $\lambda \neq (1^{p+q})$, $\lambda_1 > 1$. Replace α by $(2, 1^{p-2})$ and choose β using the proof of Lemma 2.1. If $f_{(q)} \geqslant s_\beta$ we are done. Otherwise, by Theorem 1.1, one of the following cases must occur.

- $\beta = (q-1, 1)$. Then λ/α has $q-1$ columns, one of which has two cells and the others have one cell each. The column with two cells has to be one of the first three. If the column with two cells is one of the first two columns, then incrementing its entries by 1 will give an LR-tableau of shape λ/α and weight $(q-2, 1, 1)$. If the column with two cells is the third, then changing the entry in the first column (which exists since $\lambda \supset (1^p)$) from 1 to 2, we get an LR-tableau of shape λ/α and weight $(q-2, 2)$.
- $\beta = (2, 1^{q-2})$ with q odd. In this case, λ/α has exactly two columns having $q-1$ and 1 cells respectively.

  If the first column of λ/α has $q-1$ cells, then λ/α will be a vertical strip, so we can change β to (1^q).

  If the second column of λ/α has $q-1$ cells, replace α by $(2, 2, 1^{p-4})$ and β by $(2, 2, 1^{q-4})$ as shown in the following example.
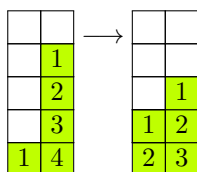
- $\beta = (1^q)$ with q even. In this case, λ/α has only one column, which has to be the first column. So $\lambda = (2, 1^{p+q-2})$. In this case it will not be true that $f_{(p)} f_{(q)} \geqslant s_\lambda$. Instead, we will show that $f_{(p,q)} \geqslant s_{(2, 1^{p+q-2})}$ using a different strategy.

  The restriction of the sign representation of $S_p \times S_q$ to $C_{(p,q)}$ is the trivial representation ($w_{(p,q)}$ is an even permutation, since both p and q are even). Therefore, $\mathrm{Ind}_{C_{(p,q)}}^{S_p \times S_q} 1$ contains the sign representation of $S_p \times S_q$. It follows that
  $$f_{(p,q)} = \mathrm{ch}\, \mathrm{Ind}_{C_{(p,q)}}^{S_{p+q}} 1 \geqslant s_{(1^p)} s_{(1^q)} \geqslant s_{(2, 1^{p+q-2})}.$$

This completes the proof of the lemma. □

LEMMA 2.6. *We have*

1. *The partition $(p, 1)$ is persistent except when $p = 3$.*
2. *The partition $(p, 2)$ is persistent when p is even. When p is odd, $f_{(p,2)} \geqslant s_\lambda$ for all partitions λ of $p + 2$, except $\lambda = (2, 2, 1^{p-2})$ and (1^{p+2}).*
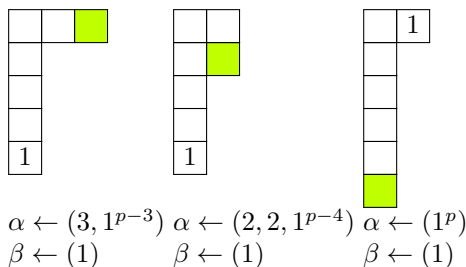3. *The partition $(p, 3)$ is persistent if $p \geqslant 6$.*

$$\alpha \leftarrow (p) \qquad \alpha \leftarrow (p-2,2) \quad \alpha \leftarrow (p-2,1,1)$$
$$\beta \leftarrow (1) \qquad \beta \leftarrow (1) \qquad \beta \leftarrow (1)$$

FIGURE 3. Cases for $\lambda \supset (p-1,1)$

Proof. In order to prove (1), assume $p > 3$ (for $p \leqslant 3$ use direct calculation) and choose $\alpha \vdash p$ such that $s_\alpha s_\beta \geqslant s_\lambda$ using Lemma 2.1 with the roles of α and β interchanged. If $f_{(p)} \geqslant s_\alpha$ then we are done since $f_{(1)} \geqslant (1)$. Otherwise one of the following cases occurs.

Case 1: $\alpha = (p-1,1)$. Given p, there are exactly three possibilities for λ, depending on where the single cell of λ/α is added to α. These possibilities are shown in Figure 3. In this figure the cells of $\alpha = (p-1,1)$ are shown in white, while the cells of λ/α are shown in green.

Case 2: $\alpha = (2, 1^{p-2})$, *with p odd.* Once again, given p, there are finitely many possibilities for λ, as shown in Figure 4 along with the replacement for α.

$$\alpha \leftarrow (3, 1^{p-3}) \quad \alpha \leftarrow (2, 2, 1^{p-4}) \quad \alpha \leftarrow (1^p)$$
$$\beta \leftarrow (1) \qquad \beta \leftarrow (1) \qquad \beta \leftarrow (1)$$

FIGURE 4. Cases for $\lambda \supset (2, 1^{p-2})$

Case 3: $\alpha = (1^p)$ *with p even.* As before, given p, there is only one possibility for $\lambda \neq (1^{p+1})$ which contains (1^p), as shown in Figure 5 along with the replacements for α and β.

In order to prove (2), choose $\beta \vdash 2$ such that $\beta \subset \lambda$ and $f_{(2)} \geqslant s_\beta$ using Lemma 2.3. Choose $\alpha \vdash p$ such that $s_\alpha s_\beta \geqslant s_\lambda$ using Lemma 2.1 with the roles of α and β interchanged. If $f_{(p)} \geqslant s_\alpha$ then we are done. Otherwise, one of the following cases occurs.

Case 1: $\alpha = (p-1,1)$. Given p, there are finitely many possibilities for λ, depending on how the two cells of λ/α are placed. These possibilities are shown in Figure 6, the coloured cells having the same significance as before.
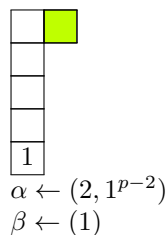
$$\alpha \leftarrow (2, 1^{p-2})$$
$$\beta \leftarrow (1)$$

FIGURE 5. Cases for $\lambda \supset (1^p)$

$$\alpha \leftarrow (p) \qquad \alpha \leftarrow (p) \qquad \alpha \leftarrow (p-2, 1, 1)$$
$$\beta \leftarrow (2) \qquad \beta \leftarrow (2) \qquad \beta \leftarrow (2)$$

$$\alpha \leftarrow (p-2, 2) \qquad \alpha \leftarrow (p-2, 1, 1) \qquad \alpha \leftarrow (p-2, 1, 1)$$
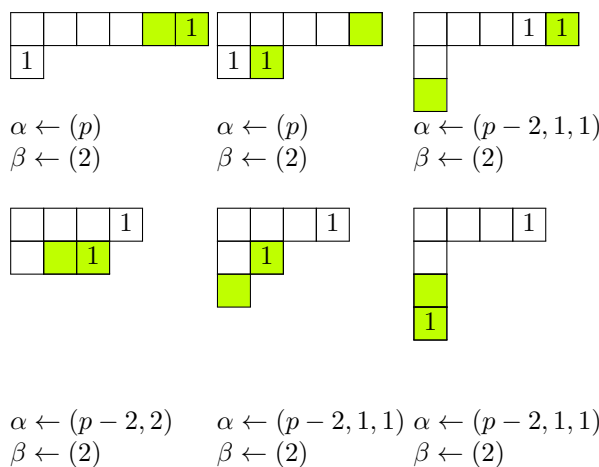$$\beta \leftarrow (2) \qquad \beta \leftarrow (2) \qquad \beta \leftarrow (2)$$

FIGURE 6. Cases for $\lambda \supset (p-1, 1)$

Case 2: $\alpha = (2, 1^{p-2})$, with p odd. Once again, given p, there are finitely many possibilities for λ, as shown in Figure 7 along with the replacements for α and β except for the fifth diagram, where $\lambda = (2, 2, 1^{p-2})$.

When $\lambda = (2, 2, 1^{p-2})$, it turns out that $f_{(p,2)} \not\geq s_\lambda$. Indeed, since p is odd, $C_{(p,2)} = D_{(p,2)}$ and so $f_{(p,2)} = f_{(p)} f_{(2)}$. We will show that $f_{(p)} f_{(2)} \not\geq f_\lambda$ by contradiction.

If $f_{(p)} f_{(2)} \geq f_\lambda$, there exist $\alpha \vdash p$ and $\beta \vdash 2$ such that $f_{(p)} \geq s_\alpha$, $f_{(2)} \geq s_\beta$ and $s_\alpha s_\beta \geq s_\lambda$. The latter condition implies that $\alpha \subset \lambda$. This gives us three possibilities for α, namely $(2^2, 1^{p-4})$, $(2, 1^{p-2})$ and (1^p). Since p is odd, $f_{(p)} \not\geq (2, 1^{p-2})$. But for the remaining two possibilities, λ/α is a single column, forcing $\beta = (1, 1)$, but $f_{(2)} \not\geq s_\beta$, contradicting our assumptions on α and β.

Hence, we have $f_{(p,2)} \not\geq s_{(2,2,1^{p-2})}$, if p is odd. This is indicated by the question marks in Figure 7.

Case 3: $\alpha = (1^p)$, with p even. Similarly, for each α of this type, there are only finitely many possibilities for λ which contains (1^p), as shown in Figure 8 along with the replacements for α and β, except for the third diagram (where $\lambda = (2, 1^p)$) in which case we have to use a different strategy. The restriction of the sign representation of $S_p \times S_2$ to $C_{(p,2)}$ is the trivial representation (since $w_{(p,2)}$ is an even permutation). Therefore, $\mathrm{Ind}_{C_{(p,2)}}^{S_p \times S_2} 1$ contains the sign representation of $S_p \times S_2$. It follows that

$$f_{(p,2)} = \mathrm{ch}\, \mathrm{Ind}_{C_{(p,2)}}^{S_{p+2}} \geq s_{(1^p)} s_{(1^2)} \geq s_{(2,2,1^{p-2})}$$
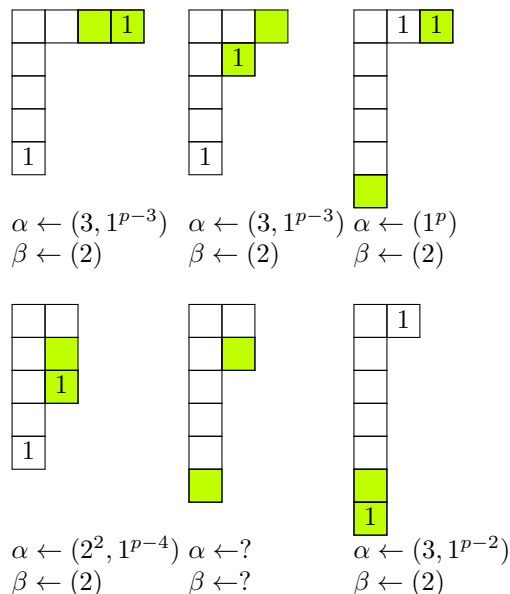
Hence $f_{(p,2)} \geq s_{(2,2,1^{p-2})}$.
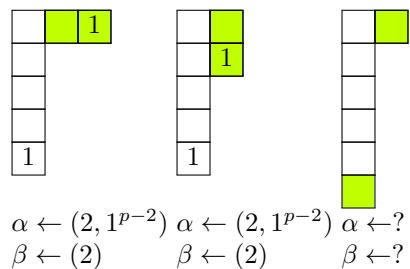
FIGURE 7. Cases for $\lambda \supset (2, 1^{p-2})$

FIGURE 8. Cases for $\lambda \supset (1^p)$

In order to prove (3), choose $\beta \vdash 3$ such that $\beta \subset \lambda$ and $f_{(3)} \geqslant s_\beta$ using Lemma 2.3. Choose $\alpha \vdash p$ such that $s_\alpha s_\beta \geqslant s_\lambda$ using Lemma 2.1 with the roles of α and β interchanged. If $f_{(p)} \geqslant s_\alpha$ then we are done. Otherwise, one of the following cases occurs.

Case 1: $\alpha = (p-1, 1)$. Given p, there are finitely many possibilities for λ, depending on how the three cells of λ/α are placed. These possibilities are shown in Figure 9. The empty cells denote the replacement for α and the cells filled by entries in $\{1, 2, 3\}$ give a skew tableau whose weight is the replacement for β such that $f_{(p)} \geqslant s_\alpha$, $f_{(q)} \geqslant s_\beta$ and $s_\alpha s_\beta \geqslant s_\lambda$.

For example, in the left-most figure of the first row, $\lambda = (p + 2, 1)$. In this case α is replaced by (p) and β is replaced by (3).

Case 2: $\alpha = (2, 1^{p-2})$, *with p odd*. Once again, for each α of this type, there are finitely many possibilities for λ, as shown in Figure 10 along with the replacements for α and β.
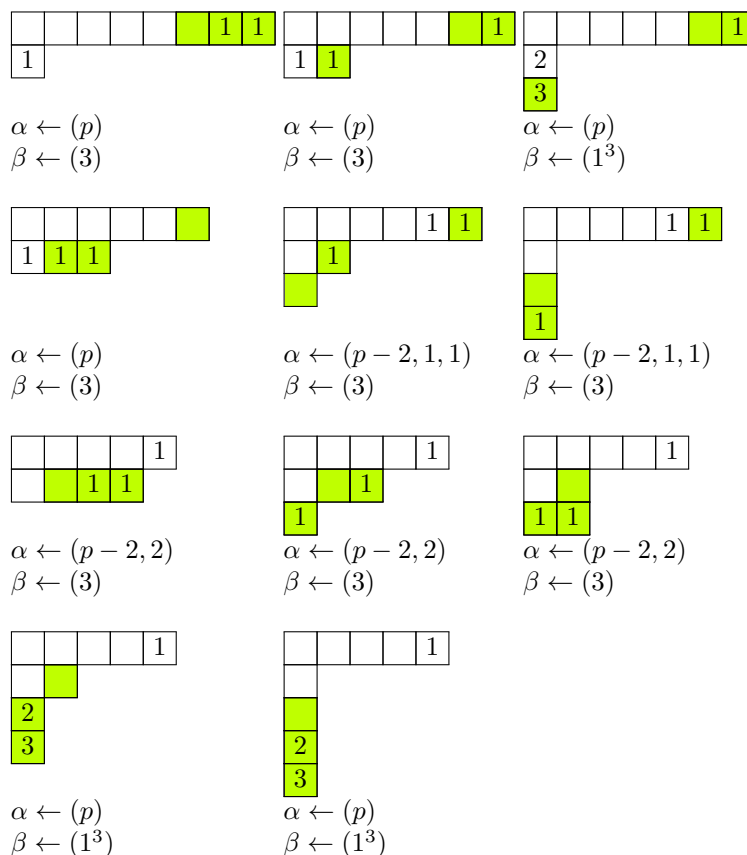
FIGURE 9. Cases for $\lambda \supset (p-1,1)$

Case 3: $\alpha = (1^p)$ *with p even.* Similarly, for each α of this type, there are only finitely many possibilities for λ which contains (1^p), as shown in Figure 11 along with the replacements for α and β. □

LEMMA 2.7. *The partitions $(1,1)$, $(2,2)$ and $(3,3)$ are persistent.*

Proof. We compute the following by hand or using SageMath [6].

- $f_{(1,1)} = s_{(2)} + s_{(1,1)}$
- $f_{(2,2)} = s_{(4)} + s_{(3,1)} + 2s_{(2,2)} + s_{(2,1,1)} + s_{(1,1,1,1)}$
- $f_{(3,3)} = s_{(6)} + s_{(5,1)} + 3s_{(4,2)} + 4s_{(4,1,1)} + s_{(3,3)} + 4s_{(3,2,1)} + 4s_{(3,1,1,1)} + s_{(2,2,2)} + 3s_{(2,2,1,1)} + s_{(2,1,1,1,1)} + s_{(1,1,1,1,1,1)}$

Evidently, the partitions $(1,1)$, $(2,2)$ and $(3,3)$ are persistent. □

Now consider partitions with three or more parts.

LEMMA 2.8. *A partition $\mu = (\mu_1, \ldots, \mu_k) \vdash n$ is persistent if the partition $\tilde{\mu}$ obtained by removing a part μ_i from μ is persistent and $n - \mu_i \geqslant 4$.*

Proof. Since $C_{\tilde{\mu}} \times C_{(\mu_i)} \leqslant C_\mu$ and $D_{\tilde{\mu}} \times D_{(\mu_i)} = D_\mu$, we have

$$\mathrm{Ind}_{C_\mu}^{D_\mu} 1 \geqslant \mathrm{Ind}_{C_{\tilde{\mu}}}^{D_{\tilde{\mu}}} 1 \otimes \mathrm{Ind}_{C_{(\mu_i)}}^{D_{(\mu_i)}} 1$$

Inducing to $S_{p_1} \times \cdots \times S_{p_k}$, and then to $S_{p_1 + \cdots + p_k}$ gives
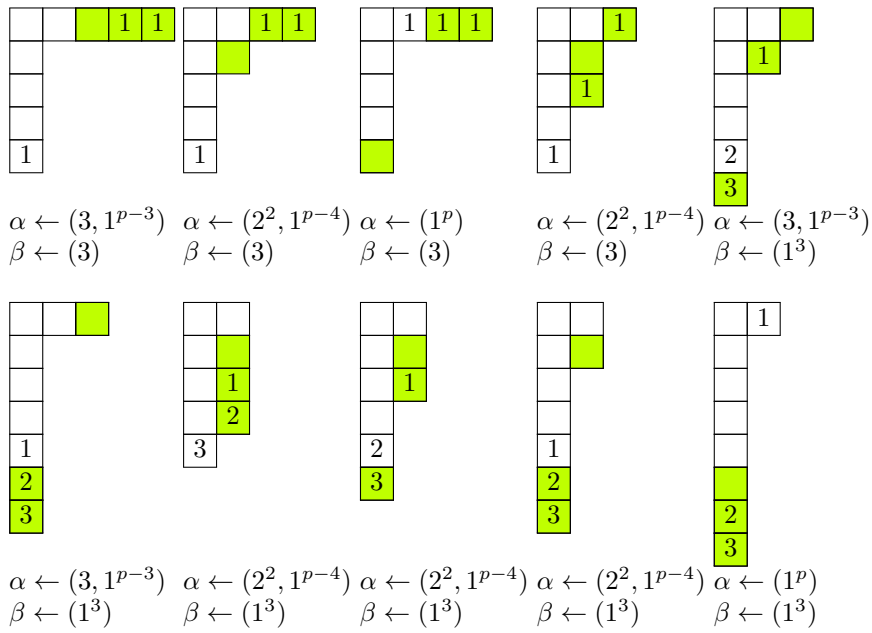
$$f_\mu \geqslant f_{\tilde{\mu}} f_{(\mu_i)}.$$

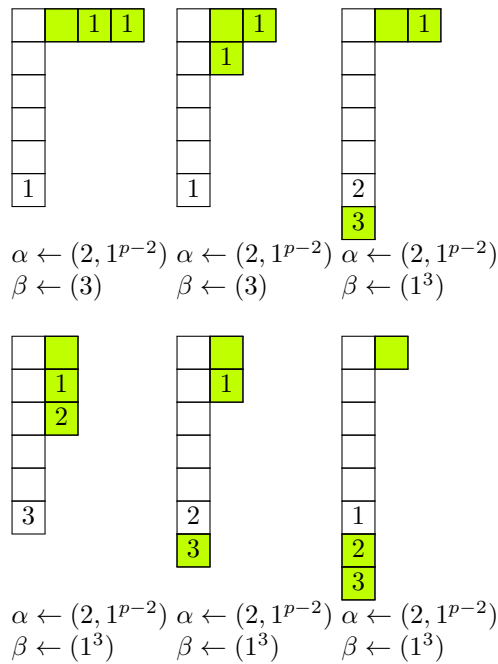FIGURE 10. Cases for $\lambda \supset (2, 1^{p-2})$

FIGURE 11. Cases for $\lambda \supset (1^p)$

Consider $\lambda \vdash n$, $\lambda \neq (1^n)$. To show $f_\mu \geqslant s_\lambda$, it suffices to show that there exist $\alpha \vdash n - \mu_i$, $\beta \vdash \mu_i$ such that $\alpha \neq (1^{n-\mu_i})$ and $s_\alpha s_\beta \geqslant s_\lambda$. Using Lemma 2.3, choose $\beta \vdash \mu_i$ such that $\beta \subset \lambda$ and $f_{(\mu_i)} \geqslant s_\beta$. Choose α using Lemma 2.1 with the roles of

α and β reversed. If $f_{\tilde{\mu}} \geqslant s_\alpha$, we are done. Otherwise, since $\tilde{\mu}$ is persistent, we must have $\alpha = (1^{n-\mu_i})$.

Since $\lambda \neq (1^n)$, we may replace α by $(2, 1^{n-\mu_i-2})$. Since $\tilde{\mu}$ is persistent, $f_{\tilde{\mu}} \geqslant s_\alpha$. Choose β using Lemma 2.1. If $f_{(\mu_i)} \geqslant s_\beta$ we are done. Otherwise, by Theorem 1.1, if $\mu_i \geqslant 3$, β must be one of $(\mu_i - 1, 1)$, $(2, 1^{\mu_i-2})$ with μ_i odd, or (1^{μ_i}) with μ_i even. We may proceed as in Case 3 of Lemma 2.5 and the third part of Lemma 2.6 for $\beta = (\mu_i - 1, 1)$ or $\beta = (2, 1^{\mu_i-2})$. If $\mu_i = 2$, then $\beta = (1, 1)$ which we will consider below. If $\mu_i = 1$, $f_{(1)} \geqslant s_\beta$ trivially.

It remains to consider $\beta = (1^{\mu_i})$ with $\mu_i \geqslant 2$ and even. In this case, λ/α has just one column, so λ must be $(2, 1^{n-2})$. Since $f_{\tilde{\mu}} \not\geqslant s_{(1^{n-\mu_i})}$, the permutation $w_{\tilde{\mu}}$ has to be an odd permutation, i.e., an odd number of the parts μ_j, $j \neq i$ is even. In particular, μ_j is even for some $j \neq i$. Let $\hat{\mu}$ denote the composition obtained from $\tilde{\mu}$ by deleting μ_j. We have:
$$f_\mu \geqslant f_{\hat{\mu}} f_{(\mu_j, \mu_i)}.$$
But both $w_{\hat{\mu}}$ and $w_{(\mu_j, \mu_i)}$ are even permutations, so
$$f_{\hat{\mu}} \geqslant s_{(1^{n-\mu_i-\mu_j})} \text{ and } f_{(\mu_j, \mu_i)} \geqslant s_{(1^{\mu_j+\mu_i})}.$$
So $f_\mu \geqslant s_{(1^{n-\mu_i-\mu_j})} s_{(1^{\mu_j+\mu_i})} \geqslant s_{(2, 1^{n-2})} = s_\lambda$. $\qquad\square$

Proof of the Main Theorem. For $\lambda = (1^n)$ we know that $f_\mu \geqslant s_\lambda$ if and only if w_μ is an even permutation. If $\lambda \neq (1^n)$, consider the following cases.

Case 1: μ has only one part. Theorem 1.1 tells us that w_μ has an invariant vector in V_λ if and only if λ is not of the form $(n-1, 1)$, (1^n), or $(2, 1^{n-2})$ with n odd, which are cases 2 and 3 of the main theorem.

Case 2: μ has two parts. By Lemmas 2.5 and 2.6 $f_{(p,q)} \geqslant s_\lambda$ unless one of the following happens.
- $\lambda = (2, 2, 1^{n-4})$ and $(p, q) = (n-2, 2)$ with n (corresponding to case 4 of the main theorem).
- $(p, q) \in \{(4, 3), (5, 3), (3, 1)\}$. Checking by direct calculation will imply that $f_{(p,q)} \geqslant s_\lambda$ unless $(p, q) = (3, 1)$ and $\lambda = (2, 2)$ or $(p, q) = (5, 3)$ and $\lambda = (2^4)$ or $(4, 4)$ corresponding to cases 5, 7, and 8 of the main theorem.

Case 3: μ has at least three parts. Assume that $n \geqslant 11$.

If $\mu_2 \geqslant 4$ then (μ_1, μ_2) is persistent by Lemma 2.5. Using Lemma 2.8 repeatedly, we can show that μ is persistent.

If $\mu_1 \geqslant 4$ and $\mu_2 < 4$ and either 2 or 3 occurs at least twice in μ, then using Lemmas 2.7 and 2.8 repeatedly, μ is persistent.

Otherwise $\mu_i \in \{1, 2, 3\}$ for $i \geqslant 2$, with 2 and 3 not repeated. If 1 occurs in μ, since $\mu_1 \geqslant 4$, $(\mu_1, 1)$ is persistent by Lemma 2.6, so μ is persistent by repeated use of Lemma 2.8. If 1 does not occur in μ, then $\mu_1 \geqslant 6$ and $\mu = (\mu_1, 3, 2)$ (since μ has at least three parts). By Lemma 2.6 $(\mu_1, 3)$ is persistent. Hence μ is persistent by repeated use of Lemma 2.8.

Finally, suppose $\mu_1 \leqslant 3$. Once again, if either 2 or 3 occurs at least twice then μ is persistent by Lemmas 2.7 and 2.8. Otherwise, 1 occurs 6 times. Since $f_{(1^6)}$ is the Frobenius characteristic of the regular representation of S_6, (1^6) is persistent. Hence μ is persistent by Lemma 2.8.

If $n < 11$, a direct computation using SageMath [6] tells us that $\mu = (3, 2, 1)$ and $\lambda = (2^3)$ (corresponding to case 6) and $\mu = (5, 3, 2)$ and $\lambda = (2^5)$ (corresponding to case 9) are the only pairs which violate $f_\mu \geqslant s_\lambda$.

This completes the proof of the main theorem. $\qquad\square$

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