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Abstract We construct atomic decompositions for crystals of type C_2 and use them to define a charge statistic, thus providing positive combinatorial formulas for the corresponding Kostka–Foulkes polynomials. Our methods include Kashiwara–Nakashima tableaux combinatorics as well as the combinatorics of string polytopes and twisted Bruhat graphs.

1. Introduction

Let $\mathfrak{g}$ be the symplectic Lie algebra $\mathfrak{sp}_4(\mathbb{C})$, i.e. the simple Lie algebra of type C_2. The irreducible $\mathfrak{g}$-modules are the highest weight modules $V(\lambda)$, with λ a dominant weight. Given an arbitrary weight μ, we denote by $d_{\lambda,\mu}$ the *weight multiplicity*, i.e. the dimension of the weight space $V(\lambda)_\mu$.

The weight multiplicity $d_{\lambda,\mu}$ admits a q-analogue, known as the Kostka–Foulkes polynomial $K_{\lambda,\mu}(q)$, so that $K_{\lambda,\mu}(1) = d_{\lambda,\mu}$. The Kostka–Foulkes polynomials have a natural representation-theoretic interpretation since their coefficients record the dimensions of the graded pieces of the Brylinski–Kostant filtration on weight spaces [4]. They also arise as structure coefficients in the theory of non-symmetric Macdonald polynomials and affine Demazure characters [7, 27]. Additionally, these polynomials are (up to renormalization) special cases of affine Kazhdan–Lusztig polynomials and have positive coefficients [10, 21, 24].

The goal of this paper is to give a positive combinatorial formula for the Kostka–Foulkes polynomials $K_{\lambda,\mu}(q)$ in type C_2. For our purposes this amounts to finding:

(1) a set $\mathcal{B}(\lambda)_\mu$ of cardinality $d_{\lambda,\mu}$ parametrizing a basis of the μ-weight space $V(\lambda)_\mu$, and
(2) a combinatorial statistic $c : \mathcal{B}(\lambda)_\mu \to \mathbb{Z}_{>0}$, called the *charge*, such that the Kostka–Foulkes polynomial $K_{\lambda,\mu}$ is a generating function of charge c on $\mathcal{B}(\lambda)_\mu$,

$$K_{\lambda,\mu}(q) = \sum_{T\in\mathcal{B}(\lambda)_\mu} q^{c(\mathtt{T})}.$$

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The set $\mathcal{B}(\lambda)_\mu$ has many known realizations, some of which are geometric, such as Littelmann paths [20], others algebro-geometric, such as Mirković–Vilonen cycles [3] and polytopes [8], and some purely combinatorial, such as Kashiwara–Nakashima tableaux [9]. An important feature that all of these models have in common is that they are endowed with a *crystal structure*, that is, for each of these models the set $\mathcal{B}(\lambda) = \bigcup \mathcal{B}(\lambda)_\mu$ has cardinality $\dim(V(\lambda))$ and is the vertex set of a colored directed graph known as a normal $\mathfrak{g}$-crystal (see Definition 3.1 and [3, 5]).

The analogous problem in type A_n was solved in 1978 by Lascoux and Schützenberger, who constructed the charge statistic using a combinatorial procedure on tableaux called cyclage [12]. There is also the fermionic formula In [13], Lascoux, Leclerc and Thibon provided another formulation of the Lascoux–Schützenberger charge statistic in terms of the crystal structure on tableaux. In [14], Lecouvey formulated a conjectural positive formula in type C_n by defining a (co)cyclage procedure on Kashiwara–Nakashima tableaux. This conjecture has been proven for one-row tableaux in [6]. In [15], Lecouvey–Lenart defined a charge statistic on King tableaux of weight zero to provide a combinatorial formula in type C_n for the Kostka–Foulkes polynomials $K_{\lambda,0}(q)$. Note that there is another charge statistic defined in [17] on tensor products of Kashiwara–Nakashima single column crystals providing a formula for the one-dimensional configuration sums, which coincide with Kostka–Foulkes polynomials only in type A_n.

In recent work by the first named author [25, 26], an alternative description of the charge statistic in type A_n was obtained through a more geometric approach, which transports the problem of finding the charge to the affine Grassmannian. In this setting, a charge statistic can be deduced after finding swapping functions combinatorially mimicking wall-crossing for hyperbolic localization. This geometric approach provides a type-independent framework which we believe can be used to find charge statistics beyond type A_n in a more uniform way. In the present paper we develop a similar strategy to construct a charge statistic in type C_2. We believe that this strategy can be extended to cover groups of higher ranks.

1.1. Charges via the affine Grassmannian. We now briefly recall the results in [25], at the heart of which lies the geometric Satake correspondence. Let Gbe a complex reductive group with Langlands dual group denoted $G^\vee$(in our setting we take $G = Sp_4(\mathbb{C})$ for which $G^\vee = SO_5(\mathbb{C})$, but we may as well state here the results in general). Let $B^\vee \subset G^\vee$ be a Borel subgroup and $T^\vee \subset B^\vee$ be a maximal torus. The affine Grassmannian $\mathcal{G}r := \mathcal{G}r_{G^\vee}$ associated to $G^\vee$ is endowed with an action of the extended torus $T^\vee_{\mathrm{ext}} := T^\vee \times \mathbb{C}^*$. (1)

Let Xbe the weight lattice, i.e. the set of cocharacters of $T^\vee$. We denote by $X_+ \subset X$ the subset of dominant weights.

For $\lambda \in X_+$, let $\overline{\mathcal{G}r^\lambda}$ denote the corresponding spherical Schubert variety in the affine Grassmannian of $G^\vee$ (cf. [25, §2.1.2.]).

Let $\widehat{X} \cong X \oplus \mathbb{Z}$ be the cocharacter lattice of $T^\vee_{\mathrm{ext}}$. We say that $\eta \in \widehat{X}$ is singular if there exists an affine root $\alpha^\vee$ of $G^\vee$ such that $\langle \eta, \alpha^\vee \rangle = 0$ (cf. [25, Definition 2.13]). We say that η is regular otherwise.

For any regular $\eta \in \widehat{X}$ and any $\mu \leqslant \lambda$ hyperbolic localization induces a functor

$$\mathrm{HL}^\eta_\mu : \mathcal{D}^b_{T^\vee_{\mathrm{ext}}}(\overline{\mathcal{G}r^\lambda}) \to \mathcal{D}^b(pt) \cong \mathrm{Vect}^{\mathbb{Z}},$$

where $\mathcal{D}^b_{T^\vee_{\mathrm{ext}}}(\overline{\mathcal{G}r^\lambda})$ is the derived category of $T^\vee_{\mathrm{ext}}$-equivariant constructible sheaves on the spherical Schubert variety $\overline{\mathcal{G}r^\lambda}$ with $\mathbb{Q}$-coefficients, and $\mathcal{D}^b(pt)$ is the derived

(1) As a guide for the reader, representation theoretic objects (e.g. $\mathcal{B}(\lambda)$) are associated with the group G, while geometric objects (e.g. $\mathcal{G}r$) always based to the Langlands dual group $G^\vee$.

category of sheaves on a point, which is equivalent to the category of graded $\mathbb{Q}$-vector spaces (cf. [25, §2.4]).

If $\eta \in X_+$, then the hyperbolic localization functors are *weight functors*, sending an intersection cohomology sheaf IC_λ to the weight space $V(\lambda)_\mu$ of the irreducible highest weight module $V(\lambda)$. In this case, as in [25, Definition 2.18], we say that η is in the *MV region*, where MV is short for Mirković–Vilonen [23]. If $\eta \in \widehat{X}$ is affine dominant, that is, the pairing between η and any positive affine root is positive, then the hyperbolic localization functors return graded vector spaces whose graded dimensions are renormalized Kostka–Foulkes polynomials. In this case, we say that η is in the *KL region*, where KL is short for Kazhdan–Lusztig.

Let $\widetilde{h} := \mathrm{grdim}(\mathrm{HL}^\eta_\mu(IC_\lambda))$. The polynomials $\widetilde{h}^\eta_{\mu,\lambda}(v)$ are called *renormalized η-Kazhdan–Lusztig polynomials*. We say that a function $r_\eta : \mathcal{B}(\lambda) \to \mathbb{Z}$ is a *recharge* for η if we have

$$\widetilde{h}^\eta_{\mu,\lambda}(q^{\frac{1}{2}}) = \sum_{\mathtt{T}\in\mathcal{B}(\lambda)_\mu} q^{r_\eta(\mathtt{T})} \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

For η_{MV} in the MV region, it is easy to construct a recharge for η_{MV} which is constant on $\mathcal{B}(\lambda)_\mu$ (cf. [25, Eq. (21)]). If η_{KL} is in the KL region and $\mu \in X_+$, then

$$K_{\mu,\lambda}(q) = \widetilde{h}^{\eta_{KL}}_{\mu,\lambda}(q^{\frac{1}{2}})q^{\frac{1}{2}\ell(\mu)}$$

is a Kostka–Foulkes polynomial by [25, Proposition 2.14], where ℓ is the Bruhat length (cf. [25, Eq.(7)]). So if r_{KL} is a recharge for η_{KL} in the KL region, we obtain a charge statistic $c : \mathcal{B}(\lambda) \to \mathbb{Z}$ by setting $c(\mathtt{T}) := r_{KL}(\mathtt{T}) + \frac{1}{2}\ell(\mathrm{wt}(\mathtt{T}))$. Notice that if $\mathrm{wt}(\mathtt{T}) \in X_+$ this is equal to $c(\mathtt{T}) = r_{KL}(\mathtt{T}) + \langle \mathrm{wt}(\mathtt{T}), \rho^\vee\rangle$.

Hyperbolic localization depends on the cocharacter η. More precisely, it can have different values in η_1 and η_2 only if they are separated by a hyperplane of the form

$$H_{\alpha^\vee} = \left\{\eta \in \widehat{X} \mid \langle \eta, \alpha^\vee\rangle = 0\right\},$$

where $\alpha^\vee$ is a positive real root for the group $G^\vee$. There is a simple rule to compute the hyperbolic localization functor after crossing such a wall. Assume that $H_{\alpha^\vee}$ is the only wall separating η_1 and η_2, with η_2 lying on its positive side. Then by [25, Proposition 2.35] we have, for $\nu = s_{\alpha^\vee}(\mu)$ such that $\mu < \nu \leqslant \lambda$:

$$\widetilde{h}^{\eta_2}_{\nu,\lambda}(v) = v^{-2}\widetilde{h}^{\eta_1}_{\nu,\lambda}(v) \text{ and}$$
$$\widetilde{h}^{\eta_2}_{\mu,\lambda}(v) = \widetilde{h}^{\eta_1}_{\mu,\lambda}(v) + (1 - v^{-2})\widetilde{h}^{\eta_1}_{\nu,\lambda}(v).$$

To track these changes combinatorially, one must construct a *swapping function* $\psi : \mathcal{B}(\lambda)_\mu \to \mathcal{B}(\lambda)_{s_{\alpha^\vee}(\mu)}$, which satisfies the condition $r_{\eta_1}(\mathtt{T}) - 1 = r_{\eta_1}(\psi(\mathtt{T}))$. Having such a swapping function ψ, we can derive r_{η_2} from r_{η_1} by swapping its values as indicated by ψ. Swapping functions are an essential ingredient to perform wall-crossing combinatorially and to modify the trivial recharge in the MV region into the desired recharge in the KL region.

In type A_n, swapping functions are given by the *modified root operators* e_α, f_α, which are defined for any positive root $\alpha \in \Phi$. This is a consequence of the *atomic decomposition* of the crystals $\mathcal{B}(\lambda)$ in type A_n given by Lecouvey–Lenart [16]. From this, it follows that the charge statistic giving the Kostka–Foulkes polynomials in type A_n is

$$\sum_{\alpha\in\Phi^+} \epsilon_\alpha(\mathtt{T}), \text{ where } \epsilon_\alpha(\mathtt{T}) = \max\{k \mid e^k_\alpha(\mathtt{T}) \neq 0\}.$$

1.2. Results. Our main results consist of the atomic decomposition of the type C_2 crystals $\mathcal{B}(\lambda)$, as well as the construction of swapping functions. As a byproduct, for any $\mu \in X_+$, we obtain the following formula for the charge statistic in type C_2

$$c : \mathcal{B}(\lambda)_\mu \to \mathbb{N}$$
$$\mathtt{T} \mapsto \epsilon_1(\mathtt{T}) + \epsilon_2(\mathtt{T}) + \epsilon_{12}(\mathtt{T}) + \widehat{\epsilon}_{21}(\mathtt{T})$$

where $\widehat{\epsilon}_{21}$ is not attached to a modified crystal operator, but rather depends on the atom in which $\mathtt{T}$ sits. This yields a positive combinatorial formula for the Kostka–Foulkes polynomials. We outline our methodology below.

1.3. Atomic decompositions and charge statistics. In [26], the first named author has shown that the LL atoms [16], where LL is short for Lecouvey–Lenart, coincide with the connected components of a graph with same vertices as $\mathcal{B}(\lambda)$, given by the f_n-closure of the W-orbits, where W denotes the Weyl group. This is one of the first obstacles which appear when considering type C_2 crystals: here the f_2-closures of the W-orbits are not atoms (cf. Definition 4.15). This calls for an alternative approach. As in [26], the language of adapted strings will be an important tool for us. Let $\varpi_1, \varpi_2 \in X$ be the fundamental weights. We first define an embedding of crystals (cf. Proposition 4.1)

$$\Phi : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda + 2\varpi_1).$$

We call the complement of Φ in $\mathcal{B}(\lambda + 2\varpi_1)$ the *principal preatom* $\mathcal{P}(\lambda + 2\varpi_1)$. If $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2$ is such that $\lambda_1 \leqslant 1$, we define $\mathcal{P}(\lambda) := \mathcal{B}(\lambda)$. The map Φ has an easy definition using the combinatorics of Kashiwara–Nakashima tableaux which allows to prove its properties directly. However, its reformulation in terms of adapted strings allows us to give equations describing the principal preatoms $\mathcal{P}(\lambda)$, which we use throughout this work. A *preatomic decomposition* of our crystal can be defined recursively. We show that the preatoms are stable under the W and f_2 action, hence naturally generalize the LL atoms. (Although we do not show it here, the preatom $\mathcal{P}(\lambda)$ is a union of one or two $\langle W, f_2\rangle$-connected components, depending on the parity of λ.) Once the preatomic decomposition of our crystal has been defined, we are ready to define its atomic decomposition. In Proposition 4.16 we show that there exists a weight-preserving injection

$$\overline{\Psi} : \mathcal{P}(\lambda) \to \mathcal{P}(\lambda + \varpi_2)$$

such that the set $\mathcal{A}(\lambda + \varpi_2) \subset \mathcal{P}(\lambda + \varpi_2)$ defined as the complement of $\mathrm{Im}(\overline{\Psi})$ if $\lambda_1 \neq 0$, respectively $\mathcal{A}(\lambda + 2\varpi_2) \subset \mathcal{P}(\lambda + 2\varpi_2)$ defined as the complement of $\mathrm{Im}(\overline{\Psi}^2)$ if $\lambda_1 = 0$, are atoms. The map $\overline{\Psi}$ is defined explicitly in terms of adapted strings. An explicit description in terms of Kashiwara–Nakashima tableaux is provided in the slightly more lengthy arXiv version of this manuscript. However, we do not need tableaux combinatorics in this paper. To show that the sets $\mathcal{A}(\lambda)$ are atoms, we resort to algebraic computations directly in the Hecke algebra. In particular, we make use of pre-canonical bases, introduced by Libedinsky–Patimo–Plaza in [18]. In analogy to the Satake isomorphism, in Proposition 4.12 we show that the ungraded character of a preatom $\mathcal{P}(\lambda)$ corresponds to the specialization at $v = 1$ of a modification $\widetilde{\mathbf{N}}^3$ of the precanonical basis $\mathbf{N}^3$ from [18, Definition 1.1]

In fact, the atomic and preatomic decompositions alone are already enough to define our charge statistic in type C_2. Let $\mathtt{T} \in \mathcal{B}(\lambda)$. We define in Definitions 4.10 and 4.25 the *atomic number* $\mathrm{at}(\mathtt{T})$and the *preatomic number* $\mathrm{pat}(\mathtt{T})$to be the positive integers such that

$$\mathtt{T} \in \mathcal{A}(\lambda - \mathrm{at}(\mathtt{T})\varpi_2 - 2\,\mathrm{pat}(\mathtt{T})\varpi_1) \subset \mathcal{P}(\lambda - 2\,\mathrm{pat}(\mathtt{T})\varpi_1) \subset \mathcal{B}(\lambda).$$

A consequence of our main result reads as follows (cf. Corollary 6.4).

Theorem. *The function $c : \mathcal{B}(\lambda)_+ \to \mathbb{Z}$ defined as*

$$c(\mathtt{T}) = \langle \lambda - \mathrm{wt}(\mathtt{T}), \rho^\vee \rangle - \mathrm{at}(\mathtt{T}) - \mathrm{pat}(\mathtt{T})$$

is a charge statistic.

1.4. Twisted Bruhat graphs and non-swappable staircases. To obtain our main result, Theorem 6.3, we first need to construct a recharge statistic r_{η_i} for each η_i in a family of cocharacters defined in Equation (42) each of which lies in a region determined by two hyperplanes, starting at the MV region and ending at the KL region. We achieve this via a careful study of the geometry of atoms in type C_2.

We consider *twisted Bruhat graphs* associated to a fixed infinite reduced expression y_∞ in the affine Weyl group. For any $m \in \mathbb{Z}_{>0}$, let y_m be the product of the first m elements of y_∞ and let $N(y_m)$ be its set of inversions. The idea is to start off by considering the Bruhat graph Γ_λ of a given dominant integral weight λ, that is, the moment graph of the spherical Schubert variety $\overline{\mathcal{G}r_\lambda}$. The vertices of the graph Γ_λ are all the weights smaller or equal than λ in the dominance order. We have an edge $\mu_1 \to \mu_2$ in Γ_λ if and only if $\mu_2 - \mu_1$ is a multiple of a root and $\mu_1 \leqslant \mu_2$. From Γ_λ we obtain our twisted Bruhat graph $\Gamma^m_\lambda$by inverting the orientation of all the arrows in Γ_λ with label in the inversion set of y_m. For $\mu \leqslant \lambda$, let $\mathrm{Arr}_m(\mu,\lambda)$ be the set of arrows pointing to μ in Γ^λ_m and let $\ell_m(\mu,\lambda) := |\,\mathrm{Arr}_m(\mu,\lambda)|$ be the number of such arrows (cf. Definition 5.1). Let t_{m+1} be the only element in $N(y_{m+1}) \setminus N(y_m)$. If $\mu < t_{m+1}\mu$ then, for the twisted Bruhat graphs in type A ([26, Prop. 2.17]) the following holds: $\ell_m(\mu,\lambda) = \ell_m(t_{m+1}\mu,\lambda) - 1$ if $\mu < t_{m+1}\mu \leqslant \lambda$. This implies that $\ell_{m+1}(\mu,\lambda) = \ell_m(t_{m+1}\mu,\lambda)$. However, as we show in Example 5.3, this property does not hold in type C_2. In Definition 5.2 we define an edge $\mu \to t_{m+1}\mu$ in Γ_λ to be *swappable* if

$$\ell_m(\mu,\lambda) = \ell_m(t_{m+1}\mu,\lambda) - 1.$$

Section 4 is dedicated to the classification of such edges. We pay particular attention to non-swappable edges and in Definition 5.28 we count the number of non-swappable edges in the following sense:

$$\mathcal{N}_m(\mu,\lambda) := |\{k \leqslant m \mid \mu < t_k\mu \leqslant \lambda \text{ and } \mu \to t_k\mu \text{ is not swappable}\}|.$$

An important property of non-swappable edges is that they will always "be swappable" in an atom isomorphic to $\mathcal{A}(\lambda - k\varpi_2)$ for large enough k. This leads to the notion of non-swappable staircases (cf. Definition 5.35). Essentially, a non-swappable staircase over (μ,λ) consists of a sequence of edges of the form $e_i := (\mu \to \mu - (n+i)\alpha)$ such that e_i is non-swappable in $\mathcal{A}(\lambda + i\varpi_2)$. We define $\widehat{\mathcal{D}}_m(\mu,\lambda)$to be the length of the longest non-swappable staircase over (μ,λ) where the label of every edge e_i is a root with label in $N(y_m)$. Moreover, in Definition 5.41 we define the following statistic, which considers only non-swappable staircases lying in a single preatom:

$$\mathcal{D}_m(\mu,\lambda,k) := \min(k, \widehat{\mathcal{D}}_m(\mu,\lambda - k\varpi_2)).$$

We are now ready to define the recharge statistics r_{η_m} (cf. Definition 6.2). For $\mathtt{T} \in \mathcal{P}(\lambda) \subset \mathcal{B}(\lambda')$ with $\mu := \mathrm{wt}(\mathtt{T})$ and $a = \mathrm{at}(\mathtt{T})$, we define

$$r_m(\mathtt{T}) := -\ell_m(\mu,\lambda - a\varpi_2) + \mathcal{N}_m(\mu,\lambda - a\varpi_2) - \mathcal{D}_m(\mu,\lambda,a) - a - 2\,\mathrm{pat}(\mathtt{T}) + \langle \lambda', \rho^\vee \rangle.$$

Our main result, from which descends our explicit formula for the charge statistic in type C_2, is the following (cf. Theorem 6.3).

Theorem. *The function $r_m : \mathcal{B}(\lambda) \to \mathbb{Z}$ is a recharge statistic for η_m for any $m \in \mathbb{N} \cup \{\infty\}$.*

1.5. Swapping functions. To prove our main theorem, we need to construct swapping functions.

The existence of non-swappable edges in type C_2 means that we cannot define swapping functions within a single atom as in type A_n. In Section 6 the swapping functions we construct involve two elements from two different atoms within the same preatom. In order to determine which are the two atoms involved we need to introduce a new quantity, which we call the *elevation* $\Omega(e)$ of an edge e that measures the height of the maximal staircases of non-swappable edges lying underneath it. For any $\mu \in X$ and any reflection $t \in W$ such that $\mu < t\mu \leqslant \lambda$ we define the swapping functions

$$\psi_{t\mu} : \mathcal{B}(\lambda)_{t\mu} \to \mathcal{B}(\lambda)_{\mu}$$

as follows. Let $\mathtt{T} \in \mathcal{B}(\lambda)_{t\mu}$ and assume that $\mathtt{T} \in \mathcal{A}(\lambda - a\varpi_2) \subset \mathcal{P}(\lambda)$. Let $e := (\mu \to t\mu) \in E(\lambda - a\varpi_2)$. Then $\psi_{t\mu}(\mathtt{T}) = \mathtt{T}'$, where $\mathtt{T}'$ is the only element of weight μ in $\mathcal{A}(\lambda - (a + \Omega(e))\varpi_2) \subset \mathcal{P}(\lambda)$. To prove Theorem 6.3 we show in Proposition 7.2, based on the results on non-swappable staircases and non-swappable edges from Section 5, that

$$r_{m+1}(\mathtt{T}) = r_{m+1}(\psi_{t\mu}(\mathtt{T})) + 1.$$

1.6. Alternative formula. In Section 6, we obtain an alternative formula for the charge statistic by focusing on a single element and counting how many times its recharge gets changed by a swapping function. The formula we obtain is in terms of the modified crystal operators, which we define in Definition 3.7.

Let $\mathtt{T}$ be an element of an atom $\mathcal{A}(\zeta)$ of highest weight $\zeta \in X_+$ and let $\mathrm{wt}(\mathtt{T}) = \mu$. Let $\widehat{\epsilon}_{21}(\mathtt{T})$ be the maximum integer such that $\mu + k\alpha_i \leqslant \zeta$. In Section 6 we show that

$$c(\mathtt{T}) = \epsilon_1(\mathtt{T}) + \epsilon_2(\mathtt{T}) + \epsilon_{12}(\mathtt{T}) + \widehat{\epsilon}_{21}(\mathtt{T})$$

is a charge statistic on $\mathcal{B}(\lambda)_\mu$, for any $\mu \in X_+$. Finally, we conjecture a formula for a charge statistic in type C_3, which is a natural generalization of our formula. We also provide an example where our statistic does not coincide with the statistic conjectured by Lecouvey [14].

2. The root system and Hecke algebra of type C_2

2.1. The root system and the affine Weyl group. Let $(X, \Phi, X^\vee, \Phi^\vee)$ be the root datum of the reductive group $Sp_4(\mathbb{C})$. The lattices X and $X^\vee$ are isomorphic to $\mathbb{Z}^2$, with bases $\{\varpi_1, \varpi_2\}$ and $\{\varpi_1^\vee, \varpi_2^\vee\}$. Let X_+ and $X_+^\vee$ be the subsets of dominant weights and dominant coweights. Sometimes we use the notation (λ_1, λ_2) to denote the weight $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2$.

The root system $\Phi \subset X$ is a root system of type C_2, with positive roots

$$\{\alpha_1, \alpha_2, \alpha_{12} := 2\alpha_1 + \alpha_2, \alpha_{21} := \alpha_1 + \alpha_2\}$$

with α_2 and α_{12} being the long roots. We have $\alpha_1 = 2\varpi_1 - \varpi_2$ and $\alpha_2 = -2\varpi_1 + 2\varpi_2$.

The coroot system $\Phi^\vee \subset X^\vee$ has positive coroots

$$\{\alpha_1^\vee, \alpha_2^\vee, \alpha_{12}^\vee := \alpha_1^\vee + \alpha_2^\vee, \alpha_{21}^\vee := \alpha_1^\vee + 2\alpha_2^\vee\}.$$

For any $i \in \{1, 2, 12, 21\}$, $\alpha_i^\vee$ is the coroot corresponding to α_i.

Let $\rho \in X$ be the half-sum of the positive roots and $\rho^\vee \in X^\vee$ be the half-sum of the positive coroots. We have $\rho = 2\alpha_1 + \frac{3}{2}\alpha_2$ and $\rho^\vee = \frac{3}{2}\alpha_1^\vee + 2\alpha_2^\vee$.

We have $X/\mathbb{Z}\Phi \cong \mathbb{Z}/2\mathbb{Z}$ and the two classes are generated by 0 and ϖ_1.

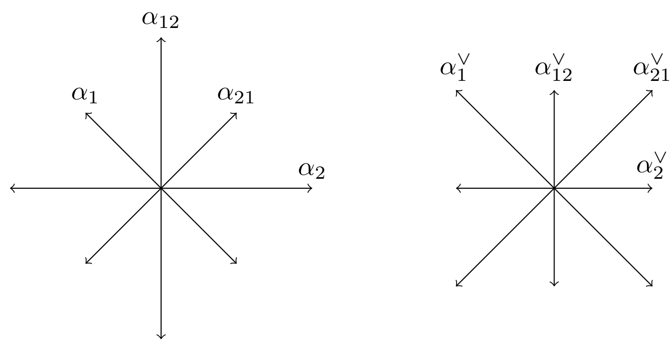

FIGURE 1. The root system Φ and the coroot system $\Phi^\vee$.

We denote by W the Weyl group of type C_2. Let $\widehat{W} := W \ltimes \mathbb{Z}\Phi$ be the affine Weyl group of type $\widetilde{C}_2$. The group $\widehat{W}$ has three simple reflections s_0, s_1, s_2 and has the following description as a Coxeter group:

$$\widehat{W} \cong \langle s_0, s_1, s_2 \mid s_0^2 = s_1^2 = s_2^2 = (s_0 s_2)^4 = (s_1 s_2)^4 = (s_0 s_1)^2 = e \rangle.$$

Notice that $\widehat{W}$ contains W as the subgroup generated by s_1 and s_2. We also consider the extended affine Weyl group $\widehat{W}_{ext} := W \ltimes X$.

Let $\widehat{X}^\vee := X^\vee \oplus \mathbb{Z}\delta$ and let $\widehat{\Phi}^\vee = \{\alpha^\vee + m\delta \mid \alpha^\vee \in \Phi^\vee, m \in \mathbb{Z}\}$ be the corresponding affine root system. The positive roots in $\widehat{\Phi}^\vee$ are

$$\widehat{\Phi}^\vee_+ = \{\alpha^\vee + m\delta \mid \alpha^\vee \in \Phi^\vee, m > 0\} \cup \Phi^\vee_+$$

and the simple roots are

$$\widehat{\Delta}^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_0^\vee := \delta - \alpha_{21}^\vee\}$$

There is a bijection between reflections in $\widehat{W}$ and positive roots $\widehat{\Phi}^\vee_+$, with simple reflections corresponding to simple roots. For a reflection $t \in \widehat{W}$ we denote by $\alpha_t^\vee$ the corresponding positive root in $\widehat{\Phi}^\vee_+$.

2.2. THE HECKE ALGEBRA AND ITS PRE-CANONICAL BASES. Recall from [11] and [21] the definition of the spherical Hecke algebra (see also [18, §2.2]). We denote by $\mathcal{H}$ the spherical Hecke algebra associated to the root system Φ. The spherical Hecke algebra is the free module over $\mathbb{Z}[v, v^{-1}]$ with standard basis $\{\mathbf{H}_\lambda\}_{\lambda \in X_+}$ and a canonical basis, the Kazhdan-Lusztig basis, which we denote by $\{\underline{\mathbf{H}}_\lambda\}_{\lambda \in X_+}$.

The spherical Hecke algebra can be thought of as a deformation of the monoid algebra $\mathbb{Z}[X_+]$, which is an abelian group is free with basis $\{e^\lambda\}_{\lambda \in X_+}$. In fact, specializing at $v = 1$, we obtain a ring homomorphism

$$(-)_{v=1} : \mathcal{H} \to \mathbb{Z}[X_+]$$
$$\mathbf{H}_\lambda \mapsto e^\lambda.$$

If $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2$ we write $\underline{\mathbf{H}}_{(\lambda_1,\lambda_2)}$ for $\underline{\mathbf{H}}_\lambda$ and similarly for $\mathbf{H}$.

For $w \in W$ and $\lambda \in X$ we denote by $w \cdot \lambda = w(\lambda + \rho) - \rho$ the dot action of w on λ. We say that a weight λ is singular if there exists $w \in W$ with $w(\lambda) = \lambda$. Clearly, a weight λ is singular if and only if $\lambda + \rho$ is singular with respect to the dot action.

We extend the definition of $\underline{\mathbf{H}}_\lambda$ to the whole X by setting $\underline{\mathbf{H}}_\lambda = 0$ if $(\lambda + \rho)$ is singular and $\underline{\mathbf{H}}_\lambda = (-1)^{\ell(w)}\underline{\mathbf{H}}_{w\cdot\lambda}$ if $w \in W$ is such that $w \cdot \lambda \in X_+$. Notice that in our setting $\lambda + \rho$ for $\lambda = (\lambda_1, \lambda_2)$ is singular if and only if $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_1 + \lambda_2 = -2$ or $\lambda_1 + 2\lambda_2 = -3$.

Recall the definition of the pre-canonical bases. We have

$$\mathbf{N}^i_\lambda = \sum_{I\subset \Phi^{\geqslant i}} (-v^2)^{|I|}\underline{\mathbf{H}}_{\lambda-\sum_{\alpha\in I}\alpha}$$

where $\Phi^{\geqslant i}$ is the subset of roots of height at least i. Notice that we have $\Phi^{\geqslant 3} = \alpha_{12} = 2\varpi_1$ and $\Phi^{\geqslant 2} = \{\alpha_{12},\alpha_{21}\} = \{2\varpi_1,\varpi_2\}$. Recall by [18, Theorem 1.2] that $\mathbf{N}^1$ is the standard basis, while $\mathbf{N}^2$ is the atomic basis $\mathbf{N}$, that is we have

$$\mathbf{N}^2_\lambda = \mathbf{N}_\lambda := \sum_{\mu\leqslant\lambda} v^{2\langle\rho^\vee,\lambda-\mu\rangle}\mathbf{H}_\mu.$$

It follows immediately from the definition that $\underline{\mathbf{H}}_\lambda = \mathbf{N}^4_\lambda$.

Example 2.1. Unfortunately, and contrary to the type A situation, the coefficients of the $\underline{\mathbf{H}}$-basis in the $\mathbf{N}^3$-basis are in general not positive. For example, we have $\mathbf{N}^3_{(0,\lambda_2)} = \underline{\mathbf{H}}_{(0,\lambda_2)} + v^2\underline{\mathbf{H}}_{(0,\lambda_2-1)}$. In particular, we get $\underline{\mathbf{H}}_{(0,1)} = \mathbf{N}^3_{(0,1)} - v^2\mathbf{N}^3_{(0,0)}$.

To recover positivity, we need to introduce a modification of the $\mathbf{N}^3$ basis. We define

$$\widetilde{\mathbf{N}}^3_\lambda = \begin{cases}\mathbf{N}^3_\lambda & \text{if } \lambda_1\neq 0\\ \underline{\mathbf{H}}_\lambda & \text{if } \lambda_1 = 0\end{cases} \tag{1}$$

Lemma 2.2. *We have*

$$\underline{\mathbf{H}}_{(\lambda_1,\lambda_2)} = \sum_{i\leqslant\lfloor\frac{\lambda_1}{2}\rfloor} v^{2i}\widetilde{\mathbf{N}}^3_{(\lambda_1-2i,\lambda_2)}$$

Proof. We prove it by induction on λ_1. The claim is clear if $\lambda_1 = 0$.

If $\lambda_1 > 0$, we have $\widetilde{\mathbf{N}}^3_\lambda = \underline{\mathbf{H}}_\lambda - v^2\underline{\mathbf{H}}_{\lambda-2\varpi_1}$. If $\lambda_1 = 1$ we have $\widetilde{\mathbf{N}}^3_\lambda = \underline{\mathbf{H}}_\lambda$ since $\lambda-2\varpi_1+\rho$ is singular. If $\lambda_1\geqslant 2$, we get $\underline{\mathbf{H}}_\lambda = \mathbf{N}^3_\lambda + v^2\underline{\mathbf{H}}_{\lambda-2\varpi_1}$ and the claim easily follows by induction. ☐

Lemma 2.3. *We have*

$$\widetilde{\mathbf{N}}^3_{(\lambda_1,\lambda_2)} = \begin{cases}\sum_{i\leqslant\lambda_2} v^{2i}\mathbf{N}^2_{(\lambda_1,\lambda_2-i)} & \text{if } \lambda_1>0\\ \sum_{i\leqslant\lfloor\frac{\lambda_2}{2}\rfloor} v^{4i}\mathbf{N}^2_{(\lambda_1,\lambda_2-2i)} & \text{if } \lambda_1 = 0.\end{cases}$$

Proof. We have $\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - v^2\mathbf{N}^3_{\lambda-\varpi_2}$. If $\lambda_1>0$ we get $\widetilde{\mathbf{N}}^3_\lambda = \mathbf{N}^3_\lambda = \mathbf{N}^2_\lambda + v^2\widetilde{\mathbf{N}}^3_{\lambda-\varpi_2}$ and the claim easily follows by induction on λ_2.

If $\lambda_1 = 0$ we have $\widetilde{\mathbf{N}}^3_\lambda = \widetilde{\mathbf{N}}^3_{(0,\lambda_2)} = \underline{\mathbf{H}}^3_{(0,\lambda_2)}$ and

$$\begin{aligned}\mathbf{N}^2_{(0,\lambda_2)} &= \underline{\mathbf{H}}_{(0,\lambda_2)} - v^2\underline{\mathbf{H}}_{(-2,\lambda_2)} - v^2\underline{\mathbf{H}}_{(0,\lambda_2-1)} + v^4\underline{\mathbf{H}}_{(-2,\lambda_2-1)}\\ &= \underline{\mathbf{H}}_{(0,\lambda_2)} + v^2\underline{\mathbf{H}}_{(0,\lambda_2-1)} - v^2\underline{\mathbf{H}}_{(0,\lambda_2-1)} - v^4\underline{\mathbf{H}}_{(0,\lambda_2-2)}\\ &= \underline{\mathbf{H}}_{(0,\lambda_2)} - v^4\underline{\mathbf{H}}_{(0,\lambda_2-2)} = \widetilde{\mathbf{N}}^3_{(0,\lambda_2)} - v^4\widetilde{\mathbf{N}}^3_{(0,\lambda_2-2)}.\end{aligned}$$

If $\lambda_2\leqslant 1$ we get $\mathbf{N}^2_{(0,\lambda_2)} = \widetilde{\mathbf{N}}^3_{(0,\lambda_2)}$. For $\lambda_2\geqslant 2$ we have $\widetilde{\mathbf{N}}^3_{(0,\lambda_2)} = \mathbf{N}^2_{(0,\lambda_2)} + v^3\widetilde{\mathbf{N}}^3_{(0,\lambda_2-2)}$ and the claim follows by induction. ☐

Remark 2.4. The decomposition of the $\underline{\mathbf{H}}$-basis in terms of the $\mathbf{N}$ basis was computed in [1, Theorem 1.1] using different methods. We prefer to reprove it using the precanonical bases since the $\widetilde{\mathbf{N}}^3$ basis has a natural combinatorial interpretation in terms of the crystal (cf. Proposition 4.12).

3. Crystals and Weyl group actions

Definition 3.1. *A (seminormal) crystal for a complex finite dimensional Lie algebra $\mathfrak{g}$ consists of a non-empty set $\mathcal{B}$ together with maps*

$$\begin{aligned} \mathrm{wt} :&\mathcal{B} \longrightarrow X \\ e_i, f_i :&B \longrightarrow B \sqcup \{0\}\,, i \in [1, \mathrm{rank}(\mathfrak{g})] \end{aligned}$$

such that for all $b, b' \in B$:

- $b' = e_i(b)$ *if and only if* $b = f_i(b')$,
- *if* $f_i(b) \neq 0$ *then* $\mathrm{wt}(f_i(b)) = \mathit{wt}(b) - \alpha_i$;
- *if* $e_i(b) \neq 0$, *then* $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$, *and*
- $\phi_i(b) - \epsilon_i(b) = \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$,

where

$$\begin{aligned} \epsilon_i(b) &= \max\{a \in \mathbb{Z}_{\geqslant 0} : e_i^a(b) \neq 0\} \text{ and} \\ \phi_i(b) &= \max\{a \in \mathbb{Z}_{\geqslant 0} : f_i^a(b) \neq 0\}. \end{aligned}$$

To each such crystal $\mathcal{B}$ is associated a crystal graph, a coloured directed graph with vertex set $\mathcal{B}$ and edges coloured by elements $i \in [1, \mathrm{rank}(\mathfrak{g})]$, where if $f_i(b) = b'$ there is an arrow $b \xrightarrow{i} b'$. A crystal is irreducible if its corresponding crystal graph is connected and finite. A seminormal crystal is called normal if it is isomorphic to the crystal of a representation of $\mathfrak{g}$. Irreducible normal crystals are thus indexed by dominant integral weights of $\mathfrak{g}$. We refer the reader to [5] *for more background on crystals.*

For a dominant weight λ we denote by $\mathcal{B}(\lambda)$ the corresponding normal crystal associated to the irreducible representation of $\mathfrak{g}$ of highest weight λ.

3.1. Crystals of Kashiwara–Nakashima tableaux.

In type C we can realize crystals using Kashiwara–Nakashima tableaux.

Definition 3.2. *Let n be a positive integer. A Kashiwara–Nakashima tableau (KN tableau for short) is a semi-standard Young tableau whose shape is a partition with at most n parts, in the alphabet*

$$\mathcal{P}_n := \left\{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\right\}$$

which satisfies the following conditions:

- *Each column is* ***admissible*** *(cf. Definition 3.3).*
- *Its* ***splitting*** *is a semi-standard Young tableau (cf. Definition 3.4).*

Definition 3.3. *Let C be a semi-standard column in the alphabet $\mathcal{P}_n$ of length at most n. Let $Z = \{z_1 > \ldots > z_m\}$ be the set of non-barred letters z in $\mathcal{P}_n$ such that both z and $\overline{z}$ both appear in C. We say that the column C is* admissible *if there exists a set $T = \{t_1 > \ldots > t_m\}$ of non-barred letters that satisfy:*

- $t_1 < z_1$ *and is maximal with the property* $t_1, \overline{t}_1 \notin C$;
- $t_i < \min(t_{i-1}, z_i)$, $t_i, \overline{t}_i \notin C$ *and is maximal with these properties.*

Definition 3.4. *The split of a column is the two-column tableau $lCrC$ where lC is the column obtained from C by replacing z_i by t_i and possibly re-ordering, and rC is obtained from C by replacing $\overline{z}_i$ by $\overline{t}_i$ and possibly re-ordering.*

The splitting of a semi-standard Young tableau consisting of admissible columns is the concatenation of the splits of its columns.

Example 3.5. Let $n=2$. The column $\begin{array}{|c|}\hline 2\\\hline \overline{2}\\\hline\end{array}$ is admissible (we have $Z=\{2\}$ and $T=\{1\}$), however, $\begin{array}{|c|}\hline 1\\\hline \overline{1}\\\hline\end{array}$ is not. Notice that although each one of its columns is admissible, the tableau $\begin{array}{|c|c|}\hline 2 & 2\\\hline \overline{2} & \overline{2}\\\hline\end{array}$ is not KN, because its split, $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 1 & 2\\\hline \overline{2} & \overline{1} & \overline{2} & \overline{1}\\\hline\end{array}$ is not semi-standard.

Definition 3.6. *Let* $\mathtt{T}$ *be a KN tableau. For* $i\in\{1,2\}$ *let* $n_i(\mathtt{T})$ *denote the number of* i*'s appearing in* $\mathtt{T}$ *and let* $n_{\overline{i}}(\mathtt{T})$ *denote the number of* $\overline{i}$*'s. Let* $t_i(\mathtt{T})=n_i(\mathtt{T})-n_{\overline{i}}(\mathtt{T})$*. Let* $\lambda_1(\mathtt{T})=t_1(\mathtt{T})-t_2(\mathtt{T})$ *and* $\lambda_2(\mathtt{T})=t_2(\mathtt{T})$*. The weight of* $\mathtt{T}$ *is defined to be* $\mathrm{wt}(\mathtt{T})=(\lambda_1(\mathtt{T}),\lambda_2(\mathtt{T}))=\lambda_1(\mathtt{T})\varpi_1+\lambda_2(\mathtt{T})\varpi_2$.

3.2. Words, signatures and crystal operators. The *word* of a KN tableau $\mathtt{T}$ is the reading of its entries, column by column, starting from the right most column and reading each column from top to bottom. We will denote the word of $\mathtt{T}$ by $\mathrm{word}(\mathtt{T})$. For example, if

$$\mathtt{T}=\begin{array}{|c|c|}\hline 1 & 2\\\hline \overline{2} & \overline{1}\\\hline\end{array} \tag{2}$$

we have $\mathrm{word}(\mathtt{T})=2\overline{1}1\overline{2}$. For each $1\leqslant i\leqslant n$, to a word $w\in\mathcal{P}_n$ we assign a labelling of the letters of w by $+,-$ or no label. For $i\leqslant n-1$, label the letters $i,\overline{i+1}$ by $+$ and the letters $i+1,\overline{i}$ by $-$. If $i=n$, label n by $+$ and $\overline{n}$ by $-$. The remaining letters remain without label. Finally, cancel out pairs of labels of the form $+-$, that is, cancel out every label $+$ with the first $-$ to its right, starting from the left-most one. For example, if the sequence of labels is $-+\square--\square++\square$ (blank box means no label), after the cancelling out process we obtain $-\square\square\square-\square++\square$. Like this, we obtain a sequence of labels which looks like this (after ignoring blank boxes):

$$(-)^r(+)^s$$

for some $r,s\in\mathbb{Z}_{\geqslant 0}$. This is the **i-signature** of w (but we also keep a record of the position of the remaining labels). We will denote it by $\sigma_i(w)$. For example, the 1-signature of $\mathrm{word}(\mathtt{T})$ as in (2) is $--++$. Its 2-signature is empty. To apply the root operator f_i to $\mathtt{T}$, we replace in $\mathtt{T}$ the letter a which is tagged by the left-most $+$ in the i-signature of $\mathrm{word}(\mathtt{T})$, by the letter $\overline{a}$, where $\overline{\overline{a}}=a$. If $s=0$, then $f_i(\mathtt{T})=0$. To apply e_i, we replace in $\mathtt{T}$ the letter a which is tagged by the right-most $-$ in the i-signature of $\mathrm{word}(\mathtt{T})$, by the letter $\overline{a}$, where $\overline{\overline{a}}=a$. If $r=0$, then $e_i(\mathtt{T})=0$.

3.3. Plactic relations for words. Note that the definition of the crystal operators and therefore of the simple reflections makes sense on arbitrary words in the alphabet $\mathcal{P}_n$. In [14] the following plactic relations (R1-3) on words are introduced.

- *R*1 $yzx\sim yxz$ for $x\leqslant y<z$ with $z\neq\overline{x}$ and $xzy\sim zxy$ for $x<y\leqslant z$ with $z\neq\overline{x}$;
- *R*2 $y\overline{x-1}(x-1)\sim yx\overline{x}$ and $x\overline{x}y\cong\overline{x-1}(x-1)y$ for $1<x\leqslant n$ and $x\leqslant y\leqslant\overline{x}$;
- *R*3 $w\sim w\smallsetminus\{z,\overline{z}\}$, where $w\in\mathcal{P}_n^*$ and $z\in[n]$ are such that w is a non-admissible column, z is the lowest non-barred letter in w such that $N(z)=z+1$ and any proper factor of w is an admissible column.

These relations define an equivalence relation $\cong$ on the word monoid $\mathcal{P}_n^*$. Each word $w\in\mathcal{P}_n^*$ is equivalent via plactic relations to the word of a unique KN tableau $\mathtt{T}(w)$. Moreover, there is the following characterization. Let $u,v\in\mathcal{P}_n^*$ and let U,V the connected components (both normal $U_q(\mathfrak{sp}(2n,\mathbb{C}))$-crystals) in which they lie. Then $u\cong v$ if and only if there exists a crystal isomorphism $\eta:U\to V$ such that $\eta(u)=v$.

3.4. Weyl group actions and modified crystal operators. Let

$$\sigma_i(\mathrm{word}(\mathtt{T})) = (-)^r(+)^s$$

be the i-signature of word($\mathtt{T}$) as defined in the previous paragraph. To apply the simple reflection s_i to $\mathtt{T}$ do the following:

- If $r = s$, then $s_i(\mathtt{T}) = \mathtt{T}$.
- If $r > s, s_i(\mathtt{T}) = e_i^{r-s}(\mathtt{T})$.
- If $s > r, s_i(\mathtt{T}) = f_i^{s-r}(\mathtt{T})$.

Let $x = s_{i_1} \cdots s_{i_r} \in W$. The action of x on a KN tableau $\mathtt{T}$ is defined by

$$x(\mathtt{T}) := s_{i_1}(\cdots(s_{i_r}(\mathtt{T}))).$$

More generally, given a crystal $\mathcal{B}$ there is an action of the Weyl group W on $\mathcal{B}$ where s_i acts by reversing the f_i-string, i.e. for $\mathtt{T} \in \mathcal{B}$ with $r = \epsilon_i(\mathtt{T})$ and $s = \phi_i(\mathtt{T})$, we define $s_i(\mathtt{T})$ as $e_i^{r-s}(\mathtt{T})$ if $r \geqslant s$ and $f_i^{s-r}(\mathtt{T})$ if $s \geqslant r$.

For a proof that this defines an action of W see [5, Proposition 2.36]. For any $x \in W$ we have $x(\mathrm{wt}(\mathtt{T})) = \mathrm{wt}(x(\mathtt{T}))$.

We now introduce the modified crystal operators. These were originally introduced in [9] and later studied in detail in [16].

Definition 3.7. *We define the* modified crystal operators $e_{12} := s_1 e_2 s_1$ *and* $f_{12} := s_1 f_2 s_1$.

Remark 3.8. Unfortunately, we cannot just define e_{21} as $s_2 e_1 s_2$ to be the modified crystal operator attached to the root α_{21}. In fact, in our inductive procedure we need the crystal operator to be constructed by conjugating the root of higher index, but it is not possible here since α_{21} and α_2 lie in different orbits under the Weyl group (α_2 is long while α_{21} is short, as shown in Figure 1). One of the main hurdles of generalizing the charge statistic from type A to type C is in fact to find an appropriate replacement for this crystal operator in the charge formula.

3.5. Adapted strings. There are two reduced expressions for the longest element w_0 of type C_2: $s_1s_2s_1s_2$ and $s_2s_1s_2s_1$. After fixing a reduced expression $\sigma = s_{i_1}s_{i_2}s_{i_3}s_{i_4}$ of w_0, an element $\mathtt{T} \in \mathcal{B}(\lambda)$ is uniquely determined by a quadruple of non-negative integers $\mathrm{str}_\sigma(\mathtt{T}) = (a, b, c, d)$, called the adapted string, such that $\mathtt{T} = f_{i_1}^a f_{i_2}^b f_{i_3}^c f_{i_4}^d (v_\lambda)$, where $v_\lambda \in \mathcal{B}(\lambda)$ is the highest weight vertex. We abbreviate $\mathrm{str}_{s_1s_2s_1s_2}$ as str_1 and $\mathrm{str}_{s_2s_1s_2s_1}$ as str_2. The adapted strings for each of the different reduced expressions form a cone, denoted by C_1 and C_2. The precise relation between these two cones has been given by Littelmann.

Theorem 3.9 ([19, Prop. 2.4]). *There exists piecewise linear mutually inverse bijections $\theta_{12} : C_1 \to C_2$ and $\theta_{21} : C_2 \to C_1$, such that $\theta_{12} \circ \mathrm{str}_1 = \mathrm{str}_2$ and $\theta_{21} \circ \mathrm{str}_2 = \mathrm{str}_1$, given by $\theta_{12}(a, b, c, d) = (a', b', c', d')$, where*

$$\begin{aligned}
a' &= \max\{d, c-b, b-a\} \\
b' &= \max\{c, a-2b+2c, a+2d\} \\
c' &= \min\{b, 2b-c+d, a+d\} \\
d' &= \min\{a, 2b-c, c-2d\},
\end{aligned}$$

and $\theta_{21}(a, b, c, d) = (a', b', c', d')$, where

$$\begin{aligned}
a' &= \max\{d, 2c-b, b-2a\} \\
b' &= \max\{c, a+d, a+2c-b\} \\
c' &= \min\{b, 2b-2c+d, d+2a\} \\
d' &= \min\{a, c-d, b-c\}.
\end{aligned}$$

Moreover, Littelmann precisely characterizes the adapted strings which occur in a given crystal $\mathcal{B}(\lambda)$.

Theorem 3.10 ([19, Corollary 2, Prop. 1.5]). *Let $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2$. Given $(a, b, c, d) \in \mathbb{Z}^4_{\geqslant 0}$, there exists $x \in \mathcal{B}(\lambda)$ with $\mathrm{str}_2(x) = (a, b, c, d)$ if and only if the following inequalities hold:*

- $b \geqslant c \geqslant d$
- $d \leqslant \lambda_1$
- $c \leqslant \lambda_2 + d$
- $b \leqslant \lambda_1 - 2d + 2c$
- $a \leqslant \lambda_2 + d - 2c + b$

4. The atomic and preatomic decompositions

In this section we introduce some important decompositions of the crystal $\mathcal{B}(\lambda)$.

4.1. Preatoms.

We start by defining the preatomic decomposition. As we note in Remark 4.5, the preatoms turn out to be a direct generalization of the LL atoms in type A, although they can contain several elements with the same weight.

Proposition 4.1. *There is an embedding of crystals $\Phi : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda + 2\varpi_1)$.*

Proof. We define the map Φ on Kashiwara-Nakashima tableaux as follows. Note that since $n = 2$, all tableaux will have at most two rows. Let $\mathtt{T}$ be a Kashiwara-Nakashima tableaux of shape a partition $[a, b]$. Then we replace the first row of $\mathtt{T}$, say $r^1 = \begin{array}{|c|c|c|}\hline r^1_1 & \dots & r^1_k \\\hline\end{array}$, by $\begin{array}{|c|c|c|c|c|}\hline 1 & r^1_1 & \dots & r^1_k & \bar{1} \\\hline\end{array}$. The resulting tableau will be denoted by $\Phi'(\mathtt{T})$.

If $\Phi'(\mathtt{T})$ contains the column $\begin{array}{|c|}\hline 1\\\hline \bar{1}\\\hline\end{array}$, we replace it with the column $\begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline\end{array}$. The new tableau will be denoted by $\Phi(\mathtt{T})$. Note that by semi-standardness, since $\mathtt{T}$ does not contain a column $\begin{array}{|c|}\hline 1\\\hline \bar{1}\\\hline\end{array}$, $\Phi'(\mathtt{T})$ can contain at most one such column.

The map Φ is well defined: the tableau $\Phi'(\mathtt{T})$ is clearly semi-standard. Assume that $\Phi'(\mathtt{T}) \neq \Phi(\mathtt{T})$. The 1 in the column $\begin{array}{|c|}\hline 1\\\hline \bar{1}\\\hline\end{array}$ of $\Phi'(\mathtt{T})$ is necessarily the right-most one, so all entries to its right in $\Phi'(\mathtt{T})$ must be strictly larger than 1. In the second row of $\Phi'(\mathtt{T})$, the $\bar{1}$ which is replaced by $\bar{2}$ to obtain $\Phi(\mathtt{T})$ has to be the left-most one, since otherwise $\Phi'(\mathtt{T})$ would contain another column $\begin{array}{|c|}\hline 1\\\hline \bar{1}\\\hline\end{array}$ which is impossible. The last thing missing to check in order to establish that $\Phi(\mathtt{T})$ is indeed a KN tableau is that it does not contain as a sub tableau $\begin{array}{|c|c|}\hline 2 & 2\\\hline \bar{2} & \bar{2}\\\hline\end{array}$. But this is impossible, because then $\Phi'(\mathtt{T})$ would necessarily have to contain $\begin{array}{|c|c|}\hline 1 & 2\\\hline \bar{1} & \bar{2}\\\hline\end{array}$ as a sub-tableau, which is not semi-standard. Note that, by construction, Φ is weight-preserving. The case $\Phi'(\mathtt{T}) = \Phi(\mathtt{T})$ is left to the reader, as the arguments are very similar to the ones above.

It remains to show that Φ is injective and that it commutes with the crystal operators. We start with a lemma.

Lemma 4.2. *Let $\mathtt{T}$ be a Kashiwara-Nakashima tableau, and let $w = \mathrm{word}(\mathtt{T})$ be its word. Then the word $\bar{1}w1$ is plactic equivalent to $\mathrm{word}(\Phi(\mathtt{T}))$.*

Proof. Let r, s be positive integers such that the second row of $\mathtt{T}$ has length s and the first row $s + r$. Let $a_1 \leqslant \cdots \leqslant a_{r+s}$ be the entries in the first row and let $b_1 \leqslant \cdots \leqslant b_s$ be the entries in the second row of $\mathtt{T}$.

Adding a $\overline{1}$ at the end of the first row of a tableau just adds a $\overline{1}$ at the beginning of its word. For a tableau $\mathtt{T}$ let $\Phi^{-\overline{1}}(\mathtt{T})$ be the tableau obtained by removing the rightmost $\overline{1}$ from $\Phi(\mathtt{T})$. Let $\mathtt{T}_s$ be tableau consisting of the first s columns of $\mathtt{T}$. Then we have

$$\operatorname{word}(\Phi(\mathtt{T})) = \overline{1}\operatorname{word}(\Phi^{-\overline{1}}(\mathtt{T})) = \overline{1}a_{r+s}\dots a_{s+1}\operatorname{word}(\Phi^{-\overline{1}}(\mathtt{T}_s)).$$

It is then enough to show that $\operatorname{word}(\Phi^{-\overline{1}}(\mathtt{T}_s)) \cong \operatorname{word}(\mathtt{T}_s)1$. We show this by induction on s.

The claim is clear if $s = 0$. For $s > 0$, let $\mathtt{U} = \mathtt{T}_{s-1}$. Notice that we have $\operatorname{word}(\Phi^{-\overline{1}}(\mathtt{U})) = a_{s-1}\operatorname{word}(\Phi(\mathtt{U})_{s-1})$ and that $\Phi(\mathtt{U})_{s-1} = \Phi(\mathtt{T})_{s-1}$. We have $\operatorname{word}(\mathtt{T}_s) = a_s b_s \operatorname{word}(\mathtt{U})$ and by induction we have

$$\operatorname{word}(\mathtt{T}_s)1 \cong a_s b_s \operatorname{word}(\Phi^{-\overline{1}}(\mathtt{U})) = a_s b_s a_{s-1}\operatorname{word}(\Phi(\mathtt{T})_{s-1})$$

Assume that $b_s \neq \overline{a_{s-1}}$. Since $a_{s-1} \leqslant a_s < b_s$ by Relation $R1$ in §3.3 we have

$$a_s b_s a_{s-1} \cong a_s a_{s-1} b_s.$$

We conclude because $a_s a_{s-1} b_s \operatorname{word}(\Phi(\mathtt{T})_{s-1}) = \operatorname{word}(\Phi^{-\overline{1}}(\mathtt{T}_s))$.

Assume now that $b_s = \overline{a_{s-1}}$. Note that $b_s = \overline{2}$ is impossible since semi-standardness alone then implies that $a_{s-1} = a_s = 2$ and $b_{s-1} = b_s = \overline{2}$ but the tableau $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \overline{2} & \overline{2} \\ \hline\end{array}$ is not KN. Therefore the only option is $b_s = \overline{1}$ and $a_{s-1} = 1$. In this case we have $a_s \in \{2, \overline{2}\}$ and Relation $R2$ tells us that

$$a_s\overline{1}1 \cong a_s 2\overline{2}.$$

Notice that the case $b_s = \overline{1}$ precisely occurs when the s-th column of $\Phi'(\mathtt{T})$ is $\begin{array}{|c|}\hline 1 \\ \hline \overline{1} \\ \hline\end{array}$ and is replaced by $\begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline\end{array}$ in $\Phi(\mathtt{T})$. Hence, we have $a_s 2\overline{2}\operatorname{word}(\Phi(\mathtt{T})_{s-1}) = \operatorname{word}(\Phi^{-\overline{1}}(\mathtt{T}_s))$ and we conclude. □

We now go back to the proof of Proposition 4.1. From Lemma 4.2 we see immediately that Φ is injective. Let $\mathtt{T}$ be a KN tableau and $w = \operatorname{word}(\mathtt{T})$. We have $\sigma_1(\overline{1}w1) = -\sigma_1(w)+$. This implies that, if f_1 is defined on w then it is also defined on $\overline{1}w1$ and

$$f_1(\overline{1}w1) = \overline{1}f_1(w)1 \tag{3}$$

Similarly, if $e_1(w)$ is defined, then $e_1(\overline{1}w1) = \overline{1}e_1(w)1$. We know by Lemma 4.2 that $\overline{1}w1 \cong \operatorname{word}(\Phi(\mathtt{T}))$ therefore

$$f_1(\operatorname{word}(\Phi(\mathtt{T}))) \cong f_1(\overline{1}w1) = \overline{1}f_1(w)1 \cong \operatorname{word}(\Phi(f_1(\mathtt{T}))). \tag{4}$$

This implies that, since $f_1(\Phi(\mathtt{T})), e_1(\Phi(\mathtt{T})), \Phi(e_1(\mathtt{T}))$ and $\Phi(f_1(\mathtt{T}))$ are KN tableaux, we have

$$f_1(\Phi(\mathtt{T})) = \Phi(f_1(\mathtt{T})) \qquad e_1(\Phi(\mathtt{T})) = \Phi(e_1(\mathtt{T})) \tag{5}$$

as desired. Now, $\sigma_2(\overline{1}w1) = \sigma_2(w)$ by definition, so e_2 and f_2 are defined on $\Phi(\mathtt{T})$ if and only if are defined on $\mathtt{T}$. Hence $f_2(\operatorname{word}(\Phi(\mathtt{T})) = \operatorname{word}(\Phi(f_2(\mathtt{T})))$ and (5) hold after replacing f_1 by f_2 and e_1 by e_2. □

Corollary 4.3. *Given a KN tableau* $\mathtt{T}$*, the new tableau* $\Phi(\mathtt{T})$ *is defined by first column inserting the letter* 1 *into* $\mathtt{T}$ *using symplectic insertion and subsequently adding a* $\overline{1}$ *at the end of the first row.*

Proof. The proof follows immediately from Lemma 4.2. □

Corollary 4.4. *The complement of* $\mathrm{Im}(\Phi)$ *is closed under the action of* W*, under* e_2 *and under outwards* e_1*, i.e. if* $\mathtt{T} \notin \mathrm{Im}(\Phi)$ *and* $\langle \mathrm{wt}(\mathtt{T}), \alpha_1^\vee \rangle \geqslant 0$ *and* $e_1(\mathtt{T}) \neq 0$*, then* $e_1(\mathtt{T}) \notin \mathrm{Im}(\Phi)$*.*

Proof. Since Φ commutes with W, the complement of its image is union of W-orbits. Let $\mathtt{T} \notin \mathrm{Im}(\Phi)$. We know that $\Phi(e_i(\mathtt{T})) = e_i(\Phi(\mathtt{T}))$ if $e_i(\mathtt{T}) \neq 0$.

Assume $e_2(\mathtt{T}) \neq 0$. If $e_2(\mathtt{T}) = \Phi(\mathtt{T}')$, then it follows from $\sigma_2(\Phi(\mathtt{T}')) = \sigma_2(\mathtt{T}')$, that $f_2(\mathtt{T}') \neq 0$ and therefore $\mathtt{T} = f_2(\Phi(\mathtt{T}')) = \Phi(f_2(\mathtt{T}'))$, which is impossible.

Assume $e_1(\mathtt{T}) \neq 0$ and $\langle \mathrm{wt}(\mathtt{T}), \alpha_1^\vee \rangle \geqslant 0$. Assume $e_1(\mathtt{T}) = \Phi(\mathtt{T}')$. Since $\langle \mathrm{wt}(\mathtt{T}'), \alpha_1^\vee \rangle = \langle \mathrm{wt}(\mathtt{T}), \alpha_1^\vee \rangle + 2 > 0$, we have $f_1(\mathtt{T}') \neq 0$, hence $\mathtt{T} = f_1(\Phi(\mathtt{T}')) = \Phi(f_1(\mathtt{T}'))$, which is impossible. □

Remark 4.5. In analogy with [26, Definition 2.17] we can consider the connected components obtained as f_2-closures of the W-orbits in the crystal graph. From Corollary 4.4 we see that preatoms are unions of the f_2-closure, and moreover, it turns out that for most λ (i.e. for $\lambda_1 > 0$) each preatom consists of exactly one or two connected components, depending on the parity of λ_1. In this sense, we can think of preatoms in type C_2 as a direct generalization of LL atoms in type A.

Definition 4.6. *For* λ *such that* $\lambda_1 \geqslant 2$*, we define the principal preatom* $\mathcal{P}(\lambda)$ *to be the complement of* $\mathrm{Im}(\Phi)$ *in* $\mathcal{B}(\lambda)$*. If* $\lambda_1 \leqslant 1$*, we define* $\mathcal{P}(\lambda) := \mathcal{B}(\lambda)$*.*

We define the preatomic decomposition *by induction on* λ_1*. If* $\lambda_1 \geqslant 2$*, let* $\mathcal{B}(\lambda - 2\varpi_1) = \bigsqcup \mathcal{P}(\mu_i)$ *be the preatomic decomposition. Then, the preatomic decomposition of* $\mathcal{B}(\lambda)$ *is*

$$\mathcal{B}(\lambda) = \mathcal{P}(\lambda) \sqcup \bigsqcup \Phi(\mathcal{P}(\mu_i)).$$

Notice that all the preatoms in $\mathcal{B}(\lambda)$ are images of a principal preatom $\mathcal{P}(\lambda - 2k\varpi_1)$ under the map Φ^k for some k. In particular, for any $\lambda \in X$ every preatom of highest weight λ is isomorphic via some power of Φ to the principal preatom $\mathcal{P}(\lambda) \subset \mathcal{B}(\lambda)$ and every preatom has a unique element of maximal weight. We now give a different characterization of preatoms using adapted strings.

Proposition 4.7. *Let* $\mathtt{T} \in \mathcal{B}(\lambda)$ *and consider* $\Phi : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda + 2\varpi_1)$*.*

(1) *If* $\mathrm{str}_1(\mathtt{T}) = (a, b, c, d)$*, we have* $\mathrm{str}_1(\Phi(\mathtt{T})) = (a+1, b+1, c+1, d)$*.*
(2) *If* $\mathrm{str}_2(\mathtt{T}) = (a, b, c, d)$ *we have* $\mathrm{str}_2(\Phi(\mathtt{T})) = (a, b+1, c+1, d+1)$*.*

Proof. If v_λ is the highest weight vector, then it follows from Lemma 4.2 that

$$\Phi(\mathtt{T}) = f_1 f_2 f_1(v_{\lambda + 2\varpi_1}). \tag{6}$$

In this case $\mathrm{str}_1(v_\lambda) = (0,0,0,0)$ so the claim follows since $(1,1,1,0)$ is an adapted string for $\Phi(\mathtt{T})$. For arbitrary $\mathtt{T} \in \mathcal{B}(\lambda)$ it follows from Proposition 4.1 that

$$\Phi(\mathtt{T}) = f_1^a f_2^b f_1^c f_2^d f_1 f_2 f_1(v_{\lambda + 2\varpi_1}). \tag{7}$$

We introduce the following notation:

$$\begin{aligned}
(a', b', c', d') &:= \mathrm{str}_1(f_2^d f_1 f_2 f_1(v_{\lambda+2\varpi_1})) = \theta_{21}(d, 1, 1, 1) \\
(a'', b'', c'', d'') &:= \mathrm{str}_1(f_1^{a'+c} f_2^{b'} f_1^{c'} f_2^{d'}(v_{\lambda+2\varpi_1})) \\
(a''', b''', c''', d''') &:= \mathrm{str}_2(f_2^{a''+b} f_1^{b''} f_2^{c''} f_1^{d''}(v_{\lambda+2\varpi_1})).
\end{aligned}$$

By Theorem 3.9, we have $(a', b', c', d') = \theta_{12}(d, 1, 1, 1) = (1, d+1, 1, 0)$. Moreover, it follows from [19, Cor. 2, ii.] that $(a'', b'', c'', d'') = (0, c+1, d+1, 1)$ and

$(a''', b''', c''', d''') = (1, b+1, c+1, d)$. Putting all of this together we get that

$$\begin{aligned}\Phi(\mathtt{T}) &= f_1^a f_2^b f_1^c f_2^d f_1 f_2 f_1 (v_{\lambda+2\varpi_1})\\ &= f_1^a f_2^b f_1^{a'+c} f_2^{b'} f_1^{c'} f_2^{d'} (v_{\lambda+2\varpi_1})\\ &= f_1^a f_2^{b+a''} f_1^{b''} f_2^{c''} f_1^{d''} (v_{\lambda+2\varpi_1})\\ &= f_1^{a+a'''} f_2^{b'''} f_1^{c'''} f_2^{d'''} (v_{\lambda+2\varpi_1}).\end{aligned}$$

Therefore

$$(a+a''', b'', c''', d''') = (a+1, b+1, c+1, d) = \mathrm{str}_1(\Phi(\mathtt{T})),$$

showing the first statement. The proof of the second statement is similar. It follows from Lemma 4.2 that

$$\Phi(v_\lambda) = f_1 f_2 f_1 (v_{\lambda+2\varpi_1}), \tag{8}$$

so that $\mathrm{str}_2(\Phi(v_\lambda)) = (0,1,1,1)$. Using [19, Prop. 2.4] we get that

$$\begin{aligned}\Phi(\mathtt{T}) &= f_2^a f_1^b f_2^c f_1^d (f_1 f_2 f_1 (v_{\lambda+2\varpi_1}))\\ &= f_2^a f_1^b f_2^c f_1^{d+1} f_2 f_1 (v_{\lambda+2\varpi_1})\\ &= f_2^a f_1^b f_1 f_2^{c+1} f_1^{d+1} (v_{\lambda+2\varpi_1})\\ &= f_2^a f_1^{b+1} f_2^{c+1} f_1^{d+1} (v_{\lambda+2\varpi_1}).\end{aligned}$$

This concludes the proof. □

Remark 4.8. Notice that one can avoid the recourse to tableaux combinatorics and use the equation in Proposition 4.7 as the definition of Φ. Then one can use the explicit description of the adapted strings in Theorem 3.9 to the check that Φ is well defined and that has the desired properties.

The description of the embedding Φ in terms of adapted strings allows us to give a convenient description of the elements in the principal preatom $\mathcal{P}(\lambda) \subset \mathcal{B}(\lambda)$.

Corollary 4.9. *There exists* $\mathtt{T} \in \mathcal{P}(\lambda)$ *with* $\mathrm{str}_2(\mathtt{T}) = (a,b,c,d)$ *if* ***and only if*** *all the inequalities in Theorem 3.10 hold and at least one of the following equations hold.*

- $d = 0$
- $d = \lambda_1$
- $b = \lambda_1 - 2d + 2c$

Proof. Let $\mathtt{T} \in \mathcal{B}(\lambda)$ with $\mathrm{str}_2(a,b,c,d)$, so all the inequalities in Theorem 3.10 hold. There exists $\mathtt{U} \in \mathcal{B}(\lambda - 2\varpi_1)$ with $\mathrm{str}_2(\mathtt{U}) = (a, b-1, c-1, d-1)$ so that $\Phi(\mathtt{U}) = \mathtt{T}$ if and only if all the inequalities in Theorem 3.10 hold for $(a, b-1, c-1, d-1)$ and $\lambda - 2\varpi_1$, which written explicitly means that $d \geqslant 1$, $d \leqslant \lambda_1 - 1$ and $b \leqslant \lambda_1 - 2d + 2c - 1$ (the others remain unchanged). The claim now easily follows for $\lambda_1 \geqslant 2$. □

Definition 4.10. *Let* $\mathtt{T} \in \mathcal{B}(\lambda)$. *Let* $\mathrm{pat}(\mathtt{T}) \in \mathbb{Z}_{\geqslant 0}$ *be such that* $\mathtt{T} \in \mathcal{P}(\lambda - 2\,\mathrm{pat}(\mathtt{T})\varpi_1) \subset \mathcal{B}(\lambda)$. *We call* $\mathrm{pat}(\mathtt{T})$ *the* preatomic number *of* $\mathtt{T}$.

In other words, $\mathrm{pat}(\mathtt{T})$ *is the maximum integer with* $\mathtt{T} \in \mathrm{Im}(\Phi^{\mathrm{pat}(\mathtt{T})})$.

We now compute the size of the preatoms using the precanonical bases from Subsection 2.2.

Definition 4.11. *Let* $\mathcal{B}^+(\lambda)$ *be the subset of* $\mathcal{B}(\lambda)$ *consisting of elements whose weight is dominant. For a subset of* $C \subset \mathcal{B}^+(\lambda)$ *we define the* ungraded character of C *as*

$$[C]_{v=1} := \sum_{c \in C} e^{\mathrm{wt}(c)} \in \mathbb{Z}[X_+]$$

More generally, for a subset $C \subset \mathcal{B}(\lambda)$ stable under the W-action we define

$$[C]_{v=1} := [C \cap \mathcal{B}^+(\lambda)]_{v=1}$$

PROPOSITION 4.12. *We have $[\mathcal{B}(\lambda)]_{v=1} = (\underline{\mathbf{H}}_\lambda)_{v=1}$ and $[\mathcal{P}(\lambda)]_{v=1} = (\widetilde{\mathbf{N}}^3_\lambda)_{v=1}$.*

Proof. The statement about $\mathcal{B}(\lambda)$ follows by the Satake isomorphism (see for example [11]). The second statement follows easily from the definition of $\widetilde{\mathbf{N}}^3_\lambda$. In fact, if $\lambda_1 \leqslant 1$ we have $\mathcal{B}(\lambda) = \mathcal{P}(\lambda)$. If $\lambda_1 \geqslant 2$ we have $\mathcal{P}(\lambda) = \mathcal{B}(\lambda) \smallsetminus \Phi(\mathcal{B}(\lambda - 2\varpi_1))$. Since Φ is weight preserving and injective, we have

$$[\mathcal{P}(\lambda)]_{v=1} = [\mathcal{B}(\lambda)]_{v=1} - [\mathcal{B}(\lambda - 2\varpi_1)]_{v=1} = (\underline{\mathbf{H}}_\lambda - \underline{\mathbf{H}}_{\lambda-2\varpi_1})_{v=1} = (\widetilde{\mathbf{N}}^3_\lambda)_{v=1}. \qquad \square$$

4.1.1. *The preatomic Z function.* In analogy with [25, Definition 1.17] we define a function Z in type C.

DEFINITION 4.13. *For $\mathtt{T} \in \mathcal{B}(\lambda)$, let $Z(\mathtt{T}) := \phi_1(\mathtt{T}) + \phi_2(\mathtt{T}) + \phi_{21}(\mathtt{T})$.*

The function Z is not constant along preatoms but nevertheless can be used to give an explicit formula for the preatomic number pat.

PROPOSITION 4.14. *Assume $\mathtt{T} \in \mathcal{B}(\lambda)$ and let $\mu := \mathrm{wt}(\mathtt{T})$. Then we have*

$$Z(\mathtt{T}) = \lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \max\left(0, \frac{|\mu_1| - \lambda_1}{2}\right) + \mathrm{pat}(\mathtt{T}). \tag{9}$$

Proof. We show the claim by induction on $\mathrm{pat}(\mathtt{T})$. We first assume $\mathrm{pat}(\mathtt{T}) = 0$, or equivalently that $\mathtt{T} \in \mathcal{P}(\lambda) \subset \mathcal{B}(\lambda)$. Let $(a, b, c, d) = \mathrm{str}_2(\mathtt{T})$.

Let $\mathbb{T} = (\mathbb{Q} \cup \{+\infty\}, \oplus, \odot)$ be the tropical semiring (cf. [22]), where $x \oplus y = \min(x, y)$ denotes the tropical addition and $x \odot y = x + y$ is the tropical multiplication. We also write fractions in $\mathbb{T}$ for the tropical division, i.e. $\frac{x}{y} = x - y$. A tropical polynomial is the function expressing the minimum of several linear functions. A tropical rational function is the difference of two tropical polynomials.

Our first goal is to reinterpret both sides of (9) as tropical rational functions in a, b, c, d, λ_1 and λ_2. For example, μ_1 can be expressed as a tropical rational function: since we have $\mu_1 = \lambda_1 + 2a + 2c - 2b - 2d$, we can write $\mu_1 = \frac{\lambda_1 \odot a^{\odot 2} \odot c^{\odot 2}}{b^{\odot 2} \odot d^{\odot 2}}$. In the rest of this proof we make the notation lighter by simply writing xy for $x \odot y$ and x^n for $x^{\odot n}$. Since $\mathrm{pat}(\mathtt{T}) = 0$ we can rewrite the RHS in (9) as

$$RHS(\mathtt{T}) := \frac{\lambda_1^2 \lambda_2^2}{bd(1 \oplus \frac{\lambda_1 ac}{bd} \oplus \frac{bd}{ac})} = \frac{ac\lambda_1^2\lambda_2^2}{a^2c^2\lambda_1 \oplus abcd \oplus b^2d^2}.$$

Expressing the LHS of (9) is unfortunately a much longer computation. We have $Z(\mathtt{T}) = \phi_2(\mathtt{T}) \odot \phi_1(\mathtt{T}) \odot \phi_{12}(\mathtt{T})$ and

- $\phi_2(\mathtt{T}) = \frac{bd\lambda_2}{ac^2}$
- $\phi_1(\mathtt{T}) = \phi_1' \circ \theta_{21}(a, b, c, d)$, where $\phi_1'(a, b, c, d) = \frac{b^2d^2\lambda_1}{ac^2}$ and θ_{21} is as in Theorem 3.9.
- $\phi_{12}(\mathtt{T}) = \phi_2 \circ \theta_{12} \circ \sigma_1 \circ \theta_{21}(a, b, c, d)$ where $\sigma_1(a, b, c, d) = (\frac{\lambda_1 b^2 d^2}{ac^2}, b, c, d)$ is the transformation expressing the action of the simple reflection s_1 on str_1.

From this, we can obtain an explicit expression of $Z(\mathtt{T})$ as a tropical rational function. However, this is a rather unfeasible task to do by hand, so we resort to the help of the computer algebra software [28]. In Sage we can simply compute $Z(\mathtt{T})$ by formally treating its three factors as ordinary rational functions in $\mathbb{Q}(a, b, c, d, \lambda_1, \lambda_2)$.

Then, to check the claim, we need to show that $Z(\mathtt{T}) = RHS(\mathtt{T})$ when $d = 0$, $d = \lambda_1$ or $b = \lambda_1 + 2d - 2c$. In other words, we need to show that, as tropical rational functions on the set of elements of the crystal, we get $Z(\mathtt{T})/RHS(\mathtt{T}) = 1$ if we specialize $d = 1$,[2] $d = \lambda_1$ or $b = \lambda_1 d^2/c^2$. Again, this can be checked with the help of SageMath. In Appendix A we attach the code that proves our claim.

Assume now $\mathrm{pat}(\mathtt{T}) > 0$, so $\mathtt{T} = \Phi(\mathtt{T}')$ for some $\mathtt{T}' \in \mathcal{B}(\lambda - 2\varpi_1)$. Since $\mathrm{pat}(\mathtt{T}) = \mathrm{pat}(\mathtt{T}')+1$, by induction it suffices to show that $Z(\mathtt{T}) = Z(\mathtt{T}')+1$. From Proposition 4.7 it follows that $\phi_1(\mathtt{T}) = \phi_1(\mathtt{T}') + 1$ and $\phi_2(\mathtt{T}) = \phi_2(\mathtt{T}')$. Moreover, we have

$$\phi_{12}(\mathtt{T}) = \phi_2(s_1(\mathtt{T})) = \phi_2(s_1(\Phi(\mathtt{T}'))) = \phi_2(\Phi(s_1(\mathtt{T}'))) = \phi_2(s_1(\mathtt{T}')) = \phi_{12}(\mathtt{T}')$$

since Φ commutes with s_1, and the claim follows. □

4.2. Atoms. The goal of this section is to describe a finer decomposition of $\mathcal{B}(\lambda)$ into atoms.

Definition 4.15. *We call a subset $A \subset \mathcal{B}(\lambda)$ an* atom *if $[A]_{v=1} = (\mathbf{N}_\mu)_{v=1}$ for some $\mu \in X_+$. This means that there exists $\mu \in X_+$ such that every weight smaller or equal than μ in X occurs exactly once as the weight of an element in A.*

An atomic decomposition *is a decomposition of $\mathcal{B}(\lambda)$ into atoms.*

Proposition 4.16. *There is an injective weight-preserving map $\overline{\Psi} : \mathcal{P}(\lambda) \hookrightarrow \mathcal{P}(\lambda + \varpi_2)$. If $\lambda_1 \neq 0$ then the set $\mathcal{A}(\lambda + \varpi_2) := \mathcal{P}(\lambda + \varpi_2) \setminus \overline{\Psi}(\mathcal{P}(\lambda))$ is an atom. If $\lambda_1 = 0$ then the set $\mathcal{A}(\lambda + 2\varpi_2) := \mathcal{P}(\lambda + 2\varpi_2) \setminus \overline{\Psi}^2(\mathcal{P}(\lambda))$ is an atom.*

We divide the proof into several steps. We begin by defining a map Ψ directly in terms of the adapted strings. The map $\overline{\Psi}$ is then obtaining by making Ψ symmetric along s_1. Then we prove injectivity in Lemma 4.20 and that the complement is an atom in Proposition 4.21.

Lemma 4.17. *Let $\mathtt{T} \in \mathcal{P}(\lambda)$ with $\mathrm{str}_2(\mathtt{T}) = (a, b, c, d)$. Then we have the following:*

1. *If $d \in \{0, \lambda_1\}$, there exists $\mathtt{U} \in \mathcal{P}(\lambda + \varpi_2)$ with $\mathrm{str}_2(\mathtt{U}) = (a, b+1, c+1, d)$;*
2. *If $d \notin \{0, \lambda_1\}$, there exists $\mathtt{U} \in \mathcal{P}(\lambda + \varpi_2)$ with $\mathrm{str}_2(\mathtt{U}) = (a, b, c+1, d+1)$.*

Proof. Assume first $d = 0$ and $d = \lambda_1$. The Littelmann inequalities for $(a, b+1, c+1, d)$ and $\lambda + \varpi_2$ are implied by the original ones for (a, b, c, d) and λ, so there exists such $\mathtt{U} \in \mathcal{B}(\lambda + \varpi_2)$. Since $d = 0$ or $d = \lambda_1$ we also see that $\mathtt{U} \in \mathcal{P}(\lambda + \varpi_2)$.

Assume now $d \neq 0$ and $d \neq \lambda_1$. Since $\mathtt{T} \in \mathcal{P}(\lambda)$ we have $b = \lambda_1 - 2d + 2c$. The Littelmann inequalities for $(a, b, c+1, d+1)$ and $\lambda + \varpi_2$ are:

- $b \geqslant c + 1 \geqslant d + 1$,
- $d + 1 \leqslant \lambda_1$,
- $c + 1 \leqslant \lambda_2 + 1 + d + 1$,
- $b \leqslant \lambda_1 - 2d + 2c$, and
- $a \leqslant \lambda_2 + d - 2c + b$.

All these inequalites are implied by the original ones (and by $d \neq \lambda_1$) except $b \geqslant c+1$. However, if $b < c+1$ then $b = c$ and $c = \lambda_1 - 2d + 2c$ or, equivalently, $d = \frac{1}{2}(c+\lambda_1)$. Since $d \leqslant c$ and $d < \lambda_1$ this is impossible. It follows that there exists $\mathtt{U} \in \mathcal{B}(\lambda + \varpi_2)$ with $\mathrm{str}_2(\mathtt{U}) = (a, b, c+1, d+1)$. Moreover, $b = \lambda_1 - 2(d+1) + 2(c+1)$, so $\mathtt{U} \in \mathcal{P}(\lambda + \varpi_2)$ □

Lemma 4.17 ensures that the following function is well defined.

Definition 4.18. *We define $\Psi : \mathcal{P}(\lambda) \to \mathcal{P}(\lambda + \varpi_2)$ as follows. Let $\mathtt{T} \in \mathcal{P}(\lambda)$ with $\mathrm{str}_2(\mathtt{T}) = (a, b, c, d)$. Then $\Psi(\mathtt{T}) = \mathtt{U}$ with*

$$\mathrm{str}_2(\mathtt{U}) = \begin{cases} (a, b+1, c+1, d) & \text{if } d = 0 \text{ or } d = \lambda_1 \\ (a, b, c+1, d+1) & \text{otherwise.} \end{cases}$$

[2] Recall that $0 \in \mathbb{Q}$ is the multiplicative unity in $\mathbb{T}$

We also define $\overline{\Psi}:\mathcal{P}(\lambda)\to\mathcal{P}(\lambda+\varpi_2)$ *as follows.*

$$\overline{\Psi}(\mathtt{T})=\begin{cases}\Psi(\mathtt{T}) & \text{if } \mathrm{wt}(\mathtt{T})_1\leqslant 0\\ s_1(\Psi(s_1(\mathtt{T}))) & \text{if } \mathrm{wt}(\mathtt{T})_1\geqslant 0\end{cases}$$

Lemma 4.19. *For* $\mathtt{T}\in\mathcal{P}(\lambda)$ *we have:*

(1) $\mathrm{wt}(\Psi(\mathtt{T}))=\mathrm{wt}(\overline{\Psi}(\mathtt{T}))=\mathrm{wt}(\mathtt{T})$
(2) $\phi_2(\Psi(\mathtt{T}))=\phi_2(\mathtt{T})$.
(3) *If* $f_2(\mathtt{T})\neq 0$ *also* $f_2(\Psi(\mathtt{T}))=\Psi(f_2(\mathtt{T}))$.
(4) *If* $e_2(\mathtt{T})\neq 0$ *also* $e_2(\Psi(\mathtt{T}))=\Psi(e_2(\mathtt{T}))$.
(5) $s_1(\overline{\Psi}(\mathtt{T}))=\overline{\Psi}(s_1(\mathtt{T}))$.

Proof. This is clear by the definition of str_2. □

Lemma 4.20. *The maps* $\Psi,\overline{\Psi}:\mathcal{P}(\lambda)\to\mathcal{P}(\lambda+\varpi_2)$ *are injective.*

Proof. It is enough to prove the statement for Ψ. Assume $\Psi(\mathtt{T})=\Psi(\mathtt{U})$ with $\mathtt{T}\neq\mathtt{U}$. Let $\mathrm{str}_2(\mathtt{T})=(a,b,c,d)$ and $\mathrm{str}_2(\mathtt{T})=(a',b',c',d')$. We can assume that $d\notin\{0,\lambda_1\}$, $d'\in\{0,\lambda_1\}$ and that

$$\mathrm{str}_2(\Psi(\mathtt{T}))=(a,b,c+1,d+1)=(a',b'+1,c'+1,d').$$

It follows that $d'=d+1$, $c'=c$ and $b'=b-1$. Since

$$b-1=b'\leqslant\lambda_1-2d'+2c'=\lambda_1-2(d+1)+2c$$

it follows that $b\leqslant\lambda_1-2d+2c-1$. But this contradicts the fact that $b=\lambda_1-2d+2c$. □

Recall the atomic basis $\mathbf{N}=\mathbf{N}^2$ of the spherical Hecke algebra from Subsection 2.2.

Proposition 4.21. *We have* $[\mathcal{A}(\lambda)]_{v=1}=(\mathbf{N}_\lambda)_{v=1}$. *In particular the set* $\mathcal{A}(\lambda)$ *is an atom.*

Proof. If $\lambda_2=0$ we have $\mathcal{A}(\lambda)=\mathcal{P}(\lambda)$, so $[\mathcal{A}(\lambda)]_{v=1}=(\widetilde{\mathbf{N}}_\lambda)_{v=1}=(\mathbf{N}_\lambda)_{v=1}$.

If $\lambda_2=1$ and $\lambda_1=0$ then we can easily check that $\mathcal{B}(\lambda)$ consists of a single atom. If $\lambda_2>1$ and $\lambda_1=0$ then we have $\mathcal{A}(\lambda)=\mathcal{P}(\lambda)\setminus\overline{\Psi}^2(\mathcal{P}(\lambda-\varpi_2))$. Since $\overline{\Psi}$ is injective and weight-preserving, we have by Lemma 2.3 that

$$[\mathcal{A}(\lambda)]_{v=1}=[\mathcal{P}(\lambda)]_{v=1}-[\mathcal{P}(\lambda-2\varpi_2)]_{v=1}=(\widetilde{\mathbf{N}}^3_\lambda-\widetilde{\mathbf{N}}^3_{\lambda-2\varpi_2})_{v=1}=(\mathbf{N}_\lambda)_{v=1}.$$

Finally, assume $\lambda_2>0$ and $\lambda_1>0$. Then, we have $\mathcal{A}(\lambda)=\mathcal{P}(\lambda)\setminus\overline{\Psi}(\mathcal{P}(\lambda-\varpi_2))$. Since $\overline{\Psi}$ is injective and weight-preserving, we have

$$[\mathcal{A}(\lambda)]_{v=1}=[\mathcal{P}(\lambda)]_{v=1}-[\mathcal{P}(\lambda-\varpi_2)]_{v=1}=(\widetilde{\mathbf{N}}^3_\lambda-\widetilde{\mathbf{N}}^3_{\lambda-\varpi_2})_{v=1}=(\mathbf{N}_\lambda)_{v=1}.$$ □

From this we can obtain an atomic decomposition of $\mathcal{B}(\lambda)$. Because we already know how to decompose $\mathcal{B}(\lambda)$ into preatoms, it is enough to decompose each preatom $\mathcal{P}(\lambda)$ into atoms. If $\lambda_2=0$ or if $\lambda=(0,1)$ we have $\mathcal{P}(\lambda)=\mathcal{A}(\lambda)$. If $\lambda_2>0$ and $\lambda\neq(0,1)$ then we have

$$\mathcal{P}(\lambda)=\begin{cases}\mathcal{A}(\lambda)\sqcup\overline{\Psi}(\mathcal{P}(\lambda-\varpi_2)) & \text{if } \lambda_1>0\\ \mathcal{A}(\lambda)\sqcup\overline{\Psi}^2(\mathcal{P}(\lambda-2\varpi_2)) & \text{if } \lambda_1=0\end{cases}$$

so, applying $\overline{\Psi}$, we obtain an atomic decomposition by induction.

Remark 4.22. It is worth noting that an atomic decomposition can also be obtained by taking the complement of Ψ rather than $\overline{\Psi}$. The advantage of using $\overline{\Psi}$ is to ensure that atoms are stable under s_1. This stability is crucial, as our approach inherently relies on s_1-symmetry, as discussed for example in Proposition 5.24. It is therefore essential to ensure that the structures we define are compatible with this symmetry.

LEMMA 4.23. *Let* $\mathtt{T} \in \mathcal{P}(\lambda)$ *with* $\mathrm{str}_2(\mathtt{T}) = (a,b,c,d)$. *Then*

$$\phi_1(\Psi(\mathtt{T})) = \begin{cases} \phi_1(\mathtt{T}) & \text{if } d = 0 \text{ and } 2a > b > 2c \text{ or } d \neq 0, \lambda_1 \text{ and } b > 2a + d \\ \phi_1(\mathtt{T}) + 1 & \text{otherwise.} \end{cases}$$

Moreover, if $\phi_1(\Psi(\mathtt{T})) = \phi_1(\mathtt{T})$ *and* $\mu_1 \leqslant 0$, *then* $\phi_1(\mathtt{T}) = 0$

Proof. Let $\pi_1 : \mathbb{Z}^4 \to \mathbb{Z}$ be the projection onto the first component. Then, we have

$$(10)\quad \phi_1(\mathtt{T}) = \pi_1(\theta_{21}(\mathrm{str}_2(\mathtt{T}))) + (\mathrm{wt}(\mathtt{T}))_1 = \lambda_1 + 2a - 2b + 2c - 2d + \max(d, 2c - b, b - 2a).$$

From here we see that, if $d = 0$ or $d = \lambda_1$, we have

$$\phi_1(\Psi(\mathtt{T})) - \phi_1(\mathtt{T}) = \max(d, 2c - b + 1, b - 2a + 1) - \max(d, 2c - b, b - 2a).$$

If $d = 0$, then $\phi_1(\Psi(\mathtt{T})) = \phi_1(\mathtt{T}))$ if and only if $2a > b > 2c$. If $d = \lambda_1$, we have $2c - b \geqslant 2d - \lambda_1 = \lambda_1$, so $\phi_1(\Psi(\mathtt{T})) - \phi_1(\mathtt{T}) = 1$.

If $0 < d < \lambda_1$ and $b = \lambda_1 - 2d + 2c$, then

$$\phi_1(\Psi(\mathtt{T})) - \phi_1(\mathtt{T}) = \max(d + 1, 2c - b + 2, b - 2a) - \max(d, 2c - b, b - 2a),$$

but $2c - b = 2d - \lambda_1 < d$, so $\phi_1(\Psi(\mathtt{T})) - \phi_1(\mathtt{T}) = \max(d + 1, b - 2a) - \max(d, b - 2a)$ and the claim easily follows. □

COROLLARY 4.24. *Let* $\mathtt{T} \in \mathcal{P}(\lambda)$ *with* $\mathrm{str}_2(\mathtt{T}) = (a,b,c,d)$. *Then*

$$\phi_{12}(\Psi(\mathtt{T})) = \begin{cases} \phi_{12}(\mathtt{T}) + 1 & \text{if } d = 0 \text{ and } 2a > b > 2c \text{ or } d \neq 0, \lambda_1 \text{ and } b > 2a + d \\ \phi_{12}(\mathtt{T}) & \text{otherwise.} \end{cases}$$

Proof. It follows from Proposition 4.14 that

$$\phi_{12}(\Psi(\mathtt{T})) - \phi_{12}(\mathtt{T}) = 1 - (\phi_1(\Psi(\mathtt{T})) - \phi_1(\mathtt{T})),$$

so we conclude by Lemma 4.23. □

DEFINITION 4.25. *Let* $\mathtt{T} \in \mathcal{B}(\lambda)$.

Let $\mathrm{at}(\mathtt{T}) \in \mathbb{Z}_{\geqslant 0}$ *be the maximum integer such that* $\mathtt{T}$ *is in the image of* $\overline{\Psi}^{\mathrm{at}(\mathtt{T})} : \mathcal{P}(\lambda - \mathrm{at}(\mathtt{T})\varpi_2) \to \mathcal{P}(\lambda)$. *We call* $\mathrm{at}(\mathtt{T})$ *the* atomic number *of* $\mathtt{T}$.

PROPOSITION 4.26. *Let* $\mathtt{T} \in \mathcal{P}(\lambda) \subset \mathcal{B}(\lambda)$ *with* $\mathrm{str}_2(\mathtt{T}) = (a,b,c,d)$ *and* $\mathrm{wt}(\mathtt{T})_1 \leqslant 0$. *We have*

$$\mathrm{at}(\mathtt{T}) = \begin{cases} \min(c, \lambda_1 + 2c - b) & \text{if } d = 0 \\ \lambda_1 + 2c - 2d - b + \min(\lambda_2 + d - c, d - 1) & \text{if } d > 0. \end{cases}$$

Proof. Notice that since $\mathrm{wt}(\mathtt{T})_1 \leqslant 0$ we have $\Psi = \overline{\Psi}$.

First recall that by Theorem 3.10, we have $0 \leqslant d \leqslant \lambda_1$. If $d = 0$, $\mathrm{at}(\mathtt{T})$ is the maximal amount we can subtract simultaneously from b and c, decreasing at the same time the value of λ_2 by the same amount, so that the inequalities and equalities mentioned in Corollary 4.9 still hold. Since $b \geqslant c$, we can focus only on c and the inequality $\lambda_1 + 2c - b \geqslant 0$, which is the only other inequality describing $\mathcal{P}(\lambda)$ which is affected after reducing b, c and λ_2 in equal amounts. Now if we decrease b and c simultaneously by the same amount, the quantity $\lambda_1 + 2c - b$ decreases by the same amount. Therefore, in this case $\mathrm{at}(\mathtt{T}) = \min(c, \lambda_1 + 2c - b)$ as desired.

Assume now $d = \lambda_1$. Recall that we need to find the maximal $\mathrm{at}(\mathtt{T})$ such that the map $\Psi^{\mathrm{at}(\mathtt{T})}(\mathtt{U}) = \mathtt{T}$ for an element $\mathtt{U} \in \mathcal{P}(\lambda - \mathrm{at}(\mathtt{T})\varpi_2)$. Recall that the definition of the map Ψ depends on the value of d. Let ψ_1 and ψ_2 be the two possible actions on adapted strings defined by Ψ, corresponding to the cases $0 < d < \lambda_1$ and $d = 0, \lambda_1$

respectively (i.e. we have $\psi_1, \psi_2 : \mathbb{Z}^4 \to \mathbb{Z}^4$ with $\psi_1(a,b,c,d) = (a,b,c+1,d+1)$ and $\psi_2(a,b,c,d) = (a,b+1,c+1,d)$). The definitions imply that we must have

$$\mathrm{str}_2(\mathtt{T}) = \psi_2^{\mathrm{at}_2(\mathtt{T})}\psi_1^{\mathrm{at}_1(\mathtt{T})}(\mathrm{str}_2(\mathtt{U}))$$

for some $\mathrm{at}_1(\mathtt{T}), \mathrm{at}_2(\mathtt{T}) \in \mathbb{N}$ with $\mathrm{at}_1(\mathtt{T}) + \mathrm{at}_2(\mathtt{T}) = \mathrm{at}(\mathtt{T})$.

Now, to calculate $\mathrm{at}_1(\mathtt{T})$ we first need to subtract the largest possible amount from b, c and λ_2 such that our inequalities and equalities stated in Corollary 4.9 will still hold. Analogously to the case $d = 0$ we can conclude that this number is $\mathrm{at}_1(\mathtt{T}) = \min(c, \lambda_1 + 2c - b - 2d)$. In this case the inequality $0 \leqslant \lambda_1 + 2c - 2d - b$ becomes $0 \leqslant 2c - \lambda_1 - b \leqslant c$ since $c \leqslant b$. Therefore $\mathrm{at}_1(\mathtt{T}) = \lambda_1 + 2c - b - 2d$. To compute $\mathrm{at}_2(\mathtt{T})$ in this case, after already reducing b, c and λ_2 by $\mathrm{at}_1(\mathtt{T})$ we need to further reduce $c' = c - \mathrm{at}_1(\mathtt{T})$ as well as d and $\lambda_2' = \lambda_2 - \mathrm{at}_1(\mathtt{T})$ by the maximal possible amount strictly smaller than d such that the preatom inequalities/equalities will still hold. This amount is

$$\mathrm{at}_2(\mathtt{T}) = \min\left(\lambda_2' + d - c', d-1\right) = \min\left(\lambda_2 + d - c, d-1\right)$$

since the inequality $\lambda_2 + d - c' \geqslant 0$ is the only preatom inequality affected by decreasing c, d and λ_2 simultaneously by the same amount. Moreover, it decreases precisely by this amount.

Finally, assume $0 < d < \lambda_1$. As in the discussion above we have

$$\mathrm{str}_2(\mathtt{T}) = \psi_2^{\mathrm{at}_2(\mathtt{T})}(\mathrm{str}_2(\mathtt{U})),$$

and thus $\mathrm{at}(\mathtt{T}) = \mathrm{at}_2(\mathtt{T})$. Moreover, if $0 < d < \lambda_1$ we have $b = \lambda_1 - 2c + 2d$ so we can also write $\mathrm{at}(\mathtt{T}) = \lambda_1 + 2c - 2d - b + \mathrm{at}_2(\mathtt{T}) = \lambda_1 + 2c - 2d - b + \mathrm{at}_2(\mathtt{T}) + \min(\lambda_2 + d - c, d-1)$. ☐

Corollary 4.27. *Let $\mathtt{U} \in \mathcal{P}(\lambda) \subset \mathcal{B}(\lambda)$ with $\mathrm{str}_2(\mathtt{U}) = (a,b,c,d)$. Then $\mathtt{U} \notin \Psi(\mathcal{P}(\lambda - \varpi_2))$ if and only if one of the following two conditions holds:*

- *$b = \lambda_1 - 2d + 2c$ and ($d \leqslant 1$ or $c = \lambda_2 + d$) ;*
- *$b < \lambda_1 - 2d + 2c$ and $c = d = 0$.*

Proof. We know that $\mathtt{U} \notin \Psi(\mathcal{P}(\lambda - \varpi_2)) \iff \mathrm{at}(\mathtt{U}) = 0$. First assume $\mathrm{at}(\mathtt{U}) = 0$. If $b = \lambda_1 - 2d + 2c$ then from Proposition 4.26 we see that either $d \leqslant 1$ or if $d > 1$, we must have $\min(\lambda_2 + d - c, d - 1) = 0$. Since $d > 1$ this implies that $\lambda_2 + d - c = 0$. If $b < \lambda_1 - 2d + 2c$ then by Proposition 4.26 $d > 0$ is impossible, so $d = 0$ necessarily. Moreover, since $\mathrm{at}(\mathtt{U}) = 0$ we must have $\min(c, \lambda_1 + 2c - b)$, but since the second term is strictly larger than zero by assumption, we conclude $c = 0$. Conversely, if $b = \lambda_1 - 2d + 2c$ and $d \leqslant 1$, it follows directly from Proposition 4.26 that $\mathrm{at}(\mathtt{U}) = 0$. If $c = \lambda_2 + d$ and $d > 1$ then $\mathrm{at}(\mathtt{U}) = 0$ also by Proposition 4.26. Now, if $b < \lambda_1 - 2d + 2c$ and $c = d = 0$ then $\mathrm{at}(\mathtt{U}) = 0$ applying the first formula in Proposition 4.26. ☐

4.3. Example: The atomic decomposition of $\mathcal{B}(k\varpi_2)$. Let $B_k := \mathcal{B}(k\varpi_2)$. By definition B_k consists of a single preatom. We describe now the atomic decomposition of B_k. Since $\lambda_1 = 0$ we have $\mathrm{str}_2(\mathtt{T}) = (a,b,c,0)$ for any $\mathtt{T} \in B_k$. By Lemma 4.23, we see that $\phi_1(\Psi(\mathtt{T})) = \phi_1(\mathtt{T})$ for any $\mathtt{T} \in B_k$, hence Ψ commutes with s_1 and we have $\Psi = \overline{\Psi}$. Then by Lemma 4.19.3, we see that Ψ also commutes with f_2.

Here we refer to the connected components under W, f_2 simply as *connected components* (cf. Remark 4.5). Notice that Ψ preserves these connected components. We claim that the crystal B_k has precisely $k+1$ connected components

$$B_k = \bigsqcup_{i=0}^{k} B_k[i].$$

and that $\Psi(B_{k-1}[i]) = B_k[i]$. In particular, it follows that $\mathcal{A}(k\varpi_2) = B_k \smallsetminus \Psi^2(B_{k-1}) = B_k[k] \sqcup B_k[k-1]$.

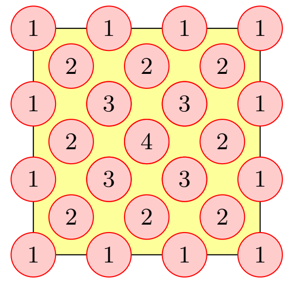

FIGURE 2. The weight multiplicities of the crystal B_3.

The crystal B_0 consists of a single element, the empty tableau, so the claim is trivial. In B_1 there are two connected components. In fact, it is easy to see that

$$B_1[0] = \left\{ \begin{array}{|c|}\hline 2 \\ \hline \bar{2} \\ \hline \end{array} \right\}$$

is fixed under the action of f_2 and s_1, and that its complement in B_1 is a connected component of cardinality 4.

The weights of the elements in $\mathcal{A}(k\varpi_2)$ form two square grid of side k and $k+1$ as shown in Figure 2, so $|\mathcal{A}(k\varpi_2)| = (k+1)^2+k^2$. From this, it follows that $|B_k|-|B_{k-1}| = (k+1)^2$.

By induction, to show our claim it is enough to show that the complement of $\Psi(B_{k-1})$ in B_k is a single connected component of cardinality $(k+1)^2$.

The complement of Ψ always contains the highest weight vector $\mathtt{T}_k \in B_k$. Then, for $0 \leqslant r \leqslant k$, the tableaux

$$f_2^r(\mathtt{T}_k) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline 2 & \cdots & 2 & \bar{2} & \cdots & \bar{2} \\ \hline \end{array},$$

in which there are r column of the form $\begin{array}{|c|}\hline 1 \\ \hline \bar{2} \\ \hline \end{array}$, are also in the same connected component as $\mathtt{T}_k$. We obtain $s_1(f_2^r(\mathtt{T}))$ from $f_2^r(\mathtt{T})$ by replacing the columns of the form $\begin{array}{|c|}\hline 1 \\ \hline \bar{2} \\ \hline \end{array}$ by columns of the form $\begin{array}{|c|}\hline 2 \\ \hline \bar{1} \\ \hline \end{array}$. The tableaux $s_1(f_2^r(\mathtt{T}))$ are the highest element in their f_2-string, and there are $k+1$ elements in their f_2-orbit, given by barring some of the 2's. So we have seen that there are at least $(k+1)^2$ elements in the connected components of $\mathtt{T}_k$. Since Ψ is an embedding and $|B_k| - |B_{k-1}| = (k+1)^2$, these are precisely all the elements in the complement of Ψ.

5. SWAPPABLE EDGES AND THEIR CLASSIFICATION

5.1. TWISTED BRUHAT GRAPHS. The Bruhat order on the weight lattice X is the order generated by the following relations

$$s_\alpha^\vee(\lambda) < \lambda \iff \begin{cases} \langle \lambda, \beta^\vee \rangle > M & \text{if } M \geqslant 0, \\ \langle \lambda, \beta^\vee \rangle < M & \text{if } M < 0. \end{cases} \tag{11}$$

where $\alpha^\vee = M\delta + \beta^\vee$, with $\beta^\vee \in \Phi_+^\vee$ and $\lambda \in X$. The set of elements smaller that λ in the Bruhat order, which we denote by $\{\leqslant \lambda\}$, can be characterized as

$$\{\leqslant \lambda\} = \mathrm{Conv}(W \cdot \lambda) \cap (\lambda + \mathbb{Z}\Phi) \tag{12}$$

(see for example [2, Chap. VIII, §7, exerc. 1]).

Let $\lambda \in X_+$. Let Γ_λ denote the moment graph of the spherical Schubert variety $\overline{\mathcal{G}r_\lambda}$. This is a directed labeled graph, also called the *Bruhat graph* of λ. We recall from [25, §2.3] the explicit description of Γ_λ. The vertices of the graph Γ_λ are all the weights in $\{\leqslant \lambda\}$. We have an edge $\mu_1 \to \mu_2$ in Γ_λ if and only if $\mu_2 - \mu_1$ is a multiple of a root $\beta \in \Phi$ and $\mu_1 \leqslant \mu_2$. In this case, the label of the edge $\mu_1 \to \mu_2$ is $m\delta - \beta^\vee$, where

$$m = -\frac{\langle \beta^\vee, \mu_1 + \mu_2 \rangle}{2}$$

(cf. [25, Lemma 2.7]). Notice that $s_{m\delta-\beta^\vee}(\mu_1) = \mu_2$. We denote by $E(\lambda)$the set of edges in Γ_λ.

Let Γ_X denote the union of all the graphs Γ_λ, for $\lambda \in X_+$ (where Γ_λ is regarded as a subgraph of $\Gamma_{\lambda'}$ if $\lambda \leqslant \lambda'$) and call it the Bruhat graph of X.

For $w \in \widehat{W}$ we denote by

$$N(w) := \{\alpha \in \widehat{\Phi}^\vee_+ \mid w^{-1}(\alpha) \in \widehat{\Phi}^\vee_-\}$$

the set of inversions. If $w = s_{i_1} \dots s_{i_k}$ is a reduced expression for w then

$$N(w) = \{\alpha^\vee_{i_1}, s_{i_1}(\alpha^\vee_{i_2}), \dots, s_{i_1}s_{i_2}\dots s_{i_{k-1}}(\alpha^\vee_{i_k})\}.$$

We say that $w = s_1 s_2 \dots s_k \dots$ is a reduced infinite expression if for any j the starting expression $w_j := s_1 s_2 \dots s_j$ is reduced. If w is a reduced infinite expression, let $N(w) = \bigcup_{j=1}^\infty N(w_j)$.

Consider $\underline{c} = s_0 s_2 s_1 s_2$. Then $y_\infty := \underline{ccc}\dots$ is an infinite reduced expression. Let y_m be the element given by the first m simple reflections in y_∞. We order the roots in $N(y_\infty)$ as follows:

$$\begin{aligned}(13)\quad & \delta - \alpha^\vee_{21} < \delta - \alpha^\vee_{12} < 2\delta - \alpha^\vee_{21} < \delta - \alpha^\vee_2 < 3\delta - \alpha^\vee_{21} < 2\delta - \alpha^\vee_{12} < \\ & \dots < M\delta - \alpha^\vee_{12} < 2M\delta - \alpha^\vee_{21} < M\delta - \alpha^\vee_2 < (2M+1)\delta - \alpha^\vee_{21} < \dots\end{aligned}$$

so that the first m roots in (13) are precisely the elements of $N(y_m)$.

We define the *m-twisted Bruhat order* $\leqslant_m$of $\widehat{W}_{ext}$ by setting

$$v \leqslant_m w \text{ if and only if } y_m^{-1} v \leqslant y_m^{-1} w,$$

and the m-twisted length by $\ell_m(v) := \ell(y_m^{-1} v)$. Recall that $X \cong \widehat{W}_{ext}/W$. Hence, the twisted Bruhat order on $\widehat{W}_{ext}$ also induces a twisted Bruhat order on X. Concretely, this means that we regard $\lambda \in X$ as a right coset in $\widehat{W}_{ext}$ and denote by $\lambda_m \in \widehat{W}_{ext}$ the element of minimal y_m-twisted length in the coset λ. Then we set $\ell_m(\lambda) := \ell_m(\lambda_m)$ and $\mu \leqslant_m \lambda$ if $\lambda_m \leqslant_m \mu_m$.

For every $m \in \mathbb{Z}_{\geqslant 0}$ we define Γ^m_λ, the *y_m-twisted Bruhat graph of* λ, to be the directed labeled graph with the same vertices of Γ_λ and where there is an edge $\mu \to \lambda$ if there exists $\alpha^\vee \in \widehat{\Phi}^\vee$ such that $s_{\alpha^\vee}(\mu) = \lambda$ and $\mu <_m \lambda$. Concretely, we can obtain Γ^m_λ from Γ_λ by inverting the orientation of all the arrows in Γ_λ with label in $N(y_m)$.

Since each graph Γ_λ has only a finite number of edges, the twisted graphs Γ^m_λ stabilize for m big enough, so we can define $\Gamma^\infty_\lambda := \Gamma^m_\lambda$ for $m \gg 0$.

For $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, we define Γ^m_X as the union of all the graphs Γ^m_λ, for $\lambda \in X_+$. The graph Γ^m_X can be obtained from Γ_X by inverting the orientation of all the arrows with label in $N(y_m)$.

Definition 5.1. *For $\mu \leqslant \lambda$, we denote by $\operatorname{Arr}_m(\mu, \lambda)$ the set of arrows pointing to μ in Γ^λ_m and by $\ell_m(\mu, \lambda) := |\operatorname{Arr}_m(\mu, \lambda)|$ the number of those arrows.*

For $i \in \{1, 2, 21, 12\}$ let $\operatorname{Arr}^i_m(\mu, \lambda)$ be arrows pointing to μ in Γ^λ_m of the form $\mu - k\alpha_i \to \mu$ for $k \in \mathbb{Z}$. Let $\ell^i_m(\mu, \lambda) = |\operatorname{Arr}^i_m(\mu, \lambda)|$.

Let $\mathrm{Arr}_\mu(\mu)$ *be the set of arrows pointing to* μ *in* Γ_X^m*. For* $i \in \{1, 2, 21, 12\}$*, the set* $\mathrm{Arr}_m^i(\mu)$ *is defined accordingly.*

Recall from [26, Lemma 2.10] that $|\mathrm{Arr}_m(\mu)| = \ell_m(\mu)$. We have

$$\mathrm{Arr}_m(\mu, \lambda) = \bigcup_{i\in\{1,2,12,21\}} \mathrm{Arr}_m^i(\mu, \lambda) \quad \text{and} \quad \ell_m(\mu, \lambda) = \sum_{i\in\{1,2,12,21\}} \ell_m^i(\mu, \lambda) \tag{14}$$

for any $\mu \leqslant \lambda$. Notice that, since there are no arrows of the form $M\delta - \alpha_1^\vee$ in $N(y_\infty)$, the set $\mathrm{Arr}_m^1(\mu, \lambda)$ does not depend on m, and does not depend on λ as long as $\mu \leqslant \lambda$. If $\mu \leqslant \lambda$, for all m by (11) we have

$$\begin{aligned} \mathrm{Arr}_m^1(\mu, \lambda) &= \{\mu - k\alpha_1 \to \mu \mid \mu - k\alpha_1 \leqslant \mu\} \\ &= \begin{cases} \{\mu - k\alpha_1 \to \mu \mid 0 < k \leqslant \mu_1\} & \text{if } \mu_1 \geqslant 0 \\ \{\mu - k\alpha_1 \to \mu \mid 0 > k > \ \mu_1\} & \text{if } \mu_1 < 0. \end{cases} \end{aligned}$$

Hence, we have

$$\ell_m^1(\mu, \lambda) = \begin{cases} \mu_1 & \text{if } \mu_1 \geqslant 0 \\ -\mu_1 - 1 & \text{if } \mu_1 < 0. \end{cases} \tag{15}$$

5.2. Swappable edges. To pass from Γ_λ^m to Γ_λ^{m+1} (and from Γ_X^m to Γ_X^{m+1}) we need to invert the arrows with label $\alpha_{t_{m+1}}^\vee$, where t_{m+1} is the reflection

$$t_{m+1} := y_{m+1} y_m^{-1} = y_m s'_{m+1} y_m^{-1}. \tag{16}$$

Here s'_{m+1} denotes the $(m+1)$-th simple reflection in y_∞. Notice that $\{\alpha_{t_{m+1}}^\vee\} = N(y_{m+1}) \setminus N(y_m)$.

If $\mu < t_{m+1}\mu$, then $\mathrm{Arr}_{m+1}(t_{m+1}\mu) \setminus \mathrm{Arr}_m(t_{m+1}\mu) = \{\mu \to t_{m+1}\mu\}$ and $\mathrm{Arr}_m(\mu)$ is in bijection with $\mathrm{Arr}_m(t_{m+1}\mu) \setminus \{\mu \to t_{m+1}\mu\}$ by [26, Lemma 2.11]. In particular, we have

$$\ell_m(\mu) = \ell_m(t_{m+1}\mu) - 1. \tag{17}$$

A property of the twisted Bruhat graphs in type A ([26, Prop. 2.17]) is that the same is true if we restrict to Γ_λ, i.e. $\ell_m(\mu, \lambda) = \ell_m(t_{m+1}\mu, \lambda) - 1$ if $\mu < t_{m+1}\mu \leqslant \lambda$. This implies that $\ell_{m+1}(\mu, \lambda) = \ell_m(t_{m+1}\mu, \lambda)$ and $\ell_m(\mu, \lambda) = \ell_{m+1}(t_{m+1}\mu, \lambda)$. However, as we will see in Example 5.3, this property does not hold in type C_2. The goal of this section is to classify the set of edges for which it holds.

Definition 5.2. *We say that an edge* $\mu \to t_{m+1}\mu$ *in* Γ_λ *is* swappable *if*

$$\ell_m(\mu, \lambda) = \ell_m(t_{m+1}\mu, \lambda) - 1. \tag{18}$$

We also say that an edge is NS *if it is not swappable. We denote by* $E^S(\lambda)$ *and* $E^N(\lambda)$ *the sets of swappable and non-swappable edges in* Γ_λ*, respectively.*

As it turns out, to determine if an edge is swappable or not, we have to solve an elementary geometric problem, which the next example illustrates.

Example 5.3. In the Figures 3 and 4 the starting points of the arrows in $\ell_m(\mu, \lambda)$ are denoted by red circles while the starting points of the arrows in $\ell_m(t_{m+1}\mu, \lambda)$ are denoted by blue squares.

Assume that $\lambda = (2, 2)$, $\mu = (2, -1)$ and that $m + 1 = 8$, i.e. that $t := t_{m+1}$ is the reflection corresponding to the root $2\delta - \alpha_2^\vee$. In Figure 3, the yellow octagon is the convex hull of $W \cdot \lambda$ while the green octagon is (the border of) the convex hull of $y_m W y_m^{-1} \cdot \mu$. As we will observe in Subsection 5.4, the arrows in $\mathrm{Arr}_m(\mu, \lambda)$ and $\mathrm{Arr}_m(t\mu, \lambda)$ can be characterized as the weights in the diagonal of the green octagon

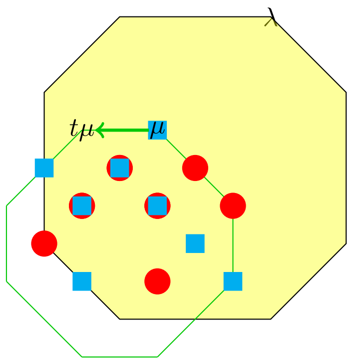

Figure 3. A swappable edge

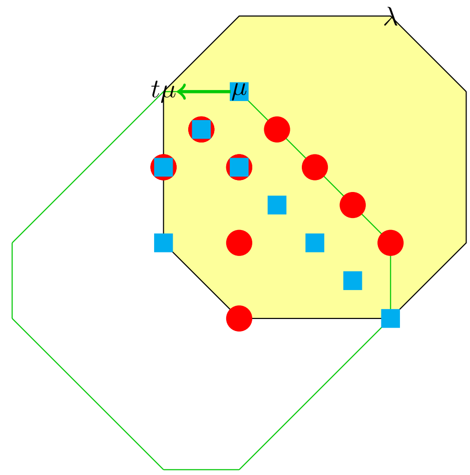

Figure 4. A non-swappable edge

which lie inside the yellow octagon. In this case we see that there are are 7 red dots and 8 blue squares, meaning that the edge $\mu \to t\mu$ is swappable.

Now assume that $\lambda = (2,2)$, $\mu = (4,-2)$ and $m+1 = 12$, i.e. that $t := t_{m+1} = s_{3\delta-\alpha_2^\vee}$. As illustrated in Figure 4, we have 9 red dots and 9 blue squares, so in this case the edge $\mu \to t\mu$ is not swappable.

5.3. Geometry of atoms. We fix $\lambda \in X_+$. Recall that $\{\leqslant \lambda\} = (\lambda + \mathbb{Z}\Phi) \cap \mathrm{Conv}(W \cdot \lambda)$.

In our situation, the convex hull $\mathrm{Conv}(W \cdot \mu)$ is an octagon with vertices as in Figure 5. We can make the actual conditions more explicit.

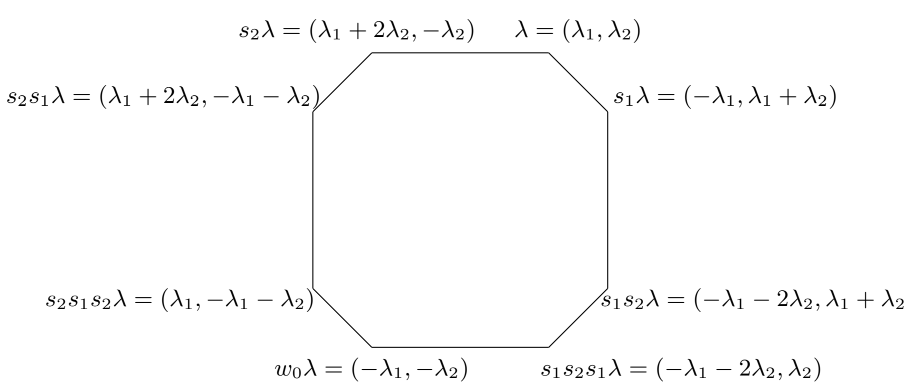

Figure 5. The W-orbit and the convex hull of λ

Lemma 5.4. *We have $\mu \leqslant \lambda$ if and only if $\mu_1 \equiv \lambda_1 \pmod 2$ and the following inequalities hold:*

$$\begin{aligned}
-\lambda_1 - 2\lambda_2 &\leqslant \mu_1 &&= \langle \mu, \alpha_1^\vee \rangle &&\leqslant \lambda_1 + 2\lambda_2 \\
-\lambda_1 - \lambda_2 &\leqslant \mu_1 + \mu_2 &&= \langle \mu, \alpha_{12}^\vee \rangle &&\leqslant \lambda_1 + \lambda_2 \\
-\lambda_1 - \lambda_2 &\leqslant \mu_2 &&= \langle \mu, \alpha_2^\vee \rangle &&\leqslant \lambda_1 + \lambda_2 \\
-\lambda_1 - 2\lambda_2 &\leqslant \mu_1 + 2\mu_2 &&= \langle \mu, \alpha_{21}^\vee \rangle &&\leqslant \lambda_1 + 2\lambda_2.
\end{aligned}$$

Proof. It is easy to see that $\mu \equiv \lambda \pmod{\mathbb{Z}\Phi}$ if and only if $\mu_1 = \lambda_1$. The inequalities can be easily deduced from Figure 5 $\square$

We introduce now some helpful quantities which evaluate the distance of a weight μ from the walls of $\mathrm{Conv}(W \cdot \lambda)$.

DEFINITION 5.5. *For $i \in \{1, 2, 21, 12\}$, let $\widehat{\phi}_i(\mu, \lambda)$ be the maximum integer k such that $\mu - k\alpha_i \leqslant \lambda$.*

LEMMA 5.6. *Let $\mu \leqslant \lambda$. We have*

(1) $\widehat{\phi}_{21}(\mu, \lambda) = \lambda_2 + \mu_2 + \min\left(\lambda_1, \frac{\lambda_1+\mu_1}{2}, \lambda_1 + \mu_1\right)$

(2) $\widehat{\phi}_{12}(\mu, \lambda) = \frac{\lambda_1+\mu_1}{2} + \min\left(\lambda_2 + \mu_2, \left\lfloor \frac{\lambda_2+\mu_2}{2} \right\rfloor, \lambda_2\right)$

(3) $\widehat{\phi}_{2}(\mu, \lambda) := \frac{\lambda_1-\mu_1}{2} + \min\left(\lambda_2 + \mu_1 + \mu_2, \left\lfloor \frac{\lambda_2+\mu_1+\mu_2}{2} \right\rfloor, \lambda_2\right)$.

Proof. We prove only the first statement, since the other two are analogous. Consider the maximal $x \in \mathbb{R}_{\geqslant 0}$ such that $\nu := \mu - x\alpha_{21} \in \mathrm{Conv}(W \cdot \lambda)$. Then $\mu - x\alpha_{21}$ belongs to the boundary of $\mathrm{Conv}(W \cdot \lambda)$ and $\widehat{\phi}_{21}(\mu, \lambda) = \lfloor x \rfloor$.

We have $(\nu_1, \nu_2) = (\mu_1, \mu_2 - x)$, hence by Lemma 5.4 the following inequalities three inequalities hold

$$\begin{aligned} -\lambda_1 - \lambda_2 &\leqslant \mu_1 + \mu_2 - x \\ -\lambda_1 - \lambda_2 &\leqslant \mu_2 - x \\ -\lambda_1 - 2\lambda_2 &\leqslant \mu_1 + 2\mu_2 - 2x \end{aligned}$$

and since we are on the boundary at least one of them must be an equality. It follows that

$$\begin{aligned} x &= \min(\mu_1 + \mu_2 + \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \mu_2, \frac{\lambda_1 + \mu_1}{2} + \lambda_2 + \mu_2) \\ &= \lambda_2 + \mu_2 + \min(\lambda_1, \frac{\lambda_1 + \mu_1}{2}, \lambda_1 + \mu_1). \end{aligned}$$ $\square$

5.4. TWISTED REFLECTION GROUPS. For $k \geqslant 0$ consider the reflection subgroup

$$W^k := y_k W y_k^{-1} \subset \widehat{W}.$$

Note that for any k we have $W^{k+1} = t_{k+1} W^k t_{k+1}$.

LEMMA 5.7. *For any $M > 0$ we have $W^{4M-3} = W^{4M-2} = W^{4M-1} = W^{4M}$. Moreover, the reflections in W^{4M} correspond to the roots*

$$\{\alpha_1^\vee, M\delta - \alpha_2^\vee, M\delta - \alpha_{12}^\vee, 2M\delta - \alpha_{21}^\vee\}.$$

Proof. We check this by induction. Recall that for any $M > 0$, t_{4M-3}, t_{4M-2}, t_{4M-1}, and t_{4M} are the reflections corresponding to the roots $(2M-1)\delta - \alpha_{21}^\vee$, $M\delta - \alpha_{12}^\vee$, $2M\delta - \alpha_{21}^\vee$, and $M\delta - \alpha_2^\vee$, respectively.

Recall that for any $M \in \mathbb{N}$ we have

$$W^{4M-3} = t_{4M-3} W^{4M-4} t_{4M-3}.$$

By induction, the reflections in W^{4M-4} correspond to the roots $\alpha_1^\vee, (M-1)\delta - \alpha_2^\vee, (M-1)\delta - \alpha_{12}^\vee$, and $2(M-1)\delta - \alpha_{21}^\vee$.

The claim follows since

$$\begin{aligned} s_{(2M-1)\delta - \alpha_{21}^\vee}(\alpha_1^\vee) &= \alpha_1^\vee \\ s_{(2M-1)\delta - \alpha_{21}^\vee}((M-1)\delta - \alpha_2^\vee) &= -M + \alpha_{12}^\vee \\ s_{(2M-1)\delta - \alpha_{21}^\vee}((M-1)\delta - \alpha_{12}^\vee) &= -M + \alpha_2^\vee \\ s_{(2M-1)\delta - \alpha_{21}^\vee}(2(M-1)\delta - \alpha_{21}^\vee) &= -2M\delta + \alpha_{21}^\vee, \end{aligned}$$

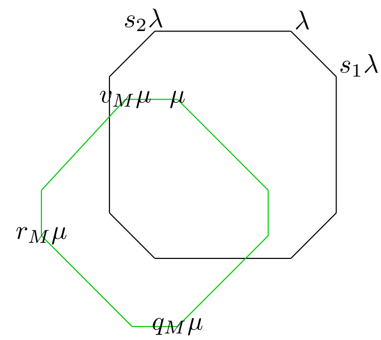

Figure 6. The green octagon is the border of the convex hull of $W^{4M} \cdot \mu$.

therefore $t_{4M-i} \in W^{4M-3}$ for $0 \leqslant i \leqslant 3$, which implies that

$$W^{4M} = W^{4M-1} = W^{4M-2} = W^{4M-3}. \qquad \square$$

There are four reflections in W^{4M}. The reflection corresponding to the root $\alpha_1^\vee$ is s_1.

Definition 5.8. *We denote the other three reflections in W^{4M} as follows.*

$$v_M := \textit{reflection corresponding to } M\delta - \alpha_2^\vee$$
$$q_M := \textit{reflection corresponding to } M\delta - \alpha_{12}^\vee$$
$$r_M := \textit{reflection corresponding to } 2M\delta - \alpha_{21}^\vee.$$

These reflection are also depicted in Figure 6. More explicitly, we have

$$v_M\mu = \mu - (\mu_2 + M)\alpha_2 = (\mu_1 + 2\mu_2 + 2M, -\mu_2 - 2M) \tag{19}$$
$$q_M\mu = \mu - (\mu_1 + \mu_2 + M)\alpha_{12} = (-\mu_1 - 2\mu_2 - 2M, \mu_2) \tag{20}$$
$$r_M\mu = \mu - (\mu_1 + 2\mu_2 + 2M)\alpha_{21} = (\mu_1, -\mu_1 - \mu_2 - 2M) \tag{21}$$

We also have $q_M = s_1 v_M s_1$ and $r_M = v_M s_1 v_M$.

We can use the twisted reflection subgroups W^m to describe the set of smaller elements with respect to twisted Bruhat order.

Lemma 5.9. *Let $\mu \in X$.*

1. *For any $m \geqslant 0$ we have $\{\leqslant_m \mu\} \subset \mathrm{Conv}(W^m \cdot \mu)$.*
2. *If $\mu_1 \geqslant 0$ and $\mu \leqslant v_M\mu$, we have*

$$\{\leqslant_{4M} \mu\} = \mathrm{Conv}(W^{4M} \cdot \mu) \cap (\mu + \mathbb{Z}\Phi) = \{\leqslant_{4M-1} v_M\mu\}$$

Proof. Let $\nu \leqslant_m \mu$. Then $y_m^{-1}\nu \leqslant y_m^{-1}\mu$, so $y_m^{-1}\nu \in \mathrm{Conv}(W \cdot y_m^{-1}\mu)$. This shows the first part. For the second part, because of (12), it is enough to show that $y_{4M}^{-1}\mu = y_{4M-1}^{-1}v_M\mu$ is dominant, since then

$$\{\leqslant_{4M} \mu\} = \{\leqslant_{4M-1} v_M\mu\} = \{\leqslant y_{4M}^{-1}\mu\} = \mathrm{Conv}(W \cdot y_{4M}^{-1}\mu) \cap (\mu + \mathbb{Z}\Phi).$$

Recall that a weight $\tau \in X$ is dominant if and only if $\tau \geqslant s_1\tau$ and $\tau \geqslant s_2\tau$. We have $s_1\mu \leqslant \mu$, and this is equivalent to $s_1\mu \leqslant_{4M} \mu$. Moreover, s_1 commutes with y_4 and therefore also with y_{4M}. It follows that $s_1 y_{4M}^{-1}\mu = y_{4M}^{-1}s_1\mu \leqslant y_{4M}^{-1}\mu$.

We have $\mu \leqslant v\mu$, and this is equivalent to $v\mu \leqslant_{4M} \mu$, so

$$y_{4M}^{-1}\mu \geqslant y_{4M}^{-1}v\mu = y_{4M-1}^{-1}\mu = s_2 y_{4M}^{-1}\mu. \qquad \square$$

Recall from Definition 5.5 the definition of $\widehat{\phi}_i(\mu,\lambda)$.

LEMMA 5.10. *Assume that $\mu \leqslant v_M\mu$. Then we have*

$$v_M\mu \nleqslant \lambda \iff M > \widehat{\phi}_2(\mu,\lambda) - \mu_2 \iff \ell^2_{4M-1}(\mu,\lambda) = \widehat{\phi}_2(\mu,\lambda), \tag{22}$$

Proof. By (19) and the definition of $\widehat{\phi}_2$ we have $v_M\mu \leqslant \lambda$ if and only if $\mu_2 + M \leqslant \widehat{\phi}_2(\mu,\lambda)$. It follows from Lemma 5.9.2) that $\mathrm{Arr}^2_{4M}(\mu)$ consists precisely of the arrows $(\mu - k\alpha_2 \to \mu)$, with $\mu - k\alpha_2$ lying on the segment between μ and $v_M\mu$. In other words, we have

$$\mathrm{Arr}^2_{4M}(\mu) = \{(\mu - k\alpha_2 \to \mu) \mid 1 \leqslant k \leqslant \mu_2 + M\}$$

If $v_M\mu \leqslant \lambda$, then $\mathrm{Arr}^2_{4M}(\mu) = \mathrm{Arr}^2_{4M}(\mu,\lambda)$, so

$$\ell^2_{4M-1}(\mu,\lambda) = \ell^2_{4M}(\mu,\lambda) - 1 = \mu_2 + M - 1 < \widehat{\phi}_2(\mu,\lambda).$$

If $v_M \nleqslant \lambda$ we have

$$\mathrm{Arr}^2_{4M}(\mu,\lambda) = \{\{(\mu - k\alpha_2) \to \mu \mid 1 \leqslant k \leqslant \widehat{\phi}_2(\mu,\lambda)\}$$

and so $\ell^2_{4M-1}(\mu,\lambda) = \ell^2_{4M}(\mu,\lambda) = \widehat{\phi}_2(\mu,\lambda)$. □

Similarly, we have

- $\mathrm{Arr}^{12}_{4M-2}(\mu) = \{(\mu - k\alpha_{12}) \to \mu \mid 1 \leqslant k \leqslant \mu_1 + \mu_2 + M\}$. and if $\mu \leqslant q_M\mu$ we have

$$q_M\mu \nleqslant \lambda \iff M > \widehat{\phi}_{12}(\mu,\lambda) - \mu_1 - \mu_2 \iff \ell^{12}_{4M-3}(\mu,\lambda) = \widehat{\phi}_{12}(\mu,\lambda) \tag{23}$$

- $\mathrm{Arr}^{21}_{4M-1}(\mu) = \{(\mu - k\alpha_{21}) \to \mu \mid 1 \leqslant k \leqslant \mu_1 + 2\mu_2 + 2M\}$ and if $\mu \leqslant r_M\mu$ we have

$$r_M\mu \nleqslant \lambda \iff 2M > \widehat{\phi}_{21}(\mu,\lambda) - \mu_1 - 2\mu_2 \iff \ell^{21}_{4M-2}(\mu,\lambda) = \widehat{\phi}_{21}(\mu,\lambda). \tag{24}$$

In the following Lemma we describe the Bruhat order on a W^{4M}-orbit.

LEMMA 5.11. *Let $\mu \in X$ and v_M, r_M, q_M as before. If $\mu < v_M\mu$ and $\mu_1 \geqslant 0$ or if $\mu < q_M\mu$ and $\mu_1 \leqslant 0$, then $v_M\mu \leqslant r_Mv_M\mu < r_M\mu$ and $q_M\mu < r_M\mu$.*

Proof. Assume first $\mu_1 \geqslant 0$ and $\mu < v_M\mu$, so $\mu_2 > -M$. We have $\langle v_M\mu, \alpha^\vee_{21}\rangle = (v_M\mu)_1 + 2(v_M\mu_2) = \mu_1 - 2M \geqslant -2M$, so $r_Mv_M\mu \geqslant v_M\mu$ by (11). We have $q_Mr_M = r_Mv_M$ and $\langle r_M\mu, \alpha^\vee_{12}\rangle = -\mu_1 - \mu_2 - 2M < -M$ so $r_Mv_M\mu < r_M\mu$. Similarly, we have $\mu < q_M\mu \leqslant v_Mq_M\mu \leqslant s_1v_Mq_M\mu = r_M\mu$. The case $\mu_1 \leqslant 0$ and $\mu < q_M\mu$ is similar. □

LEMMA 5.12. *Let $m > 0$ and assume $(t_m\mu)_1 \geqslant 0$ and $\mu \leqslant t_m\mu \leqslant \lambda$. Then $t_kt_m\mu \leqslant t_m\mu$ for all $k \leqslant m$ corresponding to roots of the form $K\delta - \alpha^\vee_2$.*

Assume instead $\mu_1 \geqslant 0$ and $\mu \leqslant t_m\mu \leqslant \lambda$. Then $t_k\mu \leqslant \lambda$ for all $k \leqslant m$ corresponding to roots of the form $K\delta - \alpha^\vee_2$.

Proof. First we prove the first part of the lemma. By assumption we have $k = 4K$, since t_k corresponds to a root of the form $K\delta - \alpha^\vee_2$. First assume that $m = 4M$, so $t_m = s_{M\delta - \alpha^\vee_2}$. Since $k \leqslant m$ we have $K \leqslant M$. By (11) we have that for $k = 4K \leqslant m = 4M$,

$$t_kt_m\mu \leqslant t_m\mu \iff \langle t_m\mu, -\alpha^\vee_2\rangle = \mu_2 + 2M > K.$$

We conclude the proof in this case since by assumption $\mu = t_mt_m\mu \leqslant t_m\mu$ and $K \leqslant M$.

Now we assume $m = 4M - 2$, so that $t_m = s_{M\delta - \alpha^\vee_{12}}$. In this case $4K \leqslant 4M - 2$, so in particular $K < M$. We have

$$t_kt_m\mu \leqslant t_m\mu \iff \langle t_m\mu, -\alpha^\vee_2\rangle = -\mu_2 > K.$$

Our assumption $\mu \leqslant t_m\mu$ implies that $\langle t_m\mu, -\alpha_{12}^\vee\rangle = \mu_1 + \mu_2 + 2M > M$ and $(t_m\mu)_1 \geqslant 0$ implies $\mu_1 + 2\mu_2 + 2M > 0$. Putting them together we obtain:

$$K < M \leqslant \mu_1 + \mu_2 + 2M \leqslant \mu_1 + 2\mu_2 + 2M - \mu_2 \leqslant -\mu_2.$$

which finishes the proof in this case.

Now we assume $m = 4M-1$, so that $t_m = s_{2M\delta - \alpha_{21}^\vee}$. In this case, we have

$$t_k t_m\mu \leqslant t_m\mu \iff \langle t_m\mu, -\alpha_2^\vee\rangle = \mu_1 + \mu_2 + 2M > K.$$

Our assumption $\mu \leqslant t_m\mu$ implies that $\langle t_m\mu, -\alpha_{21}^\vee\rangle = \mu_1 + 2\mu_2 + 4M \geqslant 2M$ and $\mu_1 = (t_m\mu)_1 \geqslant 0$. Putting them together we obtain:

$$2K < 2M \leqslant \mu_1 + 2\mu_2 + 4M \leqslant 2\mu_1 + 2\mu_2 + 4M.$$

Finally, assume that $m = 4M-3$, so that $t_m = s_{(2M-1)\delta - \alpha_{21}^\vee}$. This case follows by the same argument of the case $m = 4M-1$ since we have $2K < 2M-1$.

Now we proceed to prove the second part of the lemma, namely that, assuming $\mu_1 \geqslant 0$ and $\mu \leqslant t_m\mu \leqslant \lambda$, then $t_k\mu \leqslant \lambda$ for $k = 4K \leqslant m$. We can assume $\mu < t_k\mu$, otherwise the statement is obvious.

The case $m = 4M$ is clear since $t_k\mu$ lies on the segment between $t_m\mu$ and μ. Assume now $m = 4M-1$ or $m = 4M-3$. In both cases, we have $r_K\mu = t_{k-1}\mu \leqslant \lambda$ since it lies on the segment between μ and $t_m\mu$. We conclude by Lemma 5.11, since we get $v_K\mu \leqslant r_K v_K\mu \leqslant r_K\mu$. The last case to consider is $m = 4M-2$. Similarly, we have $q_K\mu \leqslant \lambda$ and also $s_1 q_K\mu = r_K v_K\mu \leqslant \lambda$. We conclude again by Lemma 5.11 since $v_K\mu \leqslant r_K v_K\mu$. ☐

5.5. Analysis of α_2-edges. In this section we fix $m+1 = 4M$ so that $v := v_M = t_{m+1}$ is the reflection corresponding to the affine root $M\delta - \alpha_2^\vee$, i.e. the reflection over the vertical axis $\{x \mid \langle x, \alpha_2^\vee\rangle = -M\}$. Let $r := r_M$ and $q := q_M$.

5.5.1. *Sufficient conditions for swappableness.* In this section, we assume that $\mu < v\mu \leqslant \lambda$ The goal of this section is to provide a first important constraint on an α_2-edge to be swappable (see Figure 7)

Proposition 5.13. *Assume that $q\mu \leqslant \lambda$. Then $\mu \to v\mu$ is swappable.*

We begin with a preliminary computation.

Lemma 5.14. *If $q\mu \leqslant \lambda$ and $r\mu \nleqslant \lambda$, then $-\lambda_1 \leqslant \mu_1 \leqslant \lambda_1$ and $(v\mu)_2 = -\mu_2 - 2M \leqslant -\lambda_2$.*

Proof. Observe that, since $\lambda \in X_+$, for any $\nu \in X$ we have $\nu \leqslant \lambda$ if and only if $s_1\nu \leqslant \lambda$. So we also have $s_1 r\mu = vq\mu \nleqslant \lambda$, $s_1 q\mu = qr\mu \leqslant \lambda$ and $s_1 v\mu = vr\mu \leqslant \lambda$.

If $\mu_1 > \lambda_1$ the line $\{\mu - x\alpha_{21}\}_{x \in \mathbb{R}_{>0}}$ intersects the boundary of $\mathrm{Conv}(W\cdot\lambda)$ in the segment

$$[s_2 s_1\lambda, s_2 s_1 s_2\lambda] \subset H := \{\nu \in X_{\mathbb{R}} \mid \langle \nu, \alpha_2^\vee\rangle = \langle s_2 s_1\lambda, \alpha_2^\vee\rangle = -\lambda_1 - \lambda_2\},$$

and since $r\mu \notin \mathrm{Conv}(W\cdot\lambda)$ we have

$$\langle r\mu, \alpha_2^\vee\rangle < -\lambda_1 - \lambda_2,$$

However, $qr\mu = s_1 q\mu$ lies on the same side of H as $r\mu$, since $\langle s_1 q\mu, \alpha_2^\vee\rangle = \langle r\mu, \alpha_2^\vee\rangle$. Therefore, $s_1 q\mu \notin \mathrm{Conv}(W\cdot\lambda)$, contradicting our assumption. Similarly, if $\mu_1 < -\lambda_1$, then we must have

$$\langle r\mu, \alpha_{12}^\vee\rangle < -\lambda_1 - \lambda_2,$$

which implies $vr\mu = s_1 v\mu \notin \mathrm{Conv}(W\cdot\lambda)$. We conclude that $-\lambda_1 \leqslant \mu_1 \leqslant \lambda_1$.

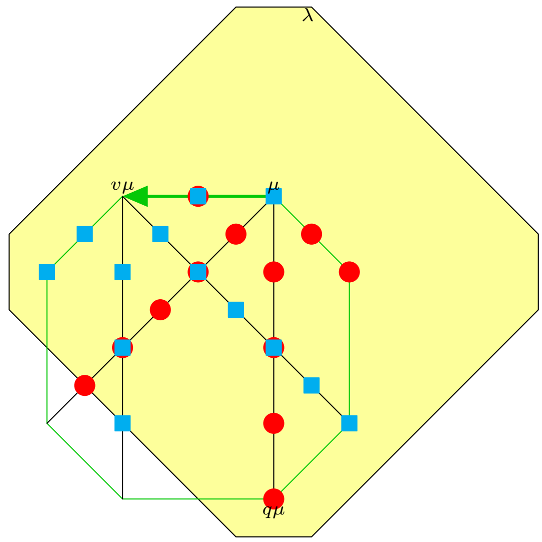

FIGURE 7. In this example $q\mu \leqslant \lambda$ and the edge $\mu \to v\mu$ is indeed swappable.

For the second part, assume that $(v\mu)_2 > -\lambda_2$, then the line $\{v\mu - x\alpha_{12}\}_{x\in\mathbb{R}_{>0}}$ intersects the segment

$$[s_2s_1s_2\lambda, w_0\lambda] \subset H' := \{\nu \in X_{\mathbb{R}} \mid \langle \nu, \alpha_{21}^\vee\rangle = \langle w_0\lambda, \alpha_{21}^\vee\rangle = -\lambda_1 - \lambda_2\}$$

forcing $\langle qv\mu, \alpha_{21}^\vee\rangle < -\lambda_1 - \lambda_2$. But since $\langle q\mu, \alpha_{21}^\vee\rangle = \langle qv\mu, \alpha_{21}^\vee\rangle$, this would contradict $q\mu \leqslant \lambda$. □

Proof of Proposition 5.13. Recall that $\ell_m(\mu) = \ell_m(v\mu) - 1$ by (17). To conclude it is enough to show that

$$\ell_m(\mu) - \ell_m(\mu, \lambda) = \ell_m(v\mu) - \ell_m(v\mu, \lambda) \tag{25}$$

The proof is divided in two cases. Assume first that $r\mu \leqslant \lambda$. In this case, since additionally $q\mu \leqslant \lambda$, the convex hull $\mathrm{Conv}(W^{m+1} \cdot \mu)$ is contained in $\mathrm{Conv}(W \cdot \lambda)$ entirely, so $\ell_m(\mu, \lambda) = \ell_m(\mu)$ and $\ell_m(v\mu) = \ell_m(v\mu, \lambda)$.

We can assume now that $r\mu \not\leqslant \lambda$. By Lemma 5.14, we have $-\lambda_1 \leqslant \mu_1 \leqslant \lambda_1$, which implies that $\lambda_1 + \mu_1 \geqslant 0$ and $\min(\lambda_1, \frac{\lambda_1+\mu_1}{2}, \lambda_1 + \mu_1) = \frac{\lambda_1+\mu_1}{2}$. It follows from Lemma 5.6 that

$$\widehat{\phi}_{21}(\mu, \lambda) = \mu_2 + \lambda_2 + \frac{\mu_1 + \lambda_1}{2}.$$

Since $q\mu \leqslant \lambda$, we have $\mathrm{Arr}^{12}_m(\mu) = \mathrm{Arr}^{12}_m(\mu, \lambda)$ and

$$\mathrm{Arr}^{21}_m(\mu) \setminus \mathrm{Arr}^{21}_m(\mu, \lambda) = \{(\mu - k\alpha_{21}) \to \mu \mid \widehat{\phi}_{21}(\mu, \lambda) < k \leqslant \mu_1 + 2\mu_2 + 2M\},$$

so we get

$$\begin{aligned} \ell_m(\mu) - \ell_m(\mu, \lambda) = \ell^{21}_m(\mu) - \ell^{21}_m(\mu, \lambda) &= 2M + \mu_1 + 2\mu_2 - \widehat{\phi}_{21}(\mu, \lambda) \\ &= 2M + \mu_2 - \lambda_2 + \frac{\mu_1 - \lambda_1}{2}. \end{aligned} \tag{26}$$

By Lemma 5.6, since $(v\mu)_2 \leqslant -\lambda_2$ we have

$$\widehat{\phi}_{12}(v\mu, \lambda) = \frac{\mu_1 + \lambda_1}{2} + \lambda_2 - M.$$

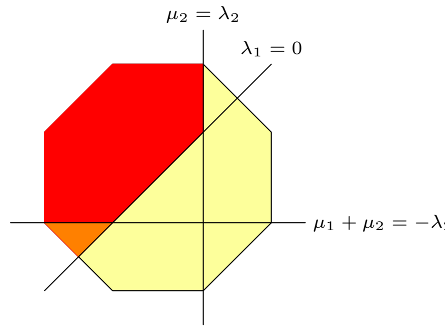

Figure 8. By Lemma 5.15 the starting point of a non-swappable edge in the α_2-direction must lie in the red or in the orange region. We further show in Proposition 5.18 that actually a starting point of a non-swappable edge can only be in the red region.

Similarly, since $rv\mu = s_1 q\mu \leqslant \lambda$ we have $\mathrm{Arr}^{21}_m(v\mu) = \mathrm{Arr}^{21}_m(v\mu,\lambda)$ and
$\mathrm{Arr}^{12}_m(v\mu) \smallsetminus \mathrm{Arr}^{12}_m(v\mu,\lambda) = \{v\mu - k\alpha_{12} \to v\mu \mid \widehat{\phi}_{12}(v\mu,\lambda) < k \leqslant (v\mu)_1 + (v\mu)_2 + M\}$
We get

$$\begin{aligned}\ell_m(v\mu) - \ell_m(v\mu,\lambda) &= \ell^{12}_m(v\mu) - \ell^{12}_m(v\mu,\lambda)\\ &= (v\mu)_1 + (v\mu)_2 + M - \widehat{\phi}_{12}(v\mu,\lambda)\\ &= \mu_1 + \mu_2 + M - \lambda_2 + M - \frac{\mu_1+\lambda_1}{2}. \end{aligned} \tag{27}$$

The claimed identity (25) now follows by comparing (26) and (27). □

As a consequence, an edge $\mu \to v\mu$ can only be not swappable if $q\mu \nleqslant \lambda$. This gives some constraint on the possible location of such weights μ (see Figure 8).

Lemma 5.15. *If $q\mu \nleqslant \lambda$, then $\mu_1 > 0$, $\mu_2 < \lambda_2$ and $r\mu \nleqslant \lambda$.*

Proof. Assume that $q\mu \nleqslant \lambda$. Then by (22) and (23) we have

$$\widehat{\phi}_2(\mu,\lambda) \geqslant \mu_2 + M > \widehat{\phi}_{12}(\mu,\lambda) - \mu_1.$$

This is equivalent to

$$\min(\lambda_2+\mu_1+\mu_2, \left\lfloor \frac{\lambda_2+\mu_1+\mu_2}{2} \right\rfloor, \lambda_2) > \min(\lambda_2+\mu_2, \left\lfloor \frac{\lambda_2+\mu_2}{2} \right\rfloor, \lambda_2). \tag{28}$$

This forces $\mu_1 > 0$. Moreover, we have $\mu_2 < \lambda_2$ otherwise both sides of (28) would be equal to λ_2.

Notice that if $q\mu \nleqslant \lambda$, also $s_1 q\mu \nleqslant \lambda$. Moreover, $r = qs_1q$ and $\langle r\mu, \alpha^\vee_{12}\rangle = -\mu_2 - 2M < -M$. By (11), we conclude that $q\mu < r\mu \nleqslant \lambda$. □

5.5.2. *Classification of swappable edges.*

Lemma 5.16. *Assume $\mu_1 \geqslant 0$. An edge $\mu \to v\mu$ is swappable if and only if*

$$2(\mu_2+M) + \ell^{12}_m(v\mu,\lambda) + \ell^{21}_m(v\mu,\lambda) = \ell^{12}_m(\mu,\lambda) + \ell^{12}_m(\mu,\lambda). \tag{29}$$

Proof. By Lemma 5.9, an arrow $(\mu - k\alpha_2 \to v\mu)$ is in $\mathrm{Arr}^2_m(\mu,\lambda)$ if and only if $0 \leqslant k < \mu_2 + M$. It follows that $\ell^2_m(\mu) = \ell^2_m(v\mu) - 1$. Moreover, by (15), we have

$$\ell^1_m(\mu,\lambda) = \ell^1_m(v\mu,\lambda) - 2(\mu_2+M).$$

The claim now follows directly from (14). □

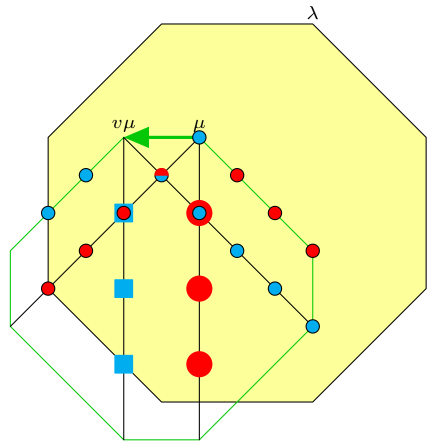

FIGURE 9. We have $\mu_1 \geqslant \lambda_1$, so to check whether $\mu \to v\mu$ is swappable we just need to count the weights below μ and $v\mu$. In this example they are both 3, hence the edge is swappable.

We now need to estimate carefully $\ell_m^{12}(\mu,\lambda)$ and $\ell_m^{21}(\mu,\lambda)$, i.e. we need to characterize the arrows in $\mathrm{Arr}_m^{12}(\mu)$ and $\mathrm{Arr}_m^{21}(\mu)$ whose starting point is contained in $\mathrm{Conv}(W\cdot\lambda)$.

We are now ready to classify all swappable α_2-edges. We have already seen that it is always swappable if $\mu_1 \leqslant 0$. Now we divide the rest into two cases: $\mu_1 \geqslant \lambda_1$ and $0 < \mu_1 < \lambda_1$. As illustrated in Figure 9, in the case $\mu_1 \geqslant \lambda_1$ it is sufficient to compare the number of weights below μ and $v\mu$ in the convex hull of $W\cdot\lambda$. We prove now this analytically.

PROPOSITION 5.17. *Let μ be such that $\mu_1 \geqslant \lambda_1$. Then $\mu \to v\mu$ is swappable if and only if*

$$\mu_2 \geqslant -\lambda_2 + 1 \text{ and } M \leqslant \left\lceil \frac{\lambda_2-\mu_2}{2} \right\rceil.$$

Proof. Since $\mu_1 \geqslant \lambda_1$ and $\mu \leqslant \lambda$, we have $\mu_2 \leqslant \lambda_2$. Since $\mu < v\mu$ we have $M+\mu_2 > 0$. We know that if $q\mu \leqslant \lambda$ then $\mu \to v\mu$ is swappable. In the other direction, if $\mu_2 \leqslant -\lambda_2$ or $\mu_2 > -\lambda_2$ and $M > \left\lceil \frac{\lambda_2-\mu_2}{2} \right\rceil$ then it follows from (23) that $q\mu \nleqslant \lambda$. So it is enough to consider the case $q\mu \nleqslant \lambda$.

We have $\lambda_1 \leqslant \mu_1 < (v\mu)_1$ and $(v\mu)_2 \leqslant \lambda_2$. In this case, we have

$$\widehat{\phi}_{21}(v\mu,\lambda) = \lambda_1 + \lambda_2 - \mu_2 - 2M = \widehat{\phi}_{21}(\mu,\lambda) - 2(\mu_2 + M). \tag{30}$$

Combining this with (29) and (23) we get that $\mu \to v\mu$ is swappable if and only if $\widehat{\phi}_{12}(\mu,\lambda) = \widehat{\phi}_{12}(v\mu,\lambda)$, which is equivalent to

$$\min\left(\lambda_2+\mu_2, \left\lfloor \frac{\lambda_2+\mu_2}{2} \right\rfloor\right) = \min\left(\lambda_2 - M, \left\lfloor \frac{\lambda_2+\mu_2}{2} \right\rfloor\right).$$

This equality holds if and only if both minima are achieved at $\left\lfloor \frac{\lambda_2+\mu_2}{2} \right\rfloor$, i.e. if

$$\lambda_2+\mu_2 \geqslant \left\lfloor \frac{\lambda_2+\mu_2}{2} \right\rfloor \leqslant \lambda_2 - M.$$

So we have $-\mu_2 < M \leqslant \left\lceil \frac{\lambda_2-\mu_2}{2} \right\rceil$, which is equivalent to $\mu_2 \geqslant -\lambda_2+1$ and the claim follows. □

Proposition 5.18. *Let $\mu \leqslant \lambda$ be such that $0 < \mu_1 < \lambda_1$. Then $\mu \to v\mu \in E^N(\lambda)$ if and only if*

$$M > \frac{\lambda_1 - \mu_1}{2} + \max\left(-\mu_2, \left\lceil \frac{\lambda_2 - \mu_2}{2} \right\rceil\right). \tag{31}$$

Proof. Notice that if the inequality (31) holds, then $q\mu \not\leqslant \lambda$ by (23). Since $\mu \to v\mu$ is swappable if $q\mu \leqslant \lambda$, we can just assume that $q\mu \not\leqslant \lambda$.

We begin by proving the following inequality.

Claim 5.19. *We have $(v\mu)_2 < -\lambda_2$.*

Proof of the claim. We have $(v\mu)_2 = -\mu_2 - 2M$. If $\mu_2 \leqslant -\lambda_2$, then $-\mu_2 - 2M < -M < \mu_2 \leqslant -\lambda_2$. If $\mu_2 \geqslant \lambda_2$, we have $-\mu_2 - 2M < -\mu_2 \leqslant -\lambda_2$.

If $-\lambda_2 < \mu_2 < \lambda_2$, then we have by (23) that

$$\begin{aligned} -\mu_2 - 2M &< \mu_2 + 2\mu_1 - 2\widehat{\phi}_{12}(\mu, \lambda) \\ &= -\lambda_1 + \mu_1 + \mu_2 - 2\left\lfloor \frac{\lambda_2 + \mu_2}{2} \right\rfloor \leqslant -\lambda_2 \end{aligned}$$ □

Assume first $\mu_1 < (v\mu)_1 \leqslant \lambda_1$, or equivalently that $M \leqslant \frac{\lambda_1 - \mu_1}{2} - \mu_2$. Since $q\mu \not\leqslant \lambda$, we have by (23) that

$$M > \frac{\lambda_1 - \mu_1}{2} - \min(\lambda_2, \left\lfloor \frac{\lambda_2 - \mu_2}{2} \right\rfloor), \tag{32}$$

forcing $\mu_2 < M - \frac{\lambda_1 - \mu_1}{2} < -\lambda_2$. Now we can easily compute both sides of (29) and check that they are both equal to $\lambda_1 + \mu_1 + 2(\lambda_2 + \mu_2)$. So $\mu \to v\mu$ is always swappable.

Assume now $\mu_1 < \lambda_1 < (v\mu)_1$, that is, that $M > \frac{\lambda_1 - \mu_1}{2} - \mu_2$. Since we assumed that $q\mu \not\leqslant \lambda$, by (23), we also have that

$$M > \frac{\lambda_1 - \mu_1}{2} + \left\lfloor \frac{\lambda_2 - \mu_2}{2} \right\rfloor. \tag{33}$$

In this case (29) is equivalent to

$$\frac{3\lambda_1 + \mu_1}{2} + 2\lambda_2 + \mu_2 - M = \lambda_1 + \mu_1 + \lambda_2 + \mu_2 + \min\left(\lambda_2 + \mu_2, \left\lfloor \frac{\lambda_2 + \mu_2}{2} \right\rfloor\right) \tag{34}$$

and we get an equality if and only if

$$M = \frac{\lambda_1 - \mu_1}{2} + \max\left(-\mu_2, \left\lceil \frac{\lambda_2 - \mu_2}{2} \right\rceil\right) \tag{35}$$

Notice that by (33) we cannot have $M < \frac{\lambda_1 - \mu_1}{2} + \left\lceil \frac{\lambda_2 - \mu_2}{2} \right\rceil$. The claim now follows. □

We can restate Proposition 5.18 in more convenient terms.

Corollary 5.20. *Assume $0 < \mu_1 < \lambda_1$. Then $(\mu \to v\mu) \in E^N(\lambda)$ if and only if $\lambda_1 < (v\mu)_1$ and $q\mu \not\leqslant \lambda$, except when $\lambda_2 \not\equiv \mu_2 \pmod 2$, $\lambda_2 + \mu_2 > 0$ and*

$$M = \frac{\lambda_1 - \mu_1}{2} + \left\lceil \frac{\lambda_2 - \mu_2}{2} \right\rceil$$

Proof. As in the proof of Proposition 5.18, we know that the only case in which $\lambda_1 < (v\mu)_1$, $q\mu \not\leqslant \lambda$ and $(\mu \to v\mu) \in E^S(\lambda)$ is for M as in (35). Since $\lambda_1 < (v\mu)_1$ then $M > \frac{\lambda_1 - \mu_1}{2} - \mu_2$. Since $q\mu \not\leqslant \lambda$ then $M > \frac{\lambda_1 - \mu_1}{2} + \left\lceil \frac{\lambda_2 - \mu_2}{2} \right\rceil$. So the equality in (35) is possible if and only if $\left\lceil \frac{\lambda_2 - \mu_2}{2} \right\rceil > \left\lfloor \frac{\lambda_2 - \mu_2}{2} \right\rfloor$ and $\left\lceil \frac{\lambda_2 - \mu_2}{2} \right\rceil > -\mu_2$, i.e. if $\lambda_2 \not\equiv \mu_2 \pmod 2$ and $\lambda_2 + \mu_2 \geqslant 1$. □

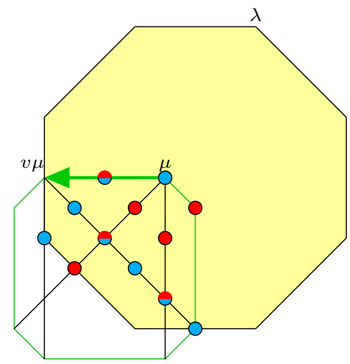

FIGURE 10. The exceptional case in which the edge $\mu \to v\mu$ is swappable even if $(v\mu)_1 > \lambda_1$.

EXAMPLE 5.21. Let $\lambda = (3,2)$, $\mu = (1,-1)$ and $M = \frac{\lambda_1-\mu_1}{2} - \left\lceil \frac{\lambda_2-\mu_2}{2} \right\rceil = 3$. As illustrated in Figure 10, in this case the edge $\mu \to v\mu$ is swappable even if $(v\mu)_1 > \lambda_1$.

5.6. ANALYSIS OF α_{12}-EDGES. Assume now that $m+1 = 4M-2$, so that $q_M := t_{m+1}$ is the reflection corresponding to the root $M\delta - \alpha_{12}^\vee$, i.e. a reflection over the hyperplane $\{\nu \mid \langle \nu, \alpha_{12}^\vee \rangle = -M\}$. Let $r := r_M$, $q := q_M$ and $v := v_M$ so that the reflections in the reflection subgroup W^{m+1} are $\{s_1, r, q, v\}$. The classification of NS-edges in the α_{12}-direction can be reduced to the known case of α_2-edges, as the Proposition 5.22 shows.

PROPOSITION 5.22. *An edge of the form $\mu \to q\mu$ is swappable if and only if $s_1\mu \to vs_1\mu$ is swappable.*

Proof. It is enough to show that

$$\ell_m(\mu,\lambda) - \ell_m(q\mu,\lambda) = \ell_{m+2}(s_1\mu,\lambda) - \ell_{m+2}(vs_1\mu,\lambda). \tag{36}$$

CLAIM 5.23. *Conjugation by s_1 induces a bijection*

$$\mathrm{Arr}_m(\mu,\lambda) \setminus \{s_1\mu \to \mu\} \cong \mathrm{Arr}_m(s_1\mu,\lambda) \setminus \{\mu \to s_1\mu\}$$

which sends $(u\mu \to \mu)$ to $(s_1u\mu \to s_1\mu)$. In particular, we have $\ell_m(\mu,\lambda) = \ell_m(s_1\mu,\lambda)+1$ if $\mu_1 > 0$.

Proof of the claim. Notice that, since $m = 4M-3$, we have $\beta^\vee \in N(y_m)$ if and only if $s_1(\beta^\vee) \in N(y_m)$. If $t\mu <_m \mu$ for $t = s_{N\delta-\alpha^\vee}$ with $\alpha^\vee \in \{\alpha_2^\vee, \alpha_{12}^\vee, \alpha_{21}^\vee\}$, then also $s_1(\alpha^\vee) \in \{\alpha_2^\vee, \alpha_{12}^\vee, \alpha_{21}^\vee\}$ and, since $\langle \mu, \alpha^\vee \rangle = \langle s_1(\mu), s_1(\alpha^\vee)\rangle$ it follows by (11) that also $s_1ts_1s_1(\mu) <_m s_1(\mu)$. If $t = s_{N\delta+\alpha_1^\vee}$ with $N \neq 0$, then $s_1ts_1 = s_{N\delta-\alpha_1^\vee}$ and we have

$$\begin{aligned} t\mu <_m \mu &\iff \mathrm{sgn}(N)\langle \mu, \alpha_1^\vee \rangle > |N| \\ &\iff \mathrm{sgn}(-N)\langle s_1(\mu), \alpha_1^\vee \rangle > |-N| \\ &\iff s_1t(\mu) <_m s_1(\mu). \end{aligned}$$
□

We assume now $\mu_1 < 0$ and let $\nu = s_1(\mu)$. Notice that $(q\mu)_1 < 0$. Since $vs_1 = s_1q$, using the claim, (36) is equivalent to

$$\ell_m(\nu,\lambda) - \ell_m(v\nu,\lambda) = \ell_{m+2}(\nu,\lambda) - \ell_{m+2}(v\nu,\lambda).$$

For any weight μ', the symmetric difference $\mathrm{Arr}_m(\mu',\lambda) \vartriangle \mathrm{Arr}_{m+2}(\mu',\lambda)$ is contained in $\{q\mu' \to \mu', r\mu' \to \mu'\}$ since these are the two only edges which we are possibly reversing. Then the claim follows since we have

$$\ell_{m+2}(\nu,\lambda) - \ell_m(\nu,\lambda) = |\{q\nu, r\nu\} \cap \{\leqslant \lambda\}| = \ell_{m+2}(v\nu,\lambda) - \ell_m(v\nu,\lambda).$$

In fact, by Lemma 5.11 we have $\nu < q\nu$, and $v\nu < rv\nu$ and $qv \leqslant \lambda \iff s_1 q\nu = rv\nu < \lambda$. Similarly, we have $\nu < r\nu$ and $v\nu < qv\nu$ and $r\nu \leqslant \lambda \iff s_1 r\nu = qr\nu < \lambda$.

The case $\mu_1 \geqslant 0$ is similar. ☐

5.7. Analysis of α_{21}-edges. We conclude the classification of swappable edges by looking at edges in the α_{21}-direction. In this case, the classification is trivial since, as it turns out, all the α_{21}-edges are swappable.

Proposition 5.24. *Any edge of the form $\mu \to \mu - k\alpha_{21}$ is swappable.*

Proof. We can assume that $\mu - k\alpha_{21} = s_{(2M-j)\delta-\alpha_{21}^\vee}(\mu)$ with $j=0$ or $j=1$. The root $(2M-j)\delta - \alpha_{21}^\vee$ is the $(4M-1-2j)$-th root occurring in (13). Let $m+1 = 4M-1-2j$ so that $r := t_{m+1} = s_{(2M-j)\delta-\alpha_{21}^\vee}$.

We have $\ell^1(\mu) = \ell^1(r\mu)$ and $\ell^{21}(\mu,\lambda) = \ell^{21}(r\mu,\lambda) - 1$, so to show that $\mu \to r\mu$ is swappable it is enough to check that

$$\ell^2(\mu,\lambda) + \ell^{12}(\mu,\lambda) = \ell^2(r\mu,\lambda) + \ell^{12}(r\mu,\lambda). \tag{37}$$

We consider first the case $j=0$, so W^m is the reflection subgroup with reflections s_1, q_M, r_M, v_M. Notice that $r = r_M$. If $\mu_1 \geqslant 0$ and $v_M\mu \geqslant \mu$ then $\mathrm{Conv}(W^m\cdot\mu) \subset \mathrm{Conv}(W\cdot\lambda)$ and the edge $\mu \to r\mu$ is swappable by the same argument as in the proof of Proposition 5.13.

Assume now $\mu_1 \geqslant 0$ and $v_M\mu < \mu$. Notice that we also have $v_M r\mu < r\mu$. If $q\mu \leqslant \lambda$, then again $\mathrm{Conv}(W^m\cdot\mu) \subset \mathrm{Conv}(W\cdot\lambda)$. If $q\mu \not\leqslant \lambda$, then also $s_1 q\mu = qr\mu \not\leqslant \lambda$ and we can rewrite (37) as

$$\ell^2_m(\mu) + \widehat{\phi}_{12}(\mu,\lambda) = \ell^2_m(r\mu) + \widehat{\phi}_{12}(r\mu,\lambda). \tag{38}$$

Claim 5.25. *We have $\mu_2 < -\lambda_2$.*

Proof of the claim. we have $\mu > v_M\mu$ so $\mu_2 + M < 0$. If $\mu_2 \geqslant -\lambda_2$ then $\widehat{\phi}_{12}(\mu,\lambda) = \lambda_1 + \mu_1 + \left\lfloor \frac{\lambda_2+\mu_2}{2} \right\rfloor$ and $q\mu \not\leqslant \lambda$ implies by (23) that $\mu_1 + \mu_2 + 2M > \lambda_1 + \lambda_2$. In particular, $\mu_1 > \lambda_1 + \lambda_2 - \mu_2 - 2M \geqslant \lambda_1$, so $\widehat{\phi}_{21}(\mu,\lambda) = \lambda_2 + \mu_2 + \lambda_1$ but this leads to a contradiction since $r\mu \leqslant \lambda$ and by (24) we get $\mu_1 + \mu_2 + 2M \leqslant \lambda_1 + \lambda_2$. ☐

We now go back to the proof of (38). We have $\ell^2_m(\mu) = -\mu_2 - M - 1$ and $\ell^2_m(r\mu) = \mu_1 + \mu_2 + M - 1$. Since $\mu_2 < -\lambda_2$ we have by Lemma 5.6 that $\widehat{\phi}_{12}(\mu,\lambda) = \frac{\mu_1+\lambda_1}{2} + \lambda_2 + \mu_2$ and $\widehat{\phi}_{12}(r\mu,\lambda) = \frac{\mu_1+\lambda_1}{2} + \lambda_2 - \mu_1 - \mu_2 - 2M$ and the claim easily follows. The case $\mu_1 < 0$ is analogous.

Consider now the case $j=1$. The proof here is similar, with the main difference being that the reflections in W^m are s_1, q_M, r_M, v_M but $r \neq r_M$. In fact, we have $r = s_{(2M-1)\delta-\alpha_{21}^\vee}$ and $r_M = s_{2M\delta-\alpha_{21}^\vee}$. In the case $\mu_1 \geqslant 0$ and $v_M\mu \geqslant \mu$, or $q_M\mu \leqslant \lambda$ then, similarly to the previous case, we have

$$\{\leqslant_m \mu\} \subset \mathrm{Conv}(W^m\cdot\mu) \smallsetminus \{r_M\mu\} \subset \mathrm{Conv}(W\cdot\lambda).$$

(In other words, the convex hull of $W^m\cdot\mu$ must lie inside $\mathrm{Conv}(W\cdot\lambda)$, except possibly for $r_M\mu$, but this does not matter since $\mu <_m r_M\mu$.) It follows that $\mu \to r\mu$ is swappable. If $\mu > v_M\mu$ and $q_M\mu \not\leqslant \lambda$ then we conclude by checking the identity (38) as before. The case $\mu_1 < 0$ is symmetric. ☐

5.8. Consequences of the classification. We can summarize the results from the previous three sections in the following proposition.

Proposition 5.26. *Assume $\mu_1 > 0$ and let t be a reflection. If the edge $\mu \to t\mu$ is not swappable, then t corresponds to a root of the form $M\delta - \alpha_2^\vee$.*

Proof. By Proposition 5.24, we know that $\mu \to t_{m+1}\mu$ cannot be in the α_{21}-direction. Since $\mu_1 \geqslant 0$, by Proposition 5.22 we also know that it cannot be in the α_{12}-direction. Hence, the only possibility is that it is an edge in the α_2 direction. □

The classification of swappable edges also allows us to easily compare swappable edges for different atoms.

Proposition 5.27. *Let $\mu \leqslant \lambda$ with $\mu_1 \geqslant 0$. Let $m = 4M$ so that $t_m = s_{M\delta-\alpha_2^\vee}$ and assume $\mu < t_m\mu \leqslant \lambda$. Consider the arrow $(\mu \to t_m\mu) \in E(\lambda)$.*

1. *If $(\mu \to t_m\mu) \in E^S(\lambda)$, then $(\mu \to t_m\mu) \in E^S(\lambda + k\varpi_2)$, for any $k \geqslant 0$.*
2. *If $(\mu \to t_m\mu) \in E^S(\lambda)$, then for any $k < m$ such that $\mu < t_k\mu \leqslant \lambda$, we also have $(\mu \to t_k\mu) \in E^S(\lambda)$.*
3. *If $(\mu \to t_m\mu) \in E^N(\lambda)$, then $\lambda - \varpi_2$ is dominant and $\mu \leqslant \lambda - \varpi_2$.*

Proof. The first two statements are clear from the explicit description given in Proposition 5.17 and Proposition 5.18.

To prove (3), first notice that if $\lambda_2 = 0$, by Lemma 5.15 and Proposition 5.18 there can be no non-swappable edges.

We now need to show the inequalities from Lemma 5.4 for μ and for $\lambda' = \lambda - \varpi_2$. In fact, by Lemma 5.15 and Proposition 5.18, we only need to establish the following inequalities, since they describe the hyperplanes delimiting the red region in Figure 8:

1. $\mu_1 + \mu_2 \leqslant \lambda_1 + \lambda_2 - 1$
2. $\mu_1 \leqslant \lambda_1 + 2\lambda_2 - 2$
3. $\mu_2 \geqslant -\lambda_1 - \lambda_2 + 1$

However note that if μ lies on either one of the hyperplanes defined by $\mu_1 = \lambda_1 + 2\lambda_2$ or $\mu_2 = -\lambda_1 - \lambda_2$ or , then $t_m\mu \not\leqslant \lambda$ since $t_m\mu$ is "on the left" of μ. Therefore the only inequality we really need to prove is 1.

We assume that the inequality is not true, that is, μ lies in the hyperplane defined by $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$. In particular, since $\mu \leqslant \lambda$, such μ must belong to the "top side" of the octagon $\mathrm{Conv}(W \cdot \lambda)$ and it must satisfy $\lambda_1 \leqslant \mu_1 \leqslant \lambda_1 + 2\lambda_2$ and $-\lambda_2 \leqslant \mu_2 \leqslant \lambda_2$. Also $(t_m\mu)_2$ must lie on the same side of the octagon, so necessarily then $(t_m\mu)_2 = -\mu_2 - 2M \geqslant -\lambda_2$, which holds if and only if

$$M \leqslant \frac{\lambda_2 - \mu_2}{2}.$$

We get a contradiction, since by Proposition 5.17 the edge $\mu \to t_m\mu$ is swappable. □

We are now ready to count the number of non-swappable edges.

Definition 5.28. *For $\mu \leqslant \lambda$ and $m \in \mathbb{N}$, we denote by*

$$\mathcal{N}_m(\mu, \lambda) := |\{k \leqslant m \mid \mu < t_k\mu \leqslant \lambda \text{ and } \mu \to t_k\mu \text{ is not swappable}\}|$$

the number of non-swappable edges in $E^N(\lambda)$ corresponding to a reflection t_k, for $k \leqslant m$, with starting point μ. Let

$$\mathcal{N}_\infty(\mu, \lambda) := |\{k \in \mathbb{N} \mid \mu < t_k\mu \leqslant \lambda \text{ and } \mu \to t_k\mu \text{ is not swappable}\}|.$$

Lemma 5.29. *If $\mu \leqslant t_m\mu \leqslant \lambda$, then $\mathcal{N}_m(t_m\mu, \lambda) = 0$.*

Proof. If $(t_m\mu)_1 \geqslant 0$, this follows directly from Lemma 5.12. The case $(t_m\mu) < 0$ is symmetric. □

Note that since Γ_λ is a finite graph, we have $\mathcal{N}_\infty(\mu,\lambda) = \mathcal{N}_m(\mu,\lambda)$ for m large enough. If $\mu_1 \geqslant 0$, the only non-swappable edges are in the α_2-direction, so in this case we have

$$\mathcal{N}_m(\mu,\lambda) = \left|\left\{1 \leqslant K \leqslant \left\lfloor \frac{m}{4} \right\rfloor \mid (\mu \to s_{K\delta-\alpha_2^\vee}\mu) \in E^N(\lambda)\right\}\right|.$$

Proposition 5.30. *Let $\widetilde{M} = \min(\left\lfloor \frac{m}{4}\right\rfloor, -\mu_2 + \widehat{\phi}_2(\mu,\lambda))$ and assume that $\mu_1 \geqslant 0$. Then we have*

$$\mathcal{N}_m(\mu,\lambda) = \begin{cases} \widetilde{M} + \min(\mu_2, \left\lfloor \frac{\mu_2-\lambda_2}{2}\right\rfloor) & \text{if } \mu_1 \geqslant \lambda_1 \\ \widetilde{M} + \frac{\mu_1-\lambda_1}{2} + \min(\mu_2, \left\lfloor \frac{\mu_2-\lambda_2}{2}\right\rfloor) & \text{if } 0 < \mu_1 < \lambda_1,\ \mu_2 \leqslant \lambda_2 \\ & \quad \text{and } \mu_1+\mu_2 \geqslant -\lambda_2 \\ 0 & \text{if } \mu_1 = 0,\ \mu_2 \geqslant \lambda_2 \\ & \quad \text{or } \mu_1+\mu_2 \leqslant -\lambda_2. \end{cases}$$

Proof. This follows directly from Propositions 5.17 and 5.18. □

If $\mu_1 \geqslant 0$, taking the limit $m \to \infty$ we get

$$(39)\quad \mathcal{N}_\infty(\mu,\lambda) = \begin{cases} \widehat{\phi}_2(\mu,\lambda) - \max(0, \left\lceil \frac{\mu_2+\lambda_2}{2}\right\rceil) & \text{if } \mu_1 \geqslant \lambda_1; \\ \widehat{\phi}_2(\mu,\lambda) + \frac{\mu_1-\lambda_1}{2} - \max(0, \left\lceil \frac{\mu_2+\lambda_2}{2}\right\rceil) & \text{if } 0 < \mu_1 < \lambda_1,\ \mu_2 \leqslant \lambda_2 \\ & \quad \text{and } \mu_1+\mu_2 \geqslant -\lambda_2; \\ 0 & \text{if } \mu_1 = 0,\ \mu_2 \geqslant \lambda_2 \\ & \quad \text{or } \mu_1+\mu_2 \leqslant -\lambda_2. \end{cases}$$

If $\mu_1 < 0$ we have $\mathcal{N}_\infty(\mu,\lambda) = \mathcal{N}_\infty(s_1(\mu),\lambda)$. In particular, we have

$$(40)\qquad \mathcal{N}_\infty(\mu,\lambda) = \begin{cases} \widehat{\phi}_{12}(\mu,\lambda) + \min\left(0, \dfrac{-\mu_1-\lambda_1}{2}\right) & \text{if } \mu_1+\mu_2 \leqslant \lambda_2 \\ \qquad - \max\left(0, \left\lceil \dfrac{\mu_1+\mu_2+\lambda_2}{2}\right\rceil\right) & \quad \text{and } \mu_2 \geqslant -\lambda_2; \\ 0 & \text{if } \mu_1+\mu_2 \geqslant \lambda_2 \\ & \quad \text{or } \mu_2 \leqslant -\lambda_2. \end{cases}$$

A remarkable property is that the number of NS edges gives exactly the correction term in (18) for non-swappable edges.

Proposition 5.31. *For any $\mu \leqslant \lambda$ with $t_{m+1}\mu \leqslant \lambda$, we have*

$$\ell_{m+1}(\mu,\lambda) - \ell_{m+1}(t_{m+1}\mu,\lambda) - 1 = \ell_m(\mu,\lambda) - \ell_m(t_{m+1}\mu,\lambda) + 1 = \mathcal{N}_{m+1}(\mu,\lambda).$$

Proof. The first equality is clear because $\mu <_m t_{m+1}\mu <_{m+1} \mu$, so we just need to show the second one.

If $\mu \to t_{m+1}\mu$ is swappable the claim is clear since $\mathcal{N}_{m+1}(\mu,\lambda) = 0$ by Proposition 5.27. We can assume $\mu_1 > 0$ and $v := t_{m+1} = s_{M\delta-\alpha_2^\vee}$, since the case $\mu_1 < 0$ and $t_{m+1} = s_{M\delta-\alpha_{12}^\vee}$ is analogous. In this case we have $q_M\mu \not\leqslant \lambda$ and $q_M v\mu \not\leqslant \lambda$.

Assume first $\mu_1 \geqslant \lambda_1$. Then

$$\begin{aligned} \ell_m(\mu,\lambda) - \ell_m(v\mu,\lambda) + 1 &= \ell^{12}_m(\mu,\lambda) - \ell^{12}_m(v\mu,\lambda) = \\ &= \min(\lambda_2+\mu_2, \left\lfloor \frac{\lambda_2+\mu_2}{2}\right\rfloor) - \min(\lambda_2 - M, \left\lfloor \frac{\lambda_2+\mu_2}{2}\right\rfloor) \\ &= M + \min(\mu_2, \left\lfloor \frac{\mu_2-\lambda_2}{2}\right\rfloor) \end{aligned}$$

In fact, since $\mu \to v\mu$ is not swappable, and $\lambda_2 + \mu_2 > \lambda_2 - M$, we have $\lambda_2 - M > \left\lfloor \frac{\lambda_2+\mu_2}{2} \right\rfloor$. The same computation also shows that the minimal K such that $\mu \to s_{K\delta-\alpha_2^\vee}\mu \in E^N(\lambda)$ is

$$K = \max(-\mu_2, \left\lceil \frac{\lambda_2 - \mu_2}{2} \right\rceil) + 1$$

so also $\mathcal{N}_{m+1}(\mu,\lambda) = M - K + 1 = M + \min(\mu_2, \left\lfloor \frac{\mu_2-\lambda_2}{2} \right\rfloor)$.

Assume now $\mu_1 < \lambda_1$. Recall that in this case we have $\mu_2 \geqslant -\lambda_2 + 1$. As in (34), we have

$$\ell_m(\mu,\lambda) - \ell_m(t_{m+1}\mu,\lambda) + 1 = M + \left\lfloor \frac{\mu_2 - \lambda_2}{2} \right\rfloor + \frac{\mu_1 - \lambda_1}{2}$$

In this case, the minimal K such that $\mu \to s_{K\delta-\alpha_2^\vee}\mu \in E^N(\lambda)$ is

$$K = \frac{\lambda_1 - \mu_1}{2} + \left\lceil \frac{\lambda_2 - \mu_2}{2} \right\rceil + 1.$$

and again $\mathcal{N}_{m+1}(\mu,\lambda) = M - K + 1$. □

We can also generalize Proposition 5.31 to the case when $t_{m+1}\mu \not\leqslant \lambda$. In this case $\ell_m(t_{m+1}\mu,\lambda)$ is not properly defined, so we first need to generalize its definition.

Definition 5.32. *Let $\mu \in X$ and assume $\mu_1 \geqslant 0$. For $m \in \mathbb{N}$ and $i \in \{2, 12, 21\}$ we define*

$$\widehat{\ell}^i_m(\mu,\lambda) := \begin{cases} \ell^i_m(\mu,\lambda) & \text{if } \mu \leqslant \lambda \\ \widehat{\phi}_i(\mu,\lambda) & \text{if } \mu \not\leqslant \lambda, \end{cases}$$

where here $\widehat{\phi}_i(\mu,\lambda)$ is to be interpreted as the function given in Lemma 5.6 (notice that $\widehat{\phi}_i$ is not properly defined if $\mu \not\leqslant \lambda$). Then we define

$$\widehat{\ell}_m(\mu,\lambda) := \ell^1(\mu) + \ell^2_m(\mu) + \widehat{\ell}^{21}_m(\mu,\lambda) + \widehat{\ell}^{12}_m(\mu,\lambda).$$

Notice that $\widehat{\ell}_m(t_m\mu,\lambda) = \ell_m(t_m\mu,\lambda)$ if $m = 4M$ and $\mu \leqslant t_m\mu \leqslant \lambda$.

Corollary 5.33. *Let $\mu \leqslant \lambda$ and $m = 4M$. Then we have*

$$\ell_m(\mu,\lambda) - \widehat{\ell}_m(t_m\mu,\lambda) - 1 = \mathcal{N}_m(\mu,\lambda)$$

Proof. Let $v := t_m$. We can assume $v\mu \not\leqslant \lambda$, otherwise the claim follows by Proposition 5.31. Notice that this forces $q_M\mu \not\leqslant \lambda$ and $r_M\mu \not\leqslant \lambda$. Notice also that $\ell^2_m(v\mu) = -\mu_2 - M - 1$ and that $\mathcal{N}_m(\mu,\lambda) = \mathcal{N}_\infty(\mu,\lambda)$.

Assume first $\mu_1 \geqslant \lambda_1$. We have

$$\begin{aligned} \ell_m(\mu,\lambda) - \widehat{\ell}_m(v\mu,\lambda) - 1 &= \widehat{\phi}_{12}(\mu,\lambda) - \widehat{\phi}_{12}(v\mu,\lambda) + \widehat{\phi}_2(\mu,\lambda) - \ell^2_m(v\mu) - 1 \\ &= M + \min(\mu_2, \left\lfloor \frac{\mu_2 - \lambda_2}{2} \right\rfloor) + \widehat{\phi}_2(\mu,\lambda) - \mu_2 - M \\ &= \widehat{\phi}_2(\mu,\lambda) + \min(0, -\left\lceil \frac{\lambda_2 + \mu_2}{2} \right\rceil). \end{aligned}$$

Assume now $\mu_1 < \lambda_1$. In this case, we have

$$\begin{aligned}\ell_m(\mu,\lambda) - \widehat{\ell}_m(v\mu,\lambda) - 1 &= M + \left\lfloor \frac{\mu_2-\lambda_2}{2} \right\rfloor + \frac{\mu_1-\lambda_1}{2} + \widehat{\phi}_2(\mu,\lambda) - \ell^2_m(v\mu) - 1 \\ &= M + \min(\mu_2, \left\lfloor \frac{\mu_2-\lambda_2}{2} \right\rfloor) + \widehat{\phi}_2(\mu,\lambda) - \mu_2 - M \\ &= \widehat{\phi}_2(\mu,\lambda) + \min(0, -\left\lceil \frac{\lambda_2+\mu_2}{2} \right\rceil) + \frac{\mu_1-\lambda_1}{2}.\end{aligned}$$

The claim follows by comparing these formulas with (39). □

5.9. Non-swappable staircases. In type A_n, swapping functions can be defined within a single atom. Unfortunately, the existence of non-swappable edges in type C_2 means that we cannot do the same, causing a relevant increase in complexity. Instead, for every non-swappable edge, the swapping functions we are going to construct in Section 6 will involve two elements from two different atoms within the same preatom. To determine which are the two atoms involved we need to introduce a new quantity, which we call the elevation of an edge and that measures the height of the maximal staircases of non-swappable edges lying underneath it.

Definition 5.34. *Let $e = (\mu \to \mu - k\alpha) \in E(\lambda)$ be an edge. We call the* elevation *of e the minimum integer $j \geqslant 0$ such that $(\mu \to \mu - (k-j)\alpha) \in E^S(\lambda - j\varpi_2)$ and we denote it by $\Omega(e)$.*

Notice that $\Omega(e) = 0$ if and only if e is swappable. The elevation of a non-swappable edge is well defined by Proposition 5.18.

In the other direction, we need a way to control how many times an element gets swapped with elements from higher atoms.

Definition 5.35. *Let $k \geqslant 0$ and let $\mu \leqslant \lambda$. A* staircase of non-swappable edges over *(μ,λ) (or* NS-staircase*, for short) is a sequence of edges $(e_i)_{1\leqslant i\leqslant a}$ such that*

- *$e_i := (\mu \to \mu - (n+i)\alpha) \in E^N(\lambda + i\varpi_2)$ for any $i = 1,\dots,a$.*
- *$n = 0$ or $e_0 := (\mu \to \mu - n\alpha) \in E^S(\lambda)$.*

We define $\widehat{\mathcal{D}}_\infty(\mu,\lambda)$ to be the length of the longest NS-staircase over (μ,λ). We define $\widehat{\mathcal{D}}_m(\mu,\lambda)$ to be the length of the longest NS-staircase over (μ,λ) where the label of every edge in e_i is a root in $N(y_m)$.

Example 5.36. Let $\lambda = (3,1)$ and $\mu = (3,0)$. Then $e_0 := \mu \to \mu - \alpha_2 = v_1\mu$ is a swappable edge, while $e_1 := (\mu \to \mu - 2\alpha_2) \in E^N(\lambda + \varpi_2)$ and $e_2 := (\mu \to \mu - 3\alpha_2) \in E^N(\lambda + 2\varpi_2)$, as illustrated in Figure 11. So (e_1, e_2) is a NS staircase of (μ,λ) and we have $\Omega(e_2) = 2$, $\Omega(e_1) = 1$ and $\Omega(e_0) = 0$.

Moreover, as illustrated in Figure 12, the staircase (e_1, e_2) cannot be extended, since $\mu - 4\alpha_2 \not\leqslant \lambda + 3\varpi_2$. Hence, we have $\widehat{\mathcal{D}}_\infty(\mu,\lambda) = 2$.

Lemma 5.37. *There exists at most one non-empty NS-staircase over (μ,λ).*

Proof. Assume that the there are two non-empty NS-staircases of the form $\mu \to \mu - (n+i)\alpha \in E^N(\lambda + i\varpi_2)$ and $\mu \to \mu - (n'+i)\beta \in E^N(\lambda + i\varpi_2)$. Now, if $\mu_1 > 0$, by Proposition 5.26 we have $\alpha = \beta = \alpha_2$ and if $\mu_1 < 0$ we have $\alpha = \beta = \alpha_{12}$, so in particular $\alpha = \beta$.

We can assume $n' < n$. Since $\mu \to \mu - n\alpha \in E^S(\lambda)$, by Proposition 5.27.1), we have that $\mu \to \mu - n\alpha \in E^S(\lambda + \varpi_2)$. With this and Proposition 5.27.2), we get $(\mu \to \mu - (n'+1)\alpha) \in E^S(\lambda + \varpi_2)$. Our second NS-staircase must therefore be empty. □

Lemma 5.38. *If $\mathcal{N}_m(\mu,\lambda) = 0$ and $\mu < t_m\mu \leqslant \lambda$, then also $\widehat{\mathcal{D}}_m(\mu,\lambda) = 0$.*

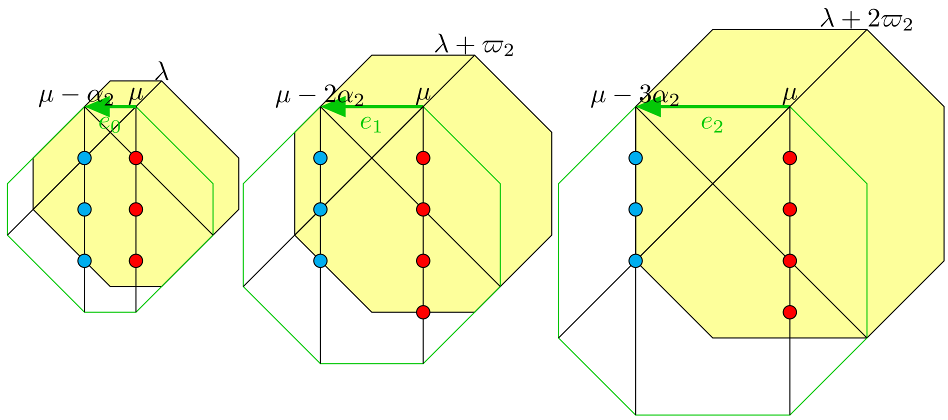

FIGURE 11. The edge e_0 is swappable while e_1 and e_2 are not. To check this, since $\mu_1 \geqslant \lambda_1$, as explained in Proposition 5.17, it is enough to compare the number of weights in the convex hull lying below μ and $v\mu$.

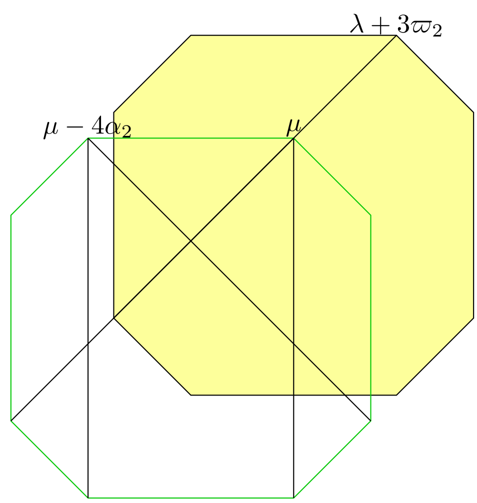

FIGURE 12. We have $\mu-4\alpha_2 \not\leqslant \lambda+3\varpi_2$ and the NS staircase (e_1, e_2) from Figure 11 cannot be extended.

Proof. Assume that $\mu_1 > 0$. If $\mu \to t_k\mu \in E^S(\lambda)$ for $k \leqslant m$, then also $\mu \to t_k\mu \in E^S(\lambda+\varpi_2)$ by Proposition 5.27. If $t_k\mu \not\leqslant \lambda$ and $(\mu \to t_k\mu) \in E^N(\lambda+\varpi_2)$, then $t_k = s_{K\delta-\alpha_2^\vee}$. But this cannot happen by Lemma 5.12.

The case $\mu_1 < 0$ is symmetric. □

PROPOSITION 5.39. *If $\mu_1 > 0$ we have*

$$\widehat{\mathcal{D}}_\infty(\mu,\lambda) = \begin{cases} \max(0, \min(\lambda_1, \mu_1) - 1) & \textit{if } -\lambda_2 \leqslant \mu_2 \leqslant \lambda_2 \\ & \textit{ and } \mu_2 \not\equiv \lambda_2 \pmod 2; \\ \max(0, \min(\mu_1, \lambda_1) + \lambda_2 + \mu_2) & \textit{if } \mu_2 < -\lambda_2; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $(e_i)_{1\leqslant i\leqslant a} = (\mu \to v_{M+i}\mu)_{1\leqslant i\leqslant a}$ be a non-empty maximal NS-staircase over (μ,λ) with $M \geqslant -\mu_2$.

Assume first $\mu_1 \geqslant \lambda_1$. We have $e_1 = (\mu \to v_{M+1}\mu) \in E^N(\lambda+\varpi_2)$, so by Proposition 5.17 we get $\mu_2 < -\lambda_2$ or $M+1 > \left\lceil \frac{\lambda_2+1-\mu_2}{2}\right\rceil$. We have either $v_M\mu = \mu$ or $e_0 = (\mu \to v_M\mu) \in E^S(\lambda)$. In the first case we get $-M = \mu_2$. In the second case we have $\mu_2 \geqslant -\lambda_2+1$, $M \leqslant \left\lceil \frac{\lambda_2-\mu_2}{2}\right\rceil$ and $M+1 > \left\lceil \frac{\lambda_2+1-\mu_2}{2}\right\rceil$, so the only possibility is

$$M = \left\lceil \frac{\lambda_2-\mu_2}{2}\right\rceil = \left\lceil \frac{\lambda_2-\mu_2+1}{2}\right\rceil,$$

which also implies $\lambda_2 \not\equiv \mu_2 \pmod 2$.

Assume further that $\mu_2 < -\lambda_2$. From the discussion above we must have $M = -\mu_2$. It is easy to check that for any $k \geqslant 1$ we have $e_k \in E^N(\lambda+k\varpi_2)$ if $v_{M+k}\mu \leqslant \lambda + k\varpi_2$ and that

$$v_{M+k}\mu \leqslant \lambda + k\varpi_2 \iff \left\lfloor \frac{\lambda_1+\lambda_2+\mu_2-k}{2}\right\rfloor \geqslant 0$$

so we get $\widehat{\mathcal{D}}_\infty(\mu,\lambda) = \max(0, \lambda_1+\lambda_2+\mu_2)$.

Assume now $\mu_2 \geqslant -\lambda_2$. If $\lambda_2 \not\equiv \mu_2 \pmod 2$ then $\widehat{\mathcal{D}}_\infty(\mu,\lambda) = 0$. If $\lambda_2 \not\equiv \mu_2 \pmod 2$ we must have $M = \left\lceil \frac{\lambda_2-\mu_2}{2}\right\rceil$. Since $v_M\mu \leqslant \lambda$, from (22) we get

$$\left\lceil \frac{\lambda_2-\mu_2}{2}\right\rceil \leqslant \left\lfloor \frac{\lambda_1+\lambda_2-\mu_2}{2}\right\rfloor,$$

so this is possible only if $\lambda_1 > 0$. It easy to check that for any $k \geqslant 1$ if $v_{M+k}\mu \leqslant \lambda + k\varpi_2$, then also $(\mu \to v_{M+k}(\mu)) \in E^N(\lambda+k\varpi_2)$. Moreover, from (22) $v_{M+k} \leqslant \lambda+k\varpi_2$ we see that is equivalent to

$$\left\lceil \frac{\lambda_2-\mu_2}{2}\right\rceil \leqslant \left\lfloor \frac{\lambda_1+\lambda_2-\mu_2-k}{2}\right\rfloor$$

which is true if and only if $k \leqslant \lambda_1 - 1$. Hence $\widehat{\mathcal{D}}_\infty(\mu,\lambda) = \max(0, \lambda_1-1)$.

The proof in the case $0 < \mu_1 < \lambda_1$ is similar. Since $e_1 \in E^N(\lambda+\varpi_2)$ we have $M+1 > \frac{\lambda_1-\mu_1}{2} + \max(-\mu_2, \left\lceil \frac{\lambda_2-\mu_2+1}{2}\right\rceil)$. We have either $v_M\mu = \mu$ or $(\mu \to v_M\mu) \in E^S(\lambda)$. However, the first case is not possible because

$$M+1 = -\mu_2+1 \leqslant \frac{\lambda_1-\mu_1}{2} + \max(-\mu_2, \left\lceil \frac{\lambda_2-\mu_2+1}{2}\right\rceil).$$

In the second case, we have $M \leqslant \frac{\lambda_1-\mu_1}{2} + \max(-\mu_2, \left\lceil \frac{\lambda_2-\mu_2}{2}\right\rceil)$, which forces

$$\max(-\mu_2, \left\lceil \frac{\lambda_2-\mu_2}{2}\right\rceil) = (-\mu_2, \left\lceil \frac{\lambda_2-\mu_2+1}{2}\right\rceil) \tag{41}$$

and $M = \frac{\lambda_1-\mu_1}{2} + \max(-\mu_2, \left\lceil \frac{\lambda_2-\mu_2}{2}\right\rceil)$. The equality in (41) can only occur if $\mu_2 < -\lambda_2$ or if $\mu_2 \geqslant -\lambda_2$ and $\lambda_2 \not\equiv \mu_2 \pmod 2$.

Assume now $\mu_2 < -\lambda_2$. Then $M = \frac{\lambda_1-\mu_1}{2} - \mu_2$. It is easy to check that for any $k \geqslant 1$ we have $e_k \in E^N(\lambda+k\varpi_2)$ if $v_{M+k}\mu \leqslant \lambda + k\varpi_2$ and that

$$v_{M+k}\mu \leqslant \lambda + k\varpi_2 \iff \left\lfloor \frac{\mu_1+\lambda_2+\mu_2-k}{2}\right\rfloor \geqslant 0$$

so we get $\widehat{\mathcal{D}}_\infty(\mu,\lambda) = \max(0, \mu_1+\lambda_2+\mu_2)$.

Finally assume $\mu_2 \geqslant -\lambda_2$. If $\lambda_2 \not\equiv \mu_2 \pmod 2$ then $\widehat{\mathcal{D}}_\infty(\mu,\lambda) = 0$. If $\lambda_2 \not\equiv \mu_2 \pmod 2$ we must have $M = \frac{\lambda_1-\mu_1}{2} + \left\lceil \frac{\lambda_2-\mu_2}{2}\right\rceil$. It easy to check that for any $k \geqslant 1$

if $v_{M+k}\mu \leqslant \lambda + k\varpi_2$, then also $(\mu \to v_{M+k}(\mu)) \in E^N(\lambda + k\varpi_2)$. Moreover, from (22) $v_{M+k} \leqslant \lambda + k\varpi_2$ we see that is equivalent to $k+1 \leqslant \mu_1$. Hence $\widehat{\mathcal{D}}_\infty(\mu,\lambda) = \max(0, \mu_1 - 1)$. □

Corollary 5.40. *If $\mu_1 < 0$ we have*

$$\widehat{\mathcal{D}}_\infty(\mu,\lambda) = \begin{cases} \max(0, \min(\lambda_1, -\mu_1) - 1) & \text{if } -\lambda_2 \leqslant \mu_1 + \mu_2 \leqslant \lambda_2 \\ & \text{and } \mu_1 + \mu_2 \not\equiv \lambda_2 \pmod 2; \\ \max(0, \min(-\mu_1, \lambda_1) + \lambda_2 + \mu_1 + \mu_2) & \text{if } \mu_1 + \mu_2 < -\lambda_2; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This immediately follows from Proposition 5.39, since by symmetry (cf. Proposition 5.22) we have $\widehat{\mathcal{D}}_\infty(\mu,\lambda) = \widehat{\mathcal{D}}_\infty(s_1(\mu),\lambda)$. □

Suppose than $\mathtt{T} \in \mathcal{A}(\lambda - k\varpi_2) \subset \mathcal{P}(\lambda)$. In our applications in Section 6, we are only interested in NS staircase over $(\mathrm{wt}(\mathtt{T}), \lambda - k\varpi_2)$ that live inside the preatom $\mathcal{P}(\lambda)$. In other words, we truncate our NS staircases $(e_i)_{1\leqslant i\leqslant a}$ so that $a \leqslant k$.

The following quantity measures the longest possible truncated NS staircase over $(\mu, \lambda - k\varpi_2)$ in a preatom of highest weight λ.

Definition 5.41. *Assume that $k \geqslant 0$ and $\mu \leqslant \lambda - k\varpi_2$. Then, for any $m \in \mathbb{N} \cup \{\infty\}$ we define*

$$\mathcal{D}_m(\mu,\lambda,k) := \min(k, \widehat{\mathcal{D}}_m(\mu, \lambda - k\varpi_2)).$$

6. The charge and recharge statistics

6.1. A family of cocharacters. Let $\widehat{X} = X \oplus \mathbb{Z}d$ be the cocharacter lattice of the extended torus $T^\vee_{\mathrm{ext}} = T^\vee \times \mathbb{C}^*$, where $T^\vee$ is the maximal torus of $G^\vee$. Let $\widehat{X}_\mathbb{Q} := \widehat{X} \otimes_\mathbb{Z} \mathbb{Q}$ and $\widehat{X}_\mathbb{R} := \widehat{X} \otimes_\mathbb{Z} \mathbb{R}$.

We recall some definitions from [25]. The *KL region* is the subset of $\widehat{X}_\mathbb{Q}$ of the elements η such that $\langle \alpha^\vee, \eta\rangle > 0$ for all $\alpha^\vee \in \widehat{\Phi}^\vee_+$. Concretely, an element in the KL region can be written as $\eta = \lambda + Cd$ where $\lambda \in X_{++}$ and $C > \langle \lambda, \beta^\vee\rangle$ for all $\beta^\vee \in \Phi^\vee_+$. The *MV region* is the subset of $\widehat{X}_\mathbb{Q}$ consisting of elements of the form $\eta = \lambda + Cd$, with $\lambda \in X_{++}$ and $C = 0$.

We call a *wall* a hyperplane in $\widehat{X}_\mathbb{R}$ of the form

$$H_{\alpha^\vee} := \{\eta \in \widehat{X}_\mathbb{R} \mid \langle \eta, \alpha^\vee\rangle = 0\} \subset \widehat{X}_\mathbb{R}$$

for $\alpha^\vee \in \widehat{\Phi}^\vee$. For $\lambda \in X_+$, we denote by $\widehat{\Phi}^\vee(\lambda)$ the set of all the labels present in the graph Γ_λ. We say that a wall $H_{\alpha^\vee}$ is a λ-wall if $\alpha^\vee \in \widehat{\Phi}^\vee(\lambda)$. We call a *$\lambda$-chamber* (or simply a chamber, if λ is clear from the context) the intersection of $\widehat{X}_\mathbb{Q}$ with a connected component of

$$\widehat{X}_\mathbb{R} \setminus \bigcup_{\alpha^\vee \in \widehat{\Phi}^\vee(\lambda)} H_{\alpha^\vee}.$$

Two chambers are *adjacent* if they are separated by a single λ-wall. The *KL chamber* is the unique chamber containing the KL region and the *MV chamber* is the unique chamber containing the MV region. We say that $\lambda \in X_\mathbb{Q}$ is *regular* if it does not lie on any wall. Otherwise, we say that λ is *singular.*

For $\lambda \in X_+$, let $\overline{\mathcal{G}r^\lambda}$ denote the corresponding spherical Schubert variety in the affine Grassmannian of $G^\vee$ (cf. [25, §2.1.2.]). For any regular $\eta \in \widehat{X}$ and any $\mu \leqslant \lambda$ the hyperbolic localization induces a functor

$$\mathrm{HL}^\eta_\mu : \mathcal{D}^b_{T^\vee \times \mathbb{C}^*}(\overline{\mathcal{G}r^\lambda}) \to \mathcal{D}^b(pt) \cong \mathrm{Vect}^\mathbb{Z},$$

where $\mathcal{D}^b_{T^\vee\times\mathbb{C}^*}(\overline{\mathcal{G}r}^\lambda)$ is the derived category of $T^\vee \times \mathbb{C}^*$-equivariant constructible sheaves on the Schubert variety $\overline{\mathcal{G}r^\lambda}$ with $\mathbb{Q}$-coefficients, and $\mathcal{D}^b(pt)$ is the derived category of sheaves on a point, which is equivalent to the category of graded $\mathbb{Q}$-vector spaces (see [25, §2.4]). In general, for any regular $\eta \in \widehat{X}_\mathbb{Q}$ we can define HL^η_μ as $\mathrm{HL}^{N\eta}_\mu$, where N is any positive integer such that $N\eta \in \widehat{X}$. By abuse of terminology, we may then refer to all the elements in $\widehat{X}_\mathbb{Q}$ as cocharacters.

Let $\widetilde{h} := \mathrm{grdim}(\mathrm{HL}^\eta_\mu(IC_\lambda))$, where IC_λ denotes the intersection cohomology sheaf of $\overline{\mathcal{G}r^\lambda}$. The polynomials $\widetilde{h}^\eta_{\mu,\lambda}(v)$ are called *renormalized η-Kazhdan–Lusztig polynomials.* We say that a function $r : \mathcal{B}(\lambda) \to \mathbb{Z}$ is a *recharge* for η if we have

$$\widetilde{h}^\eta_{\mu,\lambda}(q^{\frac{1}{2}}) = \sum_{\mathtt{T}\in\mathcal{B}(\lambda)_\mu} q^{r(\mathtt{T})} \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

If η_{KL} is in the KL chamber and $\mu \in X_+$, then

$$K_{\mu,\lambda}(q) = \widetilde{h}^{\eta_{KL}}_{\mu,\lambda}(q^{\frac{1}{2}})q^{\frac{1}{2}\ell(\mu)}$$

is a Koskta–Foulkes polynomial by [25, Proposition 2.16]. So if r_{KL} is a recharge for η_{KL} in the KL region, we obtain a charge statistic $c : \mathcal{B}(\lambda) \to \mathbb{Z}$ by setting $c(\mathtt{T}) := r_{KL}(\mathtt{T}) + \frac{1}{2}\ell(\mathrm{wt}(\mathtt{T}))$. Notice that if $\mathrm{wt}(\mathtt{T}) \in X_+$ this is equal to $c(\mathtt{T}) = r_{KL}(\mathtt{T}) + \langle \mathrm{wt}(\mathtt{T}), \rho^\vee\rangle$.

We specialize [26, Definition 3.7] to our setting.

Definition 6.1. *Let $\lambda \in X_+$. We call λ-*parabolic region *the subset of $\widehat{X}_\mathbb{Q}$ consisting of regular cocharacters η such that*

- *$\langle \eta, \beta^\vee\rangle > 0$ for every $\beta^\vee$ of the form $M\delta - \alpha_1^\vee$ with $M > 0$, or of the form $M\delta + \alpha^\vee$, with $\alpha \in \Phi_+$ and $M \geqslant 0$, and*
- *$\langle \eta, \beta^\vee\rangle < 0$ for every $\beta^\vee \in \widehat{\Phi}^\vee_+(\lambda)$ of the form $M\delta - \alpha_i^\vee$ such that $M > 0$ and $i \in \{2, 12, 21\}$.*

The walls that separate the parabolic region from the KL region are precisely

$$H_{M\delta-\alpha_i^\vee} \quad \text{with } M > 0 \text{ and } i \in \{2, 12, 21\}.$$

Every cocharacter η_P of the form

$$\eta_P = A_1\varpi_1 + A_2\varpi_2 + Cd$$

with $0 \ll A_1 \ll C \ll A_2$ lies in the parabolic region.(3)

We consider the following family of cocharacters:

$$\eta : \mathbb{Q}_{\geqslant 0} \to \widehat{X}_\mathbb{Q}, \qquad \eta(t) = \eta_P + td. \tag{42}$$

Observe that $\eta(t)$ is in the KL chamber for $t \gg 0$. We can choose t_0 such that $\eta(t_0)$ is in the KL chamber and for any i we choose $t_{i+1} < t_i$ so that $\eta(t_i)$ and $\eta(t_{i+1})$ lie in adjacent λ-chambers until we arrive at t_M in the parabolic region. We can furthermore choose $t_M = 0$ and set $t_{M+1} = \ldots = t_\infty = 0$ and $\eta_i := \eta(t_i)$ for any $i \in \mathbb{N} \cup \{\infty\}$.

6.2. Recharge statistics from the parabolic to the KL region. Our goal is to attach a recharge statistic to each of the cocharacters η_i.

Let $\mathtt{T} \in \mathcal{B}(\lambda)$. Recall that by Definitions 4.10 and 4.25 we have

$$\mathtt{T} \in \mathcal{A}(\lambda - \mathrm{at}(\mathtt{T})\varpi_2 - 2\,\mathrm{pat}(\mathtt{T})\varpi_1) \subset \mathcal{P}(\lambda - 2\varpi_1(\mathtt{T})) \subset \mathcal{B}(\lambda).$$

(3)More precisely, sufficient conditions are $0 < A_1 < C < \frac{A_2}{\gamma}$ where $\gamma = \max\{M \mid M\delta - \beta^\vee \in \Phi^\vee(\lambda)\}$.

Definition 6.2. *Assume that* $\mathtt{T} \in \mathcal{P}(\lambda) \subset \mathcal{B}(\lambda')$ *with* $\mu := \mathrm{wt}(\mathtt{T})$. *Let* $a := \mathrm{at}(\mathtt{T})$ *and* $p := \mathrm{pat}(\mathtt{T})$ *so that* $\lambda' = \lambda + 2p\varpi_1$. *We define*

$$\sigma_m(\mathtt{T}) := \ell_m(\mu, \lambda - a\varpi_2) - \mathcal{N}_m(\mu, \lambda - a\varpi_2) + \mathcal{D}_m(\mu, \lambda, a) + a + 2p.$$

Let $r_m(\mathtt{T}) := -\sigma_m(\mathtt{T}) + \langle \lambda', \rho^\vee \rangle = -\sigma_m(\mathtt{T}) + \langle \lambda, \rho^\vee \rangle + 3p$.

Our main result is the following.

Theorem 6.3. *The function* $r_m : \mathcal{B}(\lambda) \to \mathbb{Z}$ *is a recharge statistic for* η_m *for any* $m \in \mathbb{N} \cup \{\infty\}$.

The proof that r_i is a recharge for η_i is divided in two parts. We first show directly in Subsection 6.3 that r_∞ is a recharge statistic for $\eta_\infty = \eta(0)$, i.e. a recharge in the parabolic region, and then we construct for any i swapping functions between η_i and η_{i+1}. After putting everything together, this proves that $r_{KL} := r_0$ is a recharge in the KL region, and we can easily obtain from that the following formula for a charge statistic in type C_2.

Corollary 6.4. *The function* $c : \mathcal{B}(\lambda)_+ \to \mathbb{Z}$ *defined as*

$$c(\mathtt{T}) = \langle \lambda - \mathrm{wt}(\mathtt{T}), \rho^\vee \rangle - \mathrm{at}(\mathtt{T}) - \mathrm{pat}(\mathtt{T})$$

is a charge statistic.

Proof. By definition, we have $\mathcal{N}_0 = \mathcal{D}_0 = 0$ and $\ell_0 = \ell$. Hence

$$c(\mathtt{T}) = r_0(\mathtt{T}) + \frac{1}{2}\ell(\mathrm{wt}(\mathtt{T})) = \langle \lambda, \rho^\vee \rangle - \frac{1}{2}\ell(\mathrm{wt}(\mathtt{T})) - \mathrm{at}(\mathtt{T}) - 2\,\mathrm{pat}(\mathtt{T})$$

is a charge statistic. We conclude since, for $\mathtt{T} \in \mathcal{B}(\lambda)_+$, we have $\ell(\mathrm{wt}(\mathtt{T})) = 2\langle \mathrm{wt}(\mathtt{T}), \rho^\vee \rangle$. □

6.3. Recharge in the parabolic region. Let η_{MV} be a cocharacter in the MV region and η_P be in the parabolic region. The only walls separating η_{MV} from η_P are of the form $H_{M\delta - \alpha_1^\vee}$, with $M > 0$. We know from [25, Eq. (21)] that

$$r_{MV}(\mathtt{T}) = -\langle \rho^\vee, \mathrm{wt}(\mathtt{T}) \rangle.$$

is a recharge in the MV region. To construct a recharge in the parabolic region, after Levi branching, we can assume we are in rank 1 and thus compute the recharge as illustrated in [25, §3.1]. In particular, it follows from [25, Lemma 3.8] that

$$r_P(\mathtt{T}) = -\langle \rho^\vee, \mathrm{wt}(\mathtt{T}) \rangle + \phi_1(\mathtt{T}) - \ell^1(\mathrm{wt}(\mathtt{T}))$$

is a recharge in the parabolic region. It remains to show the equality between r_P and r_∞.

Let $\mathtt{T} \in \mathcal{P}(\lambda) \subset \mathcal{B}(\lambda + 2p\varpi)$ with $p = \mathrm{pat}(\mathtt{T})$ and let $a = \mathrm{at}(\mathtt{T})$ and $\mu = \mathrm{wt}(\mathtt{T})$. At $m = \infty$, we have

$$\begin{aligned} \sigma_\infty(\mathtt{T}) = \ell^1(\mu) + \sum_{i \in \{2,12,21\}} \widehat{\phi}_i(\mu, \lambda - a\varpi_2) \\ -\mathcal{N}_\infty(\mu, \lambda - a\varpi_2) + \mathcal{D}_\infty(\mu, \lambda, a) + a + 2p. \end{aligned} \tag{43}$$

Our next goal is to simplify the expression (43).

Lemma 6.5. *We have* $\widehat{\phi}_{21}(\mu, \lambda - a\varpi_2) + a = \widehat{\phi}_{21}(\mu, \lambda)$.

Proof. This follows directly from Lemma 5.6. □

Proposition 6.6. *Let* $\mu = \mathrm{wt}(\mathtt{T})$ *and assume that* $\mu_1 \leqslant 0$. *We have* $\phi_2(\mathtt{T}) = \widehat{\phi}_2(\mu, \lambda - a\varpi_2)$ *and*

$$\phi_{12}(\mathtt{T}) = \widehat{\phi}_{12}(\mu, \lambda - a\varpi_2) - \mathcal{N}_\infty(\mu, \lambda - a\varpi_2) + \mathcal{D}_\infty(\mu, \lambda, a).$$

The proof of Proposition 6.6 is rather long and technical and we postpone it to Subsection 6.4.

Lemma 6.7. *Let* $\mu = \mathrm{wt}(\mathtt{T})$. *We have*

$$\sigma_\infty(\mathtt{T}) = \ell^1(\mu) + \phi_2(\mathtt{T}) + \phi_{12}(\mathtt{T}) + \widehat{\phi}_{21}(\mu,\lambda) + 2p. \tag{44}$$

Proof. If $\mu_1 \leqslant 0$, this follows immediately from Lemma 6.5 and Proposition 6.6.

If $\mu_1 > 0$, then let $\mathtt{T}' = s_1(\mathtt{T})$. Recall that atoms are stable under s_1 by Lemma 4.19. So the element $\mathtt{T}'$ can also be characterized as the element in the same atom of $\mathtt{T}$ with weight $s_1(\mu)$. Notice that $\widehat{\phi}_{21}, \mathcal{N}_\infty$ and $\mathcal{D}_\infty$ are preserved by s_1, while $\widehat{\phi}_2(\mu, \lambda - a\varpi_2) = \widehat{\phi}_{12}(s_1(\mu), \lambda - a\varpi_2)$ and $\ell^1(\mu) = \ell^1(s_1(\mu)) - 1$. It follows that $\sigma_\infty(\mathtt{T}) = \sigma_\infty(\mathtt{T}') - 1$. On the other hand, we also have $\phi_2(\mathtt{T}) = \phi_{12}(\mathtt{T}')$ and $\phi_{12}(\mathtt{T}') = \phi_2(\mathtt{T})$, so we obtain the desired identity (44) for $\mathtt{T}$ as well. □

Proposition 6.8. *We have* $r_P(\mathtt{T}) = r_\infty(\mathtt{T})$ *for any* $\mathtt{T} \in \mathcal{B}(\lambda')$.

Proof. Let $\mu = \mathrm{wt}(\mathtt{T})$ and assume $\mathtt{T} \in \mathcal{P}(\lambda) \subset \mathcal{B}(\lambda + 2p\varpi_1)$. By Lemma 6.7 we have

$$r_\infty(\mathtt{T}) = -\ell^1(\mu) - \phi_2(\mathtt{T}) - \phi_{12}(\mathtt{T}) - \widehat{\phi}_{21}(\mu,\lambda) + \langle \lambda, \rho^\vee\rangle + p.$$

So our claim is equivalent to

$$\langle \lambda + \mu, \rho^\vee\rangle - \widehat{\phi}_{21}(\mu,\lambda) = \phi_1(\mathtt{T}) + \phi_2(\mathtt{T}) + \phi_{12}(\mathtt{T}) - p = Z(\mathtt{T}) - p.$$

By Proposition 4.14 and Lemma 5.6 we have

$$\begin{aligned}\langle \mu + \lambda, \rho\rangle - \widehat{\phi}_{21}(\mu,\lambda) &= \lambda_2 + \mu_2 + \frac{3}{2}\lambda_1 + \frac{3}{2}\mu_1 - \min(\lambda_1, \frac{\lambda_1+\mu_1}{2}, \lambda_1+\mu_1)\\ &= \lambda_2 + \mu_2 + \lambda_1 + \mu_1 - \min(\frac{\lambda_1-\mu_1}{2}, 0, \frac{\lambda_1+\mu_1}{2})\\ &= Z(\mathtt{T}) - p.\end{aligned}$$ □

6.4. Computing ϕ_2. It remains to prove the identity Proposition 6.6.

We begin by considering the case $\mathrm{at}(\mathtt{T}) = 0$. The general case will follow by induction on the atomic number.

Proposition 6.9. *For any* $\mathtt{T} \in \mathcal{P}(\lambda)$ *with* $\mathrm{wt}(\mathtt{T})_1 \leqslant 0$ *we have* $\phi_2(\mathtt{T}) = \widehat{\phi}_2(\mathrm{wt}(\mathtt{T}), \lambda - \mathrm{at}(\mathtt{T})\varpi_2)$.

Proof. Let $\mu \leqslant \lambda$. Consider the multiset

$$M_2(\mu,\lambda) := \{\phi_2(X) \mid X \in \mathcal{P}(\lambda) \text{ with } \mathrm{wt}(X) = \mu\}$$

Since $\mathcal{P}(\lambda)$ is a union of f_2-strings, we have an equality of multisets

$$M_2(\mu,\lambda) = \{\widehat{\phi}(\mu, \lambda - k\varpi_2) \mid 0 \leqslant k \leqslant \lambda_2 \text{ with } \mu \leqslant \lambda - k\varpi_2\}. \tag{45}$$

In fact, the f_2-strings contained in $\mathcal{P}(\lambda)$ which pass through an element of weight μ are in bijection with the atoms in $\mathcal{P}(\lambda)$ containing an element of weight μ.

The claim now follows by induction on λ_2. If $\lambda_2 = 0$, then $\mathcal{P}(\lambda) = \mathcal{A}(\lambda)$ and $M_2(\mu,\lambda) = \{\phi_2(\mathtt{T})\} = \{\widehat{\phi}_2(\mu,\lambda)\}$.

If $\lambda_2 > 0$, consider the embedding $\Psi : \mathcal{P}(\lambda - \varpi_2) \hookrightarrow \mathcal{P}(\lambda)$ from Definition 4.18. The map Ψ is weight-preserving and we have $\phi_2(\Psi(X)) = \phi_2(X)$ and $\mathrm{at}(\Psi(X)) = \mathrm{at}(X)+1$ for any $X \in \mathcal{P}(\lambda - \varpi_2)$ with $\mathrm{wt}(X)_1 \leqslant 0$. If $\mathtt{T} = \psi(X)$ for some $X \in \mathcal{P}(\lambda - \varpi_2)$, then $\phi_2(\mathtt{T}) = \phi_2(X) = \widehat{\phi}_2(\mu, \lambda - \varpi_2 - \mathrm{at}(X)\varpi_2)$ and the claim follows. Otherwise, we have $\mathtt{T} \in \mathcal{A}(\lambda) = \mathcal{P}(\lambda) \setminus \Psi(\mathcal{P}(\lambda - \varpi_2))$ and by (45) we see that

$$\{\phi_2(\mathtt{T})\} = M_2(\mu,\lambda) \setminus M_2(\mu, \lambda - \varpi_2) = \{\widehat{\phi}_2(\mu,\lambda)\}.$$ □

Lemma 6.10. *Let* $\mathtt{T} \in \mathcal{P}(\lambda) \subset \mathcal{B}(\lambda)$ *and let* $\mu = \mathrm{wt}(\mathtt{T})$. *Assume* $\mu_1 < 0$ *and* $\mathrm{at}(\mathtt{T}) = 0$. *Then we have* $\phi_1(\mathtt{T}) = \max(0, \mu_1 + \mu_2 - \lambda_2, -\mu_2 - \lambda_2)$.

Proof. Let $\operatorname{str}_2(\mathtt{T}) = (a, b, c, d)$. By Corollary 4.27 we have

$$\operatorname{at}(\mathtt{T}) = 0 \iff (c = d = 0) \text{ or } (b = \lambda_1 + 2c - 2d \text{ and } d \leqslant 1 \text{ or } c = \lambda_2 + d)$$

As computed in (10), we have

$$\phi_1(\mathtt{T}) = \lambda_1 + 2a - 2b + 2c - 2d + \max(d, 2c - b, b - 2a).$$

We now divide into several cases. Assume first $c = d = 0$. Then the statement is equivalent

$$\lambda_1 + 2a - b - \min(2a, b) = \max(0, \lambda_1 - b, -2\lambda_2 - b + 2a). \tag{46}$$

Since $\mu_1 = \lambda_1 - 2b + 2a < 0$ and $b \leqslant \lambda_1$, we have $2a \leqslant b$, so the LHS in (46) is $\lambda_1 - b$. Moreover, $\lambda_1 - b \geqslant 0$ and $\lambda_1 - b \geqslant 2a - b - 2\lambda_2$ otherwise we get. $\lambda_1 + 2\lambda_2 < 2a < b$. So the RHS in (46) is also equal to $\lambda_1 - b$.

We can now assume $b = \lambda_1 + 2c - 2d$, so we have $\phi_1(\mathtt{T}) = \max(-\lambda_1 + 2a - 2c + 3d, 0)$, while the RHS can be rewritten as $\max(0, d - 2c, -2\lambda_2 + d + 2a - \lambda_1)$. Moreover, we have $d - 2c \leqslant 0$ and $\mu_1 = -\lambda_1 + 2a - 2c + 2d \leqslant 0$.

So it is enough to show that

$$\max(0, \mu_1 + d) = \max(0, \mu_1 - 2(c - \lambda_2 - d) + d) \tag{47}$$

The equality is clear if $c = \lambda_2 + d$ and it also follows if $d \leqslant 1$ since that both term vanish for $\mu_1 < 0$. □

Proposition 6.11. *Let $\mathtt{T} \in \mathcal{P}(\lambda)$ and let $\mu = \operatorname{wt}(\mathtt{T})$. Assume $\mu_1 \leqslant 0$ and $\operatorname{at}(\mathtt{T}) = 0$. Then we have $\phi_{12}(\mathtt{T}) = \widehat{\phi}_{12}(\mu, \lambda) - \mathcal{N}_\infty(\mu, \lambda)$.*

Proof. If $\mu_1 = 0$, then $\phi_{12}(\mathtt{T}) = \phi_2(\mathtt{T})$, $\widehat{\phi}_{12}(\mathtt{T}) = \widehat{\phi}_2(\mathtt{T})$ and $\mathcal{N}_\infty(\mu, \lambda) = 0$, so the claim follows from Proposition 6.9. We assume in the rest of the proof $\mu_1 < 0$. We can also assume that $\mathtt{T}$ lies in the largest preatom, i.e. $\mathcal{P}(\lambda) \subset \mathcal{B}(\lambda)$. In fact, since Φ commutes with s_1 and f_2, the claim for the other preatoms easily follows by induction.

Recall now by Proposition 6.9 that $\phi_2(\mathtt{T}) = \widehat{\phi}_2(\mu, \lambda)$. We divide into three cases.

We first assume $\mu_1 + \mu_2 \leqslant \lambda_2$ and $\mu_2 \geqslant -\lambda_2$. Notice that this precisely means $\phi_1(\mathtt{T}) = 0$. By Equation (39), we have in this case

$$\widehat{\phi}_{12}(\mathtt{T}) - \mathcal{N}_\infty(\mu, \lambda) = \max(0, \left\lfloor \frac{\lambda_2 + \mu_1 + \mu_2}{2} \right\rfloor) - \min(0, \frac{-\lambda_1 - \mu_1}{2}).$$

Let $\chi := \lambda_2 + \mu_1 + \mu_2$. Then by Propositions 4.14 and 6.9 and lemma 6.10 we have

$$\begin{aligned}\phi_{12}(\mathtt{T}) &= Z(\mathtt{T}) - \phi_1(\mathtt{T}) - \phi_2(\mathtt{T}) \\ &= \frac{\lambda_1 + \mu_1}{2} + \chi + \max(0, \frac{-\mu_1 - \lambda_1}{2}) - \min(\chi, \left\lfloor \frac{\chi}{2} \right\rfloor, \lambda_2).\end{aligned}$$

Notice that $\min(0, \frac{-\lambda_1 - \mu_1}{2}) + \max(0, \frac{-\lambda_1 - \mu_1}{2}) = \frac{-\lambda_1 - \mu_1}{2}$ and that $\lambda_2 \geqslant \lfloor \frac{\chi}{2} \rfloor$. So, our claim results equivalent to the easy-to-check identity

$$\chi - \min(\chi, \left\lfloor \frac{\chi}{2} \right\rfloor) = \max(0, \left\lceil \frac{\chi}{2} \right\rceil).$$

We now assume that $\mu_1 + \mu_2 > \lambda_2$ or that $\mu_2 < -\lambda_2$. In both cases, we have from Lemma 5.4 that $\mu_1 > -\lambda_1$, so $Z(\mathtt{T}) = \lambda_1 + \lambda_2 + \mu_1 + \mu_2$. Moreover, from Proposition 5.31 we have $\mathcal{N}_\infty(\mu, \lambda) = 0$, so the claim is equivalent to $Z(\mathtt{T}) - \phi_1(\mathtt{T}) - \phi_2(\mathtt{T}) = \widehat{\phi}_{12}(\mathtt{T})$.

If $\mu_1 + \mu_2 > \lambda_2$, then $\widehat{\phi}_2(\mathtt{T}) = \frac{\lambda_1 - \mu_1}{2} + \lambda_2$ and $\widehat{\phi}_{12}(\mathtt{T}) = \frac{\lambda_1 + \mu_1}{2} + \lambda_2$, so the desired equality reduces to the identity

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 - \mu_1 - \mu_2 + \lambda_2 - \frac{\lambda_1}{2} + \frac{\mu_1}{2} - \lambda_2 = \frac{\lambda_1}{2} + \frac{\mu_1}{2} + \lambda_2.$$

Finally, if $\mu_2 < -\lambda_2$, the desired equality reduces to the identity

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_2 + \lambda_2 - \frac{\lambda_1}{2} + \frac{\mu_1}{2} - \mu_1 - \mu_2 - \lambda_2 = \frac{\lambda_1}{2} + \frac{\mu_1}{2} + \lambda_2 + \mu_2. \quad \square$$

Proposition 6.12. *Let* $\mathtt{T} \in \mathcal{P}(\lambda)$ *and let* $\mu = \mathrm{wt}(\mathtt{T})$*. Let* $A := \mathrm{at}(\mathtt{T})$*. If* $\mu_1 \leqslant 0$*, we have*

$$\phi_{12}(\mathtt{T}) = \widehat{\phi}_{12}(\mu, \lambda - A\varpi_2) - \mathcal{N}_\infty(\mu, \lambda - A\varpi_2) + \mathcal{D}_\infty(\mu, \lambda, A).$$

Proof. As in Proposition 6.11 we can assume that $\mathcal{P}(\lambda) \subset \mathcal{B}(\lambda)$. We show the claim it by induction on A. If $A = 0$, the claim immediately follows from Lemma 6.10 since $\mathcal{D}_\infty(\mu, \lambda, 0) = 0$.

If $A > 0$, then $\mathtt{T} = \Psi(\mathtt{U})$ for some $\mathtt{U} \in \mathcal{P}(\lambda - \omega_2) \subset \mathcal{B}(\lambda - \omega_2)$ with $\mathrm{at}(\mathtt{U}) = A - 1$. By induction, we have

$$\phi_{12}(\mathtt{U}) = \widehat{\phi}_{12}(\mu, \lambda - A\varpi_2) - \mathcal{N}_\infty(\mu, \lambda - A\varpi_2) + \mathcal{D}_\infty(\mu, \lambda - \varpi_2, A - 1).$$

So it suffices to show that, for any $\mathtt{U}$ in $\mathcal{P}(\lambda - \varpi_2) \subset \mathcal{B}(\lambda - \varpi_2)$ with $\mathrm{wt}(\mathtt{U}) = \mu$, we have

(48)

$$\begin{aligned}\phi_{12}(\Psi(\mathtt{U})) - \phi_{12}(\mathtt{U}) &= \mathcal{D}_\infty(\mu, \lambda, A) - \mathcal{D}_\infty(\mu, \lambda - \varpi_2, A - 1) \\ &= \min(A, \widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2)) - \min(A - 1, \widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2)).\end{aligned}$$

Let $\mathrm{str}_2(\mathtt{U}) = (a, b, c, d)$. We know from Corollary 4.24 that

$$\phi_{12}(\Psi(\mathtt{U})) - \phi_{12}(\mathtt{U}) = \begin{cases} 1 & \text{if } d = 0 \text{ and } 2a > b > 2c \text{ or } d \neq 0, \lambda_1 \text{ and } b \geqslant 2a + d \\ 0 & \text{otherwise.} \end{cases}$$

However, notice that we cannot have $d = 0$ and $2a > b > 2c$ since otherwise $\mu_1 = \lambda_1 + 2a - 2b + 2c > \lambda_1 + 2c - b \geqslant 0$. It follows that (48) is equivalent to showing that

$$\widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2) \geqslant A \iff d \neq 0, \lambda_1 \text{ and } b \geqslant 2a + d.$$

We show this in the following lemma. $\square$

Lemma 6.13. *Let* $X \in \mathcal{P}(\lambda) \subset \mathcal{B}(\lambda)$ *with* $\mu = \mathrm{wt}(X)$ *such that* $\mu_1 < 0$*. Let* $A := \mathrm{at}(X)$ *and* $\mathrm{str}_2(X) = (a, b, c, d)$*. We have*

$$\widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2) > A \iff d \neq 0, \lambda_1 \text{ and } b \geqslant 2a + d.$$

Proof. Recall from Corollary 5.40 that we have

$$\widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2) = \begin{cases} \max(0, \min(\lambda_1, -\mu_1) - 1) & \begin{array}{l}\text{if } \mu_1 + \mu_2 + \lambda_2 \geqslant A, \\ \mu_1 + \mu_2 + A \leqslant \lambda_2 \text{ and} \\ \mu_1 + \mu_2 \not\equiv \lambda_2 - A \pmod 2;\end{array} \\ \begin{array}{r}\max(0, \min(-\mu_1, \lambda_1) + \lambda_2 \\ -A + \mu_1 + \mu_2)\end{array} & \text{if } \mu_1 + \mu_2 + \lambda_2 < A; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, from Proposition 4.26 we have

$$A = \mathrm{at}(X) = \begin{cases} \min(c, \lambda_1 + 2c - b) & \text{if } d = 0 \\ \lambda_1 + 2c - 2d - b + \min(\lambda_2 + d - c, d - 1) & \text{if } d > 0. \end{cases}$$

We divide the proof into three cases.

First case: $d = 0$. We claim that in this case we actually have $\widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2) = 0$. Notice that $\mu_1 + \mu_2 + \lambda_2 = \lambda_1 + 2\lambda_2 - b \geqslant \lambda_1 + 2c - b \geqslant A$. So we can also assume that $A \leqslant \lambda_2 - \mu_1 - \mu_2 = b - \lambda_1$. Notice that this is equivalent to $c + \lambda_1 \geqslant b$ and

$A = \lambda_1 + 2c - b$. However, if $A = \lambda_1 + 2c - b$ then $\mu_1 + \mu_2 + \lambda_2 + A \equiv 0 \pmod 2$, and therefore $\widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2) = 0$.

Second case: $d = \lambda_1$. In this case we have $A = \min(\lambda_2 + c - b, 2c - b - 1)$ and $\mu_1 + \mu_2 + \lambda_2 = 2\lambda_2 - b$. It follows that $\mu_1 + \mu_2 + \lambda_2 \geqslant A$ if and only if $\lambda_2 \geqslant c$. Recall also that $b \leqslant 2c - \lambda_1$.

Assume first $\lambda_2 \geqslant c$, so that $A = 2c - b - 1$ and $\mu_1 + \mu_2 + \lambda_2 \geqslant A$. The claim now follows since $\widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2) \leqslant \lambda_1 - 1 \leqslant 2c - b - 1$.

Assume now $\lambda_2 < c$ so that $A = \lambda_2 + c - b$ and $\mu_1 + \mu_2 + \lambda_2 \leqslant A$. The claim follows because, if $\widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2) \geqslant 0$, then $\widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2) \leqslant \lambda_1 + \lambda_2 + \mu_1 + \mu_2 - A = \lambda_1 + \lambda_2 - c \leqslant \lambda_2 + c - b = A$.

Third case: $d \neq 0, \lambda_1$. In this case we have $b = \lambda_1 - 2d + 2c$. Notice that $b \geqslant 2a + d$ is equivalent to $\lambda_1 - 2a + 2c > 3d$. We also have $A = \min(\lambda_2 + d - c, d - 1)$ and $\mu_1 + \mu_2 + \lambda_2 = 2\lambda_2 - 2c + d$, so $\mu_1 + \mu_2 + \lambda_2 \geqslant A$ if and only if $\lambda_2 \geqslant c$.

Assume first $\lambda_2 \geqslant c$, so that $A = d - 1$ and $\mu_1 + \mu_2 + \lambda_2 \geqslant A$. Notice that $\lambda_1 - 1 > A$ and also

$$-\mu_1 - 1 > A \iff \lambda_1 + 2c - 2a - 2d - 1 > d - 1 \iff \lambda_1 + 2c - 2a > 3d$$

Hence, $\widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2) > A$ if and only if $\lambda_1 + 2c - 2a > 3d$.

Finally assume $\lambda_2 < c$ so that $A = \lambda_2 + d - c$ and $\mu_1 + \mu_2 + \lambda_2 < A$. In this case we have $\widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2) = \max(0, \min(0, \mu_1 + \lambda_1) + \lambda_2 + \mu_2 - A)$. We have $\mu_1 + \lambda_1 + \lambda_2 + \mu_2 - A = \lambda_1 + \lambda_2 - c > \lambda_2 + d - c = A$ and

$$\lambda_2 + \mu_2 - A > A \iff \lambda_1 + 2c - 2a > 3d.$$

It follows that $\widehat{\mathcal{D}}_\infty(\mu, \lambda - A\varpi_2) > A$ if and only if $\lambda_1 + 2c - 2a > 3d$. □

7. Swapping functions

Recall the family of cocharacters $\{\eta_m\}_{m\in\mathbb{N}}$ introduced in Subsection 6.1. The unique wall separating η_m and η_{m+1} is $H_{\alpha^\vee_{m+1}}$, where $\alpha^\vee_{m+1} \in \widehat{\Phi}^\vee_+$ is the $(m+1)$-th root occurring in the sequence (13). As in (16), let $t := t_{m+1}$ denote the corresponding reflection. For any $\mu \in X$ such that $\mu < t\mu \leqslant \lambda$ we define

$$\psi_{t\mu} : \mathcal{B}(\lambda)_{t\mu} \to \mathcal{B}(\lambda)_\mu$$

as follows. Let $\mathtt{T} \in \mathcal{B}(\lambda)_{t\mu}$ and assume that $\mathtt{T} \in \mathcal{A}(\lambda - a\varpi_2) \subset \mathcal{P}(\lambda)$ and let $e := (\mu \to t\mu) \in E(\lambda - a\varpi_2)$. Then $\psi_{t\mu}(\mathtt{T}) = \mathtt{T}'$, where $\mathtt{T}'$ is the only element of weight μ in $\mathcal{A}(\lambda - (a + \Omega(e))\varpi_2) \subset \mathcal{P}(\lambda)$.

Proposition 7.1. *The collection of maps $\psi = \{\psi_\nu\}$ is a swapping function between η_{m+1} and η_m. In particular, if r_{m+1} is a recharge for η_{m+1} then r_m is a recharge for η_m.*

To prove Proposition 7.1 we need to check that for any m and $\mathtt{T}$ we have $r_{m+1}(\mathtt{T}) = r_{m+1}(\psi_{t\mu}(\mathtt{T})) + 1$, or equivalently that $\sigma_{m+1}(\mathtt{T}) = \sigma_{m+1}(\psi_{t\mu}(\mathtt{T})) - 1$.

Proposition 7.2. *Assume $\mathtt{T} \in \mathcal{A}(\lambda - k\varpi_2) \subset \mathcal{P}(\lambda)$ with $t\mu = \mathrm{wt}(\mathtt{T})$ and that $e := (\mu \to t\mu) \in E(\lambda - a\varpi_2)$. Then, we have*

$$\sigma_{m+1}(\mathtt{T}) = \sigma_{m+1}(\psi_{t\mu}(\mathtt{T})) - 1. \tag{49}$$

Proof. By Lemma 5.29 we have $\mathcal{N}_{m+1}(t\mu, \lambda - a\varpi_2) = 0$ and by Lemma 5.38 we also get $\mathcal{D}_{m+1}(t\mu, \lambda, a) = 0$. Let $\Omega := \Omega(e)$ and recall that $\psi_{t\mu}(\mathtt{T})$ is the element of weight μ in the atom $\mathcal{A}(\lambda - (a + \Omega)\varpi_2) \subset \mathcal{P}(\lambda)$.

First assume $\Omega = 0$, or equivalently that e is swappable. Since $\mu \to t\mu$ is swappable, by definition we have $\ell_{m+1}(\mu, \lambda - a\varpi_2) = \ell_{m+1}(t\mu, \lambda - a\varpi_2) + 1$. By Proposition 5.27

we have that $\mathcal{N}_{m+1}(\mu, \lambda - a\varpi_2) = 0$ and by Lemma 5.38, we also get $\mathcal{D}_{m+1}(\mu, \lambda, a)$. The claim now easily follows.

Assume now $\Omega > 0$, so e is not swappable. Notice that $\mathcal{D}_{m+1}(\mu, \lambda, a+\Omega) = \Omega$. Combining with Lemma 5.29, our claim (49) becomes equivalent to

$$\ell_{m+1}(t\mu, \lambda - a\varpi_2) = \ell_{m+1}(\mu, \lambda - (a+\Omega)\varpi_2) - \mathcal{N}_{m+1}(\mu, \lambda - (a+\Omega)\varpi_2) + 2\Omega - 1.$$

We can assume $\mu_1 \geqslant 0$ as the case $\mu_1 < 0$ is symmetric. Because e is not swappable, we have $m+1 = 4M$, $q_M\mu \not\leqslant \lambda$ and $r_M\mu \not\leqslant \lambda$. In particular, by (23) and (24) we have $\ell^{12}_{m+1} = \widehat{\phi}_{12}$ and $\ell^{21}_{m+1} = \widehat{\phi}_{21}$. Using Corollary 5.33, our claim is then equivalent to

$$\ell_{m+1}(t\mu, \lambda - a\varpi_2) - \widehat{\ell}_{m+1}(t\mu, \lambda - (a+\Omega)\varpi_2) = 2\Omega. \tag{50}$$

By Lemma 5.6 we have

$$\widehat{\phi}_{21}(t\mu, \lambda - a\varpi_2) - \widehat{\phi}_{21}(t\mu, \lambda - (a+\Omega)\varpi_2) = \Omega$$

$$\widehat{\phi}_{12}(t\mu, \lambda - a\varpi_2) - \widehat{\phi}_{12}(t\mu, \lambda - (a+\Omega)\varpi_2) = \Omega$$

because $(t_{m+1}\mu)_2 \leqslant -\lambda_2$ (as proven in Claim 5.19) and the identity (50) now follows directly from the definition of $\widehat{\ell}_{m+1}$. □

Proof of Proposition 7.1. Proposition 7.2 precisely shows that ψ is a swapping function for r_{m+1}. This means that we can obtain a new recharge r'_m for η_m by swapping the values of r_{m+1} as indicated by ψ. It remains to show that $r_m = r'_m$. In other words, for $t = t_{m+1}$ and for any $\mu \leqslant t\mu$ we need to show that

(1) if $\mathrm{wt}(\mathtt{T}) = t\mu$ then $r_m(\mathtt{T}) = r_{m+1}(\psi(\mathtt{T})) = r_{m+1}(\mathtt{T}) - 1$;
(2) if $\mathrm{wt}(\mathtt{U}) = \mu$ and $\mathtt{U} \in \mathrm{Im}(\psi_{t\mu})$ then $r_m(\mathtt{U}) = r_{m+1}(\psi^{-1}_{t\mu}(\mathtt{U})) = r_{m+1}(\mathtt{U}) + 1$;
(3) if $\mathrm{wt}(\mathtt{U}) = \mu$ and $\mathtt{U} \notin \mathrm{Im}(\psi_{t\mu})$ then $r_m(\mathtt{U}) = r_{m+1}(\mathtt{U})$.

The first statement is clear since $r_m(\mathtt{T}) - r_{m+1}(\mathtt{T}) = \ell_{m+1}(t\mu, \tau) - \ell_m(t\mu, \tau) = -1$ by Lemmas 5.29 and 5.38. Let now $\mathtt{U} \in \mathcal{A}(\zeta)$ with $\mathrm{wt}(\mathtt{U}) = \mu$ and let $a := \mathrm{at}(\mathtt{U})$. We need to compute

$$\begin{aligned} r_m(\mathtt{U}) - r_{m+1}(\mathtt{U}) =& \ell_{m+1}(\mu,\zeta) - \ell_m(\mu,\zeta) - \mathcal{N}_{m+1}(\mu,\zeta) + \mathcal{N}_m(\mu,\zeta) + \\ &+ \mathcal{D}_{m+1}(\mu, \zeta + a\varpi_2, a) - \mathcal{D}_m(\mu, \zeta + a\varpi_2, a). \end{aligned} \tag{51}$$

If $\mathtt{U} \in \mathrm{Im}(\psi_{t\mu})$, there exists $\Omega \in \mathbb{N}$ such that $e := (\mu \to t\mu) \in E(\zeta + \Omega\varpi_2)$ and $\Omega = \Omega(e)$. If $\Omega = 0$, then e is swappable and $t\mu \leqslant \zeta$. So (51) simplifies to $r_m(\mathtt{U}) - r_{m+1}(\mathtt{U}) = \ell_{m+1}(\mu,\zeta) - \ell_m(\mu,\zeta) = 1$. If $\Omega > 0$, then we have by Proposition 5.27.1 that $(\mu \to t\mu) \in E^N(\zeta)$ or $t\mu \not\leqslant \zeta$. It follows that

$$\ell_{m+1}(\mu,\zeta) - \ell_m(\mu,\zeta) = \mathcal{N}_{m+1}(\mu,\zeta) - \mathcal{N}_m(\mu,\zeta) = \begin{cases} 1 & \text{if } t\mu \leqslant \zeta \\ 0 & \text{if } t\mu \not\leqslant \zeta, \end{cases} \tag{52}$$

so the first line in the RHS of (51) vanishes. Since $\Omega \leqslant a$, the edge e belongs to a truncated NS staircase over $(\mu, \zeta + a\varpi_2)$, hence $\mathcal{D}_{m+1}(\mu, \zeta + a\varpi_2, a) - \mathcal{D}_m(\mu, \zeta + a\varpi_2, a) = 1$.

Finally, assume that $\mathtt{U} \notin \mathrm{Im}(\psi_{t\mu})$. This means that $(\mu \to t\mu) \notin E^S(\zeta)$, so (52) holds again in this case. Moreover, there does not exists $k \leqslant a$ such that $f := (\mu \to t\mu) \in E^N(\zeta + k\varpi_2)$ with $\Omega(f) = k$, from which it follows that $\mathcal{D}_{m+1}(\mu, \zeta + a\varpi_2, a) = \mathcal{D}_m(\mu, \zeta + a\varpi_2, a)$ and (51) can be simplified to $r_m(\mathtt{U}) - r_{m+1}(\mathtt{U}) = 0$. □

7.1. Alternative formula. We can obtain an alternative formula for the charge statistic by focusing on a single element and counting how many times its recharge gets changed by a swapping function. In type A, this is discussed in [26, Remark 3.13].

Definition 7.3. *We define $\Delta^\alpha : \mathcal{B}(\lambda) \to \mathbb{Z}$, for $\alpha \in \Phi_+$ as the* total contribution *of the swapping functions along the direction α. It is defined as*

$$\Delta^\alpha = \sum r_m(\mathtt{T}) - r_{m-1}(\mathtt{T})$$

where the sum runs over all m such that the (unique) wall between the λ-chambers of η_m and η_{m-1} is of the form $H_{M\delta-\alpha^\vee}$.

We write $\Delta^i := \Delta^{\alpha_i}$ for $i \in \{1, 2, 21, 12\}$.

We have $r_{KL} - r_{MV} = \sum_{\alpha\in\Phi_+} \Delta^\alpha$. Recall that in type A for any $\alpha \in \Phi_+$ we have $\Delta^\alpha(\mathtt{T}) = \phi_\alpha(\mathtt{T}) - \ell^\alpha(\mathrm{wt}(\mathtt{T}))$. When we apply the swapping functions along the α_1-direction, to go from the MV region to the parabolic region, we regard $\mathcal{B}(\lambda)$ as a crystal of type A_1. It follows that $\Delta^1(\mathtt{T}) = \phi_1(\mathtt{T}) - \ell^1(\mathrm{wt}(\mathtt{T}))$ as in [25, Lemma 3.7]. Moreover, if $\mathtt{T} \in \mathcal{B}_+(\lambda)$, we have

$$\Delta^1(\mathtt{T}) = \phi_1(\mathtt{T}) - \langle \mathrm{wt}(\mathtt{T}), \alpha_1^\vee\rangle = \epsilon_1(\mathtt{T}). \tag{53}$$

Proposition 7.4. *For $\mathtt{T} \in \mathcal{A}(\zeta) \subset \mathcal{B}(\lambda)$ with $\mathrm{wt}(\mathtt{T}) = \mu$ and $\mathrm{at}(\mathtt{T}) = a$ we have*

1. $\Delta^{21}(\mathtt{T}) = \widehat{\phi}_{21}(\mu,\zeta) - \ell^{21}(\mu)(\mathtt{T})$.
2. $\Delta^2(\mathtt{T}) = \phi_2(\mathtt{T}) - \ell^2(\mathrm{wt}(\mathtt{T}))$
3. $\Delta^{12}(\mathtt{T}) = \phi_{12}(\mathtt{T}) - \ell^{12}(\mathrm{wt}(\mathtt{T}))$

Proof. By Proposition 5.24, the swaps in the α_{21}-direction always occur within the atom of $\mathtt{T}$, so to compute $\Delta^{21}(\mathtt{T})$ we just need to consider the string of elements in the atom of $\mathtt{T}$ of weights $\mu + k\alpha_{21}$. This means that we can compute Δ^{21} as in the rank one case and have $\Delta^{21}(\mathtt{T}) = \widehat{\phi}_{21}(\mathtt{T}) - \ell^{21}(\mu)$.

Assume first $\mu_1 \leqslant 0$. Then the swapping occurring on $\mathtt{T}$ in the α_2 direction only occur within the atom of $\mathtt{T}$, so as for Δ^{21}, we have

$$\Delta^2(\mathtt{T}) = \widehat{\phi}_2(\mu,\zeta) - \ell^2(\mu) = \phi_2(\mathtt{T}) - \ell^2(\mu),$$

where the second equality comes from Proposition 6.9.

Assume now $\mu_1 \geqslant 0$. Then by construction the number of swappable edges containing μ in the atom of $\mathtt{T}$ is $\widehat{\phi}_2(\mu,\zeta) - \mathcal{N}_\infty(\mu,\zeta)$. Of these, there are $\ell^2(\mu)$ attached to roots $M\delta + \alpha_2^\vee$, which do not correspond to any crossed wall. Moreover, $\mathtt{T}$ is also in the image of $\mathcal{D}_\infty(\mu,\zeta + a\varpi_2, a)$ swapping functions, corresponding to non-swappable edges in atoms bigger than $\mathcal{A}(\zeta)$. It follows that

$$\Delta^2(\mathtt{T}) = \widehat{\phi}_2(\mu,\zeta) - \ell^2(\mu) - \mathcal{N}_\infty(\mu,\zeta) + \mathcal{D}_\infty(\mu,\zeta + a\varpi_2, a) = \phi_2(\mathtt{T}) - \ell^2(\mu)$$

by Proposition 6.12.

The proof of the formula for Δ^{12} is symmetric. □

Assume now that $\mathtt{T} \in \mathcal{B}_+(\lambda)$. Then, as in (53), we have $\Delta^2(\mathtt{T}) = \epsilon_2(\mathtt{T})$ and $\Delta^{12}(\mathtt{T}) = \epsilon_{12}(\mathtt{T})$.

Definition 7.5. *Let $\mathtt{T} \in \mathcal{A}(\zeta)$ be such that $\mathrm{wt}(\mathtt{T}) = \mu$. We define $\widehat{\epsilon}_{21}(\mathtt{T}) := \widehat{\phi}_{21}(\mu,\zeta) - \ell^{21}(\mu)$.*

Notice that $\widehat{\epsilon}_{21}(\mathtt{T})$ can equivalently be defined as the largest integer k such that $\mathrm{wt}(\mathtt{T}) + k \leqslant \zeta$, for $\mathtt{T} \in \mathcal{A}(\zeta)$.

For $\mathtt{T} \in \mathcal{B}_+(\lambda)$, we have $r_{KL}(\mathtt{T}) - r_{MV}(\mathtt{T}) = \epsilon_1(\mathtt{T}) + \epsilon_2(\mathtt{T}) + \epsilon_{12}(\mathtt{T}) + \widehat{\epsilon}_{21}(\mathtt{T})$. Since $r_{MV}(\mathtt{T}) + \frac{1}{2}\ell(\mathrm{wt}(\mathtt{T})) = 0$ for $\mathtt{T} \in \mathcal{B}_+(\lambda)$, it follows that

$$c(\mathtt{T}) = \epsilon_1(\mathtt{T}) + \epsilon_2(\mathtt{T}) + \epsilon_{12}(\mathtt{T}) + \widehat{\epsilon}_{21}(\mathtt{T})$$

is a charge statistic on $\mathcal{B}_+(\lambda)$.

We conclude by giving a more explicit way to compute $\widehat{\epsilon}_{21}(\mathtt{T})$.

Definition 7.6. *Let* $\mathtt{T}$ *be in the biggest atom, that is we assume* $\mathtt{T} \in \mathcal{A}(\lambda) \subset \mathcal{P}(\lambda) \subset \mathcal{B}(\lambda)$ *and let* $\operatorname{str}_2(\mathtt{T}) = (a,b,c,d)$*. We define*

$$e_{21}^{\mathrm{str}}(\mathtt{T}) := \begin{cases} (a-1, b-1, c, d) & \text{if } c = d = 0 \\ (a-1, b-2, c, d+1) & \text{if } c > 0 \text{ and } d = 0 \\ (a, b, c-1, d-1) & \text{if } d > 0 \end{cases}$$

and $\overline{e}_{21}(\mathtt{T})$ *as the element in* $\mathcal{B}(\lambda)$ *such that* $\operatorname{str}_2(\overline{e}_{21}(\mathtt{T})) = e_{21}^{\mathrm{str}}(\mathtt{T})$ *if it exists, and* 0 *otherwise. Finally, define* $\widehat{e}_{12}(\mathtt{T})$ *as* $\overline{e}_{12}(\mathtt{T})$ *if* $\operatorname{wt}(\mathtt{T})_1 \leqslant 0$ *and* $s_1(\overline{e}_{12}(s_1(\mathtt{T})))$ *if* $\operatorname{wt}(\mathtt{T})_1 \geqslant 0$.

Proposition 7.7. *Let* $\mathtt{T} \in \mathcal{A}(\lambda) \subset \mathcal{B}(\lambda)$

- *If* $\widehat{e}_{21}(\mathtt{T}) \neq 0$*, then* $\widehat{e}_{21}(\mathtt{T}) \in \mathcal{A}(\lambda)$ *and* $\widehat{\epsilon}_{21}(\mathtt{T}) > 0$.
- *If* $\widehat{e}_{21}(\mathtt{T}) = 0$ *and* $\langle \operatorname{wt}(\mathtt{T}), \alpha_{21}^\vee \rangle \geqslant 0$*, then* $\widehat{\epsilon}_{21}(\mathtt{T}) = 0$.

Proof. It can be easily verified by Corollary 4.27 that if $\mathtt{T} \in \mathcal{A}(\lambda)$ and $\widehat{e}_{21}(\mathtt{T}) \neq 0$, then also $\widehat{e}_{21}(\mathtt{T}) \in \mathcal{A}(\lambda)$.

To prove the second statement, we introduce operators $f_{21}^{\mathrm{str}}, \overline{f}_{21}, \widehat{f}_{21}$, similarly to Definition 7.6, where $f_{21}^{\mathrm{str}}(\mathtt{T})$ is defined, for $\mathtt{T} \in \mathcal{A}(\lambda)$ with $\operatorname{str}_2(\mathtt{T}) = (a,b,c,d)$ as

$$f_{21}^{\mathrm{str}}(\mathtt{T}) = \begin{cases} (a+1, b+1, c, d) & \text{if } b < \lambda_1 - 2d + 2c \\ (a, b, c+1, d+1) & \text{if } d = 0 \text{ or } c = \lambda_2 + d \\ (a-1, b+2, c, d-1) & \text{if } d = 1 \text{ and } c < \lambda_2 + d \end{cases}$$

Again, one can verify via Corollary 4.27, that if $\mathtt{T} \in \mathcal{A}(\lambda)$ also $\widehat{f}_{21}(\mathtt{T}) \in \mathcal{A}(\lambda)$. If $\widehat{\epsilon}_{21}(\mathtt{T}) \neq 0$, there exists $\mathtt{U} \in \mathcal{A}(\lambda)$ with $\operatorname{wt}(\mathtt{U}) = \operatorname{wt}(\mathtt{T}) + \alpha_{21}$. Then, we have $f_{21}(\mathtt{U}) = \mathtt{T}$, from which it follows that $e_{21}(\mathtt{T}) = \mathtt{U} \neq 0$, or $\widehat{f}_{21}(\mathtt{U}) = 0$. But we cannot have $\widehat{f}_{21}(\mathtt{U}) = 0$ and $\langle \operatorname{wt}(\mathtt{U}), \alpha_{21}^\vee \rangle \geqslant 2$. For example, if $c = \lambda_2 + d$ or $d = 0$, then $\overline{f}_{21}(\mathtt{T}) = 0$ only if $a = \lambda_2 + b - 2c + 2d$, which implies $\langle \operatorname{wt}(\mathtt{U}), \alpha_{21}^\vee \rangle = \operatorname{wt}(\mathtt{U})_1 + 2\operatorname{wt}(\mathtt{U})_2 = -b \leqslant 0$. ☐

The proposition implies that $\widehat{\epsilon}_{21}$ is associated to the operator $\widehat{e}_{21}$. That is, we have $\widehat{\epsilon}_{21}(\mathtt{T}) = \max\{k \mid \widehat{e}_{21}^k(\mathtt{T}) \neq 0\}$. Similar expressions for $\widehat{e}_{21}$ on the other atoms can be obtained recursively using the embeddings Φ and $\overline{\Psi}$.

We believe that one can construct similar charge statistics in higher ranks.

Conjecture 7.8. *Assume* $\mathcal{B}$ *is a crystal of type* C_3*. Then there exists a function* $\widehat{\epsilon}_{32} : \mathcal{B} \to \mathbb{Z}_{\geqslant 0}$ *such that*

$$\begin{aligned} c(\mathtt{T}) = \epsilon_1(\mathtt{T}) + \epsilon_2(\mathtt{T}) + \epsilon_2(s_1(\mathtt{T})) + \epsilon_3(\mathtt{T}) + \epsilon_3(s_2(\mathtt{T})) + \epsilon_3(s_1 s_2(\mathtt{T})) \\ + \widehat{\epsilon}_{32}(\mathtt{T}) + \widehat{\epsilon}_{32}(s_1(\mathtt{T})) + \widehat{\epsilon}_{32}(s_2 s_1(\mathtt{T})) \end{aligned}$$

is a charge statistic on $\mathcal{B}_+(\lambda)$.

Notice that if $\operatorname{wt}(\mathtt{T}) = 0$ the conjecture predicts that $c(\mathtt{T}) = \epsilon_1(\mathtt{T}) + 2\epsilon_2(\mathtt{T}) + 3\epsilon_3(\mathtt{T}) + 3\widehat{\epsilon}_{321}(\mathtt{T})$. We have checked in many examples that such a function exists on elements of weight 0.

7.2. Comparison with the conjectural charge formula by Lecouvey. We have checked in many examples using computers and it seems safe to conjecture that our charge formula and the formula conjectured by Lecouvey in [14] coincide for $\lambda = k\varpi_1$ (in this case Lecouvey's conjecture is shown to be true in [6]). However, as the following example shows, the two statistics do not coincide in general.

If $\lambda = 2\varpi_2$, there are two tableaux of weight 0: $\mathtt{T}_1 = \begin{array}{|c|c|}\hline 1 & \overline{2} \\ \hline 2 & \overline{1} \\ \hline \end{array}$ and $\mathtt{T}_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \overline{2} & \overline{1} \\ \hline \end{array}$. We have $c(\mathtt{T}_1) = 4$ and $c(\mathtt{T}_2) = 2$ while, if we denote the Lecouvey charge statistic by Lc, we have $Lc(\mathtt{T}_1) = 2$ and $Lc(\mathtt{T}_2) = 4$, after making the appropriate alphabet conversions.

Appendix A. Proof of Proposition 4.14 with SageMath

```
R.<a,b,c,d,L1,L2>=PolynomialRing(QQ)
#L1 and L2 represent lam_1 and lam_2
K=R.fraction_field()

def theta12():
    X = [a,b,c,d]
    X[0] = 1/K(1/d + b/c + a/b)
    X[1] = 1/K(1/c + b^2/(a*c^2) + 1/(a*d^2))
    X[2] = K(b+b^2*d/c+a*d)
    X[3] = K(a+b^2/c+c/d^2)
    F(a,b,c,d) = tuple(X)
    return F

def theta21():
    X = [a,b,c,d]
    X[0] = 1/K(1/d+b/c^2+a^2/b)
    X[1] = 1/K(1/c+1/(a*d)+b/(a*c^2))
    X[2] = K(b+b^2*d/c^2+a^2*d)
    X[3] = K(a+c/d+b/c)
    F(a,b,c,d) = tuple(X)
    return F

def RRTAux(P):
#From tropical polynomials we can remove coefficients bigger than 1.
#Moreover, we are only interested in the function on positive values of
                                    a,b,c,d,L1 and L2
#we can remove monomials which are divisible by another monomial, as the
                                     minimum is never
#expressed exclusively by them.
    M = P.monomials()
    R = []
    for i in range(len(M)):
        for j in range(len(M)):
            if M[i].divides(M[j]) and i != j:
                R.append(j)
    return sum([M[j] for j in range(len(M)) if not j in R])

def RemoveRedundantTerms(X):
    return RRTAux(X.numerator())/RRTAux(X.denominator())

t12 = theta12()
t21 = theta21()
s1(a,b,c,d) = (L1*b^2*d^2/(a*c^2),b,c,d)
phi2(a,b,c,d) = L2*b*d/(a*c^2)
Phi2= K(L2*b*d/(a*c^2))
phi1aux(a,b,c,d) = L1*b^2*d^2/(a*c^2)
Phi1 = K(phi1aux(*t21))
phi12aux1 = s1(*t21)
phi12aux2 = t12(*phi12aux1)
Phi12 = K(phi2(*phi12aux2))
Z = Phi2*Phi1*Phi12
RHS = K(L1^2*L2^2/(b*d*(1+L1*a*c/(b*d)+b*d/(a*c))))
Q = Z/RHS
```

We first compute the quotient Q on the subset of elements in $\mathcal{P}(\lambda)$ such that $d = 0$.

```
f1(a,b,c,d,L1,L2) = (a,b,c,1,L1,L2)
Q1 = RemoveRedundantTerms(K(Q(*f1)(a,b,c,d,L1,L2)))
# Q(*f1) denotes composition of functions in Sage
print(Q1)
print(Q1.numerator()-Q1.denominator())
```

```
(a^3*c^3*L1 + a^2*c^4*L1 + a^2*b*c^2 + a*b*c^3 + a*b^2*c +
b^2*c^2 + b^3)/(a^3*c^3*L1 + a^2*c^4*L1 + a*c^5*L1 +
a^2*b*c^2 + a*b*c^3 + a*b^2*c*L1 + b^2*c^2 + b^3)
-a*c^5*L1 - a*b^2*c*L1 + a*b^2*c
```

There is one extra monomial (ab^2c) in the numerator which does not occur in the denominator and two additional monomials ($ac^5\lambda_1$ and $ab^2c\lambda_1$) in the denominator. However, we have

- $a + 2b + c + \lambda_1 \geqslant a + 2b + c \geqslant \min(2a + b + 2c, 3b)$
- $a + 5c + \lambda_1 \geqslant a + b + 3c$ (because $b \leqslant \lambda_1 + 2d - 2c$)).

Hence, the minimum is never expressed by these monomials. So, the quotient function Q is constantly zero on the elements of the preatom with $d = 0$.

Now we compute the quotient Q on the subset of elements in $\mathcal{P}(\lambda)$ such that $d = \lambda_1$.

```
f2(a,b,c,d,L1,L2) = (a,b,c,L1,L1,L2)
Q2 = RemoveRedundantTerms(K(Q(*f2)(a,b,c,d,L1,L2)))
print(Q2)
print(Q2.numerator()-Q2.denominator())
```

```
(a^3*c^3*L1^2 + a^2*c^4*L1 + a^2*b*c^2*L1^2 + a*b*c^3*L1 +
a*b^2*c*L1^2 + b^2*c^2*L1^2 + b^3*L1^3)/(a^3*c^3*L1^2 +
a^2*c^4*L1 + a^2*b*c^2*L1^2+ a*c^5 + a*b*c^3*L1 +
a*b^2*c*L1^2 +b^2*c^2*L1^2 + b^3*L1^3)
-a*c^5
```

There is an extra monomial in the denominator: ac^5. However, we have $a + 5c \geqslant a + 2b + c + 2\lambda_1$ (because $b \leqslant 2c - \lambda_1$)). Hence, the minimum is never expressed by this monomial, and the tropical function Q is constantly zero when $d = lam_1$. Finally, we compute Q for $b = \lambda_1 + 2c - 2d$.

```
f3(a,b,c,d,L1,L2) = (a,L1*c^2/d^2,c,d,L1,L2)
Q3 =RemoveRedundantTerms(K(Q(*f3)(a,b,c,d,L1,L2)))
print(Q3)
print(Q3.numerator()-X.denominator())
```

```
(a^3*d^4 + a^2*c*d^3 + a^2*c*d^2*L1 + a*c^2*d^2+ a*c^2*d*L1 +
c^3*d*L1 + c^3*L1^2)/(a^3*d^4 +a^2*c*d^3 + a^2*c*d^2*L1 +
a*c^2*d^2 + a*c^2*d*L1+ c^3*d*L1 + a*c^2*L1^2 + c^3*L1^2)
-a*c^2*L1^2
```

There is one extra monomial in the denominator: $ac^2\lambda_1^2$. However, we have $a + 2c + 2\lambda_1 \geqslant a + 2c + d + \lambda_1$ (because $d \leqslant \lambda_1$). This shows again that Q is zero when $b = \lambda_1 + 2d - 2c$. Hence it is always zero on the preatom, concluding the proof of Proposition 4.14 in the case $\mathrm{pat}(T) = 0$.

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