ALGEBRAIC COMBINATORICS

Andrew Frohmader & Alexander Heaton

Stable graded multiplicities for harmonics on a cyclic quiver

Volume 7, issue 6 (2024), p. 1603-1614.

https://doi.org/10.5802/alco.391

Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 7, issue 6 (2024), p. 1603–1614
https://doi.org/10.5802/alco.391

Stable graded multiplicities for harmonics on a cyclic quiver

Andrew Frohmader & Alexander Heaton

Abstract We consider Vinberg θ-groups associated to a cyclic quiver on k nodes. Let K be the product of the general linear groups associated to each node. Then K acts naturally on $\oplus\mathrm{Hom}(V_i, V_{i+1})$ and by Vinberg's theory the polynomials are free over the invariants. We therefore consider the harmonics as a representation of K, and give a combinatorial formula for the stable graded multiplicity of each K-type. A key lemma provides a combinatorial separation of variables that allows us to cancel the invariants and obtain generalized exponents for the harmonics.

1. Introduction

Consider the representations of a cyclic quiver on k nodes. Associate to each node a finite-dimensional complex vector space V_j, and to each arrow the space of linear transformations, $\mathrm{Hom}(V_j, V_{j+1})$. Set $V = V_1 \oplus \cdots \oplus V_k$ and let K be the block diagonal subgroup of $G = \mathbf{GL}(V)$ isomorphic to $\mathbf{GL}(V_1) \times \cdots \times \mathbf{GL}(V_k)$ acting on

$$\mathfrak{p} = \mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_3) \oplus \cdots \oplus \mathrm{Hom}(V_{k-1}, V_k) \oplus \mathrm{Hom}(V_k, V_1).$$

Here we let $\mathbf{GL}(U) \times \mathbf{GL}(W)$ act on $\mathrm{Hom}(U, W)$ by $(g_1, g_2) \cdot T = g_2 \circ T \circ g_1^{-1}$, as usual. For $(T_1, \ldots, T_k) \in \mathfrak{p}$, we have K-invariant functions defined by

$$\mathrm{Trace}\left[(T_1 \circ \cdots \circ T_k)^p\right]$$

for $1 \leqslant p \leqslant n = \min\{\dim V_j\}$. By a result of Le Bruyn and Procesi [13], these generate the K-invariant functions on $\mathfrak{p}$. The harmonic polynomials $\mathcal{H}$ are defined as the common kernel of all non-constant, K-invariant, constant-coefficient differential operators on $\mathfrak{p}$.

The harmonics are naturally graded by degree and we may encode the decomposition of $\mathcal{H}$ into K-irreducible representations by the q-graded character $\mathrm{char}_q(\mathcal{H})$, which places the character of the degree d invariant subspace as the coefficient of q^d. If s_K^λ is the irreducible character associated to the K-type λ, we may expand

$$\mathrm{char}_q(\mathcal{H}) = \sum_\lambda m_\lambda^{(G,K)}(q)\, s_K^\lambda.$$

Fix the K-type ν. Our main result is a combinatorial formula for $m_\nu^\infty(q, k)$, the stable multiplicity of ν in the harmonics on a cyclic quiver of length k. We will see that, for

Manuscript received 6th April 2024, accepted 12th July 2024.

Keywords. Vinberg θ-group, cyclic quiver, harmonic polynomials, graded multiplicity, crystal base.

ISSN: 2589-5486 http://algebraic-combinatorics.org/

any cyclic quiver, $m_\nu^\infty(q,k)$ is equal to $m_\nu^{(G,K)}(q)$ up to degree $\leqslant n = \min\{\dim V_j\}$ and our main Theorem 4.17 will prove

$$m_\nu^\infty(q,k) = \sum_{T\in D(\nu)} q^{\sum_{i=1}^k |\lambda_i(T)|}.$$

We sum over a certain set of *distinguished tableau* $T \in \mathcal{D}(\nu)$, and the function $\lambda_i(T)$ is computed from simple combinatorial data associated to T. The proof consists of several steps. First, we realize the cyclic quiver above as a θ-representation, or Vinberg pair (G,K), with K the fixed points of a finite order automorphism of G. A key lemma finds a combinatorial *separation of variables*, mirroring Vinberg's theorem [20] that

$$\mathbb{C}[\mathfrak{g}_1] = \mathbb{C}[\mathfrak{g}_1]^K \otimes \mathcal{H}.$$

Our combinatorial separation of variables allows us to cancel the invariants combinatorially. Other steps include constructing an action of a larger group and then restricting to K, applying a branching rule involving Littlewood-Richardson coefficients, and using the combinatorics of $\mathfrak{gl}_\infty$ crystals to translate the $c^\lambda_{\mu,\nu}$ into tableau.

2. Background

2.1. Vinberg Pairs. Let G be a connected reductive algebraic group over $\mathbb{C}$, and let $\theta : G \to G$ be an automorphism of G with finite order k, so $\theta^k = \mathrm{id}$. The group of fixed points $K = G^\theta$ acts on $\mathfrak{g}$ by restriction of the Adjoint representation. Each eigenspace of $d\theta$ is invariant. The Lie algebra splits into eigenspaces

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{k-1}.$$

In [20], Vinberg studied the representation of K on the polynomial functions on an eigenspace, and proved the following separation of variables:

$$\mathbb{C}[\mathfrak{g}_1] = \mathbb{C}[\mathfrak{g}_1]^K \otimes \mathcal{H},$$

where $\mathbb{C}[\mathfrak{g}_1]^K$ are the K-invariant functions and $\mathcal{H}$ are the harmonic polynomials. In general, for any representation of K on V the harmonics are defined as the common kernel for all invariant, non-constant, constant-coefficient differential operators $\mathcal{D}(V)^K$:

$$\mathcal{H} = \{\, f \in \mathbb{C}[V] \,:\, \partial f = 0 \text{ for all non-constant } \partial \in \mathcal{D}(V)^K\}.$$

Note that with $k=1$, Vinberg's results recover those of Kostant's paper, *Lie Group Representations on Polynomial Rings* [12]. There, Kostant proved the separation of variables

$$\mathbb{C}[\mathfrak{g}] = \mathbb{C}[\mathfrak{g}]^G \otimes \mathcal{H},$$

where G acts on its Lie algebra under the Adjoint representation, $\mathbb{C}[\mathfrak{g}]^G$ are the invariants, and $\mathcal{H}$ are the harmonics.

The harmonics are naturally graded by degree, and we may encode the decomposition of $\mathcal{H}$ into G-irreducible representations by the q-graded character $\mathrm{char}_q(\mathcal{H})$, which places the character of the degree-d invariant subspace as the coefficient of q^d. If s_G^λ denotes the character of the G-irreducible representation parametrized by λ, then we have

$$\mathrm{char}_q(\mathcal{H}) = \sum_\lambda \mathcal{K}^G_{\lambda,0}(q)\, s_G^\lambda.$$

In the Kostant setting, the polynomials $\mathcal{K}^G_{\lambda,0}(q)$ are called *generalized exponents* of G and coincide with the Lusztig q-analogues associated to the zero weight subspaces, by a theorem of Hesselink [5]. Thus,

$$\mathcal{K}^G_{\lambda,0}(q) = \sum_{w \in W} (-1)^{l(w)} P_q(w(\lambda+\rho)-\rho),$$

where ρ is the half sum of positive roots, W is the Weyl group of G, P_q is the q-Kostant partition function, and $l(w)$ is the length of $w \in W$.

Much work has been done in relation to these ideas, see [9, 14, 17] and the references within.

The separation of variables above was generalized to the linear isotropy representation for a symmetric space by Kostant and Rallis [11], and yet further to finite order automorphisms by Vinberg [20]. Vinberg's work recovers the Kostant-Rallis results when $k = 2$, which makes $\theta^2 = \mathrm{id}$ an involution, and (G, K) a symmetric pair. We may expand the q-graded character of the harmonics analogously in the Vinberg setting as

$$\mathrm{char}_q(\mathcal{H}) = \sum_{\lambda} m_\lambda^{(G,K)}(q)\, s_K^\lambda.$$

The polynomials $m_\lambda^{(G,K)}(q)$ are much less understood.

In the Kostant-Rallis setting, the graded multiplicities of an irreducible representation λ in $\mathcal{H}$ may be described in terms of the eigenvalues of a certain element of $\mathfrak{k}$, see [11, Theorem 21]. In [21], Wallach and Willenbring obtained formulas similar to Hesselink for some examples including: (GL_{2n}, Sp_{2n}), (SO_{2n+2}, SO_{2n+1}), and (E_6, F_4). Wallach and Willenbring also worked out the example of (SL_4, SO_4) explicitly and other results in special cases have appeared, [19, 10]. There are also stable results stemming from the classical restriction rules of Littlewood [7, 8, 15, 16, 24]. Recently, Frohmader developed a combinatorial formula for $(\mathbf{GL}_n, \mathbf{O}_n)$ which is expected to generalize to the other classical symmetric pairs [2].

Moving outside of the Kostant-Rallis setting, even less is known. To our knowledge the only graded result is due to Heaton [4], in which he determined the graded multiplicity for $(\mathbf{GL}_{2r}, \mathbf{GL}_2 \times \cdots \times \mathbf{GL}_2)$ by counting integral points on the intersection of polyhedra. Wallach has developed ungraded multiplicity formulas, see [22, 23]. Our contribution is a stable formula for $m_\lambda^{(G,K)}(q)$ for $(G, K) = (\mathbf{GL}_N, \mathbf{GL}_{n_1} \times \cdots \times \mathbf{GL}_{n_k})$, where $N = \sum_{i=1}^k n_i$.

2.2. Partitions, Tableaux, and $\mathbf{GL}_n$ representations. For a partition λ, let $l(\lambda)$ denote length(λ) and $|\lambda|$ the size (number of boxes) of λ. Let $\mathcal{P}_n$ denote the set of partitions with length $\leqslant n$ (including the empty partition $\varnothing$) and $\mathcal{P}$ the set of all partitions. Two bases are useful in discussing irreducible polynomial representations of $\mathbf{GL}_n$: $\epsilon_1, \ldots, \epsilon_n$ and $\omega_1, \ldots, \omega_n$, where $\omega_i = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_i$. The polynomial representations of $\mathbf{GL}_n$ are in one to one correspondence with highest weights $\lambda = a_1\epsilon_1 + \cdots + a_n\epsilon_n$, where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0$ are non-negative integers. This gives a bijection between partitions and irreducible polynomial $\mathbf{GL}_n$ representations. In terms of the ω_i basis, the highest weights are given by $\lambda = b_1\omega_1 + \cdots + b_n\omega_n$ where all $b_i \in \mathbb{Z}_{\geqslant 0}$. There are no order conditions. So the ω_i basis allows us to identify irreducible polynomial $\mathbf{GL}_n$ representations with n-tuples of non-negative integers. Computing the change of basis matrices, we see

$$\begin{aligned}
\lambda &= (a_1 - a_2)\omega_1 + \ldots (a_{n-1} - a_n)\omega_{n-1} + a_n\omega_n \\
\lambda &= (b_1 + \cdots + b_n)\epsilon_1 + (b_2 + \cdots + b_n)\epsilon_2 + \cdots + (b_{n-1} + b_n)\epsilon_{n-1} + b_n\epsilon_n.
\end{aligned}$$

In terms of partitions, ϵ_i corresponds to a box in row i and ω_i corresponds to a column of length i.

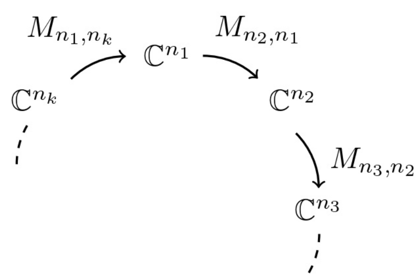

Figure 1. Cyclic quiver on k nodes

Define a partial order on $\mathcal{P}$ by $\mu \leqslant \lambda$ if $\lambda - \mu \in \mathcal{P}$. In what follows, we will view the product order on $\mathbb{Z}^\infty = \{(a_1, a_2, \dots) : a_i \in \mathbb{Z}, a_i = 0$ for all but finitely many $i\}$ as extending $\leqslant$. This is the order $(b_1, b_2, \dots) \leqslant (a_1, a_1, \dots)$ if and only if $a_i - b_i \in \mathbb{Z}_{\geqslant 0}$ for all i. To accomplish this, write $\lambda = a_1\omega_1 + \cdots + a_n\omega_n$ and $\mu = b_1\omega_1 + \cdots + b_n\omega_n$ in terms of the ω_i basis. Notice $\lambda - \mu \in \mathcal{P}$ if and only if $(a_1, \dots, a_n, 0, 0, \dots) - (b_1, \dots, b_n, 0, 0, \dots) \in \mathbb{Z}^\infty_{\geqslant 0}$ if and only if $a_i - b_i \in \mathbb{Z}_{\geqslant 0}$.

Let $SST_n(\lambda)$ be the set of semistandard tableaux on λ with entries in $\{1, \dots, n\}$ and $SST(\lambda)$ the set of semistandard tableaux on λ with entries in $\mathbb{Z}_{>0}$. We view $SST_n(\lambda)$ and $SST(\lambda)$ as $\mathfrak{gl}_n$ and $\mathfrak{gl}_\infty$ crystals, see [1, 6]. Define the weight of a tableau $T \in SST(\lambda)$ by $\mathrm{wt}(T) = k_1\epsilon_1 + \cdots + k_n\epsilon_n$ where k_i denotes the number of i's appearing in T. Writing $\mathrm{wt}(T)$ in terms of the ω_i, we see the reason for extending $\leqslant$ to Z^∞ is to enable comparison with non-dominant weights. For example, given T a tableau on a one-box shape with content 2, $\mathrm{wt}(T) = \epsilon_2 = -\omega_1 + \omega_2$ which we identify with $(-1, 1, 0, 0, \dots)$.

3. The action of K^2

Let M_{n_i,n_j} denote the space of n_i by n_j complex matrices. We have an action of $K = \mathbf{GL}_{n_1} \times \mathbf{GL}_{n_2} \times \cdots \times \mathbf{GL}_{n_k}$ on $\mathfrak{p} = M_{n_2,n_1} \oplus M_{n_3,n_2} \oplus \cdots \oplus M_{n_k,n_{k-1}} \oplus M_{n_1,n_k}$ by

$$(g_1, g_2, \dots, g_k) \cdot (X_1, X_2, \dots, X_k) = (g_2X_1g_1^{-1}, g_3X_2g_2^{-1} \dots g_1X_kg_k^{-1}).$$

This yields an action of K on $\mathbb{C}[\mathfrak{p}]$, $k \cdot f(X) = f(k^{-1} \cdot X)$ for $k \in K$ and $X \in \mathfrak{p}$. We would like to understand the graded multiplicities of this action. (Notice the indices are cyclically permuted, as in Figure 1).

We can approach the problem through branching starting from the action of $K^2 = \mathbf{GL}^2_{n_1} \times \mathbf{GL}^2_{n_2} \times \cdots \times \mathbf{GL}^2_{n_k}$ on $\mathbb{C}[\mathfrak{p}]$ by

$$(g_1, h_1, \dots, g_k, h_k) \cdot f(X_1, X_2, \dots, X_k) = f(g_2^{-1}X_1h_1, \dots g_1^{-1}X_kh_k).$$

Here $\mathbf{GL}^2_{n_i}$ denotes $\mathbf{GL}_{n_i} \times \mathbf{GL}_{n_i}$. Of course, we want to restrict this action to the diagonal subgroup $\Delta = \{(g_1, g_1, g_2, g_2, \dots, g_k, g_k)\} \cong K$. So we have two tasks: first understand the representation of the big group K^2, second understand how this representation restricts to Δ.

We begin by determining the of K^2 irreducible representations in $\mathbb{C}[\mathfrak{p}]$. First, recall,

Proposition 3.1 ([3, Proposition 4.2.5]). *The irreducible representations of $\boldsymbol{GL}_{n_1} \times \boldsymbol{GL}_{n_2} \times \cdots \times \boldsymbol{GL}_{n_l}$ are the representations $V_1 \otimes V_2 \otimes \cdots \otimes V_l$ where V_i is an irreducible representation of $\boldsymbol{GL}_{n_i}$.*

Next, notice $\mathbb{C}[\mathfrak{p}] = \mathbb{C}[M_{n_2,n_1} \oplus \cdots \oplus M_{n_1,n_k}] \cong \mathbb{C}[M_{n_2,n_1}] \otimes \cdots \otimes \mathbb{C}[M_{n_1,n_k}]$, see [3, Lemma A.1.9] and we have k commuting actions. For example, $\mathbf{GL}_{n_2} \times \mathbf{GL}_{n_1}$ acts by

$$f_1(X_1) \otimes \cdots \otimes f_k(X_k) \to f_1(g_2^{-1} X_1 h_1) \otimes \cdots \otimes f_k(X_k).$$

In fact, we can recognize this representation as the tensor product of k distinct actions, so we can decompose the actions separately.

Now recall,

Theorem 3.2 ([3, Theorem 5.6.7]). *The degree d component of $\mathbb{C}[M_{n_i,n_j}]$ under the action of $\boldsymbol{GL}_{n_i} \times \boldsymbol{GL}_{n_j}$ decomposes as follows*

$$\mathbb{C}^d[M_{n_i,n_j}] \cong \bigoplus_{\lambda} (F^{\lambda}_{n_i})^* \otimes (F^{\lambda}_{n_j})$$

with the sum over all nonnegative dominant weights λ of size d and length at most $\min\{n_i, n_j\}$.

Hence we have the following graded decomposition of the K^2 representation (Note: in all that follows we consider our indexing with respect to the cyclic quiver, i.e., mod k with representatives $1, 2, \ldots, k$):

Theorem 3.3. *The degree d component of $\mathbb{C}[\mathfrak{p}]$ under the action of K^2 decomposes as follows*

$$\begin{aligned}\mathbb{C}[\mathfrak{p}] &\cong \bigoplus_{\lambda_1,\lambda_2,\ldots,\lambda_k} [(F^{\lambda_1}_{n_2})^* \otimes F^{\lambda_1}_{n_1}] \otimes [(F^{\lambda_2}_{n_3})^* \otimes F^{\lambda_2}_{n_2}] \otimes \cdots \otimes [(F^{\lambda_k}_{n_1})^* \otimes F^{\lambda_k}_{n_k}] \\ &\cong \bigoplus_{\lambda_1,\lambda_2,\ldots,\lambda_k} \bigotimes_{i=1}^{k} [(F^{\lambda_i}_{n_{i+1}})^* \otimes F^{\lambda_i}_{n_i}]\end{aligned}$$

with the sum over all nonnegative dominant weights $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $|\lambda_1| + |\lambda_2| + \cdots + |\lambda_k| = d$ and length(λ_i) $\leqslant \min\{n_i, n_{i+1}\}$.

Proof. As discussed above, we can decompose each $\mathbb{C}^d[M_{n_i,n_j}]$ factor separately. Apply Theorem 3.2. ☐

With the action of K^2 understood, we turn to the problem of branching to the diagonal subgroup Δ.

4. Stable Multiplicities via Branching

Let $n = \min\{n_1, \ldots, n_k\}$. We work with the pairs $\mathbf{GL}^2_{n_1}$, $\mathbf{GL}^2_{n_2}$, ... , $\mathbf{GL}^2_{n_k}$ separately. Essentially, we choose to group the decomposition from Theorem 3.3 as

$$\begin{aligned}&\bigoplus_{\lambda_1,\lambda_2,\ldots,\lambda_k} [F^{\lambda_1}_{n_1} \otimes (F^{\lambda_k}_{n_1})^*] \otimes [F^{\lambda_2}_{n_2} \otimes (F^{\lambda_1}_{n_2})^*] \otimes \cdots \otimes [F^{\lambda_k}_{n_k} \otimes (F^{\lambda_{k-1}}_{n_k})^*] \\ &\cong \bigoplus_{\lambda_1,\lambda_2,\ldots,\lambda_k} \bigotimes_{i=1}^{k} [F^{\lambda_i}_{n_i} \otimes (F^{\lambda_{i-1}}_{n_i})^*]\end{aligned}$$

Recall,

Theorem 4.1 (Stable Branching Rule, [8, Theorem 2.1.4.1]). *For $l(\lambda_i) + l(\lambda_{i-1}) \leqslant n_i$,*

$$\dim \mathit{Hom}_{\boldsymbol{GL}_{n_i}}(F^{\nu^+,\nu^-}_{n_i}, F^{\lambda_i}_{n_i} \otimes (F^{\lambda_{i-1}}_{n_i})^*) = \sum_{\alpha} c^{\lambda_i}_{\alpha,\nu^+} c^{\lambda_{i-1}}_{\alpha,\nu^-}.$$

$F^{\nu^+,\nu^-}_{n_i}$ is our notation for the rational representation of $\mathbf{GL}_{n_i}$ corresponding to the tuple of partitions (ν^+, ν^-). Both ν^+ and ν^- are partitions, and if $\nu^+ =$

$(a_1, a_2, \ldots, a_\ell)$ and $\nu^- = (b_1, b_2, \ldots, b_m)$ then $F_{n_i}^{\nu^+,\nu^-}$ is the rational representation of $\mathbf{GL}_{n_i}$ with highest weight

$$(a_1, a_2, \ldots, a_\ell, 0, \ldots, 0, -b_m, -b_{m-1}, \ldots, -b_2, -b_1),$$

with the number of interior zeros arranged appropriately, see [18]. Hence we have,

Theorem 4.2. *For degree $d \leqslant n$, the degree d component of $\mathbb{C}[\mathfrak{p}]$ under the action of K decomposes as follows,*

$$\bigoplus_{\alpha_i,\lambda_i,\nu_i^\pm} \bigotimes_{i=1}^{k} c^{\lambda_i}_{\alpha_i,\nu_i^+} c^{\lambda_{i-1}}_{\alpha_i,\nu_i^-} F_{n_i}^{\nu_i^+,\nu_i^-}$$

with the sum over all $\{\alpha_i, \lambda_i, \nu_i^\pm\}_{i=1}^k$ in $\mathcal{P}_n$ such that $|\lambda_1| + \cdots + |\lambda_k| = d$. In particular, the multiplicity of the K irrep $\nu = (\nu_1^\pm, \ldots, \nu_k^\pm)$ appearing in degree d is given by

$$\sum_{\alpha_i,\lambda_i} \left(\prod_{i=1}^{k} c^{\lambda_i}_{\alpha_i,\nu_i^+} c^{\lambda_{i-1}}_{\alpha_i,\nu_i^-}\right).$$

Proof. Say $d \leqslant n$. Then for any λ_i, λ_{i-1}, $l(\lambda_i) + l(\lambda_{i-1}) \leqslant |\lambda_i| + |\lambda_{i-1}| \leqslant d \leqslant n \leqslant n_i$ so Theorem 4.1 applies and we understand the branching down to K. We also note it suffices to consider partitions in $\mathcal{P}_n$ since if a partition α_i, λ_i or $\nu_i^\pm$ has length greater than n, it contributes to a degree greater than n and so only impacts multiplicities outside the stable range. $\square$

Corollary 4.3. *The following gives the graded character $char_q(\mathbb{C}[\mathfrak{p}])$ up to degree n,*

$$\sum_{\alpha_i,\lambda_i,\nu_i^\pm} q^{\sum|\lambda_i|} \prod_{i=1}^{k} c^{\lambda_i}_{\alpha_i,\nu_i^+} c^{\lambda_{i-1}}_{\alpha_i,\nu_i^-} s_{n_i}^{\nu_i^+,\nu_i^-}$$

where $s_{n_i}^{\nu_i^+,\nu_i^-}$ is the $\boldsymbol{GL}_{n_i}$ character of $F_{n_i}^{\nu_i^+,\nu_i^-}$ and the sum is taken over all $\{\alpha_i, \lambda_i, \nu_i^\pm\}_{i=1}^k$ in $\mathcal{P}_n$.

Next, we handle the invariants, which are generated by $Tr([X_1 X_2 \ldots X_k]^i)$ for $1 \leqslant i \leqslant n$ by a result in [13].

Proposition 4.4. *We have the separation of variables*

$$\mathbb{C}[\mathfrak{p}] = \mathbb{C}[\mathfrak{p}]^K \otimes \mathcal{H}.$$

Proof. Notice that $K = G^\theta$ where $\theta : G \to G$ is given by conjugation by the diagonal matrix with entries equal to kth roots of unity $1, \zeta, \zeta^2, \ldots, \zeta^{k-1}$, each appearing with multiplicities $n_1, \ldots, n_k$. The conjugation action of K on the ζ-eigenspace is isomorphic to the action of K on $\mathfrak{p}$. The result now follows from Vinberg's theory [20]. $\square$

Hence, the graded character of $\mathcal{H}$ is given by

$$\mathrm{char}_q(\mathcal{H}) = \left[\prod_{i=1}^{n} (1 - q^{ki})\right] \mathrm{char}_q(\mathbb{C}[\mathfrak{p}]).$$

Corollary 4.5. *The following gives the graded character $char_q(\mathcal{H})$ up to degree n,*

$$\left[\prod_{i=1}^{n} (1 - q^{ki})\right] \sum_{\alpha_i,\lambda_i,\nu_i^\pm} q^{\sum|\lambda_i|} \prod_{i=1}^{k} c^{\lambda_i}_{\alpha_i,\nu_i^+} c^{\lambda_{i-1}}_{\alpha_i,\nu_i^-} s_{n_i}^{\nu_i^+,\nu_i^-}$$

where $s_{n_i}^{\nu_i^+,\nu_i^-}$ is the $\boldsymbol{GL}_{n_i}$ character of $F_{n_i}^{\nu_i^+,\nu_i^-}$ and the sum is taken over all partitions $\{\alpha_i, \lambda_i, \nu_i^\pm\}_{i=1}^k$ in $\mathcal{P}_n$.

In particular, the following formula provides the graded multiplicity of the K irrep $\nu = (\nu_1^{\pm}, \dots, \nu_k^{\pm})$ in $\mathcal{H}$, denoted $m_\nu(q)$, up to degree n,

$$[\prod_{i=1}^{n}(1-q^{ki})] \sum_{\alpha_i,\lambda_i} q^{\sum|\lambda_i|} \prod_{i=1}^{k} c^{\lambda_i}_{\alpha_i,\nu_i^+} c^{\lambda_{i-1}}_{\alpha_i,\nu_i^-}.$$

Proof. Immediate from above discussion. □

Corollary 4.6. *For ν a K irrep, if either $\sum_{i=1}^k |\nu_i^+| > n$ or $\sum_{i=1}^k |\nu_i^-| > n$, then $m_\nu(q) = 0$ in the stable range.*

Proof. Notice in the formula of Corollary 4.5, the smallest degrees come from the $q^{\sum|\lambda_i|}$ terms. Now, by basic properties of Littlewood-Richardson coefficients, if the term $q^{\sum|\lambda_i|}\prod_{i=1}^k c^{\lambda_i}_{\alpha_i,\nu_i^+} c^{\lambda_{i-1}}_{\alpha_i,\nu_i^-}$ is not zero, $|\lambda_i| \geqslant |\nu_i^+|$ for all i, but then $q^{\sum|\lambda_i|} \geqslant q^{\sum|\nu_i^+|} > q^n$. So $m_\nu(q)$ is 0 in degree less than or equal to n. □

We now turn our attention to stable multiplicities and make the following key definition.

Definition 4.7.

$$m_\nu^\infty(q,k) = [\prod_{i=1}^{\infty}(1-q^{ki})] \sum_{\alpha_i,\lambda_i} q^{\sum|\lambda_i|} \prod_{i=1}^{k} c^{\lambda_i}_{\alpha_i,\nu_i^+} c^{\lambda_{i-1}}_{\alpha_i,\nu_i^-},$$

where the sum is taken over all partitions $\{\alpha_i,\lambda_i\}_{i=1}^k$ in $\mathcal{P}$. This is the stable q-multiplicity for ν on a quiver of length k.

It is easy to see that $m_\nu^\infty(q,k) = m_\nu(q)$ up to degree n. These stable q-multiplicities will be our focus for the remainder of the paper.

We would like to cancel the $[\prod_{i=1}^\infty(1-q^{ki})]$ factor from the formula for $m_\nu^\infty(q,k)$. We recall, see [2], that $c^\lambda_{\alpha,\nu} = |\mathrm{CLR}^\lambda_{\alpha,\nu}| := |\{T \in SST(\nu) \mid \alpha \geqslant \varepsilon(T) \text{ and } \alpha + \mathrm{wt}(T) = \lambda\}|$. Here we are viewing $SST(\nu)$ as a $\mathfrak{gl}_\infty$ crystal with Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ for $i = 1, 2, \dots$ and we define $\varepsilon_i(T) = \max\{k \geqslant 0 \mid \tilde{e}_i^k T \in SST(\lambda)\}$, $\phi_i(T) = \max\{k \geqslant 0 \mid \tilde{f}_i^k T \in SST(\lambda)\}$, and

$$\phi(T) = \sum_{i=1}^{n-1} \phi_i(T)\omega_i, \qquad \varepsilon(T) = \sum_{i=1}^{n-1} \varepsilon_i(T)\omega_i.$$

In this notation, we have,

$$m_\nu^\infty(q,k) = [\prod_{i=1}^{\infty}(1-q^{ki})] \sum_{\alpha_i,\lambda_i} q^{\sum|\lambda_i|} \prod_{i=1}^{k} |\mathrm{CLR}^{\lambda_i}_{\alpha_i,\nu_i^+}||\mathrm{CLR}^{\lambda_{i-1}}_{\alpha_i,\nu_i^-}|.$$

Notice the formula for $m_\nu^\infty(q,k)$ has $\nu_i^\pm$ fixed for all i, so we are just computing various subsets of $\underline{\mathrm{SST}}(\nu) := \prod_{i=1}^k [SST(\nu_i^+) \times SST(\nu_i^-)]$. The key is to understand which $T = (T_1^+, T_1^-, \dots, T_k^+, T_k^-) \in \underline{\mathrm{SST}}(\nu)$ appear in some $\underline{\mathrm{CLR}}^\lambda_{\alpha,\nu} := \prod_{i=1}^k \mathrm{CLR}^{\lambda_i}_{\alpha_i,\nu_i^+} \times \mathrm{CLR}^{\lambda_{i-1}}_{\alpha_i,\nu_i^-}$ and with what multiplicity. In this context, $\lambda = (\lambda_1, \dots, \lambda_k), \alpha = (\alpha_1, \dots \alpha_k)$, and $\nu = (\nu_1^\pm, \dots, \nu_k^\pm)$ are tuples of partitions.

As $T_i = (T_i^+, T_i^-)$ is associated with the rational $\mathbf{GL}_{n_i}$ representation $F_{n_i}^{\nu_i^+,\nu_i^-}$, let $\mathrm{wt}(T_i) := \mathrm{wt}(T_i^+) - \mathrm{wt}(T_i^-)$. Also denote the set of all k-tuples of tableaux $\mathcal{P}^k$. We first isolate those T contributing with the following definition.

Definition 4.8. *A tuple of tableaux $T \in \underline{SST}(\nu)$ is called* distinguished *if $T \in \underline{CLR}^{\lambda}_{\alpha,\nu}$ for some $\lambda, \alpha \in \mathcal{P}^k$. We let $D(\nu)$ denote the set of all distinguished tableaux in $\underline{SST}(\nu)$.*

Lemma 4.9. *Suppose $T \in \underline{CLR}^{\lambda}_{\alpha,\nu}$. Then,*

$$\alpha_i = \lambda_1 - wt(T_i^+) + \sum_{j=2}^{i} wt(T_j),$$

$$\lambda_i = \lambda_1 + \sum_{j=2}^{i} wt(T_j),$$

$$\sum_{j=1}^{k} wt(T_j) = 0,$$

for all α_i and λ_i. In particular, α_i and λ_i are uniquely determined by λ_1 and T. With T fixed, let $\lambda(\lambda_1)$ and $\alpha(\lambda_1)$ be those elements of $\mathcal{P}^k$ determined by λ_1.

Proof. We begin by establishing the formula for λ_i. Proceed by induction. The base case is clear. Now assume the formula holds for λ_{i-1} with $1 < i \leqslant k$. From the term $\mathrm{CLR}^{\lambda_i}_{\alpha_i,\nu_i^+} \times \mathrm{CLR}^{\lambda_{i-1}}_{\alpha_i,\nu_i^-}$ we see $\alpha_i = \lambda_{i-1} - \mathrm{wt}(T_i^-)$ so by induction, $\alpha_i = \lambda_1 + \sum_{j=2}^{i-1} \mathrm{wt}(T_j) - \mathrm{wt}(T_i^-)$ and $\lambda_i = \alpha_i + \mathrm{wt}(T_i^+) = \lambda_1 + \sum_{j=2}^{i} \mathrm{wt}(T_j)$.

Next, we establish the third equality. We have,

$$\lambda_k = \lambda_1 + \sum_{j=2}^{k} \mathrm{wt}(T_j)$$

Notice from the $\mathrm{CLR}^{\lambda_1}_{\alpha_1,\nu_1^+} \times \mathrm{CLR}^{\lambda_k}_{\alpha_1,\nu_1^-}$ factor, we also have,

$$\lambda_k = \mathrm{wt}(T_1^-) + \alpha_1.$$
$$\lambda_1 = \mathrm{wt}(T_1^+) + \alpha_1.$$

Subtracting the two expressions for λ_k yields

$$\sum_{j=1}^{k} \mathrm{wt}(T_j) = 0.$$

From the term $\mathrm{CLR}^{\lambda_i}_{\alpha_i,\nu_i^+}$ we see $\alpha_i = \lambda_i - \mathrm{wt}(T_i^+) = \lambda_1 + \sum_{j=2}^{i} \mathrm{wt}(T_j) - \mathrm{wt}(T_i^+)$. □

Lemma 4.9 shows that with T fixed, there is at most a 1-parameter family of $\underline{\mathrm{CLR}}^{\lambda}_{\alpha,\nu}$ containing T. We choose to parameterize this family by λ_1, but note that any choice of a fixed λ_i or α_j uniquely constrains $\underline{\mathrm{CLR}}^{\lambda}_{\alpha,\nu}$ and could be used as parameter. The lemma below shows that the cyclic nature of the representation constrains the set of distinguished tableaux.

Lemma 4.10. *$T \in \underline{SST}(\nu)$ is distinguished if and only if $\sum_{j=1}^{k} wt(T_j) = 0$.*

Proof. Say T is distinguished. Then $T \in \underline{\mathrm{CLR}}^{\lambda}_{\alpha,\nu}$ for some λ and α so by Lemma 4.9, $0 = \sum_{j=1}^{k} \mathrm{wt}(T_j)$.

Now suppose $\sum_{i=1}^{k} \mathrm{wt}(T_i) = 0$. We must show $T \in \underline{\mathrm{CLR}}^{\lambda}_{\alpha,\nu}$ for some $\lambda, \alpha \in \mathcal{P}^k$. To do this, we require two things. First, $\alpha_i \geqslant \varepsilon(T_i^+)$ and $\alpha_i \geqslant \varepsilon(T_i^-)$ for all i. This ensures $T_i^+ \in \mathrm{CLR}^{\alpha_i + \mathrm{wt}(T_i^+)}_{\alpha_i,\nu_i^+}$ and similarly for T_i^-. Second, we have to make sure the λ_i are compatible, that is the two formulas for λ_i, $\lambda_i = \alpha_i + \mathrm{wt}(T_i^+)$ and $\lambda_i = \alpha_{i+1} + \mathrm{wt}(T_{i+1}^-)$ are equal.

By Lemma 4.9, to achieve λ_i compatibility, we must have $\lambda_i = \lambda_1 + \sum_{j=2}^{i} \mathrm{wt}(T_j)$ for $i > 1$ and $\sum_{j=1}^{k} \mathrm{wt}(T_j) = 0$, i.e. we are constrained to work within the family parameterized by λ_1. The proof comes down to showing this family is not empty by selecting a λ_1 large enough that $\alpha_i \geqslant \varepsilon(T_i^+)$ and $\alpha_i \geqslant \varepsilon(T_i^-)$ for all i. As $\alpha_i = \lambda_1 - \mathrm{wt}(T_i^+) + \sum_{j=2}^{i} \mathrm{wt}(T_j)$, this can be achieved by selecting $\lambda_1 \geqslant \sup\{\varepsilon(T_i^{\pm}) + \mathrm{wt}(T_i^+) - \sum_{j=2}^{i} \mathrm{wt}(T_j)\}_{i=1}^{k}$. Indeed, then

$$\begin{aligned}\alpha_i &= \lambda_1 - \mathrm{wt}(T_i^+) + \sum_{j=2}^{i} \mathrm{wt}(T_j)\\ &\geqslant [\varepsilon(T_i^{\pm}) + \mathrm{wt}(T_i^+) - \sum_{j=2}^{i} \mathrm{wt}(T_j)] - \mathrm{wt}(T_i^+) + \sum_{j=2}^{i} \mathrm{wt}(T_j)\\ &= \varepsilon(T_i^{\pm}).\end{aligned}$$

So we have $\alpha_i \geqslant \varepsilon(T_i^{\pm})$, which shows λ_i and α_i are partitions and hence T is contained in $\underline{\mathrm{CLR}}_{\alpha,\nu}^{\lambda}$.

□

Definition 4.11. *We isolate a least upper bound from the proof of Lemma* 4.10 *in this definition. For $T \in D(\nu)$ define $\lambda_{\min}(T) = \sup\{\varepsilon(T_i^{\pm}) + wt(T_i^+) - \sum_{j=2}^{i} wt(T_j)\}_{i=1}^{k}$.*

Note that $\lambda_{\min}$ exists. It can be explicitly constructed as follows. Notice we can work in $\mathcal{P}_N$ if we choose N large enough. Writing each $S_i^{\pm} = \varepsilon(T_i^{\pm}) + \mathrm{wt}(T_i^+) - \sum_{j=2}^{i} \mathrm{wt}(T_j)$ in terms of the ω_i basis as $S_j^{\pm} = a_{1j}^{\pm}\omega_1 = \cdots + a_{Nj}^{\pm}\omega_N$. Set $a_i = \max\{a_{i1}^{\pm}, \ldots, a_{ik}^{\pm}\}$, that is a_i is the maximum coefficient of ω_i across the $S_i^{\pm}$. Then $\lambda_{\min}(T) = a_1\omega_1 + \cdots + a_N\omega_N$. Notice also that $S_1^+ = \varepsilon(T_1^+) + \mathrm{wt}(T_1^+)$ is a partition by the tensor product rule for crystals, that is $S_1^+ = a_{11}^+\omega_1 + \cdots + a_{N1}^+\omega_N$ with $a_{1i} \in \mathbb{Z}_{\geqslant 0}$ for all i. Hence, $a_i \geqslant 0$ for all i.

Next, we give a name to the set of partitions parameterizing the $\underline{\mathrm{CLR}}_{\alpha,\nu}^{\lambda}$ containing T.

Definition 4.12. *For $T \in D(\nu)$ let S_T be the set of all $\lambda_1 \in \mathcal{P}$ such that $T \in \underline{CLR}_{\alpha(\lambda_1),\nu}^{\lambda(\lambda_1)}$.*

Lemma 4.13. *For $T \in D(\nu)$, $T \in \underline{CLR}_{\alpha(\lambda_1),\nu}^{\lambda(\lambda_1)}$ if and only if $\lambda_1 \geqslant \lambda_{\min}(T)$.*

Proof. This follows from the proof of Lemma 4.10. □

Lemma 4.14. *For $T \in D(\nu)$, $S_T = \lambda_{\min}(T) + \mathcal{P}$.*

Proof. This follows from Lemma 4.13 by observing that $\lambda_{\min}(T)$ is the unique minimal element in S_T so for any $\delta \in S_T$ we can write $\delta = \lambda_{\min}(T) + (\delta - \lambda_{\min}(T))$. □

Hence, for $T \in D(\nu)$, we have a 1-parameter family of $\underline{\mathrm{CLR}}_{\alpha,\nu}^{\lambda}$ containing T, now parameterized by $\delta \in \mathcal{P}$. We define the following functions

$$\lambda_i(T, \delta) = \lambda_{\min}(T) + \delta + \sum_{j=2}^{i} \mathrm{wt}(T_j),$$

$$\alpha_i(T, \delta) = \lambda_{\min}(T) + \delta - \mathrm{wt}(T_i^+) + \sum_{j=2}^{i} \mathrm{wt}(T_j).$$

Then this family can be written explicitly as

$$\{\prod_{i=1}^{k} \mathrm{CLR}^{\lambda_i(T,\delta)}_{\alpha_i(T,\delta),\nu_i^+} \times \mathrm{CLR}^{\lambda_{i-1}(T,\delta)}_{\alpha_i(T,\delta),\nu_i^-} \ : \ \delta \in \mathcal{P}\}$$

Lemma 4.15.

$$\lambda_i(T,\delta) = \lambda_i(T,\varnothing) + \delta$$

Proof.

$$\begin{aligned} \lambda_i(T,\delta) &= \lambda_{\min}(T) + \delta + \sum_{j=2}^{i} \mathrm{wt}(T_j), \\ &= \lambda_{\min}(T) + \varnothing + \sum_{j=2}^{i} \mathrm{wt}(T_j) + \delta, \\ &= \lambda_i(T,\varnothing) + \delta. \end{aligned}$$

□

Denote $\lambda_i(T,\varnothing)$ by $\lambda_i(T)$ for simplicity. We isolate the following key lemma which should be viewed as a combinatorial separation of variables.

Lemma 4.16.

$$\frac{1}{\prod_{i=1}^{\infty}(1-q^{ki})} m_\nu^\infty(q,k) = \sum_{\delta\in\mathcal{P}} q^{k|\delta|} \sum_{T\in D(\nu)} q^{\sum_{i=1}^{k} |\lambda_i(T)|}$$

Proof.

$$\begin{aligned} \frac{1}{\prod_{i=1}^{\infty}(1-q^{ki})} m_\nu^\infty(q,k) &= \sum_{T\in D(\nu)} \sum_{\delta\in\mathcal{P}} q^{\sum_{i=1}^{k} |\lambda_i(T,\delta)|} \\ &= \sum_{T\in D(\nu)} \sum_{\delta\in\mathcal{P}} q^{\sum_{i=1}^{k} |\lambda_i(T,\varnothing)+\delta|} \\ &= \sum_{T\in D(\nu)} \sum_{\delta\in\mathcal{P}} q^{k|\delta| \sum_{i=1}^{k} |\lambda_i(T,\varnothing)|} \\ &= \sum_{\delta\in\mathcal{P}} q^{k|\delta|} \sum_{T\in D(\nu)} q^{\sum_{i=1}^{k} |\lambda_i(T,\varnothing)|}, \end{aligned}$$

where in the second line we used Lemma 4.15. □

From this, the main theorem is immediate. Cancel $\sum_{\delta\in\mathcal{P}} q^{k|\delta|}$ with the invariants $1/\prod_{i=1}^{\infty}(1-q^{ki})$.

Theorem 4.17.

$$m_\nu^\infty(q,k) = \sum_{T\in D(\nu)} q^{\sum_{i=1}^{k} |\lambda_i(T)|}$$

Acknowledgements. Thanks to Jeb Willenbring for helpful discussions.

References

[1] Daniel Bump and Anne Schilling, *Crystal bases*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017, https://doi.org/10.1142/9876.

[2] Andrew Frohmader, *Graded multiplicities in the Kostant-Rallis setting*, 2023, https://arxiv.org/abs/2312.11295.

[3] Roe Goodman and Nolan R. Wallach, *Symmetry, representations, and invariants*, Graduate Texts in Mathematics, vol. 255, Springer, Dordrecht, 2009, https://doi.org/10.1007/978-0-387-79852-3.

[4] Alexander Heaton, *Graded multiplicity in harmonic polynomials from the Vinberg setting*, J. Lie Theory **34** (2024), no. 3, 677–692.

[5] Wim H. Hesselink, *Characters of the nullcone*, Math. Ann. **252** (1980), no. 3, 179–182, https://doi.org/10.1007/BF01420081.

[6] Jin Hong and Seok-Jin Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics, vol. 42, American Mathematical Society, Providence, RI, 2002, https://doi.org/10.1090/gsm/042.

[7] Roger Howe, Eng-Chye Tan, and Jeb F. Willenbring, *Stable branching rules for classical symmetric pairs*, Trans. Amer. Math. Soc. **357** (2005), no. 4, 1601–1626, https://doi.org/10.1090/S0002-9947-04-03722-5.

[8] Roger Howe, Eng-Chye Tan, and Jeb F. Willenbring, *The stability of graded multiplicity in the setting of the Kostant-Rallis theorem*, Transform. Groups **13** (2008), no. 3-4, 617–636, https://doi.org/10.1007/s00031-008-9030-0.

[9] Il-Seung Jang and Jae-Hoon Kwon, *Flagged Littlewood-Richardson tableaux and branching rule for classical groups*, J. Combin. Theory Ser. A **181** (2021), article no. 105419 (51 pages), https://doi.org/10.1016/j.jcta.2021.105419.

[10] Kenneth D. Johnson and Nolan R. Wallach, *Composition series and intertwining operators for the spherical principal series. I*, Trans. Amer. Math. Soc. **229** (1977), 137–173, https://doi.org/10.2307/1998503.

[11] B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809, https://doi.org/10.2307/2373470.

[12] Bertram Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404, https://doi.org/10.2307/2373130.

[13] Lieven Le Bruyn and Claudio Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc. **317** (1990), no. 2, 585–598, https://doi.org/10.2307/2001477.

[14] Cédric Lecouvey and Cristian Lenart, *Combinatorics of generalized exponents*, Int. Math. Res. Not. IMRN (2020), no. 16, 4942–4992, https://doi.org/10.1093/imrn/rny157.

[15] D. E. Littlewood, *On invariant theory under restricted groups*, Philos. Trans. Roy. Soc. London Ser. A **239** (1944), 387–417, https://doi.org/10.1098/rsta.1944.0003.

[16] Dudley E. Littlewood, *The theory of group characters and matrix representations of groups*, AMS Chelsea Publishing, Providence, RI, 2006, https://doi.org/10.1090/chel/357.

[17] Kendra Nelsen and Arun Ram, *Kostka-Foulkes polynomials and Macdonald spherical functions*, in Surveys in combinatorics, 2003 (Bangor), London Math. Soc. Lecture Note Ser., vol. 307, Cambridge Univ. Press, Cambridge, 2003, pp. 325–370.

[18] John R. Stembridge, *Rational tableaux and the tensor algebra of* gl_n, J. Combin. Theory Ser. A **46** (1987), no. 1, 79–120, https://doi.org/10.1016/0097-3165(87)90077-X.

[19] Anthony van Groningen and Jeb F. Willenbring, *The cubic, the quartic, and the exceptional group* G_2, in Developments and retrospectives in Lie theory, Dev. Math., vol. 38, Springer, Cham, 2014, pp. 385–397, https://doi.org/10.1007/978-3-319-09804-3_17.

[20] È. B. Vinberg, *The Weyl group of a graded Lie algebra*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 3, 488–526, 709.

[21] N. R. Wallach and J. Willenbring, *On some q-analogs of a theorem of Kostant-Rallis*, Canad. J. Math. **52** (2000), no. 2, 438–448, https://doi.org/10.4153/CJM-2000-020-0.

[22] Nolan R. Wallach, *An analogue of the Kostant-Rallis multiplicity theorem for* θ*-group harmonics*, in Representation theory, number theory, and invariant theory, Progr. Math., vol. 323, Birkhäuser/Springer, Cham, 2017, pp. 603–626, https://doi.org/10.1007/978-3-319-59728-7_20.

[23] Nolan R. Wallach, *Geometric invariant theory*, Universitext, Springer, Cham, 2017, https://doi.org/10.1007/978-3-319-65907-7.

[24] Jeb F. Willenbring, *An application of the Littlewood restriction formula to the Kostant-Rallis theorem*, Trans. Amer. Math. Soc. **354** (2002), no. 11, 4393–4419, https://doi.org/10.1090/S0002-9947-02-03065-9.

Andrew Frohmader, University of Wisconsin - Milwaukee, Department of Mathematics, 3200 N. Cramer St., Milwaukee, WI 53211 (USA)
E-mail : frohmad4@uwm.edu

Alexander Heaton, Lawrence University, Department of Mathematics, Computer Science, and Statistics, 711 E. John St., Appleton, WI 54911 (USA)
E-mail : alexander.m.heaton@lawrence.edu