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Mario Kummer & David Sawall

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Three results related to the half-plane property of matroids

Mario Kummer & David Sawall

ABSTRACT We settle three problems from the literature on stable and real zero polynomials and their connection to matroid theory. We disprove the weak real zero amalgamation conjecture by Schweighofer and the second author. We disprove a conjecture by Brändén and D’León by finding a relaxation of a matroid with the weak half-plane property that does not have the weak half-plane property itself. Finally, we prove that every quaternionic unimodular matroid has the half-plane property which was conjectured by Pendavingh and van Zwam.

1. INTRODUCTION

A *real zero polynomial* is a polynomial $P \in \mathbb{R}[x_1, \dots, x_n]$ with $P(0) \neq 0$ which has only real zeros when restricted to any real line through the origin. Real zero polynomials play an important role in the theory of semidefinite programming, see e.g. [18]. In this context, the second author and Schweighofer made the following conjecture:

CONJECTURE ([17, Conjecture 7.6]). *Assume that $F \in \mathbb{R}[x_1, x_2, y_1, \dots, y_m]$ and $G \in \mathbb{R}[x_1, x_2, z_1, \dots, z_n]$ are real zero polynomials. If*

$$F|_{y_1=\dots=y_m=0} = G|_{z_1=\dots=z_n=0},$$

then there is a real zero polynomial $H \in \mathbb{R}[x_1, x_2, y_1, \dots, y_m, z_1, \dots, z_n]$ such that

$$F = H|_{z_1=\dots=z_n=0} \text{ and } G = H|_{y_1=\dots=y_m=0}.$$

We provide a counterexample to this conjecture in Section 3. Using the connection between stable polynomials, a concept closely related to real zero polynomials, and the theory of matroids and polymatroids established in [5] and [2], we associate a polymatroid to every real zero polynomial. Then the key step is Theorem 3.3 which says that if two real zero polynomials satisfy the conclusion of the above conjecture, then the associated polymatroids can be *amalgamated* in the sense of Definition 2.4. Our counterexample comes from two polymatroids which cannot be amalgamated. These polymatroids are obtained by specializing the matroids F_7^{-4} and F_7^{-5} — two relaxations of the Fano matroid which have the *half-plane property*, meaning that their bases generating polynomials are stable.

More generally, a matroid has the *weak half-plane property* if its set of bases agrees with the support of a stable polynomial. The question of which matroids have the

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(weak) half-plane property has been extensively studied. In this context, Brändén and D’León offered the following conjecture:

CONJECTURE ([3, Conjecture 4.2]). *Suppose that M has the weak half-plane property. Then so does any relaxation of M .*

In Section 4 we present a counterexample to this conjecture. More precisely, we show that a relaxation P_1 of the matroid P_8 does not have the weak half-plane property. On the other hand, P_8 does have the weak half-plane property because it is representable over \mathbb{R} . For proving that P_1 does not have the weak half-plane property, we employ the techniques developed in [3] for narrowing down the space of possible coefficients of a hypothetical stable polynomial with support P_1 .

A key feature of the matroid P_1 that made us examine it for the weak half-plane property is that it is not representable — even under more general notions of representability as discussed in [14]. Matroids representable over more general structures than fields still often tend to have the weak half-plane property, see e.g. [1]. Our third result is of this flavor. Namely, we prove in Section 5 a conjecture which was attributed by Pendavingh and van Zwam to David G. Wagner:

CONJECTURE ([14, Conjecture 6.9]). *All quaternionic unimodular matroids have the half-plane property.*

The notion of quaternionic unimodular matroids is a generalization of the class of sixth root of unity matroids to the skew field of quaternions. For a precise definition see Definition 5.3.

2. PRELIMINARIES

We denote by \mathbb{N} and \mathbb{N}_0 the set of positive and nonnegative integers, respectively. In this section we let E always denote a finite set. For $i \in E$ we denote by $\delta_i \in \mathbb{R}^E$ the i th unit vector. For $x \in \mathbb{R}^E$ we write $|x| = \sum_{i \in E} |x_i|$. We further let $[n] = \{1, \dots, n\}$ and denote by $\binom{[n]}{k}$ the set of all k -element subsets of $[n]$ for every $k, n \in \mathbb{N}$. We recall the cryptomorphic definitions of M-convex sets and polymatroids.

DEFINITION 2.1. *A subset $J \subseteq \mathbb{N}_0^E$ is M-convex if for every $i \in E$ and every $\alpha, \beta \in J$ such that $\alpha_i > \beta_i$, there is $j \in E$ satisfying*

$$\alpha_j < \beta_j \text{ and } \alpha - \delta_i + \delta_j \in J.$$

DEFINITION 2.2. *A polymatroid on E is a function $r: 2^E \rightarrow \mathbb{N}_0$ such that we have for all $S, T \subseteq E$:*

- (i) $r(\emptyset) = 0$,
- (ii) $r(S) \leq r(T)$ if $S \subseteq T$, and
- (iii) $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$.

If $J \subseteq \mathbb{N}_0^E$ is an M-convex set, then we define the function $r_J: 2^E \rightarrow \mathbb{N}_0$ by

$$r_J(S) = \max\left\{\sum_{i \in S} \alpha_i \mid \alpha \in J\right\}.$$

Conversely, if $r: 2^E \rightarrow \mathbb{N}_0$ is a polymatroid, then we define the set $J_r \subseteq \mathbb{N}_0^E$ by

$$J_r = \left\{x \in \mathbb{N}_0^E \mid \sum_{i \in S} x_i \leq r(S) \text{ for all } S \subseteq E \text{ and } \sum_{i \in E} x_i = r(E)\right\}.$$

These two constructions are inverse to each other and define bijections between the set of M-convex sets in \mathbb{N}_0^E and the set of polymatroids on E , see for example [11, §4.4].

REMARK 2.3. A polymatroid $r: 2^E \rightarrow \mathbb{N}_0$ is the rank function of a matroid M on E if and only if $r(\{i\}) \leq 1$ for all $i \in E$. In this case we have

$$J_r = \left\{ \sum_{i \in B} \delta_i \mid B \text{ is a basis of } M \right\}.$$

We apply the definitions made for polymatroids to matroids by considering their rank functions as polymatroids.

2.1. AMALGAMATION OF POLYMATROIDS. There has been considerable interest in the question whether two (poly)matroids can be amalgamated in the following sense, see for example [15, 6] and also [12, §11.4].

DEFINITION 2.4. Let E_1, E_2 be two finite sets, let $E_0 = E_1 \cap E_2$ and $E_3 = E_1 \cup E_2$. Let r_i be a polymatroid on E_i for all $i = 0, 1, 2, 3$.

- (a) We say that r_0 is the restriction of r_1 to E_0 if $r_0(S) = r_1(S)$ for all $S \subseteq E_0$ and denote this by $r_0 = r_1|_{E_0}$.
- (b) We say that r_3 is an amalgam of r_1 and r_2 if $r_1 = r_3|_{E_1}$ and $r_2 = r_3|_{E_2}$.

Clearly, a necessary condition for an amalgam of r_1 and r_2 as in Definition 2.4 to exist is that $r_1|_{E_0} = r_2|_{E_0}$. However, this condition is not sufficient [12, Example 7.2.4].

DEFINITION 2.5. A polymatroid r_0 on E is called sticky if there exists an amalgam for all polymatroids r_1 and r_2 on finite sets E_1 and E_2 with $E = E_1 \cap E_2$ and such that $r_0 = r_i|_E$ for $i = 1, 2$.

In order to describe conditions for a polymatroid to be sticky, recall the following definition, generalizing the usual notion for matroids.

DEFINITION 2.6. Let r be a polymatroid on E .

- (a) A subset $F \subseteq E$ is called a flat of r if $r(F') > r(F)$ for every proper superset F' of F .
- (b) Two flats F_1, F_2 are called a modular pair if

$$r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2).$$

THEOREM 2.7 ([6]). Let r be a polymatroid on E . If every pair of flats is modular, then r is sticky. The converse holds if $|E| \leq 5$.

EXAMPLE 2.8. Let $2 \leq n \leq 5$ and $E = [n]$. One checks that

$$J = \{x \in \mathbb{N}_0^n \mid |x| = 3 \text{ and } \forall i \in E : x_i \leq 2\}$$

is M-convex. The polymatroid r_J is not sticky because the pair $\{1\}, \{2\}$ of flats is not modular.

2.2. STABLE POLYNOMIALS. We briefly recall the different stability and real zero properties of polynomials and their relation to each other. As a general reference we recommend the survey [13].

DEFINITION 2.9. Let $P \in \mathbb{R}[x_i \mid i \in E]$ be a polynomial.

- (a) The support of P is the unique subset $\text{supp}(P) \subseteq \mathbb{N}_0^E$ such that we can write

$$P = \sum_{\alpha \in \text{supp}(P)} c_\alpha x^\alpha$$

for some non-zero $c_\alpha \in \mathbb{R}$.

- (b) The polynomial P is called stable if for all $z \in \mathbb{C}^E$ such that $\text{Im}(z_i) > 0$ for all $i \in E$ we have $P(z) \neq 0$.
- (c) The polynomial P is called a real zero polynomial if for all $v \in \mathbb{R}^E$ the univariate polynomial $P(t \cdot v) \in \mathbb{R}[t]$ has only real zeros.

- (d) Let P be homogeneous and $e \in \mathbb{R}^E$. Then P is called hyperbolic with respect to e if for all $v \in \mathbb{R}^E$ the univariate polynomial $P(t \cdot e + v) \in \mathbb{R}[t]$ has only real zeros.

Taking $v = 0$ in the definition of a real zero polynomial, yields that a real zero polynomial does not vanish at the origin. Similarly, a polynomial P which is hyperbolic in direction e , satisfies $P(e) \neq 0$. Stability is preserved under taking partial derivatives in coordinate directions, under setting some variables equal to each other and under scaling variables by positive scalars. We summarize the well-known connection between the above concepts.

PROPOSITION 2.10 (see for example [13, Proposition 5.3]). Let $P \in \mathbb{R}[x_i \mid i \in E]$ be a homogeneous polynomial. The following are equivalent:

- (i) P is stable.
- (ii) P is hyperbolic with respect to every point in $\mathbb{R}_{>0}^E$.
- (iii) P is hyperbolic with respect to every point in $\mathbb{R}_{>0}^E$ and every $e \in \mathbb{R}_{\geq 0}^E$ with $P(e) \neq 0$.

LEMMA 2.11. Let $P \in \mathbb{R}[x_i \mid i \in E]$ be a homogeneous polynomial and $i \in E$. The following are equivalent:

- (i) P is hyperbolic with respect to δ_i .
- (ii) $P|_{x_i=1} \in \mathbb{R}[x_i \mid i \in E \setminus \{i\}]$ is a real zero polynomial.

Proof. We define $Q = P|_{x_i=1}$ and let $d = \deg(P)$. Assume that P is hyperbolic with respect to δ_i and let $v \in \mathbb{R}^{E \setminus \{i\}}$, $\lambda \in \mathbb{C}$ such that $Q(\lambda \cdot v) = 0$. We have to show that $\lambda \in \mathbb{R}$. Without loss of generality, we can assume that $\lambda \neq 0$. We have

$$0 = Q(\lambda \cdot v) = P(\delta_i + \sum_{j \in E \setminus \{i\}} \lambda \cdot v_j \cdot \delta_j) = \lambda^d \cdot P(\lambda^{-1} \cdot \delta_i + \sum_{j \in E \setminus \{i\}} v_j \cdot \delta_j).$$

Because P is hyperbolic with respect to δ_i , this shows that λ^{-1} and thus λ is real.

Now assume that Q is a real zero polynomial. Let $v \in \mathbb{R}^E$, $\lambda \in \mathbb{C}$ such that $P(\lambda \cdot \delta_i + v) = 0$. We have to show that $\lambda \in \mathbb{R}$. We have

$$0 = P(\lambda \cdot \delta_i + v) = P((\lambda + v_i) \cdot \delta_i + \sum_{j \in E \setminus \{i\}} v_j \cdot \delta_j).$$

Because $v_i \in \mathbb{R}$, it suffices to show that $\mu = \lambda + v_i$ is real. As above, we can assume that $\mu \neq 0$. We have

$$0 = P(\mu \cdot \delta_i + \sum_{j \in E \setminus \{i\}} v_j \cdot \delta_j) = \mu^d \cdot P(\delta_i + \sum_{j \in E \setminus \{i\}} \mu^{-1} \cdot v_j \cdot \delta_j) = Q(\mu^{-1} \cdot \sum_{j \in E \setminus \{i\}} v_j \cdot \delta_j).$$

Because Q is a real zero polynomial, this shows that μ^{-1} and thus λ is real. □

DEFINITION 2.12. Let $P \in \mathbb{R}[x_i \mid i \in E]$ be hyperbolic with respect to $e \in \mathbb{R}^E$. The hyperbolicity cone of P at e is defined as

$$\Lambda(P, e) = \{v \in \mathbb{R}^E \mid P(t \cdot e + v) \in \mathbb{R}[t] \text{ has only nonnegative zeros}\}.$$

Hyperbolicity cones are convex cones [7]. Proposition 2.10 can be rephrased in terms of hyperbolicity cones.

PROPOSITION 2.13. Let $P \in \mathbb{R}[x_i \mid i \in E]$ be a homogeneous polynomial and $e \in \mathbb{R}_{>0}^E$. The following are equivalent:

- (i) P is stable.
- (ii) P is hyperbolic with respect to e and $\delta_i \in \Lambda(P, e)$ for all $i \in E$.

The connection of stable polynomials to M-convex sets is given by the following.

THEOREM 2.14 ([2, Theorem 3.2]). *Let $P \in \mathbb{R}[x_i \mid i \in E]$ be a homogeneous stable polynomial. Then $\text{supp}(P)$ is M -convex.*

DEFINITION 2.15. *Let $J \subseteq \mathbb{N}_0^E$ be an M -convex set.*

- (a) *The generating polynomial of J is*

$$h_J = \sum_{\alpha \in J} \frac{x^\alpha}{\alpha!} \in \mathbb{R}[x_i \mid i \in E]$$

where $\alpha! = \prod_{i \in E} \alpha_i!$.

- (b) *We say that J has the half-plane property if its generating polynomial is stable.*
 (c) *We say that J has the weak half-plane property if there exists a homogeneous stable polynomial $P \in \mathbb{R}[x_i \mid i \in E]$ such that $J = \text{supp}(P)$.*
 (d) *We say that a polymatroid r on E has the (weak) half-plane property if the associated M -convex set J_r has the (weak) half-plane property.*

In the remaining part, we observe that in certain nice situations, restricting the polymatroid of a homogeneous stable polynomial corresponds to plugging in zeros for some of the variables.

DEFINITION 2.16. *Let $J \subseteq \mathbb{N}_0^E$ be M -convex and let $T \subseteq E$. We say that J is nondegenerate with respect to T if there exists $\alpha \in J$ such that for all $i \in E \setminus T$ we have $\alpha_i = 0$.*

LEMMA 2.17. *Let $J \subseteq \mathbb{N}_0^E$ be M -convex and nondegenerate with respect to $T \subseteq E$. Then we have $r_{J|_T} = r_{J'}$ where*

$$J' = \{(\alpha_i)_{i \in T} \mid \alpha \in J \text{ such that } \forall i \in E \setminus T : \alpha_i = 0\} \subseteq \mathbb{N}_0^T.$$

Proof. Let $S \subseteq T$. Then

$$r_{J|_T}(S) = r_J(S) = \max\{\sum_{k \in S} \alpha_k \mid \alpha \in J\}.$$

Let $\alpha \in J$ be a point where this maximum is attained such that $\sum_{k \in E \setminus T} \alpha_k$ is minimal. If $\sum_{k \in E \setminus T} \alpha_k = 0$, then $\alpha \in J'$ and we have $r_{J|_T}(S) = r_{J'}(S)$. Thus assume for the sake of a contradiction that $\alpha_i > 0$ for some $i \in E \setminus T$. Since J is nondegenerate with respect to $T \subseteq E$, there exists $\beta \in J$ such that for all $k \in E \setminus T$ we have $\beta_k = 0$. Because J is M -convex, there exists $j \in T$ such that $\alpha_j < \beta_j$ and $\gamma = \alpha - \delta_i + \delta_j \in J$. Then $\sum_{k \in S} \gamma_k \geq \sum_{k \in S} \alpha_k$ because $i \notin S$ and $\sum_{k \in E \setminus T} \gamma_k < \sum_{k \in E \setminus T} \alpha_k$ because $i \in E \setminus T$ and $j \in T$. This contradicts our minimality assumption. \square

COROLLARY 2.18. *Let $E = E_1 \cup E_2$ and let $P \in \mathbb{R}[x_i \mid i \in E]$ be a polynomial such that $J = \text{supp}(P)$ is M -convex. Finally, for $k = 1, 2$, we let*

$$P_k = P|_{x_i=0 \text{ for } i \notin E_k} \in \mathbb{R}[x_i \mid i \in E_k]$$

and $J_k = \text{supp}(P_k)$. If P_1, P_2 are both not identically zero, then J_1 and J_2 are M -convex and r_J is an amalgam of r_{J_1} and r_{J_2} .

Proof. The condition that P_k is not identically zero is equivalent to J being nondegenerate with respect to E_k . Now the claim follows from Lemma 2.17. \square

3. REAL ZERO AMALGAMATION

In this section we always let $E = E_1 \cup E_2$ be a finite set and $E_0 = E_1 \cap E_2$. We assume that $0 \in E_0$. For $k = 0, 1, 2$ we let $0 \neq P_k \in \mathbb{R}[x_i \mid i \in E_k]$ be homogeneous and stable such that

$$P_0 = P_k|_{x_i=0 \text{ for } i \notin E_0}$$

for $k = 1, 2$. Note that this implies that P_0, P_1 and P_2 all have the same degree d . We further denote $J_k = \text{supp}(P_k)$ and $r_k = r_{J_k}$ for $k = 0, 1, 2$. For $k = 0, 1, 2$ we also consider the polynomial H_k obtained from P_k by substituting $x_0 + x_i$ for x_i for all $i \in E_0 \setminus \{0\}$.

LEMMA 3.1. *The polynomial H_k is hyperbolic with respect to δ_0 . Its hyperbolicity cone contains δ_i for all $i \in E_k$ and the point*

$$\delta_0 - \sum_{0 \neq i \in E_0} \delta_i.$$

Proof. The statement of H_k being hyperbolic with respect to δ_0 is equivalent to P_k being hyperbolic with respect to $\sum_{i \in E_0} \delta_i$. Because of

$$P_k \left(\sum_{i \in E_0} \delta_i \right) = P_0 \left(\sum_{i \in E_0} \delta_i \right) \neq 0,$$

by stability of P_0 , this follows because P_k is stable. The hyperbolicity cone of H_k containing $\delta_i, i \in E_k$, follows from the same statement for P_k . The last statement is equivalent to δ_0 being in the hyperbolicity cone of P_k . \square

LEMMA 3.2. *Let $H \in \mathbb{R}[x_i \mid i \in E]$ be hyperbolic with respect to δ_0 such that the hyperbolicity cone of*

$$H|_{x_i=0 \text{ for } i \notin E_k}$$

agrees with the hyperbolicity cone of H_k for $k = 1, 2$. Then the hyperbolicity cone of H contains δ_i for all $i \in E$ and the point

$$\delta_0 - \sum_{0 \neq i \in E_0} \delta_i.$$

Proof. This follows immediately from Lemma 3.1. \square

Now let $E' = E \setminus \{0\}$ and likewise $E'_k = E_k \setminus \{0\}$ for $k = 0, 1, 2$. Then we consider the polynomials

$$Q_k = H_k|_{x_0=1} \in \mathbb{R}[x_i \mid i \in E'_k]$$

for $k = 0, 1, 2$. These are real zero polynomials with the property that

$$Q_0 = Q_k|_{x_i=0 \text{ for } i \notin E'_0}$$

or $k = 1, 2$. We have the following result.

THEOREM 3.3. *Assume that there is a real zero polynomial $Q \in \mathbb{R}[x_i \mid i \in E']$ with*

$$Q_k = Q|_{x_i=0 \text{ for } i \notin E'_k}$$

for $k = 1, 2$. Then r_1 and r_2 have an amalgam.

Proof. Let $d' = \text{deg}(Q)$. We define

$$H = x_0^{d'} \cdot Q \left(\frac{x_i}{x_0} \mid i \in E' \right) \in \mathbb{R}[x_i \mid i \in E].$$

Then we have

$$H|_{x_i=0 \text{ for } i \notin E'_k} = x_0^{d'-d} \cdot H_k$$

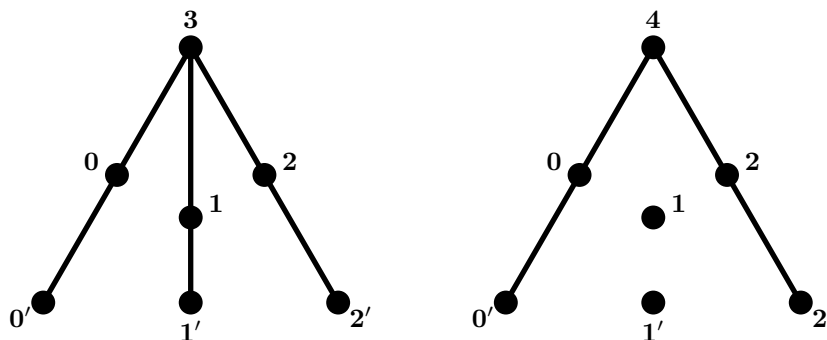


FIGURE 1. The matroids F_7^{-4} (left) and F_7^{-5} (right).

for $k = 1, 2$. Thus by Lemma 3.2 the hyperbolicity cone of H contains δ_i , $i \in E$, and the point

$$\delta_0 - \sum_{0 \neq i \in E_0} \delta_i.$$

This implies that the polynomial P obtained from H by substituting $x_i - x_0$ for x_i for all $i \in E_0 \setminus \{0\}$ is stable. We have

$$P|_{x_i=0 \text{ for } i \notin E_k} = x_0^{d'-d} \cdot P_k$$

for $k = 1, 2$. Finally, we let P' be the polynomial obtained from P by dropping all monomials that are not divisible by $x_0^{d'-d}$ and dividing the result by $x_0^{d'-d}$. Then

$$P'|_{x_i=0 \text{ for } i \notin E_k} = P_k$$

for $k = 1, 2$. The support of P' is M-convex because it agrees with the support of the stable polynomial $\frac{\partial^{d'-d}}{\partial x_0} P$. Now the claim follows from Corollary 2.18. \square

Now we are ready to disprove the real zero amalgamation conjecture from [17].

CONJECTURE 3.4 ([17, Conjecture 7.6]). *Let $E' = E'_1 \cup E'_2$ be a finite set such that $E'_0 = E'_1 \cap E'_2$ has two elements. For $k = 1, 2$ let $Q_k \in \mathbb{R}[x_i \mid i \in E'_k]$ be a real zero polynomial. If*

$$Q_1|_{x_i=0 \text{ for } i \notin E'_0} = Q_2|_{x_i=0 \text{ for } i \notin E'_0},$$

then there is a real zero polynomial $Q \in \mathbb{R}[x_i \mid i \in E']$ such that

$$Q_k = Q|_{x_i=0 \text{ for } i \notin E'_k}$$

for $k = 1, 2$.

Theorem 3.3 shows that counterexamples to Conjecture 3.4 can be constructed from polymatroids with the half-plane property that do not have an amalgam. In the next subsection we construct such a counterexample with $|E_0| = 3$.

3.1. AN EXAMPLE WITH $|E_0| = 3$. We consider the matroids F_7^{-4} and F_7^{-5} defined as in Figure 1. Here, for F_7^{-4} , every 3-element subset of $\{0, 0', 1, 1', 2, 2', 3\}$ forms a basis, except for the subsets $\{0', 0, 3\}$, $\{1', 1, 3\}$ and $\{2', 2, 3\}$. For F_7^{-5} , similarly, all 3-element subsets of $\{0, 0', 1, 1', 2, 2', 4\}$ are bases except for $\{0', 0, 4\}$ and $\{2', 2, 4\}$. These matroids both have the half-plane property, see [5, §A.2.2] for F_7^{-5} and [19]

for F_7^{-4} . Denote by e_3 the third elementary symmetric polynomials in 7 variables. This means that their generating polynomials

$$h_{F_7^{-4}} = e_3(x_0, x_{0'}, x_1, x_{1'}, x_2, x_{2'}, x_3) - x_{0'}x_0x_3 - x_{1'}x_1x_3 - x_{2'}x_2x_3$$

and

$$h_{F_7^{-5}} = e_3(x_0, x_{0'}, x_1, x_{1'}, x_2, x_{2'}, x_4) - x_{0'}x_0x_4 - x_{2'}x_2x_4$$

are stable. We let P_1 and P_2 be the polynomials obtained from $h_{F_7^{-4}}$ and $h_{F_7^{-5}}$, respectively, by setting $x_{0'} = x_0$, $x_{1'} = x_1$ and $x_{2'} = x_2$. We have

$$P_1 = 4x_1x_2x_3 + 2x_1x_2^2 + 2x_1^2x_2 + 4x_0x_2x_3 + 2x_0x_2^2 + 4x_0x_1x_3 + 8x_0x_1x_2 + 2x_0x_1^2 + 2x_0^2x_2 + 2x_0^2x_1$$

and

$$P_2 = 4x_1x_2x_4 + 2x_1x_2^2 + x_1^2x_4 + 2x_1^2x_2 + 4x_0x_2x_4 + 2x_0x_2^2 + 4x_0x_1x_4 + 8x_0x_1x_2 + 2x_0x_1^2 + 2x_0^2x_2 + 2x_0^2x_1.$$

By construction we have

$$P_1|_{x_3=0} = P_2|_{x_4=0}.$$

Hence $E_1 = \{0, 1, 2, 3\}$ and $E_2 = \{0, 1, 2, 4\}$. Note that the support of $P_0 := P_1|_{x_3=0}$ is the M-convex set from Example 2.8 for $n = 3$. We will show that the polymatroids r_1 and r_2 corresponding to P_1 and P_2 do not have an amalgam so that the real zero polynomials Q_1 and Q_2 obtained from P_1 and P_2 constitute a counterexample to Conjecture 3.4. In fact, we will even prove that for every $m \in \mathbb{N}$ the polymatroids $m \cdot r_1$ and $m \cdot r_2$ do not have an amalgam which shows that Q_1^m and Q_2^m also do not satisfy the conclusion of Conjecture 3.4 for any $m \in \mathbb{N}$.

THEOREM 3.5. *The polymatroids $m \cdot r_1$ and $m \cdot r_2$ do not have an amalgam.*

Proof. We proceed as in the proof of [15, Theorem 2]. Assume for the sake of a contradiction that the polymatroid r on $E = \{0, 1, 2, 3, 4\}$ is an amalgam of $m \cdot r_1$ and $m \cdot r_2$. We have

$$r(\{0\}) + r(\{0, 3\}) \leq r(\{0\}) + r(\{0, 3, 4\}) \leq r(\{0, 3\}) + r(\{0, 4\}).$$

By definition of r_1 and r_2 this implies that

$$2m + 2m \leq 2m + r(\{0, 3, 4\}) \leq 2m + 2m.$$

Thus we have $r(\{0, 3, 4\}) = 2m$. Likewise one shows that $r(\{2, 3, 4\}) = 2m$. Furthermore, we have that

$$4m \leq r(\{3, 4\}) + 3m \leq r(\{3, 4\}) + r(\{0, 2, 3, 4\}) \leq r(\{0, 3, 4\}) + r(\{2, 3, 4\}) = 4m.$$

This shows $r(\{3, 4\}) = m$. Finally, we have

$$m + 2m \leq r(\{3\}) + r(\{1, 3, 4\}) \leq r(\{1, 3\}) + r(\{3, 4\}) = 2m + m$$

which implies $r(\{1, 3, 4\}) = 2m$ contradicting

$$3m = r(\{1, 4\}) \leq r(\{1, 3, 4\}).$$

Hence the polymatroids $m \cdot r_1$ and $m \cdot r_2$ do not have an amalgam. □

4. WEAK HALF-PLANE PROPERTY

Let E always denote a finite set and M a matroid on E with set of bases \mathcal{B} .

DEFINITION 4.1 ([3]). *We say that four bases $B_1, B_2, B_3, B_4 \in \mathcal{B}$ form a degenerate quadrangle of M if there exists $S \subseteq E$ and pairwise different $i, j, k, l \notin S$ such that*

$$(B_1, B_2, B_3, B_4) = (S \cup \{i, k\}, S \cup \{i, l\}, S \cup \{j, l\}, S \cup \{j, k\})$$

and if at most one of $S \cup \{i, j\}$ and $S \cup \{k, l\}$ is a bases of M .

The following theorem was used in [3] to reduce the number of possible parameters when searching for a stable polynomial with support M .

THEOREM 4.2 ([2]). *For every basis $B \in \mathcal{B}$ let $0 \neq a_B \in \mathbb{R}$ be such that the multiaffine and homogeneous polynomial*

$$P = \sum_{B \in \mathcal{B}} a_B \cdot \prod_{i \in B} x_i \in \mathbb{R}[x_i \mid i \in E]$$

is stable. If B_1, B_2, B_3, B_4 form a degenerate quadrangle of M , then

$$(1) \quad a_{B_1} a_{B_3} = a_{B_2} a_{B_4}.$$

Letting $b_B := \log(|a_B|)$ for all $B \in \mathcal{B}$ we obtain from Equation (1) linear equations

$$b_{B_1} + b_{B_3} - b_{B_2} - b_{B_4} = 0$$

for all degenerate quadrangles B_1, B_2, B_3, B_4 of M . We denote by $V_M \subseteq \mathbb{R}^{\mathcal{B}}$ the linear space cut out by all such equations. By [3, Lemma 2.6] the vector space

$$W_M := \left\{ \left(\sum_{i \in B} v_i \right)_{B \in \mathcal{B}} \mid v \in \mathbb{R}^E \right\}$$

is an $(|E| - z + 1)$ -dimensional linear subspace of V_M where z is the number of connected components of M .

LEMMA 4.3. *Let U_M be a linear complement of W_M in V_M . If M has the weak half-plane property, then there exists a vector $b \in U_M$ such that*

$$\sum_{B \in \mathcal{B}} \exp(b_B) \cdot \prod_{i \in B} x_i$$

is stable.

Proof. This follows in the same way as [3, Theorem 2.3]: Scaling the variables by $x_i \mapsto \exp(v_i)x_i$ corresponds, after taking logarithms of the coefficients, to shifting by the corresponding vector from W_M . \square

For choosing a linear complement of W_M in V_M in a nice way, the following lemma might be useful.

LEMMA 4.4. *Let M be represented by a matrix $A \in \mathbb{R}^{d \times |E|}$ of rank d . For $B \in \mathcal{B}$ we denote by $A[B]$ the corresponding $d \times d$ submatrix. Then*

$$u(A) := (\log |\det(A[B])|)_{B \in \mathcal{B}} \in V_M.$$

Furthermore, if M is not regular, then $u(A) \notin W_M$.

Proof. By [5, Theorem 8.1] the polynomial

$$\sum_{B \in \mathcal{B}} \det(A[B])^2 \cdot \prod_{i \in B} x_i$$

is stable. This, together with Theorem 4.2, proves the first claim. Now assume that there exists $v \in \mathbb{R}^E$ such that $u(A) = (\sum_{i \in B} v_i)_{B \in \mathcal{B}}$. Scaling the i th column of

A by $\exp(-v_i)$ for all $i \in E$, we obtain a matrix A' representing M all of whose maximal minors are in $\{-1, 0, 1\}$. After multiplication of A' from the left by a suitable invertible matrix we can additionally assume that $A'[B_0]$ is the identity matrix for some $B_0 \in \mathcal{B}$. Then A' is a totally unimodular matrix representing M which shows that M is regular. \square

EXAMPLE 4.5. In this example, we consider the rank 4 matroid $M = P_8$ on 8 elements which is represented by the real matrix

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \end{pmatrix}$$

whose columns we label by $0, \dots, 7$. This matroid is not regular [5, §A4] and therefore $u(A)$ is in V_M but not in W_M by Lemma 4.4. Using the Macaulay2 [9] package “Matroids” [4] we compute that $\dim(V_M) = 9$. Because M is connected, this implies that the span of $u(A)$ is a linear complement of W_M in V_M . Note that P_8 has the weak half-plane property because it is representable over \mathbb{R} .

Recall that if X is a circuit-hyperplane of M , then $\mathcal{B} \cup \{X\}$ is the set of bases of a matroid M' [12, Theorem 1.5.14]. Then M' is called a *relaxation* of M .

LEMMA 4.6. *Let M' be a relaxation of M . Then every degenerate quadrangle of M' is a degenerate quadrangle of M .*

Proof. We proceed as in the proof of [8, Lemma 3.43]. Denote by X the circuit-hyperplane of M such that M' is the relaxation of M by X . We show that for all $x \in X$ and $y \in E \setminus X$ the set

$$(X \setminus \{x\}) \cup \{y\}$$

forms a basis of M (and therefore of M'). Then it immediately follows that X cannot be contained in some degenerate quadrangle of M' . Since X is a circuit, $X \setminus \{x\}$ still has rank $\text{rank}(M) - 1 = \text{rank}(M') - 1$. Then $(X \setminus \{x\}) \cup \{y\}$ has rank $\text{rank}(M')$ because X is closed and $y \notin X$. Thus $(X \setminus \{x\}) \cup \{y\}$ is a basis of M' . \square

REMARK 4.7. Assume that $\text{rank}(M) \geq 2$. Let M' be the relaxation of M by the circuit-hyperplane X and denote $\mathcal{B}' = \mathcal{B} \cup \{X\}$. Lemma 4.6 implies that the map $\mathbb{R}^{\mathcal{B}} \rightarrow \mathbb{R}^{\mathcal{B}'}$, that sends $v \in \mathbb{R}^{\mathcal{B}}$ to the vector w with $w_X = 0$ and $w_B = v_B$ for all $B \in \mathcal{B}$, maps V_M to $V_{M'}$. We thus obtain a natural embedding

$$\iota: V_M \hookrightarrow V_{M'}.$$

Furthermore, we have $\delta_X \in V_{M'}$. If U_M is a linear complement of W_M in V_M , then

$$(\iota(U_M) \oplus \mathbb{R} \cdot \delta_X) \cap W_{M'} = \{0\}.$$

However, in general $\iota(U_M)$ and δ_X do not span a linear complement of $W_{M'}$, see for example [3, Table 1].

EXAMPLE 4.8. As an illustration, we recall [3, Example 4.1]. The non-Fano matroid $M' = F_7^-$ is represented by the real matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

whose columns we label by $1, \dots, 7$. This matroid is not regular [5, §A.2.2] and therefore $u(A)$ is in $V_{M'}$ but not in $W_{M'}$ by Lemma 4.4. On the other hand, the non-Fano matroid is a relaxation of the Fano matroid F_7 by the circuit hyperplane $\{2, 4, 6\}$.

Because $\dim(V_{M'}/W_{M'}) = 1$ by [3], it follows from Remark 4.7 that $u(A) + W_{M'}$ must contain a scalar multiple of $\delta_{\{2,4,6\}}$. Indeed, one has that $u(A) = \log(2) \cdot \delta_{\{2,4,6\}}$. It was further noted in [3, Example 4.1] that by [5, Example 11.5] the polynomial

$$h_{F_7} + \mu x_2 x_4 x_6$$

is stable only for $\mu = 4$. As a side note we would like to mention that this implies in particular that

$$\sum_{S \in \binom{[7]}{3}} |\det(A[S])| \cdot \prod_{i \in S} x_i$$

is not stable, giving a negative answer to the question raised in [16, Remark 4.2].

Now we are ready to disprove the following conjecture by Brändén–D’León.

CONJECTURE 4.9 ([3, Conjecture 4.2]). *Suppose that M has the weak half-plane property. Then so does any relaxation of M .*

Our counterexample is a suitable relaxation of P_8 .

EXAMPLE 4.10. Consider again the matroid P_8 from Example 4.5. The set $X := \{3, 5, 6, 7\}$ is a circuit-hyperplane of P_8 . Following [10] we denote by P_1 the relaxation of P_8 by X . We will prove that P_1 does not have the weak half-plane property, although it is a relaxation of P_8 . Using the Macaulay2 [9] package “Matroids” [4] we compute that $\dim(V_{P_1}) = 10$. Thus by Example 4.5 and Remark 4.7 the vectors δ_X and $\iota(u(A))$ span a linear complement of W_{P_1} in V_{P_1} . We define $v = \frac{1}{\log(2)} \iota(u(A))$ — this is just for convenience to get a vector with entries in $\{0, 1, 2\}$. If P_1 has the weak half-plane property, then by Lemma 4.3 there are $a, b > 0$ such that

$$F_{a,b} = \sum_{B \text{ basis of } P_8} b^{v_B} x^B + ax_3x_5x_6x_7$$

is stable. Now consider the $(0, 1)$ -th Rayleigh difference

$$\Delta_{0,1} F_{a,b} := \partial_{x_0} F_{a,b} \partial_{x_1} F_{a,b} - F_{a,b} \partial_{x_0} \partial_{x_1} F_{a,b}.$$

By [2, Theorem 5.10] the Rayleigh difference $\Delta_{0,1} F_{a,b}$ is globally nonnegative if $F_{a,b}$ is stable. Plugging in $(1, 1, t, -1, -1, t)$ for (x_2, \dots, x_8) yields

$$(\Delta_{0,1} F_{a,b})(1, 1, t, -1, -1, t) = -abt^3 + (-ab - 4b^2 + 2a + 12b + 16)t^2 + at.$$

Because $a, b > 0$, this takes negative values for large enough t . Therefore, for no choice of $a, b > 0$ the polynomial $F_{a,b}$ is stable.

REMARK 4.11. According to [10, Proposition 4] the matroid P_1 is not representable over any field. It does not have the half-plane property because it has the matroid F_7^{-3} as a minor which does not have the half-plane property by [5, Example 11.7]. Moreover, it does not seem to be representable even in the more general context studied in [14]. This made P_1 a good candidate for being a counterexample to Conjecture 4.9 as representations over certain algebras (that are not necessarily fields) sometimes can still be used to prove the weak half-plane property. See for instance [1] where the weak half-plane property was proved for the non-Pappus and the non-Desargues matroid.

REMARK 4.12. We use the notation as in Example 4.10. The polynomial $F_{0,1} := \lim_{a \rightarrow 0} F_{a,1}$ is the basis generating polynomial of P_8 . The polynomial $F_{0,0} := \lim_{a,b \rightarrow 0} F_{a,b}$ is the basis generating polynomial of the graphical matroid $M(G_1)$ where G_1 is depicted in Figure 2 on the left. In total, the regular subdivision of the matroid polytope of P_8 defined by v_B has six maximal cells, all of which are matroid polytopes themselves. Four of the corresponding matroids are isomorphic to $M(G_1)$, the two remaining ones are isomorphic to $M(G_2)$ where G_2 is the graph on the right

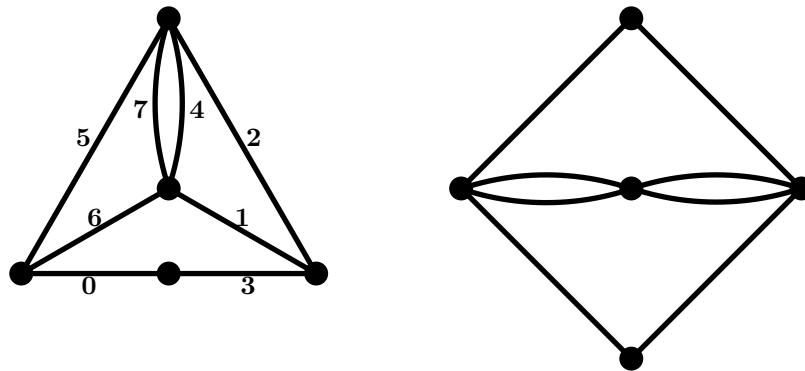


FIGURE 2. The graphs G_1 (left) and G_2 (right) from Remark 4.12.

of Figure 2. Unfortunately, this knowledge did not help us representing $F_{a,b}$ in a simple way.

5. QUATERNIONIC UNIMODULAR MATROIDS

We recall the definition of quaternionic unimodular (QU) matroids. To this end, let \mathbb{H} denote the skew field of quaternions.

DEFINITION 5.1. *Let E be a finite set and $C \subseteq \mathbb{H}^E$ a submodule of the free left \mathbb{H} -module \mathbb{H}^E . A nonzero element $x \in C$ is called an elementary chain of C if C does not contain a nonzero element whose support is strictly contained in the support of x . The submodule C is called unimodular if for every elementary chain x of C all nonzero entries of x have the same norm.*

THEOREM 5.2 ([14, Theorem 3.7]). *Let E be a finite set and $C \subseteq \mathbb{H}^E$ unimodular. The set of supports of elementary chains in C is the set of cocircuits of a matroid $M(C)$ on E .*

DEFINITION 5.3. *A matroid on a finite set E of the form $M(C)$ for some unimodular $C \subseteq \mathbb{H}^E$ is called quaternionic unimodular (QU).*

The goal of this section is to prove that every QU matroid has the half-plane property. This has been conjectured in [14, Conjecture 6.9].

DEFINITION 5.4 ([14, page 219]). *Denote by $\varphi: \mathbb{H} \rightarrow \mathbb{C}^{2 \times 2}$ the map*

$$\varphi(a + bi + cj + dk) = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

We extend φ to matrices by applying φ entry-wise. Thus for a matrix $A \in \mathbb{H}^{n \times n}$ we obtain the $2n \times 2n$ complex matrix $\varphi(A)$. We define

$$\delta(A) := \sqrt{|\det(\varphi(A))|}.$$

REMARK 5.5. We collect some basic properties of δ .

- (1) For columns $a_1, \dots, a_n \in \mathbb{H}^n$ and $\lambda \in \mathbb{R}$, we have

$$\delta(\lambda a_1 \ a_2 \ \dots \ a_n) = |\lambda| \cdot \delta(a_1 \ \dots \ a_n).$$

This follows directly from the definition.

- (2) For matrices $A, B \in \mathbb{H}^{n \times n}$ we have $\delta(AB) = \delta(A)\delta(B)$ and $\delta(A) = \delta(A^*)$. This is [14, Lemma 5.2].

For a matrix $A \in \mathbb{H}^{m \times n}$ with $m \leq n$ and $B \subseteq [n]$ of size m we denote by $A[B]$ the $m \times m$ submatrix of A consisting of the columns indexed by B . The following is a version of the Cauchy–Binet theorem over \mathbb{H} .

PROPOSITION 5.6 ([14, Theorem 5.1]). *Let $A \in \mathbb{H}^{m \times n}$ be a matrix over the quaternions and $m \leq n$. Then*

$$\delta(AA^*) = \sum_{\substack{B \subseteq [n] \\ |B|=m}} \delta(A[B]A[B]^*).$$

LEMMA 5.7. *Let E be a finite set and $C \subseteq \mathbb{H}^E$ unimodular. Let d be the rank of the matroid $M = M(C)$. There is a $d \times |E|$ matrix A over \mathbb{H} whose rows form a basis of C such that for all $B \subseteq E$ of size d we have $\delta(A[B]) = 1$ if B is a basis of M and $\delta(A[B]) = 0$ otherwise.*

Proof. Let A be a matrix over \mathbb{H} whose rows form a basis of C . By [14, Lemma 3.14] it has d rows, so $A \in \mathbb{H}^{d \times |E|}$. After multiplying A from the left by an invertible $d \times d$ matrix over \mathbb{H} , we can assume by [14, Corollary 3.26] that A is a *strong QU matrix* in the sense of [14, Definition 3.23]. Now the statement of the lemma follows from [14, Claim 5.4.1]. \square

THEOREM 5.8 ([14, Conjecture 6.9]). *Every QU matroid has the half-plane property.*

Proof. Let $E = \{1, \dots, m\}$ and $C \subseteq \mathbb{H}^m$ be unimodular. Denote by $M = M(C)$ the associated QU matroid and let $A \in \mathbb{H}^{d \times m}$ be a matrix as in Lemma 5.7. We have to show that the bases generating polynomial h_M of M is stable. This is equivalent to h_M^2 being stable. We prove this by showing that h_M^2 agrees with the stable polynomial

$$\det \left(\varphi(A) \begin{pmatrix} x_1 & & & \\ & x_1 & & \\ & & \ddots & \\ & & & x_m \\ & & & & x_m \end{pmatrix} \varphi(A)^* \right).$$

It suffices to show that these two polynomials agree on the positive orthant. We denote by $a_1, \dots, a_m \in \mathbb{H}^d$ the columns of A . Let $x \in \mathbb{R}_{>0}^m$ and write $x = (x_1^2, \dots, x_m^2)$ for $x_1, \dots, x_m \in \mathbb{R}_{>0}$. Then we have

$$\begin{aligned} (2) \quad h_M(x)^2 &= \left(\sum_{\substack{B \subseteq E \\ |B|=d}} \delta(A[B]A[B]^*) x^B \right)^2 \\ (3) \quad &= \left(\sum_{\substack{B \subseteq E \\ |B|=d}} \delta((x_1 a_1 \dots x_m a_m)[B](x_1 a_1 \dots x_m a_m)[B]^*) \right)^2 \\ (4) \quad &= \delta((x_1 a_1 \dots x_m a_m)(x_1 a_1 \dots x_m a_m)^*)^2 \\ (5) \quad &= |\det(\varphi(x_1 a_1 \dots x_m a_m)\varphi(x_1 a_1 \dots x_m a_m)^*)| \end{aligned}$$

$$\begin{aligned}
 (6) \quad &= \left| \det \left(\varphi(A) \begin{pmatrix} x_1^2 & & & \\ & x_1^2 & & \\ & & \ddots & \\ & & & x_m^2 \\ & & & & x_m^2 \end{pmatrix} \varphi(A)^* \right) \right| \\
 (7) \quad &= \det \left(\varphi(A) \begin{pmatrix} x_1^2 & & & \\ & x_1^2 & & \\ & & \ddots & \\ & & & x_m^2 \\ & & & & x_m^2 \end{pmatrix} \varphi(A)^* \right).
 \end{aligned}$$

Here we have equality in (2) by Remark 5.5 and Lemma 5.7. (3) holds by Remark 5.5. For (4) we use Proposition 5.6 and (5) follows from the definition of δ . (6) and (7) are obvious. \square

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Three results on the half-plane property

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MARIO KUMMER, Technische Universität, Dresden, Germany
E-mail : `mario.kummer@tu-dresden.de`

DAVID SAWALL, Universität Konstanz, Konstanz, Germany
E-mail : `david.sawall@uni-konstanz.de`