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Elements of minimal length and Bruhat order on fixed point cosets of Coxeter groups

Nathan Chapelier-Laget & Thomas Gobet

Abstract We study the restriction of the strong Bruhat order on an arbitrary Coxeter group W to cosets xW_L^θ, where x is an element of W and W_L^θ the subgroup of fixed points of an automorphism θ of order at most two of a standard parabolic subgroup W_L of W. When $\theta \neq \mathrm{id}$, there is in general more than one element of minimal length in a given coset, and we explain how to relate elements of minimal length. We also show that elements of minimal length in cosets are exactly those elements which are minimal for the restriction of the Bruhat order.

1. Introduction

When studying Coxeter groups, one often encounters subgroups which themselves admit a structure of Coxeter group. Although there do not seem to exist a general theory of "Coxeter subgroups" of Coxeter groups, at least several important families of subgroups are known to admit canonical structures of Coxeter groups: this includes (standard) parabolic subgroups, or more generally reflection subgroups [3, 4]. Another family is given by subgroups obtained as fixed points of automorphisms of the Coxeter-Dynkin diagram.

Let (W, S) be a Coxeter system. In the most basic of the aforementioned situations, one considers a subset $J \subseteq S$, and defines W_J as the subgroup of W generated by the elements of J. The pair (W_J, J) is again a Coxeter system, with Coxeter-Dynkin diagram obtained from the diagram of W by removing the vertices corresponding to generators in $S \backslash J$. In this situation, every coset xW_J admits two basic properties, namely

(1) There is a unique element $x^J \in xW_J$ which has minimal length among all elements in xW_J,
(2) For all $y \in xW_J$, one has $x^J \leqslant y$, where $\leqslant$ denotes the strong Bruhat order on W.

In fact, for $y \in xW_J$, one has the stronger statement that x^J is below y for the right weak order; nevertheless, there are generalizations in which this is too much to expect. For instance, Dyer extended these properties to the far more general setting of reflection subgroups of Coxeter groups [5, Theorem 1.4], that is, to the case where W_J is replaced by any subgroup W' of W generated by a subset of the set $T = \bigcup_{w \in W} wSw^{-1}$ of *reflections* of W. In this setting, property 2 is only valid for the strong Bruhat order, not for the right weak order in general.

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The purpose of this article is to study the analogue of Properties 1 and 2 above for another class of Coxeter subgroups of Coxeter groups, given by fixed points subgroups of automorphisms squaring to the identity of the Coxeter-Dynkin diagram of (a standard parabolic subgroup of) W.

To be more precise, let (W, S) be a Coxeter system and $L \subseteq S$ be a subset. Let W_L be the corresponding standard parabolic subgroup of W. Let θ be an automorphism of W_L such that $\theta(L) = L$. Then

$$W_L^\theta := \{w \in W_L \mid \theta(w) = w\}$$

admits a structure of Coxeter group; this was observed by Steinberg for finite Weyl groups [9, Section 11], and later generalized to arbitrary Coxeter systems independently by Hée [6] and Mühlherr [8] (see also Lusztig [7, Appendix]). The simple system is obtained as follows. First partition L into orbits $(J_i)_{i\in I}$ under the action of θ. Then, whenever J_i is such that the standard parabolic subgroup W_{J_i} is finite, consider its longest element. The simple system S_L^θ consists of all these elements. In what follows, we will restrict ourselves to the case where $\theta^2 = \mathrm{id}$, in which case every W_{J_i} is either of type A_1 or dihedral.

Note that, when W is irreducible and $L = S$, there may not be a lot of nontrivial automorphisms θ of the Coxeter-Dynkin diagram, but for $L \neq S$ the subgroup W_L may not be irreducible, yielding many possible automorphisms permuting irreducible components that are isomorphic as Coxeter groups. Such a situation arose in work of Chaput, Fresse and the second author [2, Section 3], where a study of the analogues of Properties 1 and 2 of cosets xW_L^θ for subsets L of a certain form was an important step in the understanding of a partial order defined on the quotient W/W_L^θ, which in type A describes a "Bruhat-like order" given by inclusion of certain nilpotent orbit closures. More precisely, while the classical Bruhat order on a finite Weyl group W describes the inclusion order of Schubert varieties, which are the closures of the B-orbits on the flag variety G/B, in the aforementioned situation the Bruhat order on W/W_L^θ describes the inclusion order of B-orbit closures for the action of B on the G-orbit of a 2-nilpotent matrix (in the case $G = \mathrm{GL}_n$). See [2, Theorem 9.1] for more details.

Unfortunately, unlike for the case of standard parabolic subgroups (or more generally reflection subgroups), there is *not* a unique element of minimal length in a given coset in general. The main results addressing the analogues of Properties 1 and 2 may be summarized as follows:

Theorem 1.1 (Relation between elements of minimal length in a given coset). *Let $u, v \in W$, $y \in W_L^\theta$ such that $v = uy$ and u, v are both of minimal length in $uW_L^\theta = vW_L^\theta$. Let $y_1y_2\cdots y_k$ be an S_L^θ-reduced expression of y in W_L^θ. Then we have*

$$\ell(u) = \ell(uy_1) = \ell(uy_1y_2) = \cdots = \ell(uy_1\cdots y_{k-1}) = \ell(v).$$

In other words, for all $i = 1, \dots, k-1$, we have that $uy_1\cdots y_i$ is of minimal length in $uW_L^\theta = vW_L^\theta$.

The main ingredient for proving Theorem 1.1 is the following proposition, which will also be useful for the proof of Theorem 1.3 below addressing the analogue of Property 2:

Proposition 1.2. *Let $u, w \in W$ such that u is of minimal length in wW_L^θ. Let $z \in W_L^\theta$ such that $w = uz$ and let $x_1x_2\cdots x_k$ be an S_L^θ-reduced expression of z. For all $i = 0, \dots, k-1$, exactly one of the following two situations occurs:*

- *either $\ell(ux_1\cdots x_i) = \ell(ux_1\cdots x_{i+1})$,*
- *or $ux_1\cdots x_i < ux_1\cdots x_{i+1}$.*

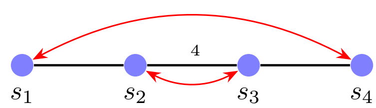
In particular, if x_{i+1} is a reflection of W, then we are in the second situation, while the first situation can only occur if x_{i+1} is not a reflection of W.

The analogue of Property 2 is then given by the following statement:

Theorem 1.3 (Elements of minimal length are minimal for the strong Bruhat order). *Let $x \in W$. There is an element $w \in W$ which is of minimal length in xW_L^θ, and such that $w \leqslant x$. In other words, the elements of minimal length in any coset xW_L^θ are precisely those elements which are minimal with respect to the restriction of the strong Bruhat order $\leqslant$ on W to xW_L^θ.*

We illustrate Theorem 1.1 with an example:

Example 1.4. Let $W = F_4$, $L = S = \{s_1, s_2, s_3, s_4\}$ and θ be the diagram automorphism of L given by the following figure:



We have $S_L^\theta = \{s_1s_4, s_2s_3s_2s_3\}$ and W_L^θ is a dihedral group of order 16. There are 72 classes in W/W_L^θ, each of them having 16 elements. Let $X \in W/W_L^\theta$ and let $u \in X$. By computational experimentations with the software SageMath we found that

$$|\mathrm{Min}(u)| \in \{1, 2, 3, 4, 5, 6, 8, 16\}.$$

We now give two examples of such classes where we see how the minimal elements are related by the elements of $S_L^\theta = \{s_1s_4, s_2s_3s_2s_3\}$ when multiplying on the right, as an illustration of Theorem 1.1. Write $x = s_1s_4$ and $y = s_2s_3s_2s_3$. For simplicity we will denote a reduced expression $s_{i_1}s_{i_2}\cdots s_{i_k}$ of an element of W simply by $i_1i_2\cdots i_k$.

(1) If $u = 42312342$ then the minimal elements of X are given in Figure 1.
(2) If $u = 343231234312$ then the minimal elements of X are given in Figure 2. Note that in this particular class, every element has minimal length in its coset. The conclusion of Theorem 1.1 is thus trivially verified in this case.

$$42312342 \xleftrightarrow{\;x\;} 42312321 \xleftrightarrow{\;y\;} 43123121 \xleftrightarrow{\;x\;} 43123412$$

Figure 1. Minimal elements of a class having 4 minimal elements.

2. Preliminaries and notation

Let (W, S) be a Coxeter system (with S finite) with set of reflections $T = \bigcup_{w \in W} wSw^{-1}$, let $L \subseteq S$, and let θ be a diagram automorphism of L. It induces an automorphism of the standard parabolic subgroup W_L, which we still denote θ. It is well-known that the subgroup

$$W_L^\theta := \{w \in W_L \mid \theta(w) = w\}$$

of θ-fixed elements of W_L admits a structure of Coxeter group (see for instance [6, 8]). The generators as Coxeter group are given by the following set. Let $K \subseteq L$ be an orbit of the action of θ. If W_K is finite, let w_0^K denote its longest element. The set of Coxeter generators of W_L^θ is given by the set of all such elements. We denote by S_L^θ the set of generators of W_L^θ as a Coxeter group. We denote by ℓ the classical length function on W.

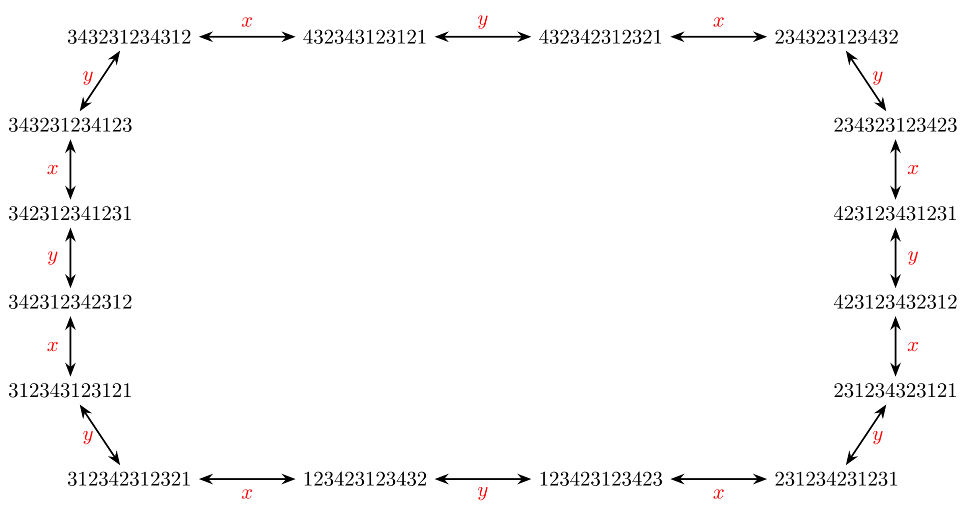


Figure 2. Minimal elements of a class having 16 minimal elements.

Note that the elements of S_L^θ are *not* elements of S in general. For instance, if W has type $A_1 \times A_1$ with Coxeter generators s and t, $L = S$ and θ exchanges s and t, then $W_L^\theta = W^\theta$ has type A_1, with Coxeter generator $st = ts$.

We will furthermore assume that θ satisfies $\theta^2 = \mathrm{id}$. In this case, the elements of S_L^θ are longest elements of finite standard parabolic subgroups of W of type A_1 or dihedral. In particular, if such elements have odd length, then they are reflections of W. It will be useful to distinguish the elements of S_L^θ depending on the parity of their length. We thus write $S_L^\theta = \Theta \,\dot\cup\, \Theta_T$, where

$$\Theta := \{x \in S_L^\theta \mid \ell(x) \text{ is even}\} = \{x \in S_L^\theta \mid x \notin T\},$$
$$\Theta_T := \{x \in S_L^\theta \mid \ell(x) \text{ is odd}\} = \{x \in S_L^\theta \mid x \in T\}.$$

For $u \in W$, we denote by $\mathrm{Min}(u) \subseteq W$ the set of elements of minimal length in uW_L^θ, that is, the set

$$\{v \in uW_L^\theta \mid \ell(w) \geqslant \ell(v) \text{ for all } w \in uW_L^\theta\}.$$

Let

$$\mathcal{M} = \bigcup_{w \in W} \mathrm{Min}(w)$$

denote the set of elements which are of minimal length in their coset. We have the following (see [8, Proposition 3.5]).

Proposition 2.1. *Let $x \in W_L^\theta$ and $x_1, x_2, \ldots, x_k \in S_L^\theta$ such that $x_1 x_2 \cdots x_k$ is an S_L^θ-reduced expression of x. Then*

$$\ell(x) = \sum_{i=1}^{k} \ell(x_i).$$

We denote by $\leqslant$ the (strong) Bruhat order on W. Recall that it is defined as the transitive closure of the relation $x < xt$ whenever $x \in W$, $t \in T$, and $\ell(x) < \ell(xt)$. One has the following characterization, which we will use extensively in the next sections (see for instance [1, Corollary 2.2.3])

Proposition 2.2. *Let $u, v \in W$. The following are equivalent*

(1) $u \leqslant v$,
(2) *There is a reduced expression of* v *admitting a reduced expression of* u *as a subword,*
(3) *Every reduced expression of* v *admits a reduced expression of* u *as a subword.*

Note that by subword we mean that some letters may not be consecutive in the bigger expression.

Also recall that, given $J \subseteq S$, the subgroup W_J of W generated by J is a Coxeter system with simple system J. Denoting

$$W^J = \{w \in W \mid \ell(ws) > \ell(w)\ \forall s \in J\},$$

every $w \in W$ admits a unique decomposition $w = w^J w_J$ with $w^J \in W^J$ and $w_J \in W_J$, and it satisfies $\ell(w) = \ell(w^J) + \ell(w_J)$. See for instance [1, Section 2.4]. In particular, every coset xW_J admits a unique element of minimal length x_0, and for every $y \in xW_J$, one has $x_0 \leqslant y$.

To each element $w \in W$, consider its set $N(w)$ of (right) inversions, which is a subset of T defined by

$$N(w) = \{t \in T \mid \ell(wt) < \ell(w)\} = \{t \in T \mid wt < w\}.$$

Recall that $|N(w)| = \ell(w)$ and that for all $x, y \in W$, we have

$$N(xy) = N(y)\Delta(y^{-1}N(x)y),$$

where Δ denotes the symmetric difference (see for instance [1, Chapter 1, Exercise 12]).

Lemma 2.3. *Let* (W,S) *be a Coxeter system and let* $t, t' \in T$ *with* $t \neq t'$ *and* $tt' = t't$. *Then* $t \notin N(t')$.

Proof. Let $s_1s_2\cdots s_{k-1}s_ks_{k-1}\cdots s_2s_1$ be a palindromic S-reduced expression of t'. Assume for contradiction that $t \in N(t')$. Then two cases can occur: either there is $1 \leqslant i < k$ such that $t = s_1s_2\cdots s_{i-1}s_is_{i-1}\cdots s_2s_1$, or there is $1 \leqslant i < k$ such that $t = s_1s_2\cdots s_ks_{k-1}\cdots s_is_{i+1}\cdots s_ks_{k-1}\cdots s_1$ (the case where $i = k$ yields $t = t'$).

In the first case, we have $t' = tt't = s_1s_2\cdots s_{i-1}s_{i+1}\cdots s_ks_{k-1}\cdots s_{i+1}s_{i-1}\cdots s_1$, which is an expression for t' in the elements of S that is of strictly smaller length than $s_1s_2\cdots s_k\cdots s_2s_1$, a contradiction, since the latter was assumed to be S-reduced.

In the second case, we have $tt' = s_1s_2\cdots s_ks_{k-1}\cdots s_{i+1}s_{i-1}\cdots s_2s_1$. But we also have $t't = s_1\cdots s_{i-1}s_{i+1}\cdots s_ks_{k-1}\cdots s_2s_1$. Since $tt' = t't$ we get $s_{i+1}\cdots s_ks_{k-1}\cdots s_i = s_i\cdots s_ks_{k-1}\cdots s_{i+1}$, yielding $s_is_{i+1}\cdots s_ks_{k-1}\cdots s_{i+1}s_i = s_{i+1}\cdots s_ks_{k-1}\cdots s_{i+1}$, contradicting again the fact that $s_1s_2\cdots s_ks_{k-1}\cdots s_2s_1$ is reduced. ☐

Remark 2.4. Lemma 2.3 can also be proven using root systems. Let Φ be the generalized root system attached to (W,S). We have $\Phi = \Phi^+ \coprod(-\Phi^+)$, where $\Phi^+ = \{\alpha_t \mid t \in T\}$ is the set of positive roots. In this setting, for $w \in W$ we have

$$N(w) = \{t \in T \mid w(\alpha_t) \in (-\Phi^+)\}.$$

Let t, t' satisfying the assumptions of Lemma 2.3 and assume for contradiction that $t \in N(t')$. Then $t'(\alpha_t) \in (-\Phi^+)$. But $t'(\alpha_t) = \pm\alpha_{t'tt'} = \pm\alpha_t$, which forces $t'(\alpha_t) = -\alpha_t$. It follows that α_t is an eigenvector of t' for the eigenvalue -1, hence it is proportional to $\alpha_{t'}$, yielding $\alpha_t = \alpha_{t'}$, a contradiction.

We now prove two Lemmatas that will be useful in the proofs of the main results:

Lemma 2.5. *For all* $x \in S_L^\theta$ *and* $u, w \in W$ *such that* $u \leqslant w$ *and* $\ell(w) = \ell(wx)$, *there is* $v \in \{u, ux\}$ *such that* $\ell(v) \leqslant \ell(u)$ *and* $v \leqslant wx$.

Proof. Let $u, w \in W$ and $x \in S_L^\theta$ such that $u \leqslant w$ and $\ell(wx) = \ell(w)$. Since $\theta^2 = \mathrm{id}$, this forces x to be the longest element in a dihedral standard parabolic subgroup W_I where $I = \{s, t\}$ and I is of type $I_2(2k)$ for some $k \geqslant 1$; indeed, in all the other cases, x has to be the longest element in a standard parabolic subgroup of type A_1 or $I_2(2k+1)$, hence it is a reflection, hence $\ell(wx) \neq \ell(w)$. There are exactly two distinct elements $w_1, w_2 \in W_I$ of length k, and they satisfy $w_1^2 = x = w_2^2$ if k is even (in which case $w_1 = w_2^{-1}$) and $w_1 w_2 = x = w_2 w_1$ if k is odd (in which case w_1 and w_2 are reflections). In all cases we have $w_1 x = w_2$ and $w_2 x = w_1$. We have $u^I \leqslant w^I$ because the map $x \mapsto x^I$ preserves the (strong) Bruhat order (see [1, Proposition 2.5.1]) .

The condition $\ell(w) = \ell(wx)$ yields $\ell(w_I x) = \ell(w_I)$, which forces w_I to lie in $\{w_1, w_2\}$, say $w_I = w_1$ (the roles of w_1 and w_2 are symmetric). Consider the decomposition $u = u^I u_I$. If $\ell(u_I) > k$, then $\ell(u_I x) < k$ and hence the unique reduced expression of $u_I x$ is a subword of the unique reduced expression of w_2. We thus have $ux = u^I u_I x \leqslant w^I w_2$, but $w^I w_2 = w^I w_1 x = w^I w_I x = wx$. We thus have the result with $v = ux$, since we also have

$$\ell(v) = \ell(ux) = \ell(u^I) + \ell(u_I x) < \ell(u^I) + k < \ell(u^I) + \ell(u_I) = \ell(u).$$

If $\ell(u_I) < k$, then the unique reduced expression of u_I is a subword of the unique reduced expression of w_2. We thus have $u = u^I u_I \leqslant w^I w_2$, but $w^I w_2 = w^I w_1 x = w^I w_I x = wx$. We thus get the result with $v = u$. It remains to treat the case where $\ell(u_I) = k$, that is, where $u_I \in \{w_1, w_2\}$. If $u_I = w_1$, then $ux = u^I u_I x = u^I w_1 x \leqslant w^I w_1 x = wx$, hence we get the result with $v = ux$ (also using that $\ell(u_I x) = \ell(u_I)$), while if $u_I = w_2$, we have that $u = u^I u_I = u^I w_2 \leqslant w^I w_2 = w^I w_1 x = wx$, hence we get the result with $v = u$. ☐

Lemma 2.6. *Let $u \in W$ and $x \in S_L^\theta$. If $\ell(u) < \ell(ux)$, then $u < ux$.*

Proof. We have $x = w_{0,I}$ for some $I \subseteq L$, where $w_{0,I}$ is the longest element in the finite standard parabolic subgroup W_I; since θ has order two, we have $|I| = 1$ or 2. If $|I| = 1$, then $x \in S$ and we have $u < ux$. Hence we can assume that $|I| = 2$, say $I = \{s, t\}$. Let $u = u^I u_I$ be the decomposition of u in $W^I W_I$. We have $ux = u^I u_I w_{0,I}$ and $\ell(u) = \ell(u^I) + \ell(u_I)$, $\ell(ux) = \ell(u^I) + \ell(u_I w_{0,I})$. Hence, setting $u_I' := u_I w_{0,I}$, we deduce from the assumption that $\ell(u_I) < \ell(u_I')$. Since $u_I, u_I' \in W_I$ which is a dihedral Coxeter group, we get $u_I < u_I'$, hence $u = u^I u_I < u^I u_I' = ux$, which concludes the proof. ☐

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We keep the notation introduced in the previous section, recalling that we always assume that θ satisfies $\theta^2 = \mathrm{id}$.

We begin by proving Proposition 1.2.

Proof of Proposition 1.2. Let $i \in \{0, 1, \ldots, k-1\}$. Set $y := x_1 x_2 \cdots x_i$. We separate the proof into two cases depending on whether x_{i+1} is a reflection or not.

• **Case where** $x_{i+1} \in \Theta_T$. Since x_{i+1} is a reflection, we want to show that $x_{i+1} \notin N(uy)$. Since $x_1 x_2 \cdots x_{i+1}$ is S_L^θ-reduced, we have that $x_1 x_2 \cdots x_i$ is also S_L^θ-reduced, hence by Proposition 2.1 we have

$$\ell(yx_{i+1}) = \ell(x_1 x_2 \cdots x_{i+1}) = \sum_{j=1}^{i+1} \ell(x_j) = \ell(x_1 \cdots x_i) + \ell(x_{i+1}) = \ell(y) + \ell(x_{i+1}).$$

It follows that $x_{i+1} \notin N(y)$. Assume for contradiction that $x_{i+1} \in N(uy)$. Then since $N(uy) = N(y) \Delta (y^{-1} N(u) y)$, we have $x_{i+1} \in y^{-1} N(u) y$, hence $t := y x_{i+1} y^{-1} \in N(u)$.

Note that $t \in W_L^\theta$ since both y and x_{i+1} lie in W_L^θ. But $t \in N(u)$ implies that $\ell(ut) < \ell(u)$, and since $ut \in uW_L^\theta$, this contradicts the fact that $u \in \mathcal{M}$.

• **Case where** $x_{i+1} \in \Theta$. Then x_{i+1} is the longest element $w_{0,I}$ of a standard finite parabolic subgroup W_I, where $I = \{s,t\}$ is such that W_I is of type $I_2(2m)$ for some $m \geqslant 1$. In particular x_{i+1} has exactly two reduced expressions $(st)^m = (ts)^m$ in W.

If $\ell(uyx_{i+1}) > \ell(uy)$, then by Lemma 2.6 we have $uyx_{i+1} > uy$, which concludes the proof in that case. It therefore suffices to show that the case where $\ell(uyx_{i+1}) < \ell(uy)$ leads to a contradiction. Hence assume that $\ell(uyx_{i+1}) < \ell(uy)$. By Lemma 2.6 again, we have $uyx_{i+1} < uy$. Setting $x := uy$, we consider the decomposition $x = x^I x_I$, where $x^I \in W^I$, $x_I \in W_I$, with respect to the standard dihedral parabolic subgroup W_I. Setting $v = xx_{i+1}$, since $x_{i+1} \in W_I$ we have $v^I = x^I$ and $v_I = x_I x_{i+1}$. We thus have

$$\ell(x^I) + \ell(x_I) = \ell(x) > \ell(v) = \ell(v^I) + \ell(v_I) = \ell(x^I) + \ell(x_I x_{i+1}),$$

from what we deduce that $\ell(x_I) > \ell(x_I x_{i+1})$. Since x_I and $x_I x_{i+1}$ have the same parity of length (because x_{i+1} has even length), we must in fact have $\ell(x_I x_{i+1}) \leqslant \ell(x_I) - 2$. In particular, we have $x_I \neq 1$, and there is $r \in I = \{s,t\}$, say $r = s$ without loss of generality, such that $x_I s < x_I$. We thus have $\ell(x_I s) = \ell(x_I) - 1$ since s is a simple reflection and we deduce that

$$\ell(x_I x_{i+1}) < \ell(x_I s) < \ell(x_I).$$

Since $x_I x_{i+1}, x_I s$ and x_I all lie in W_I which is dihedral, we deduce that

$$x_I x_{i+1} < x_I s < x_I.$$

This stays preserved when multiplying on the left by x^I, yielding

$$v = x^I x_I x_{i+1} < x^I x_I s = xs < x^I x_I = x.$$

Note that $q := sx_{i+1} \in T$ since $sx_{i+1} = s(ts)^m$. We thus have $v = xsq < xs < x$. Moreover, for length reasons, since $sq = qs$ we must have $xq < x$ (as $xq > x$ would contradict $xqs = xsq < x$ as s is simple), hence both s, q lie in $N(x)$. We now argue in a similar way as in the first case above to obtain a contradiction: since $x_1 x_2 \cdots x_{i+1} = yx_{i+1}$ satisfies $\ell(yx_{i+1}) = \sum_{j=1}^{i+1} \ell(x_j)$, then yx_{i+1} has a reduced expression obtained by concatenating a reduced expression of y and a reduced expression of $x_{i+1} = w_{0,I}$, hence $y \in W^I$. Since both s and q are reflections in W_I, we deduce that $s, q \notin N(y)$. As $x = uy$ and s, q both lie in $N(x)$ but none of them lies in $N(y)$, using $N(x) = N(uy) = (y^{-1}N(u)y)\Delta N(y)$ we deduce that both $\widetilde{s} := ysy^{-1}$ and $\widetilde{q} := yqy^{-1}$ lie in $N(u)$. We thus have $u\widetilde{s} < u$, $u\widetilde{q} < u$. We have $\widetilde{q} \notin N(\widetilde{s})$ by Lemma 2.3, as $sq = qs$ implies that $\widetilde{s}\widetilde{q} = \widetilde{q}\widetilde{s}$. Since $N(u\widetilde{s}) = (\widetilde{s}N(u)\widetilde{s})\Delta N(\widetilde{s})$ and $\widetilde{q} \notin N(\widetilde{s})$, $\widetilde{q} \in N(u)$ (hence $\widetilde{q} = \widetilde{s}\widetilde{q}\widetilde{s} \in \widetilde{s}N(u)\widetilde{s}$), we get that $\widetilde{q} \in N(u\widetilde{s})$, hence $u\widetilde{s}\widetilde{q} < u\widetilde{s} < u$. But $\widetilde{s}\widetilde{q} = ysqy^{-1} = yx_{i+1}y^{-1} \in W_L^\theta$, hence $u\widetilde{s}\widetilde{q} \in uW_L^\theta$ with $\ell(u\widetilde{s}\widetilde{q}) < \ell(u)$, contradicting $u \in \mathrm{Min}(u)$. □

We can now prove Theorem 1.1.

Proof of Theorem 1.1. We apply Proposition 1.2 with $w = v$, $z = y$, and $x_i = y_i$ for all $i = 1, \ldots, k$, which is possible since $u \in \mathcal{M}$, $v \in uW_L^\theta$, and the expression $y_1 y_2 \cdots y_k$ is S_L^θ-reduced. For all $i = 0, \ldots, k-1$, we thus get $\ell(uy_1 \cdots y_i) = \ell(uy_1 \cdots y_{i+1})$ or $uy_1 \cdots y_i < uy_1 \cdots y_{i+1}$, hence in all cases we have $\ell(uy_1 \cdots y_i) \leqslant \ell(uy_1 \cdots y_{i+1})$. We thus have

$$\ell(u) \leqslant \ell(uy_1) \leqslant \cdots \leqslant \ell(uy_1 \cdots y_i) \leqslant \cdots \leqslant \ell(uy_1 \cdots y_{k-1}) \leqslant \ell(v).$$

But since $u, v \in \mathcal{M} \cap uW_L^\theta$, we have $\ell(u) = \ell(v)$, hence all inequalities in the above sequence are in fact equalities, which concludes the proof. □

4. Proof of Theorem 1.3

Proof of Theorem 1.3. Let $u \in \mathcal{M}$. To show the result, it suffices to show the following: for all $k \geqslant 0$ and all $x_1, x_2, \dots, x_k \in S_L^\theta$ such that $x_1 x_2 \cdots x_k$ is S_L^θ-reduced, there is $w \in \mathcal{M} \cap uW_L^\theta$ such that $w \leqslant ux_1x_2\cdots x_k$. Indeed, for every $x \in W$, it then suffices to choose $u \in \mathcal{M} \cap xW_L^\theta$, to choose an S_L^θ-reduced expression $x_1x_2\cdots x_k$ of $y := u^{-1}x$ and apply the above result to $x = ux_1x_2\cdots x_k$ to find $w \in \mathcal{M} \cap xW_L^\theta$ such that $w \leqslant x$.

The advantage of the above reformulation is that it allows one to argue by induction on k. For $k = 0$ the result is trivial since one can take $w = u$.

Next, assume that $k \geqslant 1$. By induction there is $w' \in \mathcal{M} \cap uW_L^\theta$ such that $w' \leqslant ux_1x_2\cdots x_{k-1}$. By Proposition 1.2, the case where $\ell(ux_1x_2\cdots x_k) < \ell(ux_1x_2\cdots x_{k-1})$ cannot appear: we have either $ux_1x_2\cdots x_{k-1} < ux_1x_2\cdots x_k$, or $\ell(ux_1x_2\cdots x_{k-1}) = \ell(ux_1x_2\cdots x_k)$. In the first case we are done with $w := w'$, since

$$w' \leqslant ux_1x_2\cdots x_{k-1} < ux_1x_2\cdots x_k.$$

Hence assume that $\ell(ux_1x_2\cdots x_{k-1}) = \ell(ux_1x_2\cdots x_k)$. We now have $w' \leqslant ux_1x_2\cdots x_{k-1}$ and $\ell(ux_1x_2\cdots x_{k-1}) = \ell(ux_1x_2\cdots x_{k-1}x_k)$ with $x_k \in S_L^\theta$, hence by Lemma 2.5, there is $w \in \{w', w'x_k\}$ such that $w \leqslant ux_1x_2\cdots x_{k-1}x_k$ and $\ell(w) \leqslant \ell(w')$. Since w and w' lie in the same coset modulo W_L^θ and $w' \in \mathcal{M}$, we must have $w \in \mathcal{M} \cap uW_L^\theta$, which concludes the proof. □

Remark 4.1. It is natural to wonder whether the conclusions of Theorems 1.1 and 1.3 still hold without the assumption $\theta^2 = \mathrm{id}$. For Theorem 1.1, we do not know, while Theorem 1.3 does not hold in general. As a counterexample, consider a Coxeter system (W, S) of type D_4, with $S = \{s_0, s_1, s_2, s_3\}$, where s_0 is the simple reflection commuting with no other simple reflection. Let $L = \{s_1, s_2, s_3\}$ and let θ be an automorphism of (W_L, L) acting as a 3-cycle on $L := \{s_1, s_2, s_3\}$. Then W_L^θ has type A_1, with generator $s_1s_2s_3$. The coset $s_1W_L^\theta$ has two elements s_1 and s_2s_3, hence $s_2s_3W_L^\theta \cap \mathcal{M} = \{s_1\}$, but $s_1 \not\leqslant s_2s_3$.

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