ALGEBRAIC COMBINATORICS

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Volume 7, issue 5 (2024), p. 1261-1282.
https://doi.org/10.5802/alco.373

*Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing*
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

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Tree expansions of some Lie idempotents

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ABSTRACT We prove that the Catalan Lie idempotent $D_n(a, b)$, introduced in [Menous *et al.*, Adv. Appl. Math. 51 (2013), 177] can be refined by introducing n independent parameters $a_0, \dots, a_{n-1}$ and that the coefficient of each monomial is itself a Lie idempotent in the descent algebra. These new idempotents are multiplicity-free sums of subsets of the Poincaré-Birkhoff-Witt basis of the Lie module. These results are obtained by embedding noncommutative symmetric functions into the dual noncommutative Connes-Kreimer algebra, which also allows us to interpret, and rederive in a simpler way, Chapoton's results on a two-parameter tree expanded series.

1. INTRODUCTION

Lie idempotents are idempotents of the symmetric group algebra which act on words as projectors onto the free Lie algebra. Thus, they are in particular elements of the Lie module $\mathrm{Lie}(n)$, spanned by complete bracketing of standard words, such as $[[1, 3], [2, 4]]$, which can be represented as complete binary trees with leaves labelled $1, 2, \dots, n$.

Of course, these elements are not linearly independent, but the trees such that for each internal node, the smallest label is in the left subtree, and the greatest label is in the right subtree do form a basis, called the Poincaré-Birkhoff-Witt basis [19]. Such labellings are said to be admissible. These basis elements are denoted by $t(\sigma)$, where t is a complete binary tree, and σ the permutation obtained by reading its leaves from left to right.

The direct sum $\mathrm{Lie} = \bigoplus_{n\geqslant 0} \mathrm{Lie}(n)$ can be interpreted as the operad $\mathcal{L}ie$. It is also a Lie algebra for the Malvenuto–Reutenauer convolution product of permutations, which allows us to regard it as contained into **FQSym**, a permutation σ being interpreted as the basis element $\mathbf{G}_\sigma$. Then, it is (strictly) contained in the primitive Lie algebra of **FQSym**.

It turns out that the elements c_t, defined for complete binary trees t by the sum over admissible labellings

$$\mathsf{c}_t = \sum_{\sigma \text{ admissible}} t(\sigma) \tag{1}$$

Manuscript received 2nd October 2023, revised 16th April 2024, accepted 18th April 2024.

KEYWORDS. Noncommutative symmetric functions, Lie idempotents, Free Lie algebra, Dendriform algebras, PreLie algebras.

ACKNOWLEDGEMENTS. This research has been partially supported by the project CARPLO of the Agence Nationale de la recherche (ANR-20-CE40-0007).

span a Lie subalgebra $\mathfrak{C}$ of Lie, which might be called the Catalan Lie algebra.

It has proved convenient to label its basis elements by plane trees instead of binary trees: we set $C_T = \mathsf{c}_t$ where the plane tree T is the right-branch rotation of the incomplete binary tree t' obtained by removing the leaves of t (so that the maximal element of the Tamari order is the corolla).

This provides us with elements C_T of **FQSym**, labelled by plane trees. The noncommutative Connes-Kreimer algebra $\mathcal{H}_{NCK}$ [8, 9] is the free associative algebra generated by variables Y_T indexed by plane trees, endowed with the coproduct of admissible cuts. Its basis Y_F is therefore indexed by plane forests F. Its dual $\mathcal{H}_{NCK}^*$ admits a natural embedding into **FQSym**, and if X_F denotes the dual basis of Y_F, it turns out that

$$C_T = \sum_{T' \leqslant T} X_{T'} \tag{2}$$

where the sum is over the Tamari order. Moreover, if one denotes by $\tau = \bar{T}$ the underlying non-plane rooted tree of T the sums

$$x_\tau = |\mathrm{Aut}(\tau)| \sum_{\bar{T}=\tau} X_T \tag{3}$$

span a sub-preLie algebra, which is free on the generator $x_\bullet$, and x_τ coincides with the element indexed by τ in the Chapoton–Livernet basis.

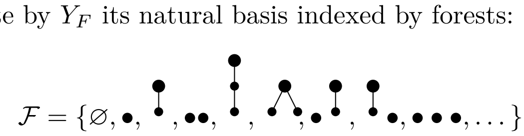
The aim of this paper is to investigate the expansions in the X and C bases of various noncommutative symmetric functions, regarded as elements of **FQSym**. Our first result concerns the family of Catalan idempotents $D_n(a, b)$. Originally introduced as noncommutative symmetric functions on the ribbon basis in [15], these elements were identified in [11] as simple weighted sums of the basis C_T. However, the calculations of [11] are rather tricky, and it is by no means obvious that such sums belong to the descent algebra. We present here a new approach, relying on the Birkhoff factorisation of a simple character of $QSym$ with values in an algebra of Laurent series. This approach produces immediately the expansion on X_F of a grouplike series $\sigma^+_{a(z)}$ which by definition belongs to the descent algebra. It is then relatively straightforward to check that the original Catalan idempotents are obtained by choosing $a(z) = \frac{a}{z} + \frac{b}{1-z}$ and taking the residue, the general case giving rise to new refined idempotents indexed by partitions of $n-1$.

Finally, we show how the embedding of $\mathcal{H}_{NCK}^*$ into **FQSym** can be used to determine the X-expansion of various noncommutative symmetric functions, including the Eulerian idempotents and the two-parameter series of Chapoton [4]

This paper is a continuation of [11], to which the reader is referred for background and notation.

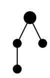
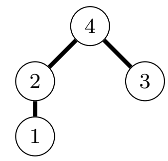
2. The noncommutative Connes-Kreimer Hopf algebra $\mathcal{H}_{NCK}$

The noncommutative Connes-Kreimer Hopf algebra $\mathcal{H}_{NCK}$, introduced by Foissy [8, 9], is as a graded vector space spanned by plane forests, the degree being the number of nodes. We denote by Y_F its natural basis indexed by forests:

(4) 

It is then freely generated by variables Y_T indexed by plane trees. The product is concatenation, and the coproduct is given by admissible cuts, which can be conveniently defined directly for the iterated coproducts in terms of labellings.

Trees will be drawn with the root at the top in this paper. The canonical labelling of a tree is obtained by visiting it in postorder, so that the labels of each subtree form an interval, with the maximum at its root.

For instance, for the tree  , we get the labelling 

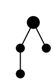
A forest F is similarly labelled, by first grafting it on a common root – that is considering the tree $T = B_+(F)$ – labelling T and removing this labelled root afterwards.

Such a labelled forest is regarded as the Hasse diagram of a poset.

With this labelling the r-iterated coproduct of an element Y_F of degree n can be described as follows:

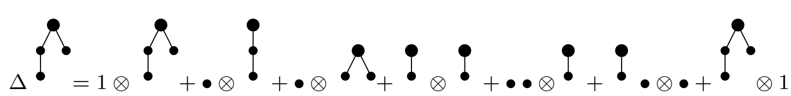
$$\Delta^r Y_F = \sum_{u \in C(F) \cap [r]^n} Y_{F_{(1)}} \otimes Y_{F_{(2)}} \cdots \otimes Y_{F_{(r)}} \tag{5}$$

where $C(F)$ is the set of words such that $i <_F j \Rightarrow u_i \leqslant u_j$, and $F_{(i)}$ is the restriction of F to vertices labelled i.

For instance, for the previous tree $T =$  and $r = 2$,

$$C(T) \cap \{1,2\}^4 = \{2222, 2212, 1222, 1122, 1212, 1112, 1111\} \tag{6}$$

give the coproduct:

(7) 

As it will be useful in the following sections, let us also recall here the polish code of a plane forest is obtained by labelling each node by the number of its descendants, and traversing it in prefix order. For the previous tree T we get the polish code 2100 and also its reverse polish code 0012.

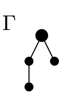
It has been shown in [12, 3.5] that $\mathcal{H}_{NCK}$ admits an embedding π into **WQSym**. It is actually an embedding into **FQSym**, given by $F \mapsto \Gamma_F(A)$, where $\Gamma_P(A)$ denotes the free generating function of a poset [6], that is, the sum of its linear extensions

$$\Gamma_P(A) = \sum_{\sigma \in L(P)} \mathbf{F}_\sigma \in \mathbf{FQSym} = \sum_{u \in C(F)} \mathbf{M}_u, \tag{8}$$

where $C(F)$ is the set of packed words u such that $i <_F j \Rightarrow u_i \leqslant u_j$. Indeed, the linear extensions of a poset are precisely those permutations σ such that $i <_P j \Rightarrow \sigma^{-1}(i) < \sigma^{-1}(j)$.

The linear extensions of such a labelled forest form an initial interval of the right weak order [1].

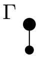
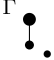
For example,

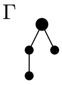
(9) Γ  $= \mathbf{F}_{3124} + \mathbf{F}_{1324} + \mathbf{F}_{1234} = \mathbf{S}^{2314} = \check{\mathbf{S}}^{3124}$

where [7, (6.4), (6.12)]

$$\mathbf{S}^\sigma = \sum_{\tau \leqslant_L \sigma} \mathbf{G}_\tau =: \check{\mathbf{S}}^{\sigma^{-1}}. \tag{10}$$

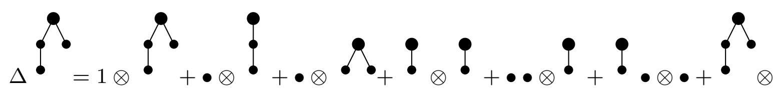
Then, Y_F can be identified with $\Gamma_F = \check{\mathbf{S}}^{\sigma_F}$, where σ_F is the maximal linear extension of F. For example

(11) Γ  $\Gamma_\bullet = \mathbf{S}^{12}\mathbf{S}^{1} = \mathbf{S}^{231} = \check{\mathbf{S}}^{312} = \Gamma$  .

As for the coproduct, Γ  $= \check{\mathbf{S}}^{3124}$, and

$$\Delta\check{\mathbf{S}}^{3124} = 1\otimes\check{\mathbf{S}}^{3124}+\check{\mathbf{S}}^{1}\otimes\check{\mathbf{S}}^{123}+\check{\mathbf{S}}^{1}\otimes\check{\mathbf{S}}^{213}+\check{\mathbf{S}}^{12}\otimes\check{\mathbf{S}}^{12}+\check{\mathbf{S}}^{21}\otimes\check{\mathbf{S}}^{12}+\check{\mathbf{S}}^{312}\otimes\check{\mathbf{S}}^{1}+\check{\mathbf{S}}^{3124}\otimes 1, \tag{12}$$

which corresponds term by term to

(13) 

Indeed, the coproduct formula [7, (6.13)]

$$\Delta\check{\mathbf{S}}^{\sigma} = \sum_{u\cdot v\leqslant\sigma} \langle\sigma | u \sqcup\!\sqcup\, v\rangle \check{\mathbf{S}}^{\mathrm{std}(u)}\otimes\check{\mathbf{S}}^{\mathrm{std}(v)}\,, \tag{14}$$

(sum over pairs of complementary subwords whose concatenation is smaller than σ in the right weak order) implies that if a value σ_i goes into v, all greater values on its right must also go into v, so as not to create new inversions. Thus, the word u and v encode admissible cuts.

3. Dual noncommutative Connes-Kreimer algebra $\mathcal{H}^*_{NCK}$

3.1. An embedding of Sym, and its dual.

Let X_F be the dual basis of Y_F. According to our description of the coproduct of Y_F, the coefficient of X_F in the product $X_{F'}X_{F''}$ is equal to the number of labellings of F by words over $\{1,2\}$, nondecreasing from bottom to top, and such that $F_{(1)} = F'$ and $F_{(2)} = F''$.

The coproduct of X_F is deconcatenation, so that trees X_T are primitive. The elements

$$\Lambda_n := X_{\bullet\bullet\cdots\bullet} \quad (n \text{ vertices}) \text{ and } \quad S_n := \sum_{|F|=n} X_F \tag{15}$$

form sequences of divided powers, and both define the same embedding of **Sym** into $\mathcal{H}^*_{NCK}$. One easily checks that, indeed,

$$\left(\sum_{n\geqslant 0}(-1)^n X_{\bullet\bullet\cdots\bullet}\right)^{-1} = \sum_{n\geqslant 0}\sum_{|F|=n} X_F. \tag{16}$$

Representing trees by their Polish codes, we have:

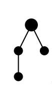
$$\begin{aligned}
R_{11} &= X_{00}\\
R_2 &= X_{00} + X_{10}\\
R_3 &= X_{000} + X_{100} + X_{010} + X_{200} + X_{110}\\
R_{21} &= 2X_{000} + X_{100} + X_{010}\\
R_{12} &= 2X_{000} + X_{100} + X_{010} + X_{200}\\
R_{111} &= X_{000}\\
R_4 &= X_{0000} + X_{0010} + X_{0100} + X_{1000} + X_{1010} + X_{0200} + X_{2000}\\
&\quad + X_{1100} + X_{0110} + X_{1110} + X_{1200} + X_{2100} + X_{2010} + X_{3000}\\
R_{31} &= (X_{1100} + X_{0110}) + 2(X_{1000} + X_{0100} + X_{0010})\\
&\quad + (X_{2000} + X_{0200}) + X_{1010} + 3X_{0000}
\end{aligned}$$

$$
\begin{aligned}
R_{22} &= (X_{1100}+X_{0110})+3(X_{1000}+X_{0100}+X_{0010})\\
&\quad +2(X_{2000}+X_{0200})+2X_{1010}+2X_{3000}+(X_{2100}+X_{2010})+5X_{0000}\\
R_{13} &= (X_{1100}+X_{0110})+2(X_{1000}+X_{0100}+X_{0010})\\
&\quad +2(X_{2000}+X_{0200})+X_{1010}+2X_{3000}+(X_{2100}+X_{2010})+X_{1200}+3X_{0000}\\
R_{211} &= (X_{1000}+X_{0100}+X_{0010})+3X_{0000}\\
R_{121} &= 2(X_{1000}+X_{0100}+X_{0010})+(X_{2000}+X_{0200})+X_{1010}+5X_{0000}\\
R_{112} &= (X_{1000}+X_{0100}+X_{0010})+(X_{2000}+X_{0200})+X_{3000}+3X_{0000}\\
R_{1111} &= X_{0000}
\end{aligned}
$$

Proposition 3.1. *The coefficient of X_F in R_I is equal to the number of linear extensions of F which are of ribbon shape I.*

Proof. The product rule of the X-basis implies immediately that the coefficient $\langle Y_F, S^I\rangle$ of X_F in S^I is the number of nondecreasing labellings of F by words of evaluation I, which is also the coefficient of M_I in $\Gamma_F(X)$. The dual map of the embedding of **Sym** into $\mathcal{H}^*_{NCK}$ is an epimorphism $\pi:\ \mathcal{H}_{NCK}\to QSym$, and $\langle Y_F, S^I\rangle$ is also the coefficient of M_I in $\pi(Y_F)$.

Thus, π is the restriction of the canonical projection (commutative image) of **FQSym** onto $QSym$, so that the coefficient of X_F in R_I is equal to the coefficient of F_I in $\pi(\Gamma_F)$, which is by definition the number of linear extensions of F of ribbon shape I. □

For instance, if $T =$  $= 2100$, we have $\Gamma_T = \mathbf{F}_{3124}+\mathbf{F}_{1324}+\mathbf{F}_{1234}$ and (3124), (1324), (1234) have respective ribbon shape 13, 22 and 4, so that X_{2100} appears with a coefficient 1 in R_{13}, R_{22} and R_4.

The product rule of the X-basis also implies that the coefficient of X_F in Λ^I is the number of strictly decreasing labellings of F by words of evaluation I.

We have therefore proved:

Theorem 3.2. *The coefficient of X_F in $\sigma_1(XA)$ is*

$$\sum_I \langle Y_F, S^I\rangle M_I(X) = \Gamma_F(X) \tag{17}$$

and that of X_F in $\lambda_1(XA)$ is

$$\sum_I \langle Y_F, \Lambda^I\rangle M_I(X) = (-1)^{|F|}\Gamma_F(-X) =: \chi_F(X). \tag{18}$$

□

Recall from [13] that the involution $X\mapsto -X$ of $QSym$ is the adjoint of $A\mapsto -A$ in **Sym**, so that

$$M_I(-X) = (-1)^{\ell(I)}\sum_{J\leqslant I} M_J(X) \quad \text{et} \quad F_I(-X) = (-1)^{|I|}F_{\bar{I}^\sim}(X). \tag{19}$$

3.2. Dendriform structure. One can also describe the product $X_{F_1}X_{F_2}$ in the following way. Endow F_1 with its canonical labelling, and F_2 with its canonical labelling shifted by the number n_1 of vertices of F_1. Then, the coefficient of X_F in the product is equal to the number of standard decreasing (from the roots towards the leaves) labellings of F whose restriction to $[1,n_1]$ is F_1, and whose restriction to $[n_1+1, n_1+n_2]$ is F_2.

This allows to define dendriform half-products: $X_{F_1} \prec X_{F_2}$ consists of the forests whose first label in the postfix order is n_1+1, and $X_{F_1} \succ X_{F_2}$ of those whose first label is 1. In particular,

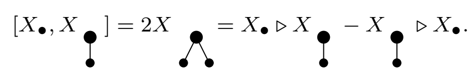
$$X_T \prec X_F = X_{FT}, \tag{20}$$

Actually, $\mathcal{H}^*_{NCK}$ can be identified with the Loday–Ronco Hopf algebra **PBT** [8, 9]. It is easily seen that its dendriform product are induced by those of **FQSym**, so that the coefficient of X_F in $\mathbf{P}_t$ is equal to the number of linear extensions of F whose decreasing tree has shape t.

3.3. PreLie and brace structures. The product rule shows that for two trees, $X_{T_1T_2}$ et $X_{T_2T_1}$ have the same coefficient in $X_{T_1}X_{T_2}$. Thus, $[X_{T_1}, X_{T_2}]$ is a linear combination of trees, and the primitive Lie algebra admits the X_T as basis.

The commutator $[X_{T_1}, X_{T_2}]$ is the difference between the sum of the X_T obtained by grafting T_1 on a node of T_2 and that of those obtained by grafting T_2 on a node of T_1. If one denotes by $X_{T_1} \triangleright X_{T_2}$ the first sum, $\triangleright$ defines then a right preLie product. There is also a left preLie product. $a \triangleleft b = b \triangleright a$.

For example,

(21) 

The preLie algebra generated by $X_\bullet$ is free. For a non-plane rooted tree τ, we set

$$x_\tau = |\mathrm{Aut}(\tau)| \sum_{\bar{T}=\tau} X_T \tag{22}$$

(where $\bar{T}$ means that the rooted tree obtained by forgetting the planar structure of T is τ) gets identified with the Chapoton-Livernet basis of the free preLie algebra on one generator.

The brace product which extends the preLie product and induces the associative product is [10]

$$\langle X_{T_1\cdots T_r}, X_T\rangle_\triangleright = \sum_{T'} X_{T'} \tag{23}$$

where the sum runs over all trees T' obtained by grafting $T_1, \ldots, T_r$ on nodes of T, respecting their order. One has then, writing $B(X_F)$ for $X_{B_+(F)}$, $B(X_F X_{F'}) = \langle X_F, B(X_{F'})\rangle$ and

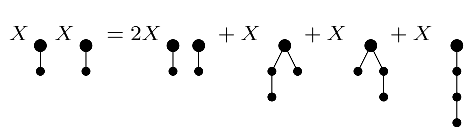
$$\langle X_F, \langle X_{F'}, X_T\rangle\rangle = \langle X_F X_{F'}, X_T\rangle. \tag{24}$$

The primitive Lie algebra of $\mathcal{H}^*_{NCK}$ is the free brace algebra on one generator.

In terms of the dendriform operations, the preLie product is $x \triangleright y = a \succ b - b \prec a$, and one has then as usual $\Lambda_n = X_\bullet \prec \Lambda_{n-1}$ and $S_n = S_{n-1} \succ X_\bullet$.

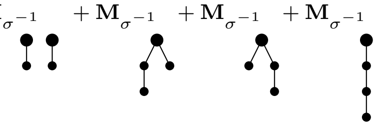
3.4. A quotient of **FQSym**. Let $\mathbf{M}_\sigma$ be the dual basis of $\mathbf{S}^\sigma$. The above embedding of $\mathcal{H}_{NCK}$ into **FQSym** allows to identify X_F with the image of $\mathbf{M}_{\sigma_F^{-1}}$ in the quotient of **FQSym** by the relations $\mathbf{M}_\sigma \equiv 0$ if σ contains the pattern 132.

For example,

(25) 

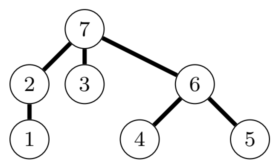
and

$$
\begin{aligned}
\mathbf{M}_{12}\mathbf{M}_{12} &= \mathbf{M}_{1234} + 2\mathbf{M}_{\mathbf{1324}} + \mathbf{M}_{\mathbf{1342}} + \mathbf{M}_{\mathbf{1423}} \\
&\quad + \mathbf{M}_{2314} + \mathbf{M}_{\mathbf{2413}} + \mathbf{M}_{3124} + \mathbf{M}_{\mathbf{3142}} + 2\mathbf{M}_{3412} \\
&\equiv 2\mathbf{M}_{3412} + \mathbf{M}_{3124} + \mathbf{M}_{2314} + \mathbf{M}_{1234} \\
&= 2\mathbf{M}_{\sigma^{-1}} + \mathbf{M}_{\sigma^{-1}} + \mathbf{M}_{\sigma^{-1}} + \mathbf{M}_{\sigma^{-1}}
\end{aligned}
\tag{26}
$$

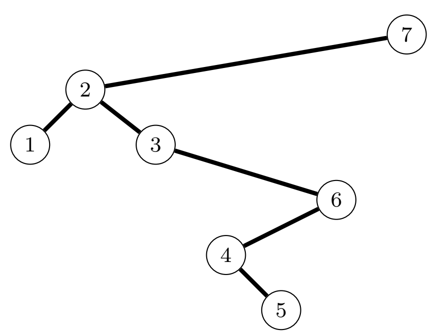


To reconstruct the forest F from its maximal linear extension σ_F, one must construct the binary search tree of its mirror image $\overline{\sigma_F}$ and take its right branch.rotation

For example, the tree 3100200



has $\sigma_T = 5463127$, and the binary search tree of $\overline{\sigma_T}$ is



4. A multivariate version of the Catalan family

4.1. A generic factorisation of σ_a. The derivation of the Catalan idempotents presented in [15, Sec. 10] can be interpreted as a Birkhoff factorisation of the character of *QSym* defined by

$$
\varphi(M_I) = \begin{cases} a^n & \text{if } I = (n) \\ 0 & \text{otherwise,} \end{cases} \tag{27}
$$

for a certain choice of a in a Rota-Baxter algebra of functions on the real line $\mathbb{R}$, the Rota-Baxter map being the the multiplication by the indicatrix of $\mathbb{R}^+$.

Let us now look at the generic factorisation of this character, for an arbitrary Rota-Baxter algebra $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$, with projectors P_+ and P_-. Under the embedding (15) of **Sym** into $\mathcal{H}^*_{NCK}$, we have

$$
\sigma_a = \sum_{n\geqslant 0} a^n S_n = \sum_{F\in\mathcal{F}} \varphi(Y_F) X_F. \tag{28}
$$

Writing the Birkhoff factorization $\varphi^+ = \varphi^- \star \varphi$ as

$$
\sigma_a^+ = \sigma_a^- \sigma_a, \quad \sigma_a^\pm = \sum_{F\in\mathcal{F}} \varphi^\pm(Y_F) X_F, \tag{29}
$$

we can easily calculate $\varphi^{\pm}$ by remarking that

$$\lambda_{-a} = \sum_{n\geqslant 0} (-a)^n X_{\underbrace{\bullet\,\bullet\,\cdots\,\bullet}_{n}} =: \sum_{F\in\mathcal{F}} \alpha(Y_F) X_F \tag{30}$$

where the character α is defined **on trees** by

$$\alpha(Y_T) = \begin{cases} -a & \text{if } T = \bullet \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

Since $\sigma_a^+ \lambda_{-a} = \sigma_a^-$, we have $\varphi^+ \star \alpha = \varphi^-$, which gives, for $T = B_+(F)$,

$$\varphi^+(Y_T) + \varphi^+(Y_F)\alpha(Y_\bullet) = \varphi^+(Y_T) - \varphi^+(Y_F)a = \varphi^-(Y_T) \tag{32}$$

an applying P_+ and P_-, we obtain the recursive formulas

$$\varphi^+(Y_T) = P_+(\varphi^+(Y_F)a), \tag{33}$$

$$\varphi^-(Y_T) = -P_-(\varphi^+(Y_F)a). \tag{34}$$

4.2. Example: polar part of a Laurent series. Let us now take $\mathcal{A} = \mathbb{C}[z^{-1}, z]]$ with $\mathcal{A}_+ = z^{-1}\,\mathbb{C}[z^{-1}]$ and $\mathcal{A}_- = \mathbb{C}[[z]]$, and let

$$a = a(z) = \sum_{n\geqslant 0} a_n z^{n-1}. \tag{35}$$

Here, $P_+(f)$ is defined as the polar part.

We have, writing $a_{i_1\cdots i_r}$ for $a_{i_1}\cdots a_{i_r}$,

$$\varphi^+(Y_\bullet) = P_+(a) = \frac{a_0}{z}$$

$$\varphi^+(Y_{\bullet\bullet}) = P_+(a)^2 = \frac{a_{00}}{z^2}$$

$$\varphi^+(Y_{\;}) = P_+\left(\frac{a_0}{z}\left(\frac{a_0}{z} + a_1 + \cdots\right)\right) = \frac{a_{00}}{z^2} + \frac{a_{01}}{z}$$

$$\varphi^+(Y_{\;}) = P_+\left(\frac{a_{00}}{z^2}\left(\frac{a_0}{z} + a_1 + a_2 z\cdots\right)\right) = \frac{a_{000}}{z^3} + \frac{a_{010}}{z^2} + \frac{a_{002}}{z}$$

$$\varphi^+(Y_{\;}) = \frac{a_{000}}{z^3} + \frac{a_{010}}{z^2} + \frac{a_{001}}{z^2} + \frac{a_{011}}{z} + \frac{a_{002}}{z}.$$

On these examples, we can observe the following explicit description:

Theorem 4.1. *For any tree T, the value of the character φ^+ on a tree Y_T is given by*

$$\varphi^+(Y_T) = \sum_{F\geqslant T} a_F z^{-r(F)} \tag{36}$$

where the sum is over the Tamari order on plane forests, $r(F)$ denotes the number of roots of F and $a_F = a_{c_1}\cdots a_{c_n}$ if $c_1\cdots c_n$ is the reverse Polish code of F.

Proof. This is an immediate consequence of the recursive description of Tamari intervals given below. ∎

4.3. A recursive description of the Tamari order.

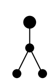
Theorem 4.2. *For a forest F, let $f(F)$ be the formal sum*

$$f(F) = \sum_{G \geqslant F} G. \tag{37}$$

Then for a tree $T = B_+(F)$,

$$f(T) = \sum_{F = F_1 F_2} f(F_1) B_+(f(F_2)). \tag{38}$$

In other words, the formal sum of the reverse Polish codes of the forests $G \geqslant B_+(F)$ is obtained by the following process: for each tree $T' = B_+(F') \geqslant T$, write down the reverse Polish code $a_{F'} = a_{c_1} \cdots a_{c_{n-1}}$, and take the sum of the words $a_{F'} a_i$ for $i = 0, \ldots, r$, where r is the number of connected components of F'. This amounts to encoding F' by $\frac{a_{F'}}{z^r}$ and taking the polar part of $\frac{a_{F'}}{z^r} a(z)$, which implies Theorem 4.1.

For example with $T =$ , the codes of the trees $T' \geqslant T$ are $0021, 0102, 0003$. The above process gives for each of them

$$\begin{aligned} &0021 \to 0020 + 0021, \\ &0102 \to 0100 + 0101 + 0102, \\ &0003 \to 0000 + 0001 + 0002 + 0003, \end{aligned}$$

which are indeed the codes of the 9 forests $G \geqslant T$.

Proof. Recall the cover relation of the Tamari order on plane trees: starting from a tree T and a vertex x that is neither its root or a leaf, the trees $T' > T$ covering T are obtained by cutting off the leftmost subtree of x and grafting it back on the left of the parent of x.

So all elements in the Tamari order above a given element are obtained by a sequence of such moves which can be encoded by a sequence of numbers recording on which node each cut is done.

We shall prove the result for a forest containing a single tree since the proof works in the same way with a general forest. We shall actually prove a stronger result: all trees above a given tree can be obtained by a sequence of cuts where no cut is done inside a subtree that was already cut.

To see that, number the internal nodes of a tree in prefix order, so that any node has a label smaller than its descendants. Now, the path from a tree to a tree above it corresponds to a word on these labels recording in which order the nodes were cut. Assume that there is somewhere a factor $1i$ where $i > 1$. Then we shall see that this factor can be rewritten either as $i1$ if i is not the leftmost child of 1 at this step or as $i11$ if it is.

First, if i is the leftmost child of the root, applying 1 then i leads to a forest containing three trees: the left subtree of i, the remaining part of the tree of root i without its left child and the remaining part of the whole tree without its left child. One easily checks that we get the same result by applying $i11$ to the tree.

Moreover, if i is not the left-most child of the root, then 1 and i commute since they work in separate parts of the tree.

So by induction, any word sending a tree T to a tree T' below it can be rewritten as a word where its 1s are at the end. The same applies to any element of the tree, whence the result. □

4.4. Grouplikes and primitives in **Sym**. By definition, the series

$$\mathsf{C} := \sigma_a^+|_{z=1} \quad \text{and} \quad \mathsf{D} = \mathrm{Res}_{z=0}\, \sigma_a^+ \tag{39}$$

are in **Sym**. We shall see that D is a multiparameter version of the Catalan idempotent of [15, 11], which is obtained by the choice $a(z) = \frac{a}{z} + \frac{b}{1-z}$.

Indeed, recall that the basis C_F of $\mathcal{H}_{NCK}^*$ is defined by

$$C_F = \sum_{G \leqslant F} X_G. \tag{40}$$

Thus,

$$\mathsf{C} = \sum_F \left(\sum_{G \geqslant F} a_G \right) X_F = \sum_G a_G \sum_{F \leqslant G} X_F = \sum_G a_G C_G, \tag{41}$$

where a_G is the (commutative) product of the code of G, and since taking the residue amounts to restricting the sum to trees,

$$\mathsf{D} = \sum_T a_T C_T \tag{42}$$

which gives back the expression obtained in [11] for $a(z) = \frac{a}{z} + \frac{b}{1-z}$.

The possible values of a_T correspond to partitions λ of $n-1$. The sums

$$\mathsf{D}_\lambda := \sum_{a_T = a_\lambda} C_T \tag{43}$$

are therefore Lie quasi-idempotents of the descent algebra.

For example,

$$\begin{aligned} \mathsf{D}_{(3)} &= C_{3000} = \bar{\Psi}_4, \\ \mathsf{D}_{(21)} &= C_{2100} + C_{2010} + C_{1200} = R_4 - R_{22} + R_{121} - R_{111} + \Psi_4 + \bar{\Psi}_4, \\ \mathsf{D}_{(111)} &= \Psi_4. \end{aligned}$$

4.5. Expansions in **Sym**. To compute the expansions of C and D on the usual bases of **Sym**, we start with the Birkhoff recurrence [14]

$$\varphi^-(x) = -P_-\big(\varphi(x) + \sum_{(x)} \varphi^-(x')\varphi(x'')\big) \tag{44}$$

$$\varphi^+(x) = P_+\big(\varphi(x) + \sum_{(x)} \varphi^-(x')\varphi(x'')\big). \tag{45}$$

This gives immediately

$$\varphi^-(M_n) = -P_-(a^n),\ \varphi^+(M_n) = P_+(a^n), \tag{46}$$

$$\varphi^-(M_{ij}) = P_-(P_-(a^i)a^j),\ \varphi^+(M_{ij}) = -P_+(P_-(a^i)a^j), \ldots \tag{47}$$

and by induction, we arrive at the proposition below.

Let a an element of a Rota-Baxter algebra $\mathcal{A}$. We set $P_\varnothing^\varnothing(a) = 1_\mathbb{K}$ and for $I = (i_1, \ldots, i_n)$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n) \in \{+,-\}^n$,

$$P_{\boldsymbol{\varepsilon}}^I(a) = P_{\varepsilon_n}\left(P_{\boldsymbol{\varepsilon}'}^{I'}(a) a^{i_n}\right) \tag{48}$$

where $I' = (i_1, \ldots, i_{n-1})$ and $\boldsymbol{\varepsilon}' = (\varepsilon_1, \ldots, \varepsilon_{n-1})$. We also write for short $P_{\boldsymbol{\varepsilon}}(a) = P_{\varepsilon_1,\ldots,\varepsilon_n}(a)$ the element of $\mathcal{A}^{\varepsilon_n}$ equal to $P_{\boldsymbol{\varepsilon}}^I(a)$ where $i_1 = i_2 = \cdots = i_n = 1$.

For instance,

$$P_{+,-,-}^{1,2,3} = P_-(P_-(P_+(a)a^2)a^3) \text{ and } P_{+,+,-}(a) = P_-(P_+(P_+(a)a)a).$$

Proposition 4.3. *Let a an element of a Rota-Baxter algebra $\mathcal{A}$. Then,*

$$\sigma_a^+ = \sum_I (-1)^{l(I)-1} P^I_{(-)^{l(I)-1},+}(a) S^I \tag{49}$$

$$= \sum_I (-1)^{|I|+l(I)} P^I_{(+)^{l(I)}}(a) \Lambda^I \tag{50}$$

$$= 1 + \sum_{\boldsymbol{\varepsilon}\in\mathcal{E}} P_{\boldsymbol{\varepsilon},+}(a) R_{\boldsymbol{\varepsilon},\bullet} \tag{51}$$

and

$$\sigma_a^- = \sum_I (-1)^{l(I)} P^I_{(-)^{l(I)}}(a) S^I \tag{52}$$

$$= \sum_I (-1)^{|I|+l(I)-1} P^I_{(+)^{l(I)-1},-}(a) \Lambda^I \tag{53}$$

$$= 1 - \sum_{\boldsymbol{\varepsilon}\in\mathcal{E}} P_{\boldsymbol{\varepsilon},-}(a) R_{\boldsymbol{\varepsilon},\bullet} \tag{54}$$

We use in the last equation the *signed ribbon basis* of **Sym** (see [15]), which is a slight modification of the noncommutative ribbon Schur functions: for any sequence of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-1}$

$$R_{\varepsilon\bullet} = (-1)^{l(I)-1} R_I \quad (R_\varnothing = 1,\ R_\bullet = R_1) \tag{55}$$

where $I = (i_1, \dots, i_r)$ is the composition of n such that

$$D(I) := \{i_1, i_1 + i_2, \dots, i_1 + \dots + i_{r-1}\} = \{1 \leqslant i \leqslant n-1\ ;\ \varepsilon_i = -\}. \tag{56}$$

Proof. The expansions on the S^I follow immediately from the recurrence (44)-(45). The other ones admit an interesting explanation in terms of the free Rota-Baxter algebra on one generator, which can be realized a subalgebra of the algebra of sequences of multivariate polynomials, with pointwise addition and product.

Let $\mathbf{x}$ be a sequence of variables $\mathbf{x} = (x_1, x_2, x_3, \dots)$ and for a sequence $\mathbf{z}$, define

$$R(\mathbf{z}) = (0, z_1, z_1 + z_2, z_1 + z_2 + z_3, \dots) \tag{57}$$

This is a Rota-Baxter operator of weight 1. It generates from $\mathbf{x}$ the free Rota-Baxter algebra $\mathfrak{A}(\mathbf{x})$ [17, 18]. Define

$$P_+ = -R, \quad P_- = I - P_+, \tag{58}$$

which are now of weight -1. Set also $V(\mathbf{z}) = (z_2, z_3, \dots)$.

The subalgebra generated by the $R(\mathbf{x}^n)$ is isomorphic to **Sym**:

$$R(\mathbf{x}^n) = (p_n(0), p_n(x_1), p_n(x_1, x_2), \dots) := \tilde{p}_n \tag{59}$$

but there is also an embedding of *QSym* in $\mathfrak{A}(\mathbf{x})_+ := P_+(\mathfrak{A}(\mathbf{x}))$ given by the same rule

$$f \mapsto \tilde{f} := (f(0), f(x_1), f(x_1, x_2), \dots) \tag{60}$$

Its image under V gives an embedding of $QSym_+$ into $\mathfrak{A}(\mathbf{x})_-$.

It is now easy to show by induction that for $I = (i_1, \dots, i_r)$,

$$\tilde{M}_I = R(\tilde{M}_{i_1 \cdots i_{r-1}} \mathbf{x}^{i_r}). \tag{61}$$

If we number the applications of R in the above expression of M_I,

for example

$$\tilde{M}_{i_1 i_2 i_3 i_4} = R_4(R_3(R_2(R_1(a^{i_1})\mathbf{x}^{i_2})\mathbf{x}^{i_3})\mathbf{x}^{i_4}) \tag{62}$$

and replace some R_j by $R' = I + R = P_-$, the result is now $\tilde{M}_I + \tilde{M}_{I'}$, where $I' = (i_1, \ldots, i_j + i_{j+1}, i_{j+2}, \ldots)$.

For example,

$$\tilde{M}_{i_1 i_2 i_3 i_4} \mapsto R_4(R_3((I+R_2)(R_1(\mathbf{x}^{i_1})\mathbf{x}^{i_2})\mathbf{x}^{i_3})\mathbf{x}^{i_4}) = \tilde{M}_{i_1 i_2 i_3 i_4} + \tilde{M}_{i_1, i_2+i_3, i_4}. \tag{63}$$

Applying this to M_{1^n}, the replacement of R_i by R'_i has the effect of removing i from the descent set of 1^n. Iterating, we see that replacing $R_{d_1}, \ldots, R_{d_k}$ by R' yields all compositions whose descent set contains the complement of $\{d_1, \ldots, d_k\}$. The result is therefore $\tilde{F}_{\bar{I}^\sim}$

Let us now look at the coefficient of S^I in $\sigma^+_{\mathbf{x}}$. For example, for $I = (i,j,k)$, this is

$$\begin{aligned}
(-1)^3 P^{ijk}_{--+}(\mathbf{x}) &= R(((I+R)((I+R)\mathbf{x}^i)\mathbf{x}^j)\mathbf{x}^k) \\
&= R(\mathbf{x}^{i+j+k} + R(R(\mathbf{x}^i)\mathbf{x}^{j+k} + R(\mathbf{x}^{i+j})\mathbf{x}^k + R(R(\mathbf{x}^i)\mathbf{x}^j)\mathbf{x}^k) \\
&= \tilde{M}_{i+j+k} + \tilde{M}_{i,j+k} + \tilde{M}_{i+j,k} + \tilde{M}_{ijk} \\
&= \sum_{J \leqslant I} \tilde{M}_J \\
&= (-1)^3 \widetilde{M_{ijk}(-X)}.
\end{aligned}$$

Thus, the Birkhoff factorisation of the character (27) of *QSym* is given by

$$\varphi^+(M_I) = \widetilde{M_I(-X)}, \quad \varphi^-(M_I) = V\widetilde{M_I(-X)}, \tag{64}$$

and

$$\varphi^+(F_I) = \tilde{F}_I(-X) = (-1)^{|I|}\tilde{F}_{\bar{I}^\sim}, \quad \varphi^-(F_I) = V\varphi_+(\tilde{F}_I). \tag{65}$$

Now, the coefficient of Λ^I in $\sigma^+_{\mathbf{x}}$ is $\tilde{M}_I(X)$, whence the second equalities in Prop. 4.3, and that of R_I is, up to sign, $\tilde{F}_{\bar{I}^\sim}(X)$, which can be expressed as $\pm P_{\boldsymbol{\varepsilon},+}(\mathbf{x})$. ☐

4.6. Combinatorial interpretation of the coefficients. Evaluating the above expression for $\varphi^+(M_I)$ on $\mathbf{x} = a(z)$, but without assuming that the a_i commute yields a set of words which can be characterized by certain inequalities involving partial sums of the subscripts. Recall that

$$\sigma^+_a = \sum_I \varphi^+(M_I) S^I, \tag{66}$$

where $\varphi^+(M_n) = P_+(a^n)$, and for $I = (I', i_p)$.

$$\begin{cases} \varphi^+(M_I) = P_+(\varphi^-(M_{I'})a^{i_p}) \\ \varphi^-(M_I) = -P_-(\varphi^-(M_{I'})a^{i_p}). \end{cases} \tag{67}$$

So if $I = (i_1, \ldots, i_p)$, the evaluation of $\varphi^+(M_I)$ amounts to computing $P_-(M_{i_1})$, then multiply by a^{i_2}, then apply P_-, then multiply the result by a^{i_3}, and so on, up to the last step where one applies a P_+ instead of P_-.

Thus, up to a global sign $(-1)^{\ell(I)-1}$, $\varphi^+(M_I)$ is a sum of monomials in z^{-1} and the a_i. Such a monomial is a product of n terms of the series a which survive the sequence of P_- and the final P_+. Writing this product as a word, considering that the a_i do not commute, and replacing a_i with i (ignoring the power of z that can be reconstituted in the end), we obtain a word $w = w_1 \ldots w_n$ over the integers such that

$$\begin{cases} w_1 + \cdots + w_{d_k} \geqslant d_k, \text{ for all } k < p, \\ w_1 + \cdots + w_{i_1+\cdots+i_p} < i_1 + \cdots + i_p, \end{cases} \tag{68}$$

where $\{d_1 = i_1, d_2 = i_1 + i_2, \ldots, d_{p-1} = i_1 + \cdots + i_{p-1}\}$ is the *descent set* $D(I)$ of I. Denote this set of words by $S(I)$.

Let us check this observation on $\varphi^+(M_{112})$, which is

$$\frac{a_3a_0^3}{z} + \frac{4a_2a_1a_0^2}{z} + \frac{2a_1^3a_0}{z} + \frac{a_2a_0^3}{z^2} + \frac{a_1^2a_0^2}{z^2}. \tag{69}$$

It is indeed obtained from the 9 words $w = w_1w_2w_3w_4$ satisfying

$$w_1 \geqslant 1, \quad w_1 + w_2 \geqslant 2, \quad w_1 + \cdots + w_4 < 4, \tag{70}$$

that are

$$3000,\ 2100,\ 2010,\ 2001,\ 1200,\ 1110,\ 1101,\ 2000,\ 1100, \tag{71}$$

by sending each value i to a_iz^{i-1}.

For a word over the integers, define

$$w_{1:k} := \sum_{i=1}^{k} w_i \tag{72}$$

and let

$$W(I) = \{w | w_{1:k} \geqslant k \text{ if } k \in D(I) \text{ and } w_{1:k} < k \text{ otherwise}\}, \tag{73}$$

so that

$$S(I) = \bigsqcup_{J \geqslant I} W(J). \tag{74}$$

Thus, if one writes as an intermediate expression for $\varphi^+(M_I)$ the sum

$$\begin{aligned} \sum_I (-1)^{\ell(I)-1} S^I \sum_{w \in S(I)} w &= \sum_I (-1)^{\ell(I)-1} S^I \sum_{J \geqslant I} \sum_{w \in W(J)} w \\ &= \sum_J \sum_{w \in W(J)} w \sum_{I \leqslant J} (-1)^{\ell(I)-1} S^I \end{aligned} \tag{75}$$

one can see that the coefficient of a word $w \in W(J)$ is, up to a sign $(-1)^{\ell(J)-1}$, the ribbon R_J.

So the expansion of σ_a^+ in the ribbon basis is obtained by listing the words $w = w_1 \ldots w_n$ satisfying $w_1 + \cdots + w_n < n$ (counted by the binomial $\binom{2n-1}{n}$). Each such w belongs to a unique $W(I)$, which determines its coefficient $(-1)^{\ell(I)-1} R_I$, and a factor $z^{w_{1:n}-n}$

For example, here are all possible words for $n = 3$ with the corresponding compositions:

(76)

000	001	010	100	002	020	200	011	101	110
3	3	3	12	3	21	111	3	12	111

For $n = 4$, here is the complete list of all words contributing to each R_I:

(77)

4	0000, 0100, 0010, 0001, 0110, 0101, 0020, 0011, 0002, 0111, 0102, 0021, 0012, 0003,
31	0120, 0030,
22	0200, 0201,
13	1000, 1010, 1001, 1011, 1002,
211	0300, 0210,
121	1020,
112	2000, 1100, 2001, 1101,
1111	3000, 2100, 2010, 1200, 1110.

We already know from previous works [15, 11] that if $a_0 = a, a_i = b$ for $i > 0$, the coefficient of a R_I is (up to a global sign) a product of Narayana polynomials. Since the coefficients in the general case are sums of monomials with the same sign, this implies that the cardinalities of the sets $W(I)$ are products of Catalan numbers. This can be seen directly as follows.

Recall the correspondence between Łukasiewicz words (Polish codes of plane trees) and Dyck paths. The code of a plane tree is obtained by labelling each node by the number of its descendants, and traversing it in prefix order.

An example would be

$$w = 40201200010 \tag{78}$$

These codes are characterized by the following property: if one forms a word u by subtracting 1 to each entry of w, the partial sums $u_{1:i}$ are all nonnegative, except for the last one which is -1.

On our example,

$$\begin{matrix} 4 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & -1 & 1 & -1 & 0 & 1 & -1 & -1 & -1 & 0 & -1 \\ 3 & 2 & 3 & 2 & 2 & 3 & 2 & 1 & 0 & 0 & -1 \end{matrix} \tag{79}$$

This characterization means that if one replaces each integer i by the word $a^i b$, one obtains a word wb, where w is a Dyck word[(1)].

On our example, this yields

$$aaaab.b.aab.b.ab.aab.b.b.b.ab \cdot b \tag{80}$$

The partial sums $u_{1:i}$ give the height of the corresponding Dyck path after the ith b.

This description can be extended to the sets $W(I)$. The word obtained by replacing each entry k by $a^k b$ in w encodes a lattice path starting at the origin, and ending at $(2n+1, -1)$. Applying the transformation $u_i = w_i - 1$ to $W(I)$ results into the set of words

$$U(I) = \{u | u_{1:k} \geqslant 0 \text{ if } k \in D(I) \text{ and } u_{1:k} < 0 \text{ otherwise}\}. \tag{81}$$

Again, the partial sums $u_{1:i}$ of such words record the heights attained by the lattice path associated with w after the ith b.

Represent a composition $I = (i_1, \dots, i_p)$ of n as a sequence of n symbols $+$ and $-$ with a $-$ in position k if k is a descent of I, and a $+$ otherwise.

For example, 312 is represented as $++--++$ and 3111 as $++---+$.

Then, the cardinality of $W(I)$ is $\prod_i C_i$ where i runs over the lengths of blocks of identical signs.

For example, $W(312)$ contains $C_2^3 = 8$ words and $W(3111)$ has $C_2 C_3 = 10$ elements.

Indeed, the blocks of symbols $+$ correspond to sections of the path associated with w lying under the horizontal axis, and the blocks of $-$ to sections where it remains above the axis. The sections of the path determined by these blocks are alternatively Dyck paths or negative of Dyck paths, whence the product of Catalan numbers. Counting them by number of peaks gives back the products of Narayana polynomials already mentioned.

(1) Here, the letter a stands for an upstep and b for a downstep.

For example, let us decompose $W(4111)$. The corresponding signed word is $+++----+$. There should be 25 such words. Let us write these as a 5×5 square where words on the same column have same first three values.

$$\begin{array}{ccccc} 0006000 & 0015000 & 0105000 & 0024000 & 0114000 \\ 0005100 & 0014100 & 0104100 & 0023100 & 0113100 \\ 0005010 & 0014010 & 0104010 & 0023010 & 0113010 \\ 0004200 & 0013200 & 0103200 & 0022200 & 0112200 \\ 0004110 & 0013110 & 0103100 & 0022110 & 0112110 \end{array} \tag{82}$$

The path corresponding to 0004200 is *bbbaaa.abaabb.b*, and that corresponding to 0112200 is *bababa.abaabb.b*. One can check that all pairs of Dyck paths are obtained. Note that in each row, the values (w_4, w_5, w_6) are the same if one replaces the fourth one by $w_4 + (w_1 + w_2 + w_3) - 3$. The sequence of these values becomes

$$300, 210, 201, 120, 111, \tag{83}$$

which is indeed the set of the first three values associated with the composition 1111, and the Polish codes of plane trees with 4 nodes except for their final 0.

5. Lie idempotents of the descent algebra

We shall now describe the expansions of several Lie idempotents of the descent algebra on the X-basis. To this aim, we shall need several versions of the $(1-q)$-transform.

Recall that in the context of ordinary symmetric functions, the alphabet $\frac{X}{1-q}$ is the set $\{q^i x_j \mid i \geqslant 0, x_j \in X\}$. It can be extended to noncommutative symmetric functions by choosing a total order of the products $q^i a_j$, which can of course be done in an infinity of ways, but only four of them are natural: take the lexicographic order on the pairs (q^i, a_j) or (a_j, q^i), keeping the original order on A and ordering the q^i in ascending or descending order of the exponents. This leads to four possible definitions of the $(1-q)$-transform as the respective inverses of the above transforms. In the sequel we shall define them directly by specifying the image of the S_n.

5.1. Dynkin.

Proposition 5.1. *The right Dynkin $\bar\Psi_n = [1, [2, [3, \dots [n-1, n] \dots]]]$ is the sum of all trees*

$$\bar\Psi_n = \sum_{|T|=n} X_T. \tag{84}$$

and the left Dynkin $\Psi_n = [\dots[[1,2],3],\dots,n]$ is the linear tree X_{L_n}

$$\Psi_n = ((X_\bullet \triangleright X_\bullet)\cdots) \triangleright X_\bullet. \tag{85}$$

Proof. We first apply Theorem 3.2 to $X = 1-q$, defined by

$$S_n((1-q)A) = (1-q)\sum_{k=0}^{n} (-q)^k R_{1^k, n-k}(A), \tag{86}$$

so that $\Psi_n(A) = \frac{1}{1-q} S_n((1-q)A)\mid_{q=1}$, and $F_I(1-q)$ is nonzero only for I of the type $(1^k, n-k)$.

Every forest with $k+1$ leaves has a unique maximal linear extension of this shape, obtained by reading its leaves from right to left and then taking the postorder reading of the remaining nodes. It has therefore $\binom{k}{i}$ linear extensions of shape $(1^i, n-i)$ for $0 \leqslant i \leqslant k$, so that $\Gamma_F(1-q) = (1-q)^k$ is divisible by $(1-q)^2$ except for $k=1$, which means that $F = L_n$ is a linear tree.

To deal with $\bar\Psi_n$, we need another version of the $1-q$ transform, denoted by $1+(-q)$, and defined[(2)] by $F_I(1+(-q)) = (1-q)(-q)^k$ if $I = (n-k, 1^k)$ and 0 otherwise,

[(2)] This strange notation is justified by the fact that addition of alphabets is not commutative, and that $X - Y$ is defined as $(-Y) + X$, *cf.* [13].

so that $\bar{\Psi}_n(A) = \frac{1}{1-q} S_n((1+(-q))A)\ |_{q=1}$. A permutation of shape $I = (n-k, 1^k)$ cannot be a linear extension of a tree, unless $k=0$, in which case it is the identity, the common linear extension of all trees. Thus, $\bar{\Psi}_n$ is the sum of all trees with n nodes. $\square$

5.2. Eulerian idempotents. Take the binomial alphabet α defined by $\sigma_1(\alpha A) = \sigma_1^\alpha$, so that $M_I(\alpha) = \binom{\alpha}{\ell(I)}$, and $F_I(\alpha) = \binom{\alpha+n-r}{n}$ where $n = |I|$ and $r = \ell(I)$. Then, the Solomon idempotent φ (often denoted by Ω, and also known as the first Eulerian idempotent) is given by

$$\varphi := \log \sigma_1 = \frac{d}{d\alpha}\Big|_{\alpha=0} \exp \alpha\varphi = \frac{d}{d\alpha}\Big|_{\alpha=0} \sigma_1(\alpha A), \tag{87}$$

so that the coefficient of X_T in φ is

$$\frac{d}{d\alpha}\Big|_{\alpha=0} \Gamma_T(\alpha). \tag{88}$$

Equivalently, with the notation of Theorem 3.2

$$\sum_F \chi_F(\alpha) X_F = \lambda_1(A)^\alpha = \exp\left\{\alpha \sum_{n\geqslant 1} (-1)^{n-1}\varphi_n\right\} \tag{89}$$

and for a forest of degree n,

$$\frac{d}{d\alpha}\Big|_{\alpha=0} \chi_F(\alpha) = (-1)^{n-1}(Y_F, \varphi_n) \tag{90}$$

so that

$$\varphi_n = (-1)^{n-1}\ \frac{d}{d\alpha}\Big|_{\alpha=0} \sum_{|T|=n} \chi_T(\alpha) X_T \tag{91}$$

which contains only trees, since φ is a Lie series.

The polynomial $\chi_T(t)$ is the evaluation of the tree T obtained by putting t in each leaf, the operator "discrete integral of the product of the subtrees"

$$\Sigma:\ t^p \mapsto \Sigma_0^t s^p \delta s = \frac{B_{p+1}(t) - B_{p+1}(0)}{p+1} \tag{92}$$

in each internal node, and multiplying the result by $(-1)^{n-1}$ (the B_k are the Bernoulli polynomials).

Indeed, if $T = B_+(T_1 \cdots T_k)$, $\chi_T(t)$ satisfies the difference equation

$$\Delta\chi_T(t) = \chi_{T_1}(t) \cdots \chi_{T_k}(t) \tag{93}$$

which can bee seen as follows. First, $\chi_T(t) = \langle Y_T, \lambda_1^t\rangle$, so that

$$\begin{aligned}
\Delta\chi_T(t) &= \langle Y_T, \lambda_1^t(\lambda_1 - 1)\rangle = \langle \Delta Y_T, \lambda_1^t \otimes (\lambda_1 - 1)\rangle \\
&= \sum_{(T)} \langle Y_{T(1)} \otimes Y_{T(2)}, \lambda_1^t \otimes (X_\bullet + X_{\bullet\bullet} + \cdots)\rangle \\
&= \langle Y_{T_1} \cdots Y_{T_k}, \lambda_1^t\rangle \quad \text{since the only nonzero term is obtained for } T(2) = \bullet \\
&= \chi_{T_1}(t) \cdots \chi_{T_k}(t).
\end{aligned}$$

This formula has been first obtained in [21] by a more complicated argument.

The coefficients of the polynomial $(-1)^{|T|}\chi_T(t)$ are given by the expansion of the other Eulerian idempotents is the forests basis. This is equivalent to the description of the "formal flow" given in [21]. The coefficient of α^k in σ_1^α is

$$\frac{1}{k!}\sum_{\ell(I)=k}\varphi^I \tag{94}$$

hence the coefficient of X_F in

$$e_n^{(k)}=\frac{1}{k!}\sum_{I\vDash n}\varphi^I \tag{95}$$

is (*cf.* Eq. (18))

$$[\alpha^k]\Gamma_F(\alpha)=(-1)^{|F|}[\alpha^k]\chi_F(\alpha). \tag{96}$$

5.3. q-idempotents and a two-parameter series. In [13], it is proved that, for the usual definition of $\frac{A}{1-q}$

$$\varphi_n(q)=\frac{1-q^n}{n}\Psi_n\left(\frac{A}{1-q}\right)=\frac{1}{n}\sum_{|I|=n}\frac{(-1)^{\ell(I)-1}}{\begin{bmatrix}n-1\\ \ell(I)-1\end{bmatrix}}\,q^{\mathrm{maj}(I)-\binom{\ell(I)}{2}}\,R_I(A) \tag{97}$$

is a Lie idempotent, interpolating between the Solomon idempotent φ_n (for $q=1$), the two Dynkin (for $q=0,\infty$) and Klyachko ($q=e^{2i\pi/n}$). Its expansion on the preLie basis x_τ (hence also on X_T) is obtained by Chapoton in [3].

One way to recover this result is to apply Theorem 3.2 to the virtual alphabet

$$\frac{1-qt|}{|1-q}=(1-qt)\times\frac{1}{1-q} \tag{98}$$

defined by [13]

$$S_n\left(\frac{1-qt|}{|1-q}A\right)=(1-qt)\sum_{k=0}^{n}(-qt)^kR_{1^k,n-k}\left(\frac{A}{1-q}\right) \tag{99}$$

so that

$$\Psi_n\left(\frac{A}{1-q}\right)=\frac{1}{1-qt}\;S_n\left(\frac{1-qt|}{|1-q}A\right)\Big|_{t=\frac{1}{q}}. \tag{100}$$

The series denoted by ♗ in [4] is essentially $\sigma_1\left(\frac{1-qt|}{|1-q}A\right)$. Actually, Chapoton takes the opposite order on the alphabet of powers of q, and to recover the same coefficients, we have to define ♗ as

$$♗=\sigma_1(X_{q,t}A):=\overrightarrow{\prod_{i\geqslant 0}}\sigma_{q^i}(A)\overleftarrow{\prod_{j\geqslant 0}}\lambda_{-q^jt}(A). \tag{101}$$

The functional equation satisfied by $f(t):=\sigma_1(X_{q,t}A)$ is then

$$f(qt)=f(t)\sigma_{qt}(A) \tag{102}$$

which is equivalent to [4, (8)] after setting $t=1+(q-1)x$.

The coefficient of $\frac{\tau}{|\mathrm{Aut}(\tau)|}$ in ♗ is thus obtained by setting $t=1+(q-1)x$ in $\Gamma_T(X_{q,t})$.

For example, with $T=10$, $\Gamma_T(A)=\mathbf{M}_{12}+\mathbf{M}_{11}$, hence $\Gamma_T(X)=M_2+M_{11}=h_2$ is a symmetric function, and

$$h_2\left(\frac{1-qt}{1-q}\right)=\frac{(1-qt)(1-q^2t)}{(1-q)(1-q^2)}=\frac{(1+qx)(1+q+q^2x)}{1+q}. \tag{103}$$

Dividing by $1+qx$, and setting $x=-1/q$, one finds $\frac{1}{1+q}$, which is indeed the coefficient of X_{10} in the series $\bar{\Omega}_q$ defined in [4, (45)].

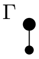
5.4. Examples. One can easily compute $\Gamma_T(A)$ by the recurrence (obvious from the definition in terms of linear extensions)

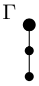
$$\Gamma_{B_+(T_1\cdots T_k)}(A) = B(\Gamma_{T_1}\cdots\Gamma_{T_k}), \tag{104}$$

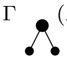
where $B(\mathbf{F}_\sigma) := \mathbf{F}_{\sigma n} = \mathbf{F}_\sigma \succ \mathbf{F}_1$ $(n = |T_1|+\cdots+|T_k|+1)$, which yields by projection onto *QSym*

$$\Gamma_{B_+(T_1\cdots T_k)}(X) = B(\Gamma_{T_1}\cdots\Gamma_{T_k}), \quad \text{where } B(F_{i_1 i_2\ldots i_r}) := F_{i_1,i_2\ldots,i_r+1} \tag{105}$$

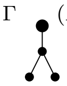
For example,

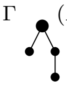
(106) Γ  $(X) = F_2 \to \binom{\alpha+1}{2}$

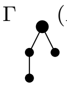
(107) Γ  $(X) = F_3 \to \binom{\alpha+2}{3}$

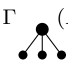
(108) Γ  $(X) = F_{12}+F_3 \to \binom{\alpha+2}{3}+\binom{\alpha+1}{3}$

(109) Γ  $(X) = F_4 \to \binom{\alpha+3}{4}$

(110) Γ  $(X) = F_{13}+F_4 \to \binom{\alpha+3}{4}+\binom{\alpha+2}{4}$

(111) Γ  $(X) = F_{22}+F_{13}+F_4 \to \binom{\alpha+3}{4}+2\binom{\alpha+2}{4}$

(112) Γ  $(X) = F_{22}+F_{13}+F_4 \to \binom{\alpha+3}{4}+2\binom{\alpha+2}{4}$

(113) Γ  $(X) = F_{112}+2F_{22}+2F_{13}+F_4 \to \binom{\alpha+3}{4}+4\binom{\alpha+2}{4}+\binom{\alpha+1}{4}$

which gives for the Eulerian idempotents

$$e_4^{(1)} = \frac{1}{4!}\left(6X_{1110}+4X_{1200}+2X_{2010}+2X_{2100}\right) = \varphi_4 \tag{114}$$

$$\begin{aligned} e_4^{(2)} = \frac{1}{4!}&(9X_{2100}+6X_{1010}+6X_{3000}+10X_{1200}+9X_{2010}\\ &+4X_{2000}+4X_{0200}+8X_{1100}+11X_{1110}+8X_{0110}) \end{aligned} \tag{115}$$

$$\begin{aligned} e_4^{(3)} = \frac{1}{4!}&(10X_{2010}+12X_{0200}+6X_{1110}+12X_{0010}+8X_{1200}+12X_{0110}\\ &+12X_{3000}+12X_{1100}+12X_{2000}+12X_{1010}+12X_{0100}+10X_{2100}+12X_{1000}) \end{aligned} \tag{116}$$

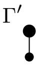
$$\begin{aligned} e_4^{(4)} = \frac{1}{4!}&(3X_{2100}+8X_{0200}+8X_{2000}+2X_{1200}+4X_{0110}+4X_{1100}\\ &+3X_{2010}+12X_{1000}+12X_{0100}+6X_{1010}+6X_{3000}+24X_{0000}+12X_{0010}+X_{1110}) \end{aligned} \tag{117}$$

To recover Chapoton's coefficients for the two-parameter series, one has to use the other version of the X-basis, defined by duality with the opposite coproduct on $\mathcal{H}_{NCK}$.

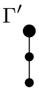
This amounts to replacing $\Gamma(X)$ by $\Gamma'(X)=\omega(\Gamma(X))$, that is,

$$\Gamma_T(X_{q,t}) = \omega(\Gamma_T)\left(\frac{1-qt|}{|1-q}\right). \tag{118}$$

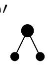
(119)

Γ'  $\left(\frac{1-qt|}{|1-q}\right) = \frac{\left(q^2x+q+1\right)(qx+1)}{q+1}$

(120)

Γ'  $\left(\frac{1-qt|}{|1-q}\right) = \frac{\left(q^3x+q^2+q+1\right)\left(q^2x+q+1\right)(qx+1)}{(q^2+q+1)(q+1)}$

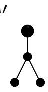
(121)

Γ'  $\left(\frac{1-qt|}{|1-q}\right) = \frac{\left(q^3x+q^2x+q^2+q+1\right)\left(q^2x+q+1\right)(qx+1)}{(q^2+q+1)(q+1)}$

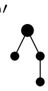
(122)

Γ'  $\left(\frac{1-qt|}{|1-q}\right) = \frac{\left(q^4x+q^3+q^2+q+1\right)\left(q^3x+q^2+q+1\right)\left(q^2x+q+1\right)(qx+1)}{(q^2+q+1)(q^2+1)(q+1)^2}$

(123)

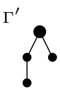
Γ'  $\left(\frac{1-qt|}{|1-q}\right) = \frac{\left(q^3x+q^2+q+1\right)\left(q^3x+q^2+1\right)\left(q^2x+q+1\right)(qx+1)}{(q^2+q+1)(q^2+1)(q+1)}$

(124)

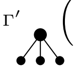
Γ'  $\left(\frac{1-qt|}{|1-q}\right) = \frac{\left(q^4x+q^3x+q^3+q^2x+q^2+q+1\right)\left(q^3x+q^2+q+1\right)\left(q^2x+q+1\right)(qx+1)}{(q^2+q+1)(q^2+1)(q+1)^2}$

(125)

(126)

Γ'  $\left(\frac{1-qt|}{|1-q}\right) = \frac{\left(q^4x+q^3x+q^3+q^2x+q^2+q+1\right)\left(q^3x+q^2+q+1\right)\left(q^2x+q+1\right)(qx+1)}{\left(q^2+q+1\right)\left(q^2+1\right)(q+1)^2}$

(127)

Γ'  $\left(\frac{1-qt|}{|1-q}\right) = \frac{\left(q^6x^2+q^5x^2+2\,q^5x+q^4x^2+2\,q^4x+q^4+3\,q^3x+q^3+2\,q^2x+2\,q^2+q+1\right)\left(q^2x+q+1\right)(qx+1)}{\left(q^2+q+1\right)\left(q^2+1\right)(q+1)}$

5.5. Appendix: noncommutative Ehrhart polynomials. In the introduction of [4], Chapoton mentions that the coefficients of the series  are q-analogues of Ehrhart polynomials (according to his definition given in [5]). These are actually specialisations of the noncommutative Ehrhart polynomials, which are defined only for the order polytopes of posets on $[n]$ [2, 20].

Recall the definition of the free generating function of a poset P

$$\Gamma_P(A) = \sum_{\sigma\in L(P)} \mathbf{F}_\sigma \in \mathbf{FQSym} \tag{128}$$

where $L(P) \subseteq \mathfrak{S}_n$ is the set of linear extensions of P. It is a morphism from the Malvenuto–Reutenauer Hopf algebra of special posets towards **FQSym**. In the sequel, we will only consider posets satisfying $i <_P j \Rightarrow i < j$.

The order polytope Q_P of P is defined by the inequalities $0 \leqslant x_i \leqslant 1$ for $i \in P$ and $i <_P j \Rightarrow x_i \leqslant x_j$.

The Ehrhart polynomial $E_Q(t)$ computes the number of integral points of nQ for $t = n$. Moreover, $(-1)^n E_T(-n)$ is the number of interior integral points.

Since nQ_P is the intersection of the cone C_P defined by $x_i \geqslant 0$ and $i <_P j \Rightarrow x_i \leqslant x_j$, and of a hypercube, one can form in **WQSym** the sum of the packed words of its integer points. The noncommutative Ehrhart polynomial of Q_P is

$$\sum_{u \in C(P)} \mathbf{M}_u = \Gamma_P(A) \tag{129}$$

where $C(P)$ is the set of packed words satisfying $i <_P j \Rightarrow u_i \leqslant u_j$, if one embeds **FQSym** into **WQSym** by

$$\mathbf{G}_\sigma(A) = \sum_{\mathrm{std}(u)=\sigma} \mathbf{M}_u. \tag{130}$$

Indeed, the linear extensions of P are precisely the permutations such that $i <_P j \Rightarrow \sigma^{-1}(i) < \sigma^{-1}(j)$.

If one specializes A to the alphabet $A_{n+1} = \{a_0, a_1 \ldots, a_n\}$, $\Gamma_P(A_{n+1})$ becomes the sum of the integral points of Q_P. Their number is therefore $E_{Q_P}(n) = \Gamma_P(n+1)$.

The change of sign $A \mapsto -A$ of the alphabet is defined on symmetric functions by means of the λ-ring structure: $p_n(-X) = -p_n(X)$, and one defines more generally, the multiplication of the alphabet by an element of binomial type $p_n(\alpha X) = \alpha p_n(X)$.

These transformations can be naturally extended to quasi-symmetric functions. One first defines them on **Sym** by setting $\sigma_t(\alpha A) = \sigma_t(A)^\alpha$, then one extends to *QSym* by defining $\sigma_t(X\alpha \cdot A) = \sigma_t(XA) * \sigma_1(\alpha A)$. These transformations can then be extended to **WQSym** by means of the internal product of $\mathbf{WQSym}^*$ [16]. One obtains

$$\mathbf{M}_u(-A) = (-1)^{\max(u)} \sum_{v \leqslant u} \mathbf{M}_v(A) \tag{131}$$

where the sum runs over the refinement order on packed words[(3)].

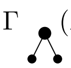
If one sets $A = \{a_0, a_1, a_2 \ldots\}$ et $A' = \{a_1, a_2, \ldots\}$, one has

$$(-1)^n \Gamma_P(-A') = \sum_{v \in \dot{C}(P)} \mathbf{M}_v(A') \tag{132}$$

where $\dot{C}(P)$ is the set of packed words satisfying $i <_P j \Rightarrow u_i < u_j$, otherwise said, of the packed words of the interior points of the cone. The interior points of the polytope nQ_P are obtained by evaluating on the alphabet $\{a_1, \ldots, a_{n-1}\}$.

The number of interior points is thus $(-1)^n \Gamma_P(1-n) = E_{Q_P}(-n)$, we have therefore in this particular case a noncommutative lift of the Ehrhart reciprocity formula.

For example,

(133) Γ  $(A) = \mathbf{F}_{123} + \mathbf{F}_{213} = \mathbf{G}_{123} + \mathbf{G}_{213}$

$$= \mathbf{M}_{123} + \mathbf{M}_{122} + \mathbf{M}_{112} + \mathbf{M}_{111} + \mathbf{M}_{213} + \mathbf{M}_{212}$$

(3) $v \leqslant u$ iff the set composition encoded by v is obtained by merging adjacent blocks of that encoded by u.

has as commutative image $F_{12} + F_3$ and as evaluation on a scalar $\binom{\alpha+2}{3} + \binom{\alpha+1}{3}$ so that the Ehrhart polynomial of the order polytope $Q = \{0 \leqslant x_1, x_2 \leqslant x_3\}$ is

$$E_Q(x) = \binom{x+3}{3} + \binom{x+2}{3} = \frac{(x+1)(x+2)(2x+3)}{6} \tag{134}$$

which is indeed the specialization $q = 1$ of

$$\Gamma(X_{q,t}) = \frac{\big(q^3x + q^2x + q^2 + q + 1\big)\big(q^2x + q + 1\big)(qx + 1)}{(q^2 + q + 1)(q + 1)} \tag{135}$$

The specialization $x = [n]_q$ gives the q-counting of the integral points of nQ by sum of the coordinates. Indeed, this amounts to setting $t = q^n$ in (101), so that by [13, Prop. 8.4]

$$♙ \mapsto \sigma_1(X_{q,q^n}A) := \overrightarrow{\prod_{0\leqslant i\leqslant n}} \sigma_{q^i}(A) = \sum_I M_I(1, q, \ldots, q^n)S^I, \tag{136}$$

that is,

$$\Gamma_P(X_{q,q^n}) = \sum_{(x_1,\ldots,x_d)\in nQ\cap\mathbb{Z}^d} q^{x_1+x_2+\cdots+x_d}. \tag{137}$$

For example, the 14 integral points of $2Q$ are

000, 001, 011, 101, 022, 111, 012, 102, 112, 022, 202, 122, 212, 222

and $\Gamma(X_{q,q^2}) = 1 + q + 3q^2 + 3q^3 + 3q^4 + 2q^5 + q^6$, as expected.

Now,

$$(-1)^3\Gamma(-A) = \mathbf{M}_{123} + \mathbf{M}_{213} + \mathbf{M}_{112} \tag{138}$$

which predicts correctly that for $n = 3$ the only interior point of $3Q$ is $(1, 1, 2)$. Setting $t = q^{-n}$ in (101) results into

$$♙ \mapsto \sigma_1(X_{q,q^{-n}}A) := \overrightarrow{\prod_{1\leqslant i\leqslant n-1}} \lambda_{-q^{-i}}(A) \tag{139}$$

so that $\Gamma_P(X_{q,q^{-n}})$ is obtained, in accordance with [5, Theorem 2.5], by evaluating $\Gamma_P(-A)$ on the alphabet $\{x_i = q^{-i} \mid i = 1, \ldots, n-1\}$. On our example, setting $x = [-3]_q$ yields $-q^{-4}$, corresponding to the interior point $(1, 1, 2)$.

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