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# Modular law through GKM theory

Tatsuya Horiguchi, Mikiya Masuda & Takashi Sato

Abstract The solution of Shareshian-Wachs conjecture by Brosnan-Chow and Guay-Paquet tied the graded chromatic symmetric functions on indifference graphs (or unit interval graphs) and the cohomology of regular semisimple Hessenberg varieties with the dot action. A similar result holds between unicellular LLT polynomials and twins of regular semisimple Hessenberg varieties. A recent result by Abreu-Nigro enabled us to prove these results by showing the modular law for the geometrical objects, and this is indeed done by Precup-Sommers and Kiem-Lee. In this paper, we give elementary and simpler proofs to the modular law through GKM theory.

## 1. Introduction

Let $n$ be a positive integer and $[n]$ the set of integers from 1 to $n$. A function $h\colon [n] \to [n]$ is called a Hessenberg function if it is non-decreasing and $h(j) \geqslant j$ for all $j \in [n]$. One may think of a Hessenberg function as a Dyck path.

A graph $G_h$ (called an indifference graph) is associated to $h$ as follows:

(1) the vertex set $V(G_h)$ is $[n]$,
(2) the edge set $E(G_h)$ is $\{\{i,j\} \mid j < i \leqslant h(j)\}$.

Let $\mathbb{P}$ be the set of positive integers. A map $\kappa\colon [n] \to \mathbb{P}$ is called a $\mathbb{P}$-coloring on $G_h$ and it is proper if $\kappa(i) \neq \kappa(j)$ whenever $\{i,j\} \in E(G_h)$. Let $z_1, z_2, \dots$ be infinitely many variables. We set $\mathbf{z}_\kappa := z_{\kappa(1)} z_{\kappa(2)} \cdots z_{\kappa(n)}$ for a coloring $\kappa$ on $G_h$. Then Stanley's chromatic symmetric function $\mathrm{csf}_h$ of $G_h$ is defined by

$$\mathrm{csf}_h := \sum_{\kappa \in PC(G_h)} \mathbf{z}_\kappa,$$

where $PC(G_h)$ denotes the set of all proper $\mathbb{P}$-colorings on $G_h$. The long-standing Stanley-Stembridge conjecture is known to be equivalent to $\mathrm{csf}_h$ being $e$-positive, that is, when $\mathrm{csf}_h$ is expressed as a polynomial in the elementary symmetric functions of $z_1, z_2, \dots$, all the coefficients are non-negative.

Shareshian-Wachs [18] introduced a graded version of $\mathrm{csf}_h$ as follows:

$$\mathrm{csf}_h(q) := \sum_{\kappa \in PC(G_h)} \mathbf{z}_\kappa q^{\mathrm{asc}(\kappa)}, \tag{1.1}$$

where

$$\mathrm{asc}(\kappa) := \#\{\{i,j\} \in E(G_h) \mid j < i,\ \kappa(j) < \kappa(i)\}.$$

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They show that the coefficients of $\mathrm{csf}_h(q)$ as a polynomial in $q$ are symmetric functions in $z_1, z_2, \dots$. A strong version of the Stanley-Stembridge conjecture is that those coefficients of $\mathrm{csf}_h(q)$ are all $e$-positive.

On the other hand, a regular semisimple Hessenberg variety $X(h)$ is associated with the Hessenberg function $h$. It is a subvariety of the flag variety $\mathrm{Fl}(n)$ defined by

$$X(h) := \{V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \in \mathrm{Fl}(n) \mid SV_i \subset V_{h(i)} \text{ for } i \in [n]\},$$

where $S$ is a linear endomorphism of $\mathbb{C}^n$ with distinct eigenvalues. It is known that $X(h)$ is nonsingular and its smooth structure is independent of the choice of $S$. Since we are only concerned with its cohomology, we suppress $S$ in the notation of $X(h)$. The cohomology $H^*(X(h))$ of $X(h)$ concentrate on even degrees and is a graded module over the symmetric group $\mathfrak{S}_n$ on $[n]$ (see Section 3).

The following remarkable fact, which ties seemingly unrelated two objects above, was conjectured by Shareshian-Wachs [18] and proved by Brosnan-Chow [6] and Guay-Paquet [10].

Theorem 1.1 ([6, 10]). *Let the situation be as above. Then*

$$\omega(\mathrm{csf}_h(q)) = \sum_{i=0}^{\infty} \mathrm{ch}\left(H^{2i}(X(h))\right) q^i,$$

*where $\omega$ denotes the involution on symmetric functions sending the $i$-th elementary symmetric function to the $i$-th complete symmetric function and* ch *denotes Frobenius characteristic sending $\mathfrak{S}_n$-modules to symmetric functions of degree $n$.*

Recently, it was noticed in [16] and [17] that a similar fact holds for unicellular LLT polynomials and the twin of $X(h)$. LLT polynomials were originally introduced by Lascoux, Leclerc, and Thibon in [15]. It can be seen as a $q$-deformation of the product of skew Schur functions and it is indexed by a tuple of skew Young diagrams. An LLT polynomial is called *unicellular* if each skew Young diagram in its index is a single box. It is observed in [7] that a unicellular LLT polynomial is associated with a Hessenberg function $h$ and can be expressed in terms of $\mathbb{P}$-coloring as

$$\mathrm{LLT}_h(q) = \sum_{\kappa \in C(G_h)} \mathbf{z}_\kappa q^{\mathrm{asc}(\kappa)},$$

where $C(G_h)$ denotes the set of all $\mathbb{P}$-colorings on $G_h$ (the properness is not required).

On the other hand, Ayzenberg-Buchstaber [5] introduced a closed smooth submanifold $Y(h)$ of $\mathrm{Fl}(n)$, which they call the twin of $X(h)$. The twin $Y(h)$ resembles $X(h)$, e.g. $H^*(Y(h))$ is isomorphic to $H^*(X(h))$ as groups and $H^*(Y(h))$ is also a graded $\mathfrak{S}_n$-module. However, they are not isomorphic as rings and as $\mathfrak{S}_n$-modules in general.

Theorem 1.2 ([16, 17]). *Let the situation be as above. Then*

$$\mathrm{LLT}_h(q) = \sum_{i=0}^{\infty} \mathrm{ch}\left(H^{2i}(Y(h))\right) q^i,$$

*where* ch *denotes Frobenius characteristic as before.*

The proof of Theorem 1.1 by Brosnan-Chow [6] uses deep results in algebraic geometry and that by Guay-Paquet [10] uses Hopf algebra on Dyck paths. The proof of Theorem 1.2 by Masuda-Sato [16] is based on Theorem 1.1. Precup-Sommers [17] prove both Theorems 1.1 and 1.2 using intersection cohomology. In fact, they prove that the right hand sides (i.e. the geometric sides) of the identities in Theorems 1.1 and 1.2 satisfy the modular law. Here a function $F$ on the set of Hessenberg functions taking values in a polynomial ring $\Lambda[q]$ over a ring $\Lambda$ is said to satisfy the modular law if

$$(1+q)F(h) = F(h_+) + qF(h_-)$$

for every modular triple $(h_-, h, h_+)$ of Hessenberg functions, see Definition 2.1 for modular triples. Abreu-Nigro [1] show that a function $F$ satisfying the modular law is uniquely determined by its "initial conditions", see Section 2 for details. In Theorems 1.1 and 1.2, the left hand sides (i.e. the algebraic sides) of the identities are known to satisfy the modular law (see Appendices), and the left-hand side and the right-hand side coincide for "initial" $h$'s (see [13, Proposition 2.15 and (2.17)] and [1, Theorem 1.1] for Theorem 1.1, and see [14, Lemma 3.11 and (3.20)] and [2, Theorem 2.4] for Theorem 1.2). Hence, once the modular law is established for the geometric sides, Theorems 1.1 and 1.2 follow.

Recently, Kiem-Lee ([13, 14]) also proved the modular law for the geometric sides. Their proofs are elementary in the sense that their main tool is blow-up. However they took different approaches between when proving Theorem 1.1 and when proving Theorem 1.2. Abreu-Nigro [3, Example 3.5] also proved Theorem 1.1, but they omitted the details.

In this paper, we prove the modular law for the geometric sides through GKM theory. Our proof is motivated by the blow-up idea of Kiem-Lee and may be regarded as graph analogue of their proofs. However ours is more elementary and simpler. Moreover, our proofs for $X(h)$ and $Y(h)$ proceed in the same way. Indeed, we blow-up the GKM graph of $X(h_+)$ (or $Y(h_+)$) along the GKM graph of $X(h_-)$ (or $Y(h_-)$). The resulting graph, which corresponds to a roof manifold in [14], is a labeled graph but not a GKM graph. We consider its graph cohomology satisfying a certain condition and compute it in two ways. The modular law for the geometric sides follows by comparing the two expressions of the graph cohomology. We also provide a simple elementary proof to the modular law for the algebraic sides for the reader's convenience in the appendix.

This paper is organized as follows. In Section 2 we explain a modular triple. We also recall the uniqueness result of Abreu-Nigro [1]. In Section 3 we briefly review GKM theory which is our main tool. In Section 4 we set up notations used for our proof of the modular law for the geometric sides. We give the proof of the modular law for $X(h)$ in Section 5. In Section 6 we point out the necessary change for the proof of the modular law for $Y(h)$. All cohomology groups are taken with $\mathbb{C}$ coefficients throughout this paper unless otherwise stated.

## 2. Modular triple and modular law

We denote the set of all Hessenberg functions on $[n]$ by $\mathcal{H}(n)$. We often express $h \in \mathcal{H}(n)$ as a vector $(h(1), \ldots, h(n))$ by listing its values. It is also convenient to visualize $h$ by drawing a configuration of the shaded boxes on a square grid of size $n \times n$, which consists of boxes in the $i$-th row and the $j$-th column satisfying $i \leqslant h(j)$. Since $h(j) \geqslant j$ for any $j \in [n]$, the essential part is the shaded boxes below the diagonal, see Figure 1 below.

If we flip the configuration of $h$ along the anti-diagonal, then the resulting one is again a configuration of a Hessenberg function, denoted by $h^t$. We call $h^t$ the transpose of $h$. For example, the two Hessenberg functions in Figure 1 are the transposes of each other.

We introduce the following terminology used in [14].

Definition 2.1 (Modular triple). *Let $h_-, h, h_+$ be elements in $\mathcal{H}(n)$. The triple $(h_-, h, h_+)$ is called a* modular triple *if it satisfies one of the following.*

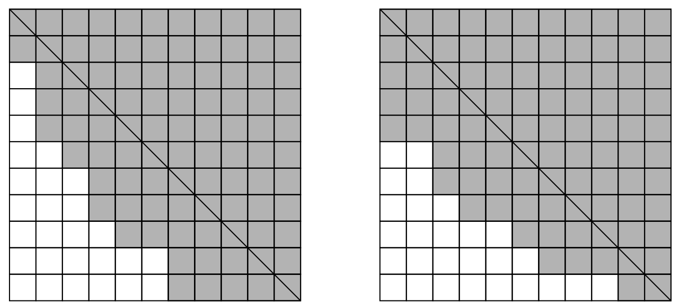


Figure 1. The configurations for $h = (2,5,6,8,9,9,11,11,11,11,11)$ and $(5,5,7,8,8,9,10,10,10,11,11)$.

(C) *If $h(d) = h(d+1)$ and $h^{-1}(d) = \{d_0\}$ for some $1 \leqslant d_0 < d < n$, then $h_-$ and $h_+$ are defined by*

$$h_-(j) = \begin{cases} d-1 & \text{for } j = d_0 \\ h(j) & \text{otherwise} \end{cases} \quad \text{and} \quad h_+(j) = \begin{cases} d+1 & \text{for } j = d_0 \\ h(j) & \text{otherwise.} \end{cases}$$

(R) *If $h(d') + 1 = h(d'+1) \neq d'+1$ and $h^{-1}(d') = \varnothing$ for some $1 \leqslant d' < n$, then $h_-$ and $h_+$ are defined by*

$$h_-(j) = \begin{cases} h(d') & \text{for } j = d'+1 \\ h(j) & \text{otherwise} \end{cases} \quad \text{and} \quad h_+(j) = \begin{cases} h(d')+1 & \text{for } j = d' \\ h(j) & \text{otherwise.} \end{cases}$$

Remark 2.2. In Definition 2.1, (C) stands for column and (R) stands for row. The statements (C) and (R) seem unrelated but they are related through $h \to h^t$, e.g. (C) for $h^t$ implies (R) for $h$.

Example 2.3. For the left $h$ in Figure 1, $(d, d_0) = (5,2), (8,4)$ satisfy (C) while $d' = 4$ satisfies (R). For the right $h$ in Figure 1, $(d, d_0) = (7,3)$ satisfies (C) while $d' = 3, 6$ satisfy (R). Since they are the transposes to each other, the $(d+1)$-th column of the left $h$ corresponds to the $d'$-th row of the right $h$ by $n + 1 - d' = d + 1$, that is, $d + d' = n$, where $n = 11$ in this case. In other words, $d' = w_0(d+1)$, where $w_0$ is the longest element of $\mathfrak{S}_n$.

Definition 2.4. *Let $\Lambda[q]$ be a polynomial ring over a ring $\Lambda$ (in fact, we take $\Lambda$ to be the ring of symmetric functions in $z_1, z_2, \dots$). We say that a function $F \colon \mathcal{H}(n) \to \Lambda[q]$ satisfies the* modular law *if*

$$(1+q)F(h) = F(h_+) + qF(h_-) \tag{2.1}$$

*for every modular triple $(h_-, h, h_+)$.*

Remark 2.5. If $F(h^t) = F(h)$ for any $h \in \mathcal{H}(n)$, then it suffices to check the modular relation (2.1) for modular triples of type (C) for $F$ by Remark 2.2.

Given $h_1 \in \mathcal{H}(n_1)$ and $h_2 \in \mathcal{H}(n_2)$, their product $h_1 h_2 \in \mathcal{H}(n_1 + n_2)$ can naturally be defined by

$$h_1 h_2(j) = \begin{cases} h_1(j) & \text{for } 1 \leqslant j \leqslant n_1 \\ h_2(j - n_1) + n_1 & \text{for } n_1 + 1 \leqslant j \leqslant n_1 + n_2. \end{cases}$$

Theorem 2.6 (Abreu-Nigro [1]). *A function $F \colon \mathcal{H}(n) \to \Lambda[q]$ which satisfies the modular law is determined by its values at products $h_1 h_2 \cdots h_r$ of all tuples $(h_1, \dots, h_r)$ such that $h_i \in \mathcal{H}(n_i)$, $h_i(j) = n_i$ for $\forall j \in [n_i]$, and $\sum_{i=1}^r n_i = n$.*

## 3. Graph cohomology of a labeled graph and GKM theory

Let $T$ be the compact torus $(S^1)^n$ where $S^1$ is the unit circle of $\mathbb{C}$. The classifying space $BT$ of $T$ is $(\mathbb{C}P^\infty)^n$. We choose a generator of $H^2(\mathbb{C}P^\infty)$ obtained as the first Chern class of the tautological line bundle over $\mathbb{C}P^\infty$ and let $t_1, \dots, t_n$ be a generator of $H^2(BT)$ coming from the factors of $BT = (\mathbb{C}P^\infty)^n$. Then $H^*(BT)$ is a polynomial ring in $t_1, \dots, t_n$.

Definition 3.1. *A* labeled graph $\Gamma = (V, E, \alpha)$ *is a graph with an edge labeling, where $V$ is a vertex set, $E$ is an edge set, and $\alpha\colon E \to H^2(BT)$ is a labeling on $E$. For $e \in E$, we denote by $e_\pm$ the endpoints of $e$. Then the (equivariant)* graph cohomology *of the labeled graph $\Gamma$ is defined by*

$$H^*_T(\Gamma) := \{f \in \mathrm{Map}(V, H^*(BT)) \mid f(e_+) \equiv f(e_-) \pmod{\alpha(e)} \quad \text{for any } e \in E\}. \tag{3.1}$$

Any constant map on $V$ taking a value $t$ in $H^*(BT)$ is an element of $H^*_T(\Gamma)$, which we also denote by $t$, and $H^*_T(\Gamma)$ is a module over $H^*(BT)$ in a natural way. We define

$$H^*(\Gamma) := H^*_T(\Gamma)/(t_1, \dots, t_n), \tag{3.2}$$

where ( ) denotes the ideal generated by the elements in it.

Graph cohomology often arises as the equivariant cohomology

$$H^*_T(X) := H^*(ET \times_T X)$$

of a nice $T$-space $X$ called a GKM manifold (see [12]), where $ET \to BT$ is the universal principal $T$-bundle and $ET \times_T X$ is the orbit space of $ET \times X$ by the diagonal $T$-action. The first projection $ET \times X \to ET$ induces a fibration

$$X \to ET \times_T X \xrightarrow{\pi} ET/T = BT,$$

so $H^*_T(X)$ is a module over $H^*(BT)$ through $\pi^*\colon H^*(BT) \to H^*_T(X)$. If $H^*(X)$ concentrate on even degrees, then the Serre spectral sequence of the fibration above collapses; so the restriction map $H^*_T(X) \to H^*(X)$ is surjective and its kernel is the ideal generated by $\pi^*(H^{>0}(BT))$, so that we obtain an isomorphism

$$H^*_T(X)/(\pi^*(t_1), \dots, \pi^*(t_n)) \cong H^*(X). \tag{3.3}$$

We say that the labeled graph $\Gamma$ is 2-*independent* if, for any $p \in V$ and edges $e_1, \dots, e_m$ incident to $p$, $\alpha(e_1), \dots, \alpha(e_m)$ are pairwise linearly independent. The equivariant cohomology ring of a GKM manifold $X$ is recovered from its fixed point set and 1-dimensional orbits, and they form a 2-independent labeled graph called a GKM graph (see [12] for details).

Recall that

$$X(h) = \{V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \in \mathrm{Fl}(n) \mid SV_i \subset V_{h(i)} \text{ for } i \in [n]\}, \tag{3.4}$$

where $S$ is a linear endomorphism of $\mathbb{C}^n$ with distinct eigenvalues. As remarked in the introduction, the diffeomorphism type of $X(h)$ is independent of the choice of $S$. We take $S$ to be a linear operator defined by a diagonal matrix with distinct eigenvalues. For later use, we assume that they are real numbers. Then $S$ commutes with the standard action of $T = (S^1)^n$ on $\mathbb{C}^n$ defined by coordinate-wise multiplication, so the induced $T$-action on $\mathrm{Fl}(n)$ leaves $X(h)$ invariant. One can easily check that the $T$-fixed point sets $X(h)^T$ and $\mathrm{Fl}(n)^T$ consist of permutation flags $V_\bullet(w)$ associated with elements $w \in \mathfrak{S}_n$;

$$V_\bullet(w) = (\langle \mathbf{e}_{w(1)} \rangle \subset \langle \mathbf{e}_{w(1)}, \mathbf{e}_{w(2)} \rangle \subset \cdots \subset \langle \mathbf{e}_{w(1)}, \dots, \mathbf{e}_{w(n)} \rangle = \mathbb{C}^n),$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ denotes the standard basis of $\mathbb{C}^n$ and $\langle\ \rangle$ denote the linear subspace of $\mathbb{C}^n$ spanned by the elements in it. In the following, we make the following identification

$$X(h)^T = \mathrm{Fl}(n)^T = \mathfrak{S}_n.$$

The GKM graph $\mathcal{G}_X(h) = (\mathfrak{S}_n, E(h), \alpha_X)$ associated to $X(h)$ with the $T$-action is given by

$$E(h) = \{\{w, w(i,j)\} \mid w \in \mathfrak{S}_n,\ j < i \leqslant h(j)\}$$

and

$$\alpha_X(\{w, w(i,j)\}) = t_{w(i)} - t_{w(j)}, \tag{3.5}$$

where $(i,j)$ denotes the transposition exchanging $i$ and $j$, see [19]. In fact, to well-define $\alpha_X$, one need to choose the sign $\pm(t_{w(i)} - t_{w(j)})$ of the label of each edge $\{w, w(i,j)\} = \{w', w'(i,j)\}$, where $w' = w(i,j)$. However the (equivariant) cohomology of $\mathcal{G}_X(h)$ does not depend on the signs. Hence we do not take care the ambiguity of the signs. Then $H_T^*(\mathcal{G}_X(h))$ consists of all $f \in \mathrm{Map}(V, H^*(BT))$ satisfying

$$f(w) - f(w(i,j)) \equiv 0 \mod (t_{w(i)} - t_{w(j)})$$

for any $i < j \leqslant h(i)$ and $w \in \mathfrak{S}_n$. Note that any edge $\{w, w(i,j)\}$ of $\mathcal{G}_X(h)$ corresponds to $\mathbb{C}P^1$ in $X(h)$ which contains $V_\bullet(w)$ and $V_\bullet(w(i,j))$, see [13, Subsection 2.2]. It is known that $H^*(X(h))$ concentrate on even degrees (see [8]), so the restriction map

$$\iota^*\colon H_T^*(X(h)) \to H_T^*(X(h)^T) = \bigoplus_{w \in \mathfrak{S}_n} H_T^*(w) = \mathrm{Map}(\mathfrak{S}_n, H^*(BT))$$

is injective. GKM theory ([9]) tells us that the image of $\iota^*$ is $H_T^*(\mathcal{G}_X(h))$, so we have an isomorphism

$$\iota^*\colon H_T^*(X(h)) \xrightarrow{\cong} H_T^*(\mathcal{G}_X(h)). \tag{3.6}$$

Through $\iota^*$, $\pi^*(t_i)$ in (3.3) corresponds to the constant map $t_i$ on $\mathfrak{S}_n$. Therefore, it follows from (3.2) and (3.3) that the isomorphism (3.6) reduces to an isomorphism

$$H^*(X(h)) \xrightarrow{\cong} H^*(\mathcal{G}_X(h)). \tag{3.7}$$

We consider an action of $\sigma \in \mathfrak{S}_n$ on $H^*(BT)$ induced by sending $t_i$ to $t_{\sigma(i)}$ for $i \in [n]$, and define an action of $\sigma \in \mathfrak{S}_n$ on $\mathrm{Map}(\mathfrak{S}_n, H^*(BT))$ by

$$(\sigma \cdot f)(w) := \sigma(f(\sigma^{-1}w)) \qquad \text{for } f \in \mathrm{Map}(\mathfrak{S}_n, H^*(BT)) \text{ and } w \in \mathfrak{S}_n. \tag{3.8}$$

This action was considered by Tymoczko [19]. It preserves not only $H_T^*(\mathcal{G}_X(h))$ but also the ideal $(t_1, \dots, t_n)$ in $H_T^*(\mathcal{G}_X(h))$, so the action descends to an action of $\mathfrak{S}_n$ on $H^*(\mathcal{G}_X(h))$. Thus we obtain actions of $\mathfrak{S}_n$ on $H_T^*(X(h))$ and $H^*(X(h))$ through the isomorphisms (3.6) and (3.7). These actions are called the *dot action*.

A similar story holds for the twin $Y(h)$ of $X(h)$ introduced by Ayzenberg-Buchstaber [5]. The twin $Y(h)$ is defined as follows. We regard $\mathrm{Fl}(n)$ as the homogeneous space $\mathrm{U}(n)/T$ where $\mathrm{U}(n)$ is the unitary group of size $n$ and $T$ is the torus consisting of diagonal matrices in $\mathrm{U}(n)$. Indeed, a unitary matrix $g = [v_1, \dots, v_n] \in \mathrm{U}(n)$ associates a flag

$$\langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \dots, v_n \rangle = \mathbb{C}^n \tag{3.9}$$

and this correspondence induces the identification $\mathrm{U}(n)/T = \mathrm{Fl}(n)$. In particular, the permutation matrix $P(w)$ corresponds to the permutation flag $V_\bullet(w)$, where the $(w(k), k)$-components of $P(w)$ are 1 for any $k$ and the others components are 0. If the flag $V_\bullet$ in (3.4) is of the form (3.9), then the condition $SV_i \subset V_{h(i)}$ for $i \in [n]$ in (3.4) can be written as

$$Sg \in gH \qquad \text{i.e.} \qquad g^{-1}Sg \in H,$$

where $H$ is the vector subspace consisting of $n \times n$ matrices $(a_{ij})$ with $a_{ij} = 0$ for $i > h(j)$. Therefore

$$X(h) = \{gT \in \mathrm{U}(n)/T \mid g^{-1}Sg \in H\}. \tag{3.10}$$

The twin $Y(h)$ of $X(h)$ is simply defined as

$$Y(h) := \{Tg \in T\backslash \mathrm{U}(n) \mid g^{-1}Sg \in H\}. \tag{3.11}$$

It is a closed smooth manifold but not necessarily an algebraic variety. The cohomology $H^*(Y(h))$ is isomorphic to $H^*(X(h))$ as groups. In particular $H^*(Y(h))$ concentrate on even degrees. However, they are not isomorphic as rings in general. See [5, Theorem 3.10 and Remark 3.11]. Note that Ayzenberg-Buchstaber write the twin as $X_h$ and the regular semisimple Hessenberg variety as $Y_h$.

Remark 3.2. By taking inverse matrices, $Y(h)$ can be defined as

$$Y(h) = \{gT \in \mathrm{U}(n)/T \mid gSg^{-1} \in H\},$$

and the second and the third authors adopted this definition in [16].

The right multiplication by $T$ on $\mathrm{U}(n)$ induces the action of $T$ on $T\backslash \mathrm{U}(n)$ which leaves $Y(h)$ invariant. One can check that

$$Y(h)^T = (T\backslash \mathrm{U}(n))^T = \{TP(w) \mid w \in \mathfrak{S}_n\} = \mathfrak{S}_n.$$

According to [5, Proposition 5.3], $Y(h)$ with the $T$-action is a GKM manifold and its GKM graph $\mathcal{G}_Y(h)$ is $(\mathfrak{S}_n, E(h), \alpha_Y)$, where the labeling $\alpha_Y \colon E(h) \to H^2(BT)$ is given by

$$\alpha_Y(\{w, w(i,j)\}) = t_i - t_j. \tag{3.12}$$

In particular, the underlying graphs of $\mathcal{G}_X(h)$ and $\mathcal{G}_Y(h)$ are the same but the labelings $\alpha_X$ and $\alpha_Y$ are different. Compare (3.5) and (3.12). The isomorphisms (3.6) and (3.7) hold for $Y(h)$ and $\mathcal{G}_Y(h)$. However, the dot action (3.8) on $\mathrm{Map}(\mathfrak{S}_n, H^*(BT))$ does not preserve $H_T^*(\mathcal{G}_Y(h))$. As for $Y(h)$ and $\mathcal{G}_Y(h)$, we consider the action of $\sigma \in \mathfrak{S}_n$ on $\mathrm{Map}(\mathfrak{S}_n, H^*(BT))$ defined by

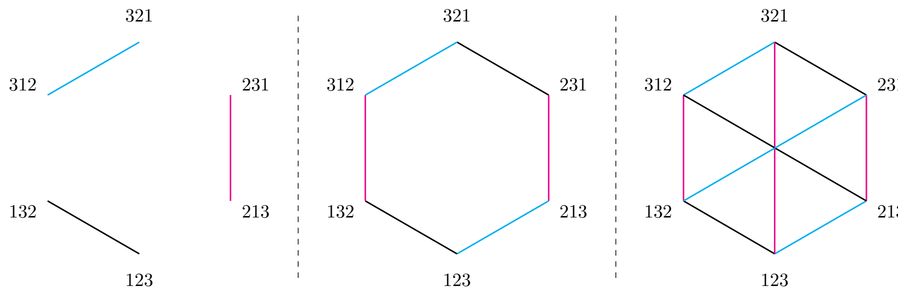
$$(\sigma \dagger f)(w) := f(\sigma^{-1}w) \qquad \text{for } f \in \mathrm{Map}(\mathfrak{S}_n, H^*(BT)) \text{ and } w \in \mathfrak{S}_n. \tag{3.13}$$

This action preserves $H_T^*(\mathcal{G}_Y(h))$ and induces actions of $\mathfrak{S}_n$ on $H_T^*(Y(h))$ and $H^*(Y(h))$, called the dagger action in [16].

Remark 3.3. Let $J$ be the anti-diagonal matrix with all entry 1. Note that the anti-diagonal transpose of a matrix $A$ is given by $J({}^tA)J$, where ${}^tA$ is the ordinary transpose of $A$. In the sense of (3.10), the correspondence $gT \mapsto gJT$ gives a diffeomorphism $X(h) \cong X(h^t)$, since $g^{-1} = {}^t\bar{g}$, $S = {}^tS$, $\bar{S} = S$, and $\bar{H} = H$. This correspondence means the orthogonal complement of flags; for orthogonal $v_1, \ldots, v_n$,

$$\begin{aligned} &(\{0\} \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \langle v_1, \ldots, v_{n-1} \rangle \subset \mathbb{C}^n) \\ &\mapsto (\mathbb{C}^n \supset \langle v_2, \ldots, v_n \rangle \supset \langle v_3, \ldots, v_n \rangle \supset \cdots \langle v_n \rangle \supset \{0\}). \end{aligned}$$

In terms of GKM graphs, an edge $\{w, w(i,j)\}$ corresponds to $\{ww_0, w(i,j)w_0\}$ and they have the same label. Then this gives an isomorphism between GKM graphs of $X(h)$ and $X(h^t)$. In particular, this induces an $H^*(BT)$-algebra isomorphism $H_T^*(X(h^t)) \to H_T^*(X(h))$ which commutes with the dot action.




The label on black edges is $t_d - t_{d+1} = t_2 - t_3$, that on cyan edges is $t_1 - t_2$, and that on magenta edges is $t_1 - t_3$.

Figure 2. GKM graphs $\mathcal{G}_-$, $\mathcal{G}$, and $\mathcal{G}_+$ when $h = (2,3,3)$ and $d = 2$.

## 4. Setting for the proof of the modular law

In this section we set up notations for the proof of the modular law for $X(h)$. The same argument works for $Y(h)$ with a little modification and we point out the modification in Section 6.

Since $H^*_T(X(h))$ is isomorphic to $H^*_T(X(h^t))$ as graded $\mathfrak{S}_n$-modules (see Remark 3.3), it suffices to check the modular relation (2.1) for modular triples $(h_-, h, h_+)$ of type (C) in Definition 2.1 (see Remark 2.5); so

$$h(d) = h(d+1) \quad \text{and} \quad h^{-1}(d) = \{d_0\} \quad \text{for some } 1 \leqslant d_0 < d < n$$

and

$$h_-(j) = \begin{cases} d-1 & \text{for } j = d_0 \\ h(j) & \text{otherwise} \end{cases} \qquad \text{and} \qquad h_+(j) = \begin{cases} d+1 & \text{for } j = d_0 \\ h(j) & \text{otherwise.} \end{cases}$$

To simplify the notations, we set

$$\mathcal{G}_- := \mathcal{G}_X(h_-), \quad \mathcal{G} := \mathcal{G}_X(h), \quad \mathcal{G}_+ := \mathcal{G}_X(h_+)$$

where the vertex sets are all $\mathfrak{S}_n$ and the edge sets are respectively

- $E(\mathcal{G}_-) = \{\{w, w(i,j)\} \mid w \in \mathfrak{S}_n,\ j < i \leqslant h_-(j)\} = E(\mathcal{G}) \setminus \{\{w, w(d,d_0)\} \mid w \in \mathfrak{S}_n\}$,
- $E(\mathcal{G}) = \{\{w, w(i,j)\} \mid w \in \mathfrak{S}_n,\ j < i \leqslant h(j)\}$,
- $E(\mathcal{G}_+) = \{\{w, w(i,j)\} \mid w \in \mathfrak{S}_n,\ j < i \leqslant h_+(j)\} = E(\mathcal{G}) \cup \{\{w, w(d+1,d_0)\} \mid w \in \mathfrak{S}_n\}$,

and the labelings on them are the same as $\alpha_X$ in (3.5), see Figure 2.

We consider two more labeled graphs ${}^\circ\mathcal{G}$ and $\widetilde{\mathcal{G}}$ defined as follows:

(1) $V({}^\circ\mathcal{G}) = \{{}^\circ w \mid w \in \mathfrak{S}_n\}$ where ${}^\circ w$ is a copy of $w$,
(2) $E({}^\circ\mathcal{G}) = \{\{{}^\circ w, {}^\circ w(i,j)\} \mid j < i \leqslant h_-(j)\} \cup \{\{{}^\circ w, {}^\circ w(d+1, d_0)\}\}$, where $w \in \mathfrak{S}_n$ and the label on $\{{}^\circ w, {}^\circ w(i,j)\}$ is $t_{w(i)} - t_{w(j)}$,

and

(1) $V(\widetilde{\mathcal{G}}) = V(\mathcal{G}) \cup V({}^\circ\mathcal{G})$,
(2) $E(\widetilde{\mathcal{G}}) = E(\mathcal{G}) \cup E({}^\circ\mathcal{G}) \cup \{\{w, {}^\circ w\} \mid w \in \mathfrak{S}_n\}$, where the label on the edge $\{w, {}^\circ w\}$ is $t_{w(d+1)} - t_{w(d)}$,

see Figure 3.

As a labeled graph, $\widetilde{\mathcal{G}}$ is considered as blowing up $\mathcal{G}_+$ along the subgraph $\mathcal{G}_-$ (cf. [11, Example 7]). Note that $\mathcal{G}, {}^\circ\mathcal{G}$, and $\mathcal{G}_-$ are full subgraphs of $\widetilde{\mathcal{G}}$. We denote

$$\tau = (d+1, d) \in \mathfrak{S}_n$$

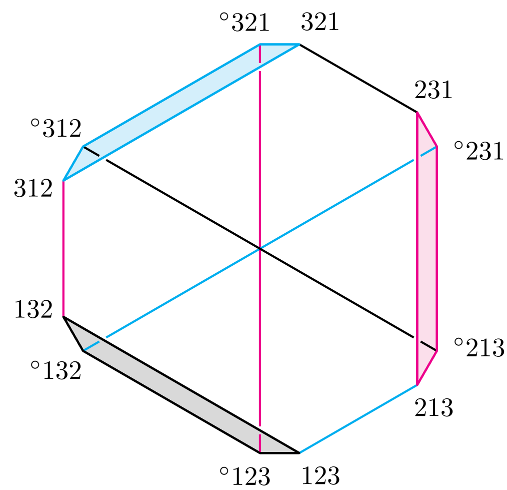



FIGURE 3. Labeled graph $\widetilde{\mathcal{G}}$ with emphasized 4-gons when $h = (2,3,3)$ and $d = 2$.

in the following.

LEMMA 4.1. *The map $\Phi\colon \mathcal{G} \to {}^\circ\mathcal{G}$ sending $w \in V(\mathcal{G})$ to ${}^\circ w\tau \in V({}^\circ\mathcal{G})$ gives a bijection between their edges preserving the labels, so $\mathcal{G}$ and ${}^\circ\mathcal{G}$ are isomorphic as labeled graphs.*

*Proof.* Noting $(i,j)\tau = \tau(\tau(i),\tau(j))$, one can easily check that $\{w, w(i,j)\}$ is an edge of $\mathcal{G}$ if and only if $\{{}^\circ w\tau, {}^\circ w(i,j)\tau\}$ is an edge of ${}^\circ\mathcal{G}$ and that the label on the edge $\{{}^\circ w\tau, {}^\circ w(i,j)\tau\}$ of ${}^\circ\mathcal{G}$ is $t_{w(i)} - t_{w(j)}$ which agrees with the label on $\{w, w(i,j)\}$ of $\mathcal{G}$. This proves the lemma. □

Unfortunately, $\widetilde{\mathcal{G}}$ is not a GKM graph of any GKM manifold since $\widetilde{\mathcal{G}}$ is not 2-independent. Indeed, the four edges

$$\{w, w\tau\},\ \{{}^\circ w, {}^\circ w\tau\},\ \{w, {}^\circ w\},\ \{w\tau, {}^\circ w\tau\},$$

which form a 4-gon, have the same label $t_{w(d+1)} - t_{w(d)}$ for each $w \in \mathfrak{S}_n$, see Figure 3. We shall assign $\widetilde{\mathcal{G}}$ an additional data. It is a map $s\colon V(\widetilde{\mathcal{G}}) \to \{\pm 1\}$ defined by

$$s(w) = +1, \quad s({}^\circ w) = -1 \quad \text{for any } w \in \mathfrak{S}_n.$$

The geometrical meaning of this sign is explained in Appendix A.

DEFINITION 4.2. *The equivariant cohomology ring of the pair $(\widetilde{\mathcal{G}}, s)$ is defined as follows. Let $K_w$ be the 4-gon with vertex set $\{w, {}^\circ w, w\tau, {}^\circ w\tau\}$. Then*

$$H_T^*(\widetilde{\mathcal{G}}, s) = \left\{ f \in H_T^*(\widetilde{\mathcal{G}}) \;\middle|\; \sum_{v \in V(K_w)} s(v) f(v) \equiv 0 \bmod (t_{w(d)} - t_{w(d+1)})^2 \ \forall w \in \mathfrak{S}_n \right\}.$$

*In other words, $H_T^*(\widetilde{\mathcal{G}}, s)$ consists of all $f \in H_T^*(\widetilde{\mathcal{G}})$ satisfying*

$$f(w) - f({}^\circ w) + f(w\tau) - f({}^\circ w\tau) \equiv 0 \mod (t_{w(d+1)} - t_{w(d)})^2, \tag{4.1}$$

*for any $w \in \mathfrak{S}_n$.*

The dot actions on $H_T^*({}^\circ\mathcal{G})$ and $H_T^*(\widetilde{\mathcal{G}}, s)$ are defined in the same way as that on $H_T^*(\mathcal{G})$, that is, $(\sigma \cdot f)({}^\circ w) = \sigma(f({}^\circ \sigma^{-1} w))$ for $\sigma \in \mathfrak{S}_n$.

Let $x_i$ be an element of $\mathrm{Map}(\mathfrak{S}_n \cup {}^\circ\mathfrak{S}_n, \mathbb{C}[t_1,\dots,t_n])$ defined by

$$x_i({}^\circ w\tau) = x_i(w) = t_{w(i)}$$

for $w \in \mathfrak{S}_n$. Then $x_i \in H_T^*(\widetilde{\mathcal{G}}, s)$. We write $x_i$ also for the restriction of $x_i$ onto $\mathcal{G}$ or ${}^\circ\mathcal{G}$. On $\mathcal{G}$, $x_i$ is the equivariant Euler class of the tautological line bundle $\{(V_\bullet, v) \mid v \in V_i/V_{i-1}\}$ over $X(h)$.

We consider the following four maps. We show in Lemma 4.3 below that the image of each map is contained in $H_T^*(\widetilde{\mathcal{G}}, s)$.

(1) $\varphi\colon H_T^*({}^\circ\mathcal{G}) \to H_T^*(\widetilde{\mathcal{G}}, s)$ is an $H^*(BT)$-algebra map defined by

$$\varphi(f)(w) := f({}^\circ w\tau), \qquad \varphi(f)({}^\circ w) := f({}^\circ w).$$

(2) $\psi_!\colon H_T^{*-2}(\mathcal{G}) \to H_T^*(\widetilde{\mathcal{G}}, s)$ is an $H^*(BT)$-module map defined by

$$\psi_!(f)(w) := ((x_{d+1} - x_d)f)(w), \qquad \psi_!(f)({}^\circ w) := 0.$$

(3) $\eta\colon H_T^*(\mathcal{G}_+) \to H_T^*(\widetilde{\mathcal{G}}, s)$ is an $H^*(BT)$-algebra map defined by

$$\eta(f)(w) = \eta(f)({}^\circ w) := f(w).$$

(4) $\rho_!\colon H_T^{*-2}(\mathcal{G}_-) \to H_T^*(\widetilde{\mathcal{G}}, s)$ is an $H^*(BT)$-module map defined by

$$\rho_!(f)(w) := ((x_d - x_{d_0})f)(w), \qquad \rho_!(f)({}^\circ w) := ((x_{d+1} - x_{d_0})f)(w).$$

One can easily check that all the maps above commute with the dot action.

Lemma 4.3. *The images of all the four maps above are contained in* $H_T^*(\widetilde{\mathcal{G}}, s)$.

*Proof.* $E(\widetilde{\mathcal{G}})$ consists of three classes $E(\mathcal{G})$, $E({}^\circ\mathcal{G})$, and $\{\{w, {}^\circ w\} \mid w \in \mathfrak{S}_n\}$. We check the congruence relation in (3.1) for each class, and the congruence relation (4.1).

(1) The congruence relation in (3.1) for $E(\mathcal{G})$ follows from Lemma 4.1 and is obvious for $E({}^\circ\mathcal{G})$ since $f \in H_T^*({}^\circ\mathcal{G})$. As for $\{w, {}^\circ w\}$, we have

$$\varphi(f)(w) - \varphi(f)({}^\circ w) = f({}^\circ w\tau) - f({}^\circ w) \equiv 0 \mod (t_{w(d+1)} - t_{w(d)})$$

because $f \in H_T^*({}^\circ\mathcal{G})$ and $\tau = (d+1, d)$. The congruence relation (4.1) follows from

$$\varphi(f)(w) - \varphi(f)({}^\circ w) + \varphi(f)(w\tau) - \varphi(f)({}^\circ w\tau) = f({}^\circ w\tau) - f({}^\circ w) + f({}^\circ w) - f({}^\circ w\tau) = 0.$$

(2) The congruence relation in (3.1) for $E(\mathcal{G})$ and $\{w, {}^\circ w\}$ is obvious and that for $E({}^\circ\mathcal{G})$ is trivial. The congruence relation (4.1) follows from

$$\begin{aligned}
&\psi_!(f)(w) - \psi_!(f)({}^\circ w) + \psi_!(f)(w\tau) - \psi_!(f)({}^\circ w\tau) \\
&= (t_{w(d+1)} - t_{w(d)})f(w) + (t_{w\tau(d+1)} - t_{w\tau(d)})f(w\tau) \\
&\equiv (t_{w(d+1)} - t_{w(d)})(f(w) - f(w\tau)) \equiv 0 \mod (t_{w(d+1)} - t_{w(d)})^2,
\end{aligned}$$

where the last congruence relation holds because $f \in H_T^*(\mathcal{G})$ and $\tau = (d+1, d)$.

(3) The congruence relations in (3.1) and (4.1) are obvious from the definition of $\eta(f)$.

(4) The congruence relations in (3.1) for $E(\mathcal{G}) \setminus \{w, w(d_0, d)\}$ and $E({}^\circ\mathcal{G}) \setminus \{{}^\circ w, {}^\circ w(d_0, d+1)\}$ are easily checked as in the case of $\psi_!$. Those for $\{w, w(d_0, d)\}$ and $\{{}^\circ w, {}^\circ w(d_0, d+1)\}$ are obvious by the definition of $\rho_!$. As for $\{w, {}^\circ w\}$, we have

$$\begin{aligned}
\rho_!(f)(w) - \rho_!(f)({}^\circ w) &= (t_{w(d)} - t_{w(d_0)})f(w) - (t_{w(d+1)} - t_{w(d_0)})f(w) \\
&= (t_{w(d)} - t_{w(d+1)})f(w) \equiv 0 \mod (t_{w(d+1)} - t_{w(d)}).
\end{aligned}$$

The congruence relation (4.1) follows from

$$\begin{aligned}
&\rho_!(f)(w) - \rho_!(f)(^\circ w) + \rho_!(f)(w\tau) - \rho_!(f)(^\circ w\tau) \\
&= (t_{w(d)} - t_{w(d+1)})f(w) + (t_{w\tau(d)} - t_{w\tau(d+1)})f(w\tau) \\
&= (t_{w(d)} - t_{w(d+1)})(f(w) - f(w\tau)) \equiv 0 \mod (t_{w(d+1)} - t_{w(d)})^2.
\end{aligned}$$

The last congruence relation holds because $f \in H_T^*(\mathcal{G}_-)$ and $\tau = (d+1, d)$.

□

## 5. Proof of the modular law for geometric sides

Under the set up in Section 4, we prove the following.

Theorem 5.1. *The homomorphisms*

$$\varphi + \psi_! \colon H_T^*(^\circ\mathcal{G}) \oplus H_T^{*-2}(\mathcal{G}) \to H_T^*(\widetilde{\mathcal{G}}, s), \qquad (\varphi + \psi_!)(f, g) = \varphi(f) + \psi_!(g)$$

*and*

$$\eta + \rho_! \colon H_T^*(\mathcal{G}_+) \oplus H_T^{*-2}(\mathcal{G}_-) \to H_T^*(\widetilde{\mathcal{G}}, s), \qquad (\eta + \rho_!)(f_+, f_-) = \eta(f_+) + \rho_!(f_-)$$

*are both isomorphisms as $\mathfrak{S}_n$-modules.*

Since $^\circ\mathcal{G}$ is isomorphic to $\mathcal{G}$ as labeled graphs by Lemma 4.1, we obtain the following corollary.

Corollary 5.2. *There is an isomorphism*

$$H^*(\mathcal{G}) \oplus H^{*-2}(\mathcal{G}) \cong H^*(\mathcal{G}_+) \oplus H^{*-2}(\mathcal{G}_-)$$

*as $\mathfrak{S}_n$-modules. Therefore $H^*(X(h))$ satisfies the modular law.*

The rest of this section is devoted to the proof of Theorem 5.1.

*Proof of the former part in Theorem 5.1.* Clearly $\varphi(H_T^*(^\circ\mathcal{G})) \cap \psi_!(H_T^{*-2}(\mathcal{G})) = \{0\}$ and $\varphi, \psi_!$ are injective, so it suffices to show the surjectivity of $\varphi + \psi_!$.

Take any element $\tilde{f} \in H_T^*(\widetilde{\mathcal{G}}, s)$ and denote its restriction to $^\circ\mathcal{G}$ by $f$. Then $f \in H_T^*(^\circ\mathcal{G})$ and $\tilde{f} - \varphi(f)$ vanishes on $V(^\circ\mathcal{G})$ by definition of $\varphi$. Therefore, there is $g \in \mathrm{Map}(V(\mathcal{G}), H^*(BT))$ such that

$$(\tilde{f} - \varphi(f))(w) = (t_{w(d+1)} - t_{w(d)})g(w) \quad \text{for } w \in V(\mathcal{G}) = \mathfrak{S}_n. \tag{5.1}$$

We show that $g \in H_T^{*-2}(\mathcal{G})$. Then (5.1) means that $\tilde{f} - \varphi(f) = \psi_!(g)$, proving the surjectivity of $\varphi + \psi_!$.

We shall check that $g$ satisfies the congruence relation in (3.1) for $\mathcal{G}$. It follows from (5.1) and the definitions of $\varphi$ and $f$ that

$$(t_{w(d+1)} - t_{w(d)})g(w) = (\tilde{f} - \varphi(f))(w) = \tilde{f}(w) - f(^\circ w\tau) = \tilde{f}(w) - \tilde{f}(^\circ w\tau). \tag{5.2}$$

Let $j < i \leqslant h(j)$ and set $v = w(i,j)$. We note that since $x_d$ and $x_{d+1} \in H_T^*(\mathcal{G})$ and $v = w(i,j)$, we have

$$\begin{aligned}
t_{v(d+1)} - t_{v(d)} &= x_{d+1}(v) - x_d(v) \\
&\equiv x_{d+1}(w) - x_d(w) = t_{w(d+1)} - t_{w(d)} \mod (t_{w(i)} - t_{w(j)}).
\end{aligned} \tag{5.3}$$

We also note that

$$\tilde{f}(w) - \tilde{f}(v) \equiv 0 \equiv \tilde{f}(^\circ w\tau) - \tilde{f}(^\circ v\tau) \mod (t_{w(i)} - t_{w(j)}) \tag{5.4}$$

since $\tilde{f} \in H_T^*(\widetilde{\mathcal{G}}, s)$ and the labels on $\{w, v\}$ and $\{{}^\circ w\tau, {}^\circ v\tau\}$ are the same, namely $t_{w(i)} - t_{w(j)}$, by Lemma 4.1. Then, it follows from (5.3), (5.2), and (5.4) that

$$\begin{aligned}(t_{w(d+1)} - t_{w(d)})(g(w) - g(v)) &\equiv (t_{w(d+1)} - t_{w(d)})g(w) - (t_{v(d+1)} - t_{v(d)})g(v) \\ &= \tilde{f}(w) - \tilde{f}({}^\circ w\tau) - (\tilde{f}(v) - \tilde{f}({}^\circ v\tau)) \\ &\equiv 0 \mod (t_{w(i)} - t_{w(j)}).\end{aligned}$$

Since $H^*(BT)$ is a polynomial ring, the congruence relation above implies

$$g(w) \equiv g(v) \mod (t_{w(i)} - t_{w(j)}) \quad \text{when } (i,j) \neq (d+1, d). \tag{5.5}$$

When $(i,j) = (d+1,d)$, we have $v = w\tau$. Then, since $(\tau(d+1), \tau(d)) = (d, d+1)$, we have

$$\begin{aligned}&(t_{w(d+1)} - t_{w(d)})(g(w) - g(w\tau)) \\ &= (t_{w(d+1)} - t_{w(d)})g(w) + (t_{w\tau(d+1)} - t_{w\tau(d)})g(w\tau) \\ &= \tilde{f}(w) - \tilde{f}({}^\circ w\tau) + \tilde{f}(w\tau) - \tilde{f}({}^\circ w) \equiv 0 \mod (t_{w(d+1)} - t_{w(d)})^2,\end{aligned}$$

where the second identity follows from (5.2) and the last congruence relation follows from (4.1) for $\tilde{f} \in H_T^*(\widetilde{\mathcal{G}}, s)$. Hence we obtain

$$g(w) \equiv g(w\tau) \mod (t_{w(d+1)} - t_{w(d)}).$$

This together with (5.5) shows that $g \in H_T^{*-2}(\mathcal{G})$. □

*Proof of the latter part in Theorem 5.1.* It is easy to see $\eta(H_T^*(\mathcal{G}_+)) \cap \rho_!(H_T^{*-2}(\mathcal{G}_-)) = \{0\}$ and $\eta, \rho_!$ are injective, so it suffices to prove the surjectivity of $\eta + \rho_!$.

Take any element $\tilde{f} \in H_T^*(\widetilde{\mathcal{G}}, s)$. Since the label on the edge $\{w, {}^\circ w\}$ is $t_{w(d+1)} - t_{w(d)}$, there is $p \in \operatorname{Map}(\mathfrak{S}_n, H^*(BT))$ such that

$$\tilde{f}(w) - \tilde{f}({}^\circ w) = (t_{w(d+1)} - t_{w(d)})p(w) \quad \text{for } w \in \mathfrak{S}_n. \tag{5.6}$$

We show that $p \in H_T^*(\mathcal{G}_-)$. The argument is the same as in the former case for $g$ in (5.1) being in $H_T^{*-2}(\mathcal{G})$.

For a transposition $(i,j)$ with $j < i \leqslant h_-(j)$, we set $v = w(i,j)$ as before. Then, since $\tilde{f} \in H_T^*(\widetilde{\mathcal{G}}, s)$, we have

$$\tilde{f}(w) - \tilde{f}(v) \equiv 0 \equiv \tilde{f}({}^\circ w) - \tilde{f}({}^\circ v) \mod (t_{w(i)} - t_{w(j)}). \tag{5.7}$$

It follows from (5.3), (5.6) and (5.7) that

$$\begin{aligned}(t_{w(d+1)} - t_{w(d)})(p(w) - p(v)) &\equiv (t_{w(d+1)} - t_{w(d)})p(w) - (t_{v(d+1)} - t_{v(d)})p(v) \\ &= \tilde{f}(w) - \tilde{f}({}^\circ w) - (\tilde{f}(v) - \tilde{f}({}^\circ v)) \\ &\equiv 0 \mod (t_{w(i)} - t_{w(j)}).\end{aligned}$$

Therefore,

$$p(w) \equiv p(v) \mod (t_{w(i)} - t_{w(j)}) \quad \text{when } (i,j) \neq (d+1, d). \tag{5.8}$$

When $(i,j) = (d+1,d)$, we have $v = w\tau$ and it follows from (5.6) and (4.1) for $\tilde{f}$ that

$$\begin{aligned}(t_{w(d+1)} - t_{w(d)})(p(w) - p(w\tau)) &= \tilde{f}(w) - \tilde{f}({}^\circ w) + \tilde{f}(w\tau) - \tilde{f}({}^\circ w\tau) \\ &\equiv 0 \mod (t_{w(d+1)} - t_{w(d)})^2.\end{aligned}$$

Hence we obtain

$$p(w) \equiv p(w\tau) \mod (t_{w(d+1)} - t_{w(d)}).$$

This together with (5.8) shows that $p \in H_T^{*-2}(\mathcal{G}_-)$.

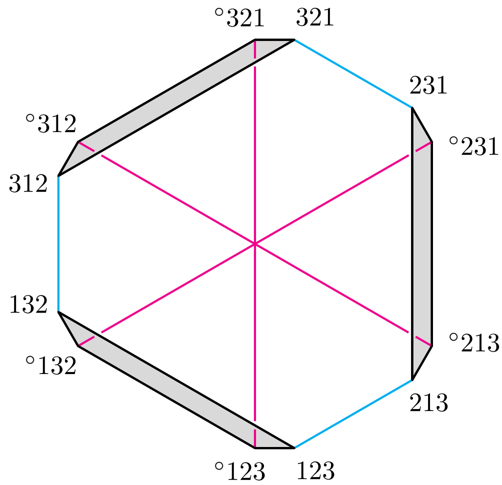



Figure 4. Labeled graph $\widetilde{\mathcal{G}}_Y$ with emphasized 4-gons when $h = (2,3,3)$ and $d = 2$.

We shall observe that $\tilde{f} + \rho_!(p) \in \eta(H_T^*(\mathcal{G}_+))$, which implies the surjectivity of $\eta + \rho_!$. It follows from the definition of $\rho_!$ that

$$\begin{aligned}(\tilde{f} + \rho_!(p))(w) &= \tilde{f}(w) + (t_{w(d)} - t_{w(d_0)})p(w),\\ (\tilde{f} + \rho_!(p))({}^\circ w) &= \tilde{f}({}^\circ w) + (t_{w(d+1)} - t_{w(d_0)})p(w)\\ &= \tilde{f}(w) - (t_{w(d+1)} - t_{w(d)})p(w) + (t_{w(d+1)} - t_{w(d_0)})p(w) \quad \text{(by (5.6))}\\ &= \tilde{f}(w) + (t_{w(d)} - t_{w(d_0)})p(w).\end{aligned}$$

Therefore,

$$(\tilde{f} + \rho_!(p))(w) = (\tilde{f} + \rho_!(p))({}^\circ w) \quad \text{for any } w \in \mathfrak{S}_n. \tag{5.9}$$

Moreover, since $\tilde{f} + \rho_!(p) \in H_T^*(\widetilde{\mathcal{G}}, s)$, $\tilde{f} + \rho_!(p)$ restricted to the subgraphs $\mathcal{G}$ and ${}^\circ\mathcal{G}$ satisfies the congruence relation for them and hence for $\mathcal{G}_+$. This together with (5.9) shows that $\tilde{f} + \rho_!(f) \in \eta(H_T^*(\mathcal{G}_+))$, proving the surjectivity of $\eta + \rho_!$. □

## 6. The case of twins

In this section we point out how our argument changes for the proof of the modular law for twins $Y(h)$.

The following remark corresponds to Remark 3.3 and it is sufficient to consider the modular triple of type (C).

Remark 6.1. The correspondence $Tg \mapsto TgJ$ gives a diffeomorphism $Y(h) \cong Y(h^t)$. In terms of GKM graphs, an edge $\{w, w(i,j)\}$ corresponds to $\{ww_0, w(i,j)w_0\}$. This gives an isomorphism between GKM graphs of $Y(h)$ and $Y(h^t)$ with a change of the labels $t_i - t_j \mapsto w_0(t_i - t_j) = t_{n+1-i} - t_{n+1-j}$. This induces a weakly $H^*(BT)$-algebra isomorphism $H_T^*(Y(h^t)) \to H_T^*(Y(h))$ which commutes with the dagger action (3.13).

We define labeled graphs ${}^\circ\mathcal{G}_Y$ and $\widetilde{\mathcal{G}}_Y$ as follows: the underlying graphs are the same as in the case of $X(h)$, and the label on each edge is changed from $t_{w(i)} - t_{w(j)}$ to $t_i - t_j$.

Lemma 6.2. *The map* $\Phi\colon w \mapsto {}^\circ w\tau$ *gives an isomorphism from* $\mathcal{G}_Y$ *to* ${}^\circ\mathcal{G}_Y$ *through the automorphism* $\tau$ *that exchanges* $t_d$ *and* $t_{d+1}$*. In particular,* $H_T^*({}^\circ\mathcal{G}_Y) \cong H_T^*(\mathcal{G}_Y)$ *as* $\mathfrak{S}_n$*-modules by* $f \mapsto \tau(f \circ \Phi)$*.*

In the case of $Y(h)$, $s$ changes to

$$s_Y(w) = (-1)^{l(w)}, \quad s_Y({}^\circ w) = (-1)^{l(w)+1} \quad \text{for any } w \in \mathfrak{S}_n,$$

where $l$ is the length function. Then we have

$$H_T^*(\widetilde{\mathcal{G}}_Y, s_Y) = \left\{ f \in H_T^*(\widetilde{\mathcal{G}}_Y) \,\middle|\, \sum_{v \in V(K_w)} s_Y(v) f(v) \equiv 0 \bmod (t_d - t_{d+1})^2 \ \forall w \in \mathfrak{S}_n \right\}.$$

In other words, $H_T^*(\widetilde{\mathcal{G}}_Y, s_Y)$ consists of all $f \in H_T^*(\widetilde{\mathcal{G}}_Y)$ satisfying

$$f(w) - f({}^\circ w) + f({}^\circ w\tau) - f(w\tau) \equiv 0 \mod (t_{d+1} - t_d)^2$$

for any $w \in \mathfrak{S}_n$.

In Section 4, we defined 4 maps $\varphi, \psi_!, \eta, \rho_!$. The necessary changes for them are replacing $x$ with $t$ and redefining $\varphi(f)(w) := \tau(f({}^\circ w\tau))$. This change arises from the change of labels. Note that $\varphi$ is a weakly $H^*(BT)$-algebra map in this case. Then all the maps commute with the dagger action. The dagger actions on $H_T^*({}^\circ\mathcal{G}_Y)$ and $H_T^*(\widetilde{\mathcal{G}}_Y, s_Y)$ are defined in the same way as that on $H_T^*(\mathcal{G}_Y)$, that is, $(\sigma \dagger f)({}^\circ w) = f({}^\circ \sigma^{-1} w)$ for $\sigma \in \mathfrak{S}_n$.

## Appendix A. Geometrical counterpart

We shall show a geometrical object corresponding to $\widetilde{\mathcal{G}}$ for geometrical understanding. It is $\widetilde{X}(h)$ defined as follows. When $V_\bullet \in X(h_+)$, the dimension of $SV_{d_0}/(V_{d-1} \cap SV_{d_0})$ is 1 or 0 since $SV_{d_0-1} \subset V_{h(d_0-1)} \subset V_{d-1}$. In particular, $\dim SV_{d_0}/(V_{d-1} \cap SV_{d_0}) = 0$ implies $SV_{d_0} \subset V_{d-1}$ and then $V_\bullet \in X(h_-)$. Hence

$$\widetilde{X}(h) = \{(V_\bullet, l) \,|\, V_\bullet \in X(h_+),\, l \in P(V_{d+1}/V_{d-1}),\, SV_{d_0} \subset l + V_{d-1}\}$$

is the blow-up of $X(h_+)$ along $X(h_-)$. Moreover, for $(V_\bullet, l)$, the correspondence

$$(V_\bullet, l) \mapsto (V_0 \subset V_1 \subset \cdots \subset V_{d-1} \subset l + V_{d-1} \subset V_{d+1} \subset \cdots \subset V_n), \tag{A.1}$$

that is, replacing $V_d$ by $l + V_{d-1}$ gives a map $\widetilde{X}(h) \to X(h)$ and it is a fiber bundle with fiber $\mathbb{C}P^1$ since $h(d) = h(d+1)$. This fiber $\mathbb{C}P^1$ means the direction of a line $V_d/V_{d-1}$ in a plane $V_{d+1}/V_{d-1}$. The torus $T$ acts on $\widetilde{X}(h)$ naturally and the fixed point set is

$$\widetilde{X}(h)^T = \left\{ (V_\bullet(w), l) \,\middle|\, w \in \mathfrak{S}_n,\, l = \langle \mathbf{e}_{w(d)} \rangle \text{ or } \langle \mathbf{e}_{w(d+1)} \rangle \right\},$$

where $V_\bullet(w)$ denotes the permutation flag associated with $w$. We denote the fixed point $(V_\bullet(w), \langle \mathbf{e}_{w(d)} \rangle)$ as $w$ and $(V_\bullet(w), \langle \mathbf{e}_{w(d+1)} \rangle)$ as ${}^\circ w$ in $\widetilde{\mathcal{G}}$. The 4-gon $K_w$ with vertex set $\{w, {}^\circ w, w\tau, {}^\circ w\tau\}$ in $\widetilde{\mathcal{G}}$ corresponds to $\mathbb{C}P^1 \times \mathbb{C}P^1$ in $\widetilde{X}(h)$ which is the inverse image of $\{V_\bullet \mid V_k = V_k(w) \text{ for } k \neq d\} = \mathbb{C}P^1 \subset X(h)$ under (A.1). See Figure 5, and note that

- taking $V_\bullet = V_\bullet(w)$ or $V_\bullet(w\tau)$ leads us to $\{0, \infty\} \times \mathbb{C}P^1$ which connects $w$ with ${}^\circ w$ and $w\tau$ with ${}^\circ w\tau$,
- taking $l = \langle \mathbf{e}_{w(d)} \rangle$ or $\langle \mathbf{e}_{w(d+1)} \rangle$ leads us to $\mathbb{C}P^1 \times \{0, \infty\}$ which connects $w$ with ${}^\circ w\tau$ and $w\tau$ with ${}^\circ w$,
- taking $l = V_d/V_{d-1}$ or $(V_d/V_{d-1})^\perp$ leads us to the "diagonal" $\mathbb{C}P^1$'s which connect $w$ with $w\tau$ and ${}^\circ w$ with ${}^\circ w\tau$.

This geometrical situation warrants the signs of vertices in (4.1).

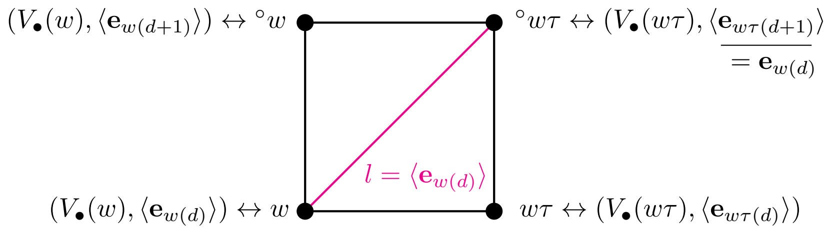



FIGURE 5. $K_w$ and the geometrical counterpart.

REMARK A.1. The labeled graph $\widetilde{\mathcal{G}}_Y$ corresponds to the blow-up $\widetilde{Y}(h)$ of $Y(h_+)$ along $Y(h_-)$, where Kiem and Lee write $Y(h)$ as $Y_h$ and $\widetilde{Y}(h)$ as $\widetilde{Y}_{\mathbf{h}}$ in [14]. See [14, (4.10) and (4.12)] for $\mathbb{C}P^1 \times \mathbb{C}P^1$ in $\widetilde{Y}(h)$ to warrant $s_Y$ in Section 6.

REMARK A.2. At the end of this appendix, we mention an interesting phenomenon. Recall that we have a fiber bundle

$$\mathbb{C}P^1 \to \widetilde{X}(h) \to X(h).$$

On the other hand, according to Kiem and Lee [14], there is a "reverse" fiber bundle

$$Y(h) \to \widetilde{Y}(h) \to \mathbb{C}P^1.$$

These fibrations can be described as homomorphisms of the corresponding labeled graphs. A graph homomorphism may collapse an edge to a vertex. Note that, for $f \in H_T^*(\widetilde{\mathcal{G}})$, we have

$$f(w) - f({}^\circ w\tau) = f(w) - f(w\tau) + f(w\tau) - f({}^\circ w\tau) \equiv 0 \mod (t_{w(d+1)} - t_{w(d)}).$$

In this remark, we redefine $\widetilde{\mathcal{G}}$ so that it has additional edges $\{\{w, {}^\circ w\tau\} \mid w \in \mathfrak{S}_n\}$ with label $t_{w(d+1)} - t_{w(d)}$. This modification does not change its (equivariant) cohomology ring. The labeled graph homomorphism corresponding to $\widetilde{X}(h) \to X(h)$ defined by (A.1) is given by

$$p\colon \widetilde{\mathcal{G}} \to \mathcal{G}, \quad p(w) = p({}^\circ w\tau) = w$$

with a fiber GKM graph which is the induced labeled subgraph on $\{w, {}^\circ w\tau\}$ (see Figure 5). The fiber is the GKM graph of $\mathbb{C}P^1 = P(\langle \mathbf{e}_{w(d)}, \mathbf{e}_{w(d+1)} \rangle)$. On the other hand, let $\Gamma_\tau$ be the GKM graph of $\mathbb{C}P^1 = P(\langle \mathbf{e}_d, \mathbf{e}_{d+1} \rangle)$ with vertices 0 and $\infty$. Then we have

$$q\colon \widetilde{\mathcal{G}}_Y \to \Gamma_\tau, \quad q(w) = 0,\ q({}^\circ w) = \infty$$

with a fiber GKM graph $\mathcal{G}_Y \cong {}^\circ\mathcal{G}_Y$.

## APPENDIX B. UNICELLULAR LLT POLYNOMIALS

It is known that unicellular LLT polynomials satisfy the modular law. For example, see [4, Proposition 18]. We give an elementary proof for readers' convenience.

THEOREM B.1. *Unicellular LLT polynomials satisfy the modular law.*

*Proof.* Let $1 \leqslant d_0 < d < n$ and $h$ be a Hessenberg function which satisfies the condition (C), that is, $h(d) = h(d+1)$ and $h^{-1}(d) = \{d_0\}$. Then $h_-$ and $h_+$ are as follows

$$h_-(j) = \begin{cases} d-1 & (j = d_0) \\ h(j) & (j \neq d_0) \end{cases} \quad \text{and} \quad h_+(j) = \begin{cases} d+1 & (j = d_0) \\ h(j) & (j \neq d_0) \end{cases}.$$

For this modular triple $(h_-, h, h_+)$, what we have to show is

$$\mathrm{LLT}_{h_+}(q) - \mathrm{LLT}_h(q) = q(\mathrm{LLT}_h(q) - \mathrm{LLT}_{h_-}(q)). \tag{B.1}$$

Let $C = C(G_{h_-})$ be the set of all colorings of $G_{h_-}$, that is, all maps $[n] \to \mathbb{P}$. Then $C$ is decomposed into the following four subsets:

- $C_{<<} = \{\kappa \in C \mid \kappa(d_0) < \kappa(d),\ \kappa(d_0) < \kappa(d+1)\}$
- $C_{<\geqslant} = \{\kappa \in C \mid \kappa(d_0) < \kappa(d),\ \kappa(d_0) \geqslant \kappa(d+1)\}$
- $C_{\geqslant<} = \{\kappa \in C \mid \kappa(d_0) \geqslant \kappa(d),\ \kappa(d_0) < \kappa(d+1)\}$
- $C_{\geqslant\geqslant} = \{\kappa \in C \mid \kappa(d_0) \geqslant \kappa(d),\ \kappa(d_0) \geqslant \kappa(d+1)\}$.

Let

$$\mathrm{Asc}_-(\kappa) = \left\{\{i,j\} \in E(G_{h_-}) \,\middle|\, j < i,\ \kappa(j) < \kappa(i)\right\}.$$

For a coloring $\kappa$ of $G_{h_-}$, we write $\mathrm{asc}_-(\kappa)$ instead of $\mathrm{asc}(\kappa)$ in Appendices to avoid confusions. By definition

$$\mathrm{asc}_-(\kappa) = |\mathrm{Asc}_-(\kappa)|.$$

Note that $\{d+1, d\}$ is an edge of $G_{h_-}$ and then

- $\kappa \in C_{<\geqslant} \Rightarrow \kappa(d) > \kappa(d+1) \Leftrightarrow \{d+1, d\} \notin \mathrm{Asc}_-(\kappa)$ and
- $\kappa \in C_{\geqslant<} \Rightarrow \kappa(d) < \kappa(d+1) \Leftrightarrow \{d+1, d\} \in \mathrm{Asc}_-(\kappa)$.

Let $\tau = (d+1, d) \in \mathfrak{S}_n$. For a coloring $\kappa$ of $G_{h_-}$, the composition $\kappa \circ \tau$ is also a coloring of $G_{h_-}$. Then the composition with $\tau$ gives a bijection $C_{<\geqslant} \to C_{\geqslant<}$. When $e \neq d, d+1$, we have

$$\{d, e\} \in \mathrm{Asc}_-(\kappa) \Leftrightarrow \{d+1, e\} \in \mathrm{Asc}_-(\kappa \circ \tau)$$

and

$$\{d+1, e\} \in \mathrm{Asc}_-(\kappa) \Leftrightarrow \{d, e\} \in \mathrm{Asc}_-(\kappa \circ \tau).$$

Hence, for $\kappa \in C_{<\geqslant}$, we have

$$\mathrm{asc}_-(\kappa) + 1 = \mathrm{asc}_-(\kappa \circ \tau) \tag{B.2}$$

since $\{d+1, d\} \notin \mathrm{Asc}_-(\kappa)$ and $\{d+1, d\} \in \mathrm{Asc}_-(\kappa \circ \tau)$. Since the subscripts of the decomposition of $C$ corresponds to the ascents of the edges $\{d, d_0\}$ and $\{d+1, d_0\}$ which are not contained in $G_{h_-}$, we have

$$\begin{aligned}
\mathrm{LLT}_{h_+}(q) &= q^2 \sum_{\kappa \in C_{<<}} z_\kappa q^{\mathrm{asc}_-(\kappa)} + q \sum_{\kappa \in C_{<\geqslant}} z_\kappa q^{\mathrm{asc}_-(\kappa)} \\
&\quad + q \sum_{\kappa \in C_{\geqslant<}} z_\kappa q^{\mathrm{asc}_-(\kappa)} + \sum_{\kappa \in C_{\geqslant\geqslant}} z_\kappa q^{\mathrm{asc}_-(\kappa)}, \\
\mathrm{LLT}_{h}(q) &= q \sum_{\kappa \in C_{<<}} z_\kappa q^{\mathrm{asc}_-(\kappa)} + q \sum_{\kappa \in C_{<\geqslant}} z_\kappa q^{\mathrm{asc}_-(\kappa)} \\
&\quad + \sum_{\kappa \in C_{\geqslant<}} z_\kappa q^{\mathrm{asc}_-(\kappa)} + \sum_{\kappa \in C_{\geqslant\geqslant}} z_\kappa q^{\mathrm{asc}_-(\kappa)}, \\
\mathrm{LLT}_{h_-}(q) &= \sum_{\kappa \in C_{<<}} z_\kappa q^{\mathrm{asc}_-(\kappa)} + \sum_{\kappa \in C_{<\geqslant}} z_\kappa q^{\mathrm{asc}_-(\kappa)} \\
&\quad + \sum_{\kappa \in C_{\geqslant<}} z_\kappa q^{\mathrm{asc}_-(\kappa)} + \sum_{\kappa \in C_{\geqslant\geqslant}} z_\kappa q^{\mathrm{asc}_-(\kappa)}.
\end{aligned}$$

Now we compute each side of (B.1):

$$\mathrm{LLT}_{h_+}(q) - \mathrm{LLT}_h(q) = (q^2 - q) \sum_{\kappa \in C_{<<}} z_\kappa q^{\mathrm{asc}_-(\kappa)} + (q-1) \sum_{\kappa \in C_{\geqslant <}} z_\kappa q^{\mathrm{asc}_-(\kappa)},$$
$$\mathrm{LLT}_h(q) - \mathrm{LLT}_{h_-}(q) = \ (q-1) \sum_{\kappa \in C_{<<}} z_\kappa q^{\mathrm{asc}_-(\kappa)} + (q-1) \sum_{\kappa \in C_{<\geqslant}} z_\kappa q^{\mathrm{asc}_-(\kappa)}.$$

By (B.2) and the bijection $C_{<\geqslant} \to C_{\geqslant <}$, we have

$$\sum_{\kappa \in C_{<\geqslant}} z_\kappa q^{\mathrm{asc}_-(\kappa)} = \sum_{\kappa \in C_{<\geqslant}} z_\kappa q^{\mathrm{asc}_-(\kappa \circ \tau) - 1} = \frac{1}{q} \sum_{\kappa \in C_{\geqslant <}} z_\kappa q^{\mathrm{asc}_-(\kappa)}. \tag{B.3}$$

This shows (B.1).

The same argument shows the modular relation for modular triples of type (R). □

Remark B.2. Our proof is in the same way as Alexandersson's proof for [4, Proposition 18] in essential.

## Appendix C. Chromatic symmetric functions

We also give an elementary, direct proof of the modular law for $\mathrm{csf}_h(q)$. As far as we know, other proofs of the modular law for $\mathrm{csf}_h(q)$ use the modular law for $\mathrm{LLT}_h(q)$ and the relation between $\mathrm{csf}_h(q)$ and $\mathrm{LLT}_h(q)$ by [7, Proposition 3.5].

Theorem C.1. *Chromatic symmetric functions satisfy the modular law.*

*Proof.* Let $d$, $d_0$, $h$, $h_-$, and $h_+$ be the same ones in the the proof of Theorem B.1. Then what we have to show is

$$\mathrm{csf}_{h_+}(q) - \mathrm{csf}_h(q) = q(\mathrm{csf}_h(q) - \mathrm{csf}_{h_-}(q)). \tag{C.1}$$

Note that $\{d+1, d\}$ is an edge of $G_{h_-}$. Let $C = PC(G_{h_-})$ be the set of all proper colorings of $G_{h_-}$. Then $C$ is decomposed into the following nine subsets.

- $C_{<<} = \{\kappa \in C \,|\, \kappa(d_0) < \kappa(d),\, \kappa(d_0) < \kappa(d+1)\}$
- $C_{<=} = \{\kappa \in C \,|\, \kappa(d_0) < \kappa(d),\, \kappa(d_0) = \kappa(d+1)\}$, $\kappa \in C_{<=} \Rightarrow \kappa(d) > \kappa(d+1)$
- $C_{<>} = \{\kappa \in C \,|\, \kappa(d_0) < \kappa(d),\, \kappa(d_0) > \kappa(d+1)\}$, $\kappa \in C_{<>} \Rightarrow \kappa(d) > \kappa(d+1)$
- $C_{=<} = \{\kappa \in C \,|\, \kappa(d_0) = \kappa(d),\, \kappa(d_0) < \kappa(d+1)\}$, $\kappa \in C_{=<} \Rightarrow \kappa(d) < \kappa(d+1)$
- $C_{==} = \{\kappa \in C \,|\, \kappa(d_0) = \kappa(d),\, \kappa(d_0) = \kappa(d+1)\} = \varnothing$
- $C_{=>} = \{\kappa \in C \,|\, \kappa(d_0) = \kappa(d),\, \kappa(d_0) > \kappa(d+1)\}$, $\kappa \in C_{=>} \Rightarrow \kappa(d) > \kappa(d+1)$
- $C_{><} = \{\kappa \in C \,|\, \kappa(d_0) > \kappa(d),\, \kappa(d_0) < \kappa(d+1)\}$, $\kappa \in C_{><} \Rightarrow \kappa(d) < \kappa(d+1)$
- $C_{>=} = \{\kappa \in C \,|\, \kappa(d_0) > \kappa(d),\, \kappa(d_0) = \kappa(d+1)\}$, $\kappa \in C_{>=} \Rightarrow \kappa(d) < \kappa(d+1)$
- $C_{>>} = \{\kappa \in C \,|\, \kappa(d_0) > \kappa(d),\, \kappa(d_0) > \kappa(d+1)\}$

Some proper colorings of $G_{h_-}$ are naturally considered as proper colorings of $G_h$ or $G_{h_+}$, that is,

$$\begin{aligned} &\text{the set of proper colorings of } G_h \text{ is } && C_{<<} \sqcup C_{<=} \sqcup C_{<>} \sqcup C_{><} \sqcup C_{>=} \sqcup C_{>>} \\ &\text{and the set of proper colorings of } G_{h_+} \text{ is } && C_{<<} \sqcup \phantom{C_{<=} \sqcup{}} C_{<>} \sqcup C_{><} \sqcup \phantom{C_{>=} \sqcup{}} C_{>>}. \end{aligned}$$

Then we have

$$\text{(C.2)} \qquad \begin{aligned}\operatorname{csf}_{h_+}(q) - \operatorname{csf}_h(q) = (q^2-q)\sum_{\kappa\in C_{<<}} z_\kappa q^{\operatorname{asc}_-(\kappa)} - q\sum_{\kappa\in C_{<=}} z_\kappa q^{\operatorname{asc}_-(\kappa)} \\ + (q-1)\sum_{\kappa\in C_{><}} z_\kappa q^{\operatorname{asc}_-(\kappa)} - \sum_{\kappa\in C_{>=}} z_\kappa q^{\operatorname{asc}_-(\kappa)}\end{aligned}$$

$$\text{(C.3)} \qquad \begin{aligned}\operatorname{csf}_{h}(q) - \operatorname{csf}_{h_-}(q) = (q-1)\sum_{\kappa\in C_{<<}} z_\kappa q^{\operatorname{asc}_-(\kappa)} + (q-1)\sum_{\kappa\in C_{<=}} z_\kappa q^{\operatorname{asc}_-(\kappa)} \\ + (q-1)\sum_{\kappa\in C_{<>}} z_\kappa q^{\operatorname{asc}_-(\kappa)} - \sum_{\kappa\in C_{=<}} z_\kappa q^{\operatorname{asc}_-(\kappa)} - \sum_{\kappa\in C_{=>}} z_\kappa q^{\operatorname{asc}_-(\kappa)}.\end{aligned}$$

By a similar argument to that for (B.3) and the bijection $C_{<=} \to C_{=<}$ given by the composition with $\tau$, we have

$$q\sum_{\kappa\in C_{<=}} z_\kappa q^{\operatorname{asc}_-(\kappa)} - \sum_{\kappa\in C_{=<}} z_\kappa q^{\operatorname{asc}_-(\kappa)} = 0.$$

Hence these terms in (C.3) are cancelled and we obtain

$$\text{(C.4)} \qquad \begin{aligned}\operatorname{csf}_{h}(q) - \operatorname{csf}_{h_-}(q) = (q-1)\sum_{\kappa\in C_{<<}} z_\kappa q^{\operatorname{asc}_-(\kappa)} - \sum_{\kappa\in C_{<=}} z_\kappa q^{\operatorname{asc}_-(\kappa)} \\ + (q-1)\sum_{\kappa\in C_{<>}} z_\kappa q^{\operatorname{asc}_-(\kappa)} - \sum_{\kappa\in C_{=>}} z_\kappa q^{\operatorname{asc}_-(\kappa)}.\end{aligned}$$

Then we obtain (C.1) from (C.2) and (C.4) by the two bijections $C_{<>} \to C_{><}$ and $C_{=>} \to C_{>=}$ given by the composition with $\tau$.

The same argument shows the modular relation for modular triples of type (R). □

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