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Normal covers of 2-arc-transitive graphs of prime-power order

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Abstract In a paper by Cai Heng Li in *Bull. London Math. Soc.* 33 (2001), it was suggested that 'non-basic' 2-arc-transitive graphs of prime-power order that occur as normal covers of smaller 2-arc-transitive graphs might be rare and difficult to construct. This note describes some of the background to Li's suggestion, and gives some examples of small valency, and then goes on to show that in fact there are infinitely many examples of valency d, for every integer $d \geqslant 2$. It is also noted that the hypercubes Q_n for $n \geqslant 4$ even, together with a 2-arc-transitive group G of index 2 in $\mathrm{Aut}(Q_n)$, show that the claims of Corollary 1.2 in the above paper by Li are not quite correct.

1. Introduction

An s-arc in a graph Γ is a sequence $(v_0, v_1, \ldots, v_s)$ of vertices of Γ such that any two consecutive v_i are adjacent, and any three consecutive v_i are distinct. A graph Γ is said to be *s-arc-transitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on the set of all s-arcs in Γ, and *s-arc-regular* if $\mathrm{Aut}(\Gamma)$ is sharply-transitive (which means it acts regularly) on the set of all s-arcs in Γ. Vertex-transitive and arc-transitive graphs (respectively) are s-arc-transitive for $s = 0$ and $s = 1$. All graphs in this paper are simple, and are finite unless specified otherwise.

The investigation of arc-transitive graphs is a classical topic in algebraic graph theory which goes back to the seminal work of Tutte [21] on connected graphs of valency 3 (often called *cubic* graphs). A very important subclass of arc-transitive graphs are those for which the stabiliser of a vertex in the automorphism group acts primitively on the neighbourhood of that vertex, and in particular, those for which this action is doubly-transitive. Note that the latter condition is equivalent to 2-arc-transitivity (when the graph is vertex-transitive). A very large amount of work has been devoted to a project on classification of finite 2-arc-transitive graphs – see for example [6, 4, 7, 9, 17], to name just a few papers – with the paper of Cai-Heng Li [8] making a notable contribution to this wider project.

The purpose of our paper is to address some of the issues that remained unresolved in [8]. In particular, we focus on Li's suggestion that *'non-basic' 2-arc-transitive graphs of prime-power order that occur as normal covers of smaller 2-arc-transitive graphs might be rare and difficult to construct.* Contrary to this suggestion, we show that infinite families of such covers exist for all valencies and feasible primes. Also we exhibit an infinite family of counter-examples to one of the main theorems (Corollary 1.2) in [8].

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Before stating our findings, we first give a brief overview of the concepts of a normal quotient and a normal cover of a graph. These concepts, the origin of which can be traced back to the work of Peter Lorimer [11], were developed and successfully used by Cheryl Praeger [17] and have now become a standard tool in the investigation of some aspects of arc-transitive graphs.

If G is a group of automorphisms of a connected graph Γ and N a normal subgroup of G, then a smaller graph Γ_N can be constructed by taking the orbits of N on $V(\Gamma)$ as vertices, and joining two such orbits Δ_1 and Δ_2 by an edge in Γ_N whenever there exists an edge $\{v_1, v_2\}$ in Γ such that $v_1 \in \Delta_1$ and $v_2 \in \Delta_2$. In this case Γ_N is said to be a *normal quotient* of Γ with respect to G. Note that the quotient group G/N acts (possibly unfaithfully) on $V(\Gamma_N)$ as a group of automorphisms, and if G acts transitively on the s-arcs of Γ, then G/N acts transitively on the s-arcs of Γ_N. Moreover, if N has at least three orbits on $V(\Gamma)$ and if G acts transitively on the 2-arcs of Γ, then the valencies of Γ and Γ_N coincide (by an easy argument showing that every 2-arc contains vertices from three different orbits of N). The graph Γ is then called a *normal cover* of Γ_N. In this case the group G/N acts faithfully on the vertices of Γ_N and the stabiliser $(G/N)_\Delta$ of a vertex $\Delta = v^N$ of Γ_N is isomorphic to the stabiliser G_v of the vertex v of Γ. We say in this case that G projects along the covering projection to G/N and that G/N lifts to the group G.

Note that normal quotients and covers of graphs allow an inductive approach to the analysis of 2-arc-transitive graphs. Given a d-valent graph Γ and a group G acting transitively on its 2-arcs, one can first check whether G contains a non-trivial normal subgroup N having at least three orbits on $V(\Gamma)$. If such a group exists, then Γ is a normal cover of a d-valent quotient Γ_N such that G projects to a 2-arc transitive group G/N of Γ_N. Moreover, by taking N to be a maximal normal subgroup subject to having at least three orbits on $V(\Gamma)$, one obtains the quotient $\bar{\Gamma} = \Gamma_N$ and a group $\bar{G} = G/N$ acting transitively on the 2-arcs of $\bar{\Gamma}$ such that every non-trivial normal subgroup of $\bar{G}$ has at most 2 orbits on $V(\bar{\Gamma})$; such a graph $\bar{\Gamma}$ is then said to be $\bar{G}$-basic.

The above observation implies that every graph admitting a 2-arc-transitive group G is a normal cover of a basic $\bar{G}$-basic graph $\bar{\Gamma}$ with $\bar{G}$ acting transitively on the 2-arcs and lifting to the group G. This suggests the following two-step approach to the analysis of 2-arc-transitive graphs: First, determine all pairs (Γ, G) where Γ is G-basic and G is 2-arc-transitive, and then as a second step, analyse normal covers of the G-basic graphs along which the 2-arc-transitive group G lifts. While the first step of this approach can often be resolved using a variant of the O'Nan-Scott theorem for (bi)-quasi-primitive permutation groups and the classification of finite simple groups, the second step usually proves to be a much harder problem.

This inductive approach can be applied to any subclass of 2-arc-transitive graphs that are closed under taking normal quotients. For example, in his 2001 paper [8] Cai Heng Li used this method to analyse 2-arc-transitive graphs of prime-power order (which are obviously closed under the operation of taking normal quotients, for every given prime). This brings us to the topic of our paper.

The first step of the above process was pursued by Li, and he left the second step as an open problem (see [8, Problem]), but he did state that he was inclined to think that "*non-basic 2-arc-transitive graphs of prime-power order would be rare and hard to construct.*"

Before proceeding, however, we note that unfortunately Li's definition of a 'non-basic' graph Γ in [8] was unclear: Did it mean that the graph Γ is not $\mathrm{Aut}(\Gamma)$-basic? or that there is some 2-arc-transitive $G \leqslant \mathrm{Aut}(\Gamma)$ for which the graph is not G-basic? or that for every such G the graph Γ is not G-basic? This distinction is quite important, because there exist 2-arc-transitive graphs that are basic with respect to one 2-arc-transitive group but non-basic with respect to another 2-arc-transitive group of automorphisms. (Consider, for example, the graphs $K_{n,n} - nK_2$ for $n \geqslant 5$, or the hypercubes Q_n for $n \geqslant 5$ odd; see Section 3.) Later in [8], Li claimed that the graph $K_{n,n} - nK_2$ is basic, and so it became clear that 'basic' means G-basic for some such G, and hence that 'non-basic' should be understood in the strictest of the three possible senses.

Definition 1.1. *A 2-arc-transitive graph Γ is said to be* non-basic *provided that for every 2-arc-transitive group G of automorphisms of Γ, the graph Γ is not G-basic.*

Li's suggestion on rarity of non-basic 2-arc-transitive graphs of prime-power order was brought to the first author's attention in a recent conference lecture by Cheryl Praeger, which appeared to be based on the assumption that it was well-founded. This piqued the first author's curiosity, and soon led him to discover two families of such graphs with small valency, the first being the obvious family of examples of cycles of prime-power order $p^s \geqslant 8$ with $s > 1$, and the other a family of 2-arc-regular 3-valent normal covers of the complete graph K_4 (see Section 2). In turn, this prompted him to invite the second author to join him in investigating the situation further, and led us to the following theorem.

Theorem 1.2. *For every prime p and every integer $d \geqslant 3$, there exists an integer m such that every 2-arc-transitive graph of valency d and order p^k such that $k \geqslant m$ is non-basic. Moreover, for every 2-arc-transitive graph Γ of prime-power order, there exists an infinite family of non-basic 2-arc-transitive normal covers of Γ with order a power of the same prime. In particular, for every integer $d \geqslant 2$, there exist infinitely many non-basic 2-arc-transitive graphs of prime-power order with valency d.*

This theorem shows that contrary to the suggestion in [8], examples of non-basic 2-arc-transitive graphs are in some sense plentiful and exist for all possible valencies and feasible primes. Moreover, the family provided in Section 2 shows that the suggestion made in [8] is not the case even when the graphs are assumed to be 2-arc-transitive but not 3-arc-transitive (or equivalently, '2-transitive' as defined in [8]).

In Section 3, we focus on the family of hypercubes Q_n for $n \geqslant 3$. We show that Q_n is a non-basic 2-arc-transitive graph if and only if $n = 3$ or n is an even integer, thus providing examples for all even valencies, and also showing that the conclusions (1) and (3) of Corollary 1.2 in [8] are not valid.

In Section 4, we extend our investigation to all valencies and prove our Theorem 1.2. In particular, we show that every hypercube graph Q_n (for even or odd $n > 1$) can be covered by infinitely many non-basic 2-arc-transitive graphs of prime-power order with valency n.

2. Examples of valency 3

The cube graph Q_3 of order 8 is a 2-arc-regular cubic graph of type 2^1 (as described in [3]), with automorphism group $S_4 \times C_2$. It is also a non-basic 2-arc-transitive graph, as it has K_4 as a normal quotient via the central subgroup of order 2 generated by the automorphism that interchanges antipodally opposite vertices. In this section we exhibit an infinite family of non-basic 2-arc-transitive normal covers of K_4 with 2-power order, and with automorphism group acting regularly on the set of 2-arcs.

To see this, note that by the theory of arc-transitive 3-valent graphs, the automorphism group G of every finite graph Γ in the class 2^1 is a quotient of the infinite group G_2^1 with presentation $G_2^1 = \langle\, h, p, a \mid h^3 = p^2 = a^2 = (ap)^2 = (hp)^2 = 1 \,\rangle$, which is isomorphic to $\mathrm{PGL}(2, \mathbb{Z})$. Conversely, every finite quotient G of this group that is 'smooth' (in the sense that the orders of h, p, a, ap and hp are preserved) is a 2-arc-regular group of automorphisms of some 3-valent graph Γ, and if Γ is not 3-arc-transitive, then $G = \mathrm{Aut}(\Gamma)$. Moreover, if Γ has order 2^n, then G has order $6|V(\Gamma)| = 3 \cdot 2^{n+1}$ for some n, and so by Burnside's $p^\alpha q^\beta$ theorem, G is soluble.

The automorphism group S_4 of K_4 is isomorphic to the quotient G_2^1/J where J is one of the two normal subgroups of index 24 in G_2^1, generated by either $(ha)^3$ and its conjugates, or $(hap)^3$ and its conjugates. (These two subgroups are interchanged by the automorphism θ of G_2^1 taking (h,p,a) to (h,p,ap), which extends the group G_2^1 to the universal group G_3 for the class 3 in [3].) By Reidemeister-Schreier theory (or its implementation as the `Rewrite` command in Magma [2]), the subgroup J is free of rank 3.

Now for any 2-power $m = 2^s$ for $s \geqslant 1$, let $L_m = J'J^{(m)}$ be the characteristic subgroup of J generated by all commutators and all mth powers of elements of J. Then L_m is a normal subgroup of index $24m^3$ in G_2^1, not preserved by the automorphism θ above, and because a finite cubic graph admitting a 2-arc-regular group of automorphisms is at most 3-arc-transitive (by Corollary 2.2 of [3]), it follows that G_2^1/L_m is isomorphic to the automorphism group of a 2-arc-regular 3-valent graph of type 2^1, with order $4m^3 = 2^{2+3s}$. Moreover, as $N = J/L_m$ is a normal subgroup of G_2^1/L_m, with quotient $(G_2^1/L_m)/(J/L_m) \cong G_2^1/J \cong \mathrm{Aut}(K_4)$, this graph is a normal cover of K_4 (and hence is a non-basic graph), as required.

Remark 2.1. In fact it can be shown that every connected 2-arc-transitive graph of 2-power order is a normal cover of K_4. This follows from some of the content of [13], for example. Furthermore, it can be shown using [13], or the theory of 2-arc-transitive cubic graphs of type 2^1 that every 2-arc-regular 3-valent graph of type 2^1 with 2-power order greater than 8 is a normal cover of Q_3 as well.

3. The hypercubes

For $n \geqslant 3$, the n-hypercube graph Q_n is the graph whose vertices may be taken as the ordered n-tuples of elements of $\mathbb{Z}_2$, with two such n-tuples adjacent if and only if they differ in exactly one position. (Equivalently, the vertices are subsets of the set $\{1, 2, \ldots, n\}$, with two such subsets adjacent if and only if one can be obtained from the other by deleting one element.) This is an n-valent regular graph of order 2^n, and is clearly 2-arc-transitive, but not 3-arc-transitive.

We should point out that the subgroup structure of $\mathrm{Aut}(Q_n)$ is rather complex and not easy to determine, even when restricted to vertex-transitive groups. For example, it was shown by Pablo Spiga in [18] that the number of non-isomorphic groups that act regularly on the vertices of Q_n grows exponentially with respect to n^2 (as n tends to infinity). This fact makes the hypercubes quite intriguing objects, especially when some control over the subgroups of their automorphism group is needed.

The aim of this section is to prove the following theorem.

Theorem 3.1. *Let $n \geqslant 3$ be an integer. Then the hypercube Q_n is a basic 2-arc-transitive graph if and only if $n > 3$ and n is odd. Equivalently, Q_n is a non-basic if and only if $n = 3$ or n is even.*

Proof. Let us start with some initial comments. Observe first that the automorphism group of Q_n is the wreath product $C_2 \wr S_n$, of order $2^n n!$, and indeed Q_n is a Cayley graph for the 'base' group $B = (C_2)^n$, with involutory generators b_j taking the 'zero' vertex $\mathbf{0} = (0, 0, \ldots, 0)$ of Q_n to the standard elementary basis vector $\mathbf{e}_j$ (with a single '1' in position j) for $1 \leqslant j \leqslant n$. Also $\mathrm{Aut}(Q_n)$ is a semi-direct product $B \rtimes H$, where $H = \mathrm{Aut}(Q_n)_{\mathbf{0}}$ is the stabiliser of the zero vertex $\mathbf{0}$, and H is isomorphic to S_n. Note that the action of H by conjugation on B corresponds to permuting the entries of the vectors in $\mathbb{Z}_2^n$ by elements of S_n in the natural way.

Two important normal subgroups of $\mathrm{Aut}(Q_n)$ that are contained in B are the 'trace' subgroup T generated by the central involution $z = b_1 b_2 \dots b_n$, and the 'augmentation' subgroup A generated by elements of B expressible as words of even length on the generating set $\{b_1, b_2, \dots, b_n\}$. Note that A contains T if and only if n is even.

We have already seen in Section 2 that Q_3 is a non-basic 2-arc-transitive graph, as claimed.

Suppose now that n is odd and $n > 3$. Let G be the subgroup of $\mathrm{Aut}(Q_n) = B \rtimes S_n$ consisting of all the elements bh, $b \in B$, $h \in S_n$, such that either $b \in A$ and $h \in A_n$, or $b \in B \setminus A$ and $h \in S_n \setminus A_n$. Then G is a subgroup of $\mathrm{Aut}(Q_n)$ of index 2 such that $G_{\mathbf{0}} = G \cap S_n = A_n$. In particular, G is vertex-transitive. Moreover, since $n \geqslant 5$, the stabiliser $G_{\mathbf{0}} = A_n$ acts doubly transitively on the neighbourhood of $\mathbf{0}$, implying that G is 2-arc-transitive.

Next suppose that Q_n is not G-basic, and let N be a non-trivial normal subgroup of G with at least three orbits on $V(Q_n)$. Then $J = N \cap A$ is a proper subgroup of A which is invariant under conjugation by A_n. The only subgroups of B that are invariant under the conjugation of A_n are 1, T, A and B, but as J has at least 3 orbits on $V(Q_n)$, it cannot be A or B. Also because n is odd, T is not a subgroup of A, implying that $J \neq T$. Hence $J = 1$. In particular, N intersects A trivially, and thus N and A centralise each other. But T is central in $\mathrm{Aut}(Q_n)$, and $B = \langle A, T\rangle$, so it follows that N centralises B. Also because B is an abelian permutation group acting regularly on $V(Q_n)$, it is equal to its own centraliser in the symmetric group of $V(Q_n)$. Thus $N \leqslant B \cap G = A$, implying that $N = J = 1$, which is a contradiction. This contradiction shows that Q_n is in fact G-basic, and hence that Q_n is a basic 2-arc-transitive graph.

Suppose now that n is even, and let G be any 2-arc-transitive subgroup of $\mathrm{Aut}(Q_n)$. (Actually in the first part of what follows, G can be any arc-transitive subgroup.) Then G_v acts transitively on the neighbourhood of v in Q_n, so $|G \cap H| = |G_v|$ is divisible by n and therefore $|G|$ is divisible by $n2^n$. Note here also that GB is a subgroup of $\mathrm{Aut}(Q_n)$ since B is normal in $\mathrm{Aut}(Q_n)$. We proceed to consider $J = G \cap B$, which is a normal 2-subgroup of G contained in B.

If J is trivial, then $|GB| = |G||B|/|J| = |G||B|$ is divisible by $n2^n2^n$, but also $|GB|$ divides $|\mathrm{Aut}(Q_n)| = 2^n n!$, and therefore 2^n divides $(n-1)!$, which is easily seen to be impossible, because the largest integer j such that 2^j divides $m!$ is at most m (and equals m if and only if m itself is a power of 2). Hence J is non-trivial.

Next, if $|B : J| > 2$, then $|J| \leqslant 2^{n-2}$ and so $G \cap B = J$ has at least $2^2 = 4$ orbits on $V(Q_n)$, which implies that $(Q_n)_J$ is a proper normal quotient of Q_n, making Q_n non G-basic. On the other hand, suppose $|B : J| = 1$ or 2, and also that n is even. If J contains the augmentation subgroup A mentioned above, then also G contains the trace subgroup T, which has 2^{n-1} orbits on $V(Q_n)$, and makes Q_n non G-basic. Otherwise $N = J \cap A$ has index 4 in B, and has at least 4 orbits on $V(Q_n)$, again making Q_n non G-basic.

Hence if n is even, then for every 2-arc-transitive subgroup $G \leqslant \mathrm{Aut}(Q_n)$, the graph Q_n is not G-basic, so Q_n itself is non-basic. Thus Q_n is non-basic for $n = 3$ (as in the previous family) and for all even $n \geqslant 4$ (by the above analysis). ☐

Hence we have another family of counter-examples to the 'rarity' suggestion in [8], this time with arbitrarily large even valency.

Next, recall that for odd $n > 3$, the 2-arc-transitive subgroup G of index 2 in $\mathrm{Aut}(Q_n)$ considered in the proof of the above theorem makes Q_n a G-basic graph.

These examples show that part (1) of Corollary 1.2 in [8] is not quite correct. The latter asserted that if Γ is a connected graph of prime-power order, and Γ is G-basic for some 2-arc-transitive subgroup G of $\mathrm{Aut}(\Gamma)$, then Γ is isomorphic to $K_{2^m,2^m} - 2^m K_2$

for some m, or Γ is $\mathrm{Aut}(\Gamma)$-basic. But Q_n is G-basic for the index 2 subgroup of $\mathrm{Aut}(Q_n)$ described above, and yet is not isomorphic to $K_{2^m,2^m} - 2^m K_2$ for any m, and also Q_n is not $\mathrm{Aut}(Q_n)$-basic, because it is an $\mathrm{Aut}(Q_n)$-normal cover of $(Q_n)_T$ where T is the trace subgroup.

They also do the same for part (3), which asserted that if Γ is a connected graph of prime-power order, and G is a 2-arc-transitive subgroup of $\mathrm{Aut}(\Gamma)$ that acts as a biprimitive affine group on $V(\Gamma)$, then either Γ is isomorphic to $K_{2^m,2^m}$ or $K_{2^m,2^m} - 2^m K_2$, or $\mathrm{soc}(G) = \mathrm{soc}(\mathrm{Aut}(Q_n))$. But if $\Gamma = Q_n$ where n is odd and $n > 3$, then clearly the first part of this conclusion does not hold, and also the second part is impossible because the trace subgroup T is a minimal normal subgroup of $\mathrm{Aut}(Q_n)$ and hence is a subgroup of $\mathrm{soc}(\mathrm{Aut}(Q_n))$, but T is not a subgroup of G and so cannot be a subgroup of $\mathrm{soc}(G)$.

Finally (in this section) we note that with some extra work (using some advanced theory of 2-transitive groups proved by Aschbacher [1], Praeger [16] and Liebeck, Praeger and Saxl [10]), it can be shown that every 2-arc-transitive subgroup of $\mathrm{Aut}(Q_d)$ contains the augmentation subgroup $A \cong C_2^{d-1}$.

4. All valencies greater than 2

After finding the above families of examples, we considered it a challenge to construct or find a non-basic example of a 2-arc-transitive graph of prime-power order with odd valency $d > 3$. This turned out to be easy, however, by taking a cover of the 5-cube.

In the automorphism group $C_2 \wr S_5$ of Q_5, the stabiliser of a vertex v is isomorphic to S_5, the stabiliser of an arc (v, w) incident with v is isomorphic to S_4, and the stabiliser of the associated edge $\{v, w\}$ is isomorphic to $S_4 \times C_2$. Hence the group $\mathrm{Aut}(Q_5)$ is a finite quotient of the amalgamated free product $F = S_5 *_{S_4} (S_4 \times C_2)$. Taking a particular transitive permutation representation of this group F on 20 points gave rise to a group G of order $2^4 \cdot |C_2 \wr S_5|$, which turned out to be the automorphism group of a non-basic 2-arc-transitive graph of order $2^4 \cdot 32$ $(= 512)$ with valency 5, covering Q_5.

Similar examples of order 2^{2d-1} and valency d can be constructed as covers of Q_d for $d = 3, 4, 6, 7, 8$ and 9, and we believe that such examples are like to exist for all $d > 2$. This, however, could be difficult to prove. But we can prove the existence of larger examples for every valency $d > 2$, using a slightly different approach.

Let us begin by recalling the following deep theorem, which is a culmination of work that stretches over a number of papers by Weiss and Trofimov, with [20] being the last in that series. A very good overview of the work involved in the proof can be found in [23].

Theorem 4.1. [Weiss and Trofimov] *For every integer $d \geqslant 3$, there exists a real constant $c_d > 0$ such that for every finite connected 2-arc-transitive graph Γ with valency d, the order of the vertex-stabiliser in $\mathrm{Aut}(\Gamma)$ is at most c_d.*

Using this, we can now prove the following.

Lemma 4.2. *For every prime p and every integer $d \geqslant 3$, there exists an integer m_d such that every 2-arc-transitive graph of valency d and order p^k such that $k \geqslant m_d$ is non-basic.*

Proof. Let $c = c_d$ be the constant from Theorem 4.1, and let m be an integer such that $m > d$ and $p^m > c!$. Also suppose that $k \geqslant m$, and let Γ be a 2-arc-transitive graph of valency d and order p^k, and let G be a subgroup of $\mathrm{Aut}(\Gamma)$ acting transitively on the set of 2-arcs of Γ. Next let P be a Sylow p-subgroup of G, and let Q be the core of P in G. By the definition of c, we see that $|G_v| \leqslant c$. Since the order of Γ is a power of p, the group P acts transitively on $V(\Gamma)$, and so $|P| \geqslant p^k > c!$. Furthermore, the index of P in G is at most equal to the order of the vertex-stabiliser G_v, which is bounded above by c. But then the index of Q in G is at most $c!$, implying that Q is non-trivial, and hence that the centre C of Q is a non-trivial abelian normal subgroup of G.

Now the subgroup N generated by all the elements of order p in C is a non-trivial elementary abelian normal p-subgroup of G. Suppose that N has just one or two orbits on $V(\Gamma)$. If N acts faithfully on each of its orbits, then Γ is a Cayley or bi-Cayley graph on N, and in particular, N is generated by at most d elements (and in fact at most $d-1$ elements if N has two orbits on $V(\Gamma)$). But then the order of Γ is at most p^d, contradicting our choice of m. On the other hand, if N acts unfaithfully on one of its orbits, say Δ, then let X be the kernel of that action. Then since N is abelian, $X = N_v$ for every $v \in \Delta$, and so X is normal in G_v for every $v \in \Delta$. Let u be a vertex in the other N-orbit Δ' such that U is not fixed by X and let v be its neighbour in Δ. Since X is normal in G_v and G_v acts doubly-transitively on the neighbourhood $\Gamma(v)$, we see that X is transitive on $\Gamma(v)$, and this implies that all vertices in $\Gamma(v)$ share the same neighbourhood. Then since Γ is connected, we conclude that $\Gamma(v) = \Delta'$. But then $p^k = |V(\Gamma)| = 2|\Gamma(v)| = 2d$, contradicting our assumptions on k.

Therefore N has at least three orbits on $V(\Gamma)$, showing that Γ is not G-basic, and hence that Γ is a non-basic 2-arc-transitive graph. ☐

The next step in the proof of our Theorem 1.2 involves an observation that every 2-arc-transitive graph of prime-power order yields an infinite family of 2-arc-transitive normal covers. We do that by using a standard tool of homological p-covers, which we now explain.

For a given finite graph Γ and prime p, let $\tilde{\Gamma}^{(p)}$ denote its p-homological cover, as defined in [12]. This is a connected graph which is characterised by a property of its covering transformation group – the group preserving each fibre – namely that covering transformation group acts regularly on each fibre, and is isomorphic to the elementary abelian p-group $\mathbb{Z}_p^\beta$ where β is the Betti number of Γ (namely the number of edges not contained in a spanning tree for Γ). It is well known (for example by [12, Proposition 6.4]) that the automorphism group G of Γ lifts to a subgroup $\tilde{G}$ of $\mathrm{Aut}(\tilde{\Gamma}^{(p)})$ of the form $\mathbb{Z}_p^\beta.\mathrm{Aut}(\Gamma)$, and that G acts transitively on the 2-arcs of Γ if and only if $\tilde{G}$ acts transitively on the 2-arcs of $\tilde{\Gamma}^{(p)}$.

Another way of defining $\tilde{\Gamma}^{(p)}$ is by first considering the universal covering projection $T_d \to \Gamma$, where T_d is the (infinite) d-regular tree. Here the group G lifts to a subgroup U of $\mathrm{Aut}(T_d)$, and there exists a normal subgroup $K \leqslant U$, isomorphic to the free group of rank β, such that $\Gamma \cong T_d/K$ and $G \cong U/K$. Now let $N = [K,K]K^{(p)}$ be the subgroup of K generated by the derived subgroup of K and the p-th powers of all elements of K. Then $\tilde{\Gamma}^{(p)}$ can be defined as the quotient graph T_d/N. Since N is characteristic in K, the group U projects to a group $\tilde{G} = U/N \leqslant \mathrm{Aut}(\tilde{\Gamma}^{(p)})$.

Note that in the above definition, instead of the prime p one could take any prime-power, say p^k, and obtain a p^k-homological cover $\tilde{\Gamma}^{(p^k)}$. Then $\mathrm{Aut}(\Gamma)$ would still lift along the corresponding covering projection $\tilde{\Gamma}^{(p^k)} \to \Gamma$ to a group $(\mathbb{Z}_{p^k})^\beta.\mathrm{Aut}(\Gamma) \leqslant \mathrm{Aut}(\tilde{\Gamma}^{(p^k)})$.

Equipped with Lemma 4.2 and the above construction, one can now easily deduce the following.

Lemma 4.3. *Suppose there exists a finite 2-arc-transitive graph Γ with valency $d \geqslant 3$ whose order is a power of some prime p. Then there exists an infinite family of d-valent non-basic 2-arc-transitive graphs, all regular covers of Γ, whose orders are powers of p.*

Proof. We define a family of graphs Γ_i recursively, by setting $\Gamma_0 = \Gamma$ and then letting Γ_j be the p-homological cover of Γ_{j-1} for every positive integer j. (Alternatively, we could let $\Gamma_j = \tilde{\Gamma}^{(p^j)}$.) Then all of these graphs Γ_i are 2-arc-transitive normal covers of Γ, with valency d, and the order of each one of them is a power of p. Moreover, the order of all except finitely many of them exceeds p^{m_d}, where m_d is the constant from Lemma 4.2, and hence all of them are non-basic. □

Noting that the d-hypercube graph Q_d is a d-valent 2-arc-transitive graph with 2^d vertices, while the complete graph K_{p^m} is a 2-arc-transitive $(p^m - 1)$-valent graph on p^m vertices for every prime p and every positive integer m, it is easy to see that the above lemma has the following straightforward consequence.

Corollary 4.4. *For every integer $d \geqslant 2$, there exist infinitely many d-valent non-basic 2-arc-transitive graphs with order a power of 2. Similarly, if p is a prime and $d = p^m - 1$ for some integer $m \geqslant 1$, then there exist infinitely many d-valent non-basic 2-arc-transitive graphs with order a power of p.*

In particular, the first part of Corollary 4.4 proves Theorem 1.2. But further, it is known that a d-valent 2-arc-transitive graph with order a power of a prime p exists if and only if either $p = 2$, or $d + 1$ is a power of p, by [8, Corollary 3.5]. Hence it follows that Corollary 4.4 deals with all possible values of the valency d and the prime p.

Finally, we point out that our proofs of Lemma 4.3 and Corollary 4.4 depend heavily on Theorem 4.1 of Trofimov and Weiss, and hence on the classification of finite simple groups. But if we want to prove only the claim of Corollary 4.4, then the full power of Theorem 4.1 is not needed. In fact, since for hypercubes Q_d as well as for complete graphs K_{d+1}, the stabiliser of a vertex v in the automorphism group induces the full symmetric group S_d on the neighbourhood of v, the following folklore and easy-to-prove special case of Theorem 4.1 suffices, thereby avoiding the need for classification of finite simple groups.

Lemma 4.5. *For every integer $d \geqslant 3$, there exists a constant c_d with the following property: If Γ is a connected d-valent graph Γ admitting a vertex-transitive group of automorphisms G such that the permutation group $G_v^{\Gamma(v)}$ induced by the action of the vertex-stabiliser G_v on the neighbourhood $\Gamma(v)$ is isomorphic to the symmetric group S_d, then $|G_v| \leqslant c_d$.*

Proof. We begin by pointing out that if $d = 3$, then a celebrated theorem of Tutte [21] implies that $|G_v| \leqslant 48$. Also the case $d = 4$ was essentially dealt with by Gardiner (see [5, Theorem 3.9]), with the bound on the order and the structure of G_v explicitly stated in [15]. Similarly if $d = 5$, then one can deduce the bound on $|G_v|$ from the work of Weiss [23], with the structure of vertex- and edge-stabilisers explicitly given in [14]. All of these facts were proved by elementary means, and do not require the classification of finite simple groups. Hence from now on we will assume that $d > 5$.

Let Γ and G be as in the statement of this lemma, let $\{u, v\}$ be an edge of Γ, and let K be the kernel of the action of the arc-stabiliser G_{uv} on $\Gamma(u) \cup \Gamma(v) \setminus \{u, v\}$. Also for each $w \in V(\Gamma)$, let $G_w^{[1]}$ denote the kernel of the action of G_w on $\Gamma(w)$. Note that G_{uv}/K acts faithfully on the set $\Gamma(u) \cup \Gamma(v) \setminus \{u, v\}$, so G_{uv}/K is isomorphic to a subgroup of $S_{d-1} \times S_{d-1}$, giving $|G_v| = d|G_{uv}| \leqslant d(d-1)!^2$. Hence it suffices to show that K is trivial.

So suppose that K is not trivial. Then by the Thompson-Wielandt theorem (see, for example, [5, Corollary 2.3] or [19, 22]), we find that K is a p-group for some prime p. Then since K is normal in G_{uv}, it is contained in the maximal normal p-subgroup $O_p(G_{uv})$ of G_{uv}. Moreover, since G_{uv} is normal in the edge-stabiliser $G_{\{u,v\}}$, and $O_p(G_{uv})$ is characteristic in G_{uv}, it follows that $O_p(G_{uv}) \lhd G_{\{u,v\}}$. Similarly, because

K is the kernel of the action of $G_v^{[1]}$ on $\Gamma(u) \setminus \{v\}$, we find that K is normal in $G_v^{[1]}$, and hence that $K \leqslant O_p(G_v^{[1]}) \lhd G_v$.

Now consider the image of the sequence of groups $O_p(G_{uv}) \lhd G_{uv} \leqslant G_v$ under the quotient projection $q\colon G \to G_v^{\Gamma(v)}$ that takes every element of G_v to the permutation it induces on $\Gamma(v)$. Clearly $q(G_v) \cong S_d$ and $q(G_{uv}) \cong S_{d-1}$, and as $O_p(G_{uv})$ is normal in G_{uv}, it follows that $q(O_p(G_{uv}))$ is either trivial or a normal p-subgroup $q(G_{uv})$. Since we are assuming that $d > 5$ and since $q(G_{uv}) \cong S_{d-1}$, we find that $q(O_p(G_{uv})) = 1$ and hence that the kernel $G_v^{[1]}$ of the projection q contains $O_p(G_{uv})$. But then because $G_v^{[1]} \leqslant G_{uv}$, and $O_p(G_{uv})$ is normal in G_{uv}, it follows that $O_p(G_{uv}) \leqslant O_p(G_v^{[1]})$. On the other hand, $G_v^{[1]}$ is clearly a normal subgroup of G_{uv}, implying that $O_p(G_v^{[1]}) \lhd G_{uv}$ and hence that $O_p(G_v^{[1]}) \leqslant O_p(G_{uv})$. Thus we have shown that $O_p(G_{uv}) = O_p(G_v^{[1]})$. Finally, recall that $O_p(G_{uv})$ is normal in $G_{\{u,v\}}$, while $O_p(G_v^{[1]})$ is normal in G_v. As these two groups coincide, it follows that they are normal in $\langle G_v, G_{\{u,v\}}\rangle$, which by connectivity of Γ equals G. In particular, $O_p(G_v^{[1]})$ is a normal subgroup of G contained in the stabiliser G_v, and hence is trivial. But this contradicts the fact that $O_p(G_v^{[1]})$ contains K and hence is non-trivial.

Thus $K = 1$, and $c_d = d(d-1)!^2$ can be taken as an upper bound on $|G_v|$, as claimed. □

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