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Corrigendum to "Tropical Positivity and Determinantal Varieties"

Marie-Charlotte Brandenburg, Georg Loho & Rainer Sinn

1. Introduction

After publication of our original paper "Tropical positivity and determinantal varieties," two flaws were pointed out by fellow colleagues. We address these issues and their implications in this note. The issues lie in subtleties for the different notions of positivity. The examples below demonstrate that they behave even more differently from each other. In particular, our results on determinantal varieties stay valid for the notion of tropical positivity based on complex Puiseux series.

2. Positivity in tropical geometry

There are two problems in Section 2 of our original paper with the same title as this section. We discuss the problems and their consequences for the published results.

2.1. Notions of positivity.

This following statement is incorrect.

Corollary 2.1 ([2, Cor 2.2]). *For hypersurfaces the positive part coincides with the combinatorially positive part, i.e.* $\mathrm{Trop}^{+}(f) = \mathrm{trop}^{+\mathcal{C}}(V(f))$ *for any polynomial* $f \in \mathcal{R}[x_1, \ldots, x_n]$ (*or more generally* $f \in \mathcal{C}[x_1, \ldots, x_n]$ *such that its coefficients are in* $\mathcal{C}_{+} \cup (-\mathcal{C}_{+})$).

The inclusion $\mathrm{trop}^{+\mathcal{C}}(V(f)) = \bigcap_{g\in\langle f\rangle} \mathrm{Trop}^{+}(g) \subset \mathrm{Trop}^{+}(f)$ holds by definition. In our proof of the other inclusion, we used the equivalence of $\mathrm{in}_w(\langle f\rangle) \cap \mathbb{R}_{\geqslant 0}[x_1, \ldots, x_n] = \{0\}$ and $\mathrm{in}_w(f) \notin \mathbb{R}_{\geqslant 0}[x_1, \ldots, x_n]$. The following example, pointed out to us by Kemal Rose and Máté Telek, shows that this equivalence does not hold in general. In particular, it is a counterexample to the statement above.

Example 2.2. Consider $f = 1 - x + x^2 + y \in \mathcal{R}[x, y]$. The Newton polytope of this polynomial is a triangle. The signs of the coefficients for all vertices is positive and along the bottom side (corresponding to the x-variable) the signs of the coefficients alternate. Since $\mathrm{in}_{(0,1)}(f) = 1 - x + x^2$, the combinatorially positive part contains the ray in direction $(0, 1)$ (in fact, it is equal to this ray). However, the polynomial has no zero $(x(t), y(t))$ over the field of complex Puiseux series such that the leading coefficients of the Puiseux series x and y are real and positive.

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Indeed, to see this, we can apply [3, Proposition 2.2]. The polynomial

$$f(1+x) = (1 - x + x^2 + y)(1+x) = 1 + x^3 + y + xy \in \langle f \rangle$$

has leading form $\mathrm{in}_{(0,1)}\left((1+x)f\right) = 1 + x^3$, which is in $\mathbb{R}_{\geqslant 0}[x,y]$.

However, the statement is still true in the following special case.

Proposition 2.3. *Let $f \in \mathbb{R}[x_1,\dots,x_n] \subset \mathcal{R}[x_1,\dots,x_n]$ be a polynomial such that the support of f intersects the boundary of the Newton polytope of f exactly in its vertices. In particular, each initial ideal corresponding to a maximal cone of* $\mathrm{trop}(f)$ *is a binomial ideal. Then we have*

$$\mathrm{Trop}^+(f) = \mathrm{trop}^{+\mathcal{C}}(V(f)).$$

Proof. By our assumption on the support of f, every cone in $\mathrm{trop}(f)$ that is contained in $\mathrm{Trop}^+(f)$ is contained in a maximal cone in $\mathrm{trop}(f)$ that is combinatorially positive, i.e. in $\mathrm{Trop}^+(f)$. Indeed, if $w \in \mathrm{Trop}^+(f)$, then $\mathrm{in}_w(f) \notin \mathbb{R}_{\geqslant 0}[x_1,\dots,x_n]$. The only terms in the initial form $\mathrm{in}_w(f)$ correspond to the vertices of the face of $\mathrm{Newt}(f)$ in direction w. Since the edge graph of this face is connected, there is an edge with a positive and a negative vertex, which dually gives a maximal cone in $\mathrm{Trop}^+(f)$. Since $\mathrm{trop}^{+\mathcal{C}}(V(f))$ is a closed polyhedral subfan of $\mathrm{trop}(f)$, it suffices to determine which maximal cones lies in $\mathrm{Trop}^+(f)$. By [3, Proposition 2.2], $w \in \mathrm{trop}^{+\mathcal{C}}(V(f))$ is equivalent to $\mathrm{in}_w(\langle f \rangle) \cap \mathbb{R}_{\geqslant 0}[x_1,\dots,x_n] = \{0\}$. By assumption, the initial form $\mathrm{in}_w(f)$ for a vector w in the relative interior of a maximal cone of $\mathrm{trop}(f)$ is a binomial so that $J = \mathrm{in}_w(\langle f \rangle)$ is a principal binomial ideal. Therefore, $\{\mathrm{in}_w(f)\}$ is a reduced Gröbner basis of J and [1, Lemma 5.7] shows that $J \cap \mathbb{R}_{\geqslant 0}[x_1,\dots,x_n] = \{0\}$ if and only if $\mathrm{in}_w(f) \notin \mathbb{R}_{\geqslant 0}[x_1,\dots,x_n]$. □

Since the Birkhoff polytope is the Newton polytope of the determinant, this statement applies to $f = \det$. This clarifies our proof of Proposition 3.1 in our paper.

In the statement of the previous proposition, it is important to assume that f has coefficients with trivial valuation. Otherwise, the leading form might not be a binomial.

Example 2.4. For $f = y - xy + x^2y + xy^2 + tx \in \mathcal{R}[x,y]$ we have $\mathrm{Trop}^+(f) = \mathrm{conv}((0,1),(0,0))$ but $\mathrm{trop}^{+\mathcal{C}}(f) = \varnothing$.

2.2. Generators of positivity. The proof of the following statement made in our paper is incomplete. The gap in the proof was pointed out to us by May Cai and Josephine Yu. Currently, we do not know a complete proof nor a counterexample to the claim.

Proposition 2.5 ([2, Prop. 2.13]). *Let $A \in \mathcal{C}^{d\times n}$ be a matrix such that the leading coefficient of every entry $A_{ij} \in \mathcal{C}$ is real. Then there exists a matrix $B \in \mathcal{R}^{d\times n}$ of real Puiseux series that has the same rank as A and the Puiseux series in every entry has the same leading term as in A, meaning that* $\mathrm{lt}(A_{ij}) = \mathrm{lt}(B_{ij})$ *holds for all* $(i,j) \in [d] \times [n]$.

The successive statements in our paper remain correct if we interpret $\left(T^r_{d,n}\right)^+$ as $\mathrm{trop}^{+\mathcal{C}}(V(I_r))$, where I_r is the ideal of $(r+1)\times(r+1)$-minors of a $d \times n$ matrix. This is in contrast to the statement made in the last paragraph on page 7 of our paper, because $\mathrm{trop}^{+\mathcal{C}}(V(I_r))$ and $\mathrm{trop}^{+\mathcal{R}}(V(I_r))$ might be different.

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