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Abstract We classify rank one 2-representations of SL_2, GL_2 and SO_3 web categories. The classification is inspired by similar results about quantum groups, given by reducing the problem to the classification of bilinear and trilinear forms, and is formulated such that it can be adapted to other web categories.

1. Introduction

We give a classification of simple transitive 2-representations of web categories on $\mathbb{C}$-vector spaces. This classification builds upon and extends results in the classification of quantum groups and Hopf algebras. The main point is that, even in this very restricted and semisimple setting, there are infinitely many such 2-representations and these are essentially impossible to classify explicitly.

1A. Motivation and results. *Classification* is a central topic in all of mathematics. In representation theory the most important classification problem is to construct and compare all simple representations. In higher representation theory, an offspring of *categorification* that originates in seminal papers such as [12, 22, 33, 51, 41], the most crucial classification problem is about the appropriate analog of simple representations. For example, given a favorite monoidal category, one can ask whether one can classify its simplest possible module categories. The favorite categories of our choice in this note are certain *diagram categories*, simplest possible will mean *simple transitive* and classification will mean *reduction* of the original problem to linear algebra.

Note, however, that linear algebra can still be arbitrarily complicated. The problem of classifying symmetric and alternating *bilinear forms* is well-known and has a very pleasant answer. Less well-known but still doable and nice is the classification of all bilinear forms. On the other hand, the classification of *trilinear forms* is tractable for small dimensions only, even if one restricts to symmetric or alternating forms: the classification problem is "wilder than wild" [6]. However, for small dimensions there is indeed a classification of trilinear forms, see e.g. [61, 11, 60].

In this note we will see a similar behavior for the following *web categories*: the category of $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{C})$ webs $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$, the category of $\mathrm{GL}_2 = \mathrm{GL}_2(\mathbb{C})$ webs $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ and the category of $\mathrm{SO}_3 = \mathrm{SO}_3(\mathbb{C})$ webs $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$, and quantum versions for which the q in the notation will appear. (That we discuss SO_3 webs and not the very similar SL_3 webs has historical reasons, see Remark 5B.2 below.) The

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classification problem we have in mind for these categories is to study the easiest form of actions of these categories on $\mathbb{C}\mathbf{Vect}$, the category of finite dimensional $\mathbb{C}$-vector spaces. In the language of [42], we want to classify *rank one simple transitive 2-representations* of these web categories. (Let us point out that rank one simple transitive 2-representations are not semisimple in general, but for web categories they are by Lemma 2.15.)

For all of these web categories we give a classification of such 2-representations. The classification takes a certain form as outlined in Classification Problem 2.19. Roughly, we construct 2-representations from linear algebra inputs such as matrices and tensors. Second, the equivalence classes of these 2-representations are given by an explicit linear algebra condition on matrices and tensors such as congruence. Finally, we argue that any rank one simple transitive 2-representation is of the form constructed in the first step.

But how explicit our classification is varies drastically:

- ▷ For $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ the classification is similar to the classification of bilinear forms and has therefore a short-and-sweet answer, see Theorem 3B.2.
- ▷ For $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ trilinear forms make their appearance. However, as we will see, the appearing trilinear forms are on $\mathbb{C}$-vector spaces of small dimensions so we still get a good answer, see Theorem 4B.2.
- ▷ For $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ honest trilinear forms appear which makes us believe that there is no good (this could e.g. mean listable) answer, see Theorem 5B.1.

These three web categories are semisimple but have infinitely many isomorphism classes of simple objects. As we will see, in all cases there are infinitely many equivalence classes of rank one simple transitive 2-representations. This is very different from the situation of semisimple categories with finitely many simple objects where some form of Ocneanu rigidity ensures that there are only finitely many simple transitive 2-representations.

In Proposition 6.2 we also show that the classification (of rank one simple transitive 2-representations) for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ (and $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$) implies the classification of bilinear forms, and in fact, the classification is a tame problem. For $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ we are not able to determine the precise characterization of the complexity of the classification problem. However, for a modification of $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ we show that the classification implies the classification of trilinear forms, see Proposition 6.3. In fact, the classification problem for the variant of $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ is strictly more difficult than any wild problem in classical representation theory, see Theorem 6.4.

1B. A few extra comments.

We finish the introduction with a few remarks.

Remark 1B.1. All web categories in this paper are monoidally equivalent to representation theoretical categories. However, since one of our main points is to use diagrammatics, we think of these as web categories instead of their representation theoretical counterparts.

Along the same lines, we would like to point out that similar results have been obtained in other fields although the translation is not completely straightforward. The connection was in fact the starting point for this note. The methods presented in these papers are different from what we do in this note; in particular, we take the diagrammatic approach and make the classification results more explicit, see e.g. Lemma 3C.13.

For example, see [8] or [23] for SL_2, [43] for GL_2, and [44] or [24] for SO_3. See also [50] and [46] for the SL_k family. ◊

REMARK 1B.2. In this remark *complexity* is meant with respect to classification of rank one simple transitive 2-representations, and we use it as an informal estimate of difficulty. We give some details later in section 6.

(a) Consider the following list:

$$\begin{array}{c}\mathrm{SL}_2,\mathrm{GL}_2\\ \begin{array}{c||c|c|c|c|c|c|c} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \dim_{\mathbb{C}} & 1 & 0 & 1 & 0 & 2 & 0 & 5 \end{array}\end{array}, \quad \begin{array}{c}\mathrm{SL}_3\\ \begin{array}{c||c|c|c|c|c|c|c} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \dim_{\mathbb{C}} & 1 & 0 & 1 & 1 & 2 & 3 & 6 \end{array}\end{array},$$

$$\begin{array}{c}\mathrm{SO}_3\\ \begin{array}{c||c|c|c|c|c|c|c} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \dim_{\mathbb{C}} & 1 & 0 & 1 & 1 & 3 & 6 & 15 \end{array}\end{array}.$$

These lists are the maximal appearing dimensions b_n^* of the hom-spaces in SL_2, SL_3 and SO_3 webs, respectively, for webs with n boundary points to the empty web. We have $b_n^{\mathrm{SL}_2} \leqslant b_n^{\mathrm{SL}_3} \leqslant b_n^{\mathrm{SO}_3}$, but on the other hand [14, Theorem 1.4.(a)] gives

$$\lim_{n\to\infty} \sqrt[n]{b_n^{\mathrm{SL}_2}} = 2 < 3 = \lim_{n\to\infty} \sqrt[n]{b_n^{\mathrm{SL}_3}} = \lim_{n\to\infty} \sqrt[n]{b_n^{\mathrm{SO}_3}}.$$

This justifies the complexity jump from SL_2 to SO_3 webs, and probably indicates that SL_3 and SO_3 webs are of the same complexity. (We note that [7] gives more precise formulas for the asymptotics of the numbers b_n^*, but we do not need them here.)

Note also that $b_n^{\mathrm{SL}_2} = b_n^{\mathrm{GL}_2}$ so their complexity is roughly the same which indeed matches what we will see in Theorem 3B.2 and Theorem 4B.2.

(b) In general we expect the complexity of SL_n (or GL_n) webs, as in e.g. [45, 10], to be equal to or higher than for SO_3, so likely unsolvable in a precise sense. However, as pointed out in [60], $(4,4,6)$ trilinear forms are classifiable and 4, 4, 6 are the dimensions of the nontrivial fundamental SL_4-representations. Thus, there might be something that can be said for SL_4 webs similar to what we do at the end of subsection 5C.

(c) All categories in this note are semisimple. A good question is to address the nonsemisimple case where one could expect *cell theory* as in e.g. [29, 41, 63] to play a role. As usual one should expect a nontrivial complexity jump from the semisimple to the nonsemisimple cases.

(In (b) we write (p,q,r) *trilinear form* for a trilinear form on $\mathbb{C}$-vector spaces of these dimensions.) ♢

REMARK 1B.3. Two technical remarks:

(a) Some calculations were done with machine help and the code used can be found on [62]. Any necessary erratum can be found on the same website.

(b) This note is readable in black-and-white since the colors we use are a visual aid only.

Please email me if you find mistakes or have comments; I aim to then update [62]. ♢

2. Background on 2-representations

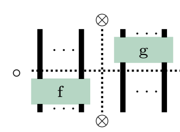
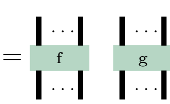
We will briefly recall some notions from 2-representation theory. Below, and throughout, we usually count objects up to isomorphism but drop the 'up to isomorphism' for brevity.

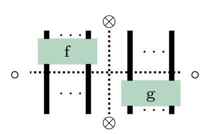
Remark 2.1. Nothing in this section is new, but reformulated compared to our main sources [21, 38, 39]. Our reformulation stems from that we only consider the semisimple case, which is just a special case of what the above literature discusses. For categories with countably many simple object see also [40] for a helpful reference.

The reader is also referred to [37, 21] for standard notions such as monoidal categories. ◊

Let $\mathbb{K}$ be a field. Everything in this section will be over $\mathbb{K}$. As basic notation we use:

- ▷ Categories are denoted using bold font such as $\mathbf{V}$ or $\mathbf{W}$, while monoidal categories ($\leftrightsquigarrow$ 2-categories with one object) are denoted by e.g. $\mathscr{C}$ or $\mathscr{D}$.
- ▷ Objects are denoted by $\mathtt{X}$, $\mathtt{Y}$ etc. and morphisms by f, g etc.
- ▷ Categorical composition is denoted by $\circ$ and monoidal composition by $\otimes$. The monoidal unit is denoted by $\mathbb{1}$, and identity morphism as $\mathrm{id}_{\mathtt{X}}$ or simplified as id. The following picture summarizes our diagrammatic composition rules:

$(\mathrm{id} \otimes \mathrm{g}) \circ (\mathrm{f} \otimes \mathrm{id}) =$  $=$  $=$

 $= (\mathrm{f} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{g}).$

- ▷ For functors calligraphic font is used, for example $\mathcal{M}$ or $\mathcal{N}$.

Remark 2.2. Before we get started, let us point out that all our higher categories and structures are strict. This turns out to be no restriction for us and the usual strictification theorems apply, see for example [28, Section 1.4] or [53, Section 4.2] for quite general strictification theorems.

In contrast, the notion of equivalence (and generally of morphism) for 2-representations is not strict. See also [57] for examples why one has to allow nonstrict equivalences. ◊

Definition 2.3. *A nonzero object of a $\mathbb{K}$-linear category is* simple *if all its subobjects are either zero objects or isomorphisms.*

By a semisimple category *we mean a $\mathbb{K}$-linear additive category such that (i) every object decomposes as a finite direct sum of simple objects, and (ii) if* $\mathrm{f}\colon \mathtt{S}_1 \to \mathtt{S}_2$ *and* $\mathrm{g}\colon \mathtt{S}_2 \to \mathtt{S}_3$ *are nonzero morphisms between simple objects* $\mathtt{S}_i$*, then* $\mathrm{g} \circ \mathrm{f}$ *is also nonzero.* ◊

A semisimple category may have infinitely many simple objects. The following will be used silently.

Lemma 2.4. *We have:*

(a) *A semisimple category is idempotent complete.*

(b) *A semisimple category is $\mathbb{K}$-linear finite length abelian.*

(c) *$\mathbb{K}$-linear functors between $\mathbb{K}$-linear additive categories are additive.*

Proof. Well-known, so we will be brief. First, (c) is an exercise, while (b) is proven in, for example, [65, Theorem C.6]. (a) follows from the following argument (due to the referee): For (a) it suffices to show that an idempotent in the endomorphism algebra of a simple object splits. And indeed, if e is a nontrivial (neither zero nor the identity) idempotent in the endomorphism algebra of a simple object $\mathtt{X}$, then so is $\mathrm{id}_{\mathtt{X}} - \mathrm{e}$. However, the composite of these two is zero, violating (ii) of Definition 2.3. □

Remark 2.5. By Lemma 2.4.(a), in the semisimple case, the theories presented in [21] on the one hand, and [38] and [39] on the other hand are essentially the same (there are subtle differences but they do not play any role for us). ♢

Definition 2.6. *An* essentially fusion category $\mathscr{C}$ *is a semisimple rigid monoidal category with countably many simple objects and such that all morphism spaces are finite-dimensional.* ♢

Example 2.7. *The category* $\mathbb{K}\mathbf{Vect}$ *of finite dimensional* $\mathbb{K}$*-vector spaces is a prototypical example of an essentially fusion category. A more exciting example is* $\mathbb{C}\mathbf{Rep}\big(\mathrm{SL}_2(\mathbb{C})\big)$*, complex finite dimensional* $\mathrm{SL}_2(\mathbb{C})$*-representations, and its relatives that we will discuss in the sections below. The simple objects, up to equivalence, in the category* $\mathbb{C}\mathbf{Rep}\big(\mathrm{SL}_2(\mathbb{C})\big)$ *are* $\{\mathtt{S}_k = \mathrm{Sym}^k(\mathbb{C}^2) | k \in \mathbb{Z}_{\geqslant 0}\}$. ♢

Remark 2.8. In the language of [39], Definition 2.6 translates to what is called a locally semisimple quasi-fiat one object 2-category in that paper, with one difference: Definition 2.6 allows countably (finite or infinite) many simple objects, whereas a locally semisimple quasi-fiat one object 2-category always has finitely many simple objects. ♢

Lemma 2.9. *An essentially fusion category* $\mathscr{C}$ *is Krull–Schmidt.*

Proof. A finite length abelian category is Krull–Schmidt, so Lemma 2.4.(b) proves the claim. □

Let $\mathscr{A} = \mathscr{A}_{\mathbb{K}}^{f}$ denote the 2-category of finitary categories, $\mathbb{K}$-linear functors and natural transformations, see [38, Definition 2.12]. For the purpose of this paper it is enough to know that $\mathbb{K}\mathbf{Vect} \in \mathscr{A}$. By $\circ$-ideal we mean an ideal with respect to the operation $\circ$, while $\circ$-$\otimes$-ideal is meant with respect to both operations $\circ$ and $\otimes$ separately.

Definition 2.10. *Let* $\mathscr{C}$ *be as in Definition 2.6.*

(a) *A* (finitary) 2-representation $\mathcal{M}$ *of* $\mathscr{C}$ *is a* $\mathbb{K}$*-linear monoidal functor* $\mathcal{M}\colon \mathscr{C} \to \mathrm{End}_{\mathscr{A}}(\mathbf{V})$ *for* $\mathbf{V} \in \mathscr{A}$.

(b) *The* rank*, denoted by* $\mathrm{rank}\,\mathcal{M}$*, of such a functor* $\mathcal{M}$ *is the number of indecomposable objects in* $\mathbf{V}$.

(c) *We call a 2-representation* semisimple *if the target category* $\mathbf{V} \in \mathscr{A}$ *is semisimple.*

(d) *Such a functor* $\mathcal{M}$ *is called* simple transitive *if it has no proper* $\mathscr{C}$*-stable* $\circ$*-ideals, meaning that every* $\circ$*-ideal* $I \subset \mathbf{V}$ *with* $\mathcal{M}(\mathtt{X})(I) \subset I$ *for all* $\mathtt{X} \in \mathscr{C}$ *is either zero or* $\mathbf{V}$.

(The 2-representation in (d) are simple, by the definition above, and transitive by Lemma 2.13 below, hence the name.) ♢

Remark 2.11. There is also the notion of a *(finitary) module category*. Similarly as for representations and modules, these notions are equivalent in the appropriate sense. We leave it to the reader to spell out the definitions, and we use them interchangeably. For example, in Definition 2.12 below the horizontal arrows are to be read as module category notation. ♢

Definition 2.12. *Two 2-representations* $\mathcal{M}\colon \mathscr{C} \to \mathrm{End}_{\mathscr{A}}(\mathbf{V})$ *and* $\mathcal{N}\colon \mathscr{C} \to \mathrm{End}_{\mathscr{A}}(\mathbf{W})$ *are* equivalent*, written* $\mathcal{M} \cong_{2rep} \mathcal{N}$*, if there is an equivalence* $\mathcal{F}\colon \mathbf{V} \to \mathbf{W} \in \mathscr{A}$ *such that*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\mathcal{M}"] \arrow[d, equal] & \mathbf{V} \arrow[d, "\mathcal{F}"] \\ \mathscr{C} \arrow[r, "\mathcal{N}"'] & \mathbf{W} \end{tikzcd}$$

is a commutative diagram up to a coherent natural isomorphism $\mathcal{N}(\mathtt{X})\big(\mathcal{F}(\mathtt{V})\big) \xRightarrow{\cong} \mathcal{F}\big(\mathcal{M}(\mathtt{X})(\mathtt{V})\big)$. ◊

Lemma 2.13. *Any simple transitive 2-representation is* transitive*, meaning generated, taking direct sums and direct summands, by the action of* $\mathcal{M}$*.*

Proof. This is [42, Lemma 4]. □

Example 2.14. *The category* $\mathbb{C}\mathbf{Rep}\big(\mathrm{SL}_2(\mathbb{C})\big)$ *acts on itself by tensoring, that is,* $\mathcal{M}(\mathtt{S}_k)$ *is the endofunctor of (left) tensoring with* $\mathtt{S}_k$*. The only* $\circ$*-*$\otimes$*-ideals in* $\mathbb{C}\mathbf{Rep}\big(\mathrm{SL}_2(\mathbb{C})\big)$ *are zero or the category itself. This follows since* $\mathbb{C}\mathbf{Rep}\big(\mathrm{SL}_2(\mathbb{C})\big)$ *is semisimple and we have* $\mathbb{C}^2 \otimes \mathtt{S}_k \cong \mathtt{S}_{k+1} \oplus \mathtt{S}_{k-1}$ *for* $k \in \mathbb{Z}_{\geqslant 1}$*. Hence,* $\mathbb{C}\mathbf{Rep}\big(\mathrm{SL}_2(\mathbb{C})\big)$ *is simple transitive. Thus, since* $\mathrm{rank}\,\mathbb{C}\mathbf{Rep}\big(\mathrm{SL}_2(\mathbb{C})\big) = \infty$*,* $\mathbb{C}\mathbf{Rep}\big(\mathrm{SL}_2(\mathbb{C})\big)$ *is an infinite rank simple transitive 2-representation of itself.* ◊

Note that 2-representations of essentially fusion categories are in general not semisimple. As an example consider $\mathbb{K}\mathbf{Vect}$ which can act on any $\mathbb{K}$-linear abelian category. (This action is unique up to the equivalence in Definition 2.12.) Hence, the following result is remarkable and key for this paper:

Lemma 2.15. *Any simple transitive 2-representation* $\mathcal{M}$ *of an essentially fusion category* $\mathscr{C}$ *with* $\mathrm{rank}\,\mathcal{M} < \infty$ *is semisimple, meaning that* $\mathbf{V}$ *is semisimple.*

Proof. A direct adaption of [39, Proposition 2.16]. □

Lemma 2.16. *Any rank one simple transitive 2-representation of an essentially fusion category* $\mathscr{C}$ *is on* $\mathbb{K}\mathbf{Vect}$*.*

Proof. Since $\mathbb{K}\mathbf{Vect}$ is the only semisimple category with one simple object, this follows from Lemma 2.15. □

Let $\mathscr{C}$ be an essentially fusion category. Recall that a *fiber functor* $\mathcal{F}\colon \mathscr{C} \to \mathbb{K}\mathbf{Vect}$ is an exact faithful monoidal functor. We write $\mathcal{F}(\mathtt{X}|\mathtt{X} \in \mathscr{C})$ for the full subcategory whose objects are direct sums of direct summands of objects of the form $\mathcal{F}(\mathtt{X})$, for $\mathtt{X} \in \mathscr{C}$. The following, very easy, lemma is another key fact:

Lemma 2.17. *We have the following.*

(a) *Any fiber functor* $\mathcal{F}\colon \mathscr{C} \to \mathbb{K}\mathbf{Vect}$ *of an essentially fusion category* $\mathscr{C}$ *gives rise to a semisimple rank one 2-representation* $\mathcal{M}$*.*

(b) *A fiber functor functor* $\mathcal{F}\colon \mathscr{C} \to \mathbb{K}\mathbf{Vect}$ *of an essentially fusion category* $\mathscr{C}$ *gives rise to a simple transitive 2-representation* $\mathcal{M}$ *if and only if* $\mathcal{F}(\mathtt{X}|\mathtt{X} \in \mathscr{C})$ *does not have any nontrivial* $\circ$*-*$\otimes$*-ideals.*

Proof. (a). For $\mathtt{X} \in \mathscr{C}$ set $\mathcal{M}(\mathtt{X})$ to be the endofunctor of tensoring with the $\mathbb{K}$-vector space $\mathcal{F}(\mathtt{X})$. One can directly verify that this defines a 2-representation.

(b). If $\mathcal{M}$ is simple transitive, then there cannot be any nontrivial $\circ$-$\otimes$-ideals by the construction of $\mathcal{M}$ in (a). Conversely, if there are no nontrivial $\circ$-$\otimes$-ideals then semisimplicity, that is Lemma 2.15, implies that $\mathcal{M}$ is simple transitive. □

With Lemma 2.17.(a) in mind, we also say *fiber 2-representation* instead of fiber functor. These are always of rank one, by definition, but the converse might be false. (A rank one 2-representation has no reason to be faithful in general.) If the condition Lemma 2.17.(b) is satisfied for a fiber functor $\mathcal{F}\colon \mathscr{C} \to \mathbb{K}\mathbf{Vect}$, then we call $\mathcal{F}$ a *simple transitive fiber 2-representation.*

Example 2.18. *The action from Example 2.14 is not a fiber 2-representation. But composition with the forgetful functor $\mathbb{C}\mathbf{Rep}\big(\mathrm{SL}_2(\mathbb{C})\big) \to \mathbb{C}\mathbf{Vect}$ defines a (simple transitive) fiber 2-representation.* $\diamondsuit$

Classification Problem 2.19. *The classification of simple transitive 2-representations of a given $\mathscr{C}$ is one of the main problems of the theory, and akin to classifying simple representations of groups or algebras. This is justified by the categorical analog of the Jordan–Hölder theorem, see* [42, Section 3.5].

For us such a classification is optimally given by:

- *The existence of certain explicitly constructed simple transitive 2-representations.* ***(Existence)***
- *The comparison of these with a computable condition.* ***(Non-redundant)***
- *A proof that all simple transitive 2-representations are of the particular form.* ***(Complete)***

In this paper we restrict to the subproblem of classifying simple transitive rank one (or fiber) 2-representations. As we will see, even this subproblem can get arbitrarily difficult, and we will sometimes only give part of the list above. $\diamondsuit$

Remark 2.20. Classification Problem 2.19 is not meant as a definition. $\diamondsuit$

Example 2.21. *Keeping Remark 2.5 in mind, the paper* [23] *classifies simple transitive 2-representations of $\mathbb{C}\mathbf{Rep}\big(\mathrm{SL}_2(\mathbb{C})\big)$ of finite rank. The classification is quite difficult, and we will discuss the much simpler classification of simple transitive fiber 2-representations $\mathbb{C}\mathbf{Rep}\big(\mathrm{SL}_2(\mathbb{C})\big) \to \mathbb{C}\mathbf{Vect}$ in section 3. It turns out that in this case all rank one simple transitive 2-representations come from fiber functors.* $\diamondsuit$

Lemma 2.22. *An essentially fusion category $\mathscr{C}$ with finitely many simple objects has only finitely many simple transitive 2-representations up to $\cong_{2rep}$.*

Proof. We point out that an essentially fusion category with finitely many simple objects is a fusion category in the usual sense as, for example, in [21, Chapter 9]. Then the claim follows from Ocneanu rigidity as e.g. in [21, Proposition 3.4.6 and Corollary 9.1.6]. $\square$

With contrast to Lemma 2.22 we have:

Theorem 2.23. *An essentially fusion category $\mathscr{C}$ can have infinitely many nonequivalent simple transitive rank one 2-representations.*

Proof. By the examples discussed in the next sections; see, for example, Theorem 3B.2. $\square$

For $\mathbb{K}\mathbf{Vect}$ we use its unique braiding (= the flip map). To finish this section, and relevant for our examples (we say braided and symmetric instead of braided monoidal and symmetric monoidal):

Definition 2.24. *Assume that the acting category $\mathscr{C}$ is braided. A fiber 2-representation is* braided *if it is given by a braided functor.* $\diamondsuit$

One should not expect fiber 2-representations to have interesting braidings:

Lemma 2.25. *A braided fiber 2-representation is given by a symmetric functor.*

Proof. We start with an auxiliary lemma (whose proof is due to a referee):

Lemma 2.26. *Let $\mathscr{C}$ be braided, and let $\mathscr{D}$ be symmetric. If there is a braided functor $\mathcal{F}\colon \mathscr{C}\to\mathscr{D}$ which is faithful, then $\mathscr{C}$ is symmetric.*

Proof. This is known, so we only give a condensed proof. The diagram

$$\begin{tikzcd}
\mathcal{F}(\mathtt{X})\otimes\mathcal{F}(\mathtt{Y}) \arrow[r, "\mathrm{b}^{\mathscr{D}}_{\mathcal{F}(\mathtt{X}),\mathcal{F}(\mathtt{Y})}"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "\mathcal{F}^2_{\mathtt{X},\mathtt{Y}}", "\cong"'] \arrow[dr, phantom, "(2)"] & \mathcal{F}(\mathtt{Y})\otimes\mathcal{F}(\mathtt{X}) \arrow[r, "\mathrm{b}^{\mathscr{D}}_{\mathcal{F}(\mathtt{Y}),\mathcal{F}(\mathtt{X})}"] \arrow[d, "\mathcal{F}^2_{\mathtt{Y},\mathtt{X}}"', "\cong"] \arrow[u, phantom, "(1)"] \arrow[dr, phantom, "(3)"] & \mathcal{F}(\mathtt{X})\otimes\mathcal{F}(\mathtt{Y}) \arrow[d, "\mathcal{F}^2_{\mathtt{X},\mathtt{Y}}"', "\simeq"] \\
\mathcal{F}(\mathtt{X}\otimes\mathtt{Y}) \arrow[r, "\mathcal{F}(\mathrm{b}_{\mathtt{X},\mathtt{Y}})"'] & \mathcal{F}(\mathtt{Y}\otimes\mathtt{X}) \arrow[r, "\mathcal{F}(\mathrm{b}_{\mathtt{Y},\mathtt{X}})"'] & \mathcal{F}(\mathtt{X}\otimes\mathtt{Y})
\end{tikzcd}$$

commutes: (1) by $\mathscr{D}$ being symmetric, (2) and (3) by coherence. We get that

$$\mathrm{End}_{\mathscr{C}}(\mathtt{X}\otimes\mathtt{Y}) \xrightarrow{\mathcal{F}_{\mathtt{X},\mathtt{Y}}} \mathrm{End}_{\mathscr{D}}\big(\mathcal{F}(\mathtt{X}\otimes\mathtt{Y})\big) \xrightarrow{(\mathcal{F}^2_{\mathtt{X},\mathtt{Y}})^{-1}\circ_{-}\circ\mathcal{F}^2_{\mathtt{X},\mathtt{Y}}} \mathrm{End}_{\mathscr{D}}\big(\mathcal{F}(\mathtt{X})\otimes\mathcal{F}(\mathtt{Y})\big)$$

sends both $\mathrm{id}_{\mathtt{X}\otimes\mathtt{Y}}$ and $\mathrm{b}_{\mathtt{Y},\mathtt{X}}\circ\mathrm{b}_{\mathtt{X},\mathtt{Y}}$ to $\mathrm{id}_{\mathcal{F}(\mathtt{X})\otimes\mathcal{F}(\mathtt{Y})}$. But this is a composite of injections, and hence it is injective. Thus, we get $\mathrm{id}_{\mathtt{X}\otimes\mathtt{Y}}=\mathrm{b}_{\mathtt{Y},\mathtt{X}}\circ\mathrm{b}_{\mathtt{X},\mathtt{Y}}$. □

The flip map defines a symmetric structure on $\mathbb{K}$**Vect**. Hence, Lemma 2.26 implies the claim. □

Assume that $\mathrm{rank}\,\mathcal{M}<\infty$ and let $K_0^{\oplus}(\mathcal{M})$ denote the additive Grothendieck group of the 2-representation $\mathcal{M}$. (By Lemma 2.15 we are in the semisimple case so the additive and the abelian Grothendieck groups agree.) For essentially fusion categories $\mathscr{C}$ one can define $K_0^{\oplus}(\mathscr{C})$ without issue by Lemma 2.9 (even though $\mathscr{C}$ is allowed to have infinitely many isomorphism classes of simple objects).

Lemma 2.27. *Write $\mathscr{M}=\mathcal{M}(\mathtt{X}|\mathtt{X}\in\mathscr{C})$. The additive Grothendieck group $K_0^{\oplus}(\mathscr{M})$ is a $K_0^{\oplus}(\mathscr{C})$-representation.*

Proof. Easy to check and omitted. □

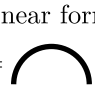
We write $\cong_{rep}$ for equivalence of $K_0^{\oplus}(\mathscr{C})$-representations.

3. Rank one 2-representations of SL2 webs

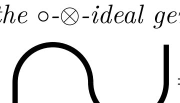
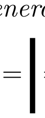
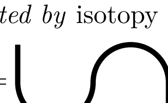
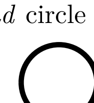
For the rest of the paper let $\mathbb{K}=\mathbb{C}$. As we will see, the main players in this section are complex bilinear forms.

3A. SL2 webs. We first recall the *Temperley–Lieb category*, or *Rumer–Teller–Weyl category*, that we will call the SL_2 web category.

Definition 3A.1. *Fix $q\in\mathbb{C}\setminus\{0\}$. Let $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ denote the $\mathbb{C}$-linear pivotal category $\otimes$-generated by the selfdual object $\mathtt{X}$, and $\circ$-$\otimes$-generated by morphisms called* caps *and* cups *(also called* bilinear form *and* coform*):*

cap = : $\mathtt{X}\otimes\mathtt{X}\to\mathbb{1}$, cup = : $\mathbb{1}\to\mathtt{X}\otimes\mathtt{X}$,

modulo the $\circ$-$\otimes$-ideal generated by isotopy *and* circle evaluation*:*

 =  = ,  $=-[2]_q=-q-q^{-1}$.

We call $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ the SL_2 web category *and its morphism* SL_2 webs*.* ◊

Remark 3A.2. In this and the following sections we work over $\mathbb{C}$ using a "generic" q instead of over $\mathbb{C}(q)$ for a variable q. The situation of $\mathbb{C}(q)$ can be discussed verbatim, but the linear algebra results used in this note need to be adjusted to $\mathbb{C}(q)$. $\diamondsuit$

Let C_k^l denote the set of crossingless matchings of k bottom and l top points, interpreted as SL_2 webs in the usual way.

Lemma 3A.3. *The set* C_k^l *is a* $\mathbb{C}$*-basis of* $\mathrm{Hom}_{\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)}(\mathtt{X}^{\otimes k}, \mathtt{X}^{\otimes l})$.

Proof. Well-known, see e.g. [16] for a self-contained argument that implies the claim. $\square$

A *nontrivial root of unity* is a $q \notin \{1,-1\}$ with $q^k = 1$ for some $k \in \mathbb{Z}_{\geqslant 0}$.

Lemma 3A.4. *We have the following.*

(a) *The simple objects of* $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ *are in one-to-one correspondence with* $\mathbb{Z}_{\geqslant 0}$.

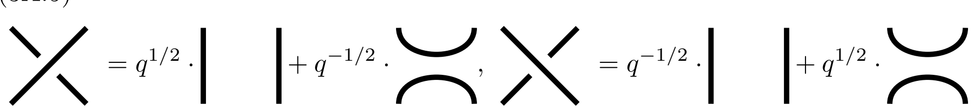
(b) $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ *is semisimple if only if* $q \in \mathbb{C} \setminus \{0\}$ *is not a nontrivial root of unity.*

(c) $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ *is an essentially fusion category if only if* $q \in \mathbb{C} \setminus \{0\}$ *is not a nontrivial root of unity.*

Proof. Recall that $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ can be defined integrally, meaning over $\mathbb{Z}[q, q^{-1}]$, and that $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ is integrally equivalent to the category of tilting modules for quantum SL_2. This is a type of folk theorem that dates back to [56], see e.g. [19, Theorem 2.58], [2, Proposition 2.3] or [58, Proposition 2.13]. The statements follow then from specialization to the complex numbers, which is well-understood on the tilting side, see e.g. [4, Section 2]. $\square$

Choose a square root $q^{1/2}$ of q. Let us define

(3A.5)



Lemma 3A.6. *The formula Equation 3A.5 endows* $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ *with the structure of a braided category.*

Proof. Well-known and easy to check. See also [32, Section 2.1]. $\square$

Notation 3A.7. *As a braided category, we consider* $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ *with the structure induced by Equation 3A.5.* $\diamondsuit$

3B. The main theorem in the SL2 case. Let $\equiv_c$ denote *matrix congruence*, that is, for complex n-by-n matrices A and B we have:

$$\big(\mathsf{A} \equiv_c \mathsf{B}\big) \Leftrightarrow \big(\exists \mathsf{P} \in \mathrm{GL}_n(\mathbb{C})\colon \mathsf{A} = \mathsf{P}^T\mathsf{B}\mathsf{P}\big).$$

Note that two congruent matrices are of the same size.

Remark 3B.1. Recall that matrix congruence is define by "$(\mathsf{A} \equiv_c \mathsf{B}) \Leftrightarrow$ (the matrices A and B represent the same bilinear form up to change-of-basis)". $\diamondsuit$

The proof of the following theorem is given in subsection 3C.

Theorem 3B.2. *Assume* $q \in \mathbb{C} \setminus \{0\}$ *is not a nontrivial root of unity.*

(a) *Let* $n \geqslant 2$. *For every* $\mathsf{N} \in \mathrm{GL}_n(\mathbb{C})$ *with* $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$ *there exists a simple transitive fiber 2-representation* $\mathcal{F}_{\mathsf{N}}^n$ *of* $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ *constructed in the proof of Lemma 3C.1.* ***(Existence)***

(b) *We have* $\mathcal{F}_{\mathsf{N}}^n \cong_{2rep} \mathcal{F}_{\mathsf{M}}^m$ *if and only if* $\mathsf{N} \equiv_c \mathsf{M}$. ***(Non-redundant)***

(c) *Every simple transitive fiber 2-representation of* $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ *is of the form* $\mathcal{F}^n_{\mathsf{N}}$*, and every simple transitive rank one 2-representation of* $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ *arises in this way.* ***(Complete)***

Moreover, there are infinitely many nonequivalent simple transitive rank one 2-representations of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$.

In fact, we will make Theorem 3B.2.(a) and (b) even more explicit. We list some $\mathcal{F}^n_{\mathsf{N}}$ for $n \in \{2,3\}$, while for $n = 4$ there are infinitely many nonequivalent $\mathcal{F}^n_{\mathsf{N}}$, see Lemma 3C.13 below for details. Moreover, Theorem 3B.2 and Lemma 3C.13 together solve Classification Problem 2.19 for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$.

Remark 3B.3. For $n = 1$ the condition $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$ becomes $1 = -[2]_q$ which has no solutions unless $q \in \{\frac{1}{2}(-1 \pm \sqrt{3})\}$. This is the monoid case, see e.g. [34], but since $\frac{1}{2}(-1 \pm \sqrt{3})$ are nontrivial roots of unity, this case is not part of Theorem 3B.2. $\diamondsuit$

Note that Theorem 3B.2 shows that the classification of simple transitive fiber 2-representations of the category $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ is equivalent to the classification of simple transitive rank one 2-representations of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$. And moreover, Theorem 3B.2 shows that both problems can be considered as a subproblem of the classification of complex bilinear forms, cf. Remark 3B.1. The latter has a nice known solution that we recall below. As we will see later, see Proposition 6.2, the converse is also true in a precise sense.

Here are a few bonus observations that accompany Theorem 3B.2.

Proposition 3B.4. *We have the following.*

(a) *We have* $K_0^{\oplus}(\mathcal{F}^n_{\mathsf{N}}) \cong_{rep} K_0^{\oplus}(\mathcal{F}^m_{\mathsf{M}})$ *as* $K_0^{\oplus}\big(\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)\big)$*-representations if and only if* $n = m$.

(b) *The fiber 2-representation* $\mathcal{F}^n_{\mathsf{N}}$ *is braided if and only if (*$q = 1$*,* $n = 2$ *and* N *is a standard solution as in Example 3C.2).*

(c) *There exist infinitely many Hopf algebras* H *with* $\mathbb{C}\mathbf{coRep}(H) \cong_{\otimes} \mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ *as monoidal categories. In particular, infinitely many of these Hopf algebras are not isomorphic to* $\mathcal{O}_q\big(\mathrm{SL}_2(\mathbb{C})\big)$.

Let us finish this section with a few (historical) remarks.

Remark 3B.5. The category $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ has been around for donkey's years and is a quantum version of the category constructed, albeit in a different language, by Rumer–Teller–Weyl [56]. Many people have worked on this category, too many to cite here, and it is not surprising that Theorem 3B.2 and Proposition 3B.4 are, in different formulations, known in the literature. Most prominently, [8] solves a related problem from which, after some work, one can get Theorem 3B.2 and Proposition 3B.4. As pointed out in [8], versions of Theorem 3B.2 and Proposition 3B.4 are probably even older. Having Remark 2.5 in mind, a similar formulation also appeared in [23], see for example [23, Section 3.2]. $\diamondsuit$

Remark 3B.6. The case of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ is one of the few web categories where the modular representation theory of the associated group is quite well-understood, see [15, Section 3.4] for a concise discussion of some of the main properties. Thus, one might hope that Theorem 3B.2 generalizes to other fields than $\mathbb{C}$, where the story is not semisimple anymore. And, indeed, the paper [52] has some very similar results. However, Remark 2.5 does not apply in the nonsemisimple case. $\diamondsuit$

Remark 3B.7. Proposition 3B.4.(c) was used in [14, Theorem 5.1] which in turn was the starting point of this paper. $\diamondsuit$

Remark 3B.8. The category $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ is cellular in the sense of [66] or [20]. The same is true for the other two web categories in this paper, by the main result of [3] or [1] and the connection to tilting modules. We however do not know how to use the cellular structure to obtain Theorem 3B.2 and its relatives later on. $\lozenge$

3C. Proof of Theorem 3B.2. The key will be the following lemma.

Lemma 3C.1. *For $n \in \mathbb{Z}_{\geqslant 2}$ let $\mathsf{N} \in \mathrm{GL}_n(\mathbb{C})$ be a matrix satisfying $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$. Then there exists an associated 2-representation $\mathcal{F}$ of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ on $\mathbb{C}\mathbf{Vect}$ with $\dim_{\mathbb{C}} \mathcal{F}(\mathtt{X}) = n$. Conversely, every 2-representation $\mathcal{F}$ of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ on $\mathbb{C}\mathbf{Vect}$ with $\dim_{\mathbb{C}} \mathcal{F}(\mathtt{X}) = n$ gives such a matrix.*

Proof. Note that a 2-representation $\mathcal{F}\colon \mathscr{W}\mathbf{eb}(\mathrm{SL}_2) \to \mathrm{End}_{\mathscr{A}}(\mathbb{C}\mathbf{Vect})$ is determined by specifying a $\mathbb{C}$-vector space $\mathcal{F}(\mathtt{X})$, a nondegenerate bilinear form $\mathcal{F}(\mathrm{cap})$ and a nondegenerate bilinear coform $\mathcal{F}(\mathrm{cup})$ satisfying the circle evaluation and the isotopy relation. From a matrix N as in the lemma we can get this data as follows. Firstly, let $\mathcal{F}(\mathtt{X}) = \mathbb{C}^n$ with fixed ordered basis $\{v_1, \dots, v_n\}$. Writing $\mathsf{N} = (m_{ij})_{1\leqslant i,j\leqslant n}$ and $\mathsf{N}^{-1} = (n_{ij})_{1\leqslant i,j\leqslant n}$ in this basis we have $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = \sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_{ij}n_{ij} = -[2]_q$. We then define $\mathcal{F}(\mathrm{cap})$ and $\mathcal{F}(\mathrm{cup})$ by

$$\mathcal{F}(\mathrm{cap})(v_i \otimes v_j) = m_{ij}, \quad \mathcal{F}(\mathrm{cup})(1) = \sum_{1\leqslant i,j\leqslant n} n_{ij} \cdot v_i \otimes v_j.$$

Since N is invertible we get that $\mathcal{F}(\mathrm{cap})$ and $\mathcal{F}(\mathrm{cup})$ are nondegenerate. They moreover satisfy the circle evaluation since $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$. Finally, they satisfy the isotopy relation since the coefficients m_{ij} defining $\mathcal{F}(\mathrm{cap})$ and the coefficients n_{ij} defining $\mathcal{F}(\mathrm{cup})$ are the entries of N and N^{-1}, respectively.

Reading the construction backwards gives a matrix $\mathsf{N} \in \mathrm{GL}_n(\mathbb{C})$ with $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$ from a 2-representation $\mathcal{F}\colon \mathscr{W}\mathbf{eb}(\mathrm{SL}_2) \to \mathrm{End}_{\mathscr{A}}(\mathbb{C}\mathbf{Vect})$. $\square$

Example 3C.2. *For $x \in \mathbb{C} \setminus \{0\}$ we call the matrices S of the form*

$$\mathsf{S}(x) = \begin{pmatrix} 0 & x \\ -qx & 0 \end{pmatrix} \text{ or } \mathsf{S}(x)' = \begin{pmatrix} 0 & x \\ -q^{-1}x & 0 \end{pmatrix}$$

the standard solutions *for* $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$. *One easily checks that $\mathsf{S}(x) \equiv_c \mathsf{S}(y)$ and $\mathsf{S}(x) \equiv_c \mathsf{S}(x)'$, and we can simply focus on $\mathsf{S} = \mathsf{S}(1)$.* $\lozenge$

Lemma 3C.3. *For every $n \in \mathbb{Z}_{\geqslant 2}$ there exists some $\mathsf{N} \in \mathrm{GL}_n(\mathbb{C})$ with $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$. For $n = 1$ there exists no such solution.*

Proof. Let id_k denote the k-by-k identity matrix. We take

$$\mathsf{N} = \left(\begin{array}{c|cc} id_{n-2} & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & x & 0 \end{array}\right). \tag{3C.4}$$

The matrix N is invertible and satisfies $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = (n-2) + x + x^{-1}$. Thus, we can let x be a solution of $x^2 + ([2]_q + n - 2)x + 1 = 0$ which exists since we work over $\mathbb{C}$.

The case $n = 1$ is discussed in Remark 3B.3. $\square$

For $n \in \mathbb{Z}_{\geqslant 2}$ let us denote by $\mathcal{F}^n_{\mathsf{N}}$ the 2-representation as constructed in the proof of Lemma 3C.1. The existence is guaranteed by Lemma 3C.3. Note also that $\mathcal{F}^n_{\mathsf{N}} \cong_{2rep} \mathcal{F}^m_{\mathsf{M}}$ implies $n = m$ and Lemma 3C.3 thus gives infinitely many nonequivalent rank one 2-representations of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$.

Lemma 3C.5. *The 2-representation $\mathcal{F}^n_{\mathsf{N}}$ is faithful, thus a fiber 2-representation.*

Proof. Using the basis of the hom-spaces of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ given by crossingless matching, see Lemma 3A.3, the lemma can be proven as follows. Firstly, for $\mathcal{F}^2_{\mathsf{S}}$ this is known by classical results, e.g. by [56, 2. Fundamentalsatz], which uses the crossingless matching basis. Indeed, this references, in modern language, also shows that $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ is the free category generated by a nondegenerate bilinear form. This in turn, by simply copying the $n = 2$ case, proves the lemma. Alternatively (and not written in 1930s German), [8, Section 4] also implies the lemma. □

Lemma 3C.6. *For any fiber 2-representation $\mathcal{M}\colon \mathscr{W}\mathbf{eb}(\mathrm{SL}_2)\to\mathrm{End}_{\mathscr{A}}(\mathbb{C}\mathbf{Vect})$ there exists a 2-representation $\mathcal{F}^n_{\mathsf{N}}$ with $\mathcal{M} \cong_{2rep} \mathcal{F}^n_{\mathsf{N}}$ as 2-representations of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$.*

Proof. From $\mathcal{M}$ we can get $\mathcal{F}^n_{\mathsf{N}}$ as follows. View $\mathcal{M}$ as a fiber functor and choose an ordered basis $\{v_1, \dots, v_n\}$ of $\mathcal{M}(\mathtt{X})$. Then we get the lexicographically ordered basis $\{v_1 \otimes v_1, \dots, v_1 \otimes v_n, \dots, v_n \otimes v_n\}$ of $\mathcal{M}(\mathtt{X}) \otimes \mathcal{M}(\mathtt{X})$. In this basis we get a 1-by-n^2 vector a determining $\mathcal{M}(\mathrm{cap})$ and a n^2-by-1 vector b determining $\mathcal{M}(\mathrm{cup})$. We then rearrange a and b into n-by-n matrices N and N^{-1} and the isotopy relation implies that these matrices, as suggested by their notation, are inverses. Moreover, the circle evaluation implies that $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$.

In total, we get a 2-representation of the form $\mathcal{F}^n_{\mathsf{N}}$. That $\mathcal{M} \cong_{2rep} \mathcal{F}^n_{\mathsf{N}}$ holds follows by construction. □

From this point onward we need to assume that we are in the semisimple case.

Lemma 3C.7. *Assume $q \in \mathbb{C} \setminus \{0\}$ is not a nontrivial root of unity. All rank one simple transitive 2-representations of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ are of the form $\mathcal{F}^n_{\mathsf{N}}$.*

Proof. By classical theory, see e.g. [64, Chapter XII] (this uses semisimplicity), we have the following property: Let $\mathbf{V}$ be any monoidal abelian category. Assume that $\mathtt{Y} \in \mathbf{V}$ has a right dual $\mathtt{Y}^\star$ and there exists an isomorphism $\mathrm{f}\colon \mathtt{Y} \to \mathtt{Y}^\star$ such that

$$\mathbb{1} \xrightarrow{ev} \mathtt{Y} \otimes \mathtt{Y}^\star \xrightarrow{\mathrm{f}\otimes\mathrm{f}^{-1}} \mathtt{Y}^\star \otimes \mathtt{Y} \xrightarrow{coev} \mathbb{1}$$

equals $-[2]_q{\cdot}\mathrm{id}_{\mathbb{1}}$. Then there exists a unique monoidal functor $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2) \to \mathbf{V}$ sending $\mathtt{X}$ to $\mathtt{Y}$.

Recall that Lemma 2.16 shows that for a rank one simple transitive 2-representation we can assume that $\mathbf{V} \cong \mathbb{C}\mathbf{Vect}$, and the proof completes. □

Lemma 3C.8. *Assume $q \in \mathbb{C} \setminus \{0\}$ is not a nontrivial root of unity. Every rank one simple transitive 2-representations of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ comes from a fiber 2-representation.*

Proof. We combine Lemma 3C.6 and Lemma 3C.7. □

Lemma 3C.9. *Assume $q \in \mathbb{C} \setminus \{0\}$ is not a nontrivial root of unity. We have $\mathcal{F}^n_{\mathsf{N}} \cong_{2rep} \mathcal{F}^m_{\mathsf{M}}$ as 2-representations of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ if and only if $\mathsf{N} \equiv_c \mathsf{M}$.*

Proof. Following the same arguments as in the proof of Lemma 3C.7, namely the characterization of monoidal functors $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2) \to \mathbf{V}$, one obtains that the datum of a rank one simple transitive 2-representation (or, alternatively, a fiber 2-representation by Lemma 3C.8) is equal to the datum of a $\mathbb{C}$-vector space and a bilinear form. In turn, bilinear forms are the same as matrix congruence, see Remark 3B.1, and the lemma follows then from the relationship of N and its associated bilinear form. □

It remains to analyze matrix congruence. Let $\mathsf{J}_n(\lambda)$ denote an n-by-n (upper triangular) Jordan block with eigenvalue $\lambda \in \mathbb{C}$. Additionally, define two matrices by

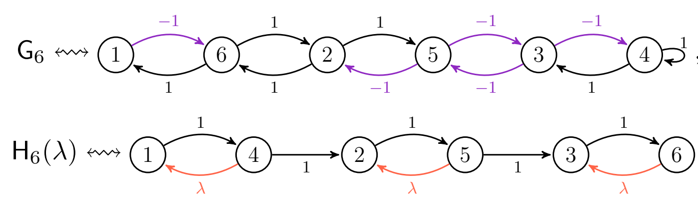
$$\mathsf{G}_n = \begin{pmatrix} & & & (\text{-}1)^n \\ & & (\text{-}1)^{n\text{-}1} & (\text{-}1)^{n\text{-}1} \\ & \iddots & \iddots & \\ \text{-}1 & \text{-}1 & & \\ 1 & 1 & & \end{pmatrix}, \quad \mathsf{H}_{2n}(\lambda) = \left(\begin{array}{c|c} 0 & id_n \\ \hline \mathsf{J}_n(\lambda) & 0 \end{array}\right).$$

The following is a normal form under $\equiv_c$ for complex n-by-n matrices $\mathsf{N} \in \mathrm{Mat}_n(\mathbb{C})$:

Lemma 3C.10. *Every $\mathsf{N} \in \mathrm{Mat}_n(\mathbb{C})$ is congruent to a direct sum of matrices of the form $\mathsf{J}_i(0)$, G_j or $\mathsf{H}_{2k}(\lambda)$ with $\lambda \notin \{0, (-1)^{k+1}\}$ determined up to $\lambda \leftrightarrow \lambda^{-1}$. Moreover, for $\mathsf{N} \in \mathrm{GL}_n(\mathbb{C})$ the matrices $\mathsf{J}_i(0)$ do not occur.*

Proof. This is [31, Theorem 1.1]. The tiny addition in the second sentence follows directly from the fact that the $\mathsf{J}_i(0)$ are degenerate. □

The matrices G_n and $\mathsf{H}_{2n}(\lambda)$ have the following associated weighted graphs with vertices labeled by the rows/columns:



We display $n = 4$ and $n = 3$ with the general picture being similar. Hence, the nondegenerate part of Lemma 3C.10 can be formulate using unions of these weighted graphs.

Example 3C.11. *Let $n = 2$ and take $x = 1$ in Example 3C.2. Then $\mathsf{S} = \mathsf{H}_2(-q)$.* ♢

Note that the Jordan blocks $\mathsf{J}_i(0)$ are all degenerate, so we can exclude them for our purposes, see the second part of Lemma 3C.10. For the remaining cases one directly checks that $\mathrm{tr}(\mathsf{G}_j^T\mathsf{G}_j^{-1}) = (-1)^{j+1}j$ and that $\mathrm{tr}\big(\mathsf{H}_{2k}(\lambda)^T\mathsf{H}_{2k}(\lambda)^{-1}\big) = k(\lambda + \lambda^{-1})$. Since $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1})$ is additive we get

$$\mathsf{C} = \bigoplus_{a=1}^{s} \mathsf{G}_{j_a} \oplus \bigoplus_{b=1}^{r} \mathsf{H}_{2k_b}(\lambda_b) \text{ satisfies } \mathrm{tr}(\mathsf{C}^T\mathsf{C}^{-1}) = \sum_{a=1}^{s}(-1)^{j_a+1}j + \sum_{b=1}^{r} k_b(\lambda_b + \lambda_b^{-1}).$$

Thus, Lemma 3C.10 gives us a list of solutions of $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$ up to $\equiv_c$. This is exactly what we want for Theorem 3B.2 to be as explicit as possible.

Example 3C.12. *For $n = 2$ we have $\mathrm{tr}(\mathsf{G}_2^T\mathsf{G}_2^{-1}) = -[2]_q$ or $\mathrm{tr}\big((\mathsf{G}_1 \oplus \mathsf{G}_1)^T(\mathsf{G}_1 \oplus \mathsf{G}_1)^{-1}\big) = -[2]_q$ if and only if $q = 1$ or $q = -1$, while $\mathrm{tr}\big(\mathsf{H}_2(\lambda)^T\mathsf{H}_2(\lambda)^{-1}\big) = -[2]_q$ if and only if $\lambda \in \{-q, -q^{-1}\}$. In particular, for $q \notin \{\pm 1\}$ we have S as an unique solution up to $\equiv_c$.* ♢

Example 3C.12 generalizes as follows:

Lemma 3C.13. *We have the following solutions of $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$ up to $\equiv_c$.*

(a) *For $n = 2$ there is the solution $\mathsf{N} = \mathsf{S}$ if $q \notin \{\pm 1\}$. For $q = 1$ has the additional solution $\mathsf{N} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and $q = -1$ has the additional solution $\mathsf{N} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.*

(b) *For $n = 3$ there are solutions of the form*

N	$\mathsf{G}_1 \oplus \mathsf{H}_1(\lambda)$	$\mathsf{G}_1 \oplus \mathsf{G}_1 \oplus \mathsf{G}_1$	$\mathsf{G}_1 \oplus \mathsf{G}_2$	G_3
#sols	*one or two*	*one for* $q \in \{\frac{1}{2}(-3 \pm \sqrt{5})\}$	*one for* $q \in \{\pm(-1)^{1/3}\}$	*one for* $q \in \{\frac{1}{2}(-3 \pm \sqrt{5})\}$

with λ a root of $x^2 + \frac{1}{2}(1 + [2]_q)x + 1$ which has two solutions unless $q \in \{\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(-5 \pm \sqrt{21})\}$.

(c) *For $n = 4$ there are infinitely many solutions.*

To get a complete list, we use the canonical forms under orthogonal congruence as found in e.g. [30].

Proof. Directly from the above discussion, and omitted. We only point out two observations.

First, note that general congruence will not keep $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$ invariant. In particular, the above needs to be combined with orthogonal congruence as in the final sentence of the lemma.

Second, that for $n \geqslant 4$ we can have $\mathsf{H}_k(\lambda) \oplus \mathsf{H}_l(\mu) \oplus \mathrm{Rest}$ appearing. Say Rest only consists of G_j summands. Then we get infinitely many solutions: Fix an arbitrary μ. Then the relevant equations for λ always have solutions since our ground field is algebraically closed. $\square$

Thus, we have proven Theorem 3B.2.

3D. Proof of Proposition 3B.4.

Lemma 3D.1. *We have $K_0^{\oplus}(\mathcal{F}^n_{\mathsf{N}}) \cong_{rep} K_0^{\oplus}(\mathcal{F}^m_{\mathsf{M}})$ as $K_0^{\oplus}\big(\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)\big)$-representations if and only if $n = m$.*

Proof. To see that we have $K_0^{\oplus}(\mathcal{F}^n_{\mathsf{N}}) \ncong_{rep} K_0^{\oplus}(\mathcal{F}^m_{\mathsf{M}})$ for $n \neq m$ we observe that $K_0^{\oplus}\big(\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)\big) \cong \mathbb{Z}[X]$ as rings via the map $[\mathtt{X}] \mapsto X$, and X acts on $K_0^{\oplus}(\mathcal{F}^n_{\mathsf{N}})$ by n. The converse follows since the 2-representations of the form $\mathcal{F}^n_{\mathsf{N}}$ are given by fiber functors and the twist of the bilinear form and coform can not be detected, cf. [21, Theorem 5.3.12]. $\square$

Lemma 3D.2. *The fiber 2-representation $\mathcal{F}^n_{\mathsf{N}}$ is braided if and only if ($q = 1$, $n = 2$ and N is a standard solution as in Example 3C.2).*

Proof. Lemma 2.26 implies that $\mathcal{F}^n_{\mathsf{N}}$ being braided implies that $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ is symmetric and that the crossing is send to the flip map. The following calculation shows that the standard solution is the only possible choice where that happens.

We view $\mathcal{F}^n_{\mathsf{N}}$ as a functor $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2) \to \mathbb{C}\mathbf{Vect}$. By the proof of Lemma 3C.1, we have that

$$\mathcal{F}^n_{\mathsf{N}}(\mathrm{cup}\circ\mathrm{cap})(v_i \otimes v_j) = m_{ij}\mathcal{F}^n_{\mathsf{N}}(\mathrm{cup})(1) = m_{ij} \sum_{1 \leqslant k,l \leqslant n} n_{kl} \cdot v_k \otimes v_l.$$

Hence, we get that

(3D.3) $\mathcal{F}^n_{\mathsf{N}}\Bigg($  $\Bigg)(v_i \otimes v_j) = q^{1/2} \cdot v_i \otimes v_j + q^{-1/2} \cdot \big(m_{ij} \sum_{1 \leqslant k,l \leqslant n} n_{kl} \cdot v_k \otimes v_l\big).$

For this to be the flip map we then need $m_{ij}n_{ij} = -q$, $m_{ij}n_{ji} = q^{1/2}$ and $n_{kl} = 0$ else. Since these have to hold for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ we therefore need $n = 2$.

For $n = 2$ a direct calculation shows that the only 2-by-2 matrices with $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$ and with Equation 3D.3 being the flip map are the standard solutions for $q = 1$. $\square$

In the two latter cases in Lemma 3D.2 the fiber 2-representation $\mathcal{F}^n_{\mathsf{N}}$ is even symmetric by Lemma 2.25.

Example 3D.4. *We again view $\mathcal{F}^n_{\mathsf{N}}$ as a functor $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2) \to \mathbb{C}\mathbf{Vect}$. Let $n = 3$ and take the matrix N as in Equation 3C.4. For $q = 1$ the variable x has to be $\frac{1}{2}(-3 \pm \sqrt{5})$. For $x = \frac{1}{2}(-3 + \sqrt{5})$ one gets (in an appropriate order of the basis*

$\{v_i \otimes v_j | 1 \leqslant i,j \leqslant n\}$) *that*

$$\mathcal{F}^3_{\mathsf{N}}\left(\right.$$  $$\left.\right) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & x & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & y & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ x^g & 0 & 0 & 0 & 0 & 1 & 0 & y^g & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \begin{aligned} x &= \tfrac{1}{2}(-3+\sqrt{5}), \\ x^g &= \tfrac{1}{2}(-3-\sqrt{5}), \\ y &= \tfrac{1}{2}(-1+\sqrt{5}), \\ y^g &= \tfrac{1}{2}(-1-\sqrt{5}), \end{aligned}$$

which squares to the identity, but is clearly not the flip map. ◊

Lemma 3D.5. *There exist infinitely many Hopf algebras H with $\mathbb{C}\mathbf{coRep}(H) \cong \mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ as monoidal categories. In particular, infinitely many of these Hopf algebras are not isomorphic to $\mathcal{O}_q\big(\mathrm{SL}_2(\mathbb{C})\big)$.*

Proof. Reconstruction theory implies that any fiber 2-functor $\mathcal{F}\colon \mathscr{W}\mathbf{eb}(\mathrm{SL}_2) \to \mathbb{C}\mathbf{Vect}$ gives rise to a Hopf algebra H being the coend of $\mathcal{F}$. The comodules over H give a monoidal category equivalent to $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$. All of this is a direct consequence of [21, Theorem 4.3.1]. Reconstruction theory moreover implies that the resulting Hopf algebras are not isomorphic whenever the used fiber functors are not equivalent. Now we use Theorem 3B.2. □

The section is complete.

4. Rank one 2-representations of GL2 webs

A lot of constructions and arguments in this section are similar to those in section 3, so we will be brief.

4A. GL2 webs.

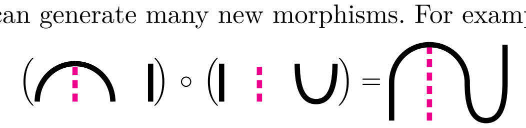
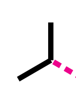
We define webs for GL_2 as follows.

Remark 4A.1. We have two types of strands in this section with the following names:

usual: , phantom: .

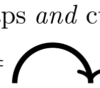
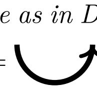
Both types carry an orientation. We omit the orientations in case they do not play a role in order to not overload the illustrations. In this case we mean any consistent orientation. ◊

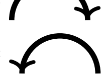
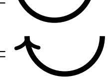
Remark 4A.2. Before reading Definition 4A.4 we remind the reader that, using isotopy, one can generate many new morphisms. For example,

(4A.3)  is isotopic to .

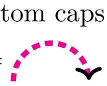
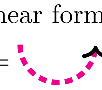
We use this silently in Definition 4A.4 below. ◊

Definition 4A.4. *Fix $q \in \mathbb{C}\setminus\{0\}$. Let $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ denote the $\mathbb{C}$-linear pivotal category $\otimes$-generated by the dual objects* $\mathtt{X}$, $\mathtt{Y}$, *and the dual objects* $\mathtt{P}$, $\mathtt{Q}$, *and $\circ$-$\otimes$-generated by morphisms called* caps *and* cups, *displayed and use as in Definition 3A.1 but oriented:*

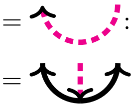
cap = : $\mathtt{X}\otimes\mathtt{Y} \to \mathbb{1}$, cup = : $\mathbb{1} \to \mathtt{Y}\otimes\mathtt{X}$,

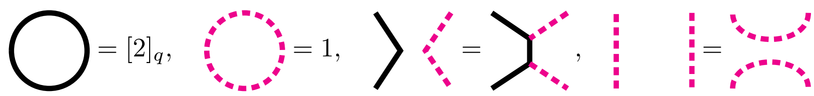
cap′ = : $\mathtt{Y}\otimes\mathtt{X} \to \mathbb{1}$, cup′ = : $\mathbb{1} \to \mathtt{X}\otimes\mathtt{Y}$,

as well as phantom caps *and* cups, phantom trilinear forms *and* coforms*:*

pcap = : $\mathtt{P}\otimes\mathtt{Q} \to \mathbb{1}$, pcup = : $\mathbb{1} \to \mathtt{Q}\otimes\mathtt{P}$,

pcap′ = : $\mathtt{Q}\otimes\mathtt{P} \to \mathbb{1}$, pcup′ = : $\mathbb{1} \to \mathtt{P}\otimes\mathtt{Q}$,

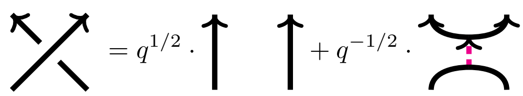
tup = : $\mathtt{X}\otimes\mathtt{Q}\otimes\mathtt{X} \to \mathbb{1}$, tdown = : $\mathbb{1} \to \mathtt{X}\otimes\mathtt{Q}\otimes\mathtt{X}$, *modulo the* $\circ$-$\otimes$*-ideal generated by* isotopy *(not illustrated; we impose all possible plane isotopies),* circle *and* phantom circle evaluation*,* H=I *and* vertical=horizontal relation *(in all consistent orientations):*

(4A.5)


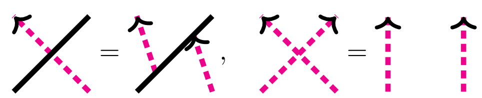
We call $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ *the* GL_2 web category *and its morphism* GL_2 webs*.* ◊

Lemma 4A.6. *We have the following.*

(a) *The 'oriented version' of Equation 3A.5 given by e.g.*

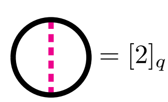


and additionally

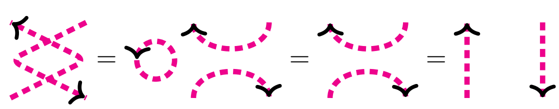


and similar formulas define a braiding on $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ *with the phantom strings being part of a symmetric structure where the Reidemeister I relations holds (the full subcategory generated by* `P` *and* `Q` *is symmetric with the phantom crossing).*

(b) *We have the* trilinear evaluation*:*

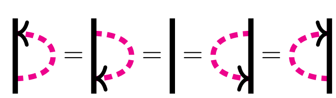


Proof. (a)—non-mixed part. Easy and we just give one calculation:



This uses the phantom circle evaluation and vertical=horizontal relation.

(a)—mixed part. One first shows that

(4A.7)


This is a direct consequence of the vertical=horizontal relation. Using this and similar formulas, one can show that the above defines a braiding.

(b). Immediately from Equation 4A.7. □

For two objects $\mathtt{A}, \mathtt{B} \in \mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ let CP_A^B denote any (fixed) choice of placement of phantom edges such that the GL_2 web obtained by removing the phantom edges corresponds to a crossingless matching.

Lemma 4A.8. *The set* CP_A^B *is a* $\mathbb{C}$*-basis of* $\mathrm{Hom}_{\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)}(\mathtt{A}, \mathtt{B})$*.*

Proof. Directly by using the braiding in Lemma 4A.6 and the usual crossingless matching basis of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$, see Lemma 3A.3. In more details, the relations involving phantom strings ensure that we have two cases. Firstly, a phantom string touches a usual string an even number of times. Then the phantom string can be unplugged from the usual string. On the other hand, if they touch an odd number of times, then the phantom string can be unplugged up to one attachment, and this attachment can be placed arbitrarily along the usual string. This in turn implies that the usual crossingless matching basis plus an arbitrary, but fixed and minimal, placement of phantom strings gives a basis. $\square$

The remainder of subsection 3A goes through for $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ with one mild change, namely Lemma 4A.9.(a). That is:

Lemma 4A.9. *We have the following.*

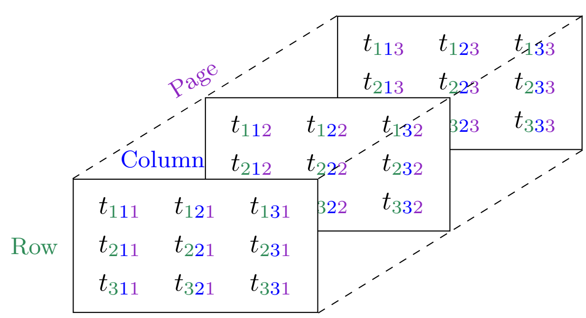
(a) *The simple objects of* $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ *are in one-to-one correspondence with* $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}$.

(b) $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ *is semisimple if only if* $q \in \mathbb{C} \setminus \{0\}$ *is not a nontrivial root of unity.*

(c) $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ *is an essentially fusion category if only if* $q \in \mathbb{C} \setminus \{0\}$ *is not a nontrivial root of unity.*

Proof. The statement follows similarly as in Lemma 3A.4. $\square$

4B. The main theorem in the GL2 case. For us an *third order tensor* is an l-by-m-by-n array of complex numbers. We represent a third order tensor by $\mathsf{T} = (t_{ijk})_{i,j,k}$ with $t_{ijk} \in \mathbb{C}$. The indexes are the rows and columns, as for usual matrices, and the *pages* k. Here is an illustration for $l = m = n = 3$:

(4B.1) 

.

As before, fix bases $\{v_1, \dots, v_a\}$ of $\mathbb{C}^a$. It is immediate that a third order tensor gives a trilinear form by

$$\mathsf{T}\colon \mathbb{C}^l \otimes \mathbb{C}^m \otimes \mathbb{C}^n \to \mathbb{C}, v_i \otimes v_j \otimes v_k \mapsto t_{ijk}.$$

In other words, T is a 1-by-lmn matrix.

Assume that we have already fixed a matrix N that corresponds to a bilinear form. We therefore get matrices associated to caps and cups. Let us call these matrices $\mathsf{N}(cap)$ and $\mathsf{N}(cup)$, respectively. Define

$$\mathsf{T}_l = (\mathsf{T} \otimes id_{\mathbb{C}^n}) \circ \big(id_{\mathbb{C}^l} \otimes id_{\mathbb{C}^m} \otimes \mathsf{N}(cup)\big),$$
$$\mathsf{T}^l = \big(id_{\mathbb{C}^l} \otimes id_{\mathbb{C}^m} \otimes \mathsf{N}(cap)\big) \circ (\mathsf{T}' \otimes id_{\mathbb{C}^n}),$$

where T' is the transpose tensor. The picture to keep in mind is Equation 4A.3 which displays the diagrammatic interpretation of T_l.

Let us denote the set of l-by-m-by-n tensors by $\mathrm{T}_{l,m,n}(\mathbb{C})$, and for elements in that set let us write $\mathsf{T} \equiv_c \mathsf{U}$ for *congruence of third order tensors* in the sense of e.g. [6, Section 4], meaning, roughly speaking, that they define the same trilinear form up to change-of-basis. Below we write $\mathsf{P}(cap) = \mathsf{P}(cup) = \mathsf{P}$ to highlight how the next display fits to Equation 4A.5.

Theorem 4B.2. *Assume* $q \in \mathbb{C} \setminus \{0\}$ *is not a nontrivial root of unity.*

(a) *Let $n \geqslant 2$. For every triple $\vec{N} = (\mathsf{N}, \mathsf{P}, \mathsf{T}) \in \mathrm{GL}_n(\mathbb{C}) \times \{\pm 1\} \times \mathrm{T}_{n,1,n}(\mathbb{C})$ with*

$$\mathrm{tr}(\mathsf{N}^T \mathsf{N}^{-1}) = [2]_q, \quad id_{\mathbb{C}^l \otimes \mathbb{C}} = \mathsf{T}^l \mathsf{T}_l, \quad id_{\mathbb{C} \otimes \mathbb{C}} = \mathsf{P}(cap)\mathsf{P}(cup)$$

there exists a simple transitive fiber 2-representation $\mathcal{F}_{\vec{N}}$ of $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ constructed in the proof of Lemma 4C.3. ***(Existence)***

(b) *We have $\mathcal{F}_{(\mathsf{N},\mathsf{P},\mathsf{T})} \cong_{2rep} \mathcal{F}_{(\mathsf{M},\mathsf{Q},\mathsf{U})}$ if and only if $\mathsf{N} \equiv_c \mathsf{M}$, $\mathsf{P} = \mathsf{Q}$ and $\mathsf{T} \equiv_c \mathsf{U}$.* ***(Non-redundant)***

(c) *Every simple transitive fiber 2-representation of $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ is of the form $\mathcal{F}^n_{\mathsf{N}}$, and every simple transitive rank one 2-representation of $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ arises in this way.* ***(Complete)***

Moreover, there are infinitely many nonequivalent simple transitive rank one 2-representations of $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$.

As before, we list some $\mathcal{F}_{\vec{N}}$ for $n \in \{2, 3\}$, while for $n = 4$ there are infinitely many nonequivalent $\mathcal{F}_{\vec{N}}$, see Lemma 4C.7 below for details (also as before, Theorem 4B.2 and Lemma 4C.7 taken together solve Classification Problem 2.19 for $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$), and the proof of Theorem 4B.2 gets its own section.

Moreover, we leave it to the reader to spell out the GL_2 analog of Proposition 3B.4 (which reads essentially the same). We rather wrap-up this section with a (historical) remark and another remark:

Remark 4B.3. $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ was first considered to construct a functorial version of Khovanov homology [9], and GL_2 webs have been studied intensively in the context of link homologies, see e.g. [18, 17, 5, 35]. Indeed, our presentation of $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ is stolen from [5]. ♢

Remark 4B.4. The reader familiar with [43] will notice that the main theorem of that paper and Theorem 4B.2 are different. This is due to us using diagrammatics that are not used in [43]. Hence, Theorem 4B.2 appears to be new in the presented form, and is the expected extension of Theorem 3B.2. ♢

4C. Proof of Theorem 4B.2.

The proof of Theorem 4B.2 is, of course, similar to the proof of Theorem 3B.2 so we will be rather brief and focus on the main differences.

Let $\mathscr{P}\mathbf{eb}(\mathrm{GL}_2) \subset \mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ denote the full subcategory generated by $\mathtt{P}, \mathtt{Q}$.

Lemma 4C.1. *We have the following.*

(a) *For $\mathsf{P} \in \{\pm 1\}$ there exists a simple transitive fiber 2-representation $\mathcal{F}_{\mathsf{P}}$ of $\mathscr{P}\mathbf{eb}(\mathrm{GL}_2)$ constructed similarly as in the proof of Lemma 3C.1.* ***(Existence)***

(b) *$\mathcal{F}_{+1}$ is not equivalent to $\mathcal{F}_{-1}$ as 2-representations of $\mathscr{P}\mathbf{eb}(\mathrm{GL}_2)$.* ***(Non-redundant)***

(c) *Every simple transitive fiber 2-representation of $\mathscr{P}\mathbf{eb}(\mathrm{GL}_2)$ is of the form $\mathcal{F}_{\pm 1}$, and every simple transitive rank one 2-representation of $\mathscr{P}\mathbf{eb}(\mathrm{GL}_2)$ arises in this way.* ***(Complete)***

Proof. The proof is similar, but much easier, than the proof of Theorem 3B.2. So let us only give the two new observations needed for the proof.

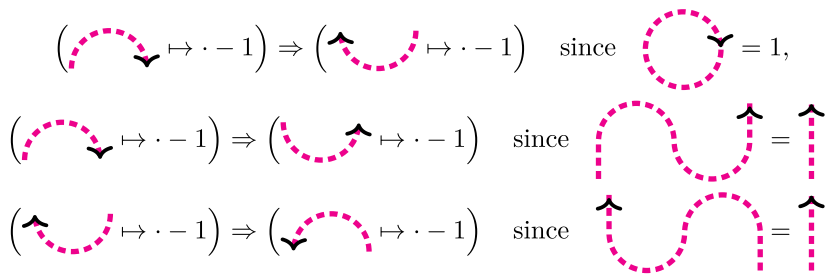
Assume that we have a one column $cap = (a_1, \dots, a_n)^T$ and a one row matrix $cup = (b_1, \dots, b_n)$. Then

$$cap \times cup = (a_1 b_1 + \dots + a_n b_n), \quad \text{the diagonal of } cup \times cap \text{ is } (a_1 b_1, \dots, a_n b_n).$$

In particular, $cap \times cup = (1)$ and $cup \times cap = id_n$ can only hold for $n = 1$. Moreover, for $n = 1$ the only possible solutions are $a_1 = b_1 = \pm 1$. Thus, a 2-representation of

$\mathscr{P}\mathbf{eb}(\mathrm{GL}_2)$ needs to send both generating objects to $\mathbb{C}$, and the phantom caps and cups to multiplication by ± 1.

It then follows from the phantom circle removal and the isotopy relations that fixing ± 1 as the value for pcap determines the other three bilinear (co)forms, so we only have ± 1 to vary. That is:

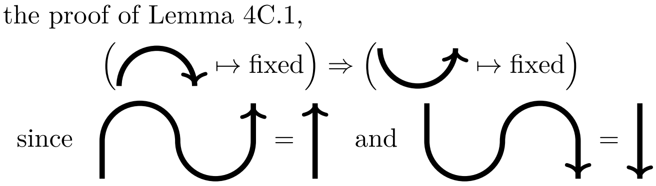


All other cases follow by symmetry. $\square$

Similarly as above, let $\mathscr{U}\mathbf{eb}(\mathrm{GL}_2) \subset \mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ denote the full subcategory generated by $\mathtt{X}, \mathtt{Y}$.

Lemma 4C.2. *Theorem 3B.2 holds verbatim for* $\mathscr{U}\mathbf{eb}(\mathrm{GL}_2)$.

Proof. As in the proof of Lemma 4C.1,



etc. (as above, the cup oriented rightwards and the circle evaluation fixes the assignment for the cap oriented leftwards, and then the zigzag fixes the assignment for the cup oriented leftwards). The rest of the proof works, mutatis mutandis, as for SL_2. $\square$

We will refer to the triples $\vec{N} = (\mathsf{N}, \mathsf{P}, \mathsf{T})$ in Theorem 4B.2 as GL_2 *triples.*

Lemma 4C.3. *For* $n \in \mathbb{Z}_{\geqslant 2}$ *let* $\vec{N} = (\mathsf{N}, \mathsf{P}, \mathsf{T})$ *be a* GL_2 *triple. Then there exists an associated 2-representation* $\mathcal{F}$ *of* $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ *on* $\mathbb{C}\mathbf{Vect}$ *with* $\dim_{\mathbb{C}} \mathcal{F}(\mathtt{X}) = \dim_{\mathbb{C}} \mathcal{F}(\mathtt{Y}) = n$ *and* $\dim_{\mathbb{C}} \mathcal{F}(\mathtt{P}) = \dim_{\mathbb{C}} \mathcal{F}(\mathtt{Q}) = 1$. *Conversely, every 2-representation* $\mathcal{F}$ *of* $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ *on* $\mathbb{C}\mathbf{Vect}$ *with* $\dim_{\mathbb{C}} \mathcal{F}(\mathtt{P}) = \dim_{\mathbb{C}} \mathcal{F}(\mathtt{Q}) = 1$ *gives such a triple.*

Proof. Very similar to the proof of Lemma 4C.3 with the following two differences. Firstly, the phantom part is taken care of by Lemma 4C.1 while the usual part is Lemma 4C.2. The two sides are related via the trilinear form and the H=I relation. Note that Lemma 4A.6 shows that the trilinear form determines the trilinear coform in exactly the same way as the bilinear form and coform are related, so we only need to specify the trilinear form. Finally, the H=I relation is part of the definition. $\square$

Lemma 4C.4. *For every* $n \in \mathbb{Z}_{\geqslant 2}$ *there exists some* GL_2 *triple. For* $n = 1$ *there exists no such triple.*

Proof. Let us take $\mathsf{P} = 1$, and let N be any matrix satisfying $\mathrm{tr}(\mathsf{N}^T \mathsf{N}^{-1}) = [2]_q$. The existence of the latter is guaranteed by (the same arguments as in) Lemma 3C.3, while the choice $\mathsf{P} = 1$ satisfies $\mathrm{tr}(\mathsf{P}^T \mathsf{P}^{-1}) = 1$ and $id = \mathsf{P}(cap)\mathsf{P}(cup)$. We may construct a trilinear form T by mapping $v_i \otimes 1 \otimes v_k \mapsto \mathsf{N}(cap)(v_i \otimes v_k)$. This form satisfies the conditions of Theorem 4B.2. On the diagrammatic side, this corresponds to ignoring the phantom edges and identifying the bilinear form with the trilinear form. The property $id = \mathsf{T}^l \mathsf{T}_l$ is then clearly satisfied.

For $n = 1$ see Remark 3B.3. $\square$

As before, let $\mathcal{F}_{\vec{N}}$ be the 2-representation constructed above.

Lemma 4C.5. *The 2-representation $\mathcal{F}_{\vec{N}}$ is faithful, thus a fiber 2-representation.*

Proof. As before, but using Lemma 4A.8 instead of Lemma 3A.3. $\square$

Lemma 4C.6. *The statements Lemma 3C.6 to Lemma 3C.9 hold mutatis mutandis for $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ as well.*

Proof. Only two things changes with respect to the proofs given in subsection 3C. Firstly, one uses the basis in Lemma 4A.8 instead of the crossingless matching basis. Second, the careful reader can copy the arguments in [64, Chapter XII] to get the analog of the result used in the proof of Lemma 3C.7. $\square$

Lemma 4C.7. GL_2 *triples, up to $\equiv_c$, are given by:*

- ▷ *The matrix* N *is classified as in Lemma 3C.13.*
- ▷ *The sign* P *can be chosen freely.*
- ▷ *The tensor* T *is classified as* N *in Lemma 3C.13 together with the choice of a sign.*

Proof. The only extra information one needs beyond Lemma 3C.13 is the classification of n-by-1-by-n trilinear forms, which is the same as the classification of n-by-n bilinear forms up to a sign. This is easy to see, but can also be found explicitly spelled out in [60, Introduction]. $\square$

Hence, taking the above together proves Theorem 4B.2.

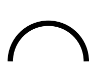
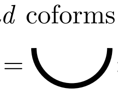
5. Rank one 2-representations of SO3 webs

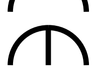
As expected, a lot of constructions and arguments in this section are similar to those in the previous sections, so we will be brief and focus on the new bits.

5A. SO3 webs.

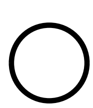
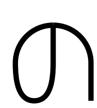
We start with a reminder on the SO_3 web category. As in the previous section we silently use (an analog of) Remark 4A.2.

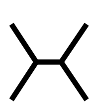
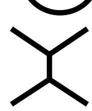
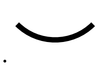
Definition 5A.1. *Fix $q \in \mathbb{C} \setminus \{0\}$ with $q^2 + q^{-2} \neq 0$. Let $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ denote the $\mathbb{C}$-linear pivotal category $\otimes$-generated by the selfdual object $\mathtt{X}$, and $\circ$-$\otimes$-generated by morphisms called* bilinear *and* trilinear forms *and* coforms*:*

cap =  $: \mathtt{X} \otimes \mathtt{X} \to \mathbb{1}$, cup =  $: \mathbb{1} \to \mathtt{X} \otimes \mathtt{X}$,

tup =  $: \mathtt{X} \otimes \mathtt{X} \otimes \mathtt{X} \to \mathbb{1}$, tdown =  $: \mathbb{1} \to \mathtt{X} \otimes \mathtt{X} \otimes \mathtt{X}$,

modulo the $\circ$-$\otimes$-ideal generated by isotopy *(not displayed; we impose all possible plane isotopies),* circle *and* bitri evaluation*, and the* H=I relation*:*

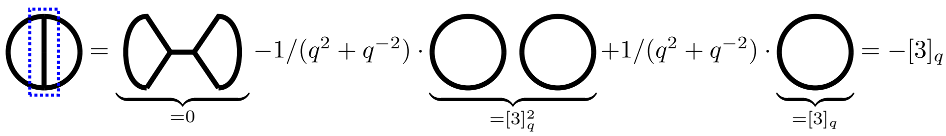
 $= [3]_q = q^2 + 1 + q^{-2}$,  $= 0$,

 $=$  $+ 1/(q^2+q^{-2}) \cdot$  $- 1/(q^2+q^{-2}) \cdot$  .

We call $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ the SO_3 web category *and its morphism* SO_3 webs. ◊

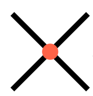
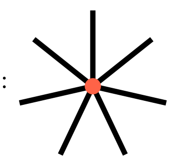
Remark 5A.2. Note that we do not define the SO_3 web category for $q^2 + q^{-2} = 0$. In particular, when talking about this category we will always assume that $q^2 + q^{-2} \neq 0$. ◊

EXAMPLE 5A.3. *The H=I relation can be used to systematically reduce faces of* SO_3 *webs in their complexity. For example,*

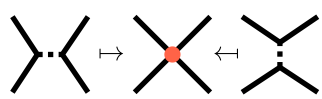


In the first picture we highlighted an I that we then replaced by H and error terms. In a similar fashion one can get relations for other faces as well. $\Diamond$

A *higher valent vertices*, exemplified, is:

4 valent:  , *7 valent*:  .

The dot is a visual aid only. Diagrams that are allowed to have these additional vertices are embedded graphs with specified bottom and top boundary. An edge of such a graph is called *inner* if it does not touch the boundary. The *contraction* operation is



where the dotted edge is contracted.

DEFINITION 5A.4. *Let* $k+l$ *be the number of boundary points of* SO_3 *web* u*. We say* u *is a* partition *(of the set* $\{1,\dots,k+l\}$*) if:*

(a) u *is one* SO_3 *web, i.e. not a nontrivial* $\mathbb{C}$*-linear combination of such diagrams.*

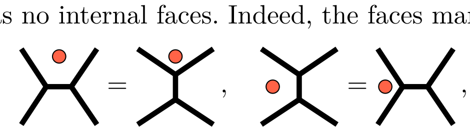
(b) u *has no internal faces.*

(c) *After a finite number of contractions,* u *is a graph without inner edges. (Here we see* u *as a trivalent graph and then apply contraction.)*

Let P_k^l *be the set of all partition* SO_3 *web diagrams with* k *bottom and* l *top boundary points.* $\Diamond$

LEMMA 5A.5. *The set* P_k^l *is a* $\mathbb{C}$*-basis of* $\mathrm{Hom}_{\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)}(\mathtt{X}^{\otimes k},\mathtt{X}^{\otimes l})$*.*

Proof. Spanning. As exemplified in Example 5A.3, the H=I relation implies that we can assume that u has no internal faces. Indeed, the faces marked with a bullet in



will have fewer edges on the right-hand sides when compared to the left-hand sides. We can repeat this operation until some internal face is a monogon and the bitri evaluation applies. Moreover, the two error terms in the H=I relation are simpler SO_3 webs since the number of vertices is smaller than for the other two SO_3 webs. In other words, internal faces can be removed recursively. Finally, the H=I relation let us get rid of inner edges, which shows that P_k^l spans.

Linear independence. There is a bijection from P_k^l to the set of all planar partitions of the set $\{1,\dots,k+l\}$ where every block has at least two parts given by associating a partition to a partition SO_3 web diagram by interpreting the connected components of the web as blocks of the partition. Let $pp(k,l)$ be the number of such partitions. Since P_k^l spans $\mathrm{Hom}_{\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)}(\mathtt{X}^{\otimes k},\mathtt{X}^{\otimes l})$, we get

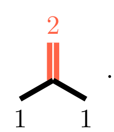
$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)}(\mathtt{X}^{\otimes k}, \mathtt{X}^{\otimes l}) \leqslant pp(k,l)$, while pivotality and [26, Lemma 4.1] imply that $pp(k,l) \leqslant \dim_{\mathbb{C}} \mathrm{Hom}_{\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)}(\mathtt{X}^{\otimes k}, \mathtt{X}^{\otimes l})$. Hence, linear independence follows. ☐

Remark 5A.6. The numbers $pp(k,l)$ are well-known in combinatorics. Without loss of generality we can consider $pp(k,l)$ for $l = 0$ and one gets

$$\{1,0,1,1,3,6,15,36,91,232,603\}, \quad pp(k,0) \text{ for } k = 0,\dots,10.$$

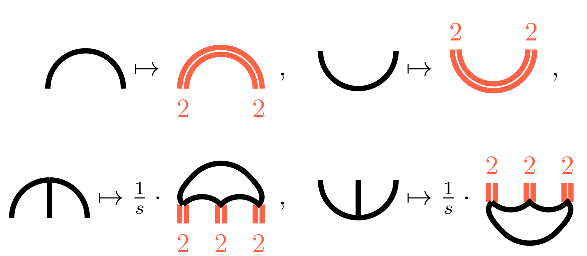
The sequence is [49, A005043]. ◊

Let us denote symmetric SL_2 webs in the sense of [55] by using labeled (and colored) edges, for example,



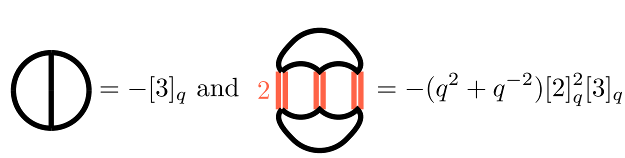
The edge labeled 1 are uncolored. Let $\mathscr{S}\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ denote the associated $\mathbb{C}$-linear pivotal category.

Lemma 5A.7. *Assume $q \in \mathbb{C} \setminus \{0\}$ is not a nontrivial root of unity. There is a faithful $\mathbb{C}$-linear pivotal functor $\mathcal{I}\colon \mathscr{W}\mathbf{eb}(\mathrm{SO}_3) \to \mathscr{S}\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ determined by*



where we choose a square root $s = \big((q^2+q^{-2})[2]_q^2\big)^{1/2}$ of $(q^2+q^{-2})[2]_q^2$.

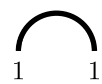
Proof. A direct verification shows that the above defines a $\mathbb{C}$-linear pivotal functor. There are only two things to note here. Firstly, the scaling which comes from the comparison of the relations



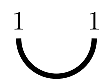
Second, to verify the defining relations hold in the image of $\mathcal{I}$ is an easy calculation.

That $\mathcal{I}$ is an embedding can be checked by using Lemma 5A.5 and the faithful representation Γ_{sym} of $\mathscr{S}\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ on symmetric powers obtained from the functor used in the proof of [55, Theorem 1.10]. The only thing the reader needs to know to verify this is the following. Fix the basis $\{v_1, v_2\}$ of $\mathbb{C}^2$. The basis elements of $\mathrm{Sym}^2\mathbb{C}^2$

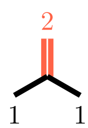
are chosen to be $\{v_1v_1, v_1v_2 = q^{-1} \cdot v_2v_1, v_2v_2\}$. Then



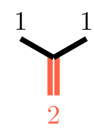
$$\mapsto \begin{cases} v_i \otimes v_i \mapsto 0, \\ v_1 \otimes v_2 \mapsto -q, v_2 \otimes v_1 \mapsto 1, \end{cases}$$



$$\mapsto (1 \mapsto v_1 \otimes v_2 - q^{-1} \cdot v_2 \otimes v_1),$$



$$\mapsto (v_i \otimes v_j \mapsto v_iv_j),$$



$$\mapsto \begin{cases} v_iv_i \mapsto [2]_q \cdot v_i \otimes v_i, \\ v_1v_2 \mapsto q^{-1}v_1 \otimes v_2 + v_2 \otimes v_1, \end{cases}$$

under Γ_{sym}, while the 2 labeled caps and cups are defined by explosion, see [55, Definition 2.18]. □

LEMMA 5A.8. *We have the following.*

(a) *The simple objects of* $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ *are in one-to-one correspondence with* $\mathbb{Z}_{\geqslant 0}$.

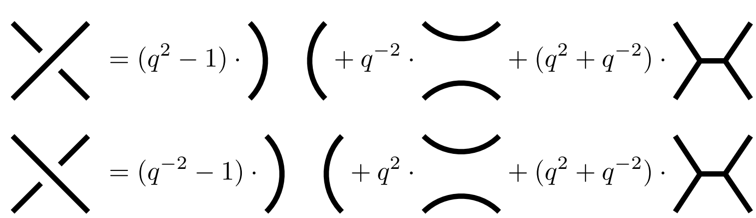
(b) $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ *is semisimple if only if* $q \in \mathbb{C} \setminus \{0\}$ *is not a nontrivial root of unity.*

(c) $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ *is an essentially fusion category if only if* $q \in \mathbb{C} \setminus \{0\}$ *is not a nontrivial root of unity.*

Proof. This follows from Lemma 3A.4 and the fact that SO_3 webs can be constructed as the full subcategory of SL_2 webs $\otimes$-generated by the diagrammatic analog of $\mathrm{Sym}^2\mathbb{C}^2$, see Lemma 5A.7. □

For SO_3 webs the crossing formulas are:

(5A.9)



Now all of Lemma 3A.6 (with Equation 5A.9) and Notation 3A.7 have the evident SO_3 analog (their formulation is omitted) and we will use these analogs freely. In particular, $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ is a braided category.

5B. THE MAIN THEOREM IN THE SO3 CASE. Recall that we introduced our notation for tensors in subsection 4B. We will use the same conventions now.

THEOREM 5B.1. *Assume* $q \in \mathbb{C} \setminus \{0\}$ *is not a nontrivial root of unity.*

(a) *Let* $n \geqslant 3$. *For every pair* $\vec{N} = (\mathsf{N}, \mathsf{T}) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{T}_{n,n,n}(\mathbb{C})$ *with* $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = [3]_q$, $\mathrm{tr}\big(\mathsf{T}(\mathsf{N}(cup) \otimes id)\big) = 0$, *and* $(id \otimes \mathsf{T}_l) \circ (\mathsf{T}^l \otimes id) = \mathsf{T}^l\mathsf{T}_l + 1/(q^2 + q^{-2}) \cdot id - 1/(q^2 + q^{-2}) \cdot \mathsf{N}(cap)\mathsf{N}(cup)$ *there exists a simple transitive fiber 2-representation* $\mathcal{F}^n_{\mathsf{N},\mathsf{T}}$ *of* $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ *constructed similarly to the proof of Lemma 3C.1.* ***(Existence)***

(b) *We have* $\mathcal{F}_{(\mathsf{N},\mathsf{T})} \cong_{2rep} \mathcal{F}_{(\mathsf{M},\mathsf{U})}$ *if and only if* $\mathsf{N} \equiv_c \mathsf{M}$ *and* $\mathsf{T} \equiv_c \mathsf{U}$. ***(Non-redundant)***

(c) *Every simple transitive fiber 2-representation of* $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ *is of the form* $\mathcal{F}_{\vec{N}}$*, and every simple transitive rank one 2-representation of* $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ *arises in this way.* ***(Complete)***

Moreover, there are infinitely many nonequivalent simple transitive rank one 2-representations of $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$*.*

We also show that for $n = 3$ there is only one possible solution.

Essentially all we said at the end of subsection 3B (before the remarks) applies for SO_3 webs as well. In particular, we leave the analog of Proposition 3B.4 to the reader, and we will only focus on the crucial difference compared to the other two cases: the appearance of (honest) trilinear forms. This might make a “huge” difference, see section 6 for a more detailed discussion.

Remark 5B.2. The category $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ was discovered in the early days of quantum topology, see [67] for the potentially earliest reference. In that paper it is effectively shown that $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ gives a diagrammatic description of SO_3-representations (this can be pieced together by comparing Lemma 5A.7 and the MathSciNet review of [67]). As far as we know, $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ is the oldest diagram category that truly deserves the name web category. Its importance stems from its connection to, for example, the chromatic polynomial and the four color theorem in graph theory. This connection originates in [59], see [25, Introduction] for a list of early appearances of this relation. $\Diamond$

Remark 5B.3. In contrast to Theorem 3B.2, a generalization of Theorem 5B.1 beyond rank one appears to be difficult. See however [24] for a related classification. $\Diamond$

Remark 5B.4. Theorem 5B.1 seems very different than [44, Theorems 1.1 and 1.2]. $\Diamond$

5C. Proof of Theorem 5B.1. A tuple $\vec{N} = (\mathsf{N}, \mathsf{T}) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{T}_{n,n,n}(\mathbb{C})$ as in Theorem 5B.1 is called an SO_3 *tuple.*

Lemma 5C.1. *For* $m \in \mathbb{Z}_{\geqslant 2}$ *let* $\mathsf{N} \in \mathrm{GL}_m(\mathbb{C})$ *be a matrix satisfying* $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$*. Then there exists an associated* SO_3 *tuple with* $n = m + 1$*.*

Proof. Recall from [55, Proof of Theorem 1.10] that $\mathscr{S}\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ is monoidally equivalent to $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ upon taking additive idempotent closures, and the equivalence is given by an explicit monoidal functor $\mathcal{F}$. In a bit more detail, the object k in $\mathscr{S}\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ corresponds to the kth Jones–Wenzl projector in $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$. In any case, we get a monoidal equivalence $\mathcal{F}\colon \mathscr{W}\mathbf{eb}(\mathrm{SL}_2)^{\oplus,\subset\oplus} \to \mathscr{S}\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)^{\oplus} \cong_{\otimes} \mathscr{S}\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)^{\oplus,\subset\oplus}$ between the additive idempotent closure of $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ and the additive closure of $\mathscr{S}\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$. We identify the two categories using $\mathcal{F}$.

Recall $\mathcal{I}$ from Lemma 5A.7, and consider the following commutative diagram:

$$\begin{tikzcd}
\mathscr{W}\mathbf{eb}(\mathrm{SO}_3) \arrow[r, hook, "\mathcal{I}"] & \mathscr{W}\mathbf{eb}(\mathrm{SL}_2)^{\oplus,\subset\oplus} \arrow[r, dotted, "\exists!\tilde{\mathcal{F}}^n_{\mathsf{N}}"] & \mathbb{C}\mathbf{Vect} \arrow[d, equal] \\
 & \mathscr{W}\mathbf{eb}(\mathrm{SL}_2) \arrow[u, hook, "\mathsf{incl.}"] \arrow[r, "\mathcal{F}^n_{\mathsf{N}}"'] & \mathbb{C}\mathbf{Vect}.
\end{tikzcd}$$

The existence of $\tilde{\mathcal{F}}^n_{\mathsf{N}}$ follows from the usual Yoga of additive and idempotent closures. Thus, we get a 2-representation $\tilde{\mathcal{F}}^n_{\mathsf{N}} \circ \mathcal{I}$ of $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$.

Note that all needed functors are given explicitly. Tracking back their definitions and a bit of calculation gives the desired matrices and tensors. $\square$

LEMMA 5C.2. *For every $n \in \mathbb{Z}_{\geqslant 3}$ there exists some* SO_3 *tuple. For $n \in \{1,2\}$ there exists no such tuples.*

Proof. By Lemma 5C.1 and the corresponding statement for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ as in Lemma 3C.3, we get the existence. The case $n = 1$ is ruled out as in Remark 3B.3, while $n = 2$ can be ruled out as in Example 3C.12. □

LEMMA 5C.3. *The statements Lemma 3C.5 to Lemma 3C.9 hold mutatis mutandis for $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ as well.*

Proof. Let us go through the lemmas one-by-one and mention what needs to be changed:

(a) For Lemma 3C.5 we first recall that Lemma 5C.2 shows that for $n = 3$ the only possible SO_3 tuple is the one coming from the standard choice (given by SO_3 acting on its defining representation), while there are no solutions for $n < 3$. Moreover, for $n = 3$ the lemma follows by using Lemma 5A.7 and then tracking the image of the basis from Lemma 5A.5 under quantum symmetric Howe duality.The general case follows by copying this for a higher dimensional target space.

(b) Lemma 3C.6 works in the same way: one chooses a basis and orders the images of the generators in corresponding matrices or tensors.

(c) In Lemma 3C.7 one replaces the reference to [64, Chapter XII] with [44, Lemma 3.4]. Indeed, the proof of [44, Lemma 3.4] can be copied as it only relies on the fusion rules of SO_3. We get the desired unique functor, up to scaling, as all generators exists uniquely, up to scaling, as maps and all relations are satisfied, for example, [44, Lemma 3.4.(2f)] is the H=I relation.

(d) Lemma 3C.8 follows as before from the previous two points.

(e) Ditto, Lemma 3C.9 follows as before from the third point above.

Details are omitted. □

We have a complete solution for matrix congruence, see subsection 3C, which is the same as equivalence of bilinear forms by the classical fact that two matrices are congruent if and only if they represent the same bilinear form up to change-of-basis.

EXAMPLE 5C.4. *For $n = 3$ Lemma 3C.13 lists all possible solutions of* $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$ *up to $\equiv_c$. The solutions of* $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = [3]_q$ *up to $\equiv_c$ are similar. That is, for q generic enough the only possible solution is*

$$\mathsf{N} = \mathsf{G}_1 \oplus \mathsf{H}_1(q^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & q^2 & 0 \end{pmatrix},$$

up to $q \leftrightarrow q^{-1}$. This is the standard solution up to permutation. ◊

Thus, we only need to worry about trilinear forms. The easiest case for us are *ternary trilinear forms*, often called $(3,3,3)$ trilinear forms, where $n = 3$. In the notation above this is the case displayed in Equation 4B.1. For $\vec{1} = (1,1,1)$, we denote the appearing 3-by-3 matrices by $\mathsf{T}_x(\vec{1})$, $\mathsf{T}_y(\vec{1})$ and $\mathsf{T}_z(\vec{1})$ in order from front to back.

Take now such a form T and write it as $\mathsf{T} = \sum_{h,i,j} t_{hij} \cdot x_h y_i z_j$, using variables. We let $\mathsf{T}_x(\vec{x}) = (\sum_h t_{hij} \cdot x_h)_{ij}$ for $\vec{x} = (x_1, x_2, x_3)$, and similarly $\mathsf{T}_y(\vec{y})$ and $\mathsf{T}_z(\vec{z})$. The determinant formula $\det\big(\mathsf{T}_x(\vec{x})\big) = 0$ is a ternary cubic that we denote by T_x. We also have T_y and T_z by using the corresponding determinant formulas. Finally, evaluation at $\vec{a} \in \mathbb{C}^3$ gives $\mathsf{T}_x(\vec{a})$. This is a complex matrix, so we can let $t_x \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ be the number of matrices $\mathsf{T}_x(\vec{a})$ with $\operatorname{rank}_{\mathbb{C}}\mathsf{T}_x(\vec{a}) = 1$ (this number can be infinite). Similarly for t_y and t_z.

Lemma 5C.5. *Any ternary cubic is projectively equivalent to one of the following:*

$$1 : x^3 = 0, \quad 2 : x^2y = 0, \quad 3 : xy(x-y) = 0, \quad 4 : xyz = 0,$$

$$5 : z(x^2 + yz) = 0, \quad 6 : x(x^2 + yz) = 0, \quad 7 : x^3 - y^2z = 0, \quad 8 : x^3 + y^3 - xyz = 0,$$

as well as 9 : *an elliptic cubic and* 10 : *a zero cubic.*

Proof. Well-known, see [61]. A more modern and detailed account can be found in many works, see for example [36, Table 1]. □

One has a complete classification of ternary trilinear forms:

Lemma 5C.6. *We have the following.*

(a) *We have* $\mathsf{T} \equiv_c \mathsf{U}$ *if and only if ((*t_x, t_y, t_z*) is equal to* (u_x, u_y, u_z) *in some order, and* $(\mathsf{T}_x, \mathsf{T}_y, \mathsf{T}_z)$ *is projective equivalent to* $(\mathsf{U}_x, \mathsf{U}_y, \mathsf{U}_z)$ *in the same order.)*

(b) *The only possible triples* (t_x, t_y, t_z)*, up to reordering, are listed in the table in Equation 5C.7 below. The only possible ternary cubics, up to projective equivalence, are listed in the same table.*

(5C.7)

	$(0,1,0)$	$(1,0,1)$	$(1,1,1)$	$(1,2,1)$	$(2,1,2)$	$(2,2,2)$	$(3,3,3)$	$(\infty,1,\infty)$	$(\infty,2,\infty)$
1			1					10	
2			2, 5	1	3	2			10
3				2					
4			4	6		4	4		
5	7	3							
6	8	4							

$(0,0,0)$: *nonzero* $\Leftrightarrow$ *all cubics are of the same projective type.*

Equation 5C.7 is to be read as follows. The projective cases of two of $(\mathsf{T}_x, \mathsf{T}_y, \mathsf{T}_z)$ *need to agree, up to order, and the list in the first column is the class for* T_x *and* T_z*. The class of* T_y *is then listed in the table, with empty entries meaning that there is no solution. The entries are as in Lemma 5C.5.*

Proof. This is [61, Theorem 12]. See also [47, Pages 2-4] for explicit matrix forms. □

Example 5C.8. *To exemplify how to read the table Equation 5C.7, let us consider the column* $(3,3,3)$*. The only nonzero possibility is that all three ternary cubics are of type* $xyz = 0$*. We thus get*

$$\mathsf{T}_x(\vec{x}) = \begin{pmatrix} t_{111}x_1+t_{211}x_2+t_{311}x_3 & t_{112}x_1+t_{212}x_2+t_{312}x_3 & t_{113}x_1+t_{213}x_2+t_{313}x_3 \\ t_{121}x_1+t_{221}x_2+t_{321}x_3 & t_{122}x_1+t_{222}x_2+t_{322}x_3 & t_{123}x_1+t_{223}x_2+t_{323}x_3 \\ t_{131}x_1+t_{231}x_2+t_{331}x_3 & t_{132}x_1+t_{232}x_2+t_{332}x_3 & t_{133}x_1+t_{233}x_2+t_{333}x_3 \end{pmatrix} \equiv_c \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}.$$

That is, we can assume that $t_{111} = t_{222} = t_{333} = 1$ *and* $t_{hij} = 0$ *otherwise.* ◊

Lemma 5C.9. *From the cases listed in Equation 5C.7 precisely ((*$0,0,0$*) and projective type 10) can be used to define an* SO_3 *tuple up to* $\equiv_c$*.*

Proof. Firstly, up to $\equiv_c$, we have

$$\mathsf{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & q^2 & 0 \end{pmatrix},$$

by Example 5C.4. The trilinear form that gives a solution is

$$t_{123} = -1, \quad t_{132} = 1, \quad t_{213} = -1, \quad t_{231} = 1, \quad t_{321} = -1, \quad t_{312} = 1,$$

where we only show the nonzero entries. That this trilinear form works is a direct calculation. This is ((0,0,0) and projective type 10) or the *Veronese cuboid.* All other nonzero cases in Equation 5C.7 can be directly ruled out. Since the trilinear form cannot be zero due to the H=I relation, the proof completes. □

Remark 5C.10. The analog of Lemma 5C.6 for higher forms that would be relevant for Theorem 5B.1, i.e. (p,p,p) trilinear forms with $p \geqslant 4$, seems to be not trackable. In fact, this problem for general p is very difficult, see e.g. [6].

[60] has some results regarding $(p,p,2p-2)$ trilinear forms, but these are not relevant for SO_3. For certain subclasses of trilinear forms a bit more can be said, see for example [13].

The paper [47] studies trilinear forms from a geometric invariant theory point of view. [47, Proposition 5] gives a numerical condition for the stability under GIT quotients of (p,q,r) trilinear forms. Another geometric treatment is given in [48], but for $(3,3,4)$ trilinear forms; in particular, the moduli space of such forms is related to the moduli space of unordered set of six points in the plane, or dually, six lines. The double cover of the plane branched along the six lines is a K3 surface, and interesting geometry appears. The analog for the (p,p,p) trilinear forms with $p \geqslant 4$ relevant for this paper appears to be out of reach. ◊

6. On the complexity of the classification problems

In this section q is allowed to be any nonzero complex number. It will play the role of a parameter.

The *rank one classification problem* for web categories, say $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$, $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ or $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$, is the classification of rank one simple transitive 2-representations of such categories for all $q \in \mathbb{C}$ at once. Here classification should be read in the sense of Classification Problem 2.19.

Remark 6.1. This is again not meant as a definition. In particular, the below are *sketchy statements* with *sketchy proofs*. We however hope that we are convincing enough so that the reader believes that making these precise (in the sense of complexity theory) is not difficult. We think that making this section precise by properly addressing the complexity questions outlined below is an interesting problem, e.g. is there some finite-tame-2-wild trichotomy for 2-representations? ◊

$\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$, and also $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ (or $\mathscr{U}\mathbf{eb}(\mathrm{GL}_2)$), is very close to be the free pivotal category generated by a bilinear form:

Proposition 6.2. *The rank one classification problem for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ implies the classification of bilinear forms. Similarly, The rank one classification problem for $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ (or $\mathscr{U}\mathbf{eb}(\mathrm{GL}_2)$) implies the classification of bilinear forms as well.*

Proof. We start by pointing out that all the statements in subsection 3C until, and including, Lemma 3C.6 work even if q is a nontrivial root of unity. Moreover, Lemma 3C.9 also holds, but needs to be adjusted as in [23, Theorem 2.3].

We assume now that the rank one classification problem for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ is solved. By the above mentioned lemmas we can associate $\mathsf{N} \in \mathrm{GL}_n(\mathbb{C})$ to a 2-representation $\mathcal{F}^n_{\mathsf{N}}$ for some $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ by choosing q appropriately. To see this, we point out that the relation

$$\bigcirc = -[2]_q$$

does not give any restriction on the appearing bilinear form if we are allowed to vary q. This can be done since $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) \in \mathbb{C}$ is some value and we can solve $\mathrm{tr}(\mathsf{N}^T\mathsf{N}^{-1}) = -[2]_q$ for $q \in \mathbb{C} \setminus \{0\}$. Thus, we obtain the classification of $\mathsf{N} \in \mathrm{GL}_n(\mathbb{C})$ up to orthogonal congruence (recall that orthogonal congruence is the congruence that preserves the trace). This problem for Hermitian matrices, by [30, Corollary 2.3] and [54, Theorem 11], is equivalent to the classification of nondegenerate bilinear form. The latter is then equivalent ot the classification of all bilinear forms, as shown in [27, Unique theorem in Section 1].

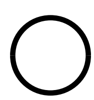
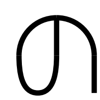
The case of $\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ (or $\mathscr{U}\mathbf{eb}(\mathrm{GL}_2)$) can be proven similarly and is omitted. □

We do not know how to deal with the $H = I$ relation, so let us ignore it. Precisely, let $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)'$ be the same as $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ but without imposing the $H = I$ relation. The category $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)'$ is close to be the free pivotal category generated by a trilinear form:

Proposition 6.3. *The rank one classification problem for $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)'$ implies the classification of trilinear forms.*

Proof. The proof strategy and arguments are almost the same as in the proof of Proposition 6.2, so let us only focus on the differences.

We want to argue that the relations

 $= [3]_q = q^2 + 1 + q^{-2}$,  $= 0$,

will not restrict the choice of trilinear form.

Similarly as in the proof of Proposition 6.2 we can vary q, eliminate the circle relation and we can assume that $\mathsf{N} \in \mathrm{GL}_n(\mathbb{C})$ (encoding the bilinear form) is arbitrary. The bitri evaluation thus does not restrict the appearing trilinear form because we can just chose the bilinear form accordingly. Although this is difficult in practice, this follows from a simple parameter count. Namely, the trilinear form has n^3 parameters, and so its kernel has $n^3 - 1$ parameters, while the bilinear form has n^2. (Note that this count does not work for $n = 1$, but classifying $(1, 1, 1)$-trilinear forms is trivial.)

The remaining steps work as at the end of the proof of Proposition 6.2 (the reduction from nondegenerate trilinear to general trilinear forms follows by copying the proof of [27, Unique theorem in Section 1]). □

Note that all of our rank one classification problems have an associated $\mathbb{C}$-vector space, i.e. the image of the generating object. Let $n \in \mathbb{Z}_{\geqslant 0}$ denote the dimension of this space.

In analogy with matrix classification problems, we call a rank one classification problem *finite* if there are only finite many equivalence classes of rank one simple transitive 2-representations for every fixed $n \in \mathbb{Z}_{\geqslant 0}$. Similarly, such a problem is *tame* if there is at most a one-parameter family of equivalences classes per n. We call such a problem *2-wild* (alternatively, *wilder than wild*) if it is strictly more difficult than any wild problem in the sense that solving it solves all wild problems, but not vice versa. (Recall that a classification problem is called *wild* if it contains the classification of indecomposables for any finite dimensional algebra.)

Theorem 6.4. *The rank one classification problem for $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)'$ is 2-wild.*

Proof. This follows from Proposition 6.3 and [6, Theorem 1.1]. □

The above, together with the easy to obtain solution of the rank one classification problem for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_1)$, implies the following. The rank one classification problem ...

(i) ... for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_1)$ is finite.

(ii) ... for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)/\mathscr{W}\mathbf{eb}(\mathrm{GL}_2)$ is tame.

(iii) ... for $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)'$ is 2-wild.

Remark 6.5.

(a) In the representation theory of finite dimensional algebras there is the notion of *finite, tame and wild representation type*. The above is inspired from these notions.

(b) Note that the categorical version of wild, that we called 2-wild, is strictly more difficult than any wild problem. In this sense one can say that categorical representation theory is more difficult than classical representation theory. However, the main caveat is that we are discussing $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)'$ and not $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ itself.

We think it is an interesting question whether the rank one classification problem for $\mathscr{W}\mathbf{eb}(\mathrm{SO}_3)$ (and with it probably for most other web categories) is wilder than wild.

Optimally, we would like to write the rank one classification problem for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_1)$ is finite, for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_2)$ it is tame and for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_3)$ it is 2-wild. (In order, no form appear, bilinear forms appear, and trilinear forms appear.) We however were not able to verify this because of the so-called square relation for $\mathscr{W}\mathbf{eb}(\mathrm{SL}_3)$. $\diamondsuit$

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