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Intersection cohomology of type-A toric varieties

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Abstract Type-A toric varieties may be obtained as GIT quotients with respect to a torus action with weights corresponding to roots of the group $SL(k)$ for some $k > 1$. These varieties appear in various important applications, in particular, as normal cones to strata in moduli spaces of vector bundles. In this paper, we describe the intersection Betti numbers of these varieties, and those of some associated projective varieties. We present an elegant combinatorial model for these numbers, and, using the work of Hausel and Sturmfels, we show that the relevant intersection cohomology groups are endowed with a canonical product structure.

1. Introduction

Let $\alpha_1, \dots, \alpha_n \in \operatorname{Hom}(T, \mathbb{C}^*)$ be n weights of a complex torus of dimension $k-1$, and let $x_1, \dots, x_n$ be the coordinates on $\mathbb{C}^n$. Considering the weights as elements of an additive group, and using the exponential notation, we can write the corresponding diagonal action of an element $q \in T$ as

$$q \cdot (x_1, \dots, x_n) \mapsto (q^{\alpha_1} x_1, \dots, q^{\alpha_n} x_n).$$

The ring of T-invariant polynomial functions on $\mathbb{C}^n$ then is the ring of functions of an affine algebraic variety, which we denote by $X = X(\mathfrak{A}, 0)$, where $\mathfrak{A} = [\alpha_1, \dots, \alpha_n]$. In our constructions, we will always assume that the sequence $\mathfrak{A}$ is unimodular, i.e. any $(k-1)$-tuple of these weights is either linearly dependent, or has determinant ± 1 with respect to the lattice $\mathbb{Z}\mathfrak{A}$ they generate. For the purposes of this introduction, we will consider the case when $n = k(k-1)$, and $\mathfrak{A}$ is the root system of the group $SL(k)$:

$$\mathfrak{A} = [\alpha_{ij} = \varepsilon_i - \varepsilon_j |\, 1 \leqslant i \neq j \leqslant k].$$

where ε_i, $i = 1, \dots k$, are the "coordinate weights" of $(\mathbb{C}^*)^k$ acting on $\mathbb{C}^k$.

A variant of this construction is obtained by fixing a weight $\theta \in \mathbb{N}\mathfrak{A}$ and considering the graded algebra

$$\bigoplus_{j=0}^{\infty} S_{j\theta},$$

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where $S_{j\theta}$ is the linear span of the monomials in $\mathbb{C}[x_1,\dots,x_n]$ of weight $j\theta$. The corresponding variety $X(\mathfrak{A},\theta)$ is smooth for generic θ, and projective over $X(\mathfrak{A},0)$, which means, in particular, that there is a canonical proper map

$$\varphi_\theta : X(\mathfrak{A},\theta) \to X(\mathfrak{A},0).$$

We note that for every $\eta \in \mathbb{Z}\mathfrak{A}$, there is a well defined line bundle L_η on $X(\mathfrak{A},\theta)$ for any θ, and the line bundle $L_\theta \to X(\mathfrak{A},\theta)$ is the polarization for the projective map ϕ_θ.

This variety appears in various contexts; in particular, it is a model for the normal cone of the singular points of the moduli spaces of semistable bundles on curves, and thus plays an important role in the calculation of the intersection cohomologies of these spaces [14, 15, 5].

The quotient variety X is usually very singular, and the topological invariants most adapted to this situation are the intersection cohomology groups $IH^*(X)$. The central problem we address in this paper, is the calculation of the associated Poincaré polynomial

$$g(k,t) = \sum_{i=0}^{[d/2]} g_i t^i = \sum_{i=0}^{[d/2]} \dim IH^{2i}(X)\cdot t^i,$$

where $d = n-k = k(k-2)$.

A related problem is the study of the topology of the projective toric variety $\widehat{X}$, associated to $\mathfrak{A}$ as follows. Let $\kappa : \mathbb{C}^* \to \mathbb{C}^*$ be the tautological weight of the group $\mathbb{C}^*$, set $\hat{T} = T \times \mathbb{C}^*$, and consider the weight data $\widehat{\mathfrak{A}} = [\alpha+\kappa \mid \alpha \in \mathfrak{A}]$ for the group $\hat{T}$. Then we set $\widehat{X} = X(\widehat{\mathfrak{A}},\kappa)$, and note that the variety X is the cone over $\widehat{X}$ associated to the line bundle L_κ, i.e. it may be obtained as the total space of L_κ with the zero-section collapsed to a point.

The properties of intersection cohomology imply that $IH^*(\widehat{X})$ satisfies Poincaré duality, and thus the corresponding Poincaré polynomial

$$h(k,t) = \sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} \dim IH^{2i}(\widehat{X})\cdot t^i$$

is palindromic.

Our notation for these polynomials is motivated by the g-and-h-polynomial calculus of Stanley [16], of which our polynomials are examples. This, in particular, implies that

$$g_i = h_i - h_{i-1} \text{ for } 0 \leqslant i \leqslant [d/2]. \tag{1}$$

Let us describe contents of our paper. In §2 we set the basic notation and describe the necessary concepts of the theory of toric varieties, while in §3 we give a brief introduction to intersection cohomology. A key result, proved in is §4 (cf. also §6.1), is that for a generic θ, the canonical map $\varphi_\theta : X(\mathfrak{A},\theta) \to X(\mathfrak{A},0)$ is *small* (cf. Theorems 4.5 and 6.1), and as a consequence, the intersection cohomology group $IH^*(X)$ is isomorphic to the cohomology of the fiber $H^*(\varphi_\theta^{-1}(0))$ as an additive group.

Remark 1.1. After this work was finished, we became aware of the preprint [12] of Mauri and Migliorini from a year earlier, who prove this in a more general context of arbitrary Lawrence toric varieties (see [12, Proposition 4.17]).

The structures introduced in our combinatorial proof are used later to give a graph-theoretic interpretation of the coefficients of the polynomial $g(k,t+1)$ as follows. Let $\mathcal{G}_k$ be the set of oriented graphs with vertex set $\{1,2,\dots,k\}$, and for $G \in \mathcal{G}_k$, denote by

- $e(G)$ the number of edges of G, and by

- $\mathrm{out}_G(m)$ the number of outgoing edges from the vertex m in G.

We show (cf. Theorem 4.5) that

Theorem 1.2. *$g(k,t+1)=\hat{g}_0+\hat{g}_1 t+\hat{g}_2 t^2+\cdots+\hat{g}_{[d/2]}t^{[d/2]}$, where*

$$\hat{g}_i=\left|\left\{G\in\mathcal{G}_k \;\middle|\; \begin{array}{l} G \text{ is acyclic and connected, with } e(G)=i+k-1,\\ \mathrm{out}_G(1)=0 \text{ and } \mathrm{out}_G(m)>0 \text{ for } m>1.\end{array}\right\}\right|.$$

For calculating the polynomial $g(k,t)$, we present the following recursion (cf. Theorem 4.19).

Theorem 1.3. *Let $p(m,t)=1+t+t^2+\cdots+t^{m-1}$, and set $g(1,t)=1$. Then*

$$g(k,t)=\sum_{\substack{J\subset\{2,\dots,k\}\\ J\neq\varnothing}}(-1)^{|J|-1}p(k-|J|,t)^{|J|}\cdot g(k-|J|,t).$$

Note that while the function g determines the function h (cf. (1)), nevertheless, going between the two functions is nontrivial, because of the truncation involved in their relation (cf. [16]). In §5 we show that for $k=3$, the h-polynomial coincides with that of the product of projective spaces (cf. Lemma 5.2), and then present an efficient recursion for the general case (cf. Theorem 5.3), which, in a certain sense, produces a resolution of $IH^*(\widehat{X})$ in terms of such products:

Theorem 1.4. *Set $h(1,t)=1$; then*

$$h(k,t)=p(k-1,t)^k-\sum_{\substack{(\lambda_1,\dots,\lambda_s)\vdash\underline{k}\\ s\geqslant 2,|\lambda_i|\geqslant 2}} t^{\sum_{i<j}|\lambda_i|\cdot|\lambda_j|}\prod_{i=1}^{s}h(|\lambda_i|,t),$$

where we denote by $\underline{k}$ the set $\{1,2,\dots,k\}$.

We also give a graph-theoretic interpretation of the h-polynomial (cf. Theorem 5.5):

Theorem 1.5. *$h(k,t+1)=\hat{h}_0+\hat{h}_1 t+\hat{h}_2 t^2+\cdots+\hat{h}_d t^d$ with*

$$\hat{h}_i=\left|\left\{G\in\mathcal{G}_k \;\middle|\; \begin{array}{l} G \text{ has an oriented cycle, } e(G)=i+k, \text{ and}\\ \text{there is a path to the vertex } 1 \text{ from any other vertex.}\end{array}\right\}\right|.$$

Finally, we note that our variety is an example of Lawrence toric varieties studied by Hausel and Sturmfels [9]. They showed, in particular, that the cohomology rings of the varieties $\varphi_\theta^{-1}(0)$ are identical for different regular values of θ, even though these varieties are not, in general, all isomorphic. This allows us to define a canonical ring structure on the intersection cohomology $IH^*(X)$ (cf. Theorem 6.3).

2. Preliminaries: toric varieties

This section contains the notation and basic facts from the theory of toric varieties. For more details, we refer the reader to [7, 17].

2.1. The quotient construction. Let $\mathfrak{g}=\bigoplus_{i=1}^n\mathbb{R}\omega_i$ be a real vector space with a fixed ordered basis, and let

$$0\to\mathfrak{a}\to\mathfrak{g}\xrightarrow{B}\mathfrak{t}\to 0 \tag{2}$$

be an exact sequence of finite dimensional real vector spaces of dimensions $n-d,n$ and d, respectively. We denote by $\Gamma_\mathfrak{g}=\bigoplus_{i=1}^n\mathbb{Z}\omega_i$ the lattice in $\mathfrak{g}$, and assume that $\Gamma_\mathfrak{g}$ intersects $\mathfrak{a}$ in a lattice $\Gamma_\mathfrak{a}$ of full rank; we denote the image $B(\Gamma_\mathfrak{g})$ in $\mathfrak{t}$ by $\Gamma_\mathfrak{t}$. The sequence (2) restricted to the lattices is also exact, as well as the dual sequence

$$0\to\mathfrak{t}^*\to\mathfrak{g}^*\xrightarrow{A}\mathfrak{a}^*\to 0$$

restricted to the dual lattices $\Gamma_{\mathfrak{t}}^*, \Gamma_{\mathfrak{g}}^*$ and $\Gamma_{\mathfrak{a}}^*$. Denoting the dual basis by $\{\omega^i\}$, we have $\mathfrak{g}^* = \bigoplus_{i=1}^n \mathbb{R}\omega^i$ and $\Gamma_{\mathfrak{g}}^* = \bigoplus_{i=1}^n \mathbb{Z}\omega^i$.

We introduce the notation α_i for the vector $A(\omega^i) \in \Gamma_{\mathfrak{a}}^*$ and consider the sequence $\mathfrak{A} = [\alpha_1, \dots, \alpha_n]$. Note that according to our assumptions, the elements of $\mathfrak{A}$ generate $\Gamma_{\mathfrak{a}}^*$ over $\mathbb{Z}$.

The complexified torus $T_{\mathfrak{a}} = \mathrm{Hom}(\Gamma_{\mathfrak{a}}^*, \mathbb{C}^*)$ acts on $\mathbb{C}^n$ diagonally with weights $(\alpha_1, \dots, \alpha_n)$. We will be interested in the quotients of this action in the sense of geometric invariant theory. Let

$$S = \mathbb{C}[x_1, \dots, x_n], \quad \deg(x_i) = \alpha_i \in \Gamma_{\mathfrak{a}}^*$$

be the ring of polynomials graded by the semigroup $\mathbb{N}\mathfrak{A} \subset \Gamma_{\mathfrak{a}}^*$. For $\theta \in \mathbb{N}\mathfrak{A}$, we denote by S_θ the vector space of homogeneous polynomials in S of degree θ. Then S_0 is a finitely generated subalgebra of homogeneous degree zero polynomials and S_θ is a module over S_0 for any $\theta \in \mathbb{N}\mathfrak{A}$.

We define the affine toric variety $X(\mathfrak{A}, 0)$ as the affine GIT quotient of $\mathbb{C}^n$ by the torus $T_{\mathfrak{a}}$ action:

$$X(\mathfrak{A}, 0) := \mathbb{C}^n /\!/^0 T_{\mathfrak{a}} = \mathrm{Spec}(S_0).$$

For any $\theta \in \mathbb{N}\mathfrak{A}$, we define the toric variety $X(\mathfrak{A}, \theta)$ as the relative projective GIT quotient of $\mathbb{C}^n$ by the $T_{\mathfrak{a}}$-action:

$$X(\mathfrak{A}, \theta) := \mathbb{C}^n /\!/^\theta T_{\mathfrak{a}} := \mathrm{Proj}(S_{(\theta)}),$$

where $S_{(\theta)}$ is the finitely generated S_0-algebra $\bigoplus_r t^r S_{r\theta}$, which is $\mathbb{N}$-graded by the degree of t.

2.2. Gale duality and toric fans. Recall that any toric variety X may be associated to a fan Σ in the lattice $\Gamma_{\mathfrak{t}} \subset \mathfrak{t}$ in such a way that each cone $\sigma \in \Sigma$ corresponds to an affine subset of X (cf. [7, §1.4]). In this section, we describe the fan which corresponds to the toric variety $X(\mathfrak{A}, \theta)$ defined above (for the result, see Proposition 2.3).

2.2.1. *Chambers in the* $\mathfrak{A}$*-picture.* We begin with the definition of some basic concepts related to our weight sequence $\mathfrak{A} = [\alpha_1, \dots, \alpha_n]$.

- For a set or sequence Z of vectors in a real vector space, denote by $\mathrm{Cone}(Z)$ the closed cone spanned by the elements of Z. By convention, the cone over the empty set is the origin of the vector space.
- Denote by $\mathrm{BInd}(\mathfrak{A})$ the set of *basis index sets*, i.e. the set of those subsets $I \subset \{1, \dots, n\}$ for which the set $\{\alpha_i\}_{i \in I}$ is a basis of $\mathfrak{a}^*$. We will use the notation $\alpha^I \subset \mathfrak{A}$ for the basis associated to $I \in \mathrm{BInd}(\mathfrak{A})$.
- Denote by $\partial\mathfrak{A}$ the union of the boundaries of the simplicial cones spanned by elements of $\mathfrak{A}$:

$$\partial\mathfrak{A} = \bigcup\{\partial\mathrm{Cone}(\alpha^I) |\, I \in \mathrm{BInd}(\mathfrak{A})\}.$$

  A connected component of the open set $\mathrm{Cone}(\mathfrak{A}) \smallsetminus \partial\mathfrak{A}$ in $\mathfrak{a}^*$ is called a *chamber*, and the set of chambers will be denoted by $Ch(\mathfrak{A})$.
- We will call $\theta \in \Gamma_{\mathfrak{a}}^*$ *generic*, if it lies in one of the chambers of $\mathfrak{A}$.
- For a chamber $\mathfrak{c} \in Ch(\mathfrak{A})$, we define $\mathrm{BInd}(\mathfrak{A}, \mathfrak{c})$ to be the set of those $I \in \mathrm{BInd}(\mathfrak{A})$ for which $\mathrm{Cone}(\alpha^I) \supset \mathfrak{c}$.

The sequence $\mathfrak{B} = [\beta_1, \dots, \beta_n]$, where be $\beta_i = B(\omega_i)$ in $\mathfrak{t}$ (cf. (2)) is called the *Gale dual* of the sequence $\mathfrak{A}$. This notion is involutive, i.e. the Gale dual of the sequence $\mathfrak{B}$ is $\mathfrak{A}$.

2.2.2. *Gale duality and fans.* A *fan* Σ on $\mathfrak{B}$ is a finite collection of cones of the form $\sigma = \operatorname{Cone}(\beta^I)$, $I \subset \{1, 2, \ldots, n\}$, satisfying some additional properties (cf. [7, §1.4]); in particular, the union of $\sigma \in \Sigma$ is the cone $\operatorname{Cone}(\mathfrak{B})$. A fan is *simplicial* if all of its cones are simplicial; further, it is *unimodular* if every maximal cone of Σ is spanned by a basis of $\Gamma_{\mathfrak{t}}$.

There is a standard construction of toric varieties $\Sigma \mapsto X(\Sigma)$ from fans in the lattice $\Gamma_{\mathfrak{t}}$ (cf. [7, §1.4]). The toric variety $X(\Sigma)$ is a *toric orbifold*, i.e. has only finite quotient singularities, if and only if Σ is simplicial. In the unimodular case, simplicial fans give rise to smooth toric varieties.

The torus $T_{\mathfrak{t}} = \operatorname{Hom}(\Gamma_{\mathfrak{t}}^*, \mathbb{C}^*)$ is embedded in $X(\Sigma)$ as an open subset, and its standard action on itself extends to an action on $X(\Sigma)$.

Notation 2.1. The orbits of the $T_{\mathfrak{t}}$-action on $X(\Sigma)$ are in bijection with the cones $\sigma \in \Sigma$. Given $\sigma \in \Sigma$, we denote by $\mathcal{O}(\sigma)$ the corresponding orbit, and by $V(\sigma)$ the orbit closure of $\mathcal{O}(\sigma)$. Note that given $\sigma, \sigma' \in \Sigma$, $\sigma \subset \sigma'$ if and only if $V(\sigma') \subset V(\sigma)$ and $\dim(\sigma) = \operatorname{codim}(V(\sigma) \subset X(\Sigma))$.

Next, we say that the fan Σ_1 is a *refinement* of Σ_2 if for every cone $\sigma_1 \in \Sigma_1$ there exists a cone $\sigma_2 \in \Sigma_2$ such that $\sigma_1 \subset \sigma_2$. In this case, we can define a map $\widehat{\psi}_\Sigma : \Sigma_1 \to \Sigma_2$ by setting $\widehat{\psi}_\Sigma(\sigma_1)$ to be the smallest cone in Σ_2 that contains σ_1, and this induces a so-called *toric* morphism $\psi : X(\Sigma_1) \to X(\Sigma_2)$.

The following lemma describes the fundamental relation between the Gale dual configurations $\mathfrak{A}$ and $\mathfrak{B}$.

Lemma 2.2. *A linear combination $\sum_i m_i \alpha_i$ vanishes if and only if there is a linear functional $l \in \mathfrak{t}^*$ such that $l(\beta_i) = m_i$.*

This relationship is instrumental in the proof of the following important statement.

Proposition 2.3. *Let $\mathfrak{A}$ be a sequence in $\mathfrak{a}^*$ and let $\mathfrak{c}$ be a chamber in $Ch(\mathfrak{A})$.*

1. *If $I \in \operatorname{BInd}(\mathfrak{A}, \mathfrak{c})$, then its complement $\overline{I} = \{1, \ldots, n\} \smallsetminus I$ is an element of $\operatorname{BInd}(\mathfrak{B})$, i.e. the set $\beta^{\overline{I}} = \{\beta_i\}_{i \in \overline{I}}$ is a basis of $\mathfrak{t}$.*
2. *The set of simplicial cones $\operatorname{Cone}(\beta^{\overline{I}})$, $I \in \operatorname{BInd}(\mathfrak{A}, \mathfrak{c})$ forms a simplicial fan $\Sigma(\mathfrak{c})$ on $\mathfrak{B}$. In particular, the interiors of these cones are disjoint and their union is $\operatorname{Cone}(\mathfrak{B})$.*
3. *Let $\theta \in \mathfrak{c}$, then the GIT quotient $X(\mathfrak{A}, \theta)$ is the toric variety associated to the fan $\Sigma(\mathfrak{c})$.*
4. *The affine toric variety $X(\mathfrak{A}, 0)$ is associated to the fan with a single top-dimensional cone: the convex polyhedral cone $\operatorname{Cone}(\mathfrak{B})$ spanned by $\mathfrak{B}$.*

Remark 2.4. Recall that for every scheme X there is a canonical morphism

$$\psi : X \to X_0$$

to the affine scheme $X_0 = \operatorname{Spec}(H^0(X, \mathcal{O}_X))$ of regular functions on X. This map for $X(\mathfrak{A}, \theta)$ is the canonical map to $X(\mathfrak{A}, 0)$, and, in the unimodular case, this is a resolution of singularities.

2.2.3. *Polar polytopes.* Finally, we note that there is a simple way to associate a fan, and thus a toric variety, to a convex polytope.

Let $P \subset \mathfrak{t}$ be a *rational* convex polytope, i.e. such that an integral multiple of its vertices $\{v_1, v_2, \ldots, v_N\}$ lie in the lattice $\Gamma_{\mathfrak{t}}$. For simplicity, first, we will assume that $\sum_{i=1}^N v_i = 0$. Then we denote by Σ_P the fan whose cones are the cones over the proper faces of P (here, we include the empty face). Following the notation from above, we denote by $X(\Sigma_P)$ the associated toric variety. If the center of mass of P is not at the origin, then we replace the polytope P by its shifted copy $P - \frac{1}{N}\sum_{i=1}^N v_i$.

More generally, if $\widetilde{P}$ is a subdivision of the boundary of P, i.e. $\widetilde{P}$ is a collection of convex polytopes whose union is the boundary of P, and the intersection of two polytopes in $\widetilde{P}$ is a polytope in $\widetilde{P}$ (cf. Figure 3), then the cones over the polytopes in $\widetilde{P}$ form a fan, which we will also denote by $\Sigma_{\widetilde{P}}$. Then $\Sigma_{\widetilde{P}}$ is a refinement of Σ_P, which induces a toric morphism $X(\Sigma_{\widetilde{P}}) \to X(\Sigma_P)$. Note that while $X(\Sigma_P)$ is projective, the variety $X(\Sigma_{\widetilde{P}})$ is not necessarily projective.

3. Intersection cohomology

In this section, we collect a few basic facts about small morphisms and the intersection cohomology of toric varieties, needed in our paper.

3.1. Small maps. Let $\widetilde{Y}$ be a connected nonsingular variety, and $\psi : \widetilde{Y} \to Y$ be a proper surjective map onto a variety Y of the same dimension. A *stratification* for ψ is a decomposition of Y into finitely many locally closed nonsingular subsets $Y = \bigsqcup_{k=0}^{n} S_k$ such that $\psi^{-1}(S_k) \to S_k$ is a topologically locally trivial fibration. The subsets S_k are called *strata*.

Denote by $d_k := \dim(\psi^{-1}(y_k))$ the dimension of the fiber of ψ over any point $y_k \in S_k$. The map ψ is called *small*, if

$$\operatorname{codim}(S_k) = \dim(Y) - \dim(S_k) > 2d_k \tag{3}$$

for every nondense stratum S_k for the map ψ.

Small maps play an important role in the calculation of the *intersection cohomology* groups of singular varieties [8]. In general, the intersection cohomology $IH^*(Y)$ of an irreducible complex projective d-dimensional variety Y is a module over the singular cohomology ring $H^*(Y)$, and satisfies Poincaré duality and the Hard Lefschetz theorem [1, Theorem 5.4.10]. The latter means that for some element $\omega \in H^2(Y)$ (the class of a hyperplane section) and for $0 \leqslant k \leqslant d$ the multiplication by ω^{d-k} mapping $IH^k(Y) \to IH^{2d-k}(Y))$ is an isomorphism of vector spaces. One can define then, for $0 \leqslant k \leqslant d$, the *primitive intersection cohomology* of Y as

$$IH^k_{\mathrm{prim}}(Y) = IH^k(Y)/\omega\, IH^{k-2}(Y).$$

The intersection cohomology of toric varieties is intimately related to the theory of convex polytopes [3, 16]. We will now briefly review this relation.

3.2. Intersection cohomology of toric varieties. Let P be a simplicial d-dimensional polytope, and denote by f_i the number of its i-dimensional faces. One associates to P its *f-polynomial*

$$f(P,t) = t^d + f_0t^{d-1} + f_1t^{d-2} + \cdots + f_{d-2}t + f_{d-1},$$

its *h-polynomial*

$$h(P,t) = h_dt^d + \cdots + h_1t + h_0 := f(P,t-1), \tag{4}$$

and its *g-polynomial*

$$g(P,t) = h_0 + (h_1 - h_0)t + (h_2 - h_1)t^2 + \cdots + (h_{[d/2]} - h_{[d/2]-1})t^{[d/2]}. \tag{5}$$

The following theorem calculates the dimension of the intersection cohomology groups of a toric orbifold.

Proposition 3.1. [7, §5.2] *Let P be a simplicial d-dimensional rational polytope with h-polynomial $h(P,t) = \sum_{k=0}^{d} h_kt^k$, and let $X(\Sigma_P)$ be the associated toric orbifold. Then*

$$\dim H^{2k}(X(\Sigma_P)) = h_k \quad \textit{and} \quad H^{2k+1}(X(\Sigma_P)) = 0 \quad \textit{for} \quad 0 \leqslant k \leqslant d. \tag{6}$$

Remark 3.2. Note that it follows from (6) and the definition (5) that the coefficient $h_k - h_{k-1}$ is the dimension of the primitive cohomology of the toric variety $X(\Sigma_P)$ in degree k, and thus, in particular, it is non-negative.

When the polytope P is not simplicial, the corresponding toric variety $X(\Sigma_P)$ will have singularities, which are not of finite quotient type. In this case, the ordinary cohomology group is not a purely combinatorial invariant, but depends also on some geometric data of the polytope [13]. A better invariant to consider is the intersection cohomology of $X(\Sigma_P)$. Then the "generalized" h-polynomial $h(P,t) = \sum \dim IH^{2k}(X(\Sigma_P))t^k$ is a purely combinatorial invariant, i.e. it can be again defined from the face lattice of the polytope P.

For a simplicial polytope P, we have $H^*(X(\Sigma_P)) = IH^*(X(\Sigma_P))$, and hence, in this case, the generalized h-polynomial of P coincides with the one defined in (4). Now, following Stanley [16], we give combinatorial definitions of the h and g polynomials for a not necessarily simplicial polytope.

Let P be a d-dimensional polytope and suppose that the h and g polynomials have been defined for all convex polytopes of dimension less than d. We set

$$h(P,t) = \sum_{F<P} g(F,t)(t-1)^{d-1-\dim(F)}, \tag{7}$$

where the sum runs over all proper faces F of P, including the empty face $\varnothing$, for which $g(\varnothing,t) = h(\varnothing,t) = 1$ and $\dim(\varnothing) = -1$. The polynomial $g(P,t)$ is defined from the polynomial $h(P,t)$ as in (5). Formulas (5) and (7) then inductively define the polynomials g and h for all polytopes.

Remark 3.3. Note that this definition agrees with definitions given in (4) and (5), since the g-polynomial of any simplex equals 1.

The following theorem calculates the dimension of the intersection cohomology groups of a toric variety.

Theorem 3.4 ([6]). *Let P be a d-dimensional rational polytope and let $X(\Sigma_P)$ be the toric variety, associated to the fan Σ_P. Then*

(i) $h(P,t) = \sum_{k=0}^{d} \dim IH^{2k}(X(\Sigma_P))t^k$ *and* $IH^{2k+1}(X(\Sigma_P)) = 0$ *for* $0 \leqslant k \leqslant d$;

(ii) $g(P,t) = \sum_{k=0}^{[d/2]} \dim IH^{2k}_{\mathrm{prim}}(X(\Sigma_P))t^k$.

Finally, we recall the following basic consequence of the *decomposition theorem* [1] applied to a small resolution of singularities of affine toric varieties.

Theorem 3.5 ([11]). *Let Σ be a fan whose cones are the cones over the faces of a rational convex polytope P, including an empty face and P itself. Let $\widetilde{\Sigma}$ be a simplicial refinement of Σ such that the morphism $\psi : X(\widetilde{\Sigma}) \to X(\Sigma)$ of the corresponding affine toric varieties is small. Given a face $F < P$, we pick a point y_F in the $T_{\mathfrak{t}}$-orbit $\mathcal{O}(\sigma)$ (see Notation 2.1), where σ is the cone over the face F in the fan Σ. Then*

$$H^{2k}(\psi^{-1}(y_F)) \simeq IH^{2k}_{\mathrm{prim}}(X(\Sigma_F)) \quad \text{for} \quad 0 \leqslant k \leqslant [\dim(X(\Sigma_F))/2].$$

In particular, for $F = P$, we have $y_P = 0$ and

$$H^{2k}(\psi^{-1}(0)) \simeq IH^{2k}(X(\Sigma)) \simeq IH^{2k}_{\mathrm{prim}}(X(\Sigma_P)).$$

4. The g-polynomial of the Gale dual type-A root polytope

In this section, we study certain generalized Gale dual type-A root polytopes. The main result of this section, Theorem 4.5, calculates the g-polynomials of these polytopes as generating functions of the number of a certain oriented graphs, graded by the number of their edges. Theorem 4.19 presents a recursive formula, which is an effective tool for the calculation of these same g-polynomials.

4.1. The type-A root polytope. Denote by $\varepsilon_1,\dots,\varepsilon_k:\mathbb{R}^k\to\mathbb{R}$ the standard coordinates on $\mathbb{R}^k$. The quotient vector space

$$\mathfrak{a}_k=\mathbb{R}^k/\mathbb{R}\cdot(1,\dots,1)$$

is naturally dual to the subspace

$$\mathfrak{a}_k^*=\{\lambda_1\varepsilon_1+\dots+\lambda_k\varepsilon_k\,|\,(\lambda_1,\dots,\lambda_k)\in\mathbb{R}^k,\ \lambda_1+\dots+\lambda_k=0\},$$

which is endowed with a full rank sublattice

$$\Gamma^*_{\mathfrak{a}_k}=\{\lambda_1\varepsilon_1+\dots+\lambda_k\varepsilon_k|\,(\lambda_1,\dots,\lambda_k)\in\mathbb{Z}^k,\ \lambda_1+\dots+\lambda_k=0\}.$$

For $1\leqslant i\neq j\leqslant k$, we define the element $\alpha_{ij}=\varepsilon_i-\varepsilon_j\in\Gamma^*_{\mathfrak{a}_k}$, and we set

$$\Phi_k=\{\alpha_{ij}\mid 1\leqslant i\neq j\leqslant k\},$$

which is the set of roots of the A_{k-1} root system.

The *type-A root polytope* is defined as the convex hull of the set of roots Φ_k in the vector space $\mathfrak{a}_k^*$. The Gale transformation converts the set Φ_k of $k(k-1)$ root vectors in $\mathfrak{a}_k^*$ into the set of $k(k-1)$ vectors in an appropriate $(k-1)^2$-dimensional space, which we will denote by Ψ_k. The *Gale dual type-A root polytope* is the convex hull of the set of vectors from Ψ_k.

Example 4.1. The Gale dual type-A root polytope, corresponding to the root system A_2, is a three-dimensional prism shown in Figure 1.

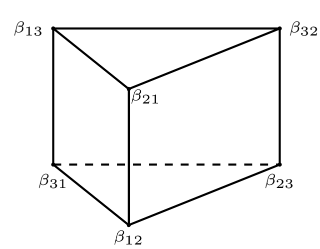

Figure 1. The Gale dual root polytope, corresponding to the root system A_2.

Remark 4.2. The Gale dual type-A polytope is a particular case of a *Lawrence polytope* [9].

In this paper, we will study the Gale dual root polytope, obtained from the type-A root system with multiplicities. For this, we fix integers $r_{ij}\in\mathbb{N}$ for $1\leqslant i<j\leqslant k$, set $r_{ji}=r_{ij}$ and consider the ordered sequence

(8)
$$\mathfrak{A}(r_{12},r_{13},\dots,r_{k-1\,k})=[\alpha_{12},\dots,\alpha_{12},\alpha_{21},\dots,\alpha_{21},\alpha_{13},\dots,\alpha_{13},\dots,\alpha_{k\,k-1}]\in\Gamma^*_{\mathfrak{a}_k}$$

of the root vectors $\alpha_{ij}\in\Phi_k$, where the vector α_{ij} is repeated r_{ij} times. Following the notation from page 577, we set

$$n:=|\mathfrak{A}|=2\sum_{1\leqslant i<j\leqslant k}r_{ij}\quad\text{and}\quad d:=n-k+1.$$

We consider the Gale dual sequence $\mathfrak{B}(r_{12},\dots,r_{k-1k})$ and denote by $\Pi(r_{12},\dots,r_{k-1k})$ the polytope obtained as the convex hull of the vectors in $\mathfrak{B}(r_{12},\dots,r_{k-1k})$. It follows from Lemma 2.2 that, similarly to the root configuration, this dual configuration is not full dimensional: $\Pi(r_{12}, \ldots, r_{k-1k})$ is a $d-1$-dimensional polytope in a d-dimensional vector space.

We can also deduce from Lemma 2.2 that the polytope $\Pi(r_{12}, \ldots, r_{k-1k})$ does not contain the origin in its interior; in fact, the toric variety $X(\mathfrak{A}(r_{12}, \ldots, r_{k-1k}), 0)$ (cf. page 578) is an affine cone over the singular projective toric variety $X(\Sigma_{\Pi(r_{12},\ldots,r_{k-1k})})$ (cf. Proposition 2.3).

Notation 4.3. The fan corresponding to the toric variety $X(\mathfrak{A}(r_{12}, \ldots, r_{k-1k}), 0)$ consists of the cones over the faces of Π, including Π itself; according to our convention, we denote the cone over the face $F < \Pi$ by $\mathrm{Cone}(F)$.

Example 4.4. A simple calculation shows that for $k = 2$ the polytope $\Pi(r_{12})$ is a product of two $(r_{12}-1)$-dimensional simplices, and thus the toric variety $X(\mathfrak{A}(r_{12}), 0)$ is an affine cone over the product of projective spaces $\mathbb{P}^{r_{12}-1} \times \mathbb{P}^{r_{12}-1}$.

4.2. Small maps and the combinatorics of the g-polynomial. Before we formulate the main result of this section, Theorem 4.5, we introduce some extra notation related to graphs.

- We denote by $\mathbb{K}_k = \mathbb{K}_k(r_{12}, \ldots, r_{k-1k})$ the directed graph with vertex set $\{1, 2, \ldots, k\}$ and with number of oriented edges from i to j equal to r_{ij}. We will use the notation $\overleftarrow{ij}$ for the edge directed from j to i.
- Denote by $\mathbb{G}_k = \mathbb{G}_k(r_{12}, \ldots, r_{k-1k})$ the graph obtained from $\mathbb{K}_k$ by deleting all edges $\overleftarrow{i1}$ for $1 < i \leqslant k$. To emphasize the break of symmetry: we will color the first vertex in $\mathbb{G}_k$ in red, and the other vertices in black.
- A directed graph is *acyclic* if it has no directed cycles.
- A directed graph is *naked* if every edge of this graph is contained in at least one directed cycle.
- We will say that a directed graph G is rooted at the i^{th} vertex if there is a directed path in G from any vertex to the i^{th} vertex.

Theorem 4.5. *Let $\Pi = \Pi(r_{12}, \ldots, r_{k-1k})$ be the Gale dual type-A root polytope defined above, and let $\hat{d} = \sum_{i<j} r_{ij} - k + 1$. Then*

$$g(\Pi, t+1) = \hat{g}_0 + t\hat{g}_1 + t^2\hat{g}_2 + \cdots + t^{\hat{d}}\hat{g}_{\hat{d}},$$

where $\hat{g}_i$ counts the number of acyclic subgraphs $G \subset \mathbb{G}_k$ with k vertices and $k-1+i$ edges, which are rooted at the first vertex.

Example 4.6. For $\Pi = \Pi(1,1,1)$ from Example 4.1 we have $\hat{g}_0 = 3$ and $\hat{g}_1 = 2$ (cf. Figure 2), hence $g(\Pi, t) = 3 + 2(t-1) = 1 + 2t$.

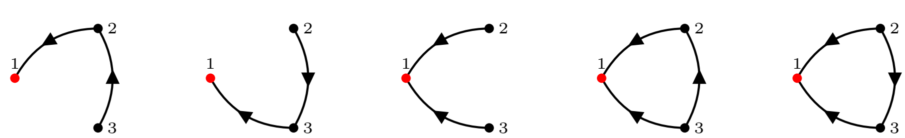

Figure 2. Acyclic subgraphs of $\mathbb{G}_3$ that contain a spanning tree rooted at the first vertex: there are 3 graphs with two edges and 2 graphs with three edges.

The strategy of the proof of Theorem 4.5 is as follows. First, we construct a toric resolution of the singularities of the affine variety $X(\mathfrak{A}(r_{12}, \ldots, r_{k-1k}), 0)$, and introduce some graph-theoretic tools based on the fact that weights of the $T_{\mathfrak{a}}$-action correspond to edges of an oriented complete graph. Next, we prove that this resolution is *small* (cf. §3.1). The Decomposition Theorem (cf. Theorem 3.5) then allows us to identify the intersection cohomology of $X(\mathfrak{A}(r_{12},\dots,r_{k-1k}),0)$ with the cohomology of the fibers of this small resolution. Finally, we describe the cohomology of the fibers using generating functions for the number of certain oriented graphs.

Lemma 4.7 ([18, Lemma 4.12]). *The simplicial cone* $\mathrm{Cone}(\{\alpha_{12},\alpha_{13},\dots,\alpha_{1k}\})$ *is a chamber in* $Ch(\mathfrak{A}(r_{12},\dots,r_{k-1k}))$.

Note that $\theta_1=(k-1,-1,\dots,-1)\in\mathrm{Cone}(\{\alpha_{12},\alpha_{13},\dots,\alpha_{1k}\})$, and consider the corresponding toric variety $X(\mathfrak{A}(r_{12},\dots,r_{k-1k}),\theta_1)$. As our fan is unimodular, this variety is smooth.

Example 4.8. Below is the chamber complex for the root system A_2 and the triangulation of the Gale dual root polytope $\Pi(1,1,1)$ (cf. Example 4.1) given by the chamber $\{\alpha_{12},\alpha_{13}\}$.

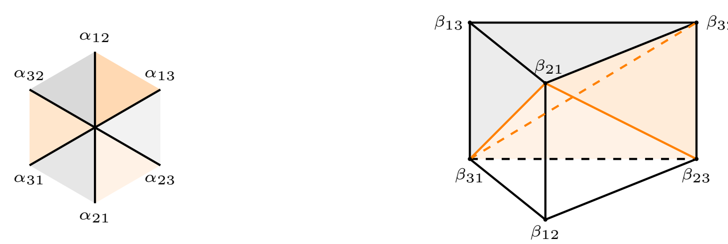

Figure 3. The chamber complex for the root system A_2 and the triangulation of the Gale dual root polytope given by the chamber $\{\alpha_{12},\alpha_{13}\}$.

To simplify the notation, from now on, we omit the dependence on $m,r_{12},\dots,r_{k-1k}$: throughout this chapter, we will use the notation

- $\mathfrak{A}:=\mathfrak{A}(r_{12},\dots,r_{k-1k})$;
- $\Pi:=\Pi(r_{12},\dots,r_{k-1k})$;
- $X:=X(\mathfrak{A}(r_{12},\dots,r_{k-1k}),0)$;
- $\widetilde{X}:=X(\mathfrak{A}(r_{12},\dots,r_{k-1k}),\theta_1)$ where $\theta_1=(k-1,-1,\dots,-1)$;
- Σ and $\widetilde{\Sigma}$ for the toric fans of X and $\widetilde{X}$, respectively;
- $\varphi:\widetilde{X}\to X$ for the canonical morphism defined in Remark 2.4; clearly, φ is a toric morphism compatible with the refinement of fans $\widehat{\varphi}_\Sigma:\widetilde{\Sigma}\to\Sigma$.

It is well known (cf. e.g. [10]) that the fiber of the toric morphism $\varphi:\widetilde{X}\to X$ over a point $x\in X$ depends only on the orbit $\mathcal{O}(\sigma)\subset X$ (cf. page 579) that contains x. We thus note that the decomposition

$$X=\bigsqcup_{F<\Pi}\mathcal{O}(\mathrm{Cone}(F)),$$

where F runs over the faces of Π, including the empty face and Π itself, is a stratification (cf. §3.1) for the canonical morphism $\varphi:\widetilde{X}\to X$.

Now we are ready to formulate our main technical result (see Remark 1.1, [12, Proposition 4.17]).

Theorem 4.9. *The canonical morphism* $\varphi:\widetilde{X}\to X$ *is small (cf.* §3.1*).*

4.3. Proof of Theorem 4.9. Clearly, we have $\dim \widetilde{X} = \dim X$, thus to prove that the map φ is small, we need to show that for any non-empty face $F < \Pi$

$$\operatorname{codim}(\mathcal{O}(\operatorname{Cone}(F)) \subset X) > 2\dim(\varphi^{-1}(y_F)), \tag{9}$$

where y_F is a point in the orbit $\mathcal{O}(\operatorname{Cone}(F))$, which is a stratum for our stratification. We start with the calculation of the codimension of $\mathcal{O}(\operatorname{Cone}(F))$ in X.

Associating to each element $\alpha_{ij} \in \mathfrak{A}$ the edge $\overleftarrow{ij}$ of the graph $\mathbb{K}_k$, we obtain a natural correspondence between subsequences $\mathcal{A} \subset \mathfrak{A}$ and subgraphs of $\mathbb{K}_k$. Applying Gale duality, we also obtain the correspondence between subgraphs of $\mathbb{K}_k$ and subsets of rays of the fan Σ (and thus rays of the fan $\widetilde{\Sigma}$ as well).

The following lemma describes the faces of the polytope Π in terms of subgraphs in $\mathbb{K}_k$.

Lemma 4.10. *Under the correspondence described above, naked graphs correspond to faces of the polytope* Π.

Proof. For any subsequence $\mathcal{A} \subset \mathfrak{A}$, we denote by $G_{\mathcal{A}}$ the graph corresponding to $\mathcal{A}$. By Gale duality, faces of Π correspond to subsequences $\mathcal{A} \subset \mathfrak{A}$ that have a positive linear combination summing up to zero:

$$\sum_{\alpha \in \mathcal{A}} m_\alpha \alpha = 0 \quad \text{ with } m_\alpha > 0. \tag{10}$$

Clearly, for any subsequence $\mathcal{C} \subset \mathcal{A}$ that corresponds to a directed cycle $G_{\mathcal{C}}$ of $G_{\mathcal{A}}$ we have $\sum_{\alpha \in \mathcal{C}} \alpha = 0$ and one can subtract a multiple of this sum from (10) to obtain a proper subsequence $\mathcal{A}' \subset \mathcal{A}$ that also has a positive linear combination summing up to zero.

Assume that $G_{\mathcal{A}}$ is not a naked graph; then, repeating the procedure described above, we arrive at a nonempty subsequence of vectors $\bar{\mathcal{A}} \subset \mathfrak{A}$ that has a positive linear combination summing up to zero, and such that the corresponding directed graph $G_{\bar{\mathcal{A}}} \subset \mathbb{K}_k$ is acyclic. Since any directed acyclic graph contains a vertex with only out-edges, we arrive at a contradiction with (10). □

Notation 4.11. We will denote the naked graph on k vertices corresponding to the face $F < \Pi$ by G_F. In particular, the naked graph G_Π has no edges and $G_\varnothing = \mathbb{K}_k$.

Example 4.12. The naked graphs from Figure 4 correspond to the following faces of the prism $\Pi(1,1,1)$ (cf. Figure 1): the two-dimensional simplex $\{\beta_{13}, \beta_{32}, \beta_{21}\}$, the square $\{\beta_{13}, \beta_{31}, \beta_{32}, \beta_{23}\}$, two edges $\{\beta_{13}, \beta_{31}\}$ and $\{\beta_{13}, \beta_{32}\}$, the vertex $\{\beta_{13}\}$.

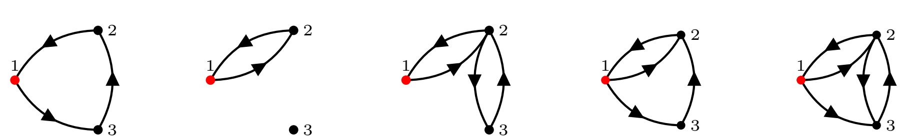

Figure 4. Naked graphs on 3 vertices.

Lemma 4.13. *Denote by* $\mathfrak{c}_1$ *the chamber* $\operatorname{Cone}(\{\alpha_{12}, \alpha_{13}, \ldots, \alpha_{1k}\})$ *in* $Ch(\mathfrak{A})$ *(cf. Lemma 4.7). Then the cones* $\operatorname{Cone}(\alpha^I)$ *for* $I \in \operatorname{BInd}(\mathfrak{A}, \mathfrak{c}_1)$ *correspond to the spanning trees in* $\mathbb{G}_k$*, rooted at the first vertex.*

Proof. First, note that the cone $\operatorname{Cone}(\alpha^I)$ is simplicial if and only if the vectors $\{\alpha_i\}_{i \in I}$ are linearly independent; this happens if and only if the corresponding graph has no (undirected) cycles, i.e. it is a tree.

Next, note that $\theta_1 = (k-1,-1,\dots,-1)$ is a point in the interior of $\mathfrak{c}_1$, hence θ_1 lies in the interior of $\mathrm{Cone}(\alpha^I)$ for each $I \in \mathrm{BInd}(\mathfrak{A},\mathfrak{c}_1)$. In other words, for each $I \in \mathrm{BInd}(\mathfrak{A},\mathfrak{c}_1)$, θ_1 can be written as a linear combination of vectors α_i, $i \in I$ with positive coefficients; this implies that the corresponding graph has at least one out-edge from the vertices $2,3,\dots,k$. Clearly, such graph cannot have an out-edge starting at the first vertex, since this would create a cycle, hence the statement follows. ☐

Since the fan $\widetilde{\Sigma}$ is simplicial, by Gale duality, we obtain the following statement.

Corollary 4.14. *For $0 \leqslant l \leqslant (2\sum_{i<j} r_{ij} - k + 1)$, there is a bijection between the codimension-l cones in the fan $\widetilde{\Sigma}$ and the connected subgraphs of $\mathbb{K}_k$ with k vertices and $k-1+l$ edges, rooted at the first vertex.*

For any graph G, we introduce the notation $e(G)$ and $s(G)$ for the number of edges and connected components of G, correspondingly. It follows from Corollary 4.14 that

$$\mathrm{codim}(\mathrm{Cone}(F)) = e(G_F) + s(G_F) - k, \tag{11}$$

and thus

$$\mathrm{codim}(\mathcal{O}(\mathrm{Cone}(F)) \subset X) = \dim(\mathrm{Cone}(F)) = 2\sum_{1\leqslant i<j\leqslant k} r_{ij} - e(G_F) - s(G_F) + 1. \tag{12}$$

Since our goal is to compare this number with the right-hand side of (9), our next step will be to analyze the fibers of φ.

In general, the combinatorics of fan refinements provides one with an explicit description of the fibers of any toric morphism (cf. [10]). To prove Theorems 4.5 and 4.9 we will only need to calculate the Betti numbers of these fibers (see Theorem 4.16 for the result).

Lemma 4.15. *Let F be a face of Π, and let G_F be the corresponding naked graph (cf. Lemma 4.10). Denote by $\mathcal{G}_F^1$ the set of connected subgraphs of $\mathbb{K}_k$ on k vertices, which have the same directed cycles as the naked graph G_F, and which are rooted at the first vertex. Then, under the morphism $\widehat{\varphi}_\Sigma$, the cone $\sigma \in \widetilde{\Sigma}$ maps to the cone $\mathrm{Cone}(F) \in \Sigma$ if and only if σ corresponds (cf. page 585) to some graph in the set $\mathcal{G}_F^1$.*

Proof. Denote by G_σ the subgraph of $\mathbb{K}_k$ that corresponds to the cone $\sigma \subset \widetilde{\Sigma}$ under the correspondence described on page 585. Repeating the argument from the proof of Lemma 4.13, we conclude that G_σ is a connected graph rooted at the first vertex.

Recall that $\widehat{\varphi}_\Sigma(\sigma) = \mathrm{Cone}(F)$ if and only if $\sigma \subset \mathrm{Cone}(F)$ and $\mathrm{Cone}(F)$ is the minimal cone in Σ that contains σ. Note that the first condition is equivalent to the fact that G_σ contains the naked graph G_F, while the second condition requires G_F to be the maximal naked graph contained in G_σ; thus G_σ should have the same directed cycles as G_F. ☐

Theorem 4.16 ([4, Corollary 4.7]). *Let $\varphi : \widetilde{X} \to X$ be as on page 584. For any face $F < \Pi$, denote by $d_l(F)$ the number of $(\dim(\mathrm{Cone}(F)) - l)$-dimensional cones $\sigma \in \widetilde{\Sigma}$ such that the (relative) interior of σ maps to the (relative) interior of the cone $\mathrm{Cone}(F) \in \Sigma$:*

$$d_l(F) = |\{\sigma \in \widetilde{\Sigma} \mid \ \widehat{\varphi}_\Sigma(\sigma) = \mathrm{Cone}(F),\ \mathrm{codim}(\sigma) - \mathrm{codim}(\mathrm{Cone}(F)) = l\}|.$$

Then, for any point y_F in $\mathcal{O}(\mathrm{Cone}(F)) \subset X$, the Poincaré polynomial of the fiber $\varphi^{-1}(y_F)$ has the form

$$\sum_{l\geqslant 0} \dim H^{2l}(\varphi^{-1}(y_F))t^{2l} = \sum_{l\geqslant 0} d_l(F)(t^2-1)^l.$$

Putting together Lemma 4.15 and equation (11), we arrive at the following interpretation of the numbers $d_l(F)$ in terms of graphs.

Lemma 4.17. *The number $d_l(F)$ defined in Theorem 4.16 is equal to the number of graphs in $\mathcal{G}^1_F$ with $e(G_F)+s(G_F)-1+l$ edges.*

In particular, it follows from Theorem 4.16 that for $y_F \in \mathcal{O}(\mathrm{Cone}(F))$ the dimension of the fiber $\varphi^{-1}(y_F)$ is bounded from above by

$$\max_{G\in\mathcal{G}^1_F}\{e(G)\} - e(G_F) - s(G_F) + 1.$$

Hence using expression (12) for the codimension of the strata $\mathcal{O}(\mathrm{Cone}(F))$ in X, we can conclude that to prove (9), it is enough to show that

$$2\sum_{1\leqslant i<j\leqslant k} r_{ij} > 2\max_{G\in\mathcal{G}^1_F}\{e(G)\} - e(G_F) - s(G_F) + 1 \tag{13}$$

for any non-empty $F < \Pi$. Note that we can think of

$$2\max_{G\in\mathcal{G}^1_F}\{e(G)\} - e(G_F)$$

as the number of edges in the graph $G \subset \mathbb{K}_k$ such that G is obtained from the naked graph G_F by adding all possible edges between the connected components of G_F. Then, trivially, we have

$$2\max_{G\in\mathcal{G}^1_F}\{e(G)\} - e(G_F) \leqslant e(\mathbb{K}_k) = 2\sum_{1\leqslant i<j\leqslant k} r_{ij},$$

with equality if and only if G_F is a disjoint union of complete graphs. Hence we obtain the inequality

$$2\max_{G\in\mathcal{G}^1_F}\{e(G)\} - e(G_F) - s(G_F) + 1 \leqslant 2\sum_{1\leqslant i<j\leqslant k} r_{ij} - s(G_F) + 1 \leqslant 2\sum_{1\leqslant i<j\leqslant k} r_{ij},$$

with equality if and only if G_F is a disjoint union of complete graphs and $s(G_F)=1$. In this latter case, G_F is connected and complete, and thus $G_F = \mathbb{K}_k$. This corresponds to the empty face $F=\varnothing$ with $\mathrm{codim}(\mathcal{O}(\mathrm{Cone}(\varnothing)))=0$, and this completes the proof of Theorem 4.9.

4.4. Proof of Theorem 4.5. Now we are ready to prove Theorem 4.5. Since the map $\varphi : \widetilde{X} \to X$ is small, it follows from Theorems 3.5 and 3.4 that the g-polynomial $g(\Pi, t^2)$ is equal to the Poincaré polynomial of the fiber $\varphi^{-1}(0)$.

By Theorem 4.16 and Lemma 4.17, the Poincaré polynomial of the fiber $\varphi^{-1}(0)$ is equal to

$$\sum_l \dim H^{2l}(\varphi^{-1}(0))t^{2l} = \sum_l d_l(\Pi)(t^2-1)^l,$$

where $d_l(\Pi)$ is the number of graphs in $\mathcal{G}^1_\Pi$ (cf. Lemma 4.15) with $e(G_\Pi)+s(G_\Pi)-1+l$ edges; after a change of variables, we arrive at the following statement:

$$g(\Pi, t^2+1) = \sum_l d_l(\Pi)t^{2l}. \tag{14}$$

By Lemma 4.10, $\mathcal{G}^1_\Pi$ is the set of acyclic subgraphs $G \subset \mathbb{K}_k$, such that G has k vertices and contains a spanning tree rooted at the first vertex; we also have $e(G_\Pi)=0$ and $s(G_\Pi)=k$.

Clearly, a graph $G \in \mathcal{G}^1_\Pi$ has no edges of the form $\overleftarrow{i1}$ for $2 \leqslant i \leqslant k$, since otherwise this edge together with a paths from i to 1 would create a directed cycle in G; hence any graph $G \in \mathcal{G}^1_\Pi$ is contained in $\mathbb{G}_k \subset \mathbb{K}_k$.

We conclude that $d_l(\Pi)$ is equal to the number of acyclic subgraphs G of $\mathbb{G}_k$ with $k-1+l$ edges and k vertices, such that G is rooted at the first vertex. Now Theorem 4.5 follows from equation (14). $\square$

Applying the same argument and using Lemma 4.15, we obtain the following statement.

Corollary 4.18. *Let F be a face of the Gale dual type-A root polytope $\Pi = \Pi(r_{12},\dots,r_{k-1k})$. Then $g(F,t+1)=\sum_{i\geqslant 0}\hat{g}_i t^i$, where $\hat{g}_i$ counts the number of graphs $G\in\mathcal{G}_F^1$ with $e(G_F)+s(G_F)-1+i$ edges and k vertices.*

Now we present a recursive formula for the g-polynomial of the Gale dual type-A root polytope.

Theorem 4.19. *Let $\Pi(r_{12},\dots,r_{k-1k})$ be a Gale dual type-A root polytope and let $p(n,t)=1+t+t^2+\dots+t^{n-1}$. For any nonempty subset $J\subset\{2,\dots,k\}$ we denote by $\overline{J}=\{1,\dots,k\}\smallsetminus J$ its complement and by $\Pi(r_{ij\in\overline{J}})$ the Gale dual root polytope corresponding to the root system $A_{|\overline{J}|-1}$ defined by the sequence of vectors $[\alpha_{ij}]_{i,j\in\overline{J}}\subset\mathfrak{A}(r_{12},r_{13},\dots,r_{k-1k})$ with α_{ij} repeated r_{ij} times. Then*

$$g(\Pi(r_{12},\dots,r_{k-1k}),t)=\sum_{\substack{J\subset\{2,\dots,k\}\\ J\neq\varnothing}}(-1)^{|J|-1}g(\Pi(r_{ij\in\overline{J}}),t)\cdot\prod_{j\in J}p\Big(\sum_{i\in\overline{J}}r_{ij},t\Big).$$

Proof. First, we perform a change of variables $t\to t+1$ in the recursion; by Theorem 4.5, the polynomial $g(\Pi(r_{12},\dots,r_{k-1k}),t+1)$ counts the number of edges in acyclic subgraphs of $\mathbb{G}_k$ with k vertices, rooted at the first vertex. Note that any such graph has a vertex $j\in\{2,3,\dots,k\}$ that has no in-edges.

Now fix a nonempty subset $J\subset\{1,2,\dots,k\}$. We claim that

$$g(\Pi(r_{ij\in\overline{J}}),t+1)\cdot\prod_{j\in J}p\Big(\sum_{i\in\overline{J}}r_{ij},t+1\Big)=\sum_{G\in\mathbb{G}_k'}t^{e(G)-k+1},\tag{15}$$

where the sum is taken over the set $\mathbb{G}_k'$ of the connected acyclic subgraphs $G\subset\mathbb{G}_k$ with k vertices, rooted at the first vertex, such that the vertices $j\in J$ of G has no in-edges.

Indeed, let G be a graph satisfying these conditions.

- A simple calculation shows that the polynomial $p\big(\sum_{i\in\overline{J}}r_{ij},t+1\big)$ counts all out-edges of G from the vertex $j\in J$.
- We denote by $G\smallsetminus J$ the graph obtained from G by deleting all vertices labelled by $j\in J$ and all edges attached to these vertices. Then clearly, $G\smallsetminus J$ is an acyclic subgraph of $\mathbb{G}_k$ with vertex set $\overline{J}$, rooted at the first vertex.
- Thus it follows from Theorem 4.5 that the edges of $G\smallsetminus J$ are counted by the polynomial $g(\Pi(r_{ij\in\overline{J}}),t+1)$.
- Noting that any edge of G is either an out-edge from some vertex $j\in J$, or is an edge of $G\smallsetminus J$, we arrive at equation (15).

Now the recursion follows from the inclusion-exclusion principle. $\square$

Example 4.20. For $k=3$, the recursive formula has the following simple form

$$\begin{aligned}g(\Pi(r_{12},r_{13},r_{23}),t)&=p(r_{12}+r_{23},t)\cdot g(\Pi(r_{13}),t)\\&\quad+p(r_{13}+r_{23},t)\cdot g(\Pi(r_{12}),t)-p(r_{12},t)\cdot p(r_{13},t).\end{aligned}$$

5. The h-polynomial of the Gale dual type-A root polytope

As explained in Example 4.4, for $k = 2$, we have $X(\Sigma_{\Pi(r_{12})}) = \mathbb{P}^{r_{12}-1} \times \mathbb{P}^{r_{12}-1}$, hence by Proposition 3.1 the h-polynomial of the polytope $\Pi(r_{12})$ is equal to the Poincaré polynomial (with $\deg(t) = 2$) of the product of two projective spaces:

$$h(\Pi(r_{12}), t) = (1 + t + t^2 + \cdots + t^{r_{12}-1})^2.$$

In this section, we calculate h-polynomials of Gale dual polytopes which are associated to the A_{k-1} root system for $k \geqslant 3$.

5.1. $k = 3$ case. We start with the calculation of the h-polynomial of the polytope $\Pi(1, 1, 1)$, shown in Figure 1. This three-dimensional prism has

- 6 faces of dimension 0,
- 9 faces of dimension 1, and
- 2 faces of dimension 2, which are all simplicial;
- there are also 3 two-dimensional nonsimplicial faces, which are squares.

Using (5), we obtain that the g-polynomial of a square equals $1 + t$. Then it follows from (7) that

$$h(\Pi(1, 1, 1), t) = (t - 1)^3 + 6(t - 1)^2 + 9(t - 1) + 2 + 3(t + 1) = (t + 1)^3.$$

Note that $(t + 1)^3$ is the Poincaré polynomial of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and by Proposition 3.1 it is equal to the h-polynomial of the octahedron, which is combinatorially dual to the three-dimensional cube.

This corresponds to the fact that the prism $\Pi(1, 1, 1)$ has a simplicial subdivision $\widetilde{\Pi}(1, 1, 1)$ which does not add any vertices and divides every two-dimensional nonsimplicial face into two simplices by adding its diagonal (see Figure 5). The resulting polytope $\widetilde{\Pi}(1, 1, 1)$ is combinatorially equivalent to the octahedron, and the map of the corresponding projective toric varieties (cf. §2.2.3)

$$f : X(\Sigma_{\widetilde{\Pi}(1,1,1)}) \to X(\Sigma_{\Pi(1,1,1)})$$

is small. More precisely, it is an isomorphism outside the three singular points of $X(\Sigma_{\Pi(1,1,1)})$ which correspond to the cones in $\Sigma_{\Pi(1,1,1)}$ over the nonsimplicial faces, and the fibers over these points are isomorphic to $\mathbb{P}^1$.

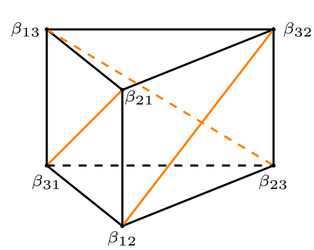

Figure 5. Refinement, combinatorially equivalent to the octahedron.

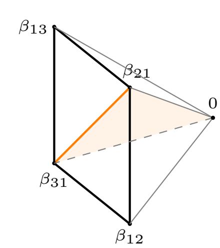

Figure 6. Calculation of the fiber over the singular point.

We note that the compact toric variety corresponding to this refinement is *not projective.*

Remark 5.1. Note that this agrees with Theorem 4.16 applied to the map

$$\varphi : X(\mathfrak{A}(1, 1, 1), (2, -1, -1)) \to X(\mathfrak{A}(1, 1, 1), 0).$$

As it is shown in the Figure 6, the corresponding morphism of fans $\widehat{\varphi}_\Sigma$ maps 2 three-dimensional and 1 two-dimensional cones in $\Sigma_{\widetilde{\Pi}(1,1,1)}$ to the interior of the cone over the square face of $\Pi(1,1,1)$. Thus by Theorem 4.16 the Poincaré polynomial of the fiber over the singular point should have the form $(t^2-1)+2$, which is indeed equal to the Poincaré polynomial of $\mathbb{P}^1$.

It turns out that the h-polynomial of any Gale dual polytope obtained from the A_2 root system has the following elegant form, which is a special case of Theorem 5.3.

Lemma 5.2. *Let $\Pi(r_{12},r_{13},r_{23})$ be a Gale dual type-A root polytope and set $r_i=\sum_{j\neq i} r_{ij}$. Then the h-polynomial of $\Pi(r_{12},r_{13},r_{23})$ is equal to the Poincaré polynomial of the product of projective spaces $\mathbb{P}^{r_1-1}\times\mathbb{P}^{r_2-1}\times\mathbb{P}^{r_3-1}$:*

$$h(\Pi(r_{12},r_{13},r_{23}),t)=\prod_{i=1}^{3}(1+t+t^2+\cdots+t^{r_i-1}).$$

5.2. Arbitrary k.

Theorem 5.3. *Let $\Pi(r_{12},\ldots,r_{k-1k})$ be a Gale dual type-A root polytope and let $r_i=\sum_{j\neq i} r_{ij}$. Given a subset $\lambda\subset\{1,\ldots,k\}$, we denote by $\Pi(r_{ij\in\lambda})$ the Gale dual root polytope corresponding to the root system $A_{|\lambda|-1}$ defined by the sequence of vectors $[\alpha_{ij}]_{i,j\in\lambda}\subset\mathfrak{A}(r_{12},\ldots,r_{k-1k})$ with α_{ij} repeated r_{ij} times. Then*

$$h(\Pi(r_{12},\ldots,r_{k-1k}),t)=$$
$$\prod_{i=1}^{k}(1+t+\cdots+t^{r_i-1})-\sum_{\substack{(\lambda_1,\ldots,\lambda_p)\vdash\{1,\ldots,k\}\\ p\geqslant 2,\,|\lambda_m|\geqslant 2}} t^{\left(\sum\limits_{m<n}\sum\limits_{i\in\lambda_m}\sum\limits_{j\in\lambda_n} r_{ij}\right)}\prod_{m=1}^{p} h(\Pi(r_{ij\in\lambda_m}),t),$$

where the sum is taken over the partitions $\underline{\lambda}=(\lambda_1,\ldots,\lambda_p)$ of the set $\underline{k}=\{1,\ldots,k\}$ that have at least 2 parts and do not have parts of cardinality 1.

Proof. First, we perform a change of variables $t\to t+1$ and rewrite the equation from Theorem 5.3 in the following form

$$\prod_{i=1}^{k}((t+1)^{r_i}-1)/t=\sum_{\substack{\underline{\lambda}\vdash\underline{k}\\ |\lambda_m|\geqslant 2}}(t+1)^{\left(\sum\limits_{m<n}\sum\limits_{i\in\lambda_m}\sum\limits_{j\in\lambda_n} r_{ij}\right)}\prod_{m=1}^{p} h(\Pi(r_{ij\in\lambda_m}),t+1). \tag{16}$$

We can interpret the left-hand side of (16) in terms of graphs as follows.

Lemma 5.4. *Let $r_i=\sum_{j\neq i} r_{ij}$, then*

$$\prod_{i=1}^{k}((t+1)^{r_i}-1)/t=\sum_{i\geqslant 0}\hat{p}_i t^i,$$

where $\hat{p}_i$ counts the number of subgraphs of $\mathbb{K}_k$ with $k+i$ edges and k vertices, which have an out-edge from each vertex.

Proof. Note that the i^{th} factor in the product counts the number of out-edges from the i^{th} vertex in a subgraph of $\mathbb{K}_k$. ☐

Now we describe the right-hand side of (16) in terms of graphs.

Theorem 5.5. *We have the following combinatorial description of the h-polynomial of the Gale dual type-A root polytope $\Pi(r_{12},\ldots,r_{k-1k})$:*

$$h(\Pi(r_{12},\ldots,r_{k-1k}),t+1)=\sum_{G\in\mathbb{K}'_k} t^{e(G)-k}, \tag{17}$$

where the sum is taken over the set $\mathbb{K}_k^{'}$ of subgraphs of the complete graph $\mathbb{K}_k$ which have k vertices, at least one cycle and which are rooted at the first vertex.

Proof. Using equations (7), (11) and Corollary 4.18, we obtain that

$$h(\Pi(r_{12},\dots,r_{k-1k}),t+1) \overset{(7)}{=} \sum_{F\lneq\Pi} g(F,t+1)t^{d-\dim(F)-1} \tag{18}$$

$$\overset{(11)}{=} \sum_{F\lneq\Pi} g(F,t+1)t^{e(G_F)+s(G_F)-k-1} \tag{19}$$

$$\overset{4.18}{=} \sum_{F\lneq\Pi}\sum_{G\in\mathcal{G}_F^1} t^{e(G)-k} = \sum_{G\subset\mathbb{K}_k^{'}} t^{e(G)-k} \tag{20}$$

as claimed. ☐

We can thus rewrite the right-hand side of equation (16) as

$$\sum_{\substack{\underline{\lambda}\vdash\underline{k}\\|\lambda_i|\geqslant 2}}\sum_{G\in\mathbb{K}_k^{''}} t^{e(G)-k}, \tag{21}$$

where the last sum is taken over the set $\mathbb{K}_k^{''}$ of subgraphs $G\subset\mathbb{K}_k$ which contain subgraphs $G_{\lambda_1},\dots,G_{\lambda_p}$ satisfying the following conditions:

- the set of vertices of G_{λ_i} is $\lambda_i\subset\{1,\dots,k\}$;
- G_{λ_i} contains a directed cycle;
- G_{λ_i} is rooted at the smallest vertex $v_i\in\lambda_i$;
- all edges in G between the sets of vertices λ_i and λ_j are in one direction: from λ_i to λ_j, if the minimum of $\lambda_i\cup\lambda_j$ is equal to the minimum of λ_i.

The last condition follows from the observation that the exponent of $(t+1)$ on the right-hand side of equation (16) is half of the number of edges in the graph $\mathbb{K}_k$ between the sets of vertices $\lambda_1,\dots,\lambda_p$.

In fact, to prove Theorem 5.3, we need to choose the root vertex of G_{λ_i} and the direction of the edges between the sets of vertices λ_i and λ_j in a more subtle way (for the result, see Lemma 5.10). The following construction allows us to perform a change of the root vertex.

Construction 5.6 (roots). *Let G be an acyclic subgraph of $\mathbb{K}_k$, which has k vertices and is rooted at the first vertex. We change the direction of all edges in all directed paths in G from the i^{th} vertex to the first vertex.*

Lemma 5.7. *This construction induces a one-to-one correspondence between the acyclic subgraphs of $\mathbb{K}_k$, which have k vertices and are rooted at the first vertex and the acyclic subgraphs of $\mathbb{K}_k$ with k vertices, which are rooted at the i^{th} vertex.*

Throughout this section, we will use the following construction.

Construction 5.8 (cycles). *Let G be a subgraph of $\mathbb{K}_k$ which contain at least one directed cycle. Removing all edges from G, that are not contained in any directed cycle, we obtain the maximal naked subgraph $G_F\subset G$. We say that a connected component of G_F is non-trivial, if it contains at least one edge; let s be the number of non-trivial connected components of G_F. We order the non-trivial components $C_1\prec\cdots\prec C_s$ in such way that for any $1\leqslant i<j\leqslant s$ the minimal vertex of $C_i\cup C_j$ is contained in C_i.*

Notation 5.9. We will refer to these ordered non-trivial connected components of G_F as *cycles of G*.

Lemma 5.10. *The right-hand side of equation* (16) *can be expressed as the sum* (21)*, where now the last sum runs over graphs $G \subset \mathbb{K}_k$, which contain subgraphs $G_{\lambda_1}, \dots, G_{\lambda_p}$ satisfying the following four conditions:*

(1) *the set of vertices of G_{λ_i} is $\lambda_i \subset \{1, \dots, k\}$;*
(2) *G_{λ_i} contains a directed cycle;*
(3) *G_{λ_i} is rooted at the smallest vertex in the smallest cycle of G contained in G_{λ_i};*
(4) *all edges in G between the sets of vertices λ_i and λ_j are given in one direction: from λ_i to λ_j, if the smallest cycle in $G_{\lambda_i} \cup G_{\lambda_j}$ is contained in G_{λ_i}.*

Proof. First, note that any cycle C_i of G contains a directed path between any two vertices, hence all vertices of C_i are contained in exactly one subgraph $G_{\lambda_j} \subset G$.

Let G_{λ_i} be a subgraph of G, such that its root vertex, the minimum v_i of λ_i, is not contained in any cycle of G; denote by C^{λ_i} the smallest cycle of G, which is contained in G_{λ_i}. We proceed as described in Lemma 5.7, with the vertices of acyclic graphs replaced by the cycles of G. More precisely, we change the direction of all paths from the vertices of C^{λ_i} to the root vertex v_i; we thus obtain a graph rooted at the smallest vertex of the smallest cycle contained in G_{λ_i}, i.e. satisfying condition (3).

Similarly, to obtain condition (4) we consider the pairs of subgraphs $G_{\lambda_i}, G_{\lambda_j}$ of G, such that the minimum of $\lambda_i \cup \lambda_j$ is not contained in any cycle of G. We pick the smallest cycle of G, which is contained in $G_{\lambda_i} \cup G_{\lambda_j}$ and change the direction of all paths from the vertices of G_{λ_i} to G_{λ_j}, if needed. □

Notation 5.11. We introduce the notation L(16) for the set of subgraphs $G \subset \mathbb{K}_k$ with k vertices that contain an out-edge from each vertex, and R(16) for the set of subgraphs $G \subset \mathbb{K}_k$ satisfying conditions (1)-(4) from Lemma 5.10.

We showed in Lemmas 5.4 and 5.10 that the left-hand and the right-hand sides of equation (16) count graphs (with fixed number of edges) from the sets L(16) and R(16), correspondingly. Hence to prove equation (16), we need to construct a bijection between two sets of graphs: L(16) and R(16).

We start by observing that any graph satisfying conditions (1)-(4) from Lemma 5.10 is clearly a subgraph of $\mathbb{K}_k$ with k vertices, and has an out-edge from each vertex, and thus R(16) is a subset of L(16).

Now let G be an element of L(16). To obtain a map from L(16) to R(16), we need to associate to G a partition $(\lambda_1, \dots, \lambda_p)$ of the set $\{1, \dots, k\}$ and find subgraphs $G_{\lambda_1}, \dots, G_{\lambda_p} \subset G$, satisfying conditions (1)-(4) from Lemma 5.10.

Let $C_1 \prec \dots \prec C_s$ be the cycles of G. We set $\lambda_0 = \varnothing$, and for $j \geqslant 0$ we iterate the following procedure to obtain a partition $(\lambda_1, \dots, \lambda_p)$ of $\{1, \dots, k\}$.

- Let λ_j be a subset of $\{1, \dots, k\}$, such that the set of vertices of each cycle C_i of G is either contained in λ_j or does not intersect λ_j. We introduce the notation $G \setminus \lambda_j$ for the graph obtained from G by deleting all vertices labelled by $\lambda_0 \cup \lambda_1 \cup \dots \cup \lambda_j$ and all edges attached to these vertices. We denote by $C^{\lambda_{j+1}}$ the smallest cycle of G, which is contained in $G \setminus \lambda_j$.
- Let $\lambda_{j+1} \subset \{1, \dots, k\}$ be the set of vertices i of the graph $G \setminus \lambda_j$, such that $C^{\lambda_{j+1}}$ is reachable from i, i.e. there is a directed path in $G \setminus \lambda_j$ from the vertex i to a vertex in a subgraph $C^{\lambda_{j+1}}$.
- As observed above, for any $1 \leqslant i \leqslant s$, the connected graph C_i contains a directed path between any two vertices, hence the set of vertices of C_i is either contained in λ_{j+1} or does not intersect λ_{j+1}. We can thus repeat the procedure, until we arrive at the empty graph $G \setminus \lambda_p$ for some p.

We make the following observation.

Lemma 5.12. *Let $\lambda_1, \dots, \lambda_p$ be subsets of $\{1, \dots, k\}$ associated to a graph G as above. Then $(\lambda_1, \dots, \lambda_p)$ is a partition of the set $\{1, \dots, k\}$.*

Proof. It follows from the construction that all vertices of non-trivial cycles of G are contained in the union $\lambda_1 \cup \dots \cup \lambda_p$. We have to show that if the maximal naked graph $G_F \subset G$ has a trivial connected component which consists of a single vertex v, then v is an element of λ_i for some $1 \leqslant i \leqslant p$.

Recall that G contains an out-edge from the vertex v, and thus there is a directed path from v to some non-trivial cycle C_j of G; then it follows from the construction that v belongs to the same subset $\lambda_i \subset \{1, \dots, k\}$ as all vertices of C_j. ☐

We denote by G_{λ_i} the subgraph of G obtained by taking vertices from $\lambda_i \subset \{1, \dots, k\}$ and all edges between them. Then clearly, the collection of subgraphs $G_{\lambda_1}, \dots, G_{\lambda_p}$ of G satisfy conditions (1)-(4) from Lemma 5.10, and thus we obtain a map from the set L(16) to the set R(16). It is easy to check that this map is a bijection, and thus Theorem 5.3 follows. ☐

6. Multiplication

In this section, we introduce a ring structure on the intersection cohomology of the affine toric variety $X(\mathfrak{A}(r_{12}, \dots, r_{k-1k}), 0)$ induced by the small resolution of singularities.

6.1. Other chambers.

It turns out that we could have carried out the arguments of §4.2-§4.4 for any chamber in the chamber complex $Ch(\mathfrak{A}(r_{12}, \dots, r_{k-1k}))$.

Proposition 6.1. *Let θ be a generic point in $\mathbb{N}\mathfrak{A}(r_{12}, \dots, r_{k-1k})$. Then the morphism of toric varieties*

$$\varphi_\theta : X(\mathfrak{A}(r_{12}, \dots, r_{k-1k}), \theta) \to X(\mathfrak{A}(r_{12}, \dots, r_{k-1k}), 0)$$

is small.

Proof. The proof follows the logic of the proof of Theorem 4.9. We repeat the argument with only minor changes, and thus we omit the details.

Let $\mathfrak{c}$ be the chamber containing θ. As explained in the proof of Lemma 4.13, each cone $\mathrm{Cone}(\alpha^I)$ for a basis index set $I \in \mathrm{BInd}(\mathfrak{A}(r_{12}, \dots, r_{k-1k}), \mathfrak{c})$ correspond to a spanning tree in $\mathbb{K}_k$; we introduce the notation $\mathrm{Trees}(\theta)$ for the set of these trees.

Let F be a face of Π, and let G_F be the corresponding naked graph (cf. Lemma 4.10). Denote by $\mathcal{G}_F^\theta$ the set of connected subgraphs of $\mathbb{K}_k$ with k vertices that have the same directed cycles as the naked graph G_F and that contain at least one tree from $\mathrm{Trees}(\theta)$. Then (cf. Theorem 4.16 and Lemma 4.17) for any point y in $\mathcal{O}(\mathrm{Cone}(F)) \subset X$ the Poincaré polynomial of the fiber $\varphi_\theta^{-1}(y)$ is equal to

$$\sum_{l \geqslant 0} d_l(F)(t^2 - 1)^l,$$

where $d_l(F)$ is the number of graphs in $\mathcal{G}_F^\theta$ with $e(G_F) + s(G_F) - 1 + l$ edges.

Repeating the dimension estimates from the proof of Theorem 4.9, we arrive at our statement. ☐

6.2. A ring structure on the intersection cohomology.

Applying Theorem 3.5 to the small morphism φ_θ from Proposition 6.1, we obtain the following statement.

Corollary 6.2. *Let θ and φ_θ be as in Proposition 6.1. There is an isomorphism*

$$\psi_\theta : H^*(\varphi_\theta^{-1}(0)) \xrightarrow{\sim} IH^*(X(\mathfrak{A}(r_{12}, \dots, r_{k-1k}), 0))$$

between the cohomology groups of the fiber $\varphi_\theta^{-1}(0)$ and the intersection cohomology groups of the affine cone $X(\mathfrak{A}(r_{12},\dots,r_{k-1k}),0)$. The isomorphism ψ_θ induces a ring structure on the intersection cohomology of $X(\mathfrak{A}(r_{12},\dots,r_{k-1k}),0)$.

The cohomology rings $H^*(\varphi_\theta^{-1}(0))$ were studied in [9]. In particular, it was shown that they do not depend on the choice of the elements $\theta\in\mathfrak{a}_k^*$ (cf. Theorem 6.3), and thus the induced ring structure on the intersection cohomology of $X(\mathfrak{A}(r_{12},\dots,r_{k-1k}),0)$ is canonical.

Theorem 6.3 ([9, Theorem 8.3]). *Let θ be a generic point in $\mathbb{N}\mathfrak{A}(r_{12},\dots,r_{k-1k})$. The cohomology ring of the fiber $\varphi_\theta^{-1}(0)$ is canonically isomorphic to the quotient of $\mathbb{C}[\varepsilon_i-\varepsilon_j, 1\leqslant i<j\leqslant k]$ by the ideal generated by the polynomials*

$$p_D(\varepsilon_1,\dots,\varepsilon_k)=\prod_{i\in D_1,j\in D_2}(\varepsilon_i-\varepsilon_j)^{r_{ij}}, \tag{22}$$

where $D=D_1\sqcup D_2$ runs over all nontrivial partitions of the set $\{1,2,\dots,k\}$.

In the remainder of this section, we explain how these relations appear in our graphical formalism.

We begin by describing the reducible variety $\varphi_\theta^{-1}(0)$. We introduce the notation $\mathrm{Top}(\theta)$ for the subset of graphs in $\mathcal{G}_\Pi^\theta$ with $\sum_{i<j}r_{ij}$ edges, i.e. with the maximal possible number of edges. Using Lemma 4.10, we can reformulate Lemma 2.1.11 from [10] in terms of graphs in the following form.

Proposition 6.4. *(i) The fiber of φ_θ over a point $y_\Pi\in X(\mathfrak{A}(r_{12},\dots,r_{k-1k}),0)$ is a connected reducible variety, whose irreducible components are toric varieties parametrized by the elements in* $\mathrm{Top}(\theta)$.

(ii) Let $G\in\mathrm{Top}(\theta)$ and denote the sequence of its edges by $\mathcal{A}_G\subset\mathfrak{A}(r_{12},\dots,r_{k-1k})$. Then the irreducible component of $\varphi_\theta^{-1}(0)$ associated to G is the toric variety $X(\mathcal{A}_G,\theta)$, which has dimension $\sum_{i<j}r_{ij}-k+1$.

(iii) For $G_1,\dots,G_m\in\mathrm{Top}(\theta)$, denote by $G_1\cap\dots\cap G_m$ the graph obtained by taking all common edges of $G_1,\dots,G_m$. Then the irreducible components $X(\mathcal{A}_{G_1},\theta),\dots,X(\mathcal{A}_{G_m},\theta)$ are glued along the toric variety $X(\mathcal{A}_{G_1\cap\dots\cap G_m},\theta)$.

Remark 6.5. Unlike their cohomology rings, the varieties $\varphi_\theta^{-1}(0)$ are not necessarily isomorphic for different θs. For example, let $\theta_1=(3,-1,-1,-1)$ and $\theta_{\mathrm{rand}}=(2,-3,2,-1)$ be two points in the chamber complex $Ch(\mathfrak{A}(1,1,1,1,1,1))$. A simple calculation shows that $\varphi_{\theta_1}^{-1}(0)$ has 6 irreducible components that have Poincaré polynomials $1+2t^2+2t^4+t^6$, while $\varphi_{\theta_{\mathrm{rand}}}^{-1}(0)$ has 6 irreducible components, two of which are isomorphic to $\mathbb{P}^3$.

To describe the ring structure on $H^*(\varphi_\theta^{-1}(0))$, we need one more ingredient: a general statement about relations in the cohomology ring of compact toric varieties. These are well-known; we will present them in the formalism of [17]. We will use the notation of §2.1-§2.2.

Let θ be a weight in $\Gamma_{\mathfrak{a}}^*$. By the Chern-Weil construction, every polynomial on $\mathfrak{a}$ gives rise to a characteristic class of the toric variety $X(\mathfrak{A},\theta)$. Thus there is a Chern-Weil homomorphism $\chi:\mathrm{Sym}(\mathfrak{a}^*)\to H^*(X(\mathfrak{A},\theta))$ from the polynomials on $\mathfrak{a}$ to the cohomology of $X(\mathfrak{A},\theta)$. In particular, for any vector $\alpha\in\Gamma_{\mathfrak{a}}^*$, $\chi(\alpha)$ is an element of $H^2(X(\mathfrak{A},\theta))$.

Now let $\mathfrak{A}$ be a sequence which lies in an open half space of the vector space $\mathfrak{a}^*$, and assume that θ is generic. Then $X(\mathfrak{A},\theta)$ is a projective orbifold, and for every polynomial $Q\in\mathrm{Sym}(\mathfrak{a}^*)$, one can write explicit formulas for the intersection numbers $\int_{X(\mathfrak{A},\theta)}\chi(Q)$ using the Jeffrey-Kirwan residue [2, 17]. In this paper, we will only need to integrate some particularly simple classes, described below.

Proposition 6.6 ([17, Proposition 2.3]). *Let $\mathfrak{A}$ be a sequence which lies in an open half space of $\mathfrak{a}^*$, and let θ be a generic point in $\Gamma^*_{\mathfrak{a}}$. Then, for any set of indices $J \subset \{1, \ldots, n\}^{n-d}$ we have*

$$\int_{X(\mathfrak{A},\theta)} \prod_{j \in J} \chi(\alpha_j) = 0,$$

if the complement of $\{\alpha_j\}_{j \in J}$ in $\mathfrak{A}$ does not span $\mathfrak{a}^*$.

Now we apply this statement to our situation. Let $\theta \in \Gamma^*_{\mathfrak{a}}$ be generic, $G \in \mathrm{Top}(\theta)$ and $\mathcal{A}_G \subset \mathfrak{A}(r_{12}, \ldots, r_{k-1k})$ be as in Proposition 6.4. Given a nontrivial partition $D = D_1 \sqcup D_2$ of the set $\{1, 2, \ldots, k\}$, denote by $\mathcal{A}[D]$ the subsequence $[\alpha_{ij}, \alpha_{ji}]_{i \in D_1, j \in D_2} \subset \mathfrak{A}(r_{12}, \ldots, r_{k-1k})$, where the element $\alpha_{ij} \in \mathcal{A}[D]$ is repeated r_{ij} times.

Clearly, the sequence $\mathcal{A}_G \cap ((\mathfrak{A}(r_{12}, \ldots, r_{k-1k}) \setminus \mathcal{A}[D])$ corresponds to a subgraph of $\mathbb{K}_k$, which doest not contain any tree on k vertices. Then it follows from Proposition 6.6 that the product

$$\prod_{\substack{i \in D_1 \\ j \in D_2}} \chi(\alpha_{ij})^{r_{ij}} \in H^*(X(\mathcal{A}_G, \theta))$$

is zero for any $G \in \mathrm{Top}(\theta)$, hence it is equal to zero in $H^*(\varphi_\theta^{-1}(0))$. We have thus reproduced the Hausel-Sturmfels relations given in (22).

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References

[1] Alexander Beilinson, Joseph Bernstein, and Pierre Deligne, *Faisceaux pervers*, in Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.

[2] Michel Brion and Michèle Vergne, *Arrangement of hyperplanes. I. Rational functions and Jeffrey-Kirwan residue*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 5, 715–741.

[3] Mark Andrea de Cataldo and Luca Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 4, 535–633.

[4] Mark Andrea de Cataldo, Luca Migliorini, and Mircea Mustaţă, *Combinatorics and topology of proper toric maps*, J. Reine Angew. Math. **744** (2018), 133–163.

[5] Camilla Felisetti, Andras Szenes, and Olga Trapeznikova, *Parabolic bundles and the intersection cohomology of moduli spaces of vector bundles on curves*, 2025, `https://arxiv.org/abs/2502.20327`.

[6] Karl-Heinz Fieseler, *Rational intersection cohomology of projective toric varieties*, J. Reine Angew. Math. **413** (1991), 88–98.

[7] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993.

[8] Mark Goresky and Robert MacPherson, *Intersection homology theory*, Topology **19** (1980), no. 2, 135–162.

[9] Tamás Hausel and Bernd Sturmfels, *Toric hyperKähler varieties*, Doc. Math. **7** (2002), 495–534.

[10] Yi Hu, Chien-Hao Liu, and Shing-Tung Yau, *Toric morphisms and fibrations of toric Calabi-Yau hypersurfaces*, Adv. Theor. Math. Phys. **6** (2002), no. 3, 457–506.

[11] Mirko Mauri, *Intersection cohomology of rank 2 character varieties of surface groups*, J. Inst. Math. Jussieu **22** (2023), no. 4, 1615–1654.

[12] Mirko Mauri and Luca Migliorini, *Hodge-to-singular correspondence for reduced curves*, J. Eur. Math. Soc. (JEMS) (2024), published online first.

[13] Mark McConnell, *The rational homology of toric varieties is not a combinatorial invariant*, Proc. Amer. Math. Soc. **105** (1989), no. 4, 986–991.

[14] Sven Meinhardt and Markus Reineke, *Donaldson-Thomas invariants versus intersection cohomology of quiver moduli*, J. Reine Angew. Math. **754** (2019), 143–178.

[15] Sergey Mozgovoy and Markus Reineke, *Intersection cohomology of moduli spaces of vector bundles over curves*, 2024, `https://arxiv.org/abs/1512.04076`.

[16] Richard Stanley, *Generalized H-vectors, intersection cohomology of toric varieties, and related results*, in Commutative algebra and combinatorics (Kyoto, 1985), Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 187–213.

[17] András Szenes and Michèle Vergne, *Toric reduction and a conjecture of Batyrev and Materov*, Invent. Math. **158** (2004), no. 3, 453–495.

[18] Hannah Winkler, *Triangulations of Gale duals of root polytopes*, Ph.D. thesis, San Francisco State University, 2014.

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