ALGEBRAIC COMBINATORICS

Pierre-Louis Curien, Bérénice Delcroix-Oger & Jovana Obradović

**Tridendriform algebras on hypergraph polytopes**

Volume 8, issue 1 (2025), p. 201-234.
https://doi.org/10.5802/alco.401



*Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing*
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

*Algebraic Combinatorics*
Volume 8, issue 1 (2025), p. 201–234
https://doi.org/10.5802/alco.401

# Tridendriform algebras on hypergraph polytopes

Pierre-Louis Curien, Bérénice Delcroix-Oger & Jovana Obradović

Abstract We extend the works of Loday–Ronco and Burgunder–Ronco on the tridendriform decomposition of the shuffle product on the faces of associahedra and permutohedra, to other families of hypergraph polytopes (or nestohedra), including simplices, hypercubes and some new families. We also extend the shuffle product to take more than two arguments, and define accordingly a new algebraic structure, that we call *polydendriform*, from which the original tridendriform equations can be crisply synthesized.

## 1. Introduction

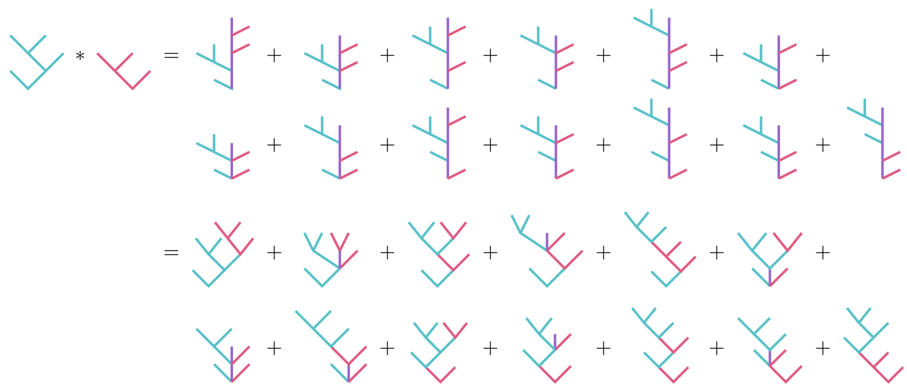
In 1998, Loday–Ronco introduced a Hopf algebra on the linear span of rooted planar binary trees [14]. This Hopf algebra is closely related to the Malvenuto– Reutenauer Hopf algebra on permutations [16]. Planar binary trees and permutations label the vertices of two well-known families of polytopes: associahedra and permutohedra. The associative products of these Hopf algebras were then extended to associative products on all faces of these polytopes labeled respectively by planar trees and surjections by Loday–Ronco [15] and Burgunder–Ronco [2]. More precisely, Loday–Ronco introduced an associative product $*$ on planar trees as a shuffle of trees, where, for two trees $T$ and $S$, $T * S$ is defined as a formal sum of trees whose nodes originate either from $T$, or from $S$, or from merging a node of $S$ with a node of $T$. Loday and Ronco remarked that it is possible to split this product $*$ according to where the roots of the resulting trees originate from, giving rise to three operations "$\prec$", "$\succ$" and "$\cdot$", with $* = (\prec) + (\succ) + (\cdot)$, forming an algebraic structure called tridendriform. For instance, the following product

*Manuscript received 15th April 2023, revised 20th March 2024, accepted 17th September 2024.*

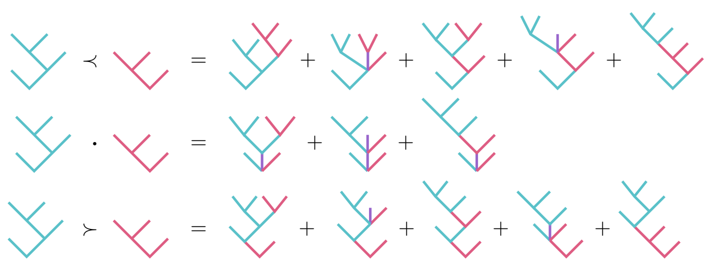
Keywords. tridendriform structure, polydendriform structure, associative product, shuffle product, hypergraph polytopes, nestohedra.

Acknowledgements. This work was partially supported by ANR projects ANR-20-CE40-0016 HighAGT, ANR-20-CE40-0007 CARPLO, ANR-20-CE48-0010 SSS and ANR-19-CE40-0006 ALCOHOL. Jovana Obradović was supported by the Science Fund of the Republic of Serbia, Grant No. 7749891, Graphical Languages - GWORDS.

ISSN: 2589-5486 



is split into



Burgunder and Ronco applied a similar ternary splitting to surjections, also known as packed words, and obtained also a tridendriform structure.

Associahedra and permutohedra are instances of polytopes called *hypergraph polytopes* [8], which are obtained by truncating some faces of simplices, and are also known as *nestohedra* [19]. The description of faces of hypergraph polytopes in terms of tree structures – called constructs – used as working definition in [7] provides an adapted framework to extend the setting of Loday–Ronco and Burgunder–Ronco to other families of polytopes.

We find it convenient to work in an "unbiased" setting, where our operations may have any finite arity (think of the product $a \times b \times c$ of three numbers $a, b, c$ as opposed to $(a \times b) \times c$ or $a \times (b \times c)$). This leads us to a reformulation of the tridendriform (actually $q$-tridendriform – see below) structure, that we call polydendriform. We exhibit conditions under which we can define such a polydendriform structure.

The underlying (binary) associative product that we obtain specializes to the associative product defined by Ronco [20] in the setting of graph associahedra [3], which are the hypergraph polytopes where the associated hypergraphs have only hyperedges of cardinality at most two (see Remark 4.25). Our results apply also to some families of hypergraph polytopes that are not graph associahedra, such as simplices, hypercubes and erosohedra.

Therefore, with respect to [20], our extension is two-fold: we describe not only an associative product, but also a tridendriform splitting of it, and our framework applies in situations that are not covered there.

The article is organized as follows. In §2, we explain in detail the case of the permutohedra, and motivate and recall Burgunder–Ronco's notion of $q$-tridendriform algebra, i.e. an algebra with operations "$\prec$", "$\succ$" and "$\cdot$", satisfying the same equations as in tridendriform algebras, but with the associated (associative) product being now defined as $* = (\prec) + (\succ) + q(\cdot)$ for an arbitrary $q \in \mathbb{K}$, where $\mathbb{K}$ is the ambient field. In §3, we recall some notions on hypergraph polytopes and constructs. In §4, we introduce our conditions for the set of faces of a family of polytopes to be endowed with a polydendriform algebra structure. We first give a so-called "strict" condition that makes it possible to define $q$-tridendriform algebras, for arbitrary $q$. We then give a weaker condition called "quasi-strict", which allows us to deal with a wider class of examples, but for which $q$ has to be $-1$. In §3 and §4, we provide a bunch of new examples that do not fit in the framework of graph associahedra, such as simplices, hypercubes and a family that we call erosohedra. We also introduce a family of graph associahedra that we call friezohedra and will serve as a running example throughout the paper.

**Warning.** We would like to mention the existence of the terms "polydendriform" [12] and "hypergraphic polytope" [1] in the literature, which designate different concepts from the ones presented in this article.

## 2. Prologue

We recall Burgunder–Ronco's shuffle product on the faces of permutohedra [2]. We set $[n] = \{1, \dots, n\}$, and identify a function $f : [n] \to X$ (for some set $X$) with the sequence $(f(1), \dots, f(n))$.

By *surjection*, we mean a function $f : [m] \to [n]$ (for some $m, n \geqslant 1$) that is surjective. For arbitrary $h : [m] \to [n]$, we can build a surjection $\mathtt{pack}(h) := \phi \circ h : [m] \to [|\mathtt{Im}(h)|]$, where $\phi$ is the unique increasing bijection $\mathtt{Im}(h) \to [|\mathtt{Im}(h)|]$. For example, we have $\mathtt{pack}(1, 4, 3, 4) = (1, 3, 2, 3)$. Surjections label the faces of permutohedra, as shown by Chapoton in [4]. Surjections also appear in [17] under the name of *packed words* as building blocks of the Hopf algebra **WQSym**, isomorphic to the one defined for permutohedra by Chapoton. The computations we make below in Burgunder–Ronco style correspond precisely to the computations and the tridendriform decomposition in **WQSym** as described in [17].

If $f : [m_1] \to [n_1]$ and $g : [m_2] \to [n_2]$ are surjections, we look for all surjections $(h, k) : [m_1 + m_2] \to [n]$, for $\max(n_1, n_2) \leqslant n \leqslant n_1 + n_2$, such that $\mathtt{pack}(h) = f$ and $\mathtt{pack}(k) = g$. Below, we do this for $f := (1, 2, 1)$ and $g := (2, 1)$, underlining the maximum elements of $h$ and of $k$.

- $n = 2$: $(1, \underline{2}, 1, \underline{2}, 1)$
- $n = 3$: $(1, \underline{2}, 1, \underline{3}, 1)$, $(1, \underline{3}, 1, \underline{3}, 2)$, $(2, \underline{3}, 2, \underline{2}, 1)$, $(1, \underline{2}, 1, \underline{3}, 2)$, $(1, \underline{3}, 1, \underline{2}, 1)$, $(2, \underline{3}, 2, \underline{3}, 1)$
- $n = 4$: $(1, \underline{2}, 1, \underline{4}, 3)$, $(1, \underline{3}, 1, \underline{4}, 2)$, $(1, \underline{4}, 1, \underline{3}, 2)$, $(2, \underline{3}, 2, \underline{4}, 1)$, $(2, \underline{4}, 2, \underline{3}, 1)$, $(3, \underline{4}, 3, \underline{2}, 1)$.

We collect those pairs in the following formal sums (cf. §1):

$$\begin{aligned}
f \prec g &:= (2, \underline{3}, 2, \underline{2}, 1) + (1, \underline{3}, 1, \underline{2}, 1) + (1, \underline{4}, 1, \underline{3}, 2) + (2, \underline{4}, 2, \underline{3}, 1) + (3, \underline{4}, 3, \underline{2}, 1) \\
&\quad\ (\max(h) > \max(k)) \\
f \cdot g &:= (1, \underline{2}, 1, \underline{2}, 1) + (1, \underline{3}, 1, \underline{3}, 2) + (2, \underline{3}, 2, \underline{3}, 1) \\
&\quad\ (\max(h) = \max(k)) \\
f \succ g &:= (1, \underline{2}, 1, \underline{3}, 1) + (1, \underline{2}, 1, \underline{3}, 2) + (1, \underline{2}, 1, \underline{4}, 3) + (1, \underline{3}, 1, \underline{4}, 2) + (2, \underline{3}, 2, \underline{4}, 1) \\
&\quad\ (\max(h) < \max(k)) \\
f * g &:= (f \prec g) + (f \cdot g) + (f \succ g).
\end{aligned}$$

The operations $\prec$, $\cdot$ and $\succ$ satisfy the following *tridendriform* equations

$$\begin{aligned}
(a \prec b) \prec c &= a \prec (b * c) && (\prec *) \\
(a \succ b) \prec c &= a \succ (b \prec c) && (\succ \prec) \\
(a * b) \succ c &= a \succ (b \succ c) && (* \succ) \\
(a \cdot b) \cdot c &= a \cdot (b \cdot c) && (\cdot ass) \\
(a \succ b) \cdot c &= a \succ (b \cdot c) && (\succ \cdot) \\
(a \prec b) \cdot c &= a \cdot (b \succ c) && (\prec \cdot \succ) \\
(a \cdot b) \prec c &= a \cdot (b \prec c) && (\cdot \prec)
\end{aligned}$$

and the operation $*$ is associative.

The tridendriform structure was first recognized and defined by Loday and Ronco [13] on Schröder trees, i.e. planar trees without unary nodes. We will denote such trees as $\bullet(T_1, \ldots, T_n)$, for $n \neq 1$, where $T_1, \ldots, T_n$ are themselves Schröder trees. The tree with only one leaf is then $\bullet()$. Schröder trees with at least two leaves label the faces of associahedra. The three tridendriform operations (already illustrated in §1) are defined as follows (with the convention that $\bullet() * S = S = S * \bullet()$):

$$\begin{aligned} \bullet(S_1, \ldots, S_n) \prec T &:= \bullet(S_1, \ldots, S_{n-1}, S_n * T) \\ S \succ \bullet(T_1, \ldots, T_n) &:= \bullet(S * T_1, T_2, \ldots, T_n) \\ \bullet(S_1, \ldots, S_m) \cdot \bullet(T_1, \ldots, T_n) &:= \bullet(S_1, \ldots, S_{m-1}, S_m * T_1, T_2 \ldots, T_n). \end{aligned}$$

Associahedra and permutohedra are examples of hypergraph polytopes, also known as nestohedra [19, 10]. Our goal is to define in this more general framework, and under suitable conditions, an associative product, with associated tridendriform decomposition, instantiating to these two examples and more.

We close this section by studying the relation between tridendriform structures and associativity more closely. Burgunder and Ronco [2] have introduced a variation of tridendriform algebras, called $q$-tridendriform algebras (for $q \in \mathbb{R}$, or more generally $q \in \Bbbk$ for some field $\Bbbk$), where the equations are the same as above, except that now the operation $\cdot$ is weighted, i.e. $a * b$ is redefined as $(a \prec b) + q(a \cdot b) + (a \succ b)$. This is justified by the following proposition.

Proposition 2.1. *Setting $a * b := \lambda_1(a \prec b) + \lambda_2(a \cdot b) + \lambda_3(a \succ b)$, if the tridendriform equations are satisfied (with this definition of $*$), then $*$ is associative if $\lambda_1 = \lambda_3 = 1$.*

*Proof.* We match

$$\begin{aligned} &\lambda_1\lambda_1 \underbrace{(a \prec b) \prec c}_{(\prec *)} + \lambda_1\lambda_2 \underbrace{(a \cdot b) \prec c}_{(\cdot\prec)} + \lambda_1\lambda_3 \underbrace{(a \succ b) \prec c}_{(\succ\prec)} + \lambda_2\lambda_1 \underbrace{(a \prec b) \cdot c}_{(\prec\cdot\succ)} \\ &+ \lambda_2\lambda_2 \underbrace{(a \cdot b) \cdot c}_{(\cdot ass)} + \lambda_2\lambda_3 \underbrace{(a \succ b) \cdot c}_{(\succ\cdot)} + \lambda_3 \underbrace{(a * b) \succ c}_{(*\succ)} \end{aligned}$$

with

$$\begin{aligned} &\lambda_1 \underbrace{a \prec (b * c)}_{(\prec *)} + \lambda_2\lambda_1 \underbrace{a \cdot (b \prec c)}_{(\cdot\prec)} + \lambda_2\lambda_2 \underbrace{a \cdot (b \cdot c)}_{(\cdot ass)} + \lambda_2 \underbrace{\lambda_3 a \cdot (b \succ c)}_{(\prec\cdot\succ)} \\ &+ \lambda_3\lambda_1 \underbrace{a \succ (b \prec c)}_{(\succ\prec)} + \lambda_3\lambda_2 \underbrace{a \succ (b \cdot c)}_{(\succ\cdot)} + \lambda_3\lambda_3 \underbrace{a \succ (b \succ c)}_{(*\succ)} \end{aligned}$$

using $(\prec *)$ (resp. $(* \succ)$, $(\prec\cdot\succ)$) and the assumption $\lambda_1 = 1$ (resp. $\lambda_1 = \lambda_3 = 1$). ☐

## 3. Hypergraph polytopes

We first recall all the needed definitions in §3.1, and then give a number of examples in §3.2.

3.1. Basic definitions. A hypergraph is given by a set $H$ of vertices (the carrier), and a subset $\mathbf{H} \subseteq \mathcal{P}(H) \backslash \varnothing$ such that $\bigcup \mathbf{H} = H$. The elements of $\mathbf{H}$ are called the *hyperedges* of $\mathbf{H}$. We always assume that $\mathbf{H}$ is *atomic*, by which we mean that $\{x\} \in \mathbf{H}$, for all $x \in H$. Identifying $x$ with $\{x\}$, $H$ can be seen as the set of hyperedges of cardinality 1, also called *vertices*. We shall use the convention to give the same name to the hypergraph and to its carrier, the former being the bold version of the latter. A hyperedge of cardinality 2 is called an *edge*. Note that any ordinary graph $(V, E)$ can be viewed as the atomic hypergraph $\{\{v\} \mid v \in V\} \cup \{e \mid e \in E\}$ (with no hyperedges of cardinality $\geqslant 3$).

If $\mathbf{H}$ is a hypergraph, and if $X \subseteq H$, we set $\mathbf{H}_X := \{Z \mid Z \in \mathbf{H} \text{ and } Z \subseteq X\}$, and $\mathbf{H} \backslash X = \mathbf{H}_{H \backslash X}$. We say that $\mathbf{H}$ is *connected* if there is no non-trivial partition $H = X_1 \cup X_2$ such that $\mathbf{H} = \mathbf{H}_{X_1} \cup \mathbf{H}_{X_2}$, and that $X \subseteq H$ is connected in $\mathbf{H}$, or that $X$

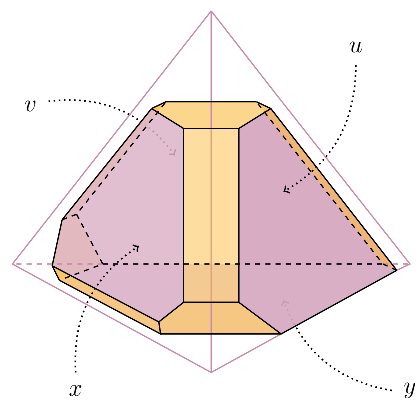



FIGURE 1. This polytope is obtained from the tetrahedron (with facets identified with $x$, $y$, $u$ and $v$, as indicated by dotted arrows) by truncating three of its vertices (as dictated by the tubes $\{x,y,u\}$, $\{x,y,v\}$, and $\{x,u,v\}$ of $\mathbf{H}$) and four of its edges (as dictated by the four hyperedges of cardinality 2 of $\mathbf{H}$).

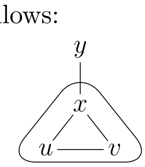
is a *tube* [(1)] of $\mathbf{H}$, if $\mathbf{H}_X$ is connected. For each finite hypergraph there exists a unique partition $H = X_1 \cup \ldots \cup X_m$ such that each $\mathbf{H}_{X_i}$ is connected and $\mathbf{H} = \bigcup(\mathbf{H}_{X_i})$. The $\mathbf{H}_{X_i}$ are the *connected components* of $\mathbf{H}$. The notation $\mathbf{H}, X \rightsquigarrow \mathbf{H}_1, \ldots, \mathbf{H}_n$ will mean that $\mathbf{H}_1, \ldots, \mathbf{H}_n$ are the connected components of $\mathbf{H}\backslash X$.

Došen and Petrić [8] have proposed the following insightful reading of the data of a finite connected hypergraph $\mathbf{H}$ as a truncated simplex: the elements of $H$ are identified with the facets (i.e. codimension 1 faces) of the $(|H|-1)$-dimensional simplex, and each $\varnothing \subsetneq X \subsetneq H$, $|X| \geqslant 2$, such that $\mathbf{H}_X$ is connected designates the intersection of the facets in $X$ as a face to be truncated. The obtained polytopes, called *hypergraph polytopes*, extend the construction of graph associahedra [3, 21], and are equivalent to nestohedra.

EXAMPLE 3.1. The hypergraph

$$\mathbf{H} = \{\{x\},\{y\},\{u\},\{v\},\{x,y\},\{x,u\},\{x,v\},\{u,v\},\{x,u,v\}\}$$

can be represented pictorially as follows:




Here, the hyperedge $\{x,u,v\}$ is represented by the circled-out area around the vertices $x$, $u$ and $v$. We show in Figure 1 the polytope encoded by $\mathbf{H}$. △

Given a finite connected hypergraph $\mathbf{H}$, the faces of the polytope obtained by performing all the truncations prescribed by $\mathbf{H}$ are labeled by non-planar trees, called *constructs*, whose nodes are decorated by non-empty subsets of $H$ and whose recursive definition is given next using the syntax introduced in [7].

Let $\varnothing \neq Y \subseteq H$. If $\mathbf{H}, Y \rightsquigarrow \mathbf{H}_1, \ldots, \mathbf{H}_n$, and if $T_1, \ldots, T_n$ are constructs of $\mathbf{H}_1, \ldots, \mathbf{H}_n$, respectively, then the tree obtained by grafting $T_1, \ldots, T_n$ on the root node decorated by $Y$, denoted by $Y(T_1, \ldots, T_n)$ (or sometimes $Y\{T_i \mid 1 \leqslant i \leqslant n\}$), is

(1) The name "tube" was originally introduced in the setting of graphs and graph associahedra [3]. We extend here its use to hypergraphs.

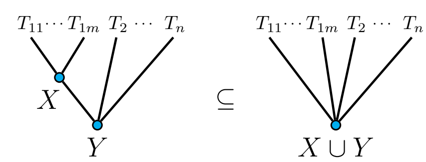
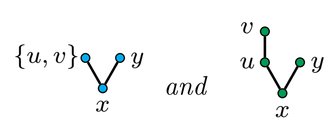



Figure 2. Immediate face inclusion

a construct of $\mathbf{H}$. We write $Y = \mathrm{root}(Y(T_1, \ldots, T_n))$. The base case is when $Y = H$ (and hence $n = 0$): then the one-node tree $H()$ (written simply $H$) is a construct. We write $T : \mathbf{H}$ to denote that $T$ is a construct of $\mathbf{H}$.

Convention 3.2. *In order to facilitate the notation for constructs, we shall represent their singleton vertices without the braces. For example, instead of* $\{x\}(\{u,v\},\{y\})$ *and* $\{x\}(\{u\}(\{v\}),\{y\})$*, we shall write* $x(\{u,v\},y)$ *and* $x(u(v),y)$*. Also, we shall freely confuse the vertices of constructs with the sets decorating them, since they are a fortiori all distinct. We shall use the following graphical representation for constructs: the constructs, say,* $x(y,\{u,v\})$ *and* $x(u(v),y)$ *will be drawn as*




*respectively.*

The description of faces as trees is particularly nice for encoding (immediate) face inclusions: by contracting an edge of a construct representing a face of dimension $p$, and merging the decorations of the two nodes related by that edge, one gets a face of dimension $p+1$, as illustrated in Figure 2. We shall not make use of the resulting partial order on faces in the present paper (see however §5), but it helps in understanding what is going on in the pictures.

Remark 3.3. Constructs as presented here are just an alternative description of the tubings and of the nested sets in the literature on graph associahedra [3] and nestohedra [19], respectively. Indeed, let us denote, for every node $Y$ of a construct $T : \mathbf{H}$, by $\uparrow_T(Y)$ the union of the labels of the descendants of $Y$ in $T$ (all the way to the leaves), including $Y$. By definition of constructs, $\uparrow_T(Y)$ is a tube of $\mathbf{H}$. We then associate with $T$ the following *nested set*, in the terminology of [19]:

$$\psi(T) = \{\uparrow_T(Y) \mid Y \text{ is a (label of a) node of } T\}.$$

Therefore, there are as many tubes in the nested set associated to a construct $T$ as there are nodes in $T$. Alternatively, the function $\psi$ is defined recursively by

$$\psi(X(T_1,\ldots,T_n)) = \{H\} \bigcup \big(\bigcup_{i=1,\ldots,n} \psi(T_i)\big).$$

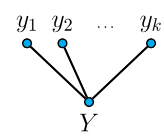
We refer to [7, Proposition 2] for an exact characterization of inductively defined constructs as nested sets. We just note here that the function $\psi$ defined above provides a bijection from constructs to nested sets. In addition, we note that, if $\mathbf{H}$ is an ordinary graph, the notion of nested set comes down to that of a *tubing* of $\mathbf{H}$, in the terminology of [3], making the above correspondence a bijection from constructs to graph tubings. We illustrate the constructs-as-tubings characterization in Example 3.10, and we come back to it again in §4.4.

3.2. Notable families of hypergraph polytopes. We next give examples of hypergraph polytopes, most of which will be revisited later in the paper. We start with the family of simplices, which are the limit cases of hypergraph polytopes for which no truncation is prescribed by the corresponding hypergraphs.

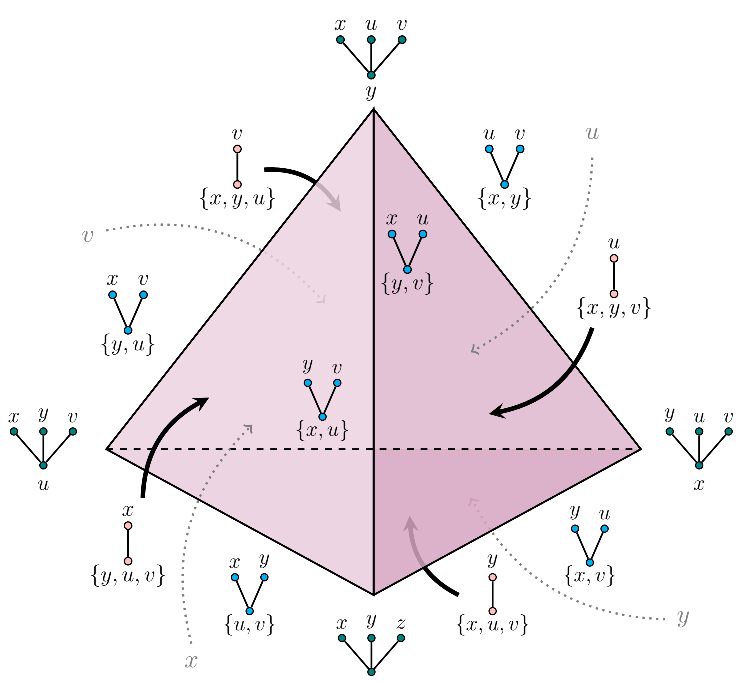
Example 3.4. **Simplices** are "encoded" by the hypergraphs

$$\mathbf{S}^X = \{\{x\} \mid x \in X\} \cup \{\{X\}\},$$

having only trivial hyperedges (i.e. the vertices and the maximal hyperedge ensuring that the hypergraph is connected). The constructs have the form




where $\varnothing \subsetneq Y \subseteq X$ and $\{y_1, \ldots, y_k\} = H \backslash Y$, and are therefore in bijection with the non-empty subsets of $X$, which can also be seen as pairs $(X, Y)$ standing for $X$ in which all elements of $Y$ have been pointed. In the picture below, we label the simplex $\mathbf{S}^{\{x,y,u,v\}}$ by the constructs associated to each face:




Here, identifying the facets of the tetrahedron with the elements of the set $\{x, y, u, v\}$ (as we do by using dotted arrows), the construct associated to the face defined as the intersection of facets contained in a subset $\varnothing \subsetneq Y \subsetneq \{x, y, u, v\}$ has the set $\{x, y, u, v\} \backslash Y$ as root vertex, and the interior of the simplex is labeled by the single-node construct

$$\{x, y, u, v\} \ .$$

In particular, the faces of dimension $k$ are labeled by constructs whose roots are the sets of size $k + 1$. △

In our next example, we illustrate how the hypergraph structure dictates truncations, and how to associate a construct to a face resulting from a truncation. The

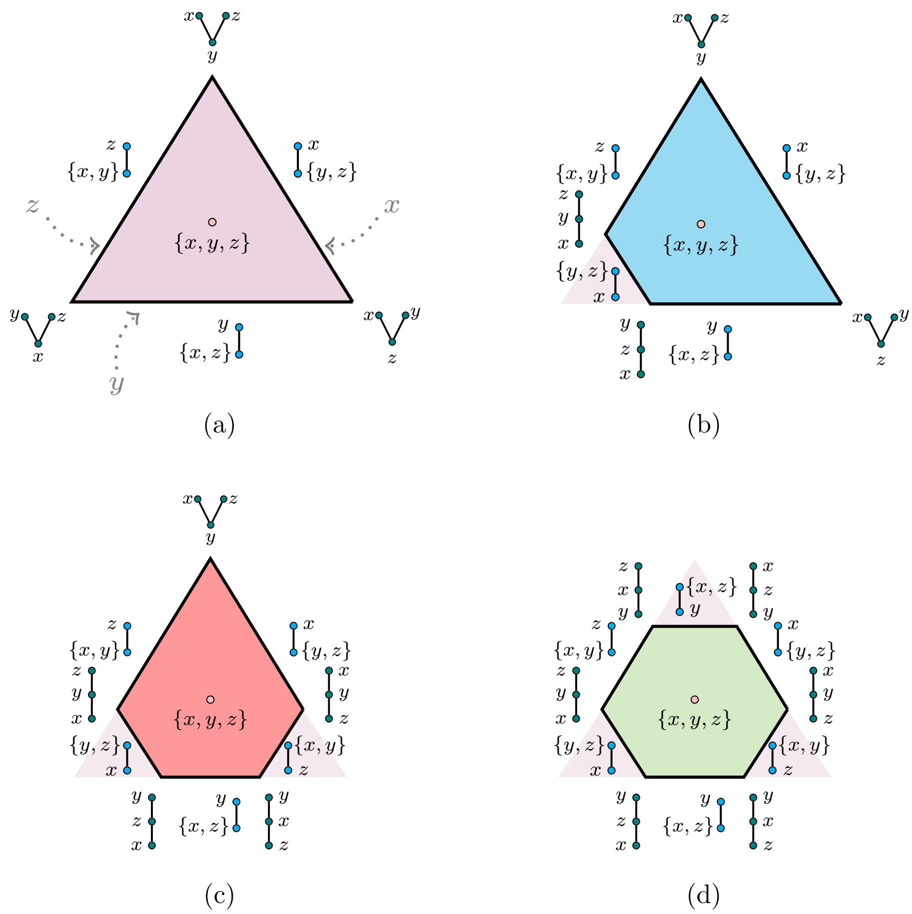
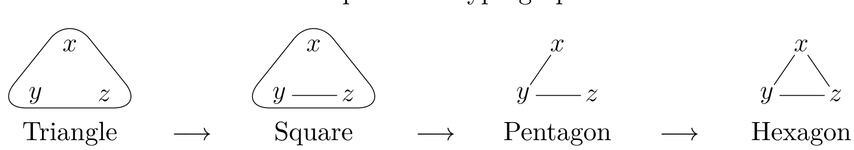



Figure 3. The sequence of 2-dimensional hypergraph polytopes

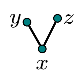
latter association is an instance of the order-isomorphism between the posets of combinatorial and geometric faces of hypergraph polytopes that has been established in [8] (see also [7, Theorem 25]).

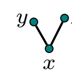
Example 3.5. We consider the sequence of hypergraphs



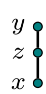

on the vertex set $\{x, y, z\}$, which starts from the hypergraph that has no hyperedges of cardinality 2, and in which each of the following hypergraphs is obtained from the previous one by adding exactly one hyperedge of cardinality 2. (Note that, in the transition from the Square to the Pentagon , the hyperedge $\{x, y, z\}$ is no longer necessary to ensure that the hypergraph is connected.) These four hypergraphs encode all possible 2-dimensional hypergraph polytopes, namely the triangle, the square, the pentagon and the hexagon, as indicated above. In Figure 3, we present the complete face lattice descriptions of these polytopes in terms of constructs.

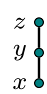
We now explain the corresponding sequence of truncations. Starting with the 2-dimensional simplex with facets identified with $x$, $y$ and $z$, the Triangle, having no hyperedges of cardinality 2, trivially dictates 0 vertex truncations, thereby leaving the simplex unaffected, the Square encodes the truncation of the bottom left vertex (the intersection of facets $y$ and $z$), the Pentagon additionally dictates the truncation of the bottom right vertex (the intersection of facets $x$ and $y$), and finally the Hexagon additionally incorporates the truncation of the top vertex (the intersection of facets $x$ and $z$), encoding thereby the truncations of all the vertices of the starting simplex.

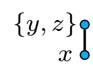
The association of constructs to geometric faces goes as follows. For the triangle, we do as described in Example 3.4. In the transition from the triangle to the square, the truncation of the bottom left vertex is reflected combinatorially by the fact that the tree 

, which is a construct of the Triangle, is *not* a construct of the Square. Indeed, since $\{y, z\}$ is connected in the Square, 

gets replaced by 3 new constructs:



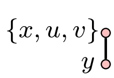
, 

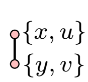
and 

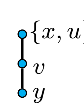
,

encoding two vertices and one edge. Note that the remaining faces of the square carry over their construct description from the triangle. Moving on to the pentagon and ultimately to the hexagon, it is now easy to associate constructs to the new faces obtained by truncations of the two remaining vertices of the starting triangle. △

Example 3.6. As a slightly more involved example, we show in Figure 4 the construct description of the 2-dimensional geometric faces of the polytope from Example 3.1. The four 2-dimensional faces that originate from the facets of the starting simplex carry over their construct description from the one given for those facets in Example 3.4. Clearly, the association of constructs to the facets of the above polytope unambiguously extends to the association of constructs to all the remaining faces of that polytope. For example, the unique common edge of the facets



and 

is given by the construct 

.

△

We next give two examples that do not fit in the framework of graph associahedra: hypercubes and erosohedra.

Example 3.7 (**Hypercubes**). For a finite ordered set $X = \{x_1 < \cdots < x_n\}$, consider the hypergraph

$$\mathbf{C}^X = \{\{x_1\}, \ldots, \{x_n\}\} \cup \{\{x_j \mid 1 \leqslant j \leqslant i\} \mid 1 \leqslant i \leqslant n\}.$$

The constructs of $\mathbf{C}^X$ are in one-to-one correspondence with the set of words of length $n$ over the alphabet $\{+, -, \bullet\}$ starting with $+$, and hence decorate the faces of an $(n-1)$-dimensional hypercube. More precisely, we recursively read a construct from such a word $(w_1 + w_2)$, where $+$ does not occur in $w_2$, as follows:

- The positions of the occurrences of $\bullet$ in $w_2$ plus the last occurrence of $+$ in $w$, form the root $R$ of the construct.
- $w_1$ encodes a construct $S$ (if not empty).

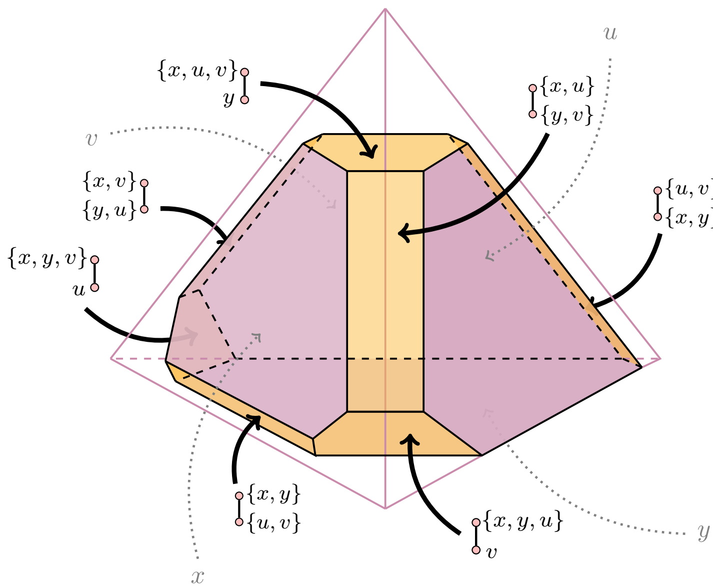
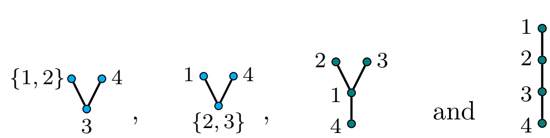



Figure 4. The association of constructs to the facets of the polytope encoded by the hypergraph $\mathbf{H} = \{\{x\},\{y\},\{u\},\{v\},\{x,y\},\{x,u\},\{x,v\},\{u,v\},\{x,y,u,v\}\}$.

- The children of $R$ in the construct are $S$ (if any), and the positions of the occurrences of $-$ in $w_2$.

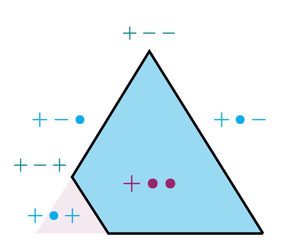
For instance, the constructs




of $\mathbf{C}^{\{1,\ldots,4\}}$ correspond to the words

$$+\bullet+- \qquad ++\bullet- \qquad +--+ \qquad \text{and} \qquad ++++ \;,$$

respectively. We have already listed all the constructs of the hypercube $\mathbf{C}^{\{y<z<x\}}$ in the top right picture of Figure 3. The corresponding words can be read from the picture below:




△

Example 3.8 (**Erosohedra**). They are obtained by truncating every vertex in the simplex. We name them so by analogy with erosion of rocks. Erosohedra in dimension 2 and 3 are represented on Figure 5. The associated hypergraphs are given by:

$$\mathbf{E}^X = \{\{x_1\},\ldots,\{x_n\}\} \cup \{\{x_j \mid j \neq i\} \mid 1 \leqslant i \leqslant n\},$$

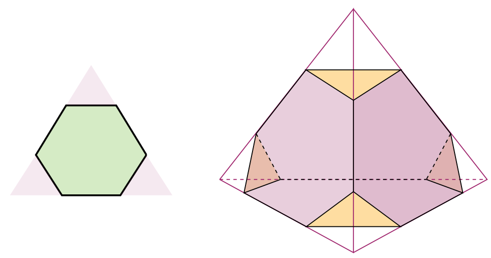
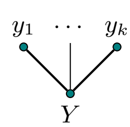


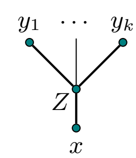
FIGURE 5. Erosohedra in dimension 2 and 3

where $X = \{x_1, \dots, x_n\}$.

The constructs of the erosohedra are of the form




(with $|Y| \geqslant 2$) or




.

△

The number of faces in the erosohedron is given as follows.

LEMMA 3.9. *The number of vertices in the erosohedron of dimension $n$ is $n(n+1)$ and the number of faces of dimension $k$ is $(n-k)\binom{n}{k+1}$. The total number of faces is thus $2^{n-1}(n+2) - 2n - 1$ for $n > 1$.*

*Proof.* The vertices of the erosohedron $\mathbf{E}^X$ correspond to constructs of the form $x(y(z_1, \dots, z_k))$, where $x, y \in X$ and $\{z_1, \dots, z_k\} = X \backslash \{x, y\}$. Faces of dimension $k > 0$ correspond to two types of constructs:

- $\{x_0, x_1, \dots, x_k\}(y_1, \dots, y_p)$, where $\{y_1, \dots, y_p\} = X \backslash \{x_0, \dots, x_k\}$, and
- $x_0(\{x_1, \dots, x_{k+1}\}(y_1, \dots, y_p))$, where $\{y_1, \dots, y_p\} = X \backslash \{x_0, \dots, x_{k+1}\}$.

Hence, there are $(n-k-1)\binom{n}{k+1} + \binom{n}{k+1}$ such faces. The total number of faces is then given by summing the previous formulas. □

EXAMPLE 3.10. We get **associahedra** and **permutohedra** from the linear and complete graphs, respectively:

$$\mathbf{K}^X = \{\{x_1\}, \dots, \{x_n\}, \{x_1, x_2\}, \dots, \{x_{n-1}, x_n\}, \{x_1, \dots, x_n\}\},$$

for $X = \{x_1 < \cdots < x_n\}$, and

$$\mathbf{P}^X = \{\{x_1\}, \dots, \{x_n\}, \{x_1, \dots x_n\}\} \cup \{\{x_i, x_j\} \mid 1 \leqslant i \neq j \leqslant n\},$$

for $X = \{x_1, \dots, x_n\}$. One can indeed check that the constructs of $\mathbf{K}^X$ (resp. $\mathbf{P}^X$) are in one-to-one correspondence with the planar trees (resp. surjections) of §2. The labeling enabling to identify planar trees with constructs of the associahedra is obtained as a generalization of the one for binary search trees: given a planar tree with root of arity $p+1$, the root is labeled by $\{x_{i_1}, \dots, x_{i_p}\}$, and each subtree $T_0, \dots, T_p$ is labeled recursively, in such a way that the condition $\max T_j < x_{i_{j+1}} < \min T_{j+1}$ for any $0 \leqslant j \leqslant p-1$ is satisfied. For permutations, note that we can arrange the data of a surjection $f : [m] \to [n]$ as the linear construct $f^{-1}(n)(f^{-1}(n-1)(\dots(f^{-1}(1))))$ of height $n$. In Figure 6, we show the planar trees (resp. surjections) that correspond to the constructs of the 2-dimensional associahedron (resp. permutohedron) from Figure 3(c) (resp. Figure 3(d)). Finally, as an example of the correspondence between con-

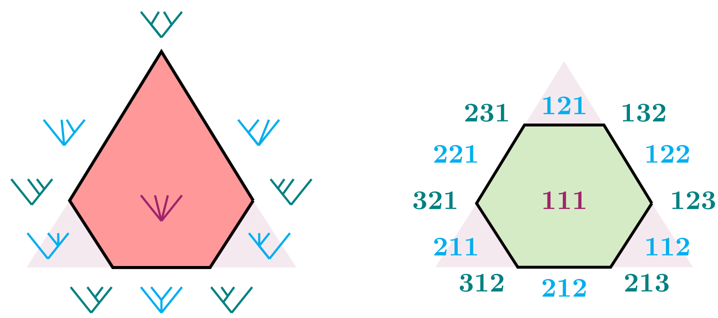



FIGURE 6. Left: the planar-tree labeling of the faces of the associahedron $\mathbf{K}^{x<y<z}$. Right: the association of surjections $f : \{x, y, z\} \to [i]$, for $1 \leqslant i \leqslant 3$, to the faces of the hexagon $\mathbf{P}^{\{x,y,z\}}$, written using the abbreviation $f(x)f(y)f(z)$ for $(f(x), f(y), f(z))$.

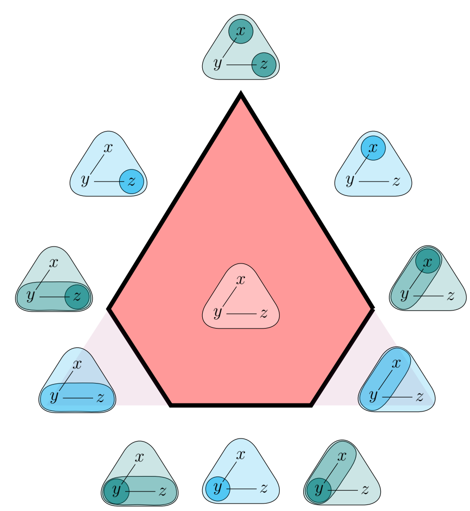


FIGURE 7. The tubing description of the associahedron $\mathbf{K}^{x<y<z}$. The tubings are presented in the standard way, as nested structures of connected subgraphs of $\mathbf{K}^{x<y<z}$ (cf. [3, Figure 1]).

structs and tubings from Remark 3.3, in Figure 7 we show the tubing description of the 2-dimensional associahedron from Figure 3(c). △

EXAMPLE 3.11. Our final example is the family of **friezohedra**. Consider the infinite graph $\mathbf{F}$ on $\mathbb{Z}$ with the set of edges $\{(x,y) || x-y| \leqslant 2\}$, and its restrictions $\mathbf{F}_X$ to finite sets $X = \{x_1 < \cdots < x_n\} \subseteq \mathbb{Z}$ such that $\mathbf{F}_X$ is connected, which we call friezohedra. Note that $\mathbf{F}_X$ is connected exactly when there is no $i$ such that $x_{i+1} - x_i > 2$,. We distinguish the *compact friezohedra*, which are the friezohedra such that $X$ is an interval in $\mathbb{Z}$ (implying a fortiori that $\mathbf{F}_X$ is connected). Families constructed from an infinite hypergraph through restrictions as in this example are called *restrictohedra* and are studied in full generality in §4.2. The name "friezohedron" comes from the shape of the hypergraphs of a compact friezohedron for $X$ sufficiently large, as illustrated in Figure 8.

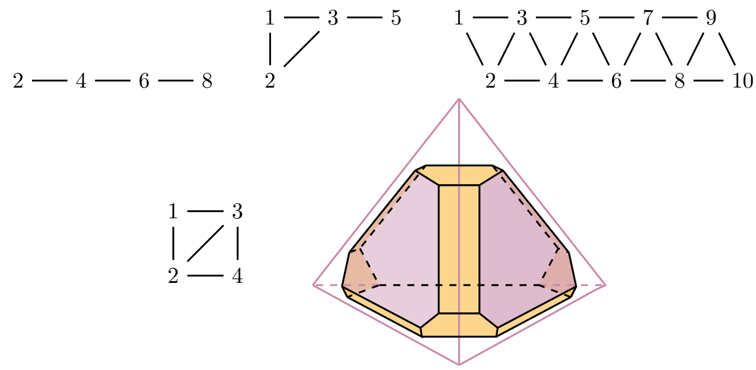



FIGURE 8. Top: three examples of friezohedra. The rightmost one is the (compact) friezohedron on $\{1, \dots, 10\}$. Bottom: the polytope for the compact friezohedron on 4 vertices.

| n \ k | 1 | 2 | 3 | 4 | 5 | Sum over k |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 1 | 1 | | | | 2 |
| 3 | 1 | 6 | 6 | | | 13 |
| 4 | 1 | 13 | 33 | 22 | | 69 |
| 5 | 1 | 25 | 119 | 188 | 94 | 427 |

FIGURE 9. Number of constructs with $k$ vertices of the compact friezohedron on $n$ vertices

We do not have at the time of writing a “simple” combinatorial interpretation of the constructs of the compact friezohedra. In Figure 9, we give the number of constructs with $k$ nodes for $|X| = n$, for low values of $k, n$. $\triangle$

More examples of truncations and constructs are to be found in [8, 7], and also in the sections that follow.

## 4. Shuffle product

In this main section, we unify the above mentioned works of Burgunder, Loday and Ronco into a notion that we call shuffle product of constructs (defined in an unbiased style, cf. §1). Towards achieving this goal, in §4.1 we introduce a general framework based on the formalism of hypergraph polytopes of §3, which will serve as “carrier” of the algebraic structure that we define by induction in §4.3. We show that the structure satisfies an equation that we call polydendriform, and derive an associative product from it. We show that associahedra and permutohedra fit in this framework, as well as all families of restrictohedra, which we define and study in §4.2. We illustrate the notions introduced with the example of friezohedra. In §4.4, we give an alternative non-recursive definition of the associated associative product. In §4.5, we further enlarge the framework in order to cover more examples.

4.1. Strict teams, clans and delegations. As announced just above, the first step towards defining the shuffle product of constructs consists in establishing a formal underlying setting. The latter consists in the notions of universe, teams, clans and delegations. Before defining those formally, we shall introduce them through the example of permutohedra.

Let us first reformulate the shuffle product described in §2 in terms of constructs, using the dictionary given in Example 3.10. We read the surjections $f = (1, 2, 1)$ and $g = (2, 1)$ as the constructs $b(\{a, c\})$ and $d(e)$ of $\mathbf{P}^{\{a,b,c\}}$ and $\mathbf{P}^{\{d,e\}}$, respectively. (Slowly, we read $f$ as a surjection from $\{a, b, c\}$ to $[2]$, and note that $f^{-1}(1) = \{a, c\}$ and $f^{-1}(2) = \{b\}$.) Under this guise, we rephrase the definitions of $f \cdot g$ and $f \prec g$ (leaving $f \succ g$ to the reader) as follows:

$$\begin{aligned} f \cdot g &= \{b, d\}(\{a,c,e\}) + \{b,d\}(e(\{a,c\})) + \{b,d\}(\{a,c\}(e)) \\ f \prec g &= b(\{a,c,d\}(e)) + b(d(\{a,c,e\})) + b(d(e(\{a,c\}))) + b(d(\{a,c\}(e))) + b(\{a,c\}(d(e))), \end{aligned}$$

where $\{b, d\}(\{a, c, e\})$ reads back as $(1, 2, 1, 2, 1)$, and so on. The algorithmic reading of a construct appearing in the shuffle product $f * g$ is as follows: we can build such a construct by choosing for its root either the union of the roots of $f$ and $g$, or only the root of $f$ (resp. of $g$), and then proceed recursively. We then obtain:

$$\begin{aligned} f \cdot g &= \{b, d\}(\{a, c\} * e) \\ f \prec g &= b(\{a, c\} * d(e)) \\ &= b(\{a, c, d\}(e)) + b(d(\{a, c\} * e)) + b(\{a, c\}(d(e))), \end{aligned}$$

and analogously for $f \succ g$.

In general, and in the unbiased setting, we shall have a finite collection $\{\mathbf{H}_a \mid a \in A\}$ of hypergraphs with disjoint sets of vertices, and constructs $C_a : \mathbf{H}_a$ for each $a \in A$ which we shall call collectively a *delegation*. Our aim will be to define the shuffle product of those constructs as a sum of constructs of some hypergraph $\mathbf{H}$ such that $H = \bigcup_{a \in A} H_a$. Above, we have $A = \{1, 2\}$, $\mathbf{H}_1 = \mathbf{P}^{\{a,b,c\}}$, $\mathbf{H}_2 = \mathbf{P}^{\{d,e\}}$, and $\mathbf{H} = \mathbf{P}^{\{a,b,c,d,e\}}$. The pair $(\{\mathbf{H}_a \mid a \in A\}, \mathbf{H})$ will be called a *preteam*, and we shall say that it is the support of the above delegation. We shall impose some connectedness conditions on our preteams to make the inductive definition of the shuffle product possible. In this paper we shall consider two styles of conditions: strict (in this section) and quasi-strict (in §4.5), leading to the notions of strict and quasi-strict *teams*, respectively. Let us zoom a bit more on what "making possible" means: in the recursive definition, we want to be able to form shuffle products of "smaller" delegations (like $\{a, c\} * e$ and $\{a, c\} * d(e)$ above), which we need to synthesize from our original delegation, and which need to have a support that is again a team, and so on. This leads us to collect teams in so-called clans, that are defined as sets of teams closed under certain operations naturally associated with the conditions defining teams among preteams. Finally, all hypergraphs involved in our clan are taken from a certain reference collection of hypergraphs, called a universe. Our results are parameterized by the choice of a universe and of a clan on this universe. We now proceed to the formal definition of these notions.

We first specify a collection (or *universe*) $\mathfrak{U}$ of connected hypergraphs. Note that some universes may contain several hypergraphs on the same set of vertices. It is for instance the case for the universe of erosohedra, which, in fact, contains both erosohedra and simplices, as we point out in Example 4.32.

A *preteam* is a pair $\tau = (\{\mathbf{H}_a \mid a \in A\}, \mathbf{H})$ of a finite set $\{\mathbf{H}_a \in \mathfrak{U} \mid a \in A\}$ of hypergraphs (for some indexing set $A$) and a hypergraph $\mathbf{H} \in \mathfrak{U}$, such that the $H_a$'s are mutually disjoint and $H = \bigsqcup_{a \in A} H_a$. We call $\mathbf{H}$ and the $\mathbf{H}_a$'s the coordinating hypergraph and the participating hypergraphs, respectively.

Example 4.1. Let us first consider the universe formed by all permutohedra $\mathbf{P}^X$. An example of preteam is given by:

$$\tau^{\mathbf{P}} = \Big(\{\mathbf{P}^{\{\heartsuit,\diamondsuit\}}, \mathbf{P}^{\{\clubsuit,\spadesuit\}}, \mathbf{P}^{\{*\}}, \mathbf{P}^{\{\star\}}\}, \mathbf{P}^{\{\heartsuit,\diamondsuit,\clubsuit,\spadesuit,*,\star\}}\Big). \qquad (4.1)$$

The reader may wonder why we do not simply take $X = [n]$ (for varying $n$), as in §2. We refer to Remark 4.13 for a discussion. △

Example 4.2. An example of preteam for the universe of friezohedra is given by

$$(\{\mathbf{H}_1 = \mathbf{F}_{\{1,3,5\}}, \mathbf{H}_2 = \mathbf{F}_{\{2,4\}}, \mathbf{H}_3 = \mathbf{F}_{\{6,7,8\}}\}, \mathbf{F}_{\{1,\dots,8\}}). \quad \triangle$$

We next move to our strict condition on preteams. The reader is advised to read it side by side with the definition of shuffle product given in §4.3. A preteam is called a *strict team* if for each choice of a subset $\varnothing \neq B \subseteq A$ and of a subset $\varnothing \neq X_b \subseteq H_b$ for each $b \in B$, inducing the decompositions $\mathbf{H}_b, X_b \rightsquigarrow \mathbf{H}_{(b,1)}, \dots, \mathbf{H}_{(b,n_b)}$ and $\mathbf{H}, \bigcup_{b\in B} X_b \rightsquigarrow \mathbf{H}^B_1, \dots, \mathbf{H}^B_{n_B}$, we have that, for each

$$\tilde{a} \in \tilde{A} := (A\backslash B) \cup \{(b,i) \mid b \in B, 1 \leqslant i \leqslant n_b\},$$

$\mathbf{H}_{\tilde{a}} \in \mathfrak{U}$, and $H_{\tilde{a}}$ is included in a connected component of $\mathbf{H}\backslash(\bigcup_{b\in B} X_b)$.

As an intuitive guide, let us anticipate as above that we have constructs $C_a$ for all $a \in A$, and suppose that $X_b$ is the root of $C_b$ for all $b \in B$. Then in the above definition we prepare the ground for defining those contributions to the shuffle product of the constructs $C_a$ that share the initial step of picking together the roots of all constructs $C_b$, for $b$ ranging over $B$.

The following is an example of a non-strict preteam.

Example 4.3. Let us consider the following preteam of hypercubes (cf. Example 3.7):

$$\left(\left\{\mathbf{H}_1 = \mathbf{C}^{\{1\}}, \mathbf{H}_2 = \mathbf{C}^{\{2,3\}}, \mathbf{H}_3 = \mathbf{C}^{\{4\}}\right\}, \mathbf{C}^{\{1,2,3,4\}}\right).$$

Choosing $B = 1$ and $X_1 = \{1\}$, we get that $\mathbf{H}_2$ is not included in a connected component of $\mathbf{H}\backslash\{1\}$. However, this preteam will fit in the extended quasi-strict setting (see Example 4.31). △

As we shall see, preteams associated respectively with associahedra, permutohedra and friezohedra are strict. On the other hand, preteams associated with simplices, erosohedra and hypercubes are not strict, as some $H_{\tilde{a}}$ are not included in a connected component of $\mathbf{H}\backslash(\bigcup_{b\in B} X_b)$: these last examples fit in the formalism of quasi-strict teams introduced in §4.5.

Lemma 4.4. *A preteam $(\{\mathbf{H}_a \mid a \in A\}, \mathbf{H})$ is strict iff, for all $a \in A$ and $e \in \mathbf{H}_a$, $e$ is a tube of $\mathbf{H}$. Also, in the above definition of strict team, it holds that $H_{\tilde{a}}$ is a tube of $\mathbf{H}$, for all $\tilde{a} \in \tilde{A}$.*

*Proof.* We shall prove the equivalence (1) ⇔ (2) ⇔ (3), where (1) is the definition of strict team given above, (2) is the characterization claimed in the statement, and (3) is the definition of team above, enhanced with the additional property claimed in the statement.

- (1) ⇒ (2). If $(\{\mathbf{H}_a \mid a \in A\}, \mathbf{H})$ is a strict team in the sense of the definition given above, then, in particular, for each $a \in A$ and $e \in \mathbf{H}_a$, taking $B = A$, $X_b = H_b$ for $b \neq a$ and $X_a = (H_a\backslash e)$, we get that $\mathbf{H}_a, X_a \rightsquigarrow \mathbf{H}_e$, and hence that $e$ is included in a connected component $K$ of $\mathbf{H}\backslash(\bigcup_{b\in B} X_b)$. But for our choice of $B$, we have $H\backslash(\bigcup_{b\in B} X_b) = e$, hence this forces $K = e$, and a fortiori $e$ is a tube of $\mathbf{H}$.

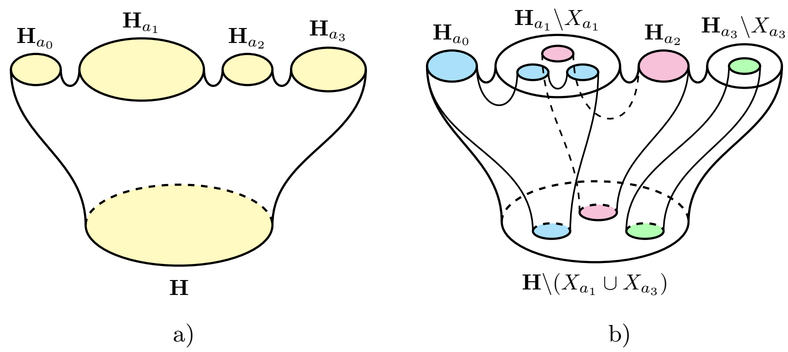



Figure 10. a) A preteam $\tau=(\{\mathbf{H}_{a_0},\mathbf{H}_{a_1},\mathbf{H}_{a_2},\mathbf{H}_{a_3}\},\mathbf{H})$ is represented as a cobordism whose upper and lower boundary disks feature the participating and coordinating hypergraphs, respectively. b) For $X_{a_1}\subseteq H_{a_1}$ and $X_{a_3}\subseteq H_{a_3}$, the decompositions $\mathbf{H}_{a_1},X_{a_1}\rightsquigarrow\mathbf{H}_{(a_1,1)},\mathbf{H}_{(a_1,2)},\mathbf{H}_{(a_1,3)}$, $\mathbf{H}_{a_3},X_{a_3}\rightsquigarrow\mathbf{H}_{(a_3,1)}$ and $\mathbf{H},(X_{a_1}\cup X_{a_2})\rightsquigarrow\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3$ are represented by "embeddings of little disks into big disks", in such a way that the little disks represent the corresponding connected components, and their complements in big disks are the removed sets. This allows us to visualize the induced preteams $\tau,X_{a_1}\cup X_{a_2}\,\underline{\rightsquigarrow}\,\tau_1,\tau_2,\tau_3$ as cobordisms in the interior of $\tau$.

- (2) ⇒ (3). Let $\tilde{a}\in\tilde{A}$ and $\tilde{e}\in\mathbf{H}_{\tilde{a}}$. Then a fortiori $\tilde{e}\in\mathbf{H}_{\pi(a)}$, where $\pi:\tilde{A}\to A$ is defined by $\pi(\tilde{a})=\tilde{a}$ if $\tilde{a}\in A\backslash B$ and $\pi(b,i)=b$. Since we assume (2), we have that $\tilde{e}$ is a tube of $\mathbf{H}$. Thus, all hyperedges of $\mathbf{H}_{\tilde{a}}$ are tubes of $\mathbf{H}$. By standard connectedness arguments, this, together with the fact that $\mathbf{H}_{\tilde{a}}$ is connected, implies that $H_{\tilde{a}}$ is a tube of $\mathbf{H}$: informally, every path of hyperedges of $\mathbf{H}_{\tilde{a}}$ witnessing the connectedness of $\mathbf{H}_{\tilde{a}}$, for arbitrary chosen vertices in $H_{\tilde{a}}$, can be turned into a path of hyperedges of $\mathbf{H}$ witnessing the connectedness of $H_{\tilde{a}}$ in $\mathbf{H}$ for the same chosen vertices.
- (3) ⇒ (1). Obvious.

□

Note that, for each $\varnothing\neq B\subseteq A$ and choice of $\varnothing\neq X_b\subseteq H_b$ for each $b\in B$, inducing the decomposition $\mathbf{H},\bigcup_{b\in B}X_b\rightsquigarrow\mathbf{H}_1,\ldots,\mathbf{H}_{n_B}$, the structure of a strict team $\tau$ implies the existence of a surjective function

$$\varphi_\tau^{B,\{X_b|b\in B\}}:\tilde{A}\to\{1,\ldots,n_B\}\quad\text{(written }\varphi_\tau^B\text{ for short)},$$

which associates to $\tilde{a}\in\tilde{A}$ the index of the connected component of $\mathbf{H}\backslash\bigcup_{b\in B}X_b$ that contains $H_{\tilde{a}}$. By Lemma 4.4, this determines preteams

$$\tau_i=(\{\mathbf{H}_{\tilde{a}}\mid\tilde{a}\in\tilde{A}\text{ and }\varphi_\tau(\tilde{a})=i\},\mathbf{H}_i^B)\quad(1\leqslant i\leqslant n_B).$$

We summarize this by the notation $\tau,\bigcup_{b\in B}X_b\,\underline{\rightsquigarrow}\,\tau_1,\ldots,\tau_{n_B}$. In order to ease the understanding of the decomposition $\tau,\bigcup_{b\in B}X_b\,\underline{\rightsquigarrow}\,\tau_1,\ldots,\tau_{n_B}$, in Figure 10, we suggest an interpretation of preteams and strict teams in terms of cobordisms.

Example 4.5. Consider the preteam in Example 4.2 :

$$(\{\mathbf{H}_1=\mathbf{F}_{\{1,3,5\}},\mathbf{H}_2=\mathbf{F}_{\{2,4\}},\mathbf{H}_3=\mathbf{F}_{\{6,7,8\}}\},\mathbf{F}_{\{1,\ldots,8\}})$$

and consider $B=\{1,2\}$, $X_1=\{3\}$ and $X_2=\{2\}$, inducing the decompositions

$$\mathbf{H}_1,X_1\rightsquigarrow\mathbf{H}_{(1,1)}=\mathbf{F}_{\{1\}},\mathbf{H}_{(1,2)}=\mathbf{F}_{\{5\}},$$

$$\mathbf{H}_2, X_2 \rightsquigarrow \mathbf{H}_{(2,1)} = \mathbf{F}_{\{4\}}$$

and

$$\mathbf{H}, \cup_{i\in B} X_i \rightsquigarrow \mathbf{H}_1^B = \mathbf{F}_{\{1\}}, \mathbf{H}_2^B = \mathbf{F}_{\{4,\dots,8\}}.$$

The map $\varphi_\tau^{B,\{X_b|b\in B\}}$ associates 1 to $(1,1)$ and 2 to the other elements. This leads to two preteams $\tau_1 = (\{\mathbf{F}_{\{1\}}\}, \mathbf{F}_{\{1\}})$ and $\tau_2 = (\{\mathbf{F}_{\{4\}}, \mathbf{F}_{\{5\}}, \mathbf{F}_{\{6,7,8\}}\}, \mathbf{F}_{\{4,\dots,8\}})$. △

A *strict clan* is a set $\Xi$ of strict teams such that, for each team $\tau \in \Xi$, and each situation $\tau, \bigcup_{b\in B} X_b \underline{\rightsquigarrow} \tau_1, \dots, \tau_n$ as above, we have that $\tau_i \in \Xi$ for all $i$.

Let us fix a strict clan $\Xi$, and some $q \in \mathbb{R}$ (our product will be parameterized by $q$, cf. end of §2). A $\Xi$-*delegation* (or delegation for short) is a pair

$$\delta = (\{C_a : \mathbf{H}_a \,|\, a \in A\}, \mathbf{H}) \quad \text{ such that } \quad \tau := (\{\mathbf{H}_a \mid a \in A\}, \mathbf{H}) \in \Xi.$$

We say that $\tau$ is the support of $\delta$, and that $C_a$ is the construct of $\delta$ at position $a$. Observe that, for $\varnothing \neq B \subseteq A$ and $\tilde{A}$ as above, assuming that $X_a$ is the root vertex of $C_a$ for each $a \in A$, there is a canonical association of a construct $C_{\tilde{a}}$ to each $\tilde{a} \in \tilde{A}$, which gives rise to delegations

$$\delta_i^B = (\{C_{\tilde{a}} : \mathbf{H}_{\tilde{a}} \,|\, \tilde{a} \in \tilde{A} \text{ and } \varphi_\tau^B(\tilde{a}) = i\}, \mathbf{H}_i^B), \tag{4.2}$$

for $1 \leqslant i \leqslant n_B$. More precisely, for $b \in B$, we set $C_b = X_b(C_{(b,1)}, \dots, C_{(b,n_b)})$ with $C_{(b,i)} : \mathbf{H}_{(b,i)}$. We summarize this by the notation $\delta, \bigcup_{b\in B} X_b \underline{\underline{\rightsquigarrow}} \delta_1^B, \dots, \delta_{n_B}^B$. All of these induced delegations feature in the definition of shuffle product given in §4.3.

Example 4.6. The strict clan associated with permutohedra is obtained by considering the set of all preteams

$$(\{\mathbf{P}^{V_i}\}_{i\in I}, \mathbf{P}^V),$$

where $\{V_i\}_{i\in I}$ forms a partition of $V$ (in the universe of permutohedra, it is easily checked that all preteams are in fact strict). △

Example 4.7. The strict clan associated with friezohedra is obtained by considering the set of all strict teams

$$(\{\mathbf{F}_{V_i}\}_{i\in I}, \mathbf{F}_V),$$

where $\{V_i\}_{i\in I}$ forms a partition of $V$ and each hypergraph $\mathbf{F}_{V_i}$ and $\mathbf{F}_V$ are connected. A delegation associated with the strict team of Example 4.2 is given by:

$$(\{3(1,5) : \mathbf{F}_{\{1,3,5\}}, 2(4) : \mathbf{F}_{\{2,4\}}, \{6,7,8\} : \mathbf{F}_{\{6,7,8\}}\}, \mathbf{F}_{\{1,\dots,8\}}).$$

Considering $B = \{1,2\}$, $X_1 = \{3\}$ and $X_2 = \{2\}$ as in Example 4.5, we get delegations

$$\delta_1^B = (\{1 : \mathbf{F}_{\{1\}}\}, \mathbf{F}_{\{1\}})$$
$$\text{and } \delta_2^B = (\{4 : \mathbf{F}_{\{4\}}, 5 : \mathbf{F}_{\{5\}}, \{6,7,8\} : \mathbf{F}_{\{6,7,8\}}\}, \mathbf{F}_{\{4,\dots,8\}}). \quad \triangle$$

We end this section by defining further conditions on clans:

- A clan $\Xi$ is *associative* if, for all

  $$\tau = (\{\mathbf{H}_a \mid a \in A\}, \mathbf{H}) \in \Xi\,,\ a_0 \in A\,,\ \tau' = (\{\mathbf{H}_{(a_0,a')} \mid a' \in A'\}, \mathbf{H}_{a_0}) \in \Xi,$$

  we have

  $$\tau'' := (\{\mathbf{H}_a \,|\, a \in A\backslash\{a_0\}\} \cup \{\mathbf{H}_{(a_0,a')} \,|\, a' \in A'\}, \mathbf{H}) \in \Xi.$$

  We shall refer to $\tau''$ as the grafting of $\tau'$ to $\tau$ along $a_0$. (Note that, again, we set the scene here for an unbiased version of associativity). We shall need this condition in order to phrase and prove the associativity of the product that we define in §4.3.

- In a different direction, we define the notion of *ordered* (strict) universe, preteam, team and clan. We suppose given an ordered set, say $\mathbb{Z}$. For $X_1, X_2 \subseteq \mathbb{Z}$, we write $X_1 < X_2$ if $\max(X_1) < \min(X_2)$. An *ordered universe* is a universe $\mathfrak{U}$ such that, for all $\mathbf{H} \in \mathfrak{U}$, $H \subseteq \mathbb{Z}$, and such that all decompositions $\mathbf{H}, X \rightsquigarrow \mathbf{H}_1, \ldots, \mathbf{H}_p$ can be indexed in such a way that $H_i < H_{i+1}$ for all $i$. An *ordered preteam* is a pair $((\mathbf{H}_1, \ldots, \mathbf{H}_p), \mathbf{H})$ such that $(\{\mathbf{H}_1, \ldots, \mathbf{H}_p\}, \mathbf{H})$ is a preteam and such that $H_1 < \cdots < H_p$. Ordered teams are teams whose underlying preteam is ordered. Note that when $\tau$ is ordered, if $\tau, \bigcup_{b \in B} X_b \underline{\rightsquigarrow} \tau_1, \ldots, \tau_{n_B}$, then each $\tau_i$ is ordered (to see this, one uses the assumption that $\mathfrak{U}$ is ordered). An ordered clan is a clan whose teams are all ordered.

4.2. Restrictohedra. Our main provision of strict clans comes from the universes of restrictohedra, that we define next. Fix a (possibly infinite) hypergraph $\mathbf{R}$, and let $\mathfrak{U}_{\mathbf{R}}$ be the universe consisting of all hypergraphs $\mathbf{R}_X$, such that $X \subseteq R$ is non-empty and finite, and $\mathbf{R}_X$ is connected: we call them the $\mathbf{R}$-*restrictohedra*, or restrictohedra for short. Let $\Xi_{\mathbf{R}}$ be the set of all pairs $(\{\mathbf{R}_{V_a} | a \in A\}, \mathbf{R}_V)$ where $V \subseteq R$, $\{V_a\}_{a \in A}$ forms a partition of $V$, and the hypergraphs $\mathbf{R}_{V_a}$ and $\mathbf{R}_V$ are all in $\mathfrak{U}_{\mathbf{R}}$. We can restrict this to an ordered setting if $\mathbf{R}$ is *order-friendly*, meaning that $R \subseteq \mathbb{Z}$ and that the connected components $\mathbf{R}_{V_1}, \ldots, \mathbf{R}_{V_p}$ of $\mathbf{R}_V$, for any finite $V \subseteq \mathbb{Z}$ such that $\mathbf{R}_V$ is not connected, can be indexed in such a way that $V_i < V_{i+1}$ for all $i$. A provision of order-friendly graphs is provided by Proposition 4.10 below.

Example 4.8. The following hypergraph is not order-friendly

$$\mathbf{R}^{nof} := \{\{a\} | a \in \mathbb{Z}\} \cup \{\{n, -n\} | n \in \mathbb{N}^*\}\}.$$

Indeed, the connected components of $\mathbf{R}^{nof}_{\{-2,1,2\}}$ are $\mathbf{R}^{nof}_{\{-2,2\}}$ and $\mathbf{R}^{nof}_{\{1\}}$. △

Proposition 4.9. *For all $\mathbf{R}$, $\Xi_{\mathbf{R}}$ is an associative clan. If $\mathbf{R}$ is order-friendly, then the restriction of $\Xi_{\mathbf{R}}$ (still denoted by $\Xi_{\mathbf{R}}$) to its ordered preteams is an ordered associative clan.*

*Proof.* We first note that every preteam $(\{\mathbf{R}_{V_a} | a \in A\}, \mathbf{R}_V)$ satisfies $\bigcup_{a \in A} \mathbf{R}_{V_a} \subseteq \mathbf{R}_V$ by definition, and hence, by Lemma 4.4, is a fortiori a strict team. Next, if (in the notation of §4.1) $\tau, \bigcup_{b \in B} X_b \underline{\rightsquigarrow} \tau_1, \ldots, \tau_{n_B}$, we have to prove that $\tau_i \in \Xi_{\mathbf{R}}$ for all $i$. This follows from the fact that, for any $V$ and $W \subseteq V$, $(\mathbf{R}_V) \backslash W = \mathbf{R}_{V \backslash W}$ and that, for all $X \subseteq R$, the connected components of $\mathbf{R}_X$ are all of the form $\mathbf{R}_Y$ for some $Y \subseteq X$. Finally, the clan is associative since $\Xi_{\mathbf{R}}$ includes all "possible" preteams in the sense that for any $X \subseteq R$ and any partition $\{X_a \mid a \in A\}$ of $X$, we have $(\{\mathbf{R}_{X_a} | a \in A\}, \mathbf{R}_X) \in \Xi_{\mathbf{R}}$ if and only if $\mathbf{R}_X$ and $\mathbf{R}_{X_a}$ (for all $a \in A$) are connected.

Suppose now that $\mathbf{R}$ is moreover order-friendly. Then it is immediate that $\mathfrak{U}_{\mathbf{R}}$ is ordered. Since we limit ourselves to ordered preteams $((\mathbf{H}_1, \ldots, \mathbf{H}_m), \mathbf{R}_V)$ with $\mathbf{H}_i = \mathbf{R}_{V_i}$ and $V_i < V_{i+1}$ for all $i$, and since $\mathbf{R}$ is order-friendly, then, for all $B \subseteq A = \{1, \ldots, m\}$, there is an induced order on $\tilde{A}$ such that, if $\tilde{a_1} < \tilde{a_2}$, then $V_{\tilde{a_1}} < V_{\tilde{a_2}}$, where $\mathbf{H}_{\tilde{a}} = \mathbf{R}_{V_{\tilde{a}}}$. This in turn implies that $(\varphi^B_\tau)^{-1}(1)$,…, $(\varphi^B_\tau)^{-1}(n_B)$ form successive intervals of $\tilde{A}$, and hence that each $\tau_i$ is ordered. □

Proposition 4.10. *For all $1 \leqslant k \in \mathbb{N} \cup \{\infty\}$, the following graph is order-friendly:*

$$\mathbf{\Gamma}^k := \{\{a\} \mid a \in \mathbb{Z}\} \cup \{\{a, a+l\} \mid a \in \mathbb{Z}, l \in \mathbb{N}, 1 \leqslant l \leqslant k\}.$$

*Proof.* A subset $V \subseteq \mathbb{Z}$ is not connected in $\mathbf{\Gamma}^k$ if and only if there is a set $X$ of at least $k$ consecutive integers in $]\min(V); \max(V)[$, which does not intersect $V$. If $X_1, \ldots, X_p$ are the sets of such maximal sequences of consecutive integers, then the interval $[\min(V), \max(V)]$ in $\mathbb{Z}$ is the union of consecutive intervals $I_0, X_1, I_1, \ldots, X_p, I_p$, and the connected components of $(\mathbf{\Gamma}^k)_V$ are $(\mathbf{\Gamma}^k)_{V\cap I_0},\dots,(\mathbf{\Gamma}^k)_{V\cap I_p}$. Then $(V\cap I_j) < (V\cap I_{j+1})$ follows a fortiori from $I_j < I_{j+1}$. $\square$

By Propositions 4.10 and 4.9, we get an induced associative ordered clan $\Xi_{\mathbf{\Gamma}^k}$. In the extreme cases $k=1$ and $k=\infty$, we have our old friends $\mathbf{\Gamma}^1_X = \mathbf{K}^X$ (for $X$ interval of $\mathbb{Z}$) and $\mathbf{\Gamma}^\infty_X = \mathbf{P}^X$ (for finite $X\subseteq\mathbb{Z}$), respectively. The teams are of the form $(\{\mathbf{\Gamma}^1_{X_1},\dots,\mathbf{\Gamma}^1_{X_p}\},\mathbf{\Gamma}^1_{\bigcup X_i})$ (where the $X_i$ are adjacent intervals) and $(\{\mathbf{\Gamma}^\infty_{X_1},\dots,\mathbf{\Gamma}^\infty_{X_p}\},\mathbf{\Gamma}^\infty_{\bigcup X_i})$ (where $X_i < X_{i+1}$ for all $i<p$), respectively. For $k=2$, we have $\mathbf{\Gamma}^2=\mathbf{F}$, and hence we recover also friezohedra as a special case.

We end the section with a characterization of universes arising as restrictohedra.

Proposition 4.11. *A universe $\mathfrak{U}$ is of the form $\mathfrak{U}_{\mathbf{R}}$, for some hypergraph $\mathbf{R}$, if and only if it satisfies the following four conditions:*

1. *For any hypergraphs $\mathbf{H}_1$ and $\mathbf{H}_2$ in $\mathfrak{U}$, if $H_1=H_2$, then $\mathbf{H}_1=\mathbf{H}_2$.*
2. *If $\mathbf{H}\in\mathfrak{U}$ and $e\in\mathbf{H}$, if $\mathbf{G}\in\mathfrak{U}$ is such that $e\subseteq G$, then $e\in\mathbf{G}$.*
3. *If $\mathbf{H}\in\mathfrak{U}$, and if $X\subseteq H$ is such that $\mathbf{H}_X$ is connected, then there exists $\mathbf{G}\in\mathfrak{U}$ such that $G=X$.*
4. *If $\mathbf{H}_1,\mathbf{H}_2\in\mathfrak{U}$ are such that $H_1\cap H_2$ is non-empty, then there exists $\mathbf{H}\in\mathfrak{U}$ such that $\mathbf{H}_1,\mathbf{H}_2\subseteq\mathbf{H}$.*

*Proof.* We first check that any universe of the form $\mathfrak{U}_{\mathbf{R}}$ satisfies the conditions in the statement. Condition (1) is immediate. Conditions (2), (3) and (4) follow immediately from the observations that, by definition, for arbitrary $X$, we have $e\in\mathbf{R}_X$ if and only if $e\in\mathbf{R}$ and $e\subseteq X$, that $(\mathbf{R}_H)_X=\mathbf{R}_X$, and that the union of two connected sets with a non-empty intersection is connected.

Conversely, suppose that $\mathfrak{U}$ satisfies the four conditions of the statement. We set $\mathbf{R}=\bigcup\{\mathbf{H}\mid\mathbf{H}\in\mathfrak{U}\}$. We shall show the following two properties, which (together with (1)) imply immediately that $\mathfrak{U}=\mathfrak{U}_{\mathbf{R}}$.

(a) If $X$ is a finite set such that there exists a hypergraph $\mathbf{H}$ such that $H=X$ and $\mathbf{H}\in\mathfrak{R}$, then there exists a hypergraph $\mathbf{H}'\in\mathfrak{U}_{\mathbf{K}}$ such that $H'=X$ and $\mathbf{H}\subseteq\mathbf{H}'$.

(b) If $X$ is a finite set such that there exists a hypergraph $\mathbf{H}'$ such that $H'=X$ and $\mathbf{H}'\in\mathfrak{U}_{\mathbf{R}}$, then there exists a hypergraph $\mathbf{H}\in\mathfrak{U}$ such that $H=X$ and $\mathbf{H}'\subseteq\mathbf{H}$.

For (a), we note that $\mathbf{H}\subseteq\mathbf{R}$ by definition of $\mathbf{R}$, hence $\mathbf{H}\subseteq\mathbf{R}_H$, so we can set $\mathbf{H}':=\mathbf{R}_H$, noticing that $\mathbf{R}_H$ is connected since it contains a connected hypergraph (namely $\mathbf{H}$) with the same set of vertices.

We now proceed to prove (b). By definition of $\mathfrak{U}_{\mathbf{R}}$, the assumptions of (b) can be rephrased as saying that $\mathbf{H}'=\mathbf{R}_X$ is connected. Also, by definition of $\mathbf{R}$, for each $e\in\mathbf{R}_X$, there exists a hypergraph $\mathbf{H}^e\in\mathfrak{U}$ such that $e\in\mathbf{H}^e$. So we have $\mathbf{R}_X\subseteq\bigcup\{\mathbf{H}^e\mid e\in\mathbf{R}_X\}$, this union being finite since $X$ is. Suppose that $\mathbf{R}_X$ has more than one hyperedge and pick $e_0\in\mathbf{R}_X$. We claim that there exists $e_1\in\mathbf{R}_X$ such that $\mathbf{H}^{e_0}\cap\mathbf{H}^{e_1}$ is non-empty. If it were not the case, then $\mathbf{R}_X$ would be the disjoint union of $\mathbf{R}_X\cap\mathbf{H}^{e_0}$ and of $\mathbf{R}_X\bigcap(\bigcup\{\mathbf{H}^e\mid e\neq e_0\})$, which would contradict the connectedness of $\mathbf{R}_X$. We can thus replace $\{\mathbf{H}^e\mid e\in\mathbf{R}_X\}$ by $\{\mathbf{H}^e\mid e\neq e_0,e_1\}\cup\{\mathbf{H}_{01}\}$, where $\mathbf{H}_{01}\in\mathfrak{U}$ is obtained from $\mathbf{H}^{e_0}$ and $\mathbf{H}^{e_1}$ by applying (4). By iterating this, we obtain a hypergraph $\mathbf{H}'\in\mathfrak{U}$ such that $\mathbf{R}_X\subseteq\mathbf{H}'$. Note that we can write this as well as $\mathbf{R}_X\subseteq\mathbf{H}'{}_X$, and, as above, we have that the connectedness of $\mathbf{R}_X$ implies the connectedness of $\mathbf{H}'{}_X$.

Our next (independent) observation is that in the presence of (2), condition (3) can be reinforced as follows. If $\mathbf{H}\in\mathfrak{U}$ and if $X\subseteq H$ is such that $\mathbf{H}_X$ is connected, then there exists $\mathbf{G}\in\mathfrak{U}$ such that $G=X$ and $\mathbf{H}_X\subseteq\mathbf{G}$. Indeed, let $\mathbf{G}$ be obtained by applying (3), and let $e \in \mathbf{H}_X$: then this latter assumption reads as $e \subseteq G$, and hence $e \in \mathbf{G}$ by (2).

Coming back to the proof of (b), we can apply the reinforced version of (3) to deduce the existence of a hypergraph $\mathbf{H} \in \mathfrak{U}$ such that $H = X$ and $\mathbf{H'}_X \subseteq \mathbf{H}$. We thus have $\mathbf{R}_X \subseteq \mathbf{H'}_X \subseteq \mathbf{H}$, which concludes the proof. □

4.3. Shuffle product of delegations of strict clans. We now define the shuffle product $*(\delta)$, for a $\Xi$-delegation $\delta$, where $\Xi$ is a strict clan. Until §4.5, we shall mostly omit the adjective "strict" for brevity, but its presence is understood.

A *linear construct* of a hypergraph $\mathbf{H}$ is an element of the vector space spanned by all the constructs of $\mathbf{H}$. We shall denote linear constructs with bold capital letters, e.g. $\mathbf{C} = \Sigma_{i\in I}\lambda_i C_i$, where $C_i : \mathbf{H}$, for each $i \in I$, and the notation $\mathbf{C} : \mathbf{H}$ will mean that $\mathbf{C}$ is a linear construct of $\mathbf{H}$. We then define $X(\mathbf{C}_1, \ldots, \sum_{i\in I} \lambda_i C^i_j, \ldots, \mathbf{C}_n)$ as $\sum_{i\in I} \lambda_i X(\mathbf{C}_1, \ldots, C^i_j, \ldots, \mathbf{C}_n)$. A *rooted linear construct* is a linear construct of the form $\mathbf{C} = X\{\mathbf{C}_a \mid a \in A\}$, and we write $\mathrm{root}(\mathbf{C}) = X$.

The *shuffle product* (or product) of a delegation $\delta = (\{C_a : \mathbf{H}_a \,|\, a \in A\}, \mathbf{H})$, with $\mathrm{root}(C_a) = X_a$ for all $a \in A$, is the linear construct of $\mathbf{H}$ defined recursively as follows (with $\delta^B_1, \ldots, \delta^B_{n_B}$ as in (4.2)):

$$*(\delta) = \sum_{\varnothing \subsetneq B \subseteq A} q^{|B|-1} *_B (\delta), \quad \text{where } *_B (\delta) = (\bigcup_{b\in B} X_b)(*(\delta^B_1), \ldots, *(\delta^B_{n_B})). \tag{4.3}$$

The instantiations of this shuffle product to associahedra and permutohedra are the ones recalled in §2. We detail the case of permutohedra in the next example.

Example 4.12. We restrict ourselves to teams with only two participating hypergraphs (which corresponds to the usual binary product on permutohedra). Then the shuffle product of a delegation

$$\delta = (\{C_1 : \mathbf{P}^Y, C_2 : \mathbf{P}^Z\}, \mathbf{P}^X), \text{ where } X = Y \cup Z, C_1 = X_1(C'_1), \text{ and } C_2 = X_2(C'_2),$$

rewrites as:

$$*(\delta) = *_{\{1\}}(\delta) + *_{\{2\}}(\delta) + q *_{\{1,2\}} (\delta),$$

where

$$\begin{aligned} *_{\{1\}}(\delta) &= X_1(*(\{C'_1 : \mathbf{P}^{Y\setminus X_1}, C_2 : \mathbf{P}^Z\}, \mathbf{P}^{X\setminus X_1})), \\ *_{\{2\}}(\delta) &= X_2(*(\{C_1 : \mathbf{P}^Y, C'_2 : \mathbf{P}^{Z\setminus X_2}\}, \mathbf{P}^{X\setminus X_2})), \\ *_{\{1,2\}}(\delta) &= (X_1 \cup X_2)\,(*(\{C'_1 : \mathbf{P}^{Y\setminus X_1}, C'_2 : \mathbf{P}^{Z\setminus X_2}\}, \mathbf{P}^{X\setminus(X_1\cup X_2)})). \end{aligned}$$

Writing $*_{\{1\}} = \prec$, $*_{\{2\}} = \succ$ and $*_{\{1,2\}} = \cdot$, and using an infix notation, the formula for the shuffle product on permutohedra on two constructs $C_1 = X_1(C'_1)$ and $C_2 = X_2(C'_2)$ writes as:

$$C_1 * C_2 = C_1 \prec C_2 + C_1 \succ C_2 + q(C_1 \cdot C_2),$$

where

$$\begin{aligned} C_1 \prec C_2 &= X_1(C'_1 * C_2), \\ C_1 \succ C_2 &= X_2(C_1 * C'_2), \\ C_1 \cdot C_2 &= (X_1 \cup X_2)\,(C'_1 * C'_2), \end{aligned}$$

with the convention that if $C'_1$ or $C'_2$ is the empty construct, then its shuffle product with another construct $C$ is $C$. It can be checked by direct induction that this definition coincides with the one given in §2. △

Remark 4.13. Let us now explain in a few words why we found convenient to consider permutohedra on a given $X$ which is not necessarily the usually considered set $\{1, \ldots, n\}$. Consider the basic example:

$$(1, 2) \cdot (2, 1) = (1, 2, 2, 1) + (1, 3, 3, 2) + (2, 3, 3, 1). \tag{4.4}$$

In terms of constructs, this rewrites as:

$$2(1)\cdot 3(4) = \{2,3\}(1*4). \tag{4.5}$$

Equation 4.5 invites us to compute products of constructs in the complete graph on $\{1,4\}$, rather than doing some renamings. This is naturally in phase with the general philosophy of species. The assignment that maps $X$ to the set of constructs of $\mathbf{P}^X$ is functorial (with respect to finite sets and bijections), giving rise to a species in the sense of Joyal.

Example 4.14. As a second example, we consider friezohedra. Consider the delegation

$$\delta = (\{2(1):\mathbf{F}_{\{1,2\}}\,,\,3(4):\mathbf{F}_{\{3,4\}},\}\,,\,\mathbf{F}_{\{1,\dots,4\}}).$$

The associated shuffle product is given by:

$$\begin{aligned}*(\delta) &= 2(1(3(4))) + 2(3(1,4)) + q2(\{1,3\}(4)) + 3(4(2(1)))\\ &\quad + 3(2(1,4)) + q3(\{2,4\}(1)) + q\{2,3\}(1,4).\end{aligned}$$

Consider now the delegation

$$(\{3(1,5):\mathbf{F}_{\{1,3,5\}}\,,\,2(4):\mathbf{F}_{\{2,4\}}\,,\,\{6,7,8\}:\mathbf{F}_{\{6,7,8\}}\}\,,\,\mathbf{F}_{\{1,\dots,8\}}).$$

of Example 4.7. The associated shuffle product is too big to be written here. Let us focus on the term associated with $B = \{1,2\}$. We have

$$*_B(\delta) = \{2,3\}(1, *(\delta_2^B)),$$

with

$$\delta_2^B = (\{4:\mathbf{F}_{\{4\}}\,,\,5:\mathbf{F}_{\{5\}}\,,\,\{6,7,8\}:\mathbf{F}_{\{6,7,8\}}\}\,,\,\mathbf{F}_{\{4,\dots,8\}})$$

(as already seen in Example 4.7). By definition, we can express $*(\delta_2^B)$ as a sum over $\varnothing \subsetneq B' \subseteq \{1,2,3\}$. Let us again make a focus, say on $B' = \{3\}$. We get

$$\begin{aligned}*_{B'}(\delta_2^B) &= \{6,7,8\}(*(\{4:\mathbf{F}_{\{4\}}\,,\,5:\mathbf{F}_{\{5\}}\}\,,\,\mathbf{F}_{\{4,5\}}))\\ &= \{6,7,8\}(4(5)) + \{6,7,8\}(5(4)) + q\,\{6,7,8\}(\{4,5\}).\end{aligned} \quad \triangle$$

When dealing with the associativity of the product in Theorem 4.17 below, we shall have to take products of (delegations made of) linear constructs, which is not a problem, as the above definitions of $*$ and $*_B$ of course extend by linearity (with the notion of delegation accordingly extended to linear constructs). The following lemmas show two situations in which the linear extension of $*_B$ still satisfies its "defining" equation 4.3 (now a property!). To see the need for such lemmas, note that the definitions of the delegations $\delta_i^B$ do depend on the root of the constructs $C_b$ ($b \in B$), which no longer exists if $C_b$ is replaced by a linear construct that is not rooted.

Lemma 4.15. *Let* $(\{\mathbf{H}_a \mid a \in A\}, \mathbf{H})$ *be a strict team, and suppose that we are given rooted linear constructs* $\mathbf{C}_a$ *for each* $a \in A$ *with root* $X_a$*, forming a delegation* $\delta$ *(in the extended sense). Let* $\varnothing \subset B \subseteq A$ *and let* $X_B = \bigcup_{b\in B} X_b$*. Then we have* $*_B(\delta) = (\bigcup_{b\in B} X_b)(*(\delta_1^B), \dots, *(\delta_{n_B}^B))$*, with the same definition of* $\delta_i^B$ *as above.*

*Proof.* We first notice that we can indeed still define $\delta_i^B$ as before, since the only information used on constructs are their roots. Let us assume for simplicity that only one of the $\mathbf{C}_a$, say $\mathbf{C}_{b_0}$, is a rooted linear construct, all the others being plain constructs, and that $b_0 \in B$, as Lemma 4.16 will a fortiori cover the case where $b_0 \notin B$. We shall also assume for simplicity that $\mathbf{C}_{b_0} = X_{b_0}\{\mathbf{C}_{b_0,i} \mid 1 \leqslant i \leqslant n_{b_0}\}$, where only one of the $\mathbf{C}_{b_0,i}$, say $\mathbf{C}_{b_0,i_0} = \sum_{k\in K} \lambda_k C_{b_0,i_0,k}$, is a linear construct, all the others being plain constructs (and we write then $\mathbf{C}_a = C_a$ for $a \neq b_0$ and $\mathbf{C}_{b_0,i} = C_{b_0,i}$

for $i \neq i_0$). Then, by "outward" linearity, we can write $\mathbf{C}_{b_0} = \sum_{k\in K} \lambda_k C_{b_0,k}$, where $C_{b_0,k} = X_{b_0}(\{C_{b_0,i} \mid i \neq i_0\} \bigcup \{C_{b_0,i_0,k}\})$. We have

$$*_B(\{C_a \mid a \in A\backslash\{b_0\}\} \bigcup \{\mathbf{C}_{b_0}\}, \mathbf{H}) = \sum_{k\in I} \lambda_k *_B (\{C_a \mid a \in A\backslash\{b_0\}\} \bigcup \{C_{b_0,k}\}, \mathbf{H}).$$

By definition, we have

$$*_B(\{C_a \mid a \in A\backslash\{b_0\}\} \bigcup \{\mathbf{C}_{b_0}\}, \mathbf{H}) = (\sum_{k\in K} \lambda_k X_k(*((\delta^k)^B_1), \ldots, *((\delta^k)^B_{n_B}))),$$

where for all $j \neq j_0 = \varphi_\tau(b_0, i_0)$, all $(\delta^k)^B_j$ are equal to $\delta^B_j$, and where the $(\delta^k)^B_{j_0}$ differ only in one (and the same) position (the one indexed by $(b_0, i_0)$), filled with $C_{b_0,i_0,k}$. Then we conclude by applying "inward" linearity. □

Lemma 4.16. *Let* $(\{\mathbf{H}_a \mid a \in A\}, \mathbf{H})$ *be a strict team, and let* $a_0 \in A$*, and suppose that we are given constructs* $C_a : \mathbf{H}_a$ *with root* $X_a$ *for all* $a \neq a_0$*, and a linear construct* $\mathbf{C}_{a_0}$*. Let* $B \subseteq A\backslash\{a_0\}$*, and let* $X_B = \bigcup_{b\in B} X_b$*. Then we have* $*_B(\delta) = (\bigcup_{b\in B} X_b)(*(\delta^B_1), \ldots, *(\delta^B_{n_B}))$*, with the same definition of the teams* $\delta^B_i$ *as above.*

*Proof.* The proof proceeds as in Lemma 4.15. The only difference is that, under the assumption that $a_0 \notin B$, no information at all is required on $\mathbf{C}_{a_0} = \sum_{k\in K} \lambda_k C_{a_0,k}$. □

So far, we have a magmatic unbiased notion of product. The following theorem establishes the associativity of the product for strict associative clans.

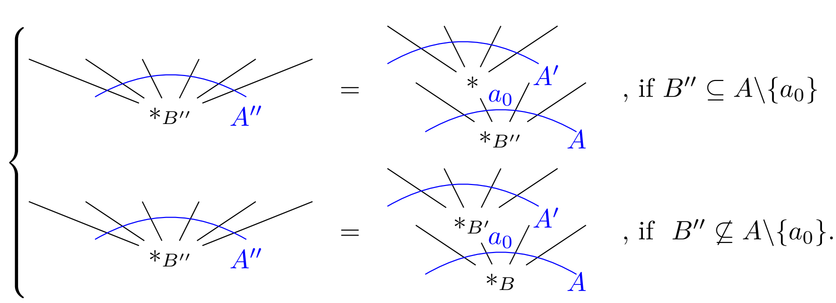
Theorem 4.17. *Let* $\Xi$ *be an associative clan, and suppose that* $\tau = (\{\mathbf{H}_a \mid a \in A\}, \mathbf{H}) \in \Xi$*,* $a_0 \in A$*, and* $\tau' = (\{\mathbf{H}_{(a_0,a')} \mid a' \in A'\}, \mathbf{H}_{a_0}) \in \Xi$*, and that we are given constructs* $C_a : \mathbf{H}_a$ *for all* $a \in A\backslash\{a_0\}$ *and constructs* $C_{(a_0,a')} : \mathbf{H}_{(a_0,a')}$ *for all all* $a' \in A'$*. Taking* $\tau''$ *to be the grafting of* $\tau'$ *to* $\tau$ *along* $a_0$ *and setting* $A'' := (A\backslash\{a_0\}) \cup \{(a_0, a') \mid a' \in A'\}$*, denote the corresponding delegations by* $\delta'' = (\tau'', \{C_{a''} \mid a'' \in A''\})$ *and* $\delta' = (\tau', \{C_{(a_0,a')} \mid a' \in A'\})$*. We then have that, for each* $\varnothing \neq B'' \subseteq A''$*, the following* polydendriform equation *holds:*

$$*^{\tau''}_{B''}(\delta'') = \begin{cases} *^{\tau}_{B''}(\{C_a \mid a \in A\backslash\{a_0\}\} \cup \{*^{\tau'}(\delta')\}), & \text{if } B'' \subseteq A\backslash\{a_0\} \\ *^{\tau}_{B}(\{C_a \mid a \in A\backslash\{a_0\}\} \cup \{*^{\tau'}_{B'}(\delta')\}), & \text{if } B'' \not\subseteq A\backslash\{a_0\} \end{cases},$$

*where the superscripts record the respective support teams, and where, in the second case,* $B = (B'' \cap (A\backslash\{a_0\})) \cup \{a_0\}$*,* $B' = \{a' \in A' \mid (a_0, a') \in B''\}$ *(both non-empty). Moreover, the polydendriform equation implies the following associativity equation:*

$$*^{\tau''}(\delta'') = *^{\tau}(\{C_a \mid a \in A\backslash\{a_0\}\} \cup \{*^{\tau'}(\delta')\}).$$

Remark 4.18. The two cases of the polydendriform equation can be drawn as:



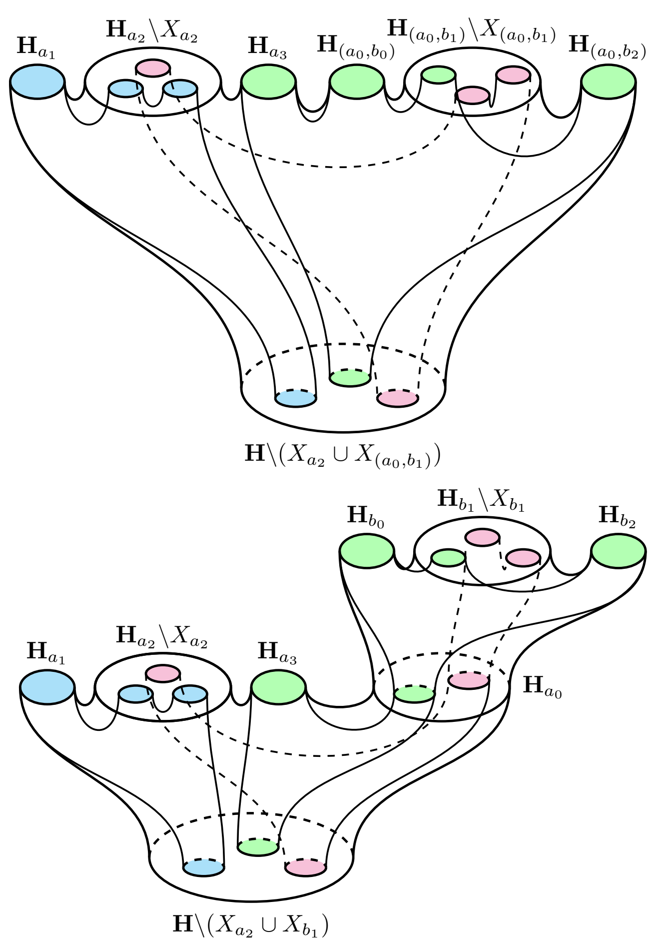



FIGURE 11. Illustration of associativity via cobordisms, with $A = \{a_0,\dots,a_3\}$, $A' = \{b_0,\dots,b_2\}$, $A'' = \{a_1,\dots,a_3,\ (a_0,b_0), (a_0,b_1),(a_0,b_2)\}$, $B'' = \{a_2,(a_0,b_1)\}$, $B' = b_1$ and $B = a_2, a_0$.

*Proof.* We set $\delta = (\tau, \{C_a \,|\, a \in A\backslash\{a_0\}\}\cup\{*^{\tau'}(\delta')\})$. We first prove the polydendriform equation, by induction on $|H|$. Denote, for each $a'' \in A''$, $X_{a''} := \mathrm{root}(C_{a''})$. By definition of the operation $*_{B''}$, supposing that $\mathbf{H}, X_{B''} \rightsquigarrow \mathbf{H}_1^{B''},\dots,\mathbf{H}_{n_{B''}}^{B''}$, where $X_{B''} = \bigcup_{b''\in B''} X_{b''}$, we have that

$$*_{B''}^{\tau''}(\delta'') = X_{B''}(*((\delta'')_1^{B''}),\dots,*((\delta'')_{n_{B''}}^{B''})),$$

where, for $1 \leqslant i \leqslant n_{B''}$,

$$(\delta'')_i^{B''} = (\{C_{\widetilde{a''}} : \mathbf{H}_{\widetilde{a''}} \,|\, \widetilde{a''} \in \widetilde{A''} \text{ and } \varphi_{\tau''}^{B''}(\widetilde{a''}) = i\}, \mathbf{H}_i^{B''}),$$

with the indexing set

$$\widetilde{A''} := A''\backslash B'' \cup \{(b'',q) \,|\, b'' \in B'' \text{ and } 1 \leqslant q \leqslant n_{b''}\}$$

arising from $\mathbf{H}_{b''}, X_{b''} \rightsquigarrow \mathbf{H}_{(b'',1)},\dots\mathbf{H}_{(b'',n_{b''})}$ ($b'' \in B''$). We examine the two cases of the statement. Figure 11 will help the reader to visualize the notations introduced in the second case.

(1) If $B'' \subseteq A\backslash\{a_0\}$, then, setting $\mathbf{C}_{a_0} := *^{\tau'}(\delta')$, we have (using Lemma 4.16):

$$*^{\tau}_{B''}(\delta) = X_{B''}(*(\delta_1^{B''}), \ldots, *(\delta_{n_{B''}}^{B''})),$$

where, for $1 \leqslant l \leqslant n_{B''}$,

$$\delta_l^{B''} = (\{C_{\tilde{a}} : \mathbf{H}_{\tilde{a}} \,|\, \tilde{a} \in \tilde{A} \text{ and } \varphi_\tau^{B''}(\tilde{a}) = l\}, \mathbf{H}_l^{B''}),$$

with the indexing set

$$\tilde{A} := A\backslash B'' \cup \{(b,p) \,|\, b \in B'' \text{ and } 1 \leqslant p \leqslant n_b\}$$

arising from $\mathbf{H}_b, X_b \rightsquigarrow \mathbf{H}_{(b,1)}, \ldots, \mathbf{H}_{(b,n_b)}$ $(b \in B'')$. Then, establishing $*^{\tau''}_{B''}(\delta'') = *^{\tau}_{B''}(\delta)$ amounts to showing that $*((\delta'')_l^{B''}) = *(\delta_l^{B''})$, for all $1 \leqslant l \leqslant n_{B''}$.

Let $\pi'' : \widetilde{A''} \to A''$ and $\pi : \tilde{A} \to A$ be the obvious projections (cf. proof of Lemma 4.4). Then it is readily seen (remembering that $H_{(a_0,a')} \subseteq H_{a_0}$) that $(\pi'')^{-1}(A\backslash\{a_0\}) = \widetilde{A''} \cap \tilde{A} = \pi^{-1}(A\backslash\{a_0\})$ and

$$\varphi_\tau^{B''}|_{\widetilde{A''}\cap\tilde{A}} = \varphi_{\tau''}^{B''}|_{\widetilde{A''}\cap\tilde{A}} \quad \text{and} \quad \varphi_\tau^{B''}(a_0) = \varphi_{\tau''}^{B''}(a_0, a'), \tag{4.6}$$

for all $a' \in A'$. It follows that for $l \neq \varphi_\tau^{B''}(a_0) := l_0$ we have that $(\delta'')_l^{B''} = \delta_l^{B''}$, while (remembering the definition of $\mathbf{C}_{a_0}$) the equality $*((\delta'')_{l_0}^{B''}) = *(\delta_{l_0}^{B''})$ follows by induction on $\mathbf{H}_{a_0}$.

(2) For $B'' \not\subseteq A\backslash\{a_0\}$, let $B := (B'' \cap (A\backslash\{a_0\})) \cup \{a_0\}$ and $B' := \{a' \in A' \,|\, (a_0, a') \in B''\}$. Let $X_{B'} := \bigcup_{b' \in B'} X_{(a_0,b')}$ and suppose that $\mathbf{H}_{a_0}, X_{B'} \rightsquigarrow (\mathbf{H}_{a_0})_1^{B'}, \ldots, (\mathbf{H}_{a_0})_{m_{B'}}^{B'}$. We have by definition

$$*^{\tau'}_{B'}(\delta') = X_{B'}(*((\delta')_1^{B'}), \ldots, *((\delta')_{m_{B'}}^{B'})),$$

where, for $1 \leqslant j \leqslant m_{B'}$,

$$(\delta')_j^{B'} = (\{C_{\widetilde{a'}} : \mathbf{H}_{\widetilde{a'}} \,|\, \widetilde{a'} \in \widetilde{A'} \text{ and } \varphi_{\tau'}^{B'}(\widetilde{a'}) = j\}, (\mathbf{H}_{a_0})_j^{B'}),$$

with the indexing set

$$\widetilde{A'} := \{(a_0, a') \,|\, a' \in A'\backslash B'\} \cup \{(a_0, b', k) \,|\, b' \in B' \text{ and } 1 \leqslant k \leqslant n_{b'}\}$$

arising from $H_{(a_0,b')}, X_{(a_0,b')} \rightsquigarrow \mathbf{H}_{(a_0,b',1)}, \ldots, \mathbf{H}_{(a_0,b',n_{b'})}$ $(b' \in B')$. Setting $\mathbf{C}_{a_0}^{B'} := X_{B'}(*((\delta')_1^{B'}), \ldots, *((\delta')_{m_{B'}}^{B'}))$, the equality that we aim to prove displays as

$$*^{\tau''}_{B''}(\delta'') = *^{\tau}_{B}(\{C_a \,|\, a \in A\backslash\{a_0\}\} \cup \{\mathbf{C}_{a_0}^{B'}\}). \tag{4.7}$$

Furthermore, by setting $X_{a_0} := X_{B'}$ and $X_B := \bigcup_{b \in B} X_b$, we can write

$$X_{B''} = (\bigcup_{b \in B\backslash\{a_0\}} X_b) \bigcup \{X_{B'}\} = (\bigcup_{b \in B\backslash\{a_0\}} X_b) \bigcup \{X_{a_0}\} = X_B.$$

We can then transform (4.7) (applying Lemma 4.15) into

$$X_B(*((\delta'')_1^{B''}), \ldots, *((\delta'')_{n_{B''}}^{B''})) = X_B(*(\delta_1^B), \ldots, *(\delta_{n_B}^B)), \tag{4.8}$$

where $\mathbf{H}, X_B \rightsquigarrow \mathbf{H}_1^B, \ldots, \mathbf{H}_{n_B}^B$, $n_B = n_{B''}$, and for $1 \leqslant l \leqslant n_B$,

$$\delta_l^B = (\{C_{\tilde{a}} : \mathbf{H}_{\tilde{a}} \,|\, \tilde{a} \in \tilde{A} \text{ and } \varphi_\tau^B(\tilde{a}) = l\}, \mathbf{H}_l^B),$$

with the indexing set

$$\tilde{A} := A\backslash B \cup \{(b,p) \,|\, b \in B \text{ and } 1 \leqslant p \leqslant n_b\}$$

arising from $\mathbf{H}_b, X_b \rightsquigarrow \mathbf{H}_{(b,1)}, \ldots, \mathbf{H}_{(b,n_b)}$ $(b \in B)$, and where

$$C_{\tilde{a}} = \begin{cases} C_a, & \text{if } \tilde{a} \in (A\backslash B), \\ *((\delta')_p^{B'}), & \text{if } \tilde{a} = (a_0, p), \\ C_{(b,p)}, & \text{if } \tilde{a} = (b,p) \quad (b \in B\backslash\{a_0\}). \end{cases}$$

Now, since $X_{B''} = X_B$, we can suppose, without loss of generality, that $H_i^{B''} = H_i^B$, for all $1 \leqslant i \leqslant n_B = n_{B''}$. Therefore, it remains to show that $*((\delta'')_i^{B''}) = *(\delta_i^B)$. Observe that, since

$$\mathbf{H}_{(a_0,1)}, \ldots, \mathbf{H}_{(a_0,n_{a_0})} \leftsquigarrow \mathbf{H}_{a_0}, X_{a_0} = \mathbf{H}_{a_0}, X_{B'} \rightsquigarrow (\mathbf{H}_{a_0})_1^{B'}, \ldots, (\mathbf{H}_{a_0})_{m_{B'}}^{B'},$$

we have that $n_{a_0} = m_{B'}$, and we can assume (without loss of generality) that $H_{(a_0,p)} = (H_{a_0})_p^{B'}$, for each $1 \leqslant p \leqslant n_{a_0}$.

Simple inspection (and standard argumentation with connected components) yields

$$\begin{aligned}\pi''^{-1}(A\backslash\{a_0\}) &= \widetilde{A''} \cap \tilde{A} = \pi^{-1}(A\backslash\{a_0\})\\ \pi''^{-1}(a_0) &= \widetilde{A'},\end{aligned}$$

where $\pi'' : \widetilde{A''} \to A''$ and $\pi : \tilde{A} \to A$ are the obvious projections, and

$$\varphi_{\tau''}^{B''}|_{\widetilde{A''}\cap\tilde{A}} = \varphi_\tau^B|_{\widetilde{A''}\cap\tilde{A}} \quad \text{and} \quad \varphi_{\tau''}^{B''}|_{\widetilde{A'}} = \tilde{\varphi}_\tau^B \circ \varphi_{\tau'}^{B'}, \tag{4.9}$$

where $\tilde{\varphi}_\tau^B(j) := \varphi_\tau^B((a_0, j))$, for each $1 \leqslant j \leqslant n_{a_0}$. We note that, thanks to (4.9), $*((\delta'')_i^{B''})$ and $*(\delta_i^B)$ look respectively like this:

$$\begin{aligned}*((\delta'')_i^{B''}) &= *(\underbrace{\ldots, C_y, \ldots}_{y\in\tilde{A}''\cap\tilde{A},\varphi_{\tau''}^{B''}(y)=i}, \ldots, \underbrace{\ldots, C_x, \ldots}_{x\in\tilde{A}',\,\varphi_{\tau'}^{B'}(x)=j\in(\tilde{\varphi}_\tau^B)^{-1}(i)}, \ldots)\\ *(\delta_i^B) &= *(\underbrace{\ldots, C_y, \ldots}_{y\in\widetilde{A''}\cap\tilde{A},\varphi_\tau^B(y)=i}, \ldots, \underbrace{*(\ldots, C_x, \ldots)}_{x\in\tilde{A}',\,\varphi_{\tau'}^{B'}(x)=j\in(\tilde{\varphi}_\tau^B)^{-1}(i)}, \ldots)\end{aligned}$$

and we conclude by applying induction to each $\mathbf{H}_i^{B''}$ (note that repeated induction, or no induction at all, may be needed for a single fixed $i$, depending on the cardinality of $\varphi_{\tau'}^{B'}(\tilde{A}) \cap (\tilde{\varphi}_\tau^B)^{-1}(i)$).

This concludes the proof of the polydendriform equation. Associativity is derived as follows. Writing $\delta_{B'}$ for $\{C_a \,|\, a \in A\backslash\{a_0\}\} \cup \{*_{B'}^{\tau'}(\delta')\}$, we have on one hand (in-lining the polydendriform equation):

$$*^{\tau''}(\delta'') = \overbrace{\sum_{\varnothing\subsetneq B''\subseteq A\backslash\{a_0\}} q^{|B''|-1} \; *_{B''}^{\tau}(\delta)}^{A_1} + \overbrace{\sum_{\varnothing\subsetneq B''\nsubseteq A\backslash\{a_0\}} q^{|B''|-1} \; *_B^{\tau}(\delta_{B'})}^{B_1}$$

with $B, B'$ determined from $B''$ as specified in the statement, and on the other hand (expanding the second summand by linearity):

$$*^{\tau}(\delta) \;=\; \underbrace{\sum_{\varnothing\subsetneq B\subseteq A\backslash\{a_0\}} q^{|B|-1} \; *_B^{\tau}(\delta)}_{A_2} \;+\; \underbrace{\sum_{\varnothing\subsetneq B\nsubseteq A\backslash\{a_0\}} \sum_{\varnothing\subsetneq B'\subseteq A'} q^{|B|+|B'|-2} \; *_B^{\tau}(\delta_{B'})}_{B_2}$$

We have $A_1 = A_2$ literally, while $B_1 = B_2$ follows by noticing that the map $B'' \mapsto ((B'' \cap (A\backslash\{a_0\})) \cup \{a_0\}, \{a' \in A' \,|\, (a_0, a') \in B''\})$ is bijective. ☐

Remark 4.19. One could formulate the polydendriform structure as an algebra over a colored operad, where the colors are hypergraphs, the operations are teams, and the carrier of the algebra for the color $\mathbf{H}$ is the set of constructs of $\mathbf{H}$.

We shall now relate the polydendriform structure to the tridendriform one, by showing that the former implies (and can be considered as the unbiased version of) the latter, in the *ordered* framework.

Let $\Xi$ be an ordered associative clan. Suppose that we have

$$\{((\mathbf{H}_1, \mathbf{H}_{2'}), \mathbf{H}), ((\mathbf{H}_2, \mathbf{H}_3), \mathbf{H}_{2'}), ((\mathbf{H}_{1'}, \mathbf{H}_3), \mathbf{H}), ((\mathbf{H}_1, \mathbf{H}_2), \mathbf{H}_{1'})\} \in \Xi.$$

Denote by $\tau_1''$ the grafting of $((\mathbf{H}_1,\mathbf{H}_2),\mathbf{H}_{1'})$ to $((\mathbf{H}_{1'},\mathbf{H}_3),\mathbf{H})$ along $1'$, and by $\tau_2''$ the grafting of $((\mathbf{H}_2,\mathbf{H}_3),\mathbf{H}_{2'})$ to $((\mathbf{H}_1,\mathbf{H}_{2'}),\mathbf{H})$ along $2'$. Note that the above teams are all of the (generic) form $((\mathbf{H}_l,\mathbf{H}_r),\mathbf{L})$. We write (cf. Example 4.12)

$$\prec := *_{\{l\}} \qquad \cdot := *_{\{l,r\}} \qquad \succ := *_{\{r\}}.$$

Proposition 4.20. *In the ordered framework, the tridendriform equations follow from the polydendriform one, relatively to the team* $\tau'' = ((\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3),\mathbf{H})$. *More precisely, Loday–Ronco's seven equations, as listed in the introduction, correspond to choosing* $B''$ *to be* $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2,3\}$, $\{2,3\}$, $\{1,3\}$, $\{1,2\}$, *respectively.*

*Proof.* As a sanity check, we first note that there are $2^3-1=7$ non-empty subsets of $\{1,2,3\}$. We check the equation $(\succ\cdot)$. Let $S:\mathbf{H}_1$, $T:\mathbf{H}_2$, $U:\mathbf{H}_3$. We have

$$*^{\tau_1''}_{\{2,3\}}(S,T,U) = *_{\{1',3\}}((*_{\{2\}}(S,T),U)) = ((S\succ T)\cdot U)$$

and

$$*^{\tau_1''}_{\{2,3\}}(S,T,U) = *^{\tau_2''}_{\{2,3\}}(S,T,U) = *_{\{2'\}}(S,(*_{\{2,3\}}(T,U))) = S\prec(T\cdot U).$$

Note that all tridendriform equations follow from the second case of the polydendriform equation, except $(\prec *)$ and $(*\succ)$ (for which we use the first case, and which are the only tridendriform equations involving $*$). ☐

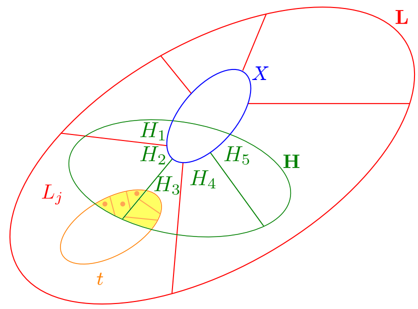
Combining the results of §4.2 and §4.3, we get a whole range of polydendriform/tridendriform structures, and in particular we get structures associated with the graphs $\mathbf{\Gamma}^k$ of Proposition 4.10. As we have seen, for the instances $k=1$ and $k=\infty$, we recover the tridendriform structures of §2, thus fulfilling our unifying goal, with a whole infinity of examples sitting "in the middle". The case k=2 is that of friezohedra.

Remark 4.21. The associative product defined here is quite different (both in form and spirit) from the algebraic structure on the constructs of all graph associahedra defined by Forcey and Ronco in [11], which provides an example of reconnectad, as defined and investigated in [9]. We first note that the definitions of [11, 9] can be straightforwardly upgraded to nestohedra. For a contrast, we briefly introduce the Forcey-Ronco products, using notation akin to the ones used here. Our teams are replaced by the data of a (hyper)graph $\mathbf{H}$ and a non empty subset $X\subseteq H$, giving rise to a "reconnecteam" $((\mathbf{H}_{\cap X},\mathbf{H}_1,\dots,\mathbf{H_n}),\mathbf{H})$, where $\mathbf{H}_{\cap X}$ is the so-called reconnected restriction of $\mathbf{H}$ to $X$ whose hyperedges are the intersections of the tubes of $\mathbf{H}$ with $X$, and where $\mathbf{H},X\rightsquigarrow\mathbf{H}_1,\dots,\mathbf{H}_n$. Now, given constructs $S:\mathbf{H}_{\cap X}$, $T_1:\mathbf{H}_1$, …, $T_n:\mathbf{H}_n$, the operad-like composition $(\circ^{\mathbf{H}}_X(S;T_1,\dots,T_n)):\mathbf{H}$ is obtained by substituting $S$ for $X$, $T_1$ for $H_1$, …, $T_n$ for $H_n$ in the codimension 1 construct $(X(H_1,\dots,H_n)):\mathbf{H}$. The underlying combinatorics in [11] is thus substitution in nodes of trees, while in our work it is that of shuffles of trees. Another key difference is that in our teams the participant hypergraphs are not determined by the coordinating hypergraph. As a final remark, the underlying "universe" of Forcey-Ronco is the collection of all (hyper)graphs, in contrast to the situation here.

4.4. A non-recursive definition of the product. In this subsection, we give an equivalent, non-recursive, definition of the product, directly inspired from [20]. Let $\mathbf{H},\mathbf{L}$ be two connected hypergraphs such that $H\subseteq L$ and such that, for all $e\in\mathbf{H}$, $e$ is a tube of $\mathbf{L}$. This entails in particular that $H$ is a tube of $\mathbf{L}$. Let $S=X(S_1,\dots,S_n)$ be a construct of $\mathbf{L}$, with $S_i:\mathbf{L}_i$ where $\mathbf{L},X\rightsquigarrow\mathbf{L}_1,\dots,\mathbf{L}_n$. Then we define a construct $S_{\restriction\mathbf{H}}:\mathbf{H}$, called the *restriction of* $S$ *to* $\mathbf{H}$, as follows. We distinguish two cases.

- If $X\cap H=\varnothing$, then there is a unique $j$ such that $H\subseteq L_j$, and we set

$$S_{\restriction\mathbf{H}} = (S_j)_{\restriction\mathbf{H}}.$$




- The compartments with red/blue border are the connected components of $\mathbf{L}\backslash X$.
- The compartments with green/red/blue border are the connected components of $\mathbf{H}\backslash X$.
- In this example, we have $(\varphi_X^{\mathbf{H},\mathbf{L}})^{-1}(j) = \{2,3\}$.
- The small yellow compartments with orange/green borders feature the tubes in $\psi(t_{\mathbf{H}})$,
- while those additionally marked with a dot are the tubes in $\psi(t_{\mathbf{H}_2})$.

FIGURE 12. Illustration of the proof of Lemma 4.23

- If $X \cap H \neq \varnothing$, let $\mathbf{H}, (X \cap H) \rightsquigarrow \mathbf{H}_1, \ldots, \mathbf{H}_p$. This determines a function $\varphi_X^{\mathbf{H},\mathbf{L}} : \{1, \ldots, p\} \to \{1, \ldots, n\}$, and we set

$$S_{\restriction \mathbf{H}} = (X \cap H)(\ldots, (S_{\varphi_X^{\mathbf{H},\mathbf{L}}(i)})_{\restriction \mathbf{H}_i}, \ldots).$$

That $S_{\restriction \mathbf{H}}$ is indeed a construct of $\mathbf{H}$ is easily seen by induction.

EXAMPLE 4.22. In the universe of friezohedra, let us consider $\mathbf{H} = \mathbf{F}_{\{1,3,5\}}$ and $\mathbf{L} = \mathbf{F}_{\{1,\ldots,5\}}$. As every edge in $\mathbf{H}$ is also in $\mathbf{L}$, the hypothesis above is satisfied. Consider the construct $S = 3(\{1,4\}(2,5))$ of $\mathbf{L}$. Then $S_{\restriction \mathbf{H}} = 3(1,5)$. △

We next give a simpler (but more "mysterious") alternative description of $S_{\restriction \mathbf{H}}$ in terms of nested sets (see Remark 3.3). In order to formulate the lemma, we need one definition (generalized from [20] to the setting of hypergraph polytopes). With each tube $t$ of $\mathbf{L}$ we associate a construct $t_{\mathbf{H}}$ as follows (note the heterogeneous nature of this definition: we go from tubes to constructs):

- If $H \subseteq t$, then we set $t_{\mathbf{H}} = H$;
- if $H\backslash t \neq \varnothing$ yielding $\mathbf{H}, (H\backslash t) \rightsquigarrow \mathbf{H}_1, \ldots, \mathbf{H}_k$, we set $t_{\mathbf{H}} = (H\backslash t)(H_1, \ldots, H_k)$.

This definition can be seen as an instantiation of our definition of $S_{\restriction \mathbf{H}}$: more precisely, we can coerce a tube $t$ of $\mathbf{L}$ to a construct $(L\backslash t)(t) : \mathbf{L}$, and we have $t_{\mathbf{H}} = ((L\backslash t)(t))_{\restriction \mathbf{H}}$.

The following proposition gives a non-recursive definition for the above restriction operation, providing a bridge between our definition and that of Ronco in [20].

PROPOSITION 4.23. *For* $\mathbf{H}, \mathbf{L}$ *and* $S$ *as above, we have* $\psi(S_{\restriction \mathbf{H}}) = \bigcup\{\psi(t_{\mathbf{H}}) \mid t \in \psi(S)\}$.

*Proof.* (Sketch) Let $S = X(S_1, \ldots, S_n)$ and $L \neq t \in \psi(S)$, i.e. $t \in \psi(S_j)$ for some $j$. Then the statement follows from the observation (illustrated in Figure 12) that, with the notation introduced above:

$$\psi(t_{\mathbf{H}}) = \bigcup\{\psi(t_{\mathbf{H}_i}) \mid \varphi_X^{\mathbf{H},\mathbf{L}}(i) = j\} \quad (j, t \in \psi(S_j) \text{ fixed}, i \text{ varying}).$$

Indeed, by definition of $\psi$, we have on one hand that $(\bigcup\{\psi(t_{\mathbf{H}}) \mid t \in \psi(S)\})\backslash\{H\}$ is the union of the sets $(\bigcup\{\psi(t_{\mathbf{H}}) \mid t \in \psi(S_j)\})$, indexed by $1 \leqslant j \leqslant n$. On the other hand, applying induction, we have that $\psi(S_{\restriction \mathbf{H}})\backslash\{H\}$ is the union of the sets $\psi(t_{\mathbf{H}_i})$, for $1 \leqslant i \leqslant p$ and $t \in \psi(S_{\varphi_X^{\mathbf{H},\mathbf{L}}(i)})$, which we can repackage as a union indexed by $j$ (gathering all $i$ such that $\varphi_X^{\mathbf{H},\mathbf{L}}(i) = j$). We then conclude by the observation. □

In particular, via the characterization of tubings as constructs, Proposition 4.23 says that the definition in terms of tubings given in [20] returns indeed a tubing.

We now come back to the promised alternative definition of the product. Let $\tau = (\{\mathbf{H}_a \mid a \in A\}, \mathbf{H})$ be a team and $U = X(U_1, \ldots, U_n)$ be a construct of $\mathbf{H}$. We associate with $U$ a "measure" $\mu^\tau(U)$ as follows (with the notation of §4.1). We set $B = \{b \in A \mid X \cap H_b \neq \varnothing\}$ and $X_b = X \cap H_b$ for each $b \in B$ (so that $n = n_B$), and we set

$$\mu^\tau(U) = (|B| - 1) + \sum_{1 \leqslant i \leqslant n_B} \mu^{\tau_i}(U_i).$$

The following proposition gives a non-inductive characterization of our product $*$.

Proposition 4.24. *Let $\Xi$ be a strict clan, and let $\delta = (\{C_a : \mathbf{H}_a \mid a \in A\}, \mathbf{H})$ be a $\Xi$-delegation of support $\tau$. Then we have:*

$$*(\delta) = \sum_{U:\mathbf{H} \text{ and } \forall a \in A, U_{\lceil \mathbf{H}_a} = C_a} q^{\mu^\tau(U)}\, U,$$

*and for each $\varnothing \neq B \subseteq A$, we have that $q^{|B|-1}(*_B(\delta))$ is the summand of the above sum where $U$ is further constrained to be such that $root(U) = X_B$.*

*Proof.* (Sketch) We use the same notations as above. By unfolding the definition of $U = X(U_1, \ldots, U_n)$, with $X = X_B$, the constraints on $U$ boil down to the constraints (for each $i$) $(U_i)_{\lceil \mathbf{H}_{\tilde{a}}} = C_{\tilde{a}}$ for all $\tilde{a} \in \tilde{A}$ such that $\varphi_B^\tau(\tilde{a}) = i$. This entails that, taking the right-hand side of the equality and its summands in the statement as a definition of $*$ and $*_B$, and noticing that

$$q^{\mu^\tau(U)}\, X_B(U_1, \ldots, U_n) = q^{|B|-1}\, X_B(\ldots, q^{\mu^{\tau_i}(U_i)}\, U_i, \ldots),$$

these definitions satisfy the equation $*_B(\delta) = (\bigcup_{b \in B} X_b)(*(\delta_1^B), \ldots, *(\delta_{n_B}^B))$. □

Remark 4.25. In the ordered setting, and in the special case where $q = 1$ and where teams have the form $((\mathbf{H}_{H_1}, \mathbf{H}_{H_2}), \mathbf{H}))$, the summation in the statement of Proposition 4.24 is the definition of associative product given in [20, Theorem 3.19]. This justifies our earlier claim that our product specializes to Ronco's setting.

Example 4.26. We consider the delegation of friezohedra

$$\delta = (\{3(1,5) : \mathbf{F}_{\{1,3,5\}}\,,\, 4(2) : \mathbf{F}_{\{2,4\}}\}, \mathbf{F}_{\{1,2,3,4,5\}}).$$

The shuffle product of $\delta$ is then given, up to some coefficients, by the sum of all the constructs $U$ of $\mathbf{F}_{\{1,\ldots,5\}}$ such that $U_{\lceil \mathbf{F}_{\{1,3,5\}}} = 3(1,5)$ and $U_{\lceil \mathbf{F}_{\{2,4\}}} = 4(2)$. The power of $q$ in the coefficient of $S = 3(\{1,4\}(2,5))$ in this sum is given by:

$$\begin{aligned}\mu^{(\{\mathbf{F}_{\{1,3,5\}}, \mathbf{F}_{\{2,4\}}\}, \mathbf{F}_{\{1,2,3,4,5\}})}(S) &= (1-1) + \mu^{(\{\mathbf{F}_{\{1\}}, \mathbf{F}_{\{5\}}, \mathbf{F}_{\{2,4\}}\}, \mathbf{F}_{\{1,2,4,5\}})}(\{1,4\}(5(2)) \\ &= (2-1) + \mu^{(\{\mathbf{F}_{\{2\}}\}, \mathbf{F}_{\{2\}})}(2) + \mu^{(\{\mathbf{F}_{\{5\}}\}, \mathbf{F}_{\{5\}})}(5) \\ &= 1 + (1-1) + (1-1) = 1. \quad \triangle\end{aligned}$$

We note that the non-recursive definition leads to another proof of the poly-dendriform equation and of associativity – that is technically simpler but geometrically less appealing than the one we gave in §4.3, based on the observation, say for $(\{\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3\}, \mathbf{H})$, $(\{\mathbf{H}_1, \mathbf{H}_2\}, \mathbf{H}_{12})$, $(\{\mathbf{H}_{12}, \mathbf{H}_3\}, \mathbf{H})$, and

$$\delta = (\{S : \mathbf{H}_1\,,\, T : \mathbf{H}_2\,,\, U : \mathbf{H}_3\}\,,\, \mathbf{H}),$$

that the data of $V : \mathbf{H}$ such that $V_{\restriction \mathbf{H_1}} = S$, $V_{\restriction \mathbf{H_2}} = T$ and $V_{\restriction \mathbf{H_3}} = U$ is equivalent to the data of $V : \mathbf{H}$ and $W : \mathbf{H}_{12}$ such that $W_{\restriction \mathbf{H_1}} = S$, $W_{\restriction \mathbf{H_2}} = T$, $V_{\restriction \mathbf{H}_{12}} = W$, and $V_{\restriction \mathbf{H_3}} = U$ [(2)].

4.5. Extending the framework. In this subsection, we enlarge the coverage of our formalism of teams and clans, and we adapt the product accordingly, in order to cover other families of polytopes like simplices, hypercubes, or erosohedra.

A preteam $\tau = (\{\mathbf{H}_a \mid a \in A\}, \mathbf{H})$ is called a *quasi-strict team* if for each choice of a subset $\varnothing \neq B \subseteq A$ and of a subset $\varnothing \neq X_b \subseteq H_b$ for each $b \in B$, we have that, for each $\tilde{a} \in \tilde{A}$,

(1) $H_{\tilde{a}}$ is included in a connected component of $\mathbf{H} \backslash (\bigcup_{b \in B} X_b)$, or
(2) $|H_{\tilde{a}}| \geqslant 2$, and, for all $x \in H_{\tilde{a}}$, $\{x\}$ is a connected component of $\mathbf{H} \backslash (\bigcup_{b \in B} X_b)$,

where $\tilde{A}$ is as in §4.1. When (2) applies (and vacuously when $|H_{\tilde{a}}| = 1$), we say that $\mathbf{H}_{\tilde{a}}$ is *dissolved* in $\mathbf{H} \backslash (\bigcup_{b \in B} X_b)$. Let us denote with $\tilde{A}_d$ the set of elements $\tilde{a}$ of $\tilde{A}$ such that case (2) applies. We define $\overline{A}$ by removing from $\tilde{A}$ all elements $\tilde{a}$ of $\tilde{A}_d$ and replacing them by the elements of $H_{\tilde{a}}$ (thus expressing the atomization of $\mathbf{H}_{\tilde{a}}$), for all $\tilde{a} \in \tilde{A}_{d'}$, i.e. $\overline{A} := (\tilde{A} \backslash \tilde{A}_d) + \cup_{\tilde{a} \in \tilde{A}_d} H_{\tilde{a}}$. The whole situation determines a partition $\overline{A} = \overline{A}_1 \cup \ldots \cup \overline{A}_{n_B}$, and $n_B$ preteams $\tau_i = (\{\mathbf{H}_{\overline{a}} \mid \overline{a} \in \overline{A}_i\}, \mathbf{H}_i)$, where $\mathbf{H}_{\overline{a}}$ is defined on the new elements $\overline{x} \in \cup_{\tilde{a} \in \tilde{A}_d} H_{\tilde{a}}$ as $\mathbf{H}_{\overline{x}} = \{\{\overline{x}\}\}$. We still use the notation $\tau, \bigcup\limits_{b \in B} X_b \rightsquigarrow \tau_1, \ldots, \tau_{n_B}$.

The definition of clan is unchanged, except that a clan now consists of quasi-strict teams and not of strict teams. The definition of the product is adapted as follows. We assign a construct $C_{\overline{a}}$ of $\mathbf{H}_{\overline{a}}$ for all $\overline{a} \in \overline{A}$, via the following adjustment with respect to the strict case: if $\overline{x}$ is an element of $H_{\tilde{a}}$ for some $\tilde{a} \in \tilde{A}_d$, then we set $C_{\overline{x}} = \{\overline{x}\}$, and we finish as in the strict case: the assignment determines delegations $\delta_i^B$ $(1 \leqslant i \leqslant n_B)$, and we define the product exactly as in (4.3), but setting $q = -1$ (see below).

We can still define a function $\varphi_\tau^B$ from $\tilde{A} \backslash \tilde{A}_d$ to $\{1, \ldots, n_B\}$, which we prefer to see as a partial function from $\tilde{A}$ to $\{1, \ldots, n_B\}$. Abusing notation, we can still write (cf. (4.2)) $\delta_i^B = (\{C_{\tilde{a}} : \mathbf{H}_{\tilde{a}} \mid \tilde{a} \in \tilde{A}$ and $\varphi_\tau^B(\tilde{a}) = i\}, \mathbf{H}_i^B)$, noticing that the participating hypergraphs of $\tau_i$ that are not the hypergraphs $\mathbf{H}_{\tilde{a}}$ with $\tilde{a} \in (\varphi_\tau^B)^{-1}(i)$ are all singleton graphs, so that the sloppy notation above extends in a unique way to the "true" definition of $\delta_i^B$.

Note however that our abuse of notation is not as innocent as it seems, since the convention relies on the fact that a singleton hypergraph $\{\{a\}\}$ admits a unique *plain* construct $a$. But the same hypergraph admits all $\lambda a$ ($\lambda \in \Bbbk$) as *linear* constructs – a fact that is stressed in the following remark.

Remark 4.27. It follows from the definitions that if $\delta_1$ and $\delta_2$ are delegations of plain constructs having the same support $\tau = (\{\mathbf{H}_a \mid a \in A\}, \mathbf{H})$, if $\delta_1$ and $\delta_2$ differ only on one participating hypergraph $\mathbf{H}_{a_0}$, if $B$ is a non-empty subset of $A$ such that $a_0 \notin B$ and $\varphi_\tau^B(a_0)$ is undefined, then $*(\delta_1) = *(\delta_2)$. Moreover, if $\delta$ is a (linear) delegation which coincides with $\delta_1$ and $\delta_2$ on all $a \in A \backslash \{a_0\}$ and has in position $a_0$ a linear construct $\sum_{i \in I} \lambda_i C_i$, then we have $*(\delta) = (\sum_{i \in I} \lambda_i) *(\delta_1)$ $(= (\sum_{i \in I} \lambda_i) *(\delta_2))$.

The notion of associative clan is unchanged. The associativity theorem still holds, but only under the assumption $q = -1$. The reason for this restriction stems from Remark 4.27 and from the following lemma.

---

[(2)]In turn, this observation relies on the composability of restrictions, i.e. one can prove that $(V_{\restriction \mathbf{H}_{12}})_{\restriction \mathbf{H_1}} = V_{\restriction \mathbf{H_1}}$.

Lemma 4.28. *If $q{=}{-}1$, then, for any delegation (in the quasi-strict setting) $\delta$ of plain constructs, the sum of all coefficients in the expansion of $*(\delta)$ as a linear combination of plain constructs is equal to* 1.

*Proof.* We prove the statement by induction on $|H|$. From the binomial expansion $(1+x)^n = \sum_{0\leqslant i\leqslant n}\binom{n}{i}x^i$ expressed as $(1+x)^n = 1+x(\sum_{1\leqslant i\leqslant n}\binom{n}{i}x^{i-1})$ and instantiated with $x = q = -1$, we readily obtain $\sum_{1\leqslant i\leqslant n}\binom{n}{i}q^{i-1} = 1$. The statement will then follow if we prove that, for each $\varnothing \subset B \subseteq A$, the sum of the coefficients in the expansion of $X_B(*(\delta_1),\ldots,*(\delta_{n_B}))$ as a linear combination of plain constructs is equal to 1. But this in turn follows by induction and by multilinearity. ☐

Theorem 4.29. *Theorem* 4.17 *extends to the quasi-strict setting for $q=-1$.*

*Proof.* Using the convention above of still defining the product by appealing to the functions $\varphi^B_\tau$, the proof of Theorem 4.17 goes through, as long as we do not use the totality of these functions. More precisely, the reasoning in case (1) unfolds without change until the equalities (4.6) included, which still hold but have now to be understood in the partial sense, i.e. the left-hand side is defined if and only if the right-hand side is defined, in which case they are equal.

Then two subcases arise.

(1a) If $\varphi^{B''}_\tau(a_0)$ is defined, then we conclude case (1) by induction as in the proof of Theorem 4.17.

(1b) Suppose (new case!) that $\varphi^{B''}_\tau(a_0)$ is undefined. Let $*^{\tau'}(\delta') = \sum_{i\in I}\lambda_i C_i$. By Lemma 4.28, we have $\sum_{i\in I}\lambda_i = 1$. Let $\delta'_i$ be the delegation obtained by replacing $*(\delta')$ by $C_i$ in $\delta$. By Remark 4.27, we have $*_{B''}(\delta'_i) = *_{B''}(\delta'_j)$ for all $i,j$, and, calling $D$ the common value, we have:

$$*_{B''}(\delta) = (\sum_{i\in I}\lambda_i)D = D.$$

On the other hand, by (4.6), we also have that $\varphi^{B''}_{\tau''}(a_0,a')$ is undefined (for all $a'\in A'$), and, again, $*^{\tau''}(\delta'')$ does not depend on the constructs $C_{(a_0,a')}$. Moreover, observing that $\delta$ and $\delta''$ coincide on the indices $a\in A\backslash\{a_0\}$, we get easily that $*_{B''}(\delta'')$ is also equal to the common value $D$, which concludes this new case in the proof of associativity.

Similarly, the reasoning in case (2) unfolds without change until the equalities (4.9) included, which again hold in the partial sense explained above. Let us repeat here the expressions for $*((\delta'')^{B''}_i)$ and for $*(\delta^B_i)$ that we wrote at this point of the proof of Theorem 4.17:

$$\begin{aligned} *((\delta'')^{B''}_i) &= *(\underbrace{\ldots,C_y,\ldots}_{y\in\tilde{A}''\cap\tilde{A},\varphi^{B''}_{\tau''}(y)=i},\ldots,\underbrace{\ldots,C_x,\ldots}_{x\in\tilde{A}',\,\varphi^{B'}_{\tau'}(x)=j\in(\tilde{\varphi}^B_\tau)^{-1}(i)},\ldots)\\ *(\delta^B_i) &= *(\underbrace{\ldots,C_y,\ldots}_{y\in\widetilde{A''}\cap\tilde{A},\varphi^{B}_{\tau}(y)=i},\ldots,\underbrace{*(\ldots,C_x,\ldots)}_{x\in\tilde{A}',\,\varphi^{B'}_{\tau'}(x)=j\in(\tilde{\varphi}^B_\tau)^{-1}(i)},\ldots)\end{aligned}$$

The first expression is still correct, as it displays (with $i$ varying) all elements $y$ and $x$ in the domain of definition $\varphi^{B''}_{\tau''}$, and all constructs involved (the ones appearing explicitly and the ones that have been dissolved) are plain constructs. The same remarks apply to the second expression, except for the fact that some dissolved constructs are not plain. Indeed, we have to look at the situations $\underbrace{\ldots,C_x,\ldots}_{x\in\tilde{A}',\,\varphi^{B'}_{\tau'}(x)=j}$, where $\tilde{\varphi}^B_\tau(j)$ is undefined. Then, by (4.9), we have that also $\varphi^{B''}_{\tau''}(x)$ is undefined for all $x\in(\varphi^{B'}_{\tau'})^{-1}(j)$, and the corresponding $C_x$ (which are plain, as stressed above) are dissolved in $*_{B''}(\delta'')$ and hence do not make their way into $(\delta'')_i^{B''}$. On the other hand, the linear constructs $*(\ldots, C_x, \ldots)$ (where $x$ ranges over $(\varphi_{\tau'}^{B'})^{-1}(j)$ for some $j$ not in the domain of definition of $\varphi_\tau^B$) appear in $*_{B'}(\delta')$, and are also dissolved. It follows that the same as what we argued about the first expression can be argued about the second one, except for the "trace" left by the constructs $*(\ldots, C_x, \ldots)$ not being plain constructs, which is taken care of by reasoning as in case (1b). Thus also the second expression is still correct, and the proof of Theorem 4.17 goes through to the end without change. □

We finish with examples of quasi-strict clans that are not strict.

Example 4.30. The universe formed by all simplices $\mathbf{S}^X$ (for a finite set $X$) gives rise to the quasi-strict clan formed by all preteams of the form $(\{\mathbf{S}^{X_a} \mid a \in A\}, \mathbf{S}^{\bigcup X_a})$ (for mutually disjoint $X_a$). That this clan is not strict is easily checked: given a delegation of constructs $C_a$ and $B \subsetneq A$, all constructs $C_a$ for $a \in A\backslash B$ are dissolved. The product instantiates as:

$$*(Y_1(\ldots), Y_2(\ldots), \ldots, Y_n(\ldots)) = \sum_{\varnothing \neq J \subseteq [n]} (\bigcup_{j\in J} Y_b)(\ldots),$$

where $(\ldots)$ is a shortcut for a tuple of singletons. We use this example to illustrate the need to choose $q = -1$ in the quasi-strict setting. Take $A = \{a_1, a_2, a_3\}$ and $Y_i \subseteq X_{a_i}$. Then, identifying constructs $Z(\ldots, z, \ldots)$ with their root $Z$, we have

$$*_{Y_1}(Y_1, Y_2, Y_3) = Y_1.$$

On the other hand, we have

$$*_{Y_1}(Y_1, *(Y_2, Y_3)) = *_{Y_1}(Y_1, Y_2) + q *_{Y_1} (Y_1, Y_2 \cup Y_3) + *_{Y_1}(Y_1, Y_3) = Y_1 + qY_1 + Y_1.$$

Therefore, the two expressions match if and only if $q = -1$. △

Example 4.31. One checks easily that the universe formed by all hypercubes $\mathbf{C}^X$ ($X = \{x_1 < \cdots < x_n\}$) is ordered, and gives rise to the quasi-strict clan formed by all preteams of the form $(\{\mathbf{C}^{X_1}, \ldots, \mathbf{C}^{X_n}\}, \mathbf{C}^{\bigcup X_i})$, where $\bigcup_{1\leqslant i\leqslant n} X_i$ is endowed with the order in which $X_1, \ldots, X_n$ form successive intervals. To illustrate the quasi-strictness, take the team $(\{\mathbf{C}^{\{x_1<x_2\}}, \mathbf{C}^{\{x_3<x_4\}}\}, \mathbf{C}^{\{x_1<x_2<x_3<x_4\}})$, and remove $x_1$. Then all of $\mathbf{C}^{\{x_3<x_4\}}$ is dissolved in $\mathbf{C}^{\{x_1<x_2<x_3<x_4\}}\backslash\{x_1\} = \mathbf{S}^{\{x_2,x_3,x_4\}}$.

In the notation introduced at the end of §3, the tridendriform structure instantiates as follows ($|v|$ stands for the length of $v$):

$$\begin{aligned} u \prec v &= u\,(-^{|v|}) \\ u \cdot (v_1 + v_2) &= u\,(-^{|v_1|}) \bullet v_2 \\ u \succ (v_1 + v_2) &= \begin{cases} (u * v_1) + v_2 & (v_1 \neq \epsilon) \\ u + v_2 & (v_1 = \epsilon) \end{cases}. \end{aligned}$$

△

As a last example in this subsection, we describe the $(-1)$-tridendriform products for erosohedra.

Example 4.32. Let us first recall that the family of erosohedra is given by:

$$\mathbf{E}^X = \{\{x_j \mid j \neq i\} \mid 1 \leqslant i \leqslant n\},$$

where $X = \{x_1, \ldots, x_n\}$, and that the constructs of $\mathbf{E}^X$ are of two shapes:

- $Y(z_1, \ldots, z_k)$, where $Y$ is a subset of $X$ of size at least 2, and
- $x(Y(z_1, \ldots, z_k))$, where $Y$ is an arbitrary subset of $X$.

Note that in the second case, $Y(z_1,\ldots,z_k)$ is a construct of a simplex, not of an erosohedron. Therefore, we take as universe the union of the families of erosohedra and of simplices. Note also that if we order our sets $X$, then we get an ordered universe (and the same is a fortiori true for the subuniverse of simplices). The products on two constructs $S$ and $T$ are given by:

$$Y(z_1,\ldots,z_k) \prec T = Y(\ldots)$$
$$x(Y(z_1,\ldots,z_k)) \prec T = x(Y(\ldots)) + x(\mathrm{root}(T)(\ldots)) - x((Y \cup \mathrm{root}(T))(\ldots))$$

$$S \succ Y(z_1,\ldots,z_k) = Y(\ldots)$$
$$S \succ x(Y(z_1,\ldots,z_k)) = x(Y(\ldots)) + x(\mathrm{root}(S)(\ldots)) - x((Y \cup \mathrm{root}(S))(\ldots))$$

$$S \cdot T = (\mathrm{root}(S) \cup \mathrm{root}(T))(\ldots),$$

where $(\ldots)$ stands for $(y_1,\ldots,y_p)$, where in turn $\{y_i\}_{i=1}^p$ is the set of elements of $X$ not appearing elsewhere in the construct. △

## 5. Further work

Work in progress. In [5], building on the material of §4.4, we extend the present work in two directions.

- We have been able to further extend our framework, so as to include in particular cyclohedra, defined as (for $X = \{x_1 < \ldots < x_n < x_1\}$)

  $$\mathbf{O}_X = \{\{x_1\},\ldots,\{x_n\},\{x_1,x_2\},\ldots,\{x_{n-1},x_n\},\{x_n,x_1\},\{x_1,\ldots,x_n\}\}.$$

  (For this example, we can indeed define an ad hoc shuffle product, based on the shuffle product for associahedra – note that for $\varnothing \neq Y \subseteq X$, the connected components of $\mathbf{O}_X \backslash Y$ are associahedra.)

  For this purpose, we relax the quasi-strictness condition to yet a weaker one, asking only (in the notation of the previous sections) that, for each $\tilde{a} \in \tilde{A}$ and each $i \in \{1,\ldots,n_B\}$, $H_{\tilde{a}} \cap H_i$ is a tube of $\mathbf{H}_{\tilde{a}}$. It is easy to see that our definition of product adapts to this *semi-strict* setting, as we call it, using in a crucial way the "technology" of restrictions of constructs defined in §4.4. We have shown that the polydendriform equation still holds in this setting.
- In [6], the first author and Guillaume Laplante-Anfossi have introduced a very natural relation on all constructs of a given hypergraph polytope (endowed with a total order order on its vertices). This relation coincides with the immediate subface relation (cf. Figure 2, and using its notation) when the edge between $X$ and $Y$ that is contracted is such that $\min Y > \max X$, while the "other half" of the relation is given by the *reverted* immediate face inclusion when $\min X > \max Y$. In turn, in [5], we prove that the transitive closure of this relation admits no cycle and therefore yields a partial order, and, in the strict setting and using and extending Proposition 4.24, we provide an equivalent definition of our products $*_B$ in terms of summations over suitable intervals in this order, generalising and unifying results from [18] and [20].

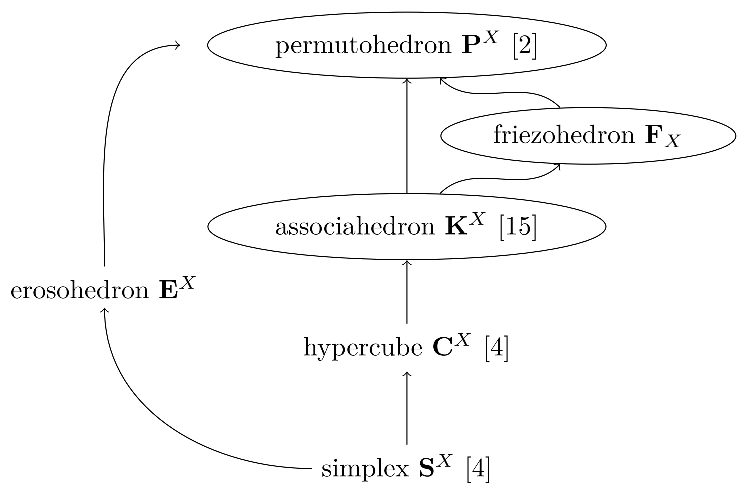
Directions for future work. We already mentioned the task of finding a nice combinatorial interpretation of the constructs of friezohedra. Here are some other questions we would like to address.

- Do there exist non-recursive definitions of the shuffle product in the quasi-strict (and in the above semi-strict) setting?
- The tridendriform algebras in our examples often satisfy more equations than the tridendrifom ones. Can we make a landscape of the corresponding operad structures?

- Hopf algebra structures are known for associahedra and permutohedra, see [2, 14]. Can we find sufficient conditions for such structures to exist on a family of polytopes?
- We also seek comparison results, in the spirit of [14, 18]: given two (families of) hypergraphs, one pointwise included in the other, what are the relations between the associated polytopes and between the associated algebras?

## Bestiary of examples

The examples emphasized in this paper are summed up in the following diagram, where we draw an arrow from $A$ to $B$ if $B$ is "more truncated" than $A$, i.e. if the connected subsets of the hypergraph generating $A$ are connected in the hypergraph generating $B$. The strict clans are circled.




## References

[1] Marcelo Aguiar and Federico Ardila, *Hopf monoids and generalized permutahedra*, Mem. Amer. Math. Soc. **289** (2023), no. 1437, vi+119.

[2] Emily Burgunder and María Ronco, *Tridendriform structure on combinatorial Hopf algebras*, J. Algebra **324** (2010), no. 10, 2860–2883.

[3] Michael P. Carr and Satyan L. Devadoss, *Coxeter complexes and graph-associahedra*, Topology Appl. **153** (2006), no. 12, 2155–2168.

[4] Frédéric Chapoton, *Algèbres de Hopf des permutahèdres, associahèdres et hypercubes*, Adv. Math. **150** (2000), no. 2, 264–275.

[5] Pierre-Louis Curien, Bérénice Delcroix-Oger, and Jovana Obradović, *Restriction, shuffles and order in nestohedra*, manuscript in progress, 2024.

[6] Pierre-Louis Curien and Guillaume Laplante-Anfossi, *Term rewriting on nestohedra*, 2024, `https://arxiv.org/abs/arXiv:2403.15987`.

[7] Pierre-Louis Curien, Jovana Obradović, and Jelena Ivanović, *Syntactic aspects of hypergraph polytopes*, J. Homotopy Relat. Struct. **14** (2019), no. 1, 235–279.

[8] Kosta Došen and Zoran Petrić, *Hypergraph polytopes*, Topology Appl. **158** (2011), no. 12, 1405–1444.

[9] Vladimir Dotsenko, Adam Keilthy, and Denis Lyskov, *Reconnectads*, Algebr. Comb. **7** (2024), no. 3, 801–842.

[10] Eva Maria Feichtner and Bernd Sturmfels, *Matroid polytopes, nested sets and Bergman fans*, Port. Math. (N.S.) **62** (2005), no. 4, 437–468.

[11] Stefan Forcey and María Ronco, *Algebraic structures on graph associahedra*, J. Lond. Math. Soc. (2) **106** (2022), no. 2, 1189–1231.

[12] Samuele Giraudo, *Pluriassociative algebras II: The polydendriform operad and related operads*, Adv. in Appl. Math. **77** (2016), 43–85.

[13] Jean-Louis Loday and María Ronco, *Trialgebras and families of polytopes*, in Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic $K$-theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 369–398.
[14] Jean-Louis Loday and María O. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. **139** (1998), no. 2, 293–309.
[15] Jean-Louis Loday and María O. Ronco, *Order structure on the algebra of permutations and of planar binary trees*, J. Algebr. Comb. **15** (2002), no. 3, 253–270.
[16] Claudia Malvenuto and Christophe Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), no. 3, 967–982.
[17] Jean-Christophe Novelli and Jean-Yves Thibon, *Polynomial realizations of some trialgebras*, 18th Formal Power Series and Algebraic Combinatorics (FPSAC'06), no. 1, 2006, pp. 243–254, `https://hal.science/hal-00622743`.
[18] Patricia Palacios and María O. Ronco, *Weak Bruhat order on the set of faces of the permutohedron and the associahedron*, J. Algebra **299** (2006), no. 2, 648–678.
[19] Alexander Postnikov, *Permutohedra, associahedra, and beyond*, Int. Math. Res. Not. **2009** (2009), no. 6, 1026–1106.
[20] María Ronco, *Generalized Tamari order*, in Associahedra, Tamari lattices and related structures, Progr. Math., vol. 299, Birkhäuser/Springer, Basel, 2012, pp. 339–350.
[21] Andrei Zelevinsky, *Nested complexes and their polyhedral realizations*, Pure Appl. Math. Q. **2** (2006), no. 3, 655–671.

Pierre-Louis Curien, Université Paris Cité, CNRS, Inria, IRIF, Picube project-team, F-75006, Paris, France
*E-mail :* `curien@irif.fr`

Bérénice Delcroix-Oger, Université de Montpellier, CNRS, IMAG, F-34090, Montpellier, France
*E-mail :* `berenice.delcroix-oger@umontpellier.fr`

Jovana Obradović, Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia
*E-mail :* `jovana@mi.sanu.ac.rs`