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Abstract We study the behavior of h-vectors associated to matroid complexes under weak maps, or inclusions of matroid polytopes. Specifically, we show that the h-vector of the order complex of the lattice of flats of a matroid is component-wise non-increasing under a weak map. This result extends to the flag h-vector. We note that the analogous result also holds for independence complexes and rank-preserving weak maps.

1. Introduction

The study of matroids and their invariants has undergone remarkable developments in recent years. In particular, many long-standing conjectures such as the Heron–Rota–Welsh conjecture [1] and the Dowling–Wilson top-heavy conjecture [5] have been resolved through the development of powerful techniques. These conjectures concern inequalities that are satisfied between certain invariants, such as the number of flats of a given rank, associated to a given matroid.

In this paper we take a different perspective and consider inequalities between invariants of *different* matroids. The set of all matroids admits a natural partial order whose relations are *weak maps*. Intuitively, if A and B are matroids and $A \to B$ is a weak map, then A is obtained from B by perturbing B to a more general position. (In terms of matroid polytopes, weak maps correspond to reverse inclusions of independence polytopes, and reverse inclusions of base polytopes if the matroids have the same rank.) Weak maps can be very complicated, even for realizable matroids: For example, weak maps of realizable matroids cannot always be realized as continuous deformations of vector configurations or as cells in a matroid subdivision. See for example [15].

In [9], Lucas gives many inequalities of matroid invariants under weak maps. It is obvious that some invariants, such as the number of independent sets of given rank and the number of flats of given rank, are non-increasing under a weak map. Less obvious is what happens to the *h-numbers* corresponding to these invariants. Given a vector of numbers called an f-vector, the h-vector is the image of the f-vector under a certain linear transformation. If the f-vector is the face vector of a simplicial complex, then the h-vector gives the numerator of the Hilbert-Poincaré series of the *Stanley-Reisner ring* of the complex.

Here, we focus on two complexes in particular: the *order complex of the lattice of flats*, also known as the *Bergman complex* of a matroid and the *independence complex*

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of a matroid. The *lattice of flats* of a matroid A is the poset whose elements are the flats of the matroid partially ordered by containment. A lattice isomorphic to the lattice of flats of some matroid is also called a *geometric lattice*. The *order complex* of this poset is the simplicial complex whose simplices are the chains of the poset. We denote this complex by $\Delta(A)$. This complex is isomorphic to that of the cones of the Bergman fan of the matroid, therefore we call it the matroid's *Bergman complex*. The *independence complex* of a matroid is the simplicial complex whose simplices are the independent sets of the matroid. We denote this complex by $\Delta_I(A)$. It is well-known that $\Delta(A)$ and $\Delta_I(A)$ are both shellable, and therefore Cohen-Macaulay.

Our main result is the following.

Theorem 1.1. *Let A and B be matroids and $A \to B$ a weak map. The following are true.*

(1) *The h-vector of $\Delta(A)$ is component-wise at least the h-vector of $\Delta(B)$.*
(2) *If A and B have the same rank, then the h-vector of $\Delta_I(A)$ is component-wise at least the h-vector of $\Delta_I(B)$.*

We note that (2) is an immediate consequence of Stanley's [13] monotonicity theorem on injections of simplicial complexes. Therefore the paper is mainly devoted to proving (1). We observe that a weak map of matroids induces a surjection of the corresponding geometric lattices, but this surjectivity alone is not enough to imply the result for general lattices (or even geometric lattices), so (1) is a special property of geometric lattices and weak maps.

Our result for (1) is actually finer, and holds for *flag h-vectors*. The flag h-vector of a graded poset with maximum chain cardinality r is a certain vector ($h_S : S \subseteq \{1, \ldots, r\}$) with the property that $\sum_{|S|=k} h_S$ is equal to h_k of the order complex of the poset. We prove the following:

Theorem 1.2. *Let A and B be matroids and let $A \to B$ a weak map. Then $h_S(\Delta(A)) \geqslant h_S(\Delta(B))$ for all S where $h_S(B)$ is defined.*

These results can be interpreted in terms of valuative invariants of matroids. The (flag) f- and h-vectors associated to the Bergman complex and independence complex of a matroid are known to be valuative invariants of the matroid (the fact that the flag f-vector of the lattice of flats is valuative was recently proven in [8]). Our results can be interpreted as saying that these invariants are monotonic with respect to inclusion of matroid polytopes. For the flag f-vector of the lattice of flats, this monotonicity was conjectured in [7]. The monotonicity of the flag f-vector was also proven independently by Elias et al. in as part of their forthcoming work on categorical valuative invariants of matroids [6]. (Our main result, the monotonicity of h-vectors, is stronger than monotonicity of f-vectors. However, a standalone proof for f-vectors can be found in Proposition 4.6.) In [7], it was also conjectured that the coefficients of the Kazhdan-Lusztig polynomials are monotonic.

Our work is inspired by previous work of Nyman and Swartz [10], where they find the component-wise maximizers and minimizers of the flag h-vector of $\Delta(A)$ over all matroids of fixed rank and size. In particular, the flag h-vector is maximized by the uniform matroid and minimized by the near-pencil matroid. All matroids have a weak map from a uniform matroid of the same size and rank, so our result recovers their maximizer. On the other hand, not all matroids have a weak map to the near-pencil matroid of the same size and rank, and in general the set of minimal matroids of given size and rank with respect to the weak map order is not well-understood.

The proof idea is as follows. Given a weak map of matroids $A \to B$, we construct a degree-preserving map from the Stanley–Reisner ring of B to a certain quotient of the Stanley–Reisner ring of A. This map is readily seen to be injective, but it is much harder to show that the map remains injective after quotienting by a linear system of parameters. We do this in an indirect way, by showing that the dual map between the corresponding dual vector spaces is surjective.

2. Matroid preliminaries

In this section we establish terminology and notation. We will assume the reader is already familiar with the basic properties of matroids and refer to [11] for further background.

Definition 2.1. *A matroid A is a (finite) ground set E together with a collection $\mathcal{I}(A)$ of subsets called* independent sets. *They have the following properties:*

(1) *A subset of an independent set is independent.*
(2) *Given two independent sets with $|I| < |J|$, there is some $x \in J \setminus I$ such that $x \cup I$ is also independent.*
(3) *$\varnothing$ is independent.*

In this paper we assume all matroids have the same ground set $[n] = \{1, \dots, n\}$.
We next define flats of a matroid:

Definition 2.2. *A* flat *of a matroid A is $F \subseteq E$ such that if I is an independent subset of F and $x \in E \setminus F$, then $I \cup \{x\}$ is independent.*

Write $\mathcal{F}(A)$ for the set of flats of A. Note that $\mathcal{I}(A)$ and $\mathcal{F}(A)$ are both posets ordered by inclusion. $\mathcal{F}(A)$ is a lattice called "the lattice of flats of A".

Proposition 2.3. *Flats have the following properties:*

(1) *An intersection of two flats is a flat.*
(2) *Given a flat F and $x \in E \setminus F$, there is a unique flat G containing x that covers F in the poset $\mathcal{F}(A)$.*
(3) *E is a flat.*

These properties may in fact be used to define a matroid:

Proposition 2.4. *Let $\mathcal{F}$ be a collection of subsets of E.*

(1) *$\mathcal{F}$ is the set of flats of some matroid A if and only if it satisfies the above three properties.*
(2) *In that case, $I \subseteq E$ is in $\mathcal{I}(A)$ if and only if for any $J \subsetneq I$, there exists $F \in \mathcal{F}$ such that $J \subseteq F$ but $I \nsubseteq F$.*

Definition 2.5. *The* rank *of a set $G \subseteq E$ with respect to a matroid A is the size of the largest independent set it contains, or, equivalently, that of the smallest flat containing it. We denote the rank of G by $\mathrm{rk}_A(G)$, or $\mathrm{rk}(G)$ if A is understood. The rank of the matroid A is defined to be $\mathrm{rk}_A(E)$. An independent set of size $\mathrm{rk}_A(E)$ is called a* basis *of A.*

Note that flats can be characterized as sets that are maximal (with respect to containment) within their rank, while independent sets are minimal within their rank. A maximal independent subset of a flat F is a *basis for F*.

Definition 2.6. *The* closure map $\mathrm{cl}_A : \mathcal{P}(E) \to \mathcal{F}(A)$ *(where $\mathcal{P}(E)$ is the power set of E) is defined so that $\mathrm{cl}_A(G)$ is the smallest flat of A containing G.*

The subscript of cl_A may be omitted when A is understood. If $\mathrm{cl}_A(G) = F$, then we say that F is the *closure* (or "A-closure") of G, or that G *spans* F.

The map cl preserves containment: If $G \subseteq G'$, then $\mathrm{cl}(G) \subseteq \mathrm{cl}(G')$.

Example 2.7. In this and all future "Example" sections of this paper, let A and B denote two specific matroids with rank 3 and ground set $E = [5]$. A will be $U_{3,5}$, the *uniform matroid*, where $\mathcal{I}(A)$ consists of all sets with $|I| \leqslant 3$, and B has bases $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{2,3,4\}$, $\{2,3,5\}$. (Since a set is independent if and only if it is a subset of a basis, this determines all independent sets.) The flats of B are $\varnothing, \{1\}, \{2\}, \{3\}, \{4,5\}, \{1,2,3\}, \{1,4,5\}, \{2,4,5\}, \{3,4,5\}$, and E.

Definition 2.8. *The* order complex *of a poset is the simplicial complex whose faces are the chains of the poset.*

Let 0_A denote the minimal flat of a matroid A. (In other words, 0_A is the set of all loops of A, and is $\varnothing$ if A is loopless.) We will write $\Delta(A)$ for the order complex of $\mathcal{F}(A) \smallsetminus \{0_A, E\}$, henceforth known as the matroid's *Bergman complex*.

3. The f- and h-vectors

The usual f- and h-vectors for a simplicial complex are defined as follows:

Definition 3.1. *Let Δ be a simplicial complex of dimension $r-1$.*

(1) *The* f-vector *of Δ is the sequence $(f_i(\Delta))_{i=0}^r$, where $f_i(\Delta)$ is the number of faces with cardinality i.*[(1)]
(2) *The* h-vector *is the sequence $(h_i(\Delta))_{i=0}^r$ satisfying*

$$\sum_{i=0}^{r} h_i x^{r-i} = \sum_{i=0}^{r} f_i (x-1)^{r-i}.$$

We now give a refinement of the f- and h-vectors for posets. Let P be a (finite) graded poset with rank function rk. We define the *rank* of P to be the maximum cardinality of a chain.[(2)] Given a chain C in P, the *flag* of C, written $\operatorname{fl}(C)$, is the set of ranks of flats in that chain. That is, $\operatorname{fl}(C) = \{\operatorname{rk}(F)\}_{F\in C}$. This is a subset of $[r]$, where r is the rank of the poset. For our purposes, the empty set will also be considered a chain, with flag $\varnothing$.

Definition 3.2. *Let P be a graded poset of rank r.*

(1) *The* flag f-vector *of P is the tuple $(f_S(P))$ taken over all $S \in \mathcal{P}([r])$, where $f_S(P)$ is the number of chains C such that $\operatorname{fl} C = S$.*
(2) *The* flag h-vector *is the tuple $(h_S(P))$ taken over all $S \subseteq \mathcal{P}([r])$, where*

$$h_S = \sum_{T\subseteq S} (-1)^{|S|-|T|} f_T.$$

While the flag vectors are defined for posets, we will abuse notation and say that $(f_S(P))$ is the flag f-vector of the order complex $\Delta(P)$.

We write $f_S(A)$ for $f_S(\Delta(A))$, $f_k(A)$ for $f_k(\Delta(A))$, and do similarly for the h-vectors.

Proposition 3.3. *The following are true for a graded poset of rank r.*

(1) $f_k = \sum_{|S|=k} f_S$.
(2) $h_k = \sum_{|S|=k} h_S$.
(3) $f_S = \sum_{T\subseteq S} h_T$.
(4) $f_r = \sum_{S\subseteq[r]} h_S$.

---

(1) For convenience, we index the f-vector by cardinality instead of dimension. We will then modify the definition of the h-vector so that it agrees with the usual indexing of the h-vector.

(2) This differs from the usual definition of rank, which is the maximum *length* of a chain.

To avoid confusion with other notions of minimality and maximality, chains of maximal length (i.e. chains of flag $[r]$) will be called *full* in this paper.

We now focus on the case when $\Delta = \Delta(A)$ where A is a matroid. There is a useful partition of the set of full chains of $\Delta(A)$ into sets of sizes h_S, as follows: Given a full chain of flats $0_A \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq E$, let $b_i = \min F_i \smallsetminus F_{i-1}$, where elements of the ground set $[n]$ are ordered in the usual way. (Here, $F_0 = 0_A$ and $F_{r+1} = E$.) The resulting string $b_1 \dots b_{r+1}$ is called the chain's *Jordan-Hölder sequence* [2]. By property (2) of Prop 2.4, any element of $F_i \smallsetminus F_{i-1}$ determines F_i given F_{i-1}. Thus there is an injection from full chains of flats to ordered sets of size $r+1$ in $[n]$.

Remark 3.4. The elements $b_i = \min F_i \smallsetminus F_{i-1}$ over all F_i and F_{i-1} form an *EL-labeling* of the poset $\mathcal{F}(A)$. Ordering the full chains lexicographically by their Jordan-Hölder sequence gives a shelling order of $\mathcal{F}(A)$. See [2].

Note that not all such ordered sets $b_1 \dots b_{r+1}$ are Jordan-Hölder sequences of full chains of flats. First, each b_i must not be in the flat spanned by $b_1, \dots, b_{i-1}$, or, equivalently, $\{b_1, \dots, b_{r+1}\}$ must be a basis for E. However, each b_i must also be the minimal element in the uniquely determined $F_i \smallsetminus F_{i-1}$. Call an ordered basis that has this latter property, and thus corresponds to a chain of flats, “valid".

Now given a string $b_1 \dots b_{r+1}$, we say the string (or its corresponding full chain, if it has one) has a descent *across* position i (or alternatively, across the corresponding F_i) if $b_i > b_{i+1}$, and that it has an *ascent* otherwise. The set of all indices across which a string (full chain) has a descent is that string's *descent set*.

Theorem 3.5 ([14]). *Let A be a matroid of rank $r+1$.*

(1) *The set of valid strings with descent sets contained in $S \subseteq [r]$ has cardinality $f_S(M)$.*
(2) *The set of valid strings with descent sets equal to $S \subseteq [r]$ has cardinality $h_S(M)$.*

Because we will use some of the constructions from the proof later, we provide a proof of this theorem.

Proof. (1) Fix $S \subseteq [r]$. We will demonstrate a bijection between chains of flag S and full chains with descent set contained in S. First, note that any non-full chain of flats $0_A = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = E$ has a unique *minimal completion* to a full chain as follows: for each interval $[F_i, F_{i+1}]$ where $\mathrm{rk}(F_{i+1}) > \mathrm{rk}(F_i) + 1$, let $F_{i,1}$ be the flat covering F_i containing $a_{i,1} := \min(F_{i+1} \smallsetminus F_i)$. Then, inductively, let $F_{i,j+1}$ be the flat covering $F_{i,j}$ containing $a_{i,j+1} := \min(F_{i+1} \smallsetminus F_{i,j})$ for $1 \leqslant j \leqslant k_i$, where $k_i = \mathrm{rk}(F_i) - \mathrm{rk}(F_{i+1}) - 1$.

Now consider the full chain

$$0_A \subsetneq F_{0,1} \subsetneq \cdots \subsetneq F_{0,k_0} \subsetneq F_1 \subsetneq F_{1,1} \subsetneq \cdots \subsetneq F_k = E.$$

By construction, $b_{i+1} = a_{i,1} < a_{i,2} < \cdots < a_{i,k_i} < \min(F_{i+1} \smallsetminus F_{i,k_i})$, so each new flat $F_{i,j}$ has an ascent across it. That is, the descent set of this chain is contained in S.

Denote by $\mu(C)$ the minimal completion of a chain C of flag S. Let ν be the map that restricts a full chain with descent set contained in S to the flats with ranks in S. We claim that ν is the inverse of μ. Clearly $\nu(\mu(C)) = C$. To show $\mu(\nu(C)) = C$, it suffices to check that the minimal completion is unique, in that $\mu(C)$ is the unique full chain containing C with no descents outside S. Suppose that D is some other full chain containing C, and let G_i be its flat of rank i. Let G_j be the first flat in which D differs from $\mu(C)$, and F_i, F_{i+1} the flats of C such that $\mathrm{rk}(F_i) < j < \mathrm{rk}(F_{i+1})$. Then G_{j-1} was constructed by the above interpolation process, while G_j was not. That is, $\min(F_{i+1} \smallsetminus G_{j-1}) \notin G_j$. Let $a = \min(F_{i+1} \smallsetminus G_{j-1})$, and let G_k be the first flat in D which contains a. We must have $k > j$. Then $\min(G_{k-1} \setminus G_{k-2}) > a = \min(G_k \setminus G_{k-1})$, so G_{k-1} is a flat of D with a descent across it, whose rank is not in S since $G_{k-1} \in [F_i, F_{i+1}]$ but is neither F_i nor F_{i+1}. This proves the uniqueness of $\mu(C)$.

Therefore $f_S(A)$, which counts the number of chains of flag S, also counts the number of valid strings with descent set $\subseteq S$.

(2) immediately follows from the identity $f_S = \sum_{T \subseteq S} h_T$. □

Example 3.6. Consider the matroid B defined in the previous example. Its nine full chains have the following associated strings, grouped by descent set:

$\varnothing$: 124

$\{1\}$: 214, 412, 314

$\{2\}$: 142, 241, 341

$\{1, 2\}$: 421, 431

From this we can directly read off the flag h-vectors, and obtain the rest of the f- and h-vectors by adding them. For instance, $h_1(B) = h_{\{1\}}(B) + h_{\{2\}}(B) = 6$.

A flat F is *minimal* in a (poset) interval $[G, H]$ if F is one of the flats generated by the interpolation process described in the above proof. That is, if $\mathrm{rk}(F) = \mathrm{rk}(G) + j$, then F contains the successively minimal elements $a_{i,1}, a_{i,2}, \ldots, a_{i,j}$ found in the inductive process described above for $F_i = G$, $F_{i+1} = H$. For any interval, there is exactly one minimal flat of each rank.

A chain of flats $0_A = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = E$ is *nonessential* if it contains at least one flat that is minimal with respect to its neighbors, that is, some F_i that is minimal in $[F_{i-1}, F_{i+1}]$. Otherwise, the chain is *essential*. Thus we can rephrase the above result as follows:

Proposition 3.7. *$h_S(A)$ counts the number of essential chains of flag S.*

Proof. From the proof of Thm. 3.5, a chain of flag S is essential if and only if the descent set of its unique minimal completion is S. □

Finally, note that since h_S counts the number of valid strings with descent set S, h_k counts the number of valid strings with exactly k descents.

4. Weak maps

We now define weak maps of matroids. We refer to [16, Chapter 8] for more information.

Definition 4.1. *Let A, B be two matroids on the same ground set. We say there is a* weak map *from A to B if $\mathcal{I}(B) \subseteq \mathcal{I}(A)$. We refer to this "map" as $A \to B$.*

The ranks of A and B are not assumed to be equal, although we clearly have $\mathrm{rk}(A) \geqslant \mathrm{rk}(B)$, since a basis for B is independent in A. It will often be useful to restrict to the case of rank-preserving weak maps, i.e. those where $\mathrm{rk}(A) = \mathrm{rk}(B)$.

When there is a weak map $A \to B$, we will often consider the map $\mathrm{cl}_B : \mathcal{F}(A) \to \mathcal{F}(B)$ which is the restriction of the closure map $\mathrm{cl}_B : \mathcal{P}(B) \to \mathcal{F}(B)$ from Definition 2.6. There is a natural extension of this map to chains.

Definition 4.2. *If $A \to B$ is a weak map, denote by $\mathrm{cl}_B : \Delta(A) \to \Delta(B)$ the map defined as follows. Given $C = (0_A \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E)$ a chain in A, take $(0_B \subseteq \mathrm{cl}_B(F_1) \subseteq \cdots \subseteq \mathrm{cl}_B(F_k) \subseteq E)$, then delete any duplicate flats to obtain $\mathrm{cl}_B(C)$.*

We now consider weak maps and flag vectors. We start with a special type of weak map, *truncations*. Given a matroid A of rank at least $r+1$, the *truncation* of A to rank $r+1$ is the rank $r+1$ matroid A_{r+1} with $\mathcal{I}(A_{r+1}) = \{I \in \mathcal{I}(A) : |I| \leqslant r+1\}$. Equivalently, we have $\mathcal{F}(A_{r+1}) = \{F \in \mathcal{F}(A) : \mathrm{rk}_A(F) < r+1\} \cup \{E\}$.

PROPOSITION 4.3. *Let $A \to A_{r+1}$ be a truncation. The following are true.*

(1) $h_S(A) = h_S(A_{r+1})$ *for all* $S \subseteq [r]$.
(2) $h_k(A) \geqslant h_k(A_{r+1})$ *for all* k.

Proof. (1) follows directly from the characterization of $\mathcal{F}(A_{r+1})$ and Definition 3.2. (2) follows from (1) since $h_k(A)$ is a nonnegative sum of $h_S(A)$ with $S \subseteq [\mathrm{rk}(A)-1]$ and $|S| = k$. □

We next observe that for general weak maps, it suffices to consider only rank-preserving weak maps. This follows from the previous proposition and the next easy proposition.

PROPOSITION 4.4. *Every weak map of matroids can be decomposed into a truncation followed by a rank-preserving weak map.*

We will prove one more combinatorial result. We first prove the following lemma:

LEMMA 4.5. *Let $A \to B$ be a weak map.*

(1) *If* $G \subseteq E$, *then* $\mathrm{rk}_A(G) \geqslant \mathrm{rk}_B(G)$.
(2) $\mathrm{cl}_B : \mathcal{F}(A) \to \mathcal{F}(B)$ *does not increase a flat's rank.*

Proof. (1) Let I be a B-basis for G. I is also independent in A, so the largest A-independent set contained in G is at least size $|I|$.

(2) Note that for all $G \subseteq E, \mathrm{rk}_B(G) = \mathrm{rk}_B(\mathrm{cl}_B(G))$. Substituting this into the inequality from (1) gives the result. □

We now show the following:

PROPOSITION 4.6. *Let $A \to B$ be a rank-preserving weak map. Then,*

(1) $\mathrm{cl}_B : \mathcal{F}(A) \to \mathcal{F}(B)$ *is surjective.*
(2) $\mathrm{cl}_B : \mathcal{F}(A) \to \mathcal{F}(B)$ *is surjective by rank: Given* $F \in \mathcal{F}(B)$, *there exists* $G \in \mathcal{F}(A)$ *with* $\mathrm{cl}_B(G) = F$ *and* $\mathrm{rk}(G) = \mathrm{rk}(F)$.
(3) $\mathrm{cl}_B : \Delta(A) \to \Delta(B)$ *is surjective by flag: Given* $C \in \Delta(B)$, *there exists* $D \in \Delta(A)$ *with* $\mathrm{cl}_B(D) = C$ *and* $\mathrm{fl}(D) = \mathrm{fl}(C)$.

Proof. It suffices to prove (3). Let $C \in \Delta(B)$ be of flag S, and let $b_1 \dots b_{r+1}$ be the ordered basis for its minimal completion. That is, if $k \in S$, then $F_k := \mathrm{cl}_B(\{b_1, \dots, b_k\})$ is the rank-k flat in C. Now let D be the chain of flag S whose rank-k flat is $G_k := \mathrm{cl}_A(\{b_1, \dots, b_k\})$. We know $\mathrm{cl}_B(G_k)$ is a B-flat which contains $\{b_1, \dots, b_k\}$, and therefore $\mathrm{cl}_B(G_k) \supseteq F_k$. However, by the lemma, $\mathrm{rk}_B(\mathrm{cl}_B(G_k)) \leqslant rk_A(G_k) = k$. Thus $\mathrm{cl}_B(G_k) = F_k$. □

(3) immediately implies the following:

COROLLARY 4.7. (1) *If $A \to B$ is a rank-preserving weak map with both matroids of rank $r+1$, then $f_S(A) \geqslant f_S(B)$ for all $S \subseteq [r]$.*
(2) *If $A \to B$ is a weak map, then $f_k(A) \geqslant f_k(B)$ for all k.*

We will strengthen this result in Thm. 8.8.

5. The independence complex

Before proceeding further on lattices of flats, we consider the independence complex.

Definition 5.1. *The* independence complex $\Delta_I(A)$ *of a matroid A is the simplicial complex whose faces are given by $\mathcal{I}(A)$.*

Denote the f- and h-vectors of $\Delta_I(M)$ by $h^I(A)$ and $f^I(A)$ respectively.

One immediate consequence of the definitions is that if $A \to B$ is a weak map, then the identity map provides an injection of $\mathcal{I}(B)$ into $\mathcal{I}(A)$, which in turn implies that $f^I_k(A) \geqslant f^I_k(B)$ for all k. However, we also have the following stronger result.

Proposition 5.2. *If $A \to B$ is a rank-preserving weak map, then $h^I_k(A) \geqslant h^I_k(B)$ for all k.*

For the proof of this, we use the following monotonicity theorem by Stanley:

Theorem 5.3 ([13]). *Let Δ' be a subcomplex of the simplicial complex Δ, where both are Cohen-Macaulay. Suppose that $e-1 = \dim \Delta' \leqslant \dim \Delta = d-1$, and that no set of $e+1$ vertices of Δ' form a face of Δ. Then $h_k(\Delta') \leqslant h_k(\Delta)$ for all k.*

See Section 6 for the definition of Cohen-Macaulay.

Proof of Proposition 5.2. By definition of a rank-preserving weak map, $\Delta_I(B)$ is a subcomplex of $\Delta_I(A)$ and they have the same dimension. In addition, independence complexes of matroids are shellable and thus Cohen-Macaulay [4]. Hence, the result follows from Theorem 5.3. □

The statement is not true for weak maps that change rank. For example, let A be the rank 2 uniform matroid on 2 elements and B the rank 1 uniform matroid on 2 elements. Then we have a weak map $A \to B$ but $h^I_1(A) = 0$, $h^I_1(B) = 1$. (This is also a *strong map*, for readers familiar with the term.)

Matt Larson notes that Proposition 5.2 can also be proved inductively, using a similar argument to [9].

6. The Stanley-Reisner ring

Fix an infinite field k. A simplicial complex Δ has an associated ring $k[\Delta]$, called the *Stanley-Reisner ring* of the complex: $k[\Delta] = k[x_{v_1}, \dots, x_{v_m}]/I_\Delta$, where $\{v_j\}$ is the set of vertices of the complex, and I_Δ is the ideal generated by monomials of the form $x_{v_{j_1}} \cdots x_{v_{j_k}}$, where $\{v_{j_1}, \dots, v_{j_k}\}$ is not a face of the complex. Note that $k[\Delta]$ is graded by degree, which we call the "coarse" grading.

Let P be a graded poset of rank r and Δ its order complex. Then $k[\Delta]$ has an $\mathbb{N}^r$-grading defined as follows: Let $v_1 < \cdots < v_r$ be a full chain in P and $d_1, \dots, d_r$ nonnegative integers. Then the degree of (the image of) the monomial $x^{d_1}_{v_1} \cdots x^{d_r}_{v_r}$ in $k[\Delta]$ is defined to be $(d_1, \dots, d_r)$. We call this the "fine" grading of $k[\Delta]$. The fine graded component of $k[\Delta]$ corresponding to a tuple α will be denoted $k[\Delta]_\alpha$. If $d_i = 0$ or 1 for all $1 \leqslant i \leqslant r$, then we say $x^{d_1}_{v_1} \cdots x^{d_r}_{v_r}$ has degree S, where $S = \{i \mid d_i = 1\}$, and analogously define $k[\Delta]_S$.

In the case we are interested in, where $\Delta = \Delta(A)$ for a matroid A, the polynomial ring is generated by variables indexed by the flats (aside from the empty flat and E), and I_Δ is generated by products of any two variables corresponding to incomparable flats. In particular, I_Δ includes all monomials except those that are the product of variables from a chain. We will write $k[A]$ for $k[\Delta(A)]$. Given a chain of flats C, write

$$x_C = \prod_{F \in C} x_F.$$

An arbitrary element $p \in k[A]_S$ can be expressed as $\sum_{\mathrm{fl}(C)=S} a_C x_C$ with $a_C \in k$.

Let A be a finitely generated graded k-algebra. A *system of parameters* for A is a sequence $\theta_1, \ldots, \theta_r \in A$ of minimal length such that $A/\langle\theta_1, \ldots, \theta_r\rangle$ is finite-dimensional over k. A system of parameters is *homogeneous* if all of its elements are homogeneous (with respect to the coarse or fine grading, depending on context), and *linear* if all of its elements have coarse degree 1. Assuming k is infinite, there always exists a homogeneous linear system of parameters.

A *regular sequence* in A is a sequence $\theta_1, \ldots, \theta_r \in R$ such that θ_i is not a zero-divisor in $A/\langle\theta_1, \ldots, \theta_{i-1}\rangle$ for all $1 \leqslant i \leqslant r$. We say that A is *Cohen-Macaulay* if every system of parameters of A is a regular sequence. The significance of this definition in combinatorics is the following observation:

Theorem 6.1. *Let Δ be a simplicial complex and assume $k[\Delta]$ is Cohen-Macaulay. Let $\theta_1, \ldots, \theta_r$ be a linear system of parameters and let $R = k[\Delta]/\langle\theta_1, \ldots, \theta_r\rangle$, which inherits the coarse grading from $k[\Delta]$. Then for all i, the dimension of the degree-i component of R is $h_i(\Delta)$.*

Now assume Δ is the order complex of a graded poset P of rank r. In this case, there is a particularly nice linear system of parameters for $k[\Delta]$, given by

$$\theta_i = \sum_{v \in P,\, \mathrm{rk}\, v = i} x_v$$

for $i = 1, \ldots, r$. Note that this system of parameters is homogeneous with respect to the fine grading.

Theorem 6.2 ([12]). *Let Δ be the order complex of a graded poset P and assume $k[\Delta]$ is Cohen-Macaulay. Let (θ_i) be as above, and let $R = k[\Delta]/\langle\theta_1, \ldots, \theta_r\rangle$, which inherits the fine grading from $k[\Delta]$. The following are true.*

1. *The dimension of the degree-S component of R is $h_S(P)$.*
2. *If $\alpha = (d_1, \ldots, d_r)$ and $d_i > 1$ for any i, then $(R_M)_\alpha = 0$.*

We now focus on the case where $\Delta = \Delta(M)$ for a matroid M of rank $r + 1$. Set $\theta_i = \sum_{F \in \mathcal{F},\, \mathrm{rk}\, F = i} x_F$ as above, and let Θ_A to be the ideal generated by the θ_i over $i \in [r]$. (We may drop the subscript A if it is clear.) Define $R_A = k[A]/\Theta_A$. By the above theorem, $\dim(R_A)_S = h_S(A)$. In fact, the images of the monomials corresponding to essential chains of flag S form a basis for $(R_A)_S$.

Example 6.3. Let B again be the matroid with $r = 2$, $n = 5$ that appears in the previous examples. Then $k[B]$ has four rank-$\{1\}$ generators $x_{\{1\}}, x_{\{2\}}, x_{\{3\}}, x_{\{4,5\}}$, and four rank-$\{2\}$ generators $x_{\{1,2,3\}}, x_{\{1,4,5\}}, x_{\{2,4,5\}}, x_{\{3,4,5\}}$, with relations given by all incomparable pairs of flats, such as $x_{\{1\}}x_{\{2\}}$ and $x_{\{1\}}x_{\{2,4,5\}}$. To get R_B, we quotient out by the ideal $(x_{\{1\}} + x_{\{2\}} + x_{\{3\}} + x_{\{4,5\}}, x_{\{1,2,3\}} + x_{\{1,4,5\}} + x_{\{2,4,5\}} + x_{\{3,4,5\}})$, resulting in a finite-dimensional algebra whose graded components have dimensions 1, 3, 3, and 2 respectively. For example, the degree-$\{1, 2\}$ component is spanned by the image of $\{x_{\{4,5\}}x_{\{2,4,5\}}, x_{\{4,5\}}x_{\{3,4,5\}}\}$.

We make one more definition before moving on.

Definition 6.4. *Let A be a matroid.*

1. *The* lexicographic order *on rank k flats in $\mathcal{F}(A)$ is defined as follows: given two flats $F \neq G$, let j be the first element of the ground set $[n]$ contained in one of F, G but not the other; if $j \in F$ but $j \notin G$, we say that $F < G$.*
2. *The* lexicographic order *on flag S chains in $\Delta(A)$ is defined as follows: given two chains $C = \{F_i\}$, $C' = \{G_i\}$, let k be the lowest rank such that $F_k \neq G_k$; if $F_k < G_k$, we say that $C < C'$.*

Note that this order is consistent with the notion of minimality in Section 3: if two flats of the same rank G, G' are both contained in the interval $[F, H]$, and G is minimal with respect to that interval, then $G \leqslant G'$.

7. Matroid maps and ring maps

Let Δ, Δ' be simplicial complexes, and let $\mathrm{cl} : \Delta \to \Delta'$ be a map of complexes (that is, $f(\sigma) \leqslant f(\tau)$ for all σ, $\tau \in \Delta$ such that $\sigma \leqslant \tau$). Then we have a map $\psi : k[\Delta'] \to k[\Delta]$ defined by

$$\psi(x_\sigma) = \sum_{\tau \in \mathrm{cl}^{-1}(\sigma)} x_\tau \tag{1}$$

for all $\sigma \in \Delta$. (Here, $x_\sigma = \prod_{v \in \sigma} x_v$.) It is straightforward to check that this gives a well-defined homomorphism $k[\Delta'] \to k[\Delta]$. Moreover, we have the following.

Proposition 7.1. cl *is surjective if and only if* ψ *is injective.*

Proof. It is clear from the definition that ψ is injective if and only if $\psi(p) \neq 0$ for all monomials $p \in k[\Delta']$. This is easily seen to be equivalent to cl being surjective. □

Given a rank-preserving weak map of matroids $A \to B$, Prop. 4.6 says we have a surjective map $\mathrm{cl}_B : \Delta(A) \to \Delta(B)$, and thus we have an injective map $\psi_B : k[B] \to k[A]$. However, this map does not preserve the fine grading of $k[A]$ and $k[B]$, as cl_B may decrease the rank of some flats. To rectify this, we introduce a new ring $k[A']$.

Given two matroids A, B on the same ground set $E = [n]$, we define the *auxiliary pseudo-matroid* A' to be the ground set E, together with the set $\mathcal{F}(A')$ of all flats $F \in \mathcal{F}(A)$ such that $\mathrm{rk}_B(\mathrm{cl}_B(F)) = \mathrm{rk}_A(F)$. Equivalently, a flat of A is in $\mathcal{F}(A')$ if and only if it has a basis which is independent in B. We call $\mathcal{F}(A')$ the flats of A', although A' is not necessarily a matroid.

Proposition 7.2. *If* $A \to B$ *is a rank-preserving weak map, then* $\mathcal{F}(A')$ *is graded by* rk_A .

Proof. What we need to show is if F, $F' \in \mathcal{F}(A')$ such that $F \subseteq F'$ and $\mathrm{rk}_A(F') > \mathrm{rk}_A(F) + 1$, then there exists $G \in \mathcal{F}(A')$ with $F \subsetneq G \subsetneq F'$. Now $\mathrm{cl}_B(F) \subsetneq \mathrm{cl}_B(F')$ since both flats maintain their ranks under cl_B, and this in turn implies $F' \nsubseteq \mathrm{cl}_B(F)$. Let $x \in F' \smallsetminus \mathrm{cl}_B(F)$, and $G = \mathrm{cl}_A(F \cup \{x\})$. Then $\mathrm{rk}_A(G) = \mathrm{rk}_A(F) + 1$, so $F \subsetneq G \subsetneq F'$. By Lemma 4.5, $\mathrm{rk}_B(\mathrm{cl}_B(G))$ is either $\mathrm{rk}_B(\mathrm{cl}_B(F))$ or $\mathrm{rk}_B(\mathrm{cl}_B(F)) + 1$. It cannot be the former, since then we would have $\mathrm{cl}_B(G) = \mathrm{cl}_B(F)$, but $\mathrm{cl}_B(G) \ni x \notin \mathrm{cl}_B(F)$. Thus $G \in \mathcal{F}(A')$. □

We define $\Delta(A')$ to be the order complex of $\mathcal{F}(A') \smallsetminus \{0_A, E\}$ and let $k[A']$ be the Stanley-Reisner ring of $\Delta(A')$. Since A' is graded, $k[A']$ is fine-graded in the sense of Section 6. Note that $k[A']$ is not necessarily Cohen-Macaulay. Since the restriction of the closure map $\mathrm{cl}_B : \mathcal{F}(A') \to \mathcal{F}(B)$ preserves containment, the induced map on chains $\Delta(A') \to \Delta(B)$ also preserves flag. Thus we have a homomorphism $\psi^A_B : k[B] \to k[A']$ as in (1). (This map will usually just be written ψ.) This map preserves the fine grading of $k[A']$ and $k[B]$, and by Prop. 4.6(3) it is injective.

Analogously to matroids, define $\theta_i \in k[A']$ as $\sum x_F$ taken over all $F \in \mathcal{F}(A')$ with $\mathrm{rk}(F) = i$, and let $\Theta_{A'}$ be the ideal generated by the θ_i. Define $R_{A'} = k[A']/\Theta_{A'}$, which inherits the fine grading of $k[A']$. Since $\psi^A_B(\Theta_B) \subseteq \Theta_{A'}$, ψ induces a well defined map $\bar{\psi}$ from R_B to $R_{A'}$.

Example 7.3. Let A and B be the matroids used in previous examples. Note that $A \to B$ is a rank-preserving weak map. $\mathcal{F}(A')$ consists of all flats of A except $\{4, 5\}$.

Then $\psi_B^A : k[B] \to k[A']$ is given by $\psi(x_{\{4,5\}}) = x_{\{4\}} + x_{\{5\}}, \psi(x_{\{1,2,3\}}) = x_{\{1,2\}} + x_{\{1,3\}} + x_{\{2,3\}}, \psi(x_{\{1,4,5\}}) = x_{\{1,4\}} + x_{\{1,5\}}$, etc.

The following result shows why we can work with A' instead of A.

Proposition 7.4. *If $A \to B$ is a rank-preserving weak map and A' its auxiliary pseudo-matroid, then $\dim(R_A)_S \geqslant \dim(R_{A'})_S$ for all S. In particular, $R_{A'}$ is finite-dimensional over k.*

Proof. $k[A']$ is equal to $k[A]/J$, where J is the ideal generated by all x_F such that $\mathrm{rk}(\mathrm{cl}_B(F)) \neq \mathrm{rk}(F)$. The induced map $R_A \to R_{A'}$ is fine degree-preserving and surjective, since the composition $k[A] \to k[A'] \to R_{A'}$ is surjective. ☐

Corollary 7.5. *Let $A \to B$ be a rank-preserving weak map such that $\bar{\psi} : R_B \to R_{A'}$ is injective. Then $h_S(A) \geqslant h_S(B)$ for all S.*

Proof. The hypothesis is equivalent to the statement that the restriction of $\bar{\psi}$ to degree S is injective for all S. Then by Thm. 6.2 and Prop. 7.4,

$$h_S(A) = \dim_k((R_A)_S) \geqslant \dim_k((R_{A'})_S) \geqslant \dim_k((R_B)_S) = h_S(B).$$ ☐

Thus, we have reduced the statement that $h_S(A) \geqslant h_S(B)$ for a rank-preserving weak map $A \to B$ to the following claim: Let $A \to B$ be a rank-preserving weak map of matroids. Then the map $\bar{\psi} : R_B \to R_{A'}$ is injective.

8. Proof of the main theorem

In this section we prove our main result, Thm. 1.2. As stated earlier, we prove that if $A \to B$ is a rank-preserving weak map, then $\bar{\psi}$ is injective. We do this by showing the induced map of the dual vector spaces is surjective, by finding preimages for each element of a basis.

Let A be a matroid of rank $r+1$. Let $k[A]^*$ denote the dual vector space to $k[A]$, and let $\Phi_A \subseteq k[A]^*$ be the annihilator of $\Theta_A \subseteq k[A]$. We have $\Phi_A = \bigoplus_{S \subseteq [r]} (\Phi_A)_S$, where $(\Phi_A)_S$ can be identified as the space of linear functionals on $k[A]_S$ which annihilate Θ_S.

Given a chain $C \in \Delta(A)$, let $\epsilon_C \in k[A]^*$ be the functional satisfying $\epsilon_C(x_C) = 1$ and $\epsilon_C(x_D) = 0$ for all $D \neq C$. Thus an arbitrary element of $k[A]^*_S$ can be written as $\sum_{\mathrm{fl}(C)=S} b_C \epsilon_C$ where $b_C \in k$ for all C.

Proposition 8.1. *A functional $f = \sum_{\mathrm{fl}(C)=S} b_C \epsilon_C$ lies in $(\Phi_A)_S$ if and only if for all $i \in S$ and all chains C with $\mathrm{fl}(C) = S \smallsetminus i$, we have $\sum_{D \supseteq C} b_D = 0$.*

Proof. The latter condition is satisfied if and only if f annihilates all elements of the form $\theta_i x_C$ with $\mathrm{fl}(C) = S \smallsetminus i$. Since these elements generate Θ_S, the result follows. ☐

Now let $A \to B$ be a rank-preserving weak map, where $\mathrm{rk}(A) = \mathrm{rk}(B) = r+1$. Let $\pi : k[A']^* \to k[B]^*$ be the map dual to ψ (i.e. it is defined by pre-composition with ψ). For each $S \subseteq [r]$, we can also view π as a map $k[A']^*_S \to k[B]^*_S$.

Theorem 8.2. *If $A \to B$ is a rank-preserving weak map of matroids, and $\pi : k[A']^* \to k[B]^*$, as well as the vector subspaces $\Phi_{A'}$ and Φ_B, are as defined above, then π maps $\Phi_{A'}$ surjectively onto Φ_B.*

Proof. We begin with the following observation.

Lemma 8.3. *The dimension of the degree-S component of Φ_A is $h_S(A)$.*

Proof. Recall that Φ is the subspace of $k[A]^*$ which annihilates Θ. Therefore, its dimension in degree S is $\dim_S k[A]/\Theta = \dim_S R_A = h_S(A)$. ☐

We now proceed to the main proof. To show surjectivity, it suffices to find preimages under π for the $h_S(B)$ members of a basis of Φ_B. We start with the case $S = [r]$. Let C be an essential full chain in $\Delta(B)$ with corresponding string $b_1 b_2 \dots b_{r+1}$. (By definition of essentiality, this string is completely descending.) Define

$$f_C = \sum_{\sigma \in S_{r+1}} \operatorname{sgn}(\sigma)\epsilon_{C_\sigma},$$

where S_k is the symmetric group on k elements and C_σ is the full B-chain

$$0_B \subsetneq \operatorname{cl}_B(\{b_{\sigma(1)}\}) \subsetneq \operatorname{cl}_B(\{b_{\sigma(1)}, b_{\sigma(2)}\}) \subsetneq \cdots \subsetneq \operatorname{cl}_B(\{b_{\sigma(1)}, b_{\sigma(2)}, \dots, b_{\sigma(r)}\}) \subsetneq E.$$

(This is a full chain because $b_1, \dots, b_{r+1}$ is a basis.) Similarly, define

$$g_C = \sum_{\sigma \in S_{r+1}} \operatorname{sgn}(\sigma)\epsilon_{D_\sigma},$$

where D_σ is the A-chain

$$0_A \subseteq \operatorname{cl}_A(\{b_{\sigma(1)}\}) \subseteq \operatorname{cl}_A(\{b_{\sigma(1)}, b_{\sigma(2)}\}) \subseteq \cdots \subseteq \operatorname{cl}_A(\{b_{\sigma(1)}, b_{\sigma(2)}, \dots, b_{\sigma(r)}\}) \subseteq E.$$

Note that $\{b_1, \dots, b_{r+1}\}$ remains a basis in A, so D_σ is a full chain with one flat of each rank. We also see that $\operatorname{cl}_B(D_\sigma) = C_\sigma$, since for an independent set I, $\operatorname{cl}(I)$ is the set of elements that form a dependent set when added to I, and therefore $\operatorname{cl}_A(I) \subseteq \operatorname{cl}_B(I)$. This also shows that each flat of D_σ is in $\mathcal{F}(A')$. Thus D_σ is a full A'-chain, and $\pi(\epsilon_{D_\sigma}) = \epsilon_{C_\sigma}$ and $\pi(g_C) = f_C$.

Next we show that the f_C, taken over all essential C, lie in Φ_B, and that the g_C lie in $\Phi_{A'}$. Fix an essential full chain C with associated string $b_1 \dots b_{r+1}$. Given $\sigma \in S_{r+1}$, and $1 \leqslant i \leqslant r$, there is at least one σ' such that $C_{\sigma'}$ differs from C_σ in rank i only, namely $\sigma \circ (i\ i+1)$. Now suppose that for some $\sigma' \in S_{r+1}$, $C_{\sigma'}$ differs from C_σ in rank i only. Then if if the rank $i-1$ and $i+1$ flats of C_σ are $F_{i-1} = \operatorname{cl}_B(H)$ and $F_{i+1} = \operatorname{cl}_B(H \cup \{b, b'\})$ respectively, where $H, \{b, b'\} \subseteq \{b_1, \dots, b_{r+1}\}$, then there are exactly two possibilities for F_i, namely $\operatorname{cl}_B(H \cup \{b\})$ and $\operatorname{cl}_B(H \cup \{b'\})$. If C_σ contains one of these two, then $C_{\sigma'}$ must contain the other one. That is, there is only one σ' satisfying the description. Then σ and σ' differ by a transposition, so ϵ_{C_σ} and $\epsilon_{C_{\sigma'}}$ will have opposite signs in the expression for f_C. As a result, the condition that, for all $i \in S$ and all chains C with $\operatorname{fl}(C) = S \setminus i$, we have $\sum_{D \supseteq C} b_D = 0$, is satisfied, and by Prop. 8.1, $f_C \in \Phi_B$. By the exact same argument, $g_C \in \Phi_{A'}$.

Next we show that the f_C are linearly independent. To do this, we first observe that $C_\sigma \leqslant C$ in the lexicographic order for all σ, with equality if and only if $\sigma = 1$. Indeed, let $\sigma \neq 1$ and let j be the smallest integer for which $\sigma(j) \neq j$. Since $b_1 > b_2 > \cdots > b_{r+1}$, it follows that C_σ's flat of rank j contains an element less than b_j, and thus comes lexicographically before C's flat of rank j. Since C and C_σ have the same flats of rank less than j, it follows that $C_\sigma < C$ lexicographically, as claimed. This means that the matrix whose rows and columns are indexed by full chains of $\Delta(B)$ in lexicographic order, with the entry in row C, column C' being the coefficient of $\epsilon_{C'}$ in f_C, is lower triangular, and all nonzero rows (i.e. those corresponding to essential C) have a 1 on the diagonal. Therefore these nonzero rows, hence the f_C themselves, are linearly independent.

Finally, we note that the number of essential full chains has already been shown to be $h_{[r]}(B)$, which is also the dimension of Φ_B in degree $[r]$. Therefore, the f_C form a basis for Φ_B in this degree. This completes the proof of the surjectivity of π in degree $[r]$.

Now let S be an arbitrary subset of $[r]$. Choose C from among the full chains of $\Delta(B)$ that have descent set S, i.e. minimal completions of essential chains of flag S.

Let $b_1 \dots b_{r+1}$ be the corresponding string. Define C_σ and D_σ as before, and let ν restrict a chain to the ranks in S. Set

$$f_C = \sum_{\sigma \in H} \operatorname{sgn}(\sigma) \epsilon_{\nu(C_\sigma)},$$

where H is the subgroup of S_{r+1} generated by the transpositions $\{(i\ i+1) \mid i \in S\}$. Analogously, set

$$g_C = \sum_{\sigma \in H} \operatorname{sgn}(\sigma) \epsilon_{\nu(D_\sigma)}.$$

As before, we have $g_C \in k[A']^*$ and $\pi(g_C) = f_C$. Now if $i \in S$, then $\sigma \in H$ if and only if $\sigma \circ (i\ i+1) \in H$, so a term corresponding to

$$0_B \subsetneq \operatorname{cl}_B(\{b_{\sigma(1)}\}) \subsetneq \cdots \subsetneq \operatorname{cl}_B(\{b_{\sigma(1)}, \dots, b_{\sigma(i-1)}, b_{\sigma(i)}\}) \subsetneq \cdots \subsetneq E$$

appears in f_C if and only if one corresponding to

$$0_B \subsetneq \operatorname{cl}_B(\{b_{\sigma(1)}\}) \subsetneq \cdots \subsetneq \operatorname{cl}_B(\{b_{\sigma(1)}, \dots, b_{\sigma(i-1)}, b_{\sigma(i+1)}\}) \subsetneq \cdots \subsetneq E$$

(that is, a chain differing from C_σ only in rank i) appears with opposite sign. That is, once again the Prop. 8.1 condition is satisfied for f_C to be in Φ_B and g_C to be in $\Phi_{A'}$.

To show that the f_C are linearly independent, it suffices to check that $\nu(C_\sigma) \leqslant \nu(C)$ in lexicographic order for all $\sigma \in H$, with equality if and only if $\sigma = 1$. Let $\sigma \in H \smallsetminus \{1\}$, and let j be the smallest integer for which $\sigma(j) \neq j$. By definition of H, we have $j \in S$. Let k be the smallest element of $[r+1] \smallsetminus S$ such that $k > j$. Then σ takes $\{j, j+1, \dots, k\}$ to itself. Since the descent set of C is S, we have $b_j > \cdots > b_k$. Hence, the rank j flat in C_σ (as well as in $\nu(C_\sigma)$) contains an element less than b_j. Thus the j-th flat of $\nu(C_\sigma)$ is less than the j-th flat of $\nu(C)$ in lexicographic order, so $\nu(C_\sigma) < \nu(C)$, as desired.

We have thus demonstrated $h_S(B)$ linearly independent elements of $(\Phi_B)_S$, a vector space of dimension $h_S(B)$; they are therefore a basis. For each one, we have found a $g_C \in (\Phi_{A'})_S$ with $\pi(g_C) = f_C$. Therefore, π is surjective in all degrees. $\square$

Remark 8.4. The top-degree component of Φ_B can be identified with the non-vanishing homology group of $\Delta(B)$ over k. The basis for $(\Phi_B)_{[r]}$ we constructed in the proof is the same basis that Björner constructs for this homology group in [3].

Example 8.5. Let A and B be the matroids appearing in previous examples. The two essential full chains of B are $C_1 = (\varnothing \subsetneq \{4,5\} \subsetneq \{2,4,5\} \subsetneq E)$ and $C_2 = (\varnothing \subsetneq \{4,5\} \subsetneq \{3,4,5\} \subsetneq E)$, with associated strings 421 and 431. From C_1, for example, we generate

$$\begin{aligned} f_{C_1} &= \epsilon_{\{4,5\}\subsetneq\{2,4,5\}} + \epsilon_{\{2\}\subsetneq\{1,2,3\}} + \epsilon_{\{1\}\subsetneq\{1,4,5\}} \\ &\quad - \epsilon_{\{4,5\}\subsetneq\{1,4,5\}} - \epsilon_{\{2\}\subsetneq\{2,4,5\}} - \epsilon_{\{1\}\subsetneq\{1,2,3\}} \\ g_{C_1} &= \epsilon_{\{4\}\subsetneq\{2,4\}} + \epsilon_{\{2\}\subsetneq\{1,2\}} + \epsilon_{\{1\}\subsetneq\{1,4\}} \\ &\quad - \epsilon_{\{4\}\subsetneq\{1,4\}} - \epsilon_{\{2\}\subsetneq\{2,4\}} - \epsilon_{\{1\}\subsetneq\{1,2\}}. \end{aligned}$$

f_{C_1} and f_{C_2} span $\Phi_{\{1,2\}}$ and lie in the image of π, so π is surjective in degree $\{1,2\}$.

Corollary 8.6. *If $A \to B$ is a rank-preserving weak map of matroids, with the rings $R_{A'}$ and R_B, as well as the map $\bar{\psi} : R_B \to R_{A'}$, defined as before, then $\bar{\psi}$ is injective.*

Proof. This follows formally from the fact that the dual map $\pi : \Theta_{A'} \to \Theta_B$ is surjective. $\square$

Corollary 7.5 thus implies the desired result:

Corollary 8.7. *If $A \to B$ is a rank-preserving weak map of matroids, then $h_S(A) \geqslant h_S(B)$ for all S.*

By combining this result with Prop. 3.3, Prop. 4.4, and Prop. 4.3, we summarize our conclusions as follows:

Theorem 8.8. *Let $A \to B$ be a weak map of matroids, with $\mathrm{rk}(B) = r + 1$. Then*

(1) $h_S(A) \geqslant h_S(B)$ *for all* $S \subseteq [r]$.
(2) $h_k(A) \geqslant h_k(B)$ *for all* k.
(3) $f_S(A) \geqslant f_S(B)$ *for all* $S \subseteq [r]$.
(4) $f_k(A) \geqslant f_k(B)$ *for all* k.

We end the paper with the following problems:

Question 8.9. *For a weak map $A \to B$, characterize for what k we have $h^I_k(A) > h^I_k(B)$ and for what S we have $h_S(A) > h_S(B)$.*

Question 8.10. *Is there a combinatorial interpretation of the inequalities in Theorem* 8.8*(a) and (b)? Is there a combinatorial interpretation of $h^I_k(A) - h^I_k(B)$, $h_k(A) - h_k(B)$, or $h_S(A) - h_S(B)$?*

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