ALGEBRAIC COMBINATORICS

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ABSTRACT A graph is *d-rigid* if for any generic realisation of the graph in $\mathbb{R}^d$ (equivalently, the *d*-dimensional sphere $\mathbb{S}^d$), there are only finitely many non-congruent realisations in the same space with the same edge lengths. By extending this definition to complex realisations in a natural way, we define $c_d(G)$ to be the number of equivalent *d*-dimensional complex realisations of a *d*-rigid graph G for a given generic realisation, and $c_d^*(G)$ to be the number of equivalent *d*-dimensional complex spherical realisations of G for a given generic spherical realisation. Somewhat surprisingly, these two realisation numbers are not always equal. Recently developed algorithms for computing realisation numbers determined that the inequality $c_2(G) \leqslant c_2^*(G)$ holds for any minimally 2-rigid graph G with 12 vertices or less. In this paper we confirm that, for any dimension d, the inequality $c_d(G) \leqslant c_d^*(G)$ holds for every *d*-rigid graph G. This result is obtained via new techniques involving coning, the graph operation that adds an extra vertex adjacent to all original vertices of the graph.

1. INTRODUCTION

A (finite simple) graph is said to be *d-rigid* if every generic realisation of the graph in *d*-dimensional Euclidean space is *rigid*, i.e. shares edge-lengths with at most finitely many other realisations in the same space modulo isometries. A *d*-rigid graph is *minimally d-rigid* if removing any edge of the graph forms a graph that is not *d*-rigid. Given a *d*-rigid graph, we would wish to know how many possible edge-length equivalent realisations exist for any given generic realisation solely from the structural properties of the graph. This is unfortunately not possible for most graphs as the number of equivalent realisations differs between different generic realisations; for example, see Figure 1. As with many problems in algebraic geometry, the solution is to extend the problem to allow complex solutions. By doing so, we can concretely define the number of equivalent complex realisations (modulo congruence) of a generic *d*-dimensional complex realisation of a graph G. We refer to this number as the *d-realisation number of G* and denote it by $c_d(G)$. (See Definition 3.5 for a rigorous definition of the concept.)

Here we must make an important technical point: **we use the definition of a graph's *d*-realisation number given by Jackson and Owen [24], and as such we consider reflections of a realisation to be congruent also**; for example, the

complete graph with $d + 1$ vertices has a d-realisation number of 1. The variant of d-realisation number used by Borcea and Streinu [7] and Capco et al. [10] does (for algebraic reasons) count reflections, and so is exactly double the d-realisation number used here. We have opted for the former definition of a d-realisation number since it preserves an important property of congruence: any two affinely spanning ordered sets of equal size with identical pairwise distances between points are congruent. We urge any reader who is cross-referencing with multiple sources to be careful about this technical point, especially since many algorithms for computing d-realisation numbers use the latter definition (e.g. [10, 15]).
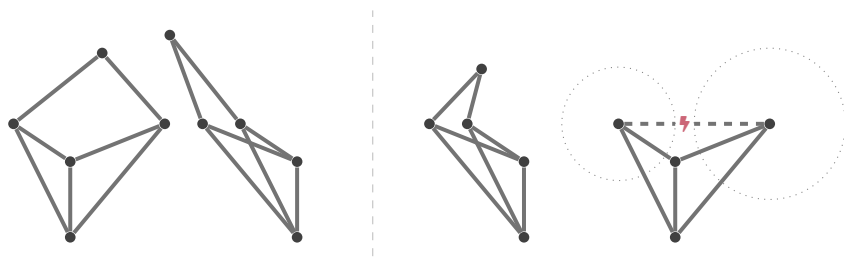

FIGURE 1. Two realisations of the same graph on 5 vertices with different numbers of equivalent real realisations. The realisation on the left has 4 non-congruent equivalent real realisations (two of which are shown), whilst the one on the right has only 2 due to the edge lengths.

Via tools such as *coning* — adding a vertex to a graph adjacent to all other vertices, a process that preserves rigidity between dimensions — Whiteley [30] proved that a graph is rigid when embedded generically on the d-dimensional sphere if and only if it is d-rigid. Because of this equivalence, it is natural to ask how many equivalent spherical realisations exist for any given generic spherical realisation of a graph. Yet again we are required to extend to complex solutions, where we also consider realisations on the complexification of the sphere. We thus define the *spherical d-realisation number* of a graph G, here denoted by $c_d^*(G)$, to be the number of equivalent complex spherical realisations (modulo congruence) of a generic d-dimensional complex spherical realisation of a graph G. (See Definition 4.5 for a rigorous definition of the concept.) As noted in Remark 4.10, this number is equal to the analogous realisation number for hyperbolic space, and so can also be considered to be the non-Euclidean d-realisation number of a graph.

In recent years, deterministic algorithms have also been constructed for computing $c_2(G)$ [10] and $c_2^*(G)$ [15] when the graph G is a minimally 2-rigid. It is also relatively easy to compute the values $c_1(G)$ and $c_1^*(G)$ for any graph G (see Propositions 3.8 and 4.9). This is, unfortunately, where the good news stops: when $d > 2$, there exist no current feasible deterministic algorithms for computing either c_d or c_d^* for general graphs. One probabilistic method for computing $c_d(G)$ and $c_d^*(G)$ when G is d-rigid involves choosing a random realisation for the graph and applying Gröbner basis computational techniques to the resulting algebraic solution set. Whilst this algorithm can be used reliably to obtain a lower bound on the d-realisation number of a graph, it is not deterministic and usually extremely slow (see [10, Section 5] for computation speed comparisons). Upper bounds can also be determined using mixed volume techniques [28] or multihomogenous Bézout bounds [4].

1.1. OUR CONTRIBUTIONS. When computing c_2 and c_2^* for various minimally 2-rigid graphs, it was observed that the two numbers occasionally will differ; see Figures 2
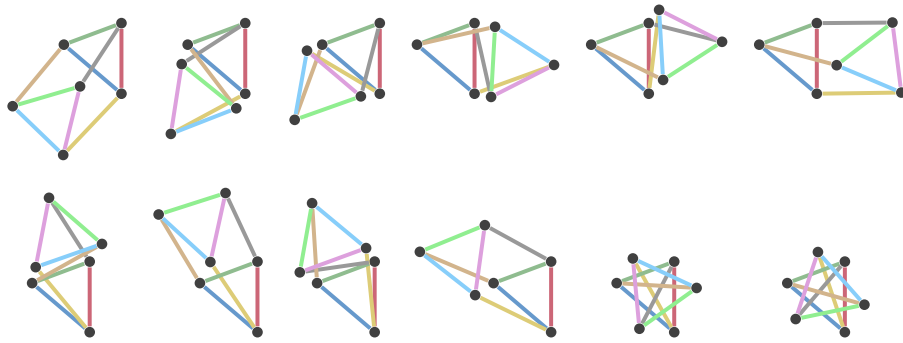
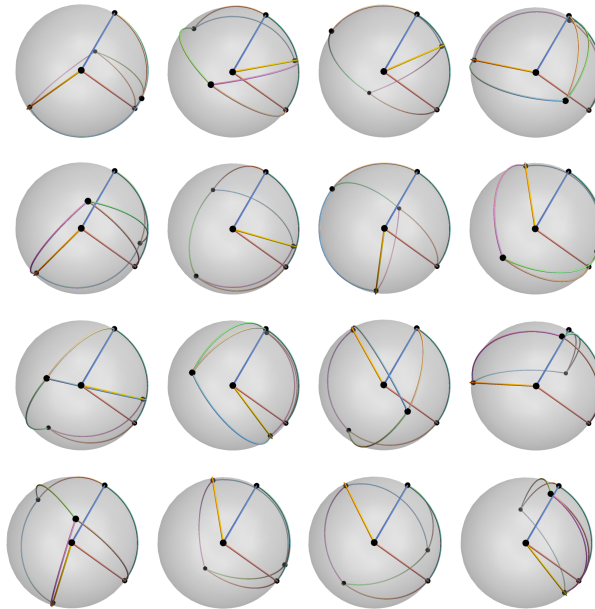FIGURE 2. 12 equivalent real realisations of the 3-prism graph.

FIGURE 3. 16 equivalent real realisations of the 3-prism graph on the sphere.

and 3 for the smallest 2-rigid graph where c_2 is strictly less than c_2^*. It was also observed that the opposite never held for any of the graphs whose realisation numbers were computed. To be exact, for every minimally 2-rigid graph with at most 12 vertices, the 2-realisation number c_2 is never greater than the spherical 2-realisation number $c_2^*(G)$: this observation can be achieved by combining implementations of the algorithms [9, 16] and the data set of all minimally 2-rigid graphs with at most 12 vertices that can be found at [11]. Similarly, randomized experiments with higher numbers of vertices also exhibited the exact same behaviour. Our main contribution of this paper is proving that this observation is indeed true for any graph in any dimension.

THEOREM 1.1. *For any graph G and any positive integer d, we have $c_d(G) \leqslant c_d^*(G)$.*

To prove this result, we first show an equivalence between the spherical d-realisation number of a graph G and the $(d+1)$-realisation number of the coned graph $G * o$, the graph formed by adding a new vertex adjacent to all other vertices of G.

Theorem 1.2. *Let d be a positive integer and let $G * o$ be a coning of a graph G. Then* $c_d^*(G) = c_{d+1}(G * o)$.

Our proof for Theorem 1.2 hinges on the observation that realisation numbers for frameworks are, in the special case of coned frameworks, projectively invariant. This is, in some sense, an extension of a result of Connelly and Whiteley [12], who proved that global rigidity (i.e. $c_d(G) = 1$ [17]) is projectively invariant. Theorem 1.1, however, requires a more specialised approach to graph coning. We actually prove that $c_d(G) \leqslant c_{d+1}(G * o)$, since Theorem 1.2 then implies the required inequality. Our technique now is as follows. We construct a very specific space of $(d+1)$-dimensional realisations of the coned graph $G * o$ which include realisations where the cone vertex o is mapped to a point at infinity. For this special class of realisations, the distance constraints stemming from edges connecting the cone vertex o to each vertex v of G are instead linear constraints that force the vertices of G to lie in a family of parallel hyperplanes. This allows us to embed our d-dimensional realisations of G into our new larger space of $(d+1)$-dimensional realisations of $G * o$. From this, we can then approximate any d-dimensional realisation of G by a sequence of $(d+1)$-dimensional realisations of $G * o$ in our larger space such that the required geometric information regarding realisation spaces (i.e. an upper bound on the number of points) is preserved.

The idea of considering a linear constraint to be a point at infinity is not new; see for example [13]. The idea of mapping realisations between Euclidean and spherical geometries to determine rigidity properties is also not new: the *central projection* [23, 26] (also known as the infinitesimal Pogorelov map or the gnomonic projection) is an example of a map between spherical and Euclidean geometry which preserves the linear information of frameworks; for example, infinitesimal rigidity, or the rank of any self-stresses of the framework. It is important to note here that the central projection does not, in general, preserve the number of configurations for a framework;[1] for more details on this, see [12, Section 8]. Although the authors cannot see any direct method for adapting the central projection to prove Theorem 1.1, a similar map is used in Section 5 to prove Theorem 1.2. An alternative approach that does preserve non-linear information is the *finite Pogorelov map* [23, 26], here denoted by T. Roughly speaking, this rational map takes pairs of d-dimensional spherical realisations p, q of a graph G and produces a single d-dimensional Euclidean realisation $T(p, q)$, with the property that the spherical realisations p and q have the same length edges if and only if the Euclidean realisations $T(p, q)$ and $T(q, p)$ have the same length edges. A major issue with the finite Pogorelov map is that, while it does preserve edge-length equivalence of pairs of realisations, it usually does not behave well when applied to the entire realisation space of a framework.[2] For example, if p, q, r are d-dimensional spherical realisations of a graph G with the same length edges, then $T(p, q)$ and $T(p, r)$ (or $T(r, p)$) do not, in general, have the same length edges. Because of this difficulty,

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[1] The notable exception to this is when a generic framework is globally rigid, since this condition is equivalent to the existence of a full-rank self-stress [17]. Using self-stresses to determine realisation numbers fails in the general case, however, for two important reasons: (i) very little can be said when every self-stress of a generic framework with n vertices has rank strictly less than $n - d - 1$ except that $c_d(G) > 1$ and $c_d^*(G) > 1$; (ii) self-stress techniques are not viable for the majority of d-rigid graphs, including the important class of minimally d-rigid graphs (which generically have no non-trivial self stresses).

[2] Yet again, the exception makes the rule: in [12, Section 7], Connelly and Whiteley's gave an alternative proof that global rigidity is equivalent for spherical and Euclidean spaces which utilises the finite Pogorelov map. This is as, to prove a framework is not globally rigid, one only needs to prove there exists one other non-congruent framework with the same length edges — a property that can be preserved using the Pogorelov map.

it does not seem to be immediate to us for how the finite Pogorelov map could be possibly applied to prove Theorem 1.1.

How our new method differs from the properties gifted by the two previously mentioned maps is that it allows us to approximate d-dimensional general realisations of a graph G by $(d+1)$-dimensional coned realisations of the cone graph $G * o$ in a way that preserves some algebraic geometric information about generic realisation spaces, not just pairs of realisations: in our particular case, the geometric information that is preserved is an upper bound on the size of the set. It is the authors' belief that our new enhanced coning technique has the potential to applied to other generic invariants related to framework rigidity (see Remark 6.5 for further discussion).

1.2. STRUCTURE OF THE PAPER. Our paper is structured as follows. In Section 2 we cover all the necessary background surrounding the topics of rigidity, spherical rigidity and coning. In Sections 3 and 4 we provide rigorous definitions for the d-realisation number and spherical d-realisation number respectively. Whilst such ideas have been alluded to in prior papers (for example [10, 24]), they have previously restricted themselves to 2-dimensional spaces only. Due to the various technicalities that occur when moving between our various higher-dimensional geometries, we have opted to include thorough proofs of all necessary background results. In Section 5 and Section 6 we prove Theorem 1.2 and Theorem 1.1 respectively. In Section 5 we also prove that repeatedly coning a graph will, in some sense, stabilise its realisation numbers (Theorem 5.2), which can be utilised to construct infinite families of graphs for each dimension with realisation numbers that can be computed deterministically. We conclude the paper by discussing various computational results we have obtained in Section 7.

2. PRELIMINARY RESULTS FOR RIGIDITY THEORY

In this section we cover the necessary background results for discussing rigidity in geometries based over the real numbers. Many of these concepts are extended to geometries based over the complex numbers in Sections 3 and 4.

2.1. EUCLIDEAN SPACE RIGIDITY. A *realisation* of a (finite simple) graph $G = (V, E)$ in $\mathbb{R}^d$ is a map $p : V \to \mathbb{R}^d$, and the linear space of all realisations of a graph is denoted by $(\mathbb{R}^d)^V$. A realisation p is said to be *generic* if the set of coordinates of p forms an algebraic independent set of $d|V|$ elements.[3] A graph-realisation pair (G, p) is said to be a *d-dimensional framework*. Two frameworks (G, p) and (G, q) in $\mathbb{R}^d$ are said to be *equivalent* if (given $\| \cdot \|$ is the standard Euclidean norm) $\|p_v - p_w\| = \|q_v - q_w\|$ holds for every edge $vw \in E$. A framework (G, p) in $\mathbb{R}^d$ is now said to be *rigid* if the following holds for some $\varepsilon > 0$: if (G, q) is an equivalent framework in $\mathbb{R}^d$ such that $\|p_v - q_w\| < \varepsilon$ for all $v \in V$, then there exists an isometry $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $q_v = f(p_v)$ for all $v \in V$.

Determining whether a framework is rigid is NP-Hard for $d \geqslant 2$ [1]. To combat this, we construct the *rigidity matrix* $R(G, p)$ of a given framework (G, p). This is the $|E| \times d|V|$ matrix with the row labelled $vw \in E$ given by

$$[0 \quad \cdots \quad 0 \quad \overbrace{p_v - p_w}^{v} \quad 0 \quad \cdots \quad 0 \quad \overbrace{p_w - p_v}^{w} \quad 0 \quad \cdots \quad 0].$$

A framework (G, p) is said to be *regular* if its rigidity matrix has maximal rank, i.e. for any d-dimensional realisation q of G we have rank $R(G, p) \geqslant$ rank $R(G, q)$. Any framework (G, p) with a surjective rigidity matrix (i.e. rank$(G, p) = |E|$) is automatically

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[3]The requirement that the algebraic independent set has $d|V|$ elements is stated to avoid the possibility of identical coordinates.

regular: in such a case we say that the framework (G, p) is *independent*. Any generic framework must also be regular, since the non-regular frameworks form an algebraic set defined by rational-coefficient polynomials.

Theorem 2.1 ([2]). *Let (G, p) be a regular framework in $\mathbb{R}^d$. Then the following properties are equivalent.*

  (i) *(G, p) is rigid.*
  (ii) *Either G has at least d vertices and* rank $R(G, p) = d|V| - \binom{d+1}{2}$, *or G is a complete graph.*

Since every generic framework is regular, Theorem 2.1 informs us that both rigidity and independence are generic properties. This motivates the following definitions.

Definition 2.2. *A graph G is said to be d-rigid (respectively, d-independent) if there exists a d-dimensional rigid (respectively, independent) generic framework (G, p). If G is rigid but $G - e$ is not for each edge $e \in E$, then G is said to be minimally d-rigid.*

One method that can be used to construct rigid/independent graphs in higher dimensions is via coning. Specifically: given a graph $G = (V, E)$, we define the *cone of G (by the new vertex o)* to be the graph $G * o := (V * o, E * o)$ where $o \notin V$, $V * o := V \cup \{o\}$ and

$$E * o := E \cup \{ov : v \in V\}.$$

We can now use the next result to move between dimensions whilst preserving rigidity and independence.

Theorem 2.3 ([30, Theorem 5]). *A graph $G = (V, E)$ is d-rigid (respectively, d-independent) if and only if $G * o$ is $(d+1)$-rigid (respectively, $(d+1)$-independent)$<$.*

2.2. Spherical space rigidity.

The *d-dimensional (unit) sphere* is the set

$$\mathbb{S}^d := \left\{ x \in \mathbb{R}^{d+1} : \|x\|^2 = 1 \right\}.$$

Similar to the Euclidean case, we define a *d-dimensional spherical realisation* to be a pair (G, p) of a graph $G = (V, E)$ and a *d-dimensional spherical realisation* $p : V \to \mathbb{S}^d$. The notions of equivalence and rigidity can be analogously defined for spherical frameworks by restricting all realisations to be spherical realisations.

An important observation is that any d-dimensional spherical realisation of a graph G can (for rigidity purposes at least) be considered to be a $(d+1)$-dimensional realisation of the cone of G. To be more specific: given a d-dimensional spherical framework (G, p), we define the $(d+1)$-dimensional framework $(G * o, p')$ by setting $p'_v = (p_v, 1)$ for all $v \in V$ and $p'_o = \mathbf{0}$. From this construction it is easy to see that the framework $(G * o, p')$ is rigid if and only if (G, p) is rigid. Slightly less obviously, this remains true if we scale each p_v by some positive scalar.

Proposition 2.4. *For a graph $G = (V, E)$, let p, q be d-dimensional realisations of the cone $G * o$ where $p_o = q_o = \mathbf{0}$. Further suppose that for each $v \in V$, there exists a scalar $r_v > 0$ such that $q_v = r_v p_v$. Then $(G * o, p)$ is rigid if and only if $(G * o, q)$ is rigid, and* rank $R(G * o, p) = $ rank $R(G * o, q)$.

Proof. That rank $R(G * o, p) = $ rank $R(G * o, q)$ follows from the observation that the rank of a rigidity matrix is projectively invariant; see [23, Theorem 1] for more details. Now choose any d-dimensional framework $(G * o, \tilde{p})$ equivalent to $(G * o, p)$ and define $\tilde{q}$ to be the realisation of $G * o$ with $\tilde{q}_v = r_v \tilde{p}_v$ for each $v \in V$. For every edge vw of

G we have

$$\|\tilde{p}_v\|^2 = \|\tilde{p}_v - \tilde{p}_o\|^2 = \|p_v - p_o\|^2 = \|p_v\|^2,$$
$$\|\tilde{p}_w\|^2 = \|\tilde{p}_w - \tilde{p}_o\|^2 = \|p_w - p_o\|^2 = \|p_w\|^2,$$
$$\tilde{p}_v \cdot \tilde{p}_v = \|\tilde{p}_v\|^2 + \|\tilde{p}_w\|^2 - \|\tilde{p}_v - \tilde{p}_w\|^2 = \|p_v\|^2 + \|p_w\|^2 - \|p_v - p_w\|^2 = p_v \cdot p_w.$$

(Remember that $ov, ow \in E * o$ and $\tilde{p}_o = p_o = \mathbf{0}$.) Using the above equalities, we see that

$$\begin{aligned}
\|\tilde{q}_v - \tilde{q}_w\|^2 &= \|r_v \tilde{p}_v - r_w \tilde{p}_w\|^2 \\
&= \|r_v \tilde{p}_v\|^2 + \|r_w \tilde{p}_w\|^2 - r_v r_w \tilde{p}_v \cdot \tilde{p}_w \\
&= \|r_v p_v\|^2 + \|r_w p_w\|^2 - r_v r_w p_v \cdot p_w \\
&= \|r_v p_v - r_w p_w\|^2 \\
&= \|q_v - q_w\|^2,
\end{aligned}$$

and hence $(G * o, q)$ and $(G * o, \tilde{q})$ are equivalent. From this it follows that $(G * o, p)$ is rigid if and only if $(G * o, q)$ is rigid. $\qquad\square$

From a combination of Theorem 2.1, Theorem 2.3 and Proposition 2.4, we can now see that a graph is rigid in d-dimensional Euclidean space if and only if it is rigid in d-dimensional spherical space.

THEOREM 2.5. *For any graph G, the following properties are equivalent.*

    (i) *G is d-rigid.*
    (ii) *Almost every (i.e. with Lebesgue measure zero complement) d-dimensional spherical framework (G, p) is rigid.*

REMARK 2.6. It was first observed by Pogorelov [26, Chapter V] that the linear space of infinitesimal motions of a d-dimensional spherical framework is isomorphic to the linear space of infinitesimal motions of the d-dimensional framework obtained by a central projection to Euclidean space. An alternative proof of Theorem 2.5 now stems from the spherical analogue to Theorem 2.1, which can be proven in an almost identical way.

3. COUNTING COMPLEX REALISATIONS

Our aim in this section is to prove that the definition of the d-realisation number for a graph alluded to in the introduction can be stated in a rigorous manner (see Definition 3.5).

3.1. COMPLEX RIGIDITY MAP. We recall that an algebraic set is a subset $A \subset \mathbb{C}^n$ of common zeroes of an ideal $I \subset \mathbb{C}[X_1, \ldots, X_n]$, and a variety is an irreducible algebraic set. If an algebraic set forms a smooth submanifold of $\mathbb{C}^n$ then it is said to be smooth. We shall refer to a subset U of an algebraic set A as a Zariski closed/open/dense subset if U is a closed/open/dense subset of A with respect to the Zariski topology.

For every point $x = (x_1, \ldots, x_d) \in \mathbb{C}^d$, we define $[x]_i := x_i$ for each $i = 1, \ldots, d$. We now also consider realisations in $\mathbb{C}^d$ by extending $(\mathbb{R}^d)^V$ to the set $(\mathbb{C}^d)^V$. Any realisation in $(\mathbb{R}^d)^V$ is said to be a *real realisation*. We extend the square of the Euclidean norm to $\mathbb{C}^d$ by defining $\|x\|^2 := \sum_{i=1}^d [x]_i^2$. We note that this is a quadratic form and not the square of the standard complex norm, and so can take complex values. Later we shall also require the bilinear map $x \cdot y := \sum_{i=1}^d [x]_i [y]_i$, where we notice that $x \cdot x = \|x\|^2$.

For any graph $G = (V, E)$ we define the *complex rigidity map* to be the multivariable map

$$f_{G,d} : (\mathbb{C}^d)^V \to \mathbb{C}^E, \ p \mapsto \left(\frac{1}{2} \| p_v - p_w \|^2 \right)_{vw \in E}.$$

We denote the Zariski closure of the image of $f_{G,d}$ by $\ell_d(G)$. Since the domain of $f_{G,d}$ is irreducible, $\ell_d(G)$ is a variety. Note that two realisations p, q of G in $\mathbb{R}^d$ are equivalent if and only if $f_{G,d}(p) = f_{G,d}(q)$. Given $O(d, \mathbb{C})$ is the group of $d \times d$ complex-valued matrices M where $M^T M = M M^T = I$, we define two realisations $p, q \in (\mathbb{C}^d)^V$ to be *congruent* (denoted by $p \sim q$) if and only if there exists $A \in O(d, \mathbb{C})$ and $x \in \mathbb{C}^d$ so that $p_v = A q_v + x$ for all $v \in V$. If the set of vertices of (G, p) affinely span $\mathbb{C}^d$, we have the following equivalent statement: two realisations p, q are congruent if and only if $f_{K_V, d}(p) = f_{K_V, d}(q)$ (see [17, Section 10] for more details). For all $p \in (\mathbb{C}^d)^V$, we define

$$C_d(G, p) := f_{G,d}^{-1}(f_{G,d}(p))/\sim$$

to be the *realisation space of* (G, p).

Our new definitions might look a little strange at first. For example, the elements of $O(d, \mathbb{C})$ are not isometries of $\mathbb{C}^d$ (i.e. they are not unitary matrices), although $\| A x \|^2 = \| x \|^2$ for all $x \in \mathbb{C}^d$. However, our previous definitions of independence and rigidity can be encoded in our new language of morphisms between complex spaces. We first recall that a morphism $f : X \to Y$ between algebraic sets $X \subset \mathbb{C}^m$ and $Y \subset \mathbb{C}^n$ (i.e. the restriction of a polynomial map $\mathbb{C}^m \to \mathbb{C}^n$) is *dominant* if $Y \smallsetminus f(X)$ is contained in a Zariski closed proper subset of Y; see Appendix A for more details regarding dominant morphisms.

LEMMA 3.1. *Let $G = (V, E)$ be any graph. Then the following are equivalent:*

    (i) *G is d-independent.*
    (ii) *The map $f_{G,d}$ is dominant.*
    (iii) *$\ell_d(G) = \mathbb{C}^E$.*

Proof. By the definition of a dominant map, (ii) and (iii) are equivalent. The map $f_{G,d}$ is dominant if and only if rank $\mathrm{d}f_{G,d}(p) = |E|$ for some $p \in (\mathbb{C}^d)^V$ (see Theorem A.1). If G is d-independent then there exists a $p \in (\mathbb{R}^d)^V$ such that rank $\mathrm{d}f_{G,d}(p) = $ rank $R(G, p) = |E|$, hence (i) implies (ii). Suppose that G is not d-independent; i.e., for each $p \in (\mathbb{R}^d)^V$ we have rank $\mathrm{d}f_{G,d}(p) < |E|$. Then the set $X := \{ p \in (\mathbb{C}^d)^V : $ rank $\mathrm{d}f_{G,d}(p) < |E| \}$ is a Zariski closed subset of $(\mathbb{C}^d)^V$ that contains $(\mathbb{R}^d)^V$. Hence, $X = (\mathbb{C}^d)^V$, and (ii) implies (i) as required. □

Let $G = (V, E)$ be a graph with at least $d + 1$ vertices and fix a sequence of d vertices $v_1, \ldots, v_d$. We now define the algebraic set

$$(1) \qquad X_{G,d} := \left\{ p \in (\mathbb{C}^d)^V : [p_{v_k}]_j = 0 \text{ if } j \geqslant k \right\}.$$

(At certain points in the paper we use vertices $w_1, \ldots, w_d$ to define $X_{G,d}$, which can be done by replacing each v_i with w_i in the above definition.) We further note that $X_{G,d}$ has dimension $d|V| - \binom{d+1}{2}$. Since $X_{G,d}$ is defined by a set of linear equations, it is irreducible. With this, we define the morphism

$$\tilde{f}_{G,d} : X_{G,d} \to \ell_d(G), \ p \mapsto f_{G,d}(p),$$

i.e., the restriction of $f_{G,d}$ to the domain $X_{G,d}$ and the codomain $\ell_d(G)$. As can be seen by the following lemma, the map $\tilde{f}_{G,d}$ allows us to more easily define the cardinality of the set $C_d(G, p)$ for most realisations p.

LEMMA 3.2. *Let $G = (V, E)$ be a graph with $|V| \geqslant d + 1$, and fix a sequence of d vertices $v_1, \ldots, v_d$. Then the image of $\tilde{f}_{G,d}$ is Zariski dense in the image of $f_{G,d}$ (and so $\tilde{f}_{G,d}$ is dominant), and*

$$\left| \tilde{f}_{G,d}^{-1} \left(f_{G,d}(p) \right) \right| = 2^d |C_d(G, p)|$$

for almost all[4] $p \in (\mathbb{C}^d)^V$.

See Appendix B for a detailed proof of Lemma 3.2.

3.2. DEFINING THE d-REALISATION NUMBER. Before stating our next result, we require the following technical result regarding dominant morphisms. The proof of this can be seen in Appendix A.

THEOREM 3.3. *Let $X \subset \mathbb{C}^m$ and $Y \subset \mathbb{C}^n$ be varieties and $f : X \to Y$ be a dominant morphism. Then the following are equivalent:*

   (i) $\dim X = \dim Y$.
   (ii) *f is* generically finite*, i.e. for a general $y \in Y$, the fibre $f^{-1}(y)$ is a finite set.*
   (iii) *There exists a $k \in \mathbb{N}$ and a non-empty Zariski open subset $U \subset X$ where $|f^{-1}(f(x))| = k$ for all $x \in U$. Furthermore, if $x \in U$ and $x' \in f^{-1}(f(x))$, then $\operatorname{rank} \mathrm{d}f(x') = \dim Y$.*

PROPOSITION 3.4. *Let $G = (V, E)$ be a graph with $|V| \geqslant d + 1$. Then the following are equivalent:*

   (i) *G is d-rigid.*
   (ii) *The map $\tilde{f}_{G,d}$ is generically finite.*
   (iii) *There exists an $n \in \mathbb{N}$ and a non-empty Zariski open subset $U \subset (\mathbb{C}^d)^V$ where $|C_d(G, p)| = n$ for all $p \in U$. Furthermore, if $p \in U$ and q is an equivalent d-dimensional realisation of G, then $\operatorname{rank} \mathrm{d}f_{G,d}(q) = d|V| - \binom{d+1}{2}$.*

Proof. By Lemma 3.2, the map $\tilde{f}_{G,d}$ is dominant. Since $\dim \ell_d(G) = \operatorname{rank} \mathrm{d}f_{G,d}(p)$ for any general realisation $p \in (\mathbb{C}^d)^V$, it follows from Theorem 2.1 that G is d-rigid if and only if

$$\dim \ell_d(G) = d|V| - \binom{d+1}{2} = \dim X_{G,d}.$$

The result now holds by applying Theorem 3.3 to the map $\tilde{f}_{G,d}$. $\qquad\square$

Using Proposition 3.4 we can now make the following well-defined definition of the d-realisation number for any graph.

DEFINITION 3.5. *The d-realisation number of a graph $G = (V, E)$ is an element of $\mathbb{N} \cup \{\infty\}$ given by*

$$c_d(G) := \begin{cases} |C_d(G, p)| \text{ for a general } p \in (\mathbb{C}^d)^V & \text{if } |V| \geqslant d + 1, \\ 1 & \text{if } |V| \leqslant d \text{ and } G \text{ is complete}, \\ \infty & \text{if } |V| \leqslant d \text{ and } G \text{ is not complete}. \end{cases}$$

It follows from Proposition 3.4 that a graph G is d-rigid if and only if $c_d(G) < \infty$.

We can relate the d-realisation number back to real realisations so long as we allow for equivalent complex realisations when counting. We first require the following technical lemma.

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[4] Here we say that a property P holds *for almost all points in* $\mathbb{C}^n$ if there exists a Zariski open subset $U \subset \mathbb{C}^n$ of points where property P holds, in which case we say that any point where property P holds is a *general point (with respect to P)*.

LEMMA 3.6. *If U is a non-empty Zariski open subset of $\mathbb{C}^n$, then $U \cap \mathbb{R}^n$ is a non-empty Zariski open subset of $\mathbb{R}^n$.*

Proof. For a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, we fix $\mathbb{F}[X^n]$ to be the set of all n-variable polynomials over $\mathbb{F}$. Define the ideal

$$F := \{f \in \mathbb{C}[X^n] : f(p) = 0 \text{ for all } p \notin U\}.$$

Since U is a Zariski open set, the zero set

$$Z(F) := \{p \in \mathbb{C}^n : f(p) = 0 \text{ for all } f \in F\}$$

is a Zariski closed set and the complement of U. As U is non-empty we immediately have that $F \neq \{0\}$. For each $f \in F$, there exist two unique polynomials $f_{\mathcal{R}}, f_{\mathcal{I}} \in \mathbb{R}[X^n]$ such that $f = f_{\mathcal{R}} + i f_{\mathcal{I}}$: to see this, note that there exists a finite subset $A \subset \mathbb{Z}_{\geqslant 0}^n$ and real values a_x, b_x for each $x \in A$ such that $f(p) = \sum_{x \in A}(a_x + ib_x)p^x$ (here $p^x = p_1^{x_1} \cdots p_n^{x_n}$ for each $x = (x_1, \ldots, x_n)$), and so $f_{\mathcal{R}}(p) = \sum_{x \in A} a_x p^x$ and $f_{\mathcal{I}}(p) = \sum_{x \in A} b_x p^x$. For each $p \in \mathbb{R}^n$ we note that $f(p) = 0$ if and only if $f_{\mathcal{R}}(p) = f_{\mathcal{I}}(p) = 0$. Fix $F_{\mathcal{R}} := \{f_{\mathcal{R}} : f \in F\}$ and $F_{\mathcal{I}} := \{f_{\mathcal{I}} : f \in F\}$, and for every set of (real or complex) polynomials $S \subset \mathbb{C}[X^n]$ fix $Z_{\mathbb{R}}(S) := \{p \in \mathbb{R}^n : f(p) = 0 \text{ for all } f \in S\}$. Then $Z_{\mathbb{R}}(F) = Z_{\mathbb{R}}(F_{\mathcal{R}}) \cap Z_{\mathbb{R}}(F_{\mathcal{I}})$. Hence, the set $Z_{\mathbb{R}}(F)$ is a Zariski closed subset of $\mathbb{R}^n$ since $Z_{\mathbb{R}}(F_{\mathcal{R}})$ and $Z_{\mathbb{R}}(F_{\mathcal{I}})$ are Zariski closed subsets of $\mathbb{R}^n$. Furthermore, the set $Z_{\mathbb{R}}(F)$ is a proper Zariski closed subset of $\mathbb{R}^n$: this follows from the observing the equivalence

$$Z_{\mathbb{R}}(F) = \mathbb{R}^n \quad \Leftrightarrow \quad F_{\mathcal{R}} = F_{\mathcal{I}} = \{0\} \quad \Leftrightarrow \quad F = \{0\},$$

with the latter property contradicting our previous observation that $F \neq \{0\}$. The result now holds as $U \cap \mathbb{R}^n = \mathbb{R}^n \smallsetminus Z_{\mathbb{R}}(F)$. $\qquad\square$

COROLLARY 3.7. *For each graph $G = (V, E)$, there exists a non-empty Zariski open subset $U_{\mathbb{R}} \subset (\mathbb{R}^d)^V$ of real d-dimensional realisations p where $|C_d(G, p)| = c_d(G)$.*

Proof. The result is trivial if G is complete and $|V| \leqslant d$. If G is not d-rigid then $c_d(G) = \infty$ and the result holds by Theorem 2.1. If G is d-rigid with $|V| \geqslant d + 1$ then we fix $U \subset (\mathbb{C}^d)^V$ to be the set defined in Proposition 3.4 and define $U_{\mathbb{R}} := U \cap (\mathbb{R}^d)^V$. The set $U_{\mathbb{R}}$ is now a Zariski open subset of $(\mathbb{R}^d)^V$ by Lemma 3.6. $\qquad\square$

It is important to note that Corollary 3.7 does not guarantee that every other equivalent realisation in $C_d(G, p)$ is also real. In fact we note that this is often not the case. For example, any minimally d-rigid graph with a vertex of degree d must have general d-dimensional realisations with complex equivalent realisations: this realisation can be chosen in a similar way to the right realisation pictured in Figure 1. Failing this, it is then natural to ask whether each graph G has at least one real generic d-dimensional realisation p where every other equivalent realisation in $C_d(G, p)$ is also real. It was proven in [24] that this too is false when $d = 2$. It does, however, remain open whether the statement is false when $d > 2$. For more on the topic of equivalent real realisations for a graph, see [3].

When $d = 1$, it is relatively easy to compute the realisation number of a graph. We first require the following terminology. A connected graph is *biconnected* if it has at least 2 vertices and the removal of any vertex will always produce a connected graph. We remark that the set of biconnected graphs is exactly the set of 2-connected graphs with the addition of the complete graph with two vertices. Given a graph $G = (V, E)$, we say that a subset $F \subset E$ is a *biconnected component* of G if the subgraph $H = (V[F], F)$ induced by F is biconnected and no subgraph of G containing H that is not H itself is biconnected. The biconnected components of a graph form a partition of its edge set.

PROPOSITION 3.8. *Let G be a graph and fix k to be the number of biconnected components of G. Then $c_1(G) < \infty$ if and only if G is connected. If G is connected with at least one edge then $c_1(G) = 2^{k-1}$.*

Proof. If we glue two connected graphs H_1 and H_2 at exactly one vertex to form a graph H then $c_1(H) = 2c_1(H_1)c_1(H_2)$. The result now follows from the observation that $c_1(G) = 1$ if G is biconnected. $\qquad\square$

As the number of biconnected components of a graph can be computed in linear time [22], it follows that there exists a linear time algorithm for computing $c_1(G)$. The current fastest deterministic algorithm for computing $c_2(G)$ when G is minimally 2-rigid runs in exponential time [10]. There are no other known deterministic algorithms for computing $c_d(G)$ outside of restricting to specific families of graphs (e.g. chordal graphs).

4. COUNTING REALISATIONS ON A COMPLEX SPHERE

Our aim in this section is to prove that the definition of the spherical d-realisation number alluded to in the introduction can be stated in a rigorous manner (see Definition 4.5).

4.1. COMPLEX SPHERICAL RIGIDITY MAP. We define

$$\mathbb{S}_{\mathbb{C}}^d := \left\{ x \in \mathbb{C}^{d+1} : \|x\|^2 = 1 \right\}$$

to be the *complexification of the d-dimensional (unit) sphere*. As $\mathbb{S}_{\mathbb{C}}^d$ is a smooth connected variety (and hence irreducible), so too is $(\mathbb{S}_{\mathbb{C}}^d)^V$ for any finite set V. Like how we chose to consider a larger set of realisations by switching from $(\mathbb{R}^d)^V$ to $(\mathbb{C}^d)^V$, we now consider spherical realisations in $\mathbb{S}_{\mathbb{C}}^d$ by extending the set $(\mathbb{S}^d)^V$ to the set $(\mathbb{S}_{\mathbb{C}}^d)^V$. Any spherical realisation in $p \in (\mathbb{S}^d)^V$ is now said to be a *real spherical realisation*. At a spherical realisation $p \in (\mathbb{S}_{\mathbb{C}}^d)^V$, the linear space

$$T_p(\mathbb{S}_{\mathbb{C}}^d)^V := \left\{ x \in (\mathbb{C}^{d+1})^V : x_v \cdot p_v = 0 \text{ for all } v \in V \right\}$$

is the tangent space of $(\mathbb{S}_{\mathbb{C}}^d)^V$ at p.

For any graph $G = (V, E)$ we define the *complex spherical rigidity map* to be the map

$$s_{G,d} \colon (\mathbb{S}_{\mathbb{C}}^d)^V \to \mathbb{C}^E, \ p \mapsto \left(\frac{1}{2}\|p_v - p_w\|^2 \right)_{vw \in E} = (1 - p_v \cdot p_w)_{vw \in E}.$$

We denote the Zariski closure of the image of $s_{G,d}$ by $\ell_d^*(G)$. Since the domain of $s_{G,d}$ is irreducible, $\ell_d^*(G)$ is a variety. The derivative of $s_{G,d}$ at a point $p \in (\mathbb{S}_{\mathbb{C}}^d)^V$ can be seen to be the linear map

$$\mathrm{d}s_{G,d}(p) \colon T_p(\mathbb{S}_{\mathbb{C}}^d)^V \to \mathbb{C}^E,$$
$$x \mapsto ((p_v - p_w) \cdot (x_v - x_w))_{vw \in E} = (-(p_v \cdot x_w + p_w \cdot x_v))_{vw \in E}.$$

We define two realisations $p, q \in (\mathbb{S}_{\mathbb{C}}^d)^V$ to be *congruent* (denoted by $p \sim q$) if and only if $p = Aq$ for some $A \in O(d+1, \mathbb{C})$. If the vertex set of p linearly spans $\mathbb{C}^{d+1}$, then two realisations p, q are congruent if and only if $s_{K_V,d}(p) = s_{K_V,d}(q)$. For all $p \in (S_{\mathbb{C}}^d)^V$, we define $C_d^*(G, p) := s_{G,d}^{-1}(s_{G,d}(p))/\sim$ to be the *spherical realisation space of (G, p)*.

We can link the infinitesimal properties of rigidity in $\mathbb{C}^d$ and in $\mathbb{S}_{\mathbb{C}}^d$ with the following result.

LEMMA 4.1. *Let $p \in (\mathbb{C}^d)^V$ and $q \in (\mathbb{S}_{\mathbb{C}}^d)^V$ be such that $[q_v]_{d+1} \neq 0$ and $[p_v]_i = [q_v]_i/[q_v]_{d+1}$ for all $v \in V$ and $i \in \{1, \ldots, d\}$. Then $\mathrm{rank}\, \mathrm{d}f_{G,d}(p) = \mathrm{rank}\, \mathrm{d}s_{G,d}(q)$.*

Proof. Define the bijective linear map

$$\varphi \colon T_q(\mathbb{S}_{\mathbb{C}}^d)^V \to (\mathbb{C}^d)^V, \quad x \mapsto \left(\frac{[x_v]_1}{[q_v]_{d+1}}, \ldots, \frac{[x_v]_d}{[q_v]_{d+1}} \right)_{v \in V}.$$

For any $u \in T_q(\mathbb{S}_{\mathbb{C}}^d)^V$ we have

$$(p_v - p_w) \cdot (\varphi(u)_v - \varphi(u)_w)$$

$$= \sum_{i=1}^d \frac{[q_v]_i[u_v]_i}{[q_v]_{d+1}^2} + \frac{[q_w]_i[u_w]_i}{[q_w]_{d+1}^2} - \frac{[q_v]_i[u_w]_i + [q_w]_i[u_v]_i}{[q_v]_{d+1}[q_w]_{d+1}}$$

$$= -\frac{[q_v]_{d+1}[u_v]_{d+1}}{[q_v]_{d+1}^2} - \frac{[q_w]_{d+1}[u_w]_{d+1}}{[q_w]_{d+1}^2}$$

$$+ \frac{[q_v]_{d+1}[u_w]_{d+1} + [q_w]_{d+1}[u_v]_{d+1}}{[q_v]_{d+1}[q_w]_{d+1}} - \frac{q_v \cdot u_w + q_w \cdot u_v}{[q_v]_{d+1}[q_w]_{d+1}}$$

$$= -\frac{q_v \cdot u_w + q_w \cdot u_v}{[q_v]_{d+1}[q_w]_{d+1}}.$$

It follows that $u \in \ker \mathrm{d}s_{G,d}(q)$ if and only if $\varphi(u) \in \ker \mathrm{d}f_{G,d}(p)$. As φ is bijective, we have rank $\mathrm{d}f_{G,d}(p) = $ rank $\mathrm{d}s_{G,d}(q)$ as required. $\square$

From Lemma 4.1 we know that a graph is rigid (respectively, independent) in $\mathbb{R}^d$ if and only if it is rigid (respectively, independent) in $\mathbb{S}^d$. Due to this, we can obtain an analogue of Lemma 3.1 for the map $s_{G,d}$.

Lemma 4.2. *Let $G = (V, E)$ be any graph. Then the following are equivalent:*

    (i) *G is d-independent.*
    (ii) *The map $s_{G,d}$ is dominant.*
    (iii) *$\ell_d^*(G) = \mathbb{C}^E$.*

Proof. By the definition of a dominant map, (ii) and (iii) are equivalent. Hence, we only need to prove that (i) and (ii) are equivalent. By Lemma 3.1, we only need to show that $s_{G,d}$ is dominant if and only if $f_{G,d}$ is dominant. This is equivalent to proving the following: there exists $p \in (\mathbb{C}^d)^V$ with rank $\mathrm{d}f_{G,d}(p) = |E|$ if and only if there exists a $q \in (\mathbb{S}_{\mathbb{C}}^d)^V$ with rank $\mathrm{d}s_{G,d}(p) = |E|$ (see Theorem A.1). Hence, the result holds by Lemma 4.1. $\square$

4.2. **Defining the spherical d-realisation number.** Let $G = (V, E)$ be a graph with at least $d+1$ vertices and fix a sequence of d vertices $v_1, \ldots, v_d$. We now define the algebraic set

$$(2) \qquad Y_{G,d} := \left\{ p \in (\mathbb{S}_{\mathbb{C}}^d)^V : [p_{v_k}]_j = 0 \text{ if } j \geqslant k+1 \text{ and } [p_{v_1}]_1 = 1 \right\}$$

Since $Y_{G,d}$ is a connected smooth manifold, it is irreducible. We further note that $Y_{G,d}$ has dimension $d|V| - \binom{d+1}{2}$. With this, we define the morphism

$$\tilde{s}_{G,d} : Y_{G,d} \to \ell_d^*(G), \quad p \mapsto s_{G,d}(p),$$

i.e., the restriction of $s_{G,d}$ to the domain $Y_{G,d}$ and the codomain $\ell_d^*(G)$. Since every d-dimensional realisation of G is equivalent to at least one element of $Y_{G,d}$ it follows that $\tilde{s}_{G,d}$ is a dominant map.

The methods utilised in Lemma 3.2 can easily be adapted to work for realisations in the d-dimensional sphere.

Lemma 4.3. *Let $G = (V, E)$ be a graph with $|V| \geqslant d+1$, and fix a sequence of d vertices $v_1, \ldots, v_d$. Then the image of $\tilde{s}_{G,d}$ is Zariski dense in the image of $s_{G,d}$ (and*

so $\tilde{s}_{G,d}$ is dominant), and

$$\left| \tilde{s}_{G,d}^{-1} \left(s_{G,d}(p) \right) \right| = 2^d |C_d^*(G, p)|$$

for almost all $p \in (\mathbb{S}_\mathbb{C}^d)^V$.

The spherical version of Proposition 3.4 is also proved in an analogous way.

PROPOSITION 4.4. *Let $G = (V, E)$ be a graph with $|V| \geqslant d + 1$. Then the following are equivalent:*

  (i) *G is d-rigid.*
  (ii) *The map $\tilde{s}_{G,d}$ is generically finite.*
  (iii) *There exists an $n \in \mathbb{N}$ and a non-empty Zariski open subset $U \subset (\mathbb{S}_\mathbb{C}^d)^V$ where $|C_d^*(G, p)| = n$ for all $p \in U$. Furthermore, if $p \in U$ and q is an equivalent d-dimensional spherical realisation of G, then $\operatorname{rank} \mathrm{d}s_{G,d}(q) = d|V| - \binom{d+1}{2}$.*

Using Proposition 4.4 we can now make the following well-defined definition of the spherical d-realisation number for a minimally d-rigid graph.

DEFINITION 4.5. *The spherical d-realisation number of a graph $G = (V, E)$ is an element of $\mathbb{N} \cup \{\infty\}$ given by*

$$c_d^*(G) := \begin{cases} |C_d^*(G, p)| \text{ for a general } p \in (\mathbb{S}_\mathbb{C}^d)^V & \text{if } |V| \geqslant d + 1, \\ 1 & \text{if } |V| \leqslant d \text{ and } G \text{ is complete}, \\ \infty & \text{if } |V| \leqslant d \text{ and } G \text{ is not complete}. \end{cases}$$

It follows from Proposition 4.4 that a graph G is d-rigid if and only if $c_d(G) < \infty$. It is worth noting that, while $c_d^*(G) < \infty$ if and only if $c_d(G) < \infty$, this does not mean the two numbers are always equal. This can be seen in the following example.

EXAMPLE 4.6. We consider the 3-prism graph. It is minimally 2-rigid with six vertices, all of which have degree three. For this graph we have $c_2(G) = 12$, which can be computed with the algorithm described in [10]. In fact, there exist edge-length assignments for which we also obtain exactly 12 real realisations (see Figure 2). However, the same graph embedded on the sphere gives $c_2^*(G) = 16$, which can be computed by the algorithm described in [15]. Again there exist edge-length assignments such that each realisation is real (see Figure 3). (Both algorithms use the alternative definitions for realisation numbers and so the outputted numbers first need to be halved.)

The natural analogue of Corollary 3.7 also holds for spheres by a similar method.

COROLLARY 4.7. *For each graph $G = (V, E)$, there exists a non-empty Zariski open subset $U_\mathbb{R} \subset (\mathbb{S}^d)^V$ of real d-dimensional spherical realisations p where $|C_d^*(G, p)| = c_d^*(G)$.*

REMARK 4.8. It is currently open whether every graph G has a real generic d-dimensional spherical realisation that is equivalent (modulo congruences) to exactly $c_d^*(G)$ real spherical realisations. The counter-example used in the previous section for real realisations in the plane required the existence of a graph H where $c_2(H)$ is odd. Similarly, any graph G where $c_2^*(G)$ is odd would be a counter-example to the result, however no such graph has yet been found.

We finish the section by observing that the spherical realisation number is always equal to the realisation number when $d = 1$, which can be proven using the same method as Proposition 3.8.

PROPOSITION 4.9. *For any graph G we have $c_1^*(G) = c_1(G)$.*

Remark 4.10. All of the results in this section can be extended from the sphere to the light cone of any pseudo-Euclidean space. To be specific, we define for each $1 \leqslant k \leqslant d$ the quadratic form

$$q_{d,k} : \mathbb{C}^{d+1} \to \mathbb{C}, \ (x_0, \dots, x_d) \mapsto \sum_{j=0}^{k} x_j^2 - \sum_{j=k+1}^{d} x_j^2,$$

and we define the light cone

$$\mathbb{H}_{d,k} := \left\{ x \in \mathbb{C}^{d+1} : q_{d,k}(x) = 0 \right\}.$$

The map

$$T : \mathbb{H}_{d,k} \to \mathbb{H}_{d,d}, \ (x_0, \dots, x_d) \mapsto (x_0, \dots, x_k, i x_{k+1}, \dots, x_d)$$

is an isometry (in the sense that $q_{d,d}(T(x) - T(y)) = q_{d,k}(x-y)$), and the space $\mathbb{H}_{d,d}$ is isomorphic to $\mathbb{S}_{\mathbb{C}}^d$. Hence, the realisation number of a graph G in any space $\mathbb{H}_{d,k}$ is equal to $c_d^*(G)$. Notably, the hyperbolic realisation number of a graph (equivalently, the space $\mathbb{H}_{d,d-1}$) is equal to the spherical realisation number of a graph.

5. Coning and the spherical realisation number

In this section we prove Theorem 1.2 by observing that the number of realisations of a framework is projectively invariant (Lemma 5.1). We then prove that $c_d(G * o) = c_d^*(G * o)$ for any coned graph $G * o$ (Theorem 5.2).

5.1. Proof of Theorem 1.2. Throughout this subsection we fix the vertices used to define our various varieties in the following way. Let $G = (V, E)$ be a graph with $|V| \geqslant d + 1$ and let $G * o$ be its cone. We fix the vertices $v_1, \dots, v_d \in V$ to define the set $X_{G,d}$ as given in (1). From this, we fix the set

$$X_{G*o,d+1} = \left\{ p' \in (\mathbb{C}^{d+1})^{V*o} : p'_o = \mathbf{0}, \ [p'_{v_k}]_{j+1} = 0 \text{ if } k \leqslant j \leqslant d \right\}.$$

(This is equivalent to defining the set $X_{G*o,d+1}$ as in (1) with the vertices $w_1, \dots, w_{d+1}$ by setting $w_1 = o$ and $w_{j+1} = v_j$ for all $j = 1, \dots, d$.) We also assume that the fixed vertices $v_1, \dots, v_d$ used to define $X_{G,d}$ and $Y_{G,d}$ (see (2)) are always identical.

Lemma 5.1. Given a graph $G = (V, E)$ with $|V| \geqslant d + 1$, choose any $p \in Y_{G,d}$ and $(r_v)_{v \in V} \in (\mathbb{C} \setminus \{0\})^V$. Define $p' \in X_{G*o,d+1}$ with $p'_v = (r_v p_v, 0)$ for each $v \in V$ and $p'_o = \mathbf{0}$. Then

$$\left| \tilde{f}_{G*o,d+1}^{-1} \left(\tilde{f}_{G*o,d+1}(p') \right) \right| = 2 \left| \tilde{s}_{G,d}^{-1} \left(\tilde{s}_{G,d}(p) \right) \right|.$$

Proof. Define $\lambda := \tilde{s}_{G,d}(p) = (1 - p_v \cdot p_w)_{vw \in E}$ and $\lambda' := \tilde{f}_{G*o,d+1}(p')$. We first note that for any $vw \in E$ we have

$$(3) \qquad \lambda'_{vw} = \frac{1}{2} \| p'_v - p'_w \|^2$$

$$= \frac{1}{2} \| p'_v \|^2 + \frac{1}{2} \| p'_w \|^2 - p'_v \cdot p'_w = \lambda'_{ov} + \lambda'_{ow} + (r_v r_w)(\lambda_{vw} - 1).$$

Define $S \subset \tilde{f}_{G*o,d+1}^{-1}(\lambda')$ to be the set of realisations $q' \in X_{G*o,d+1}$ where $[q'_{v_1}]_1 = r_{v_1}$. Note that for each $q' \in \tilde{f}_{G*o,d+1}^{-1}(\lambda')$ we have that

$$[q'_{v_1}]_1^2 = \| q'_{v_1} \|^2 = \| q'_{v_1} - q'_o \|^2 = \| p'_{v_1} - p'_o \|^2 = \| r_{v_1} p_{v_1} \|^2 = r_{v_1}^2;$$

the first equality follows from $q' \in X_{G*o,d+1}$, the second from $ov_1 \in E * o$, the third from $\tilde{f}_{G*o,d+1}(q') = \tilde{f}_{G*o,d+1}(p')$, the fourth from the construction of p' from p, and the last from $p \in Y_{G,d}$. Hence, if $q' \in \tilde{f}_{G*o,d+1}^{-1}(\lambda') \setminus S$ then we have $[q'_{v_1}]_1 = -r_{v_1}$, and so $-q' \in S$, since As $S \cap (-S) = \varnothing$ and $S \cup (-S) = \tilde{f}_{G*o,d+1}^{-1}(\lambda')$, we have that

$|\tilde{f}^{-1}_{G*o,d+1}(\lambda')| = 2|S|$ (note that this cardinality need not be finite). It now suffices to prove that $|\tilde{s}^{-1}_{G,d}(\lambda)| = |S|$.

Choose any $q \in \tilde{s}^{-1}_{G,d}(\lambda)$ and define $q' \in X_{G*o,d+1}$ by setting $q'_o = \mathbf{0}$ and $q'_v = r_v q_v$ for each $v \in V$. It is immediate that $[q'_{v_1}]_1 = r_{v_1}$ and $\frac{1}{2}\|q'_v\|^2 = \lambda'_{ov}$. For each $vw \in E$ we have

$$\frac{1}{2}\|q'_v - q'_w\|^2 = \frac{1}{2}\|q'_v\|^2 + \frac{1}{2}\|q'_w\|^2 - q'_v \cdot q'_w$$
$$= \lambda'_{ov} + \lambda'_{ow} + (r_v r_w)(\lambda_{vw} - 1)$$
$$\overset{(3)}{=} \lambda'_{vw},$$

and so $q' \in S$. Hence, $|\tilde{s}^{-1}_{G,d}(\lambda)| \leqslant |S|$.

Now choose any $q' \in S$ and define $q \in (\mathbb{C}^{d+1})^V$ by setting $q_v = q'_v/r_v$ for each $v \in V$. We first note that $\|q_v\|^2 = \|q'_v - q'_o\|^2/r_v^2 = 1$ for each $v \in V$, hence $q \in Y_{G,d}$. For each $vw \in E$ we have

$$q_v \cdot q_w = \frac{q'_v \cdot q'_w}{r_v r_w}$$
$$= \frac{\|q'_v\|^2 + \|q'_w\|^2 - \|q'_v - q'_w\|^2}{2 r_v r_w}$$
$$= \frac{\lambda'_{ov} + \lambda'_{ow} - \lambda'_{vw}}{r_v r_w}$$
$$\overset{(3)}{=} 1 - \lambda_{vw},$$

and so $q \in \tilde{s}^{-1}_{G,d}(\lambda)$. Hence, $|S| \leqslant |\tilde{s}^{-1}_{G,d}(\lambda)|$. $\qquad\square$

A nice corollary of Lemma 5.1 is that the set $C_{d+1}(G * o, p)$ is projectively invariant, in the sense that projecting each point p_v along the complex line through the points $\{p_v, p_o\}$ by a non-zero complex scalar does not alter the number of equivalent realisations.

We are now ready to prove our first main theorem.

Proof of Theorem 1.2. If $G = (V, E)$ is not d-rigid then the result follows from Theorem 2.3. If $|V| \leqslant d$ and G is complete then $|V * o| \leqslant d + 1$ and $G * o$ is complete, and so $c_{d+1}(G * o) = c^*_d(G)$. Suppose that $G = (V, E)$ is d-rigid with $|V| \geqslant d + 1$. As G is d-rigid, the coned graph $G * o$ is $(d + 1)$-rigid by Theorem 2.3. By Lemma 3.2 and Proposition 3.4, there exists a Zariski open set $U' \subset (\mathbb{C}^{d+1})^{V*o}$ such that

$$(4) \qquad \left|\tilde{f}^{-1}_{G*o,d+1}\left(\tilde{f}_{G*o,d+1}(p')\right)\right| = 2^{d+1} c_{d+1}(G * o)$$

for all $p' \in U'$. Similarly, it follows from Lemma 4.3 and Proposition 4.4 that there exists a Zariski open set $U \subset (\mathbb{S}_\mathbb{C})^d$ such that

$$(5) \qquad \left|\tilde{s}^{-1}_{G,d}\left(\tilde{s}_{G,d}(p)\right)\right| = 2^d c^*_d(G)$$

for all $p \in U$. Fix $U^* := U \times (\mathbb{C} \smallsetminus \{0\})^V$. Since U^* is a Zariski open subset of the variety $Y_{G,d} \times \mathbb{C}^V$, it is an open and dense subset of $Y_{G,d} \times \mathbb{C}^V$ with respect to the metric topology. Similarly, since U' is a Zariski open subset of the variety $X_{G*o,d+1}$, it is an open and dense subset of $X_{G*o,d+1}$ with respect to the metric topology.

Define the surjective (and hence dominant) morphism

$$\phi : Y_{G,d} \times \mathbb{C}^V \to X_{G*o,d+1},$$

where, given $p' = \phi(p, r)$, we have $p'_v = r_v p_v$ for all $v \in V$ and $p'_o = \mathbf{0}$. As ϕ is dominant and $\dim Y_{G,d} \times \mathbb{C}^V = \dim X_{G*o,d+1}$, it follows from Theorem 3.3 and

the inverse mapping theorem for holomorphic maps (see [14, Theorem 7.5]) that the image of any open dense subset of $Y_{G,d} \times \mathbb{C}^V$ (with respect to the metric topology) contains an open subset of $X_{G*o,d+1}$ (with respect to the metric topology). Hence, the set $\phi(U^*)$ has a non-empty interior with respect to the metric topology of $X_{G*o,d+1}$. Since U' is an open dense subset of $X_{G*o,d+1}$ with respect to the metric topology, $\phi(U^*) \cap U'$ contains a realisation p'. Choose $p \in U$ and $r \in (\mathbb{C} \smallsetminus \{0\})^V$ such that $\phi(p, r) = p'$. By (4), (5) and Lemma 5.1, we have that

$$2^{d+1} c_{d+1}(G * o) = \left| \tilde{f}_{G*o,d+1}^{-1} \left(\tilde{f}_{G*o,d+1}(p') \right) \right| = 2 \left| \tilde{s}_{G,d}^{-1} \left(\tilde{s}_{G,d}(p) \right) \right| = 2^{d+1} c_d^*(G),$$

and so $c_d^*(G) = c_{d+1}(G * o)$. $\qquad\square$

5.2. Repetitive coning stabilises realisation numbers. Our previous techniques of the section can be repurposed to prove the following interesting result.

Theorem 5.2. *Let d be a positive integer and let $G * o$ be a coning of a graph G. Then $c_d(G * o) = c_d^*(G * o)$.*

Proof. If G has less than d vertices then the result is trivial, hence we may suppose that G has at least d vertices. Let $H = (V', E')$ be the graph formed from $G * o$ by coning again, with $V' = V * o \cup \{o'\} = V \cup \{o, o'\}$. By Theorem 1.2, it suffices to prove that $c_d(G * o) = c_{d+1}(H)$. For our varieties $X_{G*o,d}$ and $X_{H,d+1}$, we fix $v_1 = o$ and choose $d-1$ other vertices $v_2, \dots, v_d \in V$ for $G * o$, and we fix $w_1 = o'$, $w_2 = o$ and $w_{j+1} = v_j$ for each $j \in \{2, \dots, d\}$ for H.

For any realisation $p \in X_{G*o,d}$ of $G * o$, define the realisation $p' \in X_{H,d+1}$ where $p'_o = (1, 0, \dots, 0)$ and $p'_v = (\frac{1}{2}, p_v)$ for each $v \in V$. Choose any $\bar{p} \in X_{H,d+1}$ that is equivalent to p'. Since o and o' are adjacent in H, we have that $\bar{p}_o = \pm p'_o = (\pm 1, 0, \dots, 0)$. Suppose that $\bar{p}_o = p'_o = (1, 0, \dots, 0)$. For each $v \in V$ we have

$$\|p'_v - p'_o\|^2 - \|p'_v - p'_{o'}\|^2 = ([p'_v]_1 - 1)^2 - [p'_v]_1^2 = 0,$$

and so

$$0 = \|\bar{p}_v - \bar{p}_o\|^2 - \|\bar{p}_v - \bar{p}_{o'}\|^2 = ([\bar{p}_v]_1 - 1)^2 - [\bar{p}_v]_1^2 = -2[\bar{p}_v]_1 + 1.$$

Hence, $[\bar{p}_v]_1 = 1/2$ for all $v \in V$ also. It now follows that there exists a realisation $q \in \tilde{f}_{G*o,d}^{-1}(\tilde{f}_{G*o,d}(p))$ such $q' = \bar{p}$. Hence, there exists a bijection from $\tilde{f}_{G*o,d}^{-1}(\tilde{f}_{G*o,d}(p))$ to the subset of realisations in $\tilde{f}_{H,d+1}^{-1}(\tilde{f}_{H,d+1}(p'))$ with the vertex o placed at $(1, 0, \dots, 0)$. As the remaining realisations can be obtained by a reflection in the hyperplane normal to $(1, 0, \dots, 0)$, we have $|\tilde{f}_{H,d+1}^{-1}(\tilde{f}_{H,d+1}(p'))| = 2|\tilde{f}_{G*o,d}^{-1}(\tilde{f}_{G*o,d}(p))|$ for every $p \in X_{G,d}$.

Fix the set

$$Z := \left\{ p' \in X_{H,d+1} : p'_o = (1, 0, \dots, 0), \ p'_v = (1/2, p_v) \text{ for each } v \in V \right\}$$

and define the bijective map $\psi : X_{G*o,d} \to Z$ that maps p to its unique realisation p' as defined above. Next, fix the dominant map $\phi : Z \times \mathbb{C}^{V*o} \to X_{H,d+1}$ where, given $p'' = \phi(p', r)$, we have $p'_v = r_v p_v$ for all $v \in V * o$. Now choose a non-empty Zariski set $U \subset X_{G*o,d}$ where

$$|\tilde{f}_{G*o,d}^{-1}(\tilde{f}_{G*o,d}(p))| = 2^d c_d(G * o)$$

for each $p \in U$ (Lemma 3.2). Note that the set $\psi(U)$ is a non-empty Zariski open subset of Z. By our previous work we have that for each $p' \in \psi(U)$ we have

$$|\tilde{f}_{H,d+1}^{-1}(\tilde{f}_{H,d+1}(p'))| = 2|\tilde{f}_{G*o,d}^{-1}(\tilde{f}_{G*o,d}(p))| = 2^{d+1} c_d(G * o).$$

Since the set $\psi(U) \times (\mathbb{C} \smallsetminus \{0\})^{V * o}$ is a non-empty Zariski open set, its image under the dominant map ϕ is a Zariski dense subset of $X_{H,d+1}$. It now follows from Lemma 5.1 that there exists a Zariski dense subset $U' \subset X_{H,d+1}$ where for each $q \in U'$ we have

$$|\tilde{f}_{H,d+1}^{-1}(\tilde{f}_{H,d+1}(q))| = 2^{d+1} c_d(G * o).$$

Hence, by Lemma 3.2 we have $c_{d+1}(H) = c_d(G * o)$ as required. $\qquad\square$

By combining Theorems 1.2 and 5.2, we obtain the following immediate corollary.

COROLLARY 5.3. *Let H be a graph formed from a graph G by performing $k > 0$ coning operations. Then $c_{d+k}(H) = c_{d+k}^*(H) = c_d^*(G)$.*

6. PROOF OF THEOREM 1.1

Using Theorem 1.2, we can prove Theorem 1.1 by instead proving that $c_d(G) \leqslant c_{d+1}(G * o)$. Because of this, we need to be able to evaluate d-dimensional realisations of G and $(d + 1)$-dimensional realisations of $G * o$ at the same time. One way of considering this is by fixing the cone vertex at the point at infinity so that the distance constraints between the cone vertex and all other vertices becomes a set of linear constraints forcing the vertices into a d-dimensional hyperplane. With this general idea in mind, we begin to construct such a space and the resulting variant of the rigidity map that comes from it.

We begin with the following prototype function that we will improve to give our required map. For any graph G with vertex u, fix h_u to be the morphism

$$h_u : (\mathbb{C}^{(d+1)})^V \times \mathbb{C} \to \mathbb{C}^{V \smallsetminus \{u\}}, \ (p, r) \mapsto \left(\frac{r}{2} \|p_v\|^2 - [p_v]_{d+1} \right)_{v \in V \smallsetminus \{u\}}.$$

The map h_u can be used to check for equivalent realisations for the coned graph when we invert the last coordinate of the cone vertex.

LEMMA 6.1. *Let p, q be two realisations of $G * o$ in $\mathbb{C}^{d+1}$ with $p_u = q_u = \mathbf{0}$ for some $u \in V$, $[p_o]_j = [q_o]_j = 0$ for each $j \in \{1, \dots, d\}$ and $[p_o]_{d+1} = [q_o]_{d+1} \neq 0$. Given $r := 1/[p_o]_{d+1}$, then $f_{G*o,d+1}(p) = f_{G*o,d+1}(q)$ if and only if $f_{G,d+1}(p|_V) = f_{G,d+1}(q|_V)$ and $h_u(p|_V, r) = h_u(q|_V, r)$.*

Proof. For each $v \in V$ we have

$$\begin{aligned}
\frac{1}{2} \|p_v - p_o\|^2 &= \frac{1}{2} \|p_v\|^2 - p_v \cdot p_o + \frac{1}{2} \|p_o\|^2 \\
&= \frac{1}{2} \|p_v\|^2 - \frac{1}{r}[p_v]_{d+1} + \frac{1}{2r^2} \\
&= \frac{1}{r} h_u(p|_V, r)_v + \frac{1}{2r^2},
\end{aligned}$$

and similarly $\|q_v - q_o\|^2 = \frac{1}{r} h_u(q|_V, r)_v + \frac{1}{2r^2}$. Hence, $f_{G*o,d+1}(p) = f_{G*o,d+1}(q)$ if and only if $f_{G,d+1}(p|_V) = f_{G,d+1}(q|_V)$ and $h_u(p|_V, r) = h_u(q|_V, r)$. $\qquad\square$

For the remainder of the section we fix $G = (V, E)$ to be a d-rigid graph with $|V| \geqslant d + 1$, and we also fix the distinct vertices $v_1, \dots, v_d$ of G. We first need to reformat $X_{G*o,d+1}$. Define the $((d + 1)(|V * o|) - \binom{d+2}{2})$-dimensional linear space

$$X'_{G*o,d+1} := \left\{ p \in (\mathbb{C}^{d+1})^{V*o} : \begin{array}{l} [p_{v_1}]_{d+1} = 0, \ [p_o]_k = 0, \\ [p_{v_k}]_j = 0 \text{ for all } 1 \leqslant k \leqslant j \leqslant d \end{array} \right\}.$$

Importantly, this space forces the vertex v_1 to lie on the origin and the cone vertex o to lie on the $(d+1)$-axis. We can link our new space $X'_{G*o,d+1}$ with our previously used space $X_{G*o,d+1}$ (see (1)) by the bijective linear map $L : X'_{G*o,d+1} \to X_{G*o,d+1}$ where, given $q = L(p)$, we have $[q_v]_1 = [p_v]_{d+1} - [p_o]_{d+1}$ and $[q_v]_j = [p_v]_{j-1} - [p_o]_{d+1}$ for

each $j \in \{2, \ldots, d+1\}$. Hence, any result that uses the space $X_{G*o,d+1}$ can be easily replaced by a result that uses the space $X'_{G*o,d+1}$; for example, general realisations in $X'_{G*o,d+1}$ have identical properties to general realisations in $X_{G*o,d+1}$.

Next, we define the $((d+1)|V| - \binom{d+2}{2} + d)$-dimensional linear space

$$Z_{G,d+1} := \left\{ p \in (\mathbb{C}^{d+1})^V : [p_{v_1}]_{d+1} = 0, \ [p_{v_i}]_j = 0 \text{ for all } i,j \in \{1, \ldots, d\}, \ j \geqslant i \right\}.$$

Note that the space $X_{G,d}$ embeds into $Z_{G,d+1}$ under the injective linear map

$$\lambda : X_{G,d} \to Z_{G,d+1}$$

where, given $q = \lambda(p)$ and a vertex $v \in V$, we have that $[q_v]_j = [p_v]_j$ for each $j \in \{1, \ldots, d\}$ and $[q_v]_{d+1} = 0$. Let

$$\phi : Z_{G,d+1} \times (\mathbb{C} \smallsetminus \{0\}) \to X'_{G*o,d+1}$$

be the injective open map where, given $\phi(p,r) = q$, we have $q_v = p_v$ for all $v \in V$ and $q_o = (0, \ldots, 0, 1/r)$. The only realisations in $X'_{G*o,d+1}$ not contained in the image of ϕ are those of the form p with $[p_o]_{d+1} = 0$.

With all of these spaces defined, we are now ready to define our main morphism:

$$g : Z_{G,d+1} \times \mathbb{C} \to \mathbb{C}^E \times \mathbb{C}^{V \smallsetminus \{v_1\}} \times \mathbb{C}, \ (p,r) \mapsto (f_{G,d+1}(p), h_{v_1}(p,r), r).$$

As proven by our next result, the map g is effectively identical in behaviour to $\tilde{f}_{G*o,d+1}$ over the realisations in the image of ϕ.

LEMMA 6.2. *For any $p, q \in X'_{G*o,d+1}$ where $[p_o]_{d+1}, [q_o]_{d+1} \neq 0$, the following are equivalent:*

   (i) $f_{G*o,d+1}(p) = f_{G*o,d+1}(q)$.
   (ii) $g \circ \phi^{-1}(p) = g \circ \phi^{-1}(q)$, *or* $g \circ \phi^{-1}(p) = g \circ \phi^{-1}(-q)$.
Hence, $|\tilde{f}_{G*o,d+1}^{-1}(f_{G*o,d+1}(p))| = 2|g \circ \phi^{-1}(p)|$.

Proof. It follows from our construction of $X'_{G*o,d+1}$ that $[p_o]_{d+1} = \pm[q_o]_{d+1} \neq 0$. The result now follows from Lemma 6.1. □

With our new set-up, any point $(p, 0)$ will, in some sense, behave like a realisation of the coned graph $G * o$ with the coned vertex "placed at infinity". It also forces a subset of such points (i.e. those where p is flat in the hyperplane $\mathbb{C}^d \times \{0\}$) to act like they are one dimension lower (minus the coned vertex). In fact, the map g shares many properties with the map $f_{G,d}$ when we restrict to a certain subset of elements of $Z_{G,d+1} \times \mathbb{C}$.

LEMMA 6.3. *Let $\tilde{p} \in X_{G,d}$ be a general realisation of G. Fix $p = \lambda(\tilde{p}) \in Z_{G,d+1}$. Then the following properties hold.*

   (i) *For each $(q,r) \in g^{-1}(g(p,0))$ we have $r = 0$, $f_{G,d+1}(q) = f_{G,d+1}(p)$ and $[q_v]_{d+1} = 0$ for each $v \in V$.*
   (ii) *For each $\tilde{q} \in X_{G,d}$ where $f_{G,d}(\tilde{q}) = f_{G,d}(\tilde{p})$, we have $g(\lambda(\tilde{q}), 0) = g(p, 0)$.*
   (iii) $|\tilde{f}_{G,d}^{-1}(f_{G,d}(\tilde{p}))| = |g^{-1}(g(p,0))|$.
   (iv) *For every $(q,r) \in Z_{G,d+1} \times \mathbb{C}$ we have $\operatorname{rank} \mathrm{d}g(p,0) \geqslant \operatorname{rank} \mathrm{d}g(q,r)$.*
   (v) *For every $(q,r) \in g^{-1}(g(p,0))$, the left kernels of $\mathrm{d}g(q,r)$ and $\mathrm{d}g(p,0)$ are identical.*

Proof. (i): As $g(q,r) = g(p,0)$, we have that $r = 0$ and $f_{G,d+1}(q) = f_{G,d+1}(p)$. Hence,

$$h_{v_1}(q,0) = h_{v_1}(q,r) = h_{v_1}(p,0) = 0,$$

with the latter equality holding since $[p_v]_{d+1} = 0$ for all $v \in V$. Since $h_{v_1}(q,0) = 0$, it now follows that $[q_v]_{d+1} = 0$ for each $v \in V \smallsetminus \{v_1\}$. The equality $[q_{v_1}]_{d+1} = 0$ also holds since $q \in Z_{G,d+1}$.

(ii): This follows immediately from the observation that $f_{G,d}(\tilde{q}) = f_{G,d+1}(\lambda(q))$ and $f_{G,d}(\tilde{p}) = f_{G,d+1}(p)$.

(iii): This follows from points (i) and (ii) and the observation that λ is a bijection between realisations in $X_{G,d}$ and realisations in $Z_{G,d}$ where the $(d+1)$-th coordinate of each vertex is zero.

(iv): For any $(q,r) \in Z_{G,d+1} \times \mathbb{C}$, the Jacobian of g at (q,r) will be a $(|E|+|V|) \times ((d+1)|V|+1)$ matrix of the form

$$\mathrm{d}g(q,r) = \begin{bmatrix} \mathrm{d}f_{G,d+1}(q) & 0_{|E| \times 1} \\ A & \left(\frac{1}{2}\|q_v\|^2\right)_{v \in V \smallsetminus \{v_1\}} \\ 0_{1 \times (d+1)|V|} & 1 \end{bmatrix},$$

where $0_{1 \times (d+1)|V|}$ is the $1 \times (d+1)|V|$ all zeroes matrix and A is the $(|V|-1) \times (d+1)|V|$ matrix where for the $(v,(w,i))$ entry (with $v \in V \smallsetminus \{v_1\}$ and $(w,i) \in V \times \{1,\dots,d+1\}$) we have

$$A_{v,(w,i)} = \begin{cases} r[q_v]_i & \text{if } w = v, \ i \leqslant d \\ r[q_v]_{d+1} - 1 & \text{if } w = v, \ i = d+1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the rank of $\mathrm{d}g(q,r)$ must satisfy the following upper bound:

$$\text{(6)} \qquad \operatorname{rank} \mathrm{d}g(q,r) \leqslant \dim(Z_{G,d+1} \times \mathbb{C}) = (d+1)|V * o| - \binom{d+2}{2}.$$

Now we observe the Jacobian of g when $(q,r) = (p,0)$. By moving all columns of $\mathrm{d}g(p,0)$ that correspond to the $(d+1)$-th coefficients of the vertices to the right, we obtain the matrix

$$\text{(7)} \qquad \begin{bmatrix} \mathrm{d}f_{G,d}(\tilde{p}) & 0_{|E| \times 1} & 0_{|E| \times |V|} \\ 0_{(|V|-1) \times 1} & \left(\frac{1}{2}\|p_v\|^2\right)_{v \in V \smallsetminus \{v_1\}} & -I_{|V|-1} \\ 0_{1 \times (d+1)|V|} & 1 & 0_{1 \times |V|} \end{bmatrix},$$

where $I_{|V|-1}$ is the $(|V|-1) \times (|V|-1)$ identity matrix. Since $\tilde{p}$ is a general realisation of a d-rigid graph, we have that

$$\operatorname{rank} \mathrm{d}g(p,0) = \operatorname{rank} \mathrm{d}f_{G,d}(\tilde{p}) + |V|$$
$$\text{(8)}$$
$$= d|V| - \binom{d+1}{2} + |V| = (d+1)|V * o| - \binom{d+2}{2}.$$

By combining Equations (6) and (8), we have that the derivative of g has maximal rank at $(p,0)$.

(v): It follows from (i) that $r = 0$ and there exists $\tilde{q} \in X_{G,d}$ where $\lambda(\tilde{q}) = q$ and $f_{G,d}(\tilde{q}) = f_{G,d}(\tilde{p})$. As $\tilde{p}$ is a general realisation, it follows from Proposition 3.4 that $\operatorname{rank} \mathrm{d}f_{G,d}(\tilde{q}) = \operatorname{rank} \mathrm{d}f_{G,d}(\tilde{p})$, which is maximal over all points in $(\mathbb{C}^d)^V$. By reformatting the matrix $\mathrm{d}g(q,0)$ into the same format as the matrix in Equation (7), we see that the left kernels of $\mathrm{d}g(q,0)$ and $\mathrm{d}g(p,0)$ agree if and only if left kernels of $\mathrm{d}f_{G,d}(\tilde{p})$ and $\mathrm{d}f_{G,d}(\tilde{q})$ agree. Since $\tilde{p}$ is a general realisation, its image is a smooth point in the Zariski closure $\ell_d(G)$ of the image of $f_{G,d}$: indeed if this was not true then $\tilde{p}$ would be contained in the preimage of a proper algebraic subset of $\ell_d(G)$ under the morphism $f_{G,d}$, contradicting that it is a general realisation. It follows from the inverse mapping theorem for holomorphic maps (see for example [14, Chapter I, Theorem 7.5]) that the tangent space of $\ell_d(G)$ at $f_{G,d}(\tilde{p})$ (respectively, $f_{G,d}(\tilde{q})$) is exactly the left kernel of $\mathrm{d}f_{G,d}(\tilde{p})$ (respectively, $\mathrm{d}f_{G,d}(\tilde{q})$). Since $f_{G,d}(\tilde{p}) = f_{G,d}(\tilde{q})$, the left kernels of $\mathrm{d}f_{G,d}(\tilde{p})$ and $\mathrm{d}f_{G,d}(\tilde{q})$ are equal. This now concludes the proof. $\qquad\square$

Lemmas 6.2 and 6.3 indicate that our map g can be used to consider both the d-realisation number of G and the $(d+1)$-realisation number of $G * o$ by observing the fibres of either specific or general elements of the domain of g respectively. Our next lemma allows us to compare the fibre sizes of these two types of elements.

LEMMA 6.4. *Let $f : \mathbb{C}^m \to \mathbb{C}^n$ be a holomorphic map and let $x \in \mathbb{C}^n$ be a point in $\mathbb{C}^n$ where:*

(i) *$|f^{-1}(f(x))| < \infty$,*
(ii) *$\operatorname{rank} \mathrm{d}f(x) \geqslant \operatorname{rank} \mathrm{d}f(y)$ for all $y \in \mathbb{C}^m$, and,*
(iii) *the left kernels of $\mathrm{d}f(x)$ and $\mathrm{d}f(z)$ are identical for all $z \in f^{-1}(f(x))$.*

Then there exists an open neighbourhood $U \subset \mathbb{C}^m$ of x (with respect to the metric topology for $\mathbb{C}^n$) where $\operatorname{rank} \mathrm{d}f(y) = \operatorname{rank} \mathrm{d}f(x)$ and $|f^{-1}(f(x))| \leqslant |f^{-1}(f(y))|$ for all $y \in U$.

Proof. Label the points in $f^{-1}(f(x))$ as $x_1, \ldots, x_k$, with $x_1 = x$. As a consequence of the inverse mapping theorem for holomorphic maps (see for example [14, Chapter I, Theorem 7.5]), pairwise disjoint open sets $U_1, \ldots, U_k \subset \mathbb{C}^m$ and smooth manifolds $V_1, \ldots, V_k \subset \mathbb{C}^n$ so that for each $j \in \{1, \ldots, k\}$ the following properties hold: (i) $x_j \in U_j$, (ii) the set V_j is an open neighbourhood of $f(x_j)$ in the image of f, (iii) the tangent space of V_j is the left kernel of $\mathrm{d}f(x_j)$, and (iv) the restricted map $f|_{U_j}^{V_j}$ is biholomorphic (i.e. invertible with holomorphic inverse). Since the left kernel of each Jacobian $\mathrm{d}f(x_j)$ is identical, we may choose our sets so that $V_1 = \ldots = V_k$. Choose any $y \in U_1$. Since $f|_{U_1}^{V_1}$ is biholomorphic, $\operatorname{rank} \mathrm{d}f(y) = \operatorname{rank} \mathrm{d}f(x)$. For each $j \in \{1, \ldots, k\}$, we observe that, since $V_j = V_1$, there exists a unique point $y_j \in U_j$ so that $f(y_j) = f(y)$. As the sets $U_1, \ldots, U_k$ are pairwise disjoint, $|f^{-1}(f(y))| \geqslant k$. $\square$

We are now ready to use the map g to prove our second main theorem of the paper.

Proof of Theorem 1.1. By Theorem 1.2, it suffices to prove that $c_d(G) \leqslant c_{d+1}(G * o)$. By Theorem 2.3, it also suffices for us to prove the specific case where G is d-rigid (and hence $G * o$ is $(d+1)$-rigid) with at least $d+1$ vertices. All notation we now use is in line with our prior notation for the section.

Choose a general realisation $\tilde{p} \in X_{G,d}$ of G and define $p = \lambda(\tilde{p}) \in Z_{G,d+1}$. It follows from Lemma 6.3(iv), Lemma 6.3(v) and Lemma 6.4 that there exists an open neighbourhood $U \subset Z_{G,d+1} \times \mathbb{C}$ of $(p, 0)$ (with respect to the metric topology) where for all $(p', r') \in U$ we have $|g^{-1}(g(p, 0))| \leqslant |g^{-1}(g(p', r'))|$. Since the map ϕ is an injective open map with dense image, it follows that there exists a general realisation $q \in X'_{G*o,d+1}$ where $\phi(q) \in U$. Hence, by Lemma 6.2, Lemma 6.3(iii) and Lemma 3.2, we have that

$$
\begin{aligned}
2^{d+1} c_d(G) &= 2|\tilde{f}_{G,d}^{-1}(f_{G,d}(\tilde{p}))| \\
&= 2|g^{-1}(g(p, 0))| \leqslant 2|g^{-1}(g(\phi(q)))| \\
&= |\tilde{f}_{G*o,d+1}^{-1}(f_{G*o,d+1}(q))| \\
&= 2^{d+1} c_{d+1}(G * o).
\end{aligned}
$$

Thus $c_d(G) \leqslant c_{d+1}(G * o)$ as required. $\square$

REMARK 6.5. The map $p \mapsto |C_d(G, p)|$ has two important properties: (i) it is constant over the set of general realisations, and (ii) it is locally minimized at every realisation of G that is mapped to a smooth point of $\ell_d(G)$ by the rigidity map. Theoretically, any algebraic property that satisfies points (i) and (ii) will be amenable to the techniques given in this section.

7. How and when do spherical and planar realisation numbers differ?

In this section we compare the d-realisation number and spherical d-realisation numbers of graphs using computational means. Since we are utilising computational methods, we will, for the most part, restrict to results for minimally d-rigid graphs. The reason for this is two-fold: (i) all known deterministic algorithms for computing realisation numbers require that the graph is minimally 2-rigid, and (ii) Gröbner basis computational methods are too expensive even for comparably small graphs.

A common operation we use in this section is the *(d-dimensional) 0-extension*. This is the graph operation where a new vertex is added to a graph and is set to be adjacent to exactly d vertices of the original graph. It is well-known that, given a graph G and a graph G' formed from G by a d-dimensional 0-extension, the graph G' is (minimally) d-rigid if and only if G is (minimally) d-rigid (see for example [29, Proposition 5.1]). As can be seen by the following result, this operation also behaves very well with realisation numbers. (See Appendix C for a proof of this result.)

Lemma 7.1. *If G' is a 0-extension of a graph G, then $c_d(G') = 2c_d(G)$ and $c_d^*(G') = 2c_d^*(G)$.*

In what follows we use the combinatorial algorithms from [10, 15] respectively their implementations [9, 16, 18]. Note again that the results from these computations are the double of the realisation numbers of what we consider in this paper. We include computations for all minimally rigid graphs with at most 12 vertices. The respective numbers of realisations for the plane can be found in [11].

7.1. Ratio of spherical to planar realisation numbers.

In this subsection we evaluate the ratio of spherical d-realisation number to d-realisation number for minimally d-rigid graphs. We begin by first proving that the minimum possible ratio is, due to our previous results, not an especially interesting value to look into.

Proposition 7.2. *For any pair of positive integers $n > d$,*

$$\min\{c_d^*(G)/c_d(G) : G \text{ is minimally } d\text{-rigid with } n \text{ vertices}\} = 1.$$

Proof. By Theorem 1.1, the minimum for any given value of n is at least 1. Define G_{d+1} to be the complete graph with $d + 1$ vertices. Then $c_d(G_{d+1}) = c_d^*(G_{d+1}) = 1$. Now construct a sequence $G_{d+1}, G_{d+2}, \dots$ of minimally d-rigid graphs, whereby G_{n+1} is formed from G_n by a 0-extension. It now follows from Lemma 7.1 that for each $n > d$ we have $c_d(G_n) = c_d^*(G_n) = 2^{n-d-1}$. $\square$

We now turn from the minimum of the ratio to its maximum. For any positive integers $n > d$, define

$$\theta_d(n) := \max\left\{c_d^*(G)/c_d(G) : G \text{ is minimally } d\text{-rigid with } n \text{ vertices}\right\}.$$

Here things seem to be possibly more interesting. For example, it follows from Figures 2 and 3 that $\theta_2(6) \geqslant 16/12 = 4/3$. In fact we have $\theta_2(6) = 4/3$, since every other minimally 2-rigid graph G with 6 vertices has $c_2^*(G) = c_2(G) = 8$.

Our first easy observation about $\theta_d(n)$ is that it is increasing: this is an immediate corollary of Lemma 7.1. Our next (slightly less easy) observation about $\theta_d(n)$ is that it is bounded above by an exponential function.

Proposition 7.3. *For each positive integer d, there exists a constant $\alpha \geqslant 1$ such that $\theta_d(n) = O(\alpha^n)$.*

Proof. As shown in [6], there exists a constant $\alpha > 1$ such that

$$\max\left\{c_d^*(G) : G \text{ is minimally } d\text{-rigid with } n \text{ vertices}\right\} = O(\alpha^n).$$

Since $c_d(G) \geqslant 1$ for any minimally d-rigid graph G with at least $d+1$ vertices, it follows that $\theta_d(G) = O(\alpha^n)$ also. □

Following from Proposition 7.3, we define α_d to be the infimum of all values $\alpha \geqslant 1$ such that $\theta_d(n) = O(\alpha^n)$. By Proposition 7.2, we have $\alpha_d \geqslant 1$ for any positive integer d. An immediate corollary to Proposition 4.9 is that $\alpha_1 = 1$. To prove that $\alpha_d > 1$ for any $d \geqslant 2$, it in fact suffices to prove that $\theta_d(n) > 1$ for some positive integer n.

PROPOSITION 7.4. *Suppose that there exists a minimally d-rigid graph G where $c_d^*(G)/c_d(G) > 1$. Then $\alpha_d > 1$.*

Proof. Fix H to be any minimally d-rigid graph where $c_d^*(H)/c_d(H) > 1$. If H has d vertices or less then $c_d^*(H) = c_d(H) = 1$, thus H has at least $d+1$ vertices and at least one edge. Choose any edge $v_0 v_1$ of H. Given $H_1 = H$, we inductively define the graphs $H_1, \ldots, H_d$ by constructing H_{j+1} from H_j by adding a new vertex v_{j+1} adjacent to $v_0, \ldots, v_j$ and $d - j - 1$ other vertices. Now fix $G := H_d$. Since G is formed from H by a sequence of d-dimensional 0-extensions, it is also minimally d-rigid. By Lemma 7.1 we have that

$$\frac{c_d^*(G)}{c_d(G)} = \frac{2^{d-1}c_d^*(H)}{2^{d-1}c_d(H)} = \frac{c_d^*(H)}{c_d(H)} > 1.$$

We now fix $S := \{v_0, \ldots, v_d\}$ to be the constructed clique in G.

Fix v and e to be the number of vertices and edges of G respectively. We now construct for each positive integer k the $n = (v-d)k + d$ vertex graph G_k by gluing k copies of G at the clique S; see Figure 4 for an example of G_4 when G is the minimally 2-rigid 3-prism.
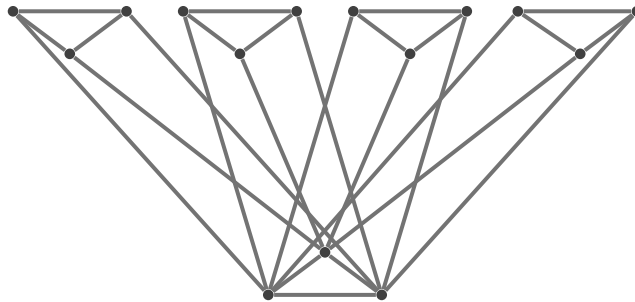

FIGURE 4. Gluing four copies of the 3-prism at a common triangle subgraph.

We first claim that each graph G_k is minimally d-rigid. It is relatively intuitive that G_k is d-rigid (for a rigorous proof of this statement see [20, Theorem 2.5.2]). Since each graph G_k has $k(v - d - 1) + d + 1$ vertices and

$$k\left(e - \binom{d+1}{2}\right) + \binom{d+1}{2} = k\left(\left(dv - \binom{d+1}{2}\right) - \binom{d+1}{2}\right) + \binom{d+1}{2}$$

$$= d(k(v-d-1)+d+1) - \binom{d+1}{2}$$

edges, G_k is also minimally d-rigid.

We next claim that $c_d(G_k) = c_d(G)^k$ and $c_d^*(G_k) = c_d^*(G)^k$. In lieu of a technical proof, we sketch a proof of this claim as follows. Choose a general realisation p of

G in $\mathbb{C}^d$ (respectively, $\mathbb{S}^d$). If we fix the vertices $v_0, \ldots, v_d$ and count the number of equivalent realisations of (G, p), then we see that (G, p) has $c_d(G)$ (respectively, $c_d^*(G)$) such equivalent realisations. Hence, each time we glue another copy of G to go from G_i to G_{i+1}, we must scale the number of realisations by $c_d(G)$ (respectively, $c_d^*(G)$).

Given the graph G_k has $n = k(v - d - 1) + d + 1$ vertices, we see that

$$\frac{c_d^*(G_k)}{c_d(G_k)} = \left(\frac{c_d^*(G)}{c_d(G)} \right)^k = \left(\frac{c_d^*(G)}{c_d(G)} \right)^{\frac{n-d-1}{v-d-1}} = \left(\frac{c_d^*(G)}{c_d(G)} \right)^{\frac{-d-1}{v-d-1}} \left(\left(\frac{c_d^*(G)}{c_d(G)} \right)^{\frac{1}{v-d-1}} \right)^n.$$

Hence, $\alpha_d \geqslant (c_d^*(G)/c_d(G))^{\frac{1}{v-d-1}} > 1$. $\qquad\square$

COROLLARY 7.5. $(4/3)^{3/8} \leqslant \alpha_2 \leqslant 2 \cdot 3^{1/2}$.

Proof. The upper bound for α_2 follows from the method employed in Proposition 7.3 with the upper bound for c_2^* over all n vertex minimally 2-rigid graphs being given by

$$8 \cdot 3^{-7/2} \cdot (2 \cdot 3^{1/2})^{n-2};$$

see [5, Theorem 18] (remember that our d-realisation number is exactly half of the defined d-realisation number used in [5]).

It follows from the proof of Proposition 7.4 that we can maximise our lower bound for α_2 by searching for minimally 2-rigid graphs G with v vertices that contain a triangle where the value $(c_2^*(G)/c_2(G))^{\frac{1}{v-3}}$ is high. In Table 1 we have collected some examples where this value is high. The name of each graph comes from an integer representation of its adjacency matrix, where we take the entries of the upper triangular part (since we always have loop-free graphs) and consider the sequence of row-wise entries as binary digits. For example, the triangle graph K_3 can be written as $(1,1,1)_2 \hat{=} 7$, and the 3-prism (Figure 2) can be written as $(1,1,1,1,0,1,1,1,0,1,1,0,0)_2 \hat{=} 7916$. See [19] or [11] for more details. The graphs in the table are those that achieve the highest value for $(c_2^*/c_2)^{1/(v-3)}$ with the respective number of vertices.

v	G	c_2	c_2^*	c_2^*/c_2	$(c_2^*/c_2)^{1/(v-3)}$
6	7916	12	16	1.3333	1.10064
7	1256267	24	32	1.3333	1.07457
8	170957470	64	96	1.5	1.08447
9	2993854888	160	288	1.8	1.10292
10	4847160401729	400	768	1.92	1.09767
11	5366995734673421	864	2048	2.3704	1.11391
12	37615476241376327552	2016	5120	2.5397	1.10911

TABLE 1. Graphs and the ratios we obtain from their number of realisations.

Fix G to be the 11 vertex graph 5366995734673421, pictured in Figure 5. From observation of the table, G has the highest value for $(c_2^*/c_2)^{\frac{1}{v-3}}$ at $(2048/864)^{\frac{1}{11-3}} \approx 1.11391$. Hence, $\alpha_2 \geqslant (4/3)^{3/8} = (2048/864)^{1/8}$ as required. (Note that there can be graphs with more vertices which give a better bound but have not been computed yet.) $\qquad\square$

By rounding the lower bound down and the upper bound up, Corollary 7.5 informs us that $1.1139 \leqslant \alpha_2 \leqslant 3.4642$. It is conjectured in [19] that $c_2(G) \geqslant 2^{|V|-3}$. If this conjecture is true, we could immediately improve our upper bound to be roughly 1.7321.

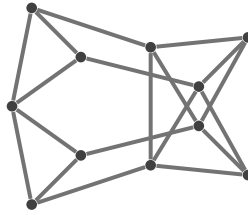We conclude the subsection by making the following conjecture.

FIGURE 5. A graph (known as 5366995734673421) with 864 realisations in the plane and 2048 on the sphere which gives a bound for the triangle-fan of $(2048/864)^{1/8} = (4/3)^{3/8}$.

CONJECTURE 7.6. *For every $d > 1$ we have $\alpha_d > 1$.*

As can be seen from Proposition 7.4, it suffices for us to find a single minimally d-rigid graph in each dimension $d > 2$ where $c_d^*(G) > c_d(G)$. It follows from Corollary 5.3 that the problem cannot be solved by merely finding a suitable graph for one dimension and then coning to obtain similar suitable graphs in higher dimensions. Since no feasible deterministic algorithm exists for higher dimensions, we could only use polynomial system solving tools (e.g. Gröbner basis) with random edge lengths. As well as only being able to give a probabilistic answer, this method is computationally infeasible and can only be done for small numbers of vertices. For the graphs we could compute using this method, we saw that $c_d^*(G) = c_d(G)$ always, however this is most likely because such graphs were too small for the two values to differ.

7.2. EXPLORING DATA SETS. The next computational question we approach is the following: how often do the spherical d-realisation number and the d-realisation number agree for minimally d-rigid graphs? Interestingly, these two numbers seem to differ more than they agree. Figure 6 shows the amount of minimally 2-rigid graphs for which the two realisation numbers differ. We solely consider those minimally 2-rigid graphs with minimal degree 3 since the removal of a degree 2 vertex alters both the spherical 2-realisation number and the 2-realisation number by a factor of $1/2$.

	# graphs with $c_2 = c_2^*$	# graphs with $c_2 \neq c_2^*$	
$n = 6$	1	1	2
$n = 7$	1	3	4
$n = 8$	7	25	32
$n = 9$	42	222	264
$n = 10$	330	2859	3189
$n = 11$	3063	43614	46677
$n = 12$	32855	781020	813875

FIGURE 6. The number of minimally 2-rigid graphs (up to isomorphism) with n vertices and minimum degree 3 for which $c_2(G)$ is the same as/different from $c_2^*(G)$.

In light of this experimental data, we believe that the following conjecture is true.

CONJECTURE 7.7. *Let d be an integer greater than 1. Let $A_{n,d}$ (respectively, $B_{n,d}$) be the set of all minimally d-rigid graphs (up to isomorphism) with n vertices and minimum degree $2d-1$ where $c_d^*(G) = c_d(G)$ (respectively, $c_d^*(G) \neq c_d(G)$). Then $|A_{n,d}|/|B_{n,d}| \to 0$ as $n \to \infty$.*

Now let us investigate how the realisation counts can differ. Figure 7 shows this relation for all 813875 minimally 2-rigid graphs with 12 vertices and minimum degree 3. We observe that there are 9916 different pairs $(c_2(G), c_2^*(G))$ (ignoring repetition) that occur. Only five of those pairs have equal coordinates (i.e. $c_2(G) = c_2^*(G)$): $(512, 512)$, $(768, 768)$, $(869, 869)$, $(960, 960)$ and $(1024, 1024)$. The majority (30789) of graphs for which $c_2(G) = c_2^*(G)$ have 512 realisations. In the figure we colour-coded the amount of occurrences of the pairs $(c_2(G), c_2^*(G))$, with blue indicating few occurrences and red implying many occurrences. The most frequent pair is $(768, 1024)$, which occurs for 76025 graphs. The pair with the largest difference is $(2496, 6144)$, which is obtained by a single graph. Including the graph indicated in the last row of Table 1, there are 20 graphs with the pair $(2016, 5120)$, which gives the largest ratio.

In Figure 8 we analyse more deeply the ratios $c_2^*(G)/c_2(G)$ for minimally 2-rigid graphs with minimal degree 3. Here we can see that the minimum achievable ratio is 1, confirming Proposition 7.2. The maximum ratio in the range is achieved by some graphs with 12 vertices, one of which is given in Table 1. Interestingly, the median ratio varies between 1.39 and 1.4 depending on vertex number. Although the maximum achievable ratio is increasing exponentially (see Corollary 7.5), the range of the quartiles do not seem to vary much as the number of vertices is increased. Note, however, that the size of the graphs considered is rather limited.
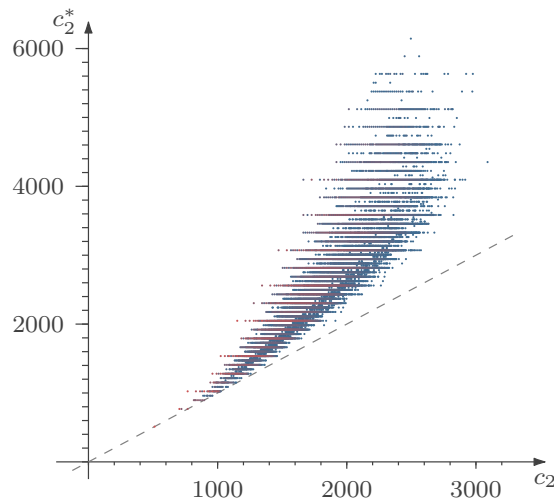

FIGURE 7. The distribution of pairs $(c_2(G), c_2^*(G))$ for all minimally 2-rigid graphs G with 12 vertices and minimum degree three. We use colour to represent the number of occurrences of a given pair: the more often a point appears in the list of all possible pairs, the more red its coordinate representation is in our plot. All points lie above the dashed line representing the equality $c_2(G) = c_2^*(G)$, showing the main result of the paper.
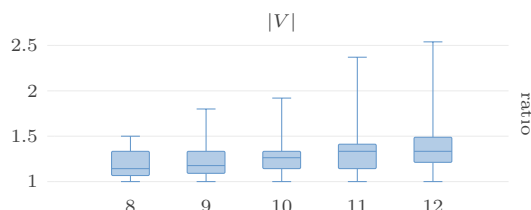
Acknowledgements

FIGURE 8. A box plot of the distribution of ratios $c_2^*(G)/c_2(G)$ for minimally 2-rigid graphs with between 8 to 12 vertices and minimal degree 3.

APPENDIX A. DOMINANT MORPHISMS

Dominant morphisms can be defined in a variety of different but equivalent ways.

THEOREM A.1 ([8, Section AG, Theorem 17.3]). *Let $X \subset \mathbb{C}^m$ be an algebraic set and $Y \subset \mathbb{C}^n$ be a variety. Then the following are equivalent for any morphism $f : X \to Y$:*

  (i) *f is dominant.*
 (ii) *For some irreducible component X' of X, there exists a point $x \in X'$ such that x is a non-singular point of X' and $\operatorname{rank} \mathrm{d}f(x) = \dim Y$.*
(iii) *There exists a Zariski open subset $U \subset X$ where for each $x \in U$, x is a non-singular point of X and $\operatorname{rank} \mathrm{d}f(x) = \dim Y$.*

It follows immediately from Theorem A.1 that for any varieties X, Y, the existence of a dominant morphism from X to Y implies $\dim X \geqslant \dim Y$. As can be seen by Theorem 3.3, more powerful statements can be attained relating to f if $\dim X = \dim Y$. To prove Theorem 3.3, we require the following two results.

COROLLARY A.2 ([25, Section 8, Corollary 1]). *Let $X \subset \mathbb{C}^m$ and $Y \subset \mathbb{C}^n$ be varieties and $f : X \to Y$ be a dominant morphism. Then there exists a non-empty Zariski open subset $U \subset Y$ such that $U \subset f(X)$, and for every $y \in U$, every irreducible component of the algebraic set $f^{-1}(y)$ has dimension $\dim X - \dim Y$.*

COROLLARY A.3 ([21, Proposition 7.16]). *Let $X \subset \mathbb{C}^m$ and $Y \subset \mathbb{C}^n$ be varieties and $f : X \to Y$ be a dominant morphism. Suppose that there exists a non-empty Zariski open subset $U \subset Y$ where $|f^{-1}(y)| < \infty$ for every $y \in Y$. Then there exists a $k \in \mathbb{N}$ and a non-empty Zariski open subset $U' \subset U$ where $|f^{-1}(y)| = k$ for every $y \in U'$.*

Proof of Theorem 3.3. It is immediate that (iii) implies (ii). Fix $U \subset Y$ to be the Zariski open set from Corollary A.2. An algebraic set is a finite set if and only if it is zero dimensional. Hence, (i) and (ii) are equivalent. Suppose that $\dim X = \dim Y$. Fix $U' \subset U$ to be the non-empty Zariski open subset from Corollary A.3. Define the set

$$C := \{x \in X : x \text{ is singular or } \operatorname{rank} \mathrm{d}f(x) < n\}.$$

By Theorem A.1, C is a proper Zariski closed subset of X. As $\dim C < \dim X = \dim Y$, the set $f(C)$ is not dense in Y. It now follows that the complement of the Zariski closure of $f(C)$ in U is a non-empty Zariski open subset of Y. Hence, (i) implies (iii), concluding the proof. □

APPENDIX B. PROOF OF LEMMA 3.2

We begin with the following technical lemma describing when we can "rotate" a framework into our set $X_{G,d}$.

LEMMA B.1. *Let $G = (V, E)$ be a graph with $|V| \geqslant d + 1$, and fix a sequence of d vertices $v_1, \ldots, v_d$. For each realisation $p \in (\mathbb{C}^d)^V$, define the $(d - 1) \times (d - 1)$ symmetric matrix*

$$\mathbb{G}(p) := \begin{bmatrix} (p_{v_2} - p_{v_1})^T \\ \vdots \\ (p_{v_d} - p_{v_1})^T \end{bmatrix} \begin{bmatrix} p_{v_2} - p_{v_1} & \cdots & p_{v_d} - p_{v_1} \end{bmatrix}.$$

Then there exists a realisation $q \in X_{G,d}$ congruent to p if $\mathbb{G}(p)$ only has non-zero leading principal minors.[5]

The conditions stated in Lemma B.1 seem rather bizarre, especially since no such conditions are required if we restrict ourselves to real realisations. It is also rather easy to see they are not necessary either: simply choose any realisation $p \in X_{G,d}$ where $p_{v_i} = \mathbf{0}$ for each $i \in \{1, \ldots, d\}$. To see why we need to be so cautious, take $G = (V, E)$ to be the complete graph with 4 vertices v_1, v_2, v_3, v_4, and let p be the 3-dimensional realisation of G where

$$p_{v_1} = (0, 0, 0), \qquad p_{v_2} = (1, 0, 0), \qquad p_{v_3} = (2, 1, i), \qquad p_{v_4} = (0, 1, 0).$$

In this specific case, there are no realisations in $X_{G,d}$ that are congruent to p. The framework (G, p) also has the interesting properties that its vertices affinely span $\mathbb{C}^3$ and $\|p_v - p_w\|^2 > 0$ for every edge $vw \in E$. Since the matrix

$$\mathbb{G}(p) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & i \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

has rank 1, our constructed realisation is not a counter-example to Lemma B.1.

Proof. Fix $p \in (\mathbb{C}^d)^V$ with the stipulated properties. With this, define $p^1 \in (\mathbb{C}^d)^V$ to be the realisation where $p_v^1 = p_v - p_{v_1}$ for each $v \in V$. If $d = 1$ then $p^1 \in X_{G,1}$, so suppose that $d \geqslant 2$. We now observe that the matrix $\mathbb{G}(p^1) = \mathbb{G}(p)$. In fact, a much stronger statement is true: if q is a congruent realisation then $\mathbb{G}(q) = \mathbb{G}(p)$.

We now form the sequence of congruent realisations $p^1, \ldots, p^d$ in the following inductive way. Fix $n \in \{2, \ldots, d\}$, and suppose that $p^1, \ldots, p^{n-1}$ have already been constructed such that $[p_{v_k}^{n-1}]_j = 0$ if $k \leqslant \min\{j, n - 1\}$. For each $k \in \{1, \ldots, d\}$, fix e_k to be the vector with $[e_k]_j = 1$ if $j = k$ and $[e_k]_j = 0$ otherwise. We observe here that the linear space spanned by the vectors $e_1, \ldots, e_{n-2}$ contains the points $p_{v_2}^{n-1}, \ldots, p_{v_{n-1}}^{n-1}$. Fix z to be the vector formed from $p_{v_n}^{n-1}$ by replacing its first $n - 2$ coordinates with zeroes. It is immediate that $z \cdot e_j = 0$ for all $j \in \{1, \ldots, n - 2\}$.

Suppose for contradiction that $\|z\|^2 = 0$, and fix $y = p_{v_n}^{n-1} - z$. Set $A = [p_{v_2}^{n-1} \cdots p_{v_{n-1}}^{n-1}]$. The determinant of $A^T A$ is the leading principal minor of order $n - 2$ of $\mathbb{G}(p) = \mathbb{G}(p^{n-1})$, and so is non-zero. Hence, the column span of A is exactly the linear space spanned by $e_1, \ldots, e_{n-2}$; importantly, this implies that y is contained in the column span of A. The leading principal minor of order $n - 1$ of

---

[5]A *leading principal minor (of order n)* of a square matrix is the determinant of the matrix formed by taking the first n rows and columns.

$\mathbb{G}(p) = \mathbb{G}(p^{n-1})$ is the determinant of the $(n-1) \times (n-1)$ matrix

$$
\begin{aligned}
B &= \begin{bmatrix} (p_{v_2}^{n-1})^T \\ \vdots \\ (p_{v_n}^{n-1})^T \end{bmatrix} \begin{bmatrix} p_{v_2}^{n-1} \cdots p_{v_n}^{n-1} \end{bmatrix} \\
&= \begin{bmatrix} A^T A & A^T(y+z) \\ (y+z)^T A & (y+z)^T(y+z) \end{bmatrix} \\
&= \begin{bmatrix} A^T A & A^T y \\ y^T A & y^T y \end{bmatrix} \\
&= \begin{bmatrix} A^T \\ y^T \end{bmatrix} \begin{bmatrix} A \ y \end{bmatrix}.
\end{aligned}
$$

As y is contained in the column span of A, the matrix $[A \ y]$ has rank at most $n-2$. However, this implies $\det B = 0$, contradicting our leading principal minor assumption for $\mathbb{G}(p)$. Hence, $\|z\|^2 \neq 0$.

Fix $x := z/\|z\|^2$. Define the $d \times d$ matrix $\tilde{M}$ where $\tilde{M}^T = [e_1 \ \cdots \ e_{n-2} \ x \ \mathbf{0} \ \cdots \ \mathbf{0}]$. We note that the matrix $\tilde{M}$ is an isometry (in the quadratic form sense) from the linear subspace of $\mathbb{C}^d$ spanned by $p_{v_2}^{n-1}, \ldots, p_{v_n}^{n-1}$ to the linear subspace $\mathbb{C}^{n-1} \times \{0\}^{d-n+1}$. By Witt's theorem (see, for example, [27, Theorem 11.15]), there exists a matrix $M \in O(d, \mathbb{C})$ where $M p_{v_j}^{n-1} = \tilde{M} p_{v_j}^{n-1}$ for each $j \in \{1, \ldots, n\}$. With this we now fix $p_v^n = M p_v^{n-1}$ for each $v \in V$. The result now follows from completing the inductive argument and fixing $q = p^d$. □

With this, we can prove Lemma 3.2.

Proof of Lemma 3.2. For every $p \in (\mathbb{C}^d)^V$, let $\mathbb{G}(p)$ be the matrix defined in the statement of Lemma B.1. With this, we fix the proper algebraic set

$$
Z := \left\{ p \in (\mathbb{C}^d)^V : \begin{array}{l} \mathbb{G}(p) \text{ has a zero leading principal minor,} \\ \text{or } p \text{ does not affinely span } \mathbb{C}^d \end{array} \right\}.
$$

Since Z contains hypersurfaces in $(\mathbb{C}^d)^V$ and $Z \neq (\mathbb{C}^d)^V$ it has dimension $d|V| - 1$. By Lemma B.1, the set $f_{G,d}((\mathbb{C}^d)^V \setminus Z)$ is a subset of the image of $\tilde{f}_{G,d}$. Hence, the image of $\tilde{f}_{G,d}$ is Zariski dense in the image of $f_{G,d}$.

Since $\dim \ell_d(G) = \operatorname{rank} df_{G,d}(p)$ for any general realisation $p \in (\mathbb{C}^d)^V$, it follows from Theorem 2.1 that G is d-rigid if and only if $\dim \ell_d(G) = d|V| - \binom{d+1}{2}$. Suppose that G is not d-rigid. The extension of Theorem 2.1 to complex realisations gives that the set $C_d(G, p)$ contains infinitely many points for almost all realisations p; in particular, it can be used to prove that for almost all $p \in (\mathbb{C}^d)^V \setminus Z$, there exists a sequence $(p^n)_{n \in \mathbb{N}}$ of realisations where p^n is not congruent to either p^m or p for each $m \neq n$, and $p^n \to p$ as $n \to \infty$ in the standard complex norm for $(\mathbb{C}^d)^V$ (note that this is an actual metric and not the quadratic form $\|\cdot\|^2$). An application of Lemma B.1 to both p and each p^n gives that $\tilde{f}_{G,d}^{-1}(f_{G,d}(p))$ also contains infinitely many points as is required.

We now suppose that G is d-rigid, i.e., $\dim \ell_d(G) = d|V| - \binom{d+1}{2}$. We split the proof into three cases.

(Case 1: $f_{G,d}(Z)$ is Zariski dense in $\ell_d(G)$ and $|V| \geqslant d + 2$.) In this case, the restriction of the map $f_{G,d}$ to Z and $\ell_d(G)$, which we now denote by $g : Z \to \ell_d(G)$, is dominant. Hence, by Theorem A.1, there exists a non-empty Zariski open subset $U \subset Z$ where

$$
\operatorname{rank} dg(p) = \dim \ell_d(G) = d|V| - \binom{d+1}{2}
$$

for each $p \in U$. Since $\operatorname{rank} \mathrm{d}g(p) \leqslant \operatorname{rank} \mathrm{d}f_{G,d}(p)$, we have that $\operatorname{rank} \mathrm{d}f_{G,d}(p) = d|V| - \binom{d+1}{2}$ for each $p \in U$. As Z has dimension $d|V| - 1$, it contains an irreducible component of dimension $d|V| - 1$. It follows that the nullity of $\mathrm{d}g(p)$ is $\binom{d+1}{2} - 1$ for each $p \in U$. If q is congruent to $p \in Z$ then $\mathbb{G}(q) = \mathbb{G}(p)$ and q is affinely spanning if and only if p is, hence Z is closed under congruences. This implies that for each $p \in U$, the map from the affine congruences of $\mathbb{C}^d$ to the corresponding congruent realisation of p in Z is not injective, as otherwise the nullity of $\mathrm{d}g(p)$ would be at least $\binom{d+1}{2}$ for each $p \in U$. Hence, any realisation in U cannot affinely span $\mathbb{C}^d$. The set of realisations of G that do not affinely span $\mathbb{C}^d$ is the intersection of $|V| - d \geqslant 2$ pairwise-distinct hypersurfaces, and so U is contained in a $(d|V| - 2)$-dimensional subvariety in Z. However, this now contradicts that U is a non-empty Zariski open subset of the $(d|V| - 1)$-dimensional algebraic set Z.

(Case 2: $f_{G,d}(Z)$ is not Zariski dense in $\ell_d(G)$ and $|V| \geqslant d + 2$.) Fix the non-empty Zariski open set

$$X := \left\{ p \in X_{G,d} : \tilde{f}_{G,d}(p) \text{ is not contained in the closure of } f_{G,d}(Z) \right\}.$$

Choose any $p \in X$. By Lemma B.1, for each realisation $p' \in (\mathbb{C}^d)^V$ that is equivalent to p, there exists some other realisation $p'' \in X_{G,d}$ that is congruent to p'. Hence, it now suffices to prove that p is congruent to exactly 2^d realisations (including itself) in $X_{G,d}$.

Choose any $q \in X_{G,d}$ congruent to p, and let $A \in O(d, \mathbb{C})$ be the matrix that maps the vertices of p to the vertices of q. As $p, q \in X_{G,d}$, we have

$$\sum_{n=1}^{k} A_{j,n}[p_{v_{k+1}}]_n = [q_{v_{k+1}}]_j = 0$$

for each $1 \leqslant k < j \leqslant d$, hence $A_{j,k} = 0$ for each $1 \leqslant k < j \leqslant d$. Since A^T maps the vertices of q to the vertices of p, we similarly have $A_{j,k} = 0$ for each $1 \leqslant j < k \leqslant d$. As the vertices of p linearly span $\mathbb{C}^d$, the set $\tilde{f}_{G,d}^{-1}(f_{G,d}(p))$ is in one-to-one correspondence with the diagonal orthogonal matrices. The set of diagonal orthogonal matrices is exactly the set of diagonal matrices A where $A_{j,j} = \pm 1$ for each $j \in \{1, \ldots, d\}$, hence there are 2^d of them. This now concludes the proof.

(Case 3: $|V| = d + 1$.) Since G is d-rigid, it can be easily verified that G must be the complete graph on $d + 1$ vertices. Furthermore, every element $p \in (\mathbb{C}^d)^V \smallsetminus Z$ is only equivalent to congruent realisations ([17, Corollary 8]), and each equivalent realisation is also contained in $(\mathbb{C}^d)^V \smallsetminus Z$. We now define $X = X_{G,d} \smallsetminus Z$ and repeat the same technique as was utilised in Case 2. This now concludes the proof. $\qquad\square$

Appendix C. The effect of 0-extensions on realisation numbers

In this section we prove Lemma 7.1. The specific case where $d = 2$ was originally proven in [7]. We restrict to the non-spherical case throughout this section since the proof is almost identical.

LEMMA C.1. *Let $p_0, \ldots, p_d \in \mathbb{C}^d$ be affinely independent points where $[p_j]_k = 0$ for all $1 \leqslant j \leqslant k \leqslant d$. Then there exists exactly one point $x \in \mathbb{C}^d \smallsetminus \{p_0\}$ which is a solution to the set of equations*

$$(9) \qquad \qquad \|x - p_j\|^2 = \|p_0 - p_j\|^2, \qquad j \in \{1, \ldots, d\}.$$

Proof. First note that we must have that $[p_0]_d \neq 0$ for $p_0, \ldots, p_d \in \mathbb{C}^d$ to be affinely independent. Let $x \in \mathbb{C}^d$ be a solution to the equations in (9). Then the set of points $\{x, p_1, \ldots, p_d\}$ have the same set of pairwise distances between them as the set of points $\{p_0, p_1, \ldots, p_d\}$. Hence, there exists a map $M \in O(d, \mathbb{C})$, where $Mp_j = p_j$ for

each $j \in \{1, \ldots, d\}$ and $Mp_0 = x$; see [17, Section 10] for more details. Since M is invariant over $p_1, \ldots, p_d$, it follows that M is either the identity matrix or M is the diagonal matrix with $M_{jj} = 1$ for each $j < d$ and $M_{dd} = -1$. The result now follows immediately. $\qquad\square$

Proof of Lemma 7.1. For the linear spaces $X_{G,d}$ and $X_{G',d}$, fix the vertices $v_1, \ldots, v_d \in V$ to be the vertices adjacent to the new vertex u that is appended during the 0-extension operation. Fix a general realisation $p \in X_{G,d}$ of G. For each $q \in \tilde{f}_{G,d}^{-1}(f_{G,d}(p))$, define the dominant morphism

$$f_q : \mathbb{C}^d \to \mathbb{C}^d, \ x \mapsto \left(\|x - q_{v_j}\|^2\right)_{j=1}^d$$

and the non-empty Zariski open set $U_q \subset \mathbb{C}^d$ of points not contained in the affine span of $q_{v_1}, \ldots, q_{v_d}$. Since each map f_q is dominant, it follows from Corollary A.2 that there exists a non-empty Zariski open set $U \subset \mathbb{C}^d$ such that

$$U \subset \bigcap \left\{ f_q(\mathbb{C}^d) : q \in \tilde{f}_{G,d}^{-1}(f_{G,d}(p)) \right\}.$$

From this we note that the set

$$U' := \bigcap \left\{ U_q \cap f_q^{-1}(U) : q \in \tilde{f}_{G,d}^{-1}(f_{G,d}(p)) \right\}$$

is a non-empty Zariski open set. Hence, there exists a general realisation $p' \in X_{G',d}$ of G' with $p'_v = p_v$ for all $v \in V$ and $p'_u \in U'$. By applying Lemma C.1 to every realisation in $X_{G',d}$ that is equivalent to p', we see that

$$\left| \tilde{f}_{G',d}^{-1}\left(f_{G',d}(p')\right) \right| = 2 \left| \tilde{f}_{G,d}^{-1}\left(f_{G,d}(p)\right) \right|.$$

The result now follows from Lemma 3.2. $\qquad\square$

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