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Abstract We prove the existence of signed combinatorial interpretations for several large families of structure constants. These families include standard bases of symmetric and quasisymmetric polynomials, as well as various bases in Schubert theory. The results are stated in the language of computational complexity, while the proofs are based on the effective Möbius inversion.

1. Introduction

1.1. Foreword. Back in 1969, the renowned British philosophers Jagger and Richards made a profound if not wholly original observation, that "you can't always get what you want" [48]. Only recently, this piece of conventional wisdom started to gain acceptance in algebraic combinatorics, at least when it comes to *combinatorial interpretations*, cf. [72].

The interest in combinatorial interpretations is foundational in the area, and goes back to the early papers by Young, where he defined what we now call standard Young tableaux, to count degrees of irreducible S_n characters. After over a century of progress and numerous successes finding combinatorial interpretations, a dozen or so open problems remain a thorn in our side (see e.g. [88] and [73]).

Negative results started to arrive a few years ago, all of the form "a combinatorial interpretation of [some numbers] implies a collapse of the polynomial hierarchy", see [19, 44, 45]. Informally, these conditional results show that a major conjecture in theoretical computer science, a close relative to $\mathsf{P} \neq \mathsf{NP}$, implies that combinatorial interpretations *do not exist* in some cases.

Setting aside the precise technical meaning of a "combinatorial interpretation" (see below), one can ask what's the best one can do? What should be the right replacement of a "combinatorial interpretation"? We suggest that *signed combinatorial interpretations* are a perfect answer to these questions.

To understand why, first note that having signs is unavoidable when the numbers can be both positive and negative, and when the sign is hard to compute. A prototypical example is the S_n character value $\chi^\lambda(\mu)$ given by the Murnaghan–Nakayama rule (see below). This is why it's so puzzling to see signed combinatorial interpretations

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for nonnegative structure constants when no (unsigned) combinatorial interpretation is known, e.g. the Kronecker and Schubert coefficients.

In this paper, we make a systematic study and prove signed combinatorial interpretations for structure constants for many families of symmetric functions, their relatives, and generalizations. We present signed combinatorial interpretations in all cases without fail, leading to the following meta observation:

In algebraic combinatorics, all integral constants have signed combinatorial interpretations.

In conclusion, let us mention that signed combinatorial interpretations are useful for many applications, ranging from upper bounds to asymptotics, from analysis of algorithms to computational complexity, see below. As it turns out, "if you try sometime you'll find you get what you need", fully consistent with the philosophy in [48].

1.2. Signed combinatorial interpretations. We postpone the discussion of computational complexity classes until later in this section. Let us briefly remind the reader of the complexity classes NP, $\#\mathsf{P}$, and GapP.

Denote $W := \{0,1\}^*$ and $W_n := \{0,1\}^n$ be sets of *words*. The length $|w|$ is called the *size* of $w \in W$, so $|w| = n$ for all $w \in W_n$. A *language* is defined as $A \subseteq W$. Denote $A_n := A \cap W_n$. We say that A is in NP if the membership $[w \in^? A]$ can be decided in polynomial time (in the size $|w|$), by a Turing Machine. We think of A_n as the set of *combinatorial objects* of size n. For example, partitions, Young tableaux, permutations etc. are all combinatorial objects when encoded appropriately.

Let $f : W \to \mathbb{N}$ be an integer function. We say that f has a *combinatorial interpretation* (also called a *combinatorial formula*), if $f \in \#\mathsf{P}$. This means that there is a language $B \subseteq W^2$ in NP, such that for all $w \in W$ we have $f(w) = \#\{u : (w,u) \in B\}$. Informally, we say that the language B *counts the numbers* $\{f(w)\}$. Note that the decision problem $[f(w) >^? 0]$ is also in NP. We refer to [73] for the background and many examples.

In algebraic combinatorics, all standard combinatorial interpretations are in $\#\mathsf{P}$. These include *character degrees* $f^\lambda = \chi^\lambda(1)$, *Kostka numbers* $K_{\lambda\mu}$, and the *Littlewood–Richardson coefficients* $c^\lambda_{\mu\nu}$, see e.g. [63, 84, 89]. Indeed, in each case these are counted by certain Young tableaux, the validity of which can be verified in polynomial time.[(1)]

Let $f : \{0,1\}^* \to \mathbb{Z}$ be an integer function. We say that function f has a *signed combinatorial interpretation* (also called a *signed combinatorial formula*), if $f = g - h$ for some $g, h \in \#\mathsf{P}$. The set of such functions is denoted $\mathsf{GapP} := \#\mathsf{P} - \#\mathsf{P}$. This class is well studied in the computational complexity literature, see e.g. [32, 41]. We use $\mathsf{GapP}_{\geqslant 0}$ to denote the subset of nonnegative $f \in \mathsf{GapP}$. Note that $[f(w) >^? 0]$ is not always in NP (see below).

In algebraic combinatorics, there are many natural examples of signed combinatorial interpretations. As we mentioned above, these include *character values* $\chi^\lambda(\mu)$ via the *Murnaghan–Nakayama rule* (see e.g. [49, 84, 89]). Another example is the *inverse Kostka numbers* $K^{-1}_{\lambda\mu}$ defined as the entry in the inverse Kostka matrix $(K_{\lambda\mu})^{-1}$, given by the *Eğecioğlu–Remmel rule* [26]. In both cases, the rules subtract the number of certain rim hook tableaux, some with a positive sign and some with a negative, where the sign is easily computable.

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(1) There is a subtlety here in the encoding of partitions. Here and throughout the paper we use *unary encoding*. Although some of our results generalize to the binary encoding, this would require more involved arguments, see §9.5.

As we mentioned above, there are many examples of signed combinatorial interpretations for nonnegative functions, some of which are collected in [73]. Famously, *Kronecker coefficients* $g(\lambda, \mu, \nu)$ are given by the large signed summation of the numbers of 3-dimensional contingency arrays, see [16, 21, 75]. Another celebrated example is the *Schubert coefficients* (also called *Schubert structure constants*) $c^{\gamma}_{\alpha\beta}$ given by the *Postnikov–Stanley formula* in terms of the number of chains in the Bruhat order [81, Cor. 17.13].

1.3. Main results. Let R be a ring and let $\Upsilon := \{\xi_\alpha\}$ be a linear basis in R, where the indices form a set $A = \{\alpha\}$ of combinatorial objects. The *structure constants* $\big\{c(\alpha, \beta, \gamma)\big\}$ for Υ are defined as

$$\xi_\alpha \cdot \xi_\beta \, = \, \sum_{\gamma \in A} c(\alpha, \beta, \gamma)\, \xi_\gamma \quad \text{where} \quad \alpha, \beta \in A.$$

When the structure constants are integral, one can ask whether the function $\mathbf{c} : A^3 \to \mathbb{Z}$ is in GapP, i.e. has a signed combinatorial interpretation. Additionally, when they are nonnegative, one can ask if $\mathbf{c}$ is in $\#\mathsf{P}$, i.e. has (the usual) combinatorial interpretation.

We postpone the definitions of various rings and bases to later sections, moving straight to results both known and new. The following result is routine, well-known, and is included both for contrast and for completeness:

Theorem 1.1 (classic structure constants). *Let $\Lambda_n = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ denote symmetric polynomials in n variables. The following bases in Λ_n have structure constants in* $\#\mathsf{P}$:

- *Schur polynomials* $\{s_\lambda : \ell(\lambda) \leqslant n\}$,
- *monomial symmetric polynomials* $\{m_\lambda : \ell(\lambda) \leqslant n\}$,
- *power sum symmetric polynomials* $\{p_\lambda : \lambda_1 \leqslant n\}$,
- *elementary symmetric polynomials* $\{e_\lambda : \lambda_1 \leqslant n\}$, *and*
- *complete homogeneous symmetric polynomials* $\{h_\lambda : \lambda_1 \leqslant n\}$.

The last four of these items are completely straightforward and follow directly from the definition. On the other hand, the first item corresponding to the Littlewood–Richardson (LR) coefficients is highly nontrivial. Rather than reprove it, we give an elementary proof of a weaker claim, that LR-coefficients are in GapP. This sets us up for several generalizations.

First, we consider deformations of Schur polynomials. Let $q, t, \alpha \in \mathbb{Q}$ be fixed rational numbers.

Theorem 1.2. *The following bases in Λ_n have structure coefficients in* $\mathsf{GapP}/\mathsf{FP}$:

- *Jack symmetric polynomials* $\{P_\lambda(x; \alpha) : \ell(\lambda) \leqslant n\}$, *where* $\alpha > 0$,
- *Hall–Littlewood polynomials* $\{P_\lambda(x; t) : \ell(\lambda) \leqslant n\}$, *where* $0 \leqslant t < 1$, *and*
- *Macdonald symmetric polynomials* $\{P_\lambda(x; q, t) : \ell(\lambda) \leqslant n\}$, *for* $0 \leqslant q, t < 1$.

Here $\mathsf{GapP}/\mathsf{FP}$ is a class of rational functions which can be written as f/g where $f \in \mathsf{GapP}$ and $g \in \mathsf{FP}$ is a function which can be computed in polynomial time. Essentially, we prove that these structure constants are rational, the numerators have signed combinatorial interpretations, and the denominators have a nice product formula.

Second, we consider quasisymmetric polynomials which are somewhat intermediate between symmetric and general polynomials:

Theorem 1.3 (quasisymmetric structure constants). *Let $\mathrm{QSym}_n \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be the ring of quasisymmetric polynomials in n variables. The following bases have structure constants in* $\#\mathsf{P}$:

- *monomial quasisymmetric polynomial* $\{M_\alpha\}$*, and*
- *fundamental quasisymmetric polynomials* $\{F_\alpha\}$*.*

The following bases have structure constants in GapP :

- *dual immaculate polynomials* $\{\mathfrak{S}^*_\alpha\}$*, and*
- *quasisymmetric Schur polynomials* $\{\mathcal{S}_\alpha\}$*.*

The following bases have structure constants in $\#\mathsf{P}/\mathsf{FP}$:

- *type 1 quasisymmetric power sum* $\{\Psi_\alpha\}$*,*
- *type 2 quasisymmetric power sum* $\{\Phi_\alpha\}$*, and*
- *combinatorial quasisymmetric power sum* $\{\mathfrak{p}_\alpha\}$*.*

Here we have α *is a composition into at most* n *positive parts.*

The first two items go back to Gessel [33], while the rest are new. Next, recall that Schubert polynomials mentioned above generalize Schur polynomials and form a linear basis in the ring of *all* polynomials. The following result can be viewed as the analogue of Theorem 1.1 in this more general setting.

Theorem 1.4 (Polynomial structure constants). *In the polynomial ring* $\mathbb{C}[x_1, \ldots, x_n]$*, the following bases have structure constants in* $\#\mathsf{P}$:

- *monomial slide polynomials* $\{\mathfrak{M}_\alpha\}$*, and*
- *fundamental slide polynomials* $\{\mathfrak{F}_\alpha\}$*.*

The following bases have structure constants in GapP :

- *Demazure atoms* $\{\mathsf{atom}_\alpha\}$*,*
- *key polynomials* $\{\kappa_\alpha\}$*,*
- *Schubert polynomials* $\{\mathfrak{S}_\alpha\}$*,*
- *Lascoux polynomials* $\{\mathfrak{L}_\alpha\}$*, and*
- *Grothendieck polynomials* $\{\mathfrak{G}_\alpha\}$*.*

Here we have $\alpha \in \mathbb{N}^n$ *is a composition into* n *nonnegative parts.*

As we mentioned above, the result for Schubert polynomials follows from the work of Postnikov and Stanley [81]. We reprove this result in a simpler (but closely related) way, leading to results in other cases.

Finally, we consider *plethysm*. This is a general notion (see [69]), which is especially natural in the ring of symmetric functions, where it can be defined as follows. Let $\pi : \mathrm{GL}(V) \to \mathrm{GL}(W)$ and $\rho : \mathrm{GL}(W) \to \mathrm{GL}(U)$ be polynomial representations of the general linear group. One can define $\rho[\pi] := \rho \circ \pi$ to be the composition of these representations (cf. [63, §I.8,App. I.A] and [49, §5.4]). At the level of characters, the composition above corresponds to *plethysm* of symmetric functions, and gives *plethysm coefficients*:

$$f[g] = \sum_\lambda a^\lambda_{\mu\nu}\, s_\lambda \quad \text{where} \quad f, g \in \Lambda,$$

and Λ is the inverse limit of Λ_n in the category of graded rings (see e.g. [63, §I.2]). It was shown by Fischer and Ikenmeyer [30, §9] that plethysm coefficients for $s_\lambda[s_\mu]$ are GapP-complete, so in particular they are in GapP, see also §9.1. The following result is a generalization to other bases listed in Theorem 1.1.

Theorem 1.5 (plethysm coefficients). *Let* $\{f_\lambda\}$ *and* $\{g_\lambda\}$ *be families of symmetric polynomials from the following list of linear bases*:

$$\{s_\lambda\}, \quad \{m_\lambda\}, \quad \{p_\lambda\}, \quad \{e_\lambda\}, \quad \{h_\lambda\}.$$

Then the corresponding plethysm coefficients $\{a^\lambda_{\mu\nu}\}$ *are in* GapP.

1.4. Background and motivation. Structure constants are fundamental in algebraic combinatorics, reflecting both the advances and the challenges posed by the nature of symmetric objects. They are a succinct trove of information reducing algebraic structures to explicit combinatorial objects.

Having a combinatorial interpretation of numbers does more than put a face to the name. It reveals a combinatorial structure which in turn is a shadow of a rich but non-quantitative geometric or algebraic structure. Different combinatorial interpretations give different shadows, helping to understand the big picture.

For example, on a technical level, standard Young tableaux are the leading terms in a natural linear basis of irreducible S_n modules, a simple bookkeeping tool for a large data structure (cf. [49, 84]). But on a deeper level, they are a byproduct of the *branching rule*, which in turn comes from S_n having a long subgroup chain (cf. [70]).

It would be impossible to overstate the impact of combinatorial interpretations in algebraic combinatorics and symmetric function theory, as they completely permeate the area (see e.g. [63, 89]). The LR-coefficients alone have over 15 different combinatorial interpretations (see [73, §11.4]), and are the subject of hundreds of papers.

Additionally, having a combinatorial interpretation does wonders for applications of all kinds. For example, for the LR-coefficients, these include the *saturation theorem* [52], efficient algorithm for positivity $[c^{\lambda}_{\mu\nu} >^? 0]$, see [23, 17], and various lower and upper bounds, see [77]. Even when combinatorial interpretations are known only in special cases, the remarkable applications follow. For example, for the Kronecker coefficients, these include unimodality [74], and NP-hardness of the positivity $[g(\lambda, \mu, \nu) >^? 0]$ proved in [43].

Naturally, a signed combinatorial interpretation is inherently less powerful than the usual (unsigned) one. And yet, this is usually the best known tool to obtain any results at all. For example, for the Kronecker coefficients, the signed combinatorial interpretation mentioned above gives both a fast algorithm to compute the numbers, see [75], and a sharp upper bound in some cases, see [76]. Of course, lower bounds cannot be obtained this way, which is why finding a good lower bound for the Kronecker coefficients remains a major open problem, see e.g. [79].

An interesting case study is the Murnaghan–Nakayama (MN) rule for the S_n characters values $\chi^{\lambda}(\mu)$, defined as the signed sum over certain rim hook (ribbon) tableaux (see e.g. [49, 84]). The rule was used in [55, 82] to obtain upper bounds for character values, which in turn give upper bounds for the mixing times of random walks on S_n generated by conjugacy classes.

In a surprising development, when the conjugacy class μ is a rectangle, it is known that all rim hook tableaux given by the MN rule have the same sign, see [90]. This observation led to rich combinatorial developments, including combinatorial proofs of character orthogonality [94, 93], applications in probability [12], tilings [71], and the LLT polynomials describing representations of Hecke algebra at roots of unity [57] (cf. Remark 8.2).

In a different direction, a signed combinatorial interpretation coming from the *Frobenius formula* (that is somewhat different from the MN rule), was used to show that deciding positivity $[\chi^{\lambda}(\mu) > 0]$ of the character value is PH-hard [45, Thm 1.1.5]. Moreover, the authors show that character absolute value has no combinatorial interpretation [45, Thm 1.1.3]. More precisely, $|\chi^{\lambda}(\mu)|$ is not in $\#\mathsf{P}$ unless $\Sigma_2^{\mathsf{p}} = \mathsf{PH}$. In other words, one should not expect the (usual) combinatorial interpretation for the character values, unless one believes that the polynomial hierarchy collapses.

In a warning to the reader, we should emphasize that some natural combinatorial formulas defining the numbers above are not, in fact, GapP formulas. For example, the definition of Kronecker coefficients gives:

$$g(\lambda,\mu,\nu) \,:=\, \langle \chi^\lambda, \chi^\mu \cdot \chi^\nu \rangle \,=\, \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\, \chi^\mu(\sigma)\, \chi^\nu(\sigma),$$

for all $\lambda, \mu, \nu \vdash n$. This only shows that $\{g(\lambda,\mu,\nu)\}$ are in $\mathsf{GapP}/\mathsf{FP}$.

Indeed, while the summation above is in GapP via the MN rule, the division by $n!$ is not allowed in GapP. The same issue appears also when applying *Billey's formula* to compute Schubert coefficients [9], the *Féray–Śniady formula* [29, Thm 4] for the characters, and Hurwitz's original formula for the *double Hurwitz numbers*, see [34] and further references in [73, §8.5]. All these formulas involve divisions, thus they only prove that the corresponding integral functions are in $\mathsf{GapP}/\mathsf{FP}$.

We should mention that there are cases when the integrality was established algebraically. For example, the integrality of S_n character values: $\chi^\lambda(\mu) \in \mathbb{Z}$, follows from the fact that σ and σ^a are conjugate, for all $\sigma \in S_n$ and $(a, \mathrm{ord}(\sigma)) = 1$, see e.g. [87, §13.1]. The proof is based on a Galois theoretic argument combined with a calculation of cyclotomic polynomials, and cannot be easily translated to a GapP formula. In other words, being in GapP can be a significantly stronger result of independent interest.

Finally, we note that the literature on structure constants discussed above is much too large to be reviewed in this paper. We include some additional references in Section 9. Our general point stands: we obtain signed combinatorial interpretations in many cases where none is known, and new simple signed combinatorial interpretations in a handful of cases where some are known.

1.5. Proof ideas. We start by observing that all structure constants in Theorems 1.3 and 1.4 (excluding $\{\Psi_\alpha, \Phi_\alpha, \mathfrak{p}_\alpha\}$) are known to be integral. Furthermore, the proofs that these are integral involve *unitriangular* changes of bases. This is best illustrated by the *Kostka matrix* $(K_{\lambda\mu})$, which satisfies:

$$K_{\lambda\lambda} \,=\, 1 \quad \text{and} \quad K_{\lambda\mu} = 0 \;\text{ unless }\; \lambda \trianglelefteq \mu.$$

From there, the problem is reduced to making the *Möbius inversion* effective. This, in turn, hinges on the fact that partial orders such as "$\triangleleft$" have polynomial height (cf. §9.3). In the case of Kostka numbers, this gives an alternative signed combinatorial interpretation of the inverse Kostka numbers $K^{-1}_{\lambda\mu}$ that is different from that in [26] and [25].

Now, for the bases in Theorems 1.3 and 1.4, the heavy lifting of establishing the unitriangular property was done in a series of previous papers, some of them very recent. Our approach gives a simple to use tool to extend the unitriangular property to a GapP result for the structure constants. This is done in Section 3. In later Sections 4–6, we derive the results one by one.

Finally, we prove Theorem 1.5 on plethysm coefficients in a short Section 7. The proof uses a simple argument again involving GapP formulas for (generalized) Kostka numbers.

2. Basic definitions and notations

We use $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $[n] = \{1, \ldots, n\}$. To simplify the notation, for a set X and an element $x \in X$, we write $X - x := X \smallsetminus \{x\}$. Similarly, we write $X + y := X \cup \{y\}$.

Let $\mathcal{P} = (X, \prec)$ be a poset on the ground set X with a partial order "$\prec$". Function $h : X^2 \to \mathbb{R}$ is called *triangular* w.r.t. $\mathcal{P}$ if $h(x,y) = 0$ unless $x \preccurlyeq y$ for all $x, y \in X$, and $h(x,x) \neq 0$ for all $x \in X$. Similarly, function $h : X^2 \to \mathbb{R}$ is called *unitriangular* w.r.t. $\mathcal{P}$ if it is triangular and $h(x,x) = 1$ for all $x \in X$.

Fix n. An *integer partition* λ of k, denoted $\lambda \vdash k$, is a sequence of weakly decreasing nonnegative integers $(\lambda_1, \ldots, \lambda_n)$ which sum up to k. Denote by $U_{n,k}$ the set of these partitions, and let $\mathrm{U}_n = \cup_k \mathrm{U}_{n,k}$. Similarly a *composition* (sometimes called *weak composition*) α of k, denoted $\alpha \vDash k$, is a sequence of nonnegative integers $(\alpha_1, \ldots, \alpha_n)$ which sum up to k. Let $\mathrm{V}_{n,k} := \{\alpha \in \mathbb{N}^n : \alpha \vDash k\}$, and let $\mathrm{V}_n := \cup_k \mathrm{V}_{n,k}$ be sets of compositions. A *strong composition* $\alpha \vDash k$ has all parts strictly positive. Let $\mathrm{W}_{n,k} := \{\alpha \in \mathbb{N}^m_{\geqslant 1} : \alpha \vDash k, m \leqslant n\}$, and let $\mathrm{W}_n := \cup_k \mathrm{W}_{n,k}$ be sets of strong compositions. Clearly, every partition of n is also a composition. Let $D(\alpha) := \{(i,j) : i \leqslant \alpha_j\}$ denote the *diagram* of α.

We write $|\alpha| := \alpha_1 + \ldots + \alpha_n$ the size of the composition, and $\ell(\alpha)$ the number of parts in α. Denote $z_\alpha := 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots$, where $m_i = m_i(\alpha)$ denotes the multiplicity of i in α. For two compositions $\alpha, \beta \vDash k$, the *dominance order* is defined as follows:

$$\alpha \trianglelefteq \beta \iff \alpha_1 + \ldots + \alpha_i \geqslant \beta_1 + \ldots + \beta_i \quad \text{for all } i.$$

Note that compositions of different integers are incomparable.

For a permutation $\sigma \in S_n$, the *Lehmer code* is a sequence $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{N}^n$ given by $c_i(\sigma) := \#\{j > i : \sigma_i \geqslant \sigma_j\}$. Note that $|\mathbf{c}| = \mathrm{inv}(\sigma)$ is the *number of inversions* in σ. Denote by S_∞ the set of bijections $w : \mathbb{N}_{\geqslant 1} \to \mathbb{N}_{\geqslant 1}$ which eventually stabilize: $w(m) = m$ for m large enough. We view such w as an infinite word, and note that the number of inversions is well defined on S_∞.

Young diagram of shape λ, denoted $[\lambda]$, is a set of squares $\{(i,j) : 1 \leqslant i \leqslant \ell(\lambda), 1 \leqslant j \leqslant \lambda_i\}$. A *semistandard Young tableau* of shape λ is a map $A : [\lambda] \to \mathbb{N}$, which is weakly increasing in rows: $A(i,j) \leqslant A(i,j+1)$ and strictly increasing in columns: $A(i,j) < A(i+1,j)$. The *content* of A is a sequence $(m_1, m_2, \ldots)$, where m_k is the number of k in the multiset $\{A(i,j)\}$. Let $\mathrm{SSYT}(\lambda, \mu)$ denote the set of semistandard Young tableaux of shape λ and content μ.

Consider the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ in n variables. For an integer sequence $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, denote $\boldsymbol{x}^{\mathbf{a}} := x_1^{a_1} \cdots x_n^{a_n}$. We use $[\boldsymbol{x}^{\mathbf{a}}]F$ to denote the coefficient of $\boldsymbol{x}^{\mathbf{a}}$ in the polynomial F. A polynomial $F \in \mathbb{C}[x_1, \ldots, x_n]$ is *symmetric* if its coefficients satisfy

$$[x_1^{a_1} \cdots x_n^{a_n}]F = [x_{\sigma(1)}^{a_1} \cdots x_{\sigma(n)}^{a_n}]F \quad \text{for all } \sigma \in S_n \text{ and } (a_1, \ldots, a_n) \in \mathbb{N}^n.$$

The ring of symmetric polynomials is denoted by Λ_n. Similarly, F is *quasisymmetric* if

$$[x_1^{a_1} \cdots x_k^{a_k}]F = [x_{i_1}^{a_1} \cdots x_{i_k}^{a_n}]F \text{ for all } i_1 < \ldots < i_k,\ k \in [n], \text{ and } (a_1, \ldots, a_k) \in \mathbb{N}^k_{\geqslant 1}.$$

The ring of quasisymmetric polynomials is denoted by QSym_n. By definition, we have:

$$\Lambda_n \subset \mathrm{QSym}_n \subset \mathbb{C}[x_1, \ldots, x_n].$$

Class FP is a class of functions computable in poly-time. Class $\#\mathsf{P}$ is closed under addition and multiplication:

$$f, g \in \#\mathsf{P} \implies f + g,\ f \cdot g \in \#\mathsf{P}.$$

Class $\mathsf{GapP} := \#\mathsf{P} - \#\mathsf{P}$ is closed under addition, subtraction and multiplication:

$$f, g \in \mathsf{GapP} \implies f \pm g,\ f \cdot g \in \mathsf{GapP}.$$

Class $\mathsf{GapP}/\mathsf{FP}$ is a class of rational functions which can be written as f/g where $f \in \mathsf{GapP}$ and $g \in \mathsf{FP}$. Clearly,

$$\#\mathsf{P} \subset \mathsf{GapP} \subset \mathsf{GapP}/\mathsf{FP}.$$

3. Effective Möbius inversion

Let $X = \cup X_n$, where $X_n \subseteq \{0,1\}^n$, be a family of combinatorial objects. Let $\mathcal{P} := (X, \prec)$ be a poset such that $x \prec y$ only if $x, y \in X_n$ for some n. We use $\mathcal{P}_n = (X_n, \prec)$ to denote a subposet of $\mathcal{P}$. The *height* of a poset $\mathcal{Q}$, denoted $\operatorname{height}(\mathcal{Q})$, is the size of the maximal chain in $\mathcal{Q}_n$. We say that $\mathcal{P}$ has *polynomial height*, if $\operatorname{height}(\mathcal{P}_n) \leqslant C n^c$, for some fixed $C, c > 0$.

Let $\delta : X^2 \to \{0,1\}$ be the *delta function* defined as $\delta(x,y) = 1$ if $x = y$, and $\delta(x,y) = 0$ otherwise. Let $\xi : X^2 \to \{0,1\}$ be the *incidence function* defined as $\xi(x,y) = 1$ if $x \preccurlyeq y$ and $\xi(x,y) = 0$ otherwise. We say that ξ is *poly-time computable*, if for all $x, y \in X_n$ the decision problem $[x \preccurlyeq^? y]$ can be decided in $O(n^c)$ time, for some fixed $c > 0$.

The *Möbius inverse* is a function $\mu(x,y) : X^2 \to \mathbb{Z}$, such that

$$\sum_{z \in X_n} \xi(x,z) \cdot \mu(z,y) \, = \, \delta(x,y) \quad \text{for all} \quad x, y \in X_n \, .$$

Proposition 3.1. *Let $\mathcal{P} := (X, \prec)$ be a poset with polynomial height, and suppose that the incidence function ξ is poly-time computable. Then the Möbius inverse function μ is in* GapP.

Proof. For all $x, y \in X$, denote by $\mathcal{P}_{xy}$ the interval in $\mathcal{P}$, and let $h := \operatorname{height}\big(\mathcal{P}_{xy}\big)$. Denote by $\mathcal{C}_\ell(x,y)$ the set of chains $x \to z_1 \to \ldots \to z_{\ell-1} \to y$ in $\mathcal{P}_{xy}$ of length ℓ. By *Hall's theorem* (see e.g. [89, Prop. 3.8.5]), we have:

$$\mu(x,y) \, = \, \sum_{\ell=0}^{h} (-1)^\ell \, \big|\mathcal{C}_\ell(x,y)\big|.$$

We conclude: $\mu(x,y) \, = \, \mu_+(x,y) \, - \, \mu_-(x,y)$, where

$$\mu_+(x,y) := \sum_{i=0}^{\lfloor h/2 \rfloor} \big|\mathcal{C}_{2i}(x,y)\big| \quad \text{and} \quad \mu_-(x,y) := \sum_{i=0}^{\lfloor h/2 \rfloor} \big|\mathcal{C}_{2i+1}(x,y)\big|.$$

Since h is polynomial, we have $\mu_\pm \in \#\mathsf{P}$ by definition. This completes the proof. □

Let $\eta : X^2 \to \mathbb{Z}$ be unitriangular w.r.t. $\mathcal{P}$, i.e. $\eta(x,x) = 1$ for all $x \in X$, and $\eta(x,y) \neq 0$ implies $x \preccurlyeq y$, $x, y \in X_n$ for some n. The *inverse* of η (in the incidence algebra), is a function $\rho(x,y) : X^2 \to \mathbb{Z}$, such that

$$\sum_{z \in X_n} \eta(x,z) \cdot \rho(z,y) \, = \, \delta(x,y) \quad \text{for all} \quad x, y \in X_n \, .$$

Proposition 3.2. *Let $\mathcal{P} := (X, \prec)$ be a poset with polynomial height, and suppose that the incidence function ξ is poly-time computable. Suppose function η is in* GapP. *Then the inverse function of η is also in* GapP.

Proof. Denote by ρ the inverse function as in the theorem. We similarly have:

$$\rho(x,y) \, = \, \sum_{\ell=0}^{h} \, \sum_{(x \to z_1 \to \ldots \to z_{\ell-1} \to y) \in \mathcal{C}_\ell(x,y)} (-1)^\ell \, \eta(x,z_1) \cdot \eta(z_1,z_2) \, \cdots \, \eta(z_{\ell-1},y).$$

The result follows. □

4. Symmetric polynomials

Let $\Lambda_m = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ be the ring of symmetric polynomials. Denote by $\mathcal{Q}_m$ the poset on partitions $\lambda \vdash m$ with dominance order $\lambda \trianglelefteq \mu$, for all $\lambda, \mu \vdash m$. It is known that the dominance order is a lattice, see §9.2. Clearly, $\operatorname{height}(\mathcal{Q}_m) = O(m^2)$. In fact, it was shown in [36], that $\operatorname{height}(\mathcal{Q}_m) = \Theta(m^{3/2})$.

4.1. Standard bases. *Monomial symmetric polynomials* $\{m_\lambda : \lambda \in \mathrm{U}_n\}$ are defined as

$$m_\lambda(x_1, \ldots, x_n) := \sum_w \boldsymbol{x}^{w(\lambda)},$$

where the summation is over all $w \in S_n$ giving different reorderings $w(\lambda)$ of λ.

Power sum symmetric polynomials $\{p_\lambda : \lambda \in \mathrm{U}_n\}$ are defined by

$$p_\lambda(x_1, \ldots, x_n) := p_{\lambda_1}(x_1, \ldots, x_n) \cdots p_{\lambda_n}(x_1, \ldots, x_n),$$

where

$$p_k(x_1, \ldots, x_n) := x_1^k + \ldots + x_n^k.$$

Observe that

$$p_\lambda(x_1, \ldots, x_n) = \sum_\mu P(\lambda, \mu)\, m_\mu(x_1, \ldots, x_n),$$

where $P(\lambda, \mu)$ is the number of nonnegative integer-valued matrices with column sum λ and row sums μ, where each column contains at most one nonzero entry.

Similarly, *elementary symmetric polynomials* $\{e_\lambda : \lambda \in \mathrm{U}_n\}$ are are defined by

$$e_\lambda(x_1, \ldots, x_n) := e_{\lambda_1}(x_1, \ldots, x_n) \cdots e_{\lambda_n}(x_1, \ldots, x_n),$$

where

$$e_k(x_1, x_2, \ldots, x_n) := \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} x_{i_1} \cdots x_{i_k}.$$

Observe that

$$e_\lambda(x_1, \ldots, x_n) = \sum_\mu E(\lambda, \mu)\, m_\mu(x_1, \ldots, x_n),$$

where $E(\lambda, \mu)$ is the number of $\{0,1\}$-matrices with column sums λ and row sums μ.

Complete homogeneous symmetric polynomials $\{h_\lambda : \lambda \in \mathrm{U}_n\}$ are defined by

$$h_\lambda(x_1, \ldots, x_n) := h_{\lambda_1}(x_1, \ldots, x_n) \cdots h_{\lambda_n}(x_1, \ldots, x_n),$$

where

$$h_k(x_1, x_2, \ldots, x_n) := \sum_{\mu \vdash k} m_\mu.$$

Observe that

$$h_\lambda(x_1, \ldots, x_n) = \sum_\mu H(\lambda, \mu)\, m_\mu(x_1, \ldots, x_n),$$

where $H(\lambda, \mu)$ is the number of nonnegative integer-valued matrices with column sums λ and row sums μ.

Finally, *Schur polynomials* $\{s_\lambda : \lambda \in \mathrm{U}_n\}$ can be defined as

$$s_\lambda(x_1, \ldots, x_n) := \sum_\mu K_{\lambda\mu}\, m_\mu(x_1, \ldots, x_n),$$

where the *Kostka numbers* $K_{\lambda\mu}$ compute the number of semistandard Young tableaux of shape λ and content μ. The *Littlewood–Richardson coefficients* $c^\lambda_{\mu\nu}$ are defined by

$$s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda.$$

Recall that $\{c^\lambda_{\mu\nu}\}$ are given as the number of LR-tableaux (we omit the definition), a subset of semistandard Young tableaux, see e.g. [63, 89].

4.2. Structure constants. We start with a traditional approach to structure constants, which we outline for completeness.

Proof of Theorem 1.1. The result for $\{p_\lambda\}$, $\{e_\lambda\}$ and $\{h_\lambda\}$ is trivial. The definition of $\{m_\lambda\}$ gives their structure coefficients

$$T(\lambda,\mu,\nu) \,:=\, \#\big\{(u(\lambda),w(\mu))\,:\,u(\lambda)+w(\mu)=\nu\big\},$$

implying that they are in $\#\mathsf{P}$. Finally, the LR-coefficients $\big\{c^{\lambda}_{\mu\nu}\big\}$ are in $\#\mathsf{P}$ by the definition of LR-tableaux. $\square$

We now prove a weaker result, using the effective Möbius inversion.

Proposition 4.1. *The inverse Kostka numbers* $\{K^{-1}_{\lambda\mu}\}$ *and the LR-coefficients* $\big\{c^{\lambda}_{\mu\nu}\big\}$ *are in* GapP.

Proof. Recall $\big(K_{\lambda\mu}\big)$ is unitriangular w.r.t. to the dominance order, so

$$s_\lambda \,=\, \sum_{\mu\,\trianglelefteq\,\lambda} K_{\lambda\mu}\, m_\mu\,.$$

Thus, the *inverse Kostka numbers* are given by

$$m_\lambda \,=\, \sum_{\mu\,\trianglelefteq\,\lambda} K^{-1}_{\lambda\mu}\, s_\mu\,. \tag{4.1}$$

Now Proposition 3.2 implies that $\{K^{-1}_{\lambda\mu}\}\in\mathsf{GapP}$. We have:

$$\begin{aligned} s_\mu\cdot s_\nu \,&=\, \Big(\sum_{\tau\trianglelefteq\mu} K_{\mu\tau}\, m_\tau\Big)\cdot\Big(\sum_{\varkappa\trianglelefteq\nu} K_{\nu\varkappa}\, m_\varkappa\Big)\\ &=\, \sum_{\tau\trianglelefteq\mu}\,\sum_{\varkappa\trianglelefteq\nu}\,\sum_{\sigma} K_{\mu\tau}\,K_{\nu\varkappa}\,T(\tau,\varkappa,\sigma)\cdot m_\sigma\\ &=\, \sum_{\tau\,\trianglelefteq\,\mu}\,\sum_{\varkappa\,\trianglelefteq\,\nu}\,\sum_{\sigma}\,\sum_{\lambda\,\trianglelefteq\,\sigma} K_{\mu\tau}\,K_{\nu\varkappa}\,T(\tau,\varkappa,\sigma)\,K^{-1}_{\sigma\lambda}\cdot s_\lambda \end{aligned}$$

Since $\{K^{-1}_{\lambda\mu}\}\in\mathsf{GapP}$ and $\{T(\lambda,\mu,\nu)\}\in\#\mathsf{P}$, this implies that LR-coefficients are in GapP. $\square$

4.3. (q,t) deformations. Following [64], *Macdonald symmetric polynomials* P_λ can be defined in terms of semistandard Young tableaux:

$$P_\lambda(\boldsymbol{x};q,t) \,:=\, \sum_{\mu} m_\mu(\boldsymbol{x}) \sum_{T\,\in\,\mathrm{SSYT}(\lambda,\mu)} \psi_T(q,t)\,,$$

where $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\psi_T(q,t)$ is a explicit rational function given by a product formula. In particular, for fixed $q,t\in\mathbb{Q}$ such that $0\leqslant q,t<1$, this function $\psi:T\to\mathbb{Q}$ is in FP/FP.

The *Hall–Littlewood polynomials* $P_\lambda(\boldsymbol{x};t)$, see [61], specialize Macdonald symmetric polynomials by taking $q=0$. Similarly, the *Jack symmetric polynomials* $P_\lambda(\boldsymbol{x};\alpha)$, see [47], specialize Macdonald symmetric polynomials in another direction:

$$P_\lambda(\boldsymbol{x};\alpha) \,:=\, \lim_{t\to1} P_\lambda(\boldsymbol{x};t^\alpha,t).$$

It follows from the definition above and the explicit form of ψ_T that Macdonald symmetric polynomials are unitriangular in the monomial symmetric polynomials:

$$P_\lambda(\boldsymbol{x};q,t) \,=\, \sum_{\mu\,\trianglelefteq\,\lambda} K_{\lambda\mu}(q,t)\, m_\mu(\boldsymbol{x}),$$

where $K_{\lambda,\lambda}(q,t)=1$, see [63, Thm 2.3]. Using the argument in the proof of Proposition 4.1 gives the last part of Theorem 1.2. By the specialization to Hall–Littlewood polynomials and Jack polynomials, we obtain the remaining two parts of the theorem. We omit the details.

4.4. Schur P-polynomials. For a partition λ with distinct parts, the *Schur P-polynomial* is given by

$$P_\lambda(\boldsymbol{x}) \,:=\, P_\lambda(\boldsymbol{x}; -1).$$

This specialization of the Hall–Littlewood polynomials was defined by Schur (1911) in the study of projective representation theory of S_n. They are also called *Q-functions*, see [63, §III.8].

Note that Schur P-polynomials span a subring of Λ_n. The following result follows verbatim the proof of Theorem 1.2.

Corollary 4.2. *Schur P-polynomials have structure constants in* GapP*.*

4.5. (q,t) analogues. In Theorem 1.2, we consider deformations of Schur polynomials, viewed as bases in Λ. One can also view (q,t) as variables and extend the results in this direction. For the Hall–Littlewood polynomials $P_\lambda(\boldsymbol{x};t) \in \Lambda[t]$, the corresponding Kostka polynomials $K_{\lambda,\mu}(t) \in \mathbb{N}[t]$ are the coefficients of their expansion in Schur polynomials. They have a known combinatorial interpretation by Lascoux and Schützenberger (see e.g. [63, §III.6]). Using the LR rule, we conclude:

Proposition 4.3. *Hall–Littlewood polynomials $\{P_\lambda(t)\}$ have structure constants in* #P*.*

Here structure constants form a family of polynomials $\left\{c^\lambda_{\mu\nu}(t) \in \mathbb{N}[t]\right\}$. The proposition states that there is a #P function $f : \{(\lambda, \mu, k)\} \to \mathbb{N}$, such that

$$c^\lambda_{\mu\nu}(t) \,=\, \sum_{k\in\mathbb{N}} f(\lambda,\mu,k)\, t^k\,.$$

More generally, recall the *modified Macdonald polynomials* $\widetilde{H}_\mu(\boldsymbol{x};q,t) \in \Lambda[q,t]$, see e.g. [37, Thm 2.8]. They are defined so that the corresponding *(q,t)-Kostka polynomials* $\widetilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$ become the coefficients of their expansion in Schur polynomials. A combinatorial interpretation for the (q,t)-Kostka polynomials remains open (see e.g. [92, §4.1]).

On the other hand, a *signed* combinatorial interpretation of $\widetilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$ follows immediately from *Haglund's monomial formula* [37, App. A], giving a combinatorial interpretation for coefficients of their expansion in Schur functions, combined with a GapP formula for the (usual) inverse Kostka numbers. Using the LR rule again, we conclude:

Proposition 4.4. *Modified Macdonald polynomials $\{\widetilde{H}_\mu(q,t)\}$ have structure constants in* GapP*.*

Remark 4.5. It follows from the argument above, finding a combinatorial interpretation for the (q,t)-Kostka numbers would easily imply that a combinatorial interpretation for the modified Macdonald polynomials. It would be interesting to find an unconditional proof of this claim.

5. Quasisymmetric bases

5.1. Posets of interest. Recall that $\mathrm{W}_{n,k} := \{\alpha \in \mathbb{N}^m_{\geqslant 1} \,:\, \alpha \vDash k,\, m \leqslant n\}$, and let $\mathrm{W}_n := \cup_k \mathrm{W}_{n,k}$ be sets of strong compositions. Denote by $\mathcal{Z}_{n,k} = (\mathrm{W}_{n,k}, \lhd)$ a poset on strong compositions w.r.t. the dominance order. Let $\mathcal{Z}_n = \cup_k \mathcal{Z}_{n,k}$. Clearly, we have $\operatorname{height}(\mathcal{Z}_{n,k}) = O(kn)$.

For $\alpha \in \mathbb{N}^m$, define $\mathsf{sort}(\alpha)$ as the partition formed by listing α in weakly decreasing order. For $\alpha, \beta \in \mathrm{W}_{n,k}$ we say β is a *refinement* of α if one can obtain α by adding consecutive parts of β. For example, $\beta = (1,2,2,1,1)$ refines $\alpha = (3,3,1)$, but β does not refine $\gamma = (4,1,1,1)$. This defines the *refinement order* "$\preccurlyeq$" on $\mathrm{W}_{n,k}$. Denote by $\mathcal{D}_{n,k} = (\mathrm{W}_{n,k}, \trianglelefteq')$ a poset on $\mathrm{W}_{n,k}$ where

$$\alpha \trianglelefteq' \beta\,, \alpha, \beta \in \mathrm{W}_{n,k} \quad \Longleftrightarrow \quad \begin{cases} \mathsf{sort}(\beta) \lhd \mathsf{sort}(\alpha) & \text{if } \ \mathsf{sort}(\beta) \neq \mathsf{sort}(\alpha), \\ \beta \trianglelefteq \alpha & \text{if } \ \mathsf{sort}(\beta) = \mathsf{sort}(\alpha). \end{cases}$$

Observe that $\operatorname{height}\big(\mathcal{D}_{n,k}\big) = O(kn^3)$.

5.2. Integral quasisymmetric bases. Let $\mathrm{QSym}_n \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be the ring of quasisymmetric polynomials in n variables. The *monomial quasisymmetric polynomials* $\{M_\alpha \,:\, \alpha \in \mathrm{W}_n\}$ are defined as

$$M_\alpha(x_1, \ldots, x_n) \,:=\, \sum_{1 \leqslant i_1 < \ldots < i_\ell \leqslant n} x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell}\,,$$

where $\ell = \ell(\alpha) \leqslant n$. Clearly, $\{M_\alpha\}$ is a linear basis in QSym_n.

Following [33], the *fundamental quasisymmetric polynomials* $\{F_\alpha \,:\, \alpha \in \mathrm{W}_n\}$ are defined by

$$F_\alpha(x_1, x_2, \ldots, x_n) \,:=\, \sum_{\beta \preccurlyeq \alpha} M_\beta(x_1, x_2, \ldots, x_n).$$

Following [7], the *dual immaculate polynomials* $\{\mathfrak{S}^*_\alpha \,:\, \alpha \in \mathrm{W}_n\}$ can be defined by

$$\mathfrak{S}^*_\alpha(x_1, x_2, \ldots, x_n) \,:=\, \sum_\beta K^I_{\alpha,\beta}\, M_\beta(x_1, x_2, \ldots, x_n),$$

where $K^I_{\alpha,\beta}$ is the number of fillings of the diagram $D(\alpha) = \{(i,j) \,:\, i \leqslant \alpha_j\}$ with content β such that entries weakly increase along rows and strictly increase down the *leftmost* column.

Following [39], the *quasisymmetric Schur polynomials* $\{\mathcal{S}_\alpha \,:\, \alpha \in \mathrm{W}_n\}$ can be defined by

$$\mathcal{S}_\alpha(x_1, x_2, \ldots, x_n) \,:=\, \sum_\beta K^S_{\alpha,\beta}\, M_\beta(x_1, x_2, \ldots, x_n),$$

where $K^S_{\alpha,\beta}$ is the number of fillings T of $D(\alpha)$ with content β such that:

- entries in T weakly decrease across rows,
- entries in T strictly increase down the leftmost column, and
- the triple rule holds. That is, embed T in an $\ell(\alpha) \times \max(\alpha)$ rectangle, filling each newly added box with 0. Call this T'. Then for $1 \leqslant i < j \leqslant \ell(\alpha)$ and $2 \leqslant k \leqslant \max(\alpha)$, we have

$$T'(i,k) \leqslant T'(j,k) \neq 0 \quad \Longrightarrow \quad T'(i,k-1) < T'(j,k).$$

Observe that $\{F_\alpha\}$, $\{\mathfrak{S}^*_\alpha\}$, and $\{\mathcal{S}_\alpha\}$ are linear bases in QSym_n.

Example 5.1. Let $\alpha = (2,2) \vDash 4$. We have:

$$\begin{aligned} F_{2,2} \,&=\, M_{1,1,1,1} \,+\, M_{1,1,2} \,+\, M_{2,1,1} \,+\, M_{2,2}\,, \\ \mathfrak{S}^*_{2,2} \,&=\, 3M_{1,1,1,1} \,+\, 2M_{1,1,2} \,+\, 2M_{1,2,1} \,+\, M_{1,3} \,+\, M_{2,1,1} \,+\, M_{2,2}\,, \text{ and} \\ \mathcal{S}_{2,2} \,&=\, 2M_{1,1,1,1} \,+\, M_{1,1,2} \,+\, M_{1,2,1} \,+\, M_{2,1,1} \,+\, M_{2,2}\,. \end{aligned}$$

Let $\beta = (1,2,1) \vDash 4$. Then $K^I_{\alpha,\beta} = 2$ as given by the following tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 2 \\ \hline \end{array}\,.$$

Similarly, we have $K^S_{\alpha,\beta} = 1$ as given by the following tableau:

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 1 \\ \hline \end{array}\,.$$

5.3. Unitriangular property. Observe that the refinement order is a coarsening of dominance order. By the definition of fundamental quasisymmetric functions, we thus have unitriangular property for $\{F_\alpha\}$ w.r.t. the dominance order.

We now prove the corresponding result for the other two bases of quasisymmetric polynomials.

Proposition 5.2. *Dual immaculate polynomials $\{\mathfrak{S}^*_\alpha\}$ have unitriangular property w.r.t. the dominance order:*

$$\mathfrak{S}^*_\alpha \;=\; \sum_{\alpha \trianglelefteq \beta} K^I_{\alpha,\beta}\, M_\beta \quad \text{and} \quad K^I_{\alpha,\alpha} \;=\; 1\,,$$

for all $\alpha \in \mathrm{W}_{n,k}$.

Proof. By the definition of $\mathfrak{S}^*_\alpha$, we have $K^I_{\alpha,\beta} = 0$ if $|\beta| \neq k$. It was shown in [7, Prop. 3.15], that $K^I_{\alpha,\beta} = 0$ unless α precedes β in lexicographic order, and that $K^I_{\alpha,\alpha} = 1$. The entries increasing conditions in the definition of $K^I_{\alpha,\beta}$ implies that if i appears in row j in a tableau, then $i \leqslant j$. Thus $K^I_{\alpha,\beta} = 0$ unless $\alpha \trianglelefteq \beta$. This completes the proof. $\square$

Proposition 5.3. *Quasisymmetric Schur polynomials $\{\mathcal{S}_\alpha\}$ have unitriangular property w.r.t. the order $\trianglelefteq'$:*

$$(*) \qquad \mathcal{S}_\alpha \;=\; \sum_{\alpha \trianglelefteq' \beta} K^S_{\alpha,\beta}\, M_\beta \quad \text{and} \quad K^S_{\alpha,\alpha} \;=\; 1\,,$$

for all $\alpha \in \mathrm{W}_{n,k}$.

Proof. Suppose a tableau T is counted by $K^S_{\alpha,\alpha}$. The second and third tableau conditions ensure that no entries in T may repeat within a column. Since the diagrams are left-aligned, this implies $K^S_{\alpha,\beta} = 0$ unless $\alpha \trianglelefteq' \beta$. Now, it was shown in [39, Prop. 6.7] that $(*)$ holds for the lexicographic order (which strengthens $\trianglelefteq'$), and that $K^S_{\alpha,\alpha} = 1$. This completes the proof. $\square$

Proof of Theorem 1.3, first and second part. The first part is straightforward:

$$M_\alpha \cdot M_\beta \;=\; \sum_\tau c(\alpha,\beta,\tau)\, M_\tau\,,$$

where $c(\alpha,\beta,\tau)$ is the number of ways to write

$$(\tau_1,\tau_2,\ldots) \;=\; (\alpha_1,\alpha_2,\ldots) \;+\; (0,\ldots,0,\beta_1,0,\ldots,0,\beta_2,\ldots).$$

For the fundamental quasisymmetric polynomials $\{F_\alpha\}$, a combinatorial interpretation for the structure constants is given in [33, §4]. In [5, Cor. 5.12], this combinatorial interpretation is restated (and reproved). It follows from there that the corresponding structure constants are in $\#\mathsf{P}$.

For the second part, we include details only for the dual immaculate polynomials $\{\mathfrak{S}^*_\alpha\}$. The result for $\{\mathcal{S}_\alpha\}$ follows by the same argument, replacing the dominance order $\trianglelefteq$ and with $\trianglelefteq'$.

To simplify the notation, write $K_{\alpha,\beta}$ for the *immaculate Kostka numbers* $K^I_{\alpha,\beta}$. By Proposition 5.2 and their combinatorial interpretation, $\{K_{\alpha,\beta}\}$ are in $\#\mathsf{P}$. The inverse coefficients $K^{-1}_{\alpha,\beta}$ are defined by

$$M_\alpha \;=\; \sum_{\beta \trianglelefteq \alpha} K^{-1}_{\alpha,\beta}\, \mathfrak{S}^*_\beta\,.$$

Now Proposition 3.2 implies that $\{K^{-1}_{\alpha,\beta}\}$ are in GapP. Denote by $c^{\gamma}_{\alpha\beta}$ the structure constants defined by

$$\mathfrak{S}^*_\alpha \cdot \mathfrak{S}^*_\beta \, = \, \sum_{\gamma} c^{\gamma}_{\alpha\beta} \, \mathfrak{S}^*_\gamma .$$

This gives:

$$\begin{aligned}
\mathfrak{S}^*_\alpha \, \cdot \, \mathfrak{S}^*_\beta \, &= \, \Big(\sum_{\rho \trianglelefteq \alpha} K_{\alpha,\rho} \, M_\rho \Big) \cdot \Big(\sum_{\omega \trianglelefteq \beta} K_{\beta,\omega} \, M_\omega \Big) \\
&= \sum_{\rho \trianglelefteq \alpha} \sum_{\omega \trianglelefteq \beta} \sum_{\tau} K_{\alpha,\rho} \, K_{\beta,\omega} \, c(\rho,\omega,\tau) \, M_\tau \\
&= \sum_{\rho \trianglelefteq \alpha} \sum_{\omega \trianglelefteq \beta} \sum_{\tau} \sum_{\gamma \trianglelefteq \tau} K_{\alpha,\rho} \, K_{\beta,\omega} \, c(\rho,\omega,\tau) \, K^{-1}_{\tau,\gamma} \, \mathfrak{S}^*_\gamma .
\end{aligned}$$

Thus, the structure constants $\{c^{\gamma}_{\alpha\beta}\}$ are also in GapP. $\square$

Remark 5.4. It is easy to see that structure constants for dual immaculate and quasisymmetric Schur polynomials can be negative. Thus, Theorem 1.3 proving their signed combinatorial interpretation is optimal in this case.

5.4. Rational quasisymmetric bases. Following [3], *combinatorial quasisymmetric power sums* $\{\mathfrak{p}_\alpha \, : \, \alpha \in \mathrm{W}_n\}$ are defined as

$$\mathfrak{p}_\alpha \, := \, \sum_{\beta} K^{\mathfrak{p}}_{\alpha,\beta} \, M_\beta \, ,$$

where $K^{\mathfrak{p}}_{\alpha,\beta}$ is the number of $\ell(\beta) \times \ell(\alpha)$ matrices (r_{ij}) with entries in $\mathbb{N}$, such that:

- $r_{i1} + \ldots + r_{i\,\ell(\alpha)} = \beta_i$ for all $1 \leqslant i \leqslant \ell(\beta)$,
- $\mathsf{sort}(\alpha)_j$ is the only nonzero entry in column j, for all $1 \leqslant j \leqslant \ell(\alpha)$, and
- the word obtained by reading entries top to bottom, left to right is α.

For $\alpha \preccurlyeq \beta$, denote by $\alpha^{(i)}$ the corresponding parts in α which sum to β_i. For example, for $\alpha = (1,2,2,1,1)$ and $\beta = (3,3,1)$, we have $\alpha^{(1)} = (1,2)$, $\alpha^{(2)} = (2,1)$, and $\alpha^{(3)} = (1)$.

Following [6], *type 1 quasisymmetric power sums* $\{\Psi_\alpha \, : \, \alpha \in \mathrm{W}_n\}$ are defined as

$$\Psi_\alpha \, := \, z_\alpha \sum_{\alpha \preccurlyeq \beta} \frac{1}{p(\alpha,\beta)} \, M_\beta \quad \text{where} \quad p(\alpha,\beta) \, := \, \prod_{i=1}^{\ell(\beta)} \Big(\prod_{j=1}^{\ell(\alpha^{(i)})} \sum_{k=1}^{j} \alpha^{(i)}_k \Big).$$

Similarly, *type 2 quasisymmetric power sums* $\{\Phi_\alpha \, : \, \alpha \in \mathrm{W}_n\}$ are defined as:

$$\Phi_\alpha \, := \, z_\alpha \sum_{\alpha \preccurlyeq \beta} \frac{1}{s(\alpha,\beta)} \, M_\beta \quad \text{where} \quad s(\alpha,\beta) \, := \, \prod_{i=1}^{\ell(\beta)} \Big(\ell(\alpha^{(i)})! \prod_{j=1}^{\ell(\alpha^{(i)})} \alpha^{(i)}_j \Big).$$

Note that $\mathfrak{p}_\alpha$ have integer coefficients, while Ψ_α and Φ_α have rational coefficients.

Example 5.5. Let $\alpha = (1,1,2)$. We have:

$$\begin{aligned}
\mathfrak{p}_{1,1,2} \, &= \, 2 M_{1,1,2} \, + \, M_{2,2}, \\
\Psi_{1,1,2} \, &= \, 2 M_{1,1,2} \, + \, M_{2,2} \, + \, \tfrac{4}{5} M_{1,3} \, + \, \tfrac{1}{3} M_4, \ \text{and} \\
\Phi_{1,1,2} \, &= \, 2 M_{1,1,2} \, + \, M_{2,2} \, + \, \tfrac{4}{3} M_{1,3} \, + \, \tfrac{1}{2} M_4.
\end{aligned}$$

For $\beta = \alpha$, we have $K^{\mathfrak{p}}_{\alpha,\beta} = 2$, given by the following matrices:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Proposition 5.6. *The following bases have triangular property w.r.t. the dominance order:*

- *combinatorial quasisymmetric power sum* $\{\mathfrak{p}_\alpha\}$,
- *type 1 quasisymmetric power sum* $\{\Psi_\alpha\}$, *and*
- *type 2 quasisymmetric power sum* $\{\Phi_\alpha\}$.

Proof. It is proved in [3, Prop. 5.15], that $K^{\mathfrak{p}}_{\alpha,\beta} = 0$ unless $\alpha \preccurlyeq \beta$. By definition of Ψ_α and Φ_α, the corresponding Kostka constants $K^{\Psi}_{\alpha,\beta} = K^{\Phi}_{\alpha,\beta} = 0$ unless $\alpha \preccurlyeq \beta$. Since the refinement order is a coarsening of the dominance order, this proves the first part of the triangular property.

For the second part, it was shown in [3, Prop. 5.15] and [6], that

$$K^{\mathfrak{p}}_{\alpha,\alpha} = \prod_{i=1}^{n} m_i! \qquad \text{and} \qquad K^{\Psi}_{\alpha,\alpha} = K^{\Phi}_{\alpha,\alpha} = z_\alpha \prod_{i=1}^{\ell(\alpha)} \frac{1}{\alpha_i}\,.$$

This completes the proof. □

Proof of Theorem 1.3, third part. The proof of Theorem 1.3 (first and second part) extends similarly to show these structure coefficients are in $\mathsf{GapP}/\mathsf{FP}$.

However, by [6, Prop. 3.16] and [6, Eq. (26)], $\{\Psi_\alpha\}$ and $\{\Phi_\alpha\}$ have structure constants in $\#\mathsf{P}/\mathsf{FP}$. Similarly [3, Thm. 5.6] implies structure constants of $\{\mathfrak{p}_\alpha\}$ are in $\#\mathsf{P}/\mathsf{FP}$. □

6. Polynomial bases

6.1. Posets of interest. For $k \in \mathbb{Z}_{>0}$, recall $\mathrm{V}_{n,k} := \{\alpha \in \mathbb{N}^n \,:\, \alpha \vDash k\}$. Denote by $\mathcal{I}_{n,k} = (\mathrm{V}_{n,k}, \lhd)$ the poset on weak compositions with respect to dominance order. Clearly, we have $\mathrm{height}(\mathcal{I}_{n,k}) = O(kn)$. Similarly, let $\mathrm{V}_n = \bigcup_k \mathrm{V}_{n,k}$ and denote by $\mathcal{I}_n = (\mathrm{V}_n, \lhd)$ the poset on weak compositions with respect to the dominance order. Clearly, we have $\mathrm{height}(\mathcal{I}_n) = O(n^3)$.

To all $\alpha \in V_{n,k}$, we can uniquely associate a word $w \in S_\infty$ s.t. $\mathrm{inv}(w) = k$, via the inverse map to the Lehmer code: $\alpha = \mathsf{code}(w)$. Define the set

$$L_{n,k} := \{\alpha \in \mathrm{V}_{n,k} \,:\, \alpha_i \leqslant n - i \ \text{ for all } \ 1 \leqslant i \leqslant n\}.$$

By the definition of the Lehmer code, we have

$$L_{n,k} = \{\mathsf{code}(w) \,:\, w \in S_n \ \text{ s.t. } \ \mathrm{inv}(w) = k\}.$$

Define the *Lehmer poset* $\mathcal{L}_{n,k} := (L_{n,k}, \lhd)$.

6.2. Polynomial bases, first batch. For $\alpha \in \mathrm{V}_n$, let $\mathrm{flat}(\alpha)$ be the strong composition formed by removing 0's in α. Following [42, §3.4] (see also [5, 10]), the *monomial slide polynomials* $\{\mathfrak{M}_\alpha \,:\, \alpha \in \mathrm{V}_n\}$ and the *fundamental slide polynomial* $\{\mathfrak{F}_\alpha \,:\, \alpha \in \mathrm{V}_n\}$ are defined as

$$\mathfrak{M}_\alpha(\boldsymbol{x}) := \sum_{\substack{\beta \unrhd \alpha \\ \mathrm{flat}(\beta) = \mathrm{flat}(\alpha)}} \boldsymbol{x}^\beta \qquad \text{and} \qquad \mathfrak{F}_\alpha(\boldsymbol{x}) := \sum_{\substack{\beta \unrhd \alpha \\ \mathrm{flat}(\beta) \preccurlyeq \mathrm{flat}(\alpha)}} \boldsymbol{x}^\beta\,.$$

We consider certain (semistandard) tableaux $T : D(\alpha) \to \mathbb{N}_{\geqslant 1}$ of shape $D(\alpha)$, which we define below. For a tableau T, define the *augmented tableau* $\hat{T}$ to be the tableaux formed by adding a box right before each row of $D(\alpha)$, filling the new box in row i with entry i. Following [65], the *Demazure atoms* $\{\mathsf{atom}_\alpha \,:\, \alpha \in \mathrm{V}_n\}$ introduced in [60], can be defined as

$$\mathsf{atom}_\alpha(\boldsymbol{x}) := \sum_{\beta} K^{\mathsf{atom}}_{\alpha,\beta}\, \boldsymbol{x}^\beta.$$

Here $K^{\mathsf{atom}}_{\alpha,\beta}$ is the number of augmented tableaux T of shape $D(\alpha)$ which satisfy:

- column entries are distinct
- entries weakly decrease across rows
- for all $i < j$ one of the following holds:
  - $c < b < a$,
  - $a \leqslant c < b$,
  - $b < a \leqslant c$,

  where $c \leftarrow T(i, k-1),\ a \leftarrow T(i,k),\ b \leftarrow T(j,k)$ if $\alpha_i \geqslant \alpha_j$,
  and $b \leftarrow T(i, k-1),\ c \leftarrow T(j, k-1),\ a \leftarrow T(j,k)$ if $\alpha_i < \alpha_j$.

Example 6.1. For $\alpha = (0,2,1)$, we have $\mathsf{atom}_\alpha(x_1,x_2,x_3) = x_1x_2x_3 + x_2^2x_3$, with monomials corresponding to the following augmented tableaux:

$$\begin{array}{|c|cc}\hline 1 & & \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \\ \hline \end{array} \qquad \begin{array}{|c|cc}\hline 1 & & \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \\ \hline \end{array}\ .$$

Here the entries added to each row are drawn in gray.

6.3. Polynomial bases, second batch. Let $D \subseteq [n] \times [n]$ be a square diagram where entries (boxes) are labelled with $\bullet$ or $\circ$. We defined two types of moves on these labeled diagrams as follows.

Take row $i \in [n]$ and box $(i,j) \in D$ that is rightmost with label $\bullet$. Let $i' := \max\{r \in [i] : (r,j) \notin D\}$. Suppose each $(r,j) \in D$ with $i'+1 \leqslant r \leqslant i$ has label $\bullet$. Define:

- the *Kohnert move* on D at (i,j) outputs the diagram $D' = D - (i,j) + (i',j)$. Let the new box (i',j) have label $\bullet$.
- the *K-Kohnert move* on D at (i,j) outputs the diagram $D' = D + (i',j)$. Let the new box (i',j) have label $\bullet$, and the box (i,j) have reassigned label $\circ$.

Let $\mathsf{Koh}(D)$ denote the set of all diagrams obtainable through applying successive Kohnert moves on diagram D. Similarly, following [83], let $\mathsf{KKoh}(D)$ denote the set of all diagrams obtainable through applying successive Kohnert and K-Kohnert moves on D.

For a subset $S \subseteq [n] \times [n]$, let $\mathtt{wt}(S) \in \mathbb{N}^n$ denote the *weight* of S, defined by

$$\mathtt{wt}(S)_i \ := \ \#\{(i,j) \in S \ : \ j \in [n]\}.$$

It was shown in [54], that the *key polynomials* $\{\kappa_\alpha : \alpha \in \mathrm{V}_n\}$ introduced in [60], can be computed combinatorially as follows:

$$\kappa_\alpha(\boldsymbol{x}) \ := \sum_{S \in \mathsf{Koh}(D(\alpha))} \boldsymbol{x}^{\mathtt{wt}(S)},$$

where $D(\alpha) := \big\{(i,j) \in [n] \times [n] \ : \ \alpha_i \geqslant j\big\}$. Similarly, it was shown in [83, 78], that the *Lascoux polynomials* $\{\mathcal{L}_\alpha : \alpha \in \mathrm{V}_n\}$ introduced in [56], can be computed combinatorially as follows:

$$\mathcal{L}_\alpha(\boldsymbol{x}) \ := \sum_{S \in \mathsf{KKoh}(D(\alpha))} (-1)^{|\alpha| - \#S}\, \boldsymbol{x}^{\mathtt{wt}(S)}.$$

Example 6.2. For $\alpha = (0,2,1)$, we have:

$$\begin{aligned} \kappa_{(0,2,1)} \ &= \ x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3, \quad \text{and} \\ \mathcal{L}_{(0,2,1)} \ &= \ \kappa_{(0,2,1)} \ - \ (x_1^2x_2^2 + 2x_1^2x_2x_3 + 2x_1x_2^2x_3) \ + \ x_1^2x_2^2x_3\,. \end{aligned}$$

For example, the degree 4 terms of $\mathcal{L}_{(0,2,1)}$ correspond to the following diagrams:

$$
\begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & \circ \\ \hline & \\ \hline \end{array}
\qquad
\begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & \\ \hline \circ & \\ \hline \end{array}
\qquad
\begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \circ & \\ \hline \bullet & \\ \hline \end{array}
\qquad
\begin{array}{|c|c|}\hline \bullet & \\ \hline \bullet & \bullet \\ \hline \circ & \\ \hline \end{array}
\qquad
\begin{array}{|c|c|}\hline & \bullet \\ \hline \bullet & \circ \\ \hline \bullet & \\ \hline \end{array}\,.
$$

6.4. Polynomial bases, third batch. Let $D \subset [n] \times [n]$ be a diagram and let $(i,j) \in D$ be a box in the diagram. The *ladder move* is a transformation $D \to D - (i,j) + (i-k, j+1)$ and the *K-ladder move* is a transformation $D \to D + (i-k, j+1)$, allowed only when the following are satisfied:

- $(i, j+1) \notin D$,
- $(i-k, j), (i-k, j+1) \notin D$ for some $0 < k < i$, and
- $(i-l, j), (i-l, j+1) \in D$ for all $0 < l < k$.

Recall that the *Lehmer code* $\mathsf{code}(w) \in \mathbb{N}^n$ uniquely determines $w \in S_\infty$. Let $\mathsf{rPipes}(w)$ denote the set of diagrams obtainable through successive ladder moves, starting from $D(\mathsf{code}(w))$, where $w \in S_n$. Similarly take $\mathsf{Pipes}(w)$ to be the set of diagrams obtainable through successive ladder and K-ladder moves, starting from $D(\mathsf{code}(w))$, where $w \in S_n$.

It was proved in [8], that the *Schubert polynomials* $\{\mathfrak{S}_w : w \in S_n\}$ introduced in [58], can be defined as:

$$\mathfrak{S}_w(\boldsymbol{x}) := \sum_{P \in \mathsf{rPipes}(w)} \boldsymbol{x}^{\mathtt{wt}(P)}.$$

Similarly, it was proved in [31], that the *Grothendieck polynomials* $\{\mathfrak{G}_w : w \in S_n\}$ introduced in [59], can be defined as:

$$\mathfrak{G}_w(\boldsymbol{x}) = \sum_{P \in \mathsf{Pipes}(w)} (-1)^{|\alpha| - \#P} \, \boldsymbol{x}^{\mathtt{wt}(P)}.$$

Since $\mathsf{code}(w)$ uniquely determines w, we may write $\mathfrak{S}_\alpha := \mathfrak{S}_{\mathsf{code}^{-1}(\alpha)}$ and $\mathfrak{G}_\alpha := \mathfrak{G}_{\mathsf{code}^{-1}(\alpha)}$.

Example 6.3. Let $w = 2143$ and $\alpha := \mathsf{code}(w) = (1,0,1,0)$. We have:

$$
\begin{aligned}
\mathfrak{S}_\alpha(x_1, x_2, x_3) &= x_1^2 + x_1x_2 + x_1x_3\,, \quad \text{and} \\
\mathfrak{G}_\alpha(x_1, x_2, x_3) &= \mathfrak{S}_\alpha - (x_1^2x_2 + x_1^2x_3 + x_1x_2x_3) + x_1^2x_2x_3\,.
\end{aligned}
$$

The degree 3 terms of $\mathfrak{G}_\alpha$ correspond to the following diagrams:

$$
\begin{array}{|c|c|c|c|}\hline + & & + & \\ \hline & & & \\ \hline + & & & \\ \hline & & & \\ \hline \end{array}
\qquad
\begin{array}{|c|c|c|c|}\hline + & & & \\ \hline & + & & \\ \hline + & & & \\ \hline & & & \\ \hline \end{array}
\qquad
\begin{array}{|c|c|c|c|}\hline + & & + & \\ \hline & + & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}\,.
$$

6.5. Unitriangular property. First, we consider homogeneous bases:

Proposition 6.4. *The following linear bases in $\mathbb{C}[x_1, \ldots, x_n]$ have unitriangular property w.r.t. the dominance order $\lhd$ on V_n :*

- *monomial slide polynomials $\{\mathfrak{M}_\alpha\}$,*
- *fundamental slide polynomials $\{\mathfrak{F}_\alpha\}$,*
- *Demazure atoms $\{\mathsf{atom}_\alpha\}$,*
- *key polynomials $\{\kappa_\alpha\}$,*
- *Schubert polynomials $\{\mathfrak{S}_\alpha\}$,*
- *Lascoux polynomials $\{\mathfrak{L}_\alpha\}$, and*
- *Grothendieck polynomials $\{\mathfrak{G}_\alpha\}$.*

Proof. The results for $\{\mathfrak{M}_\alpha\}$ and $\{\mathfrak{F}_\alpha\}$ follow directly from their definition. For the Demazure atoms polynomials, consider a tableau T counted by $K^{\mathsf{atom}}_{\alpha,\beta}$. An entry i in T must lie in a row weakly below row i. If $\alpha \trianglelefteq \beta$ fails, this condition must be broken in every such tableaux, so $K^{\mathsf{atom}}_{\alpha,\beta} = 0$. Similarly, we have $K^{\mathsf{atom}}_{\alpha,\alpha} = 1$, since the unique valid tableau has all i's in row i.

The result for $\{\kappa_\alpha\}$ follows from the observation, that applying a Kohnert move produces a monomial higher in dominance order than α. Similarly, for the Schubert polynomials, note that the corresponding *Schubert–Kostka numbers* $K_{\alpha,\beta} = 0$ unless $\alpha \trianglelefteq \beta$. Indeed, applying a ladder move produces a monomial higher in dominance order than α.

Finally, the result for $\{\mathfrak{L}_\alpha\}$ follows since applying a Kohnert or a K-Kohnert move produces a monomial higher in dominance order than α. Similarly, the result for $\{\mathfrak{G}_\alpha\}$ follows since applying a ladder or K-ladder move produces a monomial higher in dominance order than α. ☐

The proof below follows the idea of the proof of Proposition 4.1.

Proof of Theorem 1.4. The first results follow from Theorems 5.5 and 5.11 in [5]. For the remaining results, we include details for the Schubert polynomials $\{\mathfrak{S}_\alpha\}$. The result for other bases follows by the same argument. By Proposition 6.4, we have:

$$\mathfrak{S}_\alpha(\boldsymbol{x}) \, = \sum_{\alpha \trianglelefteq \omega} K_{\alpha,\omega} \, \boldsymbol{x}^\omega.$$

By the *pipe dream* combinatorial interpretation above, the *Schubert–Kostka numbers* $\{K_{\alpha,\omega}\}$ are in $\#\mathsf{P}$. The *inverse Schubert–Kostka numbers* are defined by

$$\boldsymbol{x}^\alpha \, = \sum_{\alpha \trianglelefteq \omega} K^{-1}_{\alpha,\omega} \, \mathfrak{S}_\omega.$$

Again Proposition 3.2 implies that $K^{-1} \in \mathsf{GapP}$.

Recall that the *Schubert coefficients* $\{c^\gamma_{\alpha\beta}\}$ are defined by

$$\mathfrak{S}_\alpha \cdot \mathfrak{S}_\beta \, = \sum_\gamma c^\gamma_{\alpha\beta} \, \mathfrak{S}_\gamma \, .$$

We have:

$$\begin{aligned}\mathfrak{S}_\alpha \cdot \mathfrak{S}_\beta \, &= \Big(\sum_{\alpha \trianglelefteq \omega} K_{\alpha,\omega} \, \boldsymbol{x}^\omega \Big) \cdot \Big(\sum_{\beta \trianglelefteq \rho} K_{\beta,\rho} \, \boldsymbol{x}^\rho \Big) \, = \sum_{\alpha \trianglelefteq \omega} \sum_{\beta \trianglelefteq \rho} K_{\alpha,\omega} \, K_{\beta,\rho} \, \boldsymbol{x}^{\omega+\rho} \\ &= \sum_{\alpha \trianglelefteq \omega} \sum_{\beta \trianglelefteq \rho} \sum_{\omega+\rho \trianglelefteq \gamma} K_{\alpha,\omega} \, K_{\beta,\rho} \, K^{-1}_{\omega+\rho,\gamma} \cdot \mathfrak{S}_\gamma \, .\end{aligned}$$

Thus Schubert coefficients $\{c^\gamma_{\alpha\beta}\}$ are in GapP, as desired. ☐

Remark 6.5. The poset of monomials is also unitriangular w.r.t. the reverse lexicographic order, so the Möbius inversion can also be used in this setting. However, the height of the resulting poset is exponential, so Proposition 3.2 is not applicable. For example, for $w \in S_{2n}$ such that $\mathsf{code}(w) = (0^{n-1}, n, 0^n)$, the monomial support of $\mathfrak{S}_w$ corresponds to all weak compositions of n, the number of which is exponential in n.

Remark 6.6. In some cases, the (generalized) inverse Kostka numbers is computed explicitly. Notably, in [68, §5], the authors use the lattice structure and the Crosscut Theorem (see §9.3) to prove that for the fundamental slide polynomials, all inverse Kostka numbers are in $\{0, \pm 1\}$, cf. §9.2. See also inverse Kostka numbers for *forest polynomials* given in [67, Prop. 10.16].

REMARK 6.7. It is easy to see that the structure constants for Demazure atoms, key, and Lascoux polynomials can be negative without predictable signs. Thus, Theorem 1.3 proving their signed combinatorial interpretation is optimal in this case. For Schubert polynomials the structure constants are always positive, and conjectured to be not in #P by the first author [73, Conj. 10.1] (cf. [88, Problem 11]). For Grothendieck polynomials the structure constants have predictable signs (see e.g. [14]). Adjusting for signs, whether these have a combinatorial interpretation also remains a major open problem.

7. Plethysm

The following proof streamlines and extends the argument in [30, §9].

Proof of Theorem 1.5. Since $\{f_\lambda\}$, $\{g_\mu\}$ are linear bases in Λ_n, we can write

$$f_\lambda[g_\mu] \, = \, \sum_\nu d^\nu_{\lambda\mu}\, m_\nu\,.$$

By (4.1), the result follows once we show that coefficients $\{d^\nu_{\lambda\mu}\}$ are in GapP.

Using combinatorial interpretations in Section 4, let $Y^{(g)}_\mu = \{\tau, \tau', \tau'', \ldots\}$ be a set of monomials in g_μ :

$$g_\mu \, = \, m_\tau \, + \, m_{\tau'} \, + \, m_{\tau''} \, + \, \ldots$$

Denote $K^{(f)}_{\lambda,\rho} := |Y^{(f)}_\lambda| = [\boldsymbol{x}^\rho]\, f_\lambda$. Then:

$$f_\lambda[g_\mu] \, = \, f_\lambda\big(\boldsymbol{x}^\tau, \boldsymbol{x}^{\tau'}, \ldots\big) \, = \, \sum_\rho K^{(f)}_{\lambda,\rho}\, m_\rho\big(\boldsymbol{x}^\tau, \boldsymbol{x}^{\tau'}, \ldots\big)$$

$$= \, \sum_\rho K^{(f)}_{\lambda,\rho} \sum_w \boldsymbol{x}^{w(\rho)_1\,\tau} \cdot \boldsymbol{x}^{w(\rho)_2\,\tau'} \, \cdots$$

Taking coefficients in $\boldsymbol{x}^\nu$ on both sides implies that $\{d^\nu_{\lambda\mu}\} \in \#\mathsf{P}$. This completes the proof. □

REMARK 7.1. It remains a major open problem whether there is a combinatorial interpretation for plethysm coefficients, even in a special case when $\lambda = (r^{n/r})$ is a rectangle, see [88, Problem 9]. The first author conjectured that these are not in #P [73, Conj 8.8].

8. Further applications

We conclude with one additional application relating two type of structure constants: into symmetric and quasisymmetric polynomials.

THEOREM 8.1. *Let $A = \cup A_n$ be a family of combinatorial objects, and let $\{G_w(\boldsymbol{x}) : w \in A\}$ be a family of symmetric polynomials such that*

$$G_w(\boldsymbol{x}) \, = \, \sum_{\alpha \in \mathrm{V}_n} c_{w\alpha}\, F_\alpha(\boldsymbol{x}),$$

where the coefficients $\{c_{w\alpha}\}$ are in GapP. Consider the coefficients defined by

$$G_w(\boldsymbol{x}) \, = \, \sum_{\lambda \in \mathrm{U}_n} d_{w\lambda}\, s_\lambda(\boldsymbol{x}).$$

Then $\{d_{w\lambda}\}$ are also in GapP. Furthermore, the result holds when $\{F_\alpha\}$ are replaced with $\{M_\alpha\}$.

For example, this theorem gives another proof of Proposition 4.4.

Proof of Theorem 8.1. Take $\lambda \vdash m$. By [27, Thm 11], we have:

$$d_{w\lambda} \,=\, \sum_{\alpha \vDash m} \sum_{\beta \preccurlyeq \alpha} K(\beta,\lambda)\, c_{w\alpha}\,,$$

where $\{K(\beta,\lambda)\}$ are in GapP. In fact, $K(\beta,\lambda)$ have an explicit signed combinatorial interpretation as a signed sum of certain rim-hook tableaux. This proves the first part.

For the second part, consider the inverse coefficients $K^{-1}_{\alpha,\beta}$ defined by

$$M_\alpha(\boldsymbol{x}) \,=\, \sum_{\beta \trianglelefteq \alpha} K^{-1}_{\alpha,\beta}\, F_\beta(\boldsymbol{x}).$$

By Proposition 3.2, coefficients $\{K^{-1}_{\alpha,\beta}\}$ are in GapP. The result now follows from the first part. ☐

Remark 8.2. Combining Theorem 8.1 with [38, Eq. (82)] gives a GapP formula for the Schur expansion of *LLT polynomials*. While this expansion has been proven to be Schur-positive [40], there is no known (unsigned) combinatorial interpretation for this expansion (see e.g. [2, §7]).

9. Final remarks and open problems

9.1. Historical notes. Many different generalizations of Kostka numbers (weight multiplicities) and the LR–coefficients (structure constants) have been studied across the area, too many to review here. In connection to our Theorem 1.2, the most notable (and most general) are generalizations of the LR rule to Hall–Littlewood polynomials [86], to Macdonald polynomials [97], and to Koornwinder polynomials [95] (see also references therein).

A generalization of the *inverse Kostka numbers* to the Hall–Littlewood polynomials is given in [18], a quasisymmetric version is given in [27], and a noncommutative version is given in [4]. A curious signed combinatorial interpretation for the Kerov character polynomials was given in [28].

For Schubert polynomials and their structure constants, the literature is again much too large to review. We refer to [51] for an overview, and to [53] for a recent breakthrough which includes a new type of signed combinatorial interpretation in a special case.

For the plethysm, the literature is again much too large to review. We refer to [20, 62, 96] for algorithms implying that several plethysm coefficients are in GapP. We warn the reader that a formula in [24, Thm 4.1] does not give a GapP formula, at least not without effort. We refer to [22] for a recent overview of the area.

9.2. Dominance order. It was shown by Bogart (unpublished) and Brylawski [15], that the Möbius function for the dominance order $\mathcal{Q}_n$ satisfies

$$\mu(\alpha,\beta) \,\in\, \{0,\pm 1\} \quad \text{for all} \quad \alpha,\beta \vdash n.$$

This was reproved and generalized in a series of papers, including [11, 35, 50, 91]. The proofs also give easy $\operatorname{poly}(n)$ time algorithms for computing $\mu(\alpha,\beta)$. See also Remark 6.6 for connections to the inverse Kostka numbers.

There seem to be no closed formula for the Möbius function on $\mathcal{Z}_{n,k}$ (cf. [85] for a different order on strong compositions). It would be interesting to see if this Möbius function is bounded. We note that $\mathbb{P}[\lambda \lhd \mu] \to 0$ for uniform $\lambda,\mu \vdash n$, as $n \to \infty$, [80]. Only recently, it was shown that $\mathbb{P}[\lambda \lhd \mu] = n^{\Theta(1)}$ [66], but the lower and upper bounds remain far apart. It would also be interesting to find the corresponding result for $\mathcal{Z}_{n,k}$.

9.3. Möbius inversion for lattices. As we mentioned above, it is well known that the dominance order $\mathcal{Q}_m$ is a lattice. This holds for other posets we consider in this paper. The proof of the following result is straightforward and will be omitted.

Proposition 9.1. *Partial orders* $\mathcal{Z}_{n,k}$, $\mathcal{I}_{n,k}$, *and* $\mathcal{L}_{n,k}$ *are lattices.*

For lattices of polynomial width, an alternative approach to the effective Möbius inversion is given by the *Crosscut Theorem* [89, Cor. 3.9.4]. Since the lattices in Proposition 9.1 have exponential width, this approach does not give a new family of GapP formulas for the structure constants.

9.4. Signed combinatorial interpretations. Although our GapP formulas tend to be rather complicated, in most cases they give the first signed combinatorial interpretations for these structure constants. In fact, one can simplify one part of it as follows. Suppose $f = g - h$ is a GapP formula, $g, h \in \#\mathsf{P}$, so that $h(w) = \#\{u : (w,u) \in B\}$ for some NP language $B \subseteq W^2$. Let $a, C \in \mathbb{N}_{\geqslant 1}$ be s.t. $|u| \leqslant C|w|^a$ in the definition of $h(w)$. Observe that $2^{Cn^a} - h = \#\{u : (w,u) \notin B\} \in \#\mathsf{P}$, for all $|w| = n$. We conclude that $f = \big[g + (2^{Cn^a} - h)\big] - 2^{Cn^a}$ is a GapP formula with the negative part in FP. In other words, adding a sufficiently large power of two can turn a signed combinatorial interpretation into the usual combinatorial interpretation.

9.5. Binary encoding. Both the Kostka and the LR-coefficients are in $\#\mathsf{P}$ in binary when written as Gelfand–Tseitlin patterns. It follows from the signed combinatorial interpretation of the inverse Kostka numbers given in [26], that $K^{-1} \in \mathsf{GapP}$ in binary. This can also be derived from an argument in the proof of Proposition 4.1, but an extra effort is needed. The same applies to other structure constants in the paper.

9.6. Quantum complexity. By analogy with GapP, one can similarly ask whether the structure constants considered in the paper are in the quantum complexity classes $\#\mathsf{BQP}$ and $\#\mathsf{BQP}/\mathsf{FP}$. This was proved for Kronecker and plethysm coefficients in [46] (see also [13]).

9.7. Combinatorial interpretations. Finding the right place for a combinatorial function is the first step towards understanding its true nature. Theorems 1.1–1.5 give the best inclusions we know for combinatorial interpretations of the structure constants (see Remarks 4.5, 5.4, 6.7 and 7.1). Restricting to integral functions, from the computational complexity point of view our examples of fall into four buckets.

First, there are GapP functions where the sign is computationally hard to determine. Second, there are $\#\mathsf{P}$ functions where the sign is easy to compute and the absolute value is in $\#\mathsf{P}$. Third, there are nonnegative GapP functions which have no $\#\mathsf{P}$ formula (modulo standard complexity assumptions). Finally, there are nonnegative GapP functions which have no *known* $\#\mathsf{P}$ formula.

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