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On the automorphism group of a putative Conway 99-graph

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Abstract Let Γ be a Conway 99-graph, that is, a strongly regular graph with parameters $(99, 14, 1, 2)$. Existence of such a graph remains an elusive open problem, however various authors have made significant contributions by analyzing the structure of the automorphism group $G = \mathrm{Aut}(\Gamma)$. In this paper we duplicate many results of our predecessors (e.g. Behbahani & Lam, Crnković & Maksimović), but crucially, we accomplish this without the aid of a computer. Specifically, we give computer-free proofs that divisibility of $|G|$ by 2 implies $|G|$ divides 6 while divisibility of $|G|$ by 7 implies $G \cong \mathbb{Z}_7$.

1. Introduction

The question of existence of a strongly regular graph with parameters $(99, 14, 1, 2)$ is a longstanding open problem. Its possible existence was first suggested by Norman Biggs in 1969 [2]. According to the account given by Richard Guy in [9], John H. Conway worked on this problem as early as 1975. Later, Conway would offer a \$1000 prize to anyone who could solve it (see [6], where it is listed as problem 2 among five posed open problems). From that point onward, a strongly regular graph having these parameters came to be known colloquially as a *Conway 99-graph*.

In [6] Conway gave an alternate formulation of the existence problem, which we here reproduce:

> Is there a graph with 99 vertices in which every edge (i.e. pair of joined vertices) belongs to a unique triangle and every nonedge (pair of unjoined vertices) to a unique quadrilateral?

Famously, A.A. Makhnev and I.M. Minakova proved in [12] that a strongly regular graph with parameters $(v, k, 1, 2)$ can only exist if $k = u^2 + u + 2$ where $u \in \{1, 3, 4, 10, 31\}$. Such graphs are known to exist for $u \in \{1, 4\}$ but not much is known about the remaining cases. Should a Conway 99-graph exist, it would correspond to the case $u = 3$.

In [12, Theorem 2.7] it is asserted that the order of the automorphism group G of a Conway 99-graph must divide $2 \cdot 3^3 \cdot 7 \cdot 11$. In [11, Corollary 2.4] Makhnev asserts the following:

If G contains an involution t, then one of the following holds:

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(i) If $|G|$ is divided by 7 then $|G|$ divides 42, $[O(G), t] = 1$, and in the case $|G| = 42$ the subgroup $O(G)$ is nonabelian. (Here $O(G)$ denotes the maximal odd-order normal subgroup of G.)
(ii) $|G|$ divides 6.

Below we summarize results of various authors on the automorphism group of a Conway-99 graph that were obtained since publication of [11, 12] and which were brought to our attention by an anonymous referee (see [5, p.16]):

(a) In [1] Behbahani & Lam showed that any automorphism of prime order must have order 2 or 3.
(b) In [7] Crnković & Maksimović ruled out groups of order 6 or 9.

In the present paper, we prove the following:

(1) If 2 divides $|G|$, then $|G|$ divides 6.
(2) If $|G|$ is divisible by 7, then G is isomorphic to $\mathbb{Z}_7$.

Importantly, we stress that results (1) and (2) are obtained without the aid of a computer. To the best of our knowledge, this has not been accomplished in any previous published work on the problem.

Our choice of notation and terminology are standard. We refer the reader to [3, 4, 5, 8, 13] as excellent sources of background material.

Our paper is organized as follows. In Section 2 we establish a labeling scheme for the vertices of a putative Conway 99-graph Γ under the assumption that 7 divides $|G|$. This allows us to embed G in the symmetric group of degree 14.

In Section 3 we prove that Γ admits no order 14 automorphisms. This is a crucial step toward showing that divisibility by 2 implies $|G|$ divides 6.

In Section 4 we show that divisibility by 7 implies G is isomorphic to either $\mathbb{Z}_7$ or $Frob(21)$ (the Frobenius group of order 21). Assuming the latter, we are next able to construct a unique feasible G-orbit partition of $V(\Gamma)$ which we subsequently show cannot exist. As a result, divisibility by 7 implies $G \cong \mathbb{Z}_7$.

2. Preliminaries

Throughout this paper, Γ will denote a putative Conway 99-graph, i.e. a strongly regular graph with parameters $(99, 14, 1, 2)$. We denote its automorphism group by G. Although Γ is not vertex transitive, a result first proved by Wilbrink [14], one can always "hang" the graph from any vertex $x \in V(\Gamma)$ whereby vertices are grouped together in accordance with their distance from x. This is a property of distance-regular graphs in general and strongly regular graphs in particular. In such case, we refer to x as the "root vertex".

As is customary, we denote by $\Gamma_1(x)$ and $\Gamma_2(x)$ the set of neighbors and non-neighbors of x respectively, commonly referred to as the first and second subconstituents of Γ. When x is understood from context, we will abbreviate these sets by Γ_1 and Γ_2. In Figure 1 we depict the distance distribution diagram for Γ. Note that the diagram indicates that $|\Gamma_1| = 14$ and $|\Gamma_2| = 84$.

For the moment we assume $x \in V(\Gamma)$ is an arbitrary vertex, however in due course our choice of x will carry special significance. We label the 14 neighbors of x by iX, $1 \leqslant i \leqslant 7$, $X \in \{L, R\}$, see Figure 2. Since $\lambda = 1$, we may assume without loss of generality that $\{iL, iR\}$ is an edge for every $1 \leqslant i \leqslant 7$.

Observe that for any $\{X, Y\} = \{L, R\}$, the vertices $iX, jY \in \Gamma_1$ are adjacent if and only if $i = j$. As $\mu = 2$, each pair of nonadjacent vertices $iX, jY \in \Gamma_1$ must have a unique common neighbor in Γ_2. We label this common neighbor $ijXY$. As there are $\binom{14}{2} - 7 = 84$ pairs of nonadjacent vertices in Γ_1, we see that each of the 84 vertices in Γ_2 receives a unique label, again due to $\mu = 2$.

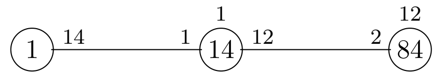

Figure 1. Distance distribution diagram of Γ

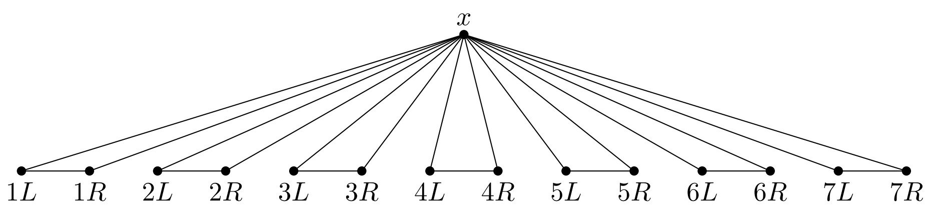

Figure 2. A labeling scheme for Γ_1

Lemma 2.1. *Suppose $g \in G$ fixes $\Gamma_1 \cup \{x\}$ pointwise. Then g fixes Γ_2 pointwise, i.e. g is the identity automorphism.*

Proof. Evident from our labeling scheme and the fact that $\mu = 2$. □

Lemma 2.2. *Suppose there exists $s \in G$ with $|s| = 7$. Then s fixes a unique vertex in $V(\Gamma)$, hence the $\langle s\rangle$-orbit structure on $V(\Gamma)$ is $[1, 7^{14}]$ (i.e. one orbit of size 1 and 14 orbits of size 7).*

Proof. As $|V(\Gamma)| = 99 \equiv 1 \pmod 7$ we deduce that s must have at least one fixed vertex which we may choose to fulfill the role of root vertex x of Γ. This establishes that both Γ_1 and Γ_2 are s-invariant. In particular, the orbit structure of $\langle s\rangle$ on Γ_1 is $[7^2]$, $[1^7, 7]$, or $[1^{14}]$. As iL is a fixed point of s if and only if iR is, the number of fixed points must be even, i.e. the $\langle s\rangle$-orbit structure is either $[7^2]$ or $[1^{14}]$. However, Lemma 2.1 rules out $[1^{14}]$, leaving $[7^2]$ as the orbit structure on Γ_1. Now suppose a vertex $ijXY \in \Gamma_2$ is fixed by s. As $\mu = 2$ and $|s|$ is odd, s must fix the μ-graph $\Gamma_1(x) \cap \Gamma_1(ijXY) = \{iX, jY\}$ vertex-wise, a contradiction. This proves the $\langle s\rangle$-orbit structure on Γ_2 is $[7^{12}]$. The result follows. □

Remark 2.3. Note that one consequence of our labeling scheme, together with the assumption in Lemma 2.2, is that the automorphism group $\mathrm{Aut}(\Gamma)$ embeds in $\mathrm{Sym}(\Gamma_1) \cong S_{14}$. By relabeling the vertices of Γ_1, we may assume that $s = s_L s_R$ where $s_L = (1L, 2L, \ldots, 7L)$ and $s_R = (1R, 2R, \ldots, 7R)$. The two $\langle s\rangle$-orbits on Γ_1 are now transparent. They are $\{1L, 2L, \ldots, 7L\}$ and $\{1R, 2R, \ldots, 7R\}$.

Lemma 2.4. *Every vertex of Γ_2 lies on five 3-cycles wholly inside Γ_2. Thus there are 140 3-cycles in Γ_2.*

Proof. Clearly, every vertex in Γ lies on seven 3-cycles. Given a vertex $v \in \Gamma_2$ it has precisely two Γ_1-neighbors u and w. As $\lambda = 1$, each of vu and vw must be an edge in a unique 3-cycle. Moreover, these two 3-cycles cannot coincide. Indeed, this would require that u and w be adjacent, whence uwx would be a second 3-cycle on the edge uw where x is the root vertex. This proves the remaining five 3-cycles on v lie entirely inside Γ_2. But now we have that the total number of 3-cycles in Γ_2 is $\frac{84\cdot 5}{3} = 140$ as claimed. □

3. Consequences of divisibility by 2

In this section, we prove $|G|$ must divide 6 under the assumption that $|G|$ is even. A critical role here is played by Theorem 3.11 in which we prove that G cannot have any order 14 automorphisms.

Lemmas 3.1 and 3.2 set the groundwork for the rest of this section.

Lemma 3.1. *Suppose G contains a cyclic subgroup K of order 14. Then $K = \langle st\rangle$ where $s = (1L, 2L, \dots, 7L)\,(1R, 2R, \dots, 7R)$ and $t = (1L, 1R)\,(2L, 2R)\,\dots\,(7L, 7R)$.*

Proof. Let $C_S(s) = (\langle s_L\rangle \times \langle s_R\rangle) \rtimes \langle t\rangle$ denote the centralizer of s in $S = \mathrm{Sym}(\Gamma_1)$. There are seven involutions in $C_S(s)$ which take the form

$$t^{(s_R)^i} = (1L, (1+i)R)\,(2L, (2+i)R)\dots(7L, iR),$$

$0 \leqslant i \leqslant 6$, and in each case $\langle st^{(s_R)^i}\rangle$ is a cyclic subgroup of $C_S(s)$ of order 14. However, since $t^{(s_R)^i}$ maps $1L$ to $(1+i)R$ and $1R$ to $(1-i)L$, adjacency is preserved only if $i = 0$. Thus t is the unique involution in $\mathrm{Aut}(\Gamma_1)$ whereby $K = \langle st\rangle$ is the unique cyclic subgroup of order 14 in $\mathrm{Aut}(\Gamma_1)$. □

Lemma 3.2. *$K = \langle st\rangle$ fixes the root vertex x of Γ but has no other fixed points. Thus each of the remaining seven K-orbits on Γ has size 14.*

Proof. Recall from Lemma 2.2 that x is the unique vertex fixed by $s \in K$. As t commutes with s we have that x^t is fixed by s, whence $x^t = x$. Thus x is the unique vertex fixed by K.

We now consider K-orbits on $V(\Gamma) \setminus \{x\}$. Clearly, K fuses the two $\langle s\rangle$-orbits in Γ_1 so we are left to consider the orbit structure of K on Γ_2. Suppose there exists an $\langle s\rangle$-orbit $\mathcal{O}$ that is K-invariant. Then as $|t| = 2$ and $|\mathcal{O}| = 7$, there must be a vertex $v \in \mathcal{O}$ fixed by t. Let iX be a Γ_1-neighbor of v. Then $iY = (iX)^t$ must also be a Γ_1-neighbor of v. But this violates $\lambda = 1$ as iX and iY are adjacent. We conclude that all K-orbits of Γ_2 have size 14. □

Remark 3.3. It is easy to see that a set of orbit representatives in the action of K on Γ_2 is given by $\{12LL,\ 13LL,\ 14LL,\ 12LR,\ 13LR,\ 14LR\}$. Moreover, each numerical coordinate i occurs exactly four times in each orbit. For example, in the orbit with representative $12LL$, the coordinate 3 occurs in each of $23LL$, $34LL$, $23RR$, $34RR$. However, these four vertices are distributed evenly into pairs in the sense that $23LL$, $34LL$ are neighbors of $3L$ while $23RR$, $34RR$ are neighbors of $3R$.

Consider the equitable partition π induced by the K-orbits $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_6$ on Γ_2. As is customary, we refer to π as a *K-orbit partition.* We denote by b_{ij} the number of $\mathcal{O}_j$-neighbors of any fixed vertex in $\mathcal{O}_i$ and refer to it as an *orbit valency* (or simply *valency*) due to what occurs naturally in the quotient graph Γ_2/π. Orbit valencies for which $i = j$ will be called *internal* and expressed as b_i rather than b_{ii}. All other valencies will be referred to as *external.* Pictorially, b_{ij} appears as a label of an arc from $\mathcal{O}_i$ to O_j, however in our case this arc is an edge (i.e. $b_{ij} = b_{ji}$ for all i, j) since all orbits $\mathcal{O}_i$ have the same size.

At present we have that st is an order 14 automorphism of the subgraph of Γ induced on $\Gamma_1 \cup \{x\}$. We wish to show st cannot extend to an automorphism of Γ. Our first step toward this objective is to count in two ways the cardinality of the set

$$S = \{uvw : uvw \text{ is a 2-path with } w \in \mathcal{O}_i\},$$

where u is a fixed vertex in $\mathcal{O}_i$ (see Figure 3).

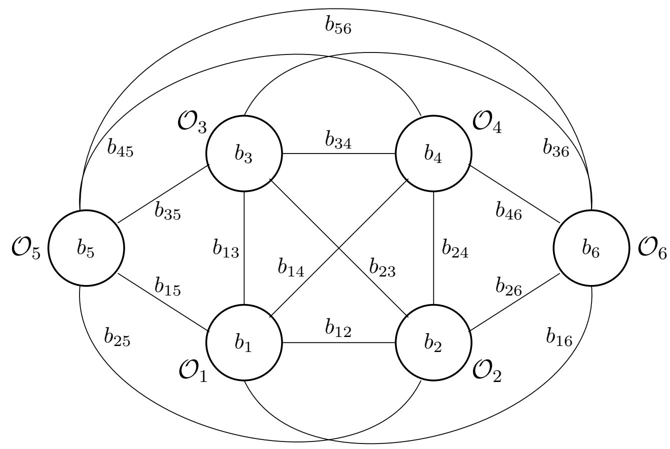

FIGURE 3. A general K-orbit partition on Γ_2 where $K = \langle st \rangle \cong \mathbb{Z}_{14}$

For each of the b_i neighbors of u in $\mathcal{O}_i$ there is a unique 2-path from u to w (since $\lambda = 1$). Similarly, for each of the $13 - b_i$ non-neighbors of u in $\mathcal{O}_i$ there are two 2-paths from u to w (since $\mu = 2$). Thus $|S| = b_i \cdot 1 + (13 - b_i) \cdot 2 = 26 - b_i$.

On the other hand, we may condition our count on the location of the intermediate vertex v. For v in $\mathcal{O}_j$ there are $b_{ij}\,(b_{ij} - 1)$ such 2-paths. In addition, u has exactly two neighbors in Γ_1 each of which has a unique neighbor $w \in \mathcal{O}_i \setminus \{u\}$ (cf. Remark 3.3). This gives $|S| = \sum_{j=1}^{6} b_{ij}\,(b_{ij} - 1) + 2$. Equating these two expressions for $|S|$, we obtain $26 - b_i = \sum_{j=1}^{6} b_{ij}\,(b_{ij} - 1) + 2$. But due to the fact that $\sum_{j=1}^{6} b_{ij} = 12$, this simplifies to

$$36 - \left(b_i^2 + b_i\right) = \sum_{j \neq i} b_{ij}^2 \tag{1}$$

We divide our analysis into cases based on an assumed value for b_i. Once a choice of b_i is made, we find all ways of expressing $36 - \left(b_i^2 + b_i\right)$ as a sum of five squares while maintaining the valency requirement $\sum_{j=1}^{6} b_{ij} = 12$. Note that $b_i \leqslant 5$ since otherwise the value of $36 - \left(b_i^2 + b_i\right)$ would be negative.

All solution sets are provided in the lemma below. Verification of the list is straightforward, so is left to the reader. Note that each solution set is expressed as an ordered pair of the form

$$\left(b_i, \left\{a_2, a_3, a_4, a_5, a_6\right\}\right).$$

This is because unlike the internal valency b_i which remains fixed, the values $a_2, a_3, \ldots, a_6$ may be assigned to the external valencies b_{ij} in any specified manner. Thus, there are multiple solutions corresponding to each solution set achieved by suitably permuting the members of the multiset $\left\{a_2, a_3, a_4, a_5, a_6\right\}$. Below, we list these members in decreasing order.

LEMMA 3.4. *For $b_i \in \{1, 3, 5\}$ there are no solutions to equation* 1. *For other values of b_i the solutions are listed as follows:*

(a) $\left(0, \left\{4, 3, 3, 1, 1\right\}\right)$ *and* $\left(0, \left\{3, 3, 3, 3, 0\right\}\right)$ *when* $b_i = 0$.
(b) $\left(2, \left\{4, 3, 2, 1, 0\right\}\right)$ *when* $b_i = 2$,
(c) $\left(4, \left\{3, 2, 1, 1, 1\right\}\right)$ *and* $\left(4, \left\{2, 2, 2, 2, 0\right\}\right)$ *when* $b_i = 4$.

For future reference, it is convenient to designate these solution sets by type, e.g.

$$\text{I. } \big(0,\{4,3,3,1,1\}\big), \text{II. } \big(0,\{3,3,3,3,0\}\big), \text{III. } \big(2,\{4,3,2,1,0\}\big)$$
$$\text{IV. } \big(4,\{3,2,1,1,1\}\big), \text{V. } \big(4,\{2,2,2,2,0\}\big)$$

We also extend this terminology to orbits, saying an orbit is *of type* T if its set of valencies correspond to a solution set of type $\mathrm{T} \in \{\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV},\mathrm{V}\}$.

We next count in two ways the number of 2-paths starting from a fixed vertex u in $\mathcal{O}_i$ and ending at some vertex w in $\mathcal{O}_j$, $j \neq i$. Here u has exactly b_{ij} neighbors in $\mathcal{O}_j$, and as $\lambda = 1$ there exists a unique 2-path starting at u and ending at w for each neighbor w of u in $\mathcal{O}_j$. Similarly, u has $14-b_{ij}$ non-neighbors in $\mathcal{O}_j$, and as $\mu = 2$ there are exactly two 2-paths starting at u and ending at each non-neighbor w of u in $\mathcal{O}_j$. Thus in total there are $b_{ij}\cdot 1 + (14-b_{ij})\cdot 2 = 28 - b_{ij}$ such 2-paths from u into $\mathcal{O}_j$ when $j \neq i$.

For the second count, we focus on the location of an intermediate vertex v in each such 2-path. Here v can occur in any of the six K-orbits on Γ_2 as well as in Γ_1. In the case of K-orbits on Γ_2, there are b_{ik} choices for v in $\mathcal{O}_k$, and for each such v there are b_{kj} choices for w in $\mathcal{O}_j$. This gives $b_{ik}b_{kj}$ 2-paths of desired type. In addition, there are two choices for v in Γ_1 each of which has two neighbors w in $\mathcal{O}_j$. This produces four more 2-paths. Thus, in total there are precisely $\sum_{k=1}^{6} b_{ik}b_{kj} + 4$ paths of the type in question when $j \neq i$.

Equating these two counts yields $28 - b_{ij} = \sum_{k=1}^{6} b_{ik}b_{kj} + 4$, or equivalently

$$24 - b_{ij} = \sum_{k=1}^{6} b_{ik}b_{kj}. \tag{2}$$

Lemma 3.5. *In a K-orbit partition of Γ_2 (cf. Figure 3) we have the following:*

(a) *The number of orbits of type I is at most* 2.
(b) *The number of orbits of type II is at most* 1.
(c) *The number of orbits of type III is at most* 4.
(d) *The number of orbits of type IV is at most* 4.
(e) *The number of orbits of type V is at most* 1.

Proof. (a): Suppose there exist two orbits $\mathcal{O}_i$ and $\mathcal{O}_j$ of type I. Then since $b_i = b_j = 0$, equation 2 reduces to $24 - b_{ij} = \sum_{k\neq i,j} b_{ik}b_{kj}$. Note that this equation is satisfied only if $b_{ij} = 4$, which results in the solution $20 = 3^2+3^2+1^2+1^2$. As an orbit of type I has only one external valency 4, there cannot be a third orbit of this type.

(b): Let $\mathcal{O}_i$ and $\mathcal{O}_j$ be two orbits of type II. Since $b_i = b_j = 0$, equation 2 again reduces to $24 - b_{ij} = \sum_{k\neq i,j} b_{ik}b_{kj}$. But regardless of how one chooses $b_{ij} \in \{0,3\}$ and reorders the corresponding multisets, this equation is never satisfied. Thus there is at most one orbit of type II.

(c): Let $\mathcal{O}_i$ and $\mathcal{O}_j$ be two orbits of type III. Since $b_i = b_j = 2$, equation 2 becomes $24 - 5b_{ij} = \sum_{k\neq i,j} b_{ik}b_{kj}$. In this case one has $b_{ij} \in \{0,1,2,3,4\}$. There are no solutions if $b_{ij} \in \{1,2\}$, however every remaining choice of b_{ij} works. Specifically, if $b_{ij} = 0$ one gets $24 = 4\cdot 1 + 3\cdot 4 + 2\cdot 3 + 1\cdot 2$ as a solution. For $b_{ij} = 3$ one gets $9 = 4\cdot 0 + 2\cdot 4 + 1\cdot 1 + 0\cdot 2$, while for $b_{ij} = 4$ one gets $4 = 3\cdot 0 + 2\cdot 1 + 1\cdot 2 + 0\cdot 3$. Having only three allowable external valencies adjoining pairs of type III orbits, it is not possible to have a fifth orbit of this type.

(d): Given any two orbits $\mathcal{O}_i$ and $\mathcal{O}_j$ of type IV, we have that $b_{ij} \in \{1,2,3\}$. As $b_1 = b_2 = 4$, equation 2 becomes $24 - 9b_{ij} = \sum_{k\neq i,j} b_{ik}b_{kj}$. Clearly $b_{ij} = 1$ leads to a solution, namely $15 = 1^2+1^2+2^2+3^2$, but other choices of b_{ij} fail. Since a type IV orbit has only three available external valencies equal to 1, there can be at most four orbits of this type.

	I	II	III	IV	V
I	2	0	4	0	0
II	0	1	4	4	1
III	2	1	4	2	1
IV	0	1	4	4	1
V	0	1	4	4	1

TABLE 1. Bounds on the number of orbits of mixed type

(e): Finally, suppose there are two orbits $\mathcal{O}_i$ and $\mathcal{O}_j$ of type V. Then since $b_i = b_j = 4$, equation 2 becomes $24 - 9b_{ij} = \sum_{k\neq i,j} b_{ik}b_{kj}$. Here $b_{ij} \in \{0, 2\}$ and neither choice leads to a solution. This proves there is at most one orbit of type V. □

In the above, we applied equation 2 to bound the number of orbits of a given type that can occur in a K-orbit partition of Γ_2. We now do the same for orbits of mixed type. Note that we do not strive to obtain sharp bounds at this stage. Our goal is simply to eliminate several possibilities in an expedient manner.

LEMMA 3.6. *Let* $\mathrm{T_i}, \mathrm{T_j} \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}\}$. *Then the* $(\mathrm{T_i}, \mathrm{T_j})$*-entry in Table* 1 *bounds from above the number of orbits of type* $\mathrm{T_j}$ *that can coexist with a fixed orbit of type* $\mathrm{T_i}$ *in a* K*-orbit partition of* Γ_2.

Proof. Note that the diagonal entries in Table 1 were previously confirmed in Lemma 3.5. Moreover, one need not check any entry $(\mathrm{T_i}, \mathrm{T_j})$ that is equal to the diagonal entry $(\mathrm{T_j}, \mathrm{T_j})$ since the latter is the maximum allowable number of orbits of type $\mathrm{T_j}$ in any K-orbit partition of Γ_2.

Case 1. (I, II) = (II, I) = 0 : Clearly, the only option for the shared valency is $b_{ij} = 3$. As $b_i = b_j = 0$, equation 2 reduces to $21 = \sum_{k\neq i,j} b_{ik}b_{kj}$ where $b_{ik} \in \{4, 3, 1, 1\}$ and $b_{kj} \in \{3, 3, 3, 0\}$. It is easy to see that no permutation of multisets leads to a solution, i.e. (I, II) = 0. (For (II, I) = 0, the only change to the above is $b_{ik} \in \{3, 3, 3, 0\}$ and $b_{kj} \in \{4, 3, 1, 1\}$.)

Case 2. (I, IV) = (IV, I) = 0 : In this case $b_i = 0$ and $b_j = 4$, so equation 2 reduces to $21 - 5b_{ij} = \sum_{k\neq i,j} b_{ik}b_{kj}$. Here there are two options for b_{ij}. If $b_{ij} = 1$ then we get $16 = \sum_{k\neq i,j} b_{ik}b_{kj}$ where $b_{ik} \in \{4, 3, 3, 1\}$ and $b_{kj} \in \{3, 2, 1, 1\}$ and no permutation of multisets leads to a solution. For the second option $b_{ij} = 3$, we get $6 = \sum_{k\neq i,j} b_{ik}b_{kj}$ where $b_{ik} \in \{4, 3, 1, 1\}$ and $b_{kj} \in \{2, 1, 1, 1\}$. Again no permutation of multisets gives a solution. Thus (I, IV) = (IV, I) = 0.

Case 3. (I, V) = (V, I) = 0 : Here the multisets $\{4, 3, 3, 1, 1\}$ and $\{2, 2, 2, 2, 0\}$ are disjoint, so there is no possible valency that can adjoin two orbits of these respective types.

Case 4. (III, IV) = 2 : In this case, $b_i = 2$ and $b_j = 4$ so equation 2 becomes $24 - 7b_{ij} = \sum_{k\neq i,j} b_{ik}b_{kj}$. There are three choices for b_{ij}, namely $b_{ij} \in \{3, 2, 1\}$. If $b_{ij} = 3$ the equation reduces to $3 = \sum_{k\neq i,j} b_{ik}b_{kj}$ where $b_{ik} \in \{4, 2, 1, 0\}$ and $b_{kj} \in \{2, 1, 1, 1\}$, and it is immediate that there is no solution. If $b_{ij} = 2$ we obtain $10 = \sum_{k\neq i,j} b_{ik}b_{kj}$ where $b_{ik} \in \{4, 3, 1, 0\}$ and $b_{kj} \in \{3, 1, 1, 1\}$. Here there is a unique solution, namely $10 = 4\cdot 1 + 3\cdot 1 + 1\cdot 3 + 0\cdot 1$. Finally, if $b_{ij} = 1$ we obtain $17 = \sum_{k\neq i,j} b_{ik}b_{kj}$ where $b_{ik} \in \{4, 3, 2, 0\}$ and $b_{kj} \in \{3, 2, 1, 1\}$. In this case there are three solutions, namely $17 = 4\cdot3+3\cdot1+2\cdot1+0\cdot2 = 4\cdot1+3\cdot3+2\cdot2+0\cdot1 = 4\cdot2+3\cdot1+2\cdot3+0\cdot1$. In any case, there are just two possibilities for the valency adjoining an orbit of fixed type III to an orbit of type IV. We conclude that (III, IV) = 2.

As all cases in the lemma statement have been treated, the proof is complete. □

The reader will note that the relation in Lemma 3.6 is not generally symmetric.

Let us write $[\mathrm{I}^{\mathrm{a}}, \mathrm{II}^{\mathrm{b}}, \mathrm{III}^{\mathrm{c}}, \mathrm{IV}^{\mathrm{d}}, \mathrm{V}^{\mathrm{e}}]$ to indicate a K-orbit partition of Γ_2 having a orbits of type I, b orbits of type II, and so on. (If an orbit of specific type does not occur in the partition, we simply omit that type from the above partition notation.)

Lemma 3.7. *There is no K-orbit partition of the form* $[\mathrm{III}^{\mathrm{c}}, \mathrm{IV}^{6-\mathrm{c}}]$ *for any c.*

Proof. By Lemma 3.5, one has $2 \leqslant c \leqslant 4$. As a type III orbit has a single valency of 4 and a type IV orbit has none, there are $c/2$ external valencies of 4 in the partition. This means c must be even. However, by Lemma 3.6 the existence of a type III orbit requires that there be at most two type IV orbits. This implies $6-c \leqslant 2$ whence $c=2$ is prohibited. Thus $c=4$.

As shown in the proof of Lemma 3.5, each pair of type III orbits must share an external valency of 0, 3 or 4. But as there are four type III orbits, every such valency gets used. On the other hand, we showed in Lemma 3.5 that two orbits of type IV must share an external valency of 1. This leaves a type IV orbit with an unusable valency 3, again a contradiction. □

One conclusion of Lemma 3.7 is that a viable K-orbit partition of Γ_2 must contain an orbit of type I, II or V. By way of the next two lemmas, we narrow this down considerably.

Lemma 3.8. *There is no K-orbit partition that contains an orbit of type* I.

Proof. By Lemma 3.6, the only possible orbit partition containing a type I orbit is $[\mathrm{I}^2, \mathrm{III}^4]$. However, we have seen that no pair of type III orbits can share an external valency of 2 (cf. proof of Lemma 3.5(c)). As no type I orbit admits such a valency, we see that each orbit of type III has an unusable external valency of 2. The result follows. □

Lemma 3.9. *A K-orbit partition of Γ_2 must be of the form* $[\mathrm{II}, \mathrm{IV}^4, \mathrm{V}]$.

Proof. By Lemmas 3.7 and 3.8, a K-orbit partition must contain at least one orbit of type II or V. Suppose there exists a type II orbit in the partition. Then by Lemma 3.5 there cannot be a second orbit of type II, nor can there be more than one orbit of type V. Thus the partition is of the form $[\mathrm{II}, \mathrm{III}^{\mathrm{c}}, \mathrm{IV}^{\mathrm{d}}, \mathrm{V}^{\mathrm{e}}]$ where $c+d+e=5$ and $0 \leqslant e \leqslant 1$. Also, as in the proof of Lemma 3.7, c must be even.

Suppose first that $c=2$. Then by Lemma 3.6 we have $d \leqslant 2$, whence the partition must be of the form $[\mathrm{II}, \mathrm{III}^2, \mathrm{IV}^2, \mathrm{V}]$. In the proof of Lemma 3.6(d), we saw that any two orbits of respective types III and IV must be adjoined by a valency of either 1 or 2. As there are two type IV orbits, both these external valencies get used adjoining a type III orbit to these type IV orbits. But this prohibits the existence of a type V orbit since such an orbit has four external valencies of 2 and there are no longer any valencies of 2 in a type III orbit to accommodate this.

For $c=4$, the argument is similar. Each pair of two type III orbits must share a valency of $0, 3$ or 4. As there are four type III orbits in this case, every such valency gets used adjoining orbits of this type. However, a type II orbit has four external valencies of 3. As a type III orbit no longer has a valency of 3 to accommodate this, we reach a contradiction This proves $c=0$ when a type II orbit is assumed to occur in the partition.

Our only remaining case is to assume a type V orbit occurs in the partition. Here the four external valencies of 2 in such an orbit get used adjoining it to four other orbits, be they of type III or IV. But this implies there must be an orbit of type II in the partition, a case we have already treated. We conclude that a K-orbit partition of Γ_2 must indeed be of the form $[\mathrm{II}, \mathrm{IV}^4, \mathrm{V}]$. □

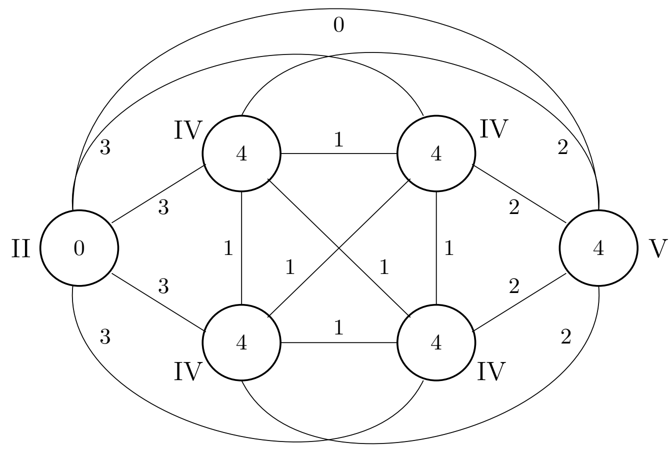

FIGURE 4. The lone surviving K-orbit partition of Γ_2 (cf. Lemma 3.9)

We depict the K-orbit partition of type $[\mathrm{II}, \mathrm{IV}^4, \mathrm{V}]$ in Figure 4. It too will be shown to not exist in due course.

LEMMA 3.10. *Neither a type* IV *orbit nor a type* V *orbit can contain a* 3*-cycle.*

Proof. First observe that type IV and type V orbits have internal valency 4. In any orbit of either type we may fix a vertex v and denote its neighbors by v^p, v^r, v^s, v^t where $p, r, s, t \in K$. However $v^{p^{-1}}, v^{r^{-1}}, v^{s^{-1}}, v^{t^{-1}}$ must also be neighbors of v. This means, with one exception, the automorphisms p, r, s, t must come in pairs. The one exception is if two or four of p, r, s, t are involutory. However, this cannot be the case since K contains a unique involution. Therefore, without loss of generality we may assume $r = p^{-1}$ and $t = s^{-1}$. Moreover, we know that $|r|, |t| \in \{7, 14\}$. Three broad cases can arise here, namely $|r| = |t| = 7$, $|r| = |t| = 14$ and $|r| = 7$, $|t| = 14$.

Case 1. $|r| = |t| = 7$: In this case the orbit is comprised of two connected components but that won't affect our argument. Since K contains a unique subgroup of order 7, we must have $r = t^m$ for some integer $m \in \{2, 3\}$. However, both subcases violate $\lambda = 1$ as depicted in Figure 5. Specifically, if $r = t^2$ then the edge $v^t v^{t^2}$ lies on two 3-cycles, while if $r = t^3$ the edge vv^{t^4} suffers the same fate. (Note that the subcases $m = 4, 5$ are identical to $m = 3, 2$ respectively.)

Case 2. $|r| = |t| = 14$: Here we have $r = t^m$ where $m \in \{3, 5\}$. Both subcases violate $\mu = 2$ as indicated in Figure 6. Specifically, if $r = t^3$ then vertices v^{t^3} and v^{t^5} have v^{t^2}, v^{t^4}, v^{t^6} as common neighbors, whereas if $r = t^5$ then vertices v and v^{t^4} have v^{t^5}, v^{t^9}, $v^{t^{-1}}$ as common neighbors.

Case 3. $|r| = 7, |t| = 14$: Here there are three subcases to consider, namely $r \in \{t^2, t^4, t^6\}$ as indicated in Figure 6. The first and last of these subcases lead to violations. Specifically, $r = t^2$ violates $\lambda = 1$ since the edge vv^t lies on two 3-cycles with respective antipodal vertices v^{t^2} and $v^{t^{-1}}$. In contrast, $r = t^6$ violates $\mu = 2$ since the vertices v^{t^4} and $v^{t^{11}}$ have v^{t^3}, $v^{t^{10}}$, $v^{t^{12}}$ as common neighbors (as well as v^{t^5}). Curiously, the case $r = t^4$ does not lead to any λ or μ violations, however it produces no 3-cycles either. This completes the proof of the lemma. □

THEOREM 3.11. *G does not contain any order* 14 *elements.*

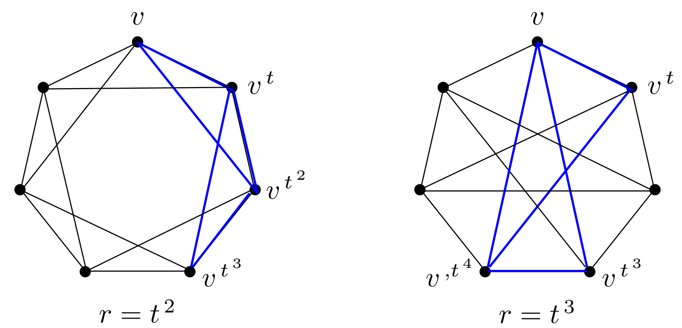

Figure 5. The case $|r| = |t| = 7$ of Lemma 3.10

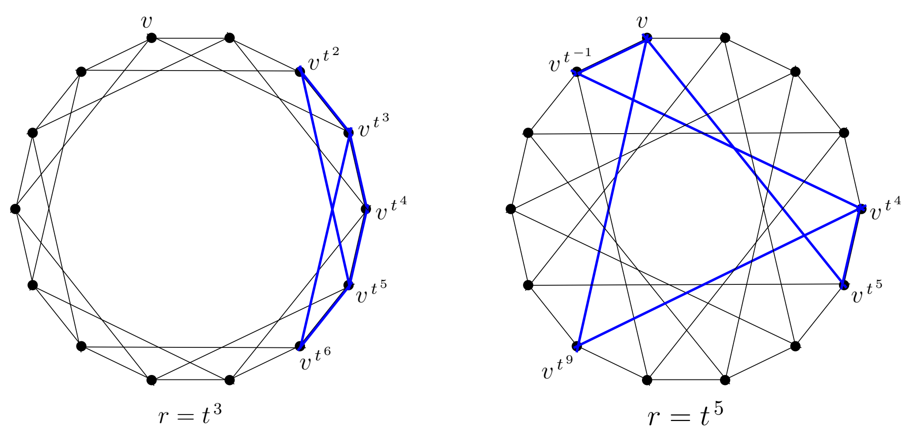

Figure 6. The case $|r| = |t| = 14$ of Lemma 3.10

Proof. By way of contradiction, suppose G contains an element of order 14. By Lemma 3.1 we may assume this element is st, and as usual we set $K = \langle st \rangle$. Lemma 3.9 now asserts that a K-orbit partition must be of the form $[\mathrm{II}, \mathrm{IV}^4, \mathrm{V}]$ as depicted in Figure 4. We claim this orbit partition cannot occur.

For this purpose, it behooves us to restrict the action on $V(\Gamma)$ from $K \cong \mathbb{Z}_{14}$ to $\langle s \rangle \cong \mathbb{Z}_7$. This latter action yields 15 orbits: one fixed vertex (the root x) and 14 orbits of size 7. Let B be the 15×15 quotient matrix corresponding to this action. We compute the trace of B in two different ways.

First, let A be the adjacency matrix of Γ. We know that the characteristic polynomial of B must divide that of A. Further, we know that the spectrum of A is $\{14, 3^{(54)}, -4^{(44)}\}$. Since each row sum of B is 14, we conclude that 14 is an eigenvalue of B. The spectrum of B is therefore $\{14, 3^{(a)}, -4^{(14-a)}\}$ for some nonnegative integer $a \leqslant 14$, and accordingly the trace of B is $14 + 3a - 4(14 - a) = 7a - 42$.

Note that the diagonal entries of B correspond to the internal valencies of orbits. Thus the trace of B is the sum of these internal valencies. In the action of $\langle s \rangle$ on $\Gamma_1 \cup \{x\}$, we have the fixed vertex x (obviously with internal valency zero) and two orbits of size 7, each having internal valency zero so yielding no contribution to the trace. In Γ_2 we have 12 $\langle s \rangle$-orbits of size 7 which fuse in pairs to form the six K-orbits of Γ_2. Five of these six K-orbits have internal valency 4, which, as a direct consequence of Lemma 3.10, decompose into pairs of $\langle s \rangle$-orbits each having internal valency 2. The sixth K-orbit has internal valency 0 so contributes nothing to the trace.

It follows that the trace of B is $10(2) = 20$. Equating our two computations now yields $20 = 7a - 42$, an obvious contradiction since a must be integral. We conclude that a putative Conway 99-graph cannot admit an automorphism of order 14. ☐

Proposition 3.12. *$|G|$ is not divisible by 14.*

Proof. Suppose 14 divides $|G|$. Then by [12] one has $|G| \in \{14, 42\}$. Recall from [11] that $[O(G), t] = 1$, where $O(G)$ is the maximal odd order normal subgroup of G and $t \in G$ is an involution. This alone rules out D_{14}, D_{42}, $D_{14} \times \mathbb{Z}_3$ and $Frob(42)$ as possible isomorphism types of G. The only remaining possibilities are $\mathbb{Z}_{14}$, $\mathbb{Z}_{42}$, $\mathbb{Z}_7 \times S_3$ and $Frob(21) \times \mathbb{Z}_2$. However, each of these groups has an element of order 14 so is ruled out by Theorem 3.11. ☐

Corollary 3.13. *If 2 divides $|G|$, then $|G|$ divides 6.*

Proof. This is an immediate consequence of Proposition 3.12 and [11, Corollary 2.4]. ☐

4. Consequences of divisibility by 7

In this section, we prove $G \cong \mathbb{Z}_7$ under the assumption that $|G|$ is divisible by 7. Our first step in this direction is to show that $|G|$ must divide 21. This leads to just two possible isomorphism types for G, namely $\mathbb{Z}_7$ and the Frobenius group $Frob(21)$. Ultimately, we rule out $Frob(21)$ to obtain our desired result.

Lemma 4.1. *Suppose 7 divides $|G|$, and let P_7 be a Sylow 7-subgroup of G. Then P_7 is normal in G.*

Proof. By [12] and Proposition 3.12, $|G|$ must divide $3^3 \cdot 7 \cdot 11$. By Sylow's Theorem, the number n_7 of Sylow 7-subgroups must satisfy $n_7 = [G : N_G(P_7)]$ and $n_7 \equiv 1 \pmod 7$. It is straightforward to deduce that $P_7 \trianglelefteq G$ for all orders of G except possibly $3^2 \cdot 7 \cdot 11$ and $3^3 \cdot 7 \cdot 11$. However, in these two cases one has $P_{11} \trianglelefteq G$ where P_{11} is a Sylow 11-subgroup of G. As P_7 does not embed in $\mathrm{Aut}(P_{11}) \cong \mathbb{Z}_{10}$ it follows that $[P_7, P_{11}] = 1$. But then $P_{11} \leqslant N_G(P_7)$ whence $n_7 = [G : N_G(P_7)] \in \{1, 9, 27\}$. We now conclude from the congruence $n_7 \equiv 1 \pmod 7$ that $n_7 = 1$. Hence $P_7 \trianglelefteq G$ as claimed. ☐

Proposition 4.2. *Divisibility by 7 implies $|G|$ divides 21.*

Proof. By our labeling scheme in Section 2 and Remark 2.3, it is clear that $P_7 = \langle s \rangle$ where $s = (1L, 2L, \ldots, 7L)\,(1R, 2R, \ldots, 7R)$. Furthermore, as $\langle s \rangle \trianglelefteq G$ by Lemma 4.1, we have that G embeds in the normalizer $N_S(\langle s \rangle)$ where $S = \mathrm{Sym}(\Gamma_1) \cong S_{14}$. Thus the order of G must simultaneously divide $|N_S(\langle s \rangle)| = 2^2 \cdot 3 \cdot 7^2$ and $3^3 \cdot 7 \cdot 11$. Obviously this implies $|G|$ divides 21. ☐

Corollary 4.3. *If 7 divides $|G|$, then G is isomorphic to either $\mathbb{Z}_7$ or $Frob(21)$.*

Proof. By Proposition 4.2, $|G| \in \{7, 21\}$. There are only two groups of order 21 up to isomorphism, namely $\mathbb{Z}_{21}$ and $Frob(21)$. But 3 does not divide the order of the centralizer $C_S(s) = (\langle s_L \rangle \times \langle s_R \rangle) \rtimes \langle t \rangle \cong (\mathbb{Z}_7 \times \mathbb{Z}_7) \rtimes \mathbb{Z}_2$ where $S = \mathrm{Sym}(\Gamma_1)$. Thus G cannot contain an element of order 21, i.e. $G \ncong \mathbb{Z}_{21}$. ☐

In what follows, we gather detailed information about a putative Conway 99-graph Γ under the assumption that $G \cong Frob(21)$. Throughout, we assume $G = \langle s, r \rangle$ with $|r| = 3$. Since r normalizes $\langle s \rangle$, it is clear that r fixes the root vertex x whence Γ_1 and Γ_2 are G-invariant. Still, we have yet to pin down the precise structure of r. This is remedied below.

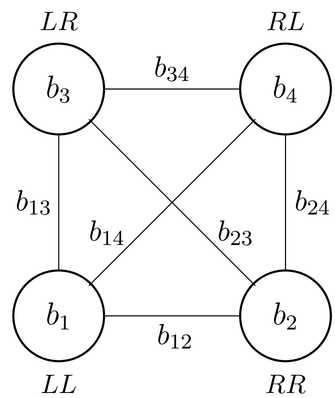

Figure 7. General form of a G-orbit partition of Γ_2

Lemma 4.4. *With notation as above, we may assume*

$$r = (1L, 2L, 4L)\,(3L, 6L, 5L)\,(1R, 2R, 4R)\,(3R, 6R, 5R).$$

Proof. It is immediate that r is an element in the normalizer $N_S(\langle s\rangle)$ in $S = \mathrm{Sym}(\Gamma_1)$. Applying standard group theoretic arguments, one deduces that there are 98 elements of order 3 in $N_S(\langle s\rangle)$ and these take the form

$$\Big(\big((1L, 2L, 4L)(3L, 6L, 5L)\big)^{(s_L)^i}\big((1R, 2R, 4R)(3R, 6R, 5R)\big)^{(s_R)^j}\Big)^{\pm 1}$$

where $0 \leqslant i, j \leqslant 6$. However, it is easily verified that such an element is an automorphism of Γ_1 if and only if $i = j$. Thus, the 14 elements of order 3 in G are precisely

$$\big((1L, 2L, 4L)\,(3L, 6L, 5L)\,(1R, 2R, 4R)\,(3R, 6R, 5R)\big)^{\pm\, s^i}$$

from which the result follows. □

Lemma 4.5. *The G-orbit structure on Γ is $[1, 7^2, 21^4]$.*

Proof. As 21 does not divide 14, the two $\langle s\rangle$-orbits on Γ_1 cannot fuse under the action of G. Thus the G-orbit structure on Γ_1 is $[7^2]$.

Now let $\mathcal{O}$ be an arbitrary G-orbit on Γ_2. We claim G acts regularly on $\mathcal{O}$ from which the desired result will follow. To this end, suppose $ijXY \in \mathcal{O}$ is fixed by some $g \in G$, where $X, Y \in \{L, R\}$. It follows that $|g| = 3$, since s has no fixed points in Γ_2. As $\lambda = 1$, we have $i \neq j$. Thus since $\mu = 2$, g must either fix or interchange the two Γ_1-neighbors of $ijXY$, i.e. $\{iX^g, jY^g\} = \{iX, jY\}$. But as G has odd order, these vertices must be fixed by g, that is, $iX^g = iX$ and $jY^g = jY$. As g preserves adjacency, we must also have $(iX^C)^g = iX^C$ and $(jY^C)^g = jY^C$ where $\{X, X^C\} = \{Y, Y^C\} = \{L, R\}$. In every instance, we obtain $iL^g = iL$ and $jL^g = jL$. By transitivity of $\langle s\rangle$ on the orbit containing iL, we get $jL = iL^z$ for some $z \in \langle s\rangle$. This gives $jL^{[z,g]} = iL^{g^{-1}zg} = iL^{zg} = jL^g = jL$, i.e. jL is a fixed point of $[z, g]$. But $[z, g] \in \langle s\rangle$ because $\langle s\rangle$ is normal in G. This implies $[z, g] = 1$, a contradiction since G is nonabelian. We conclude that the G-orbit structure on Γ_2 is $[21^4]$ as claimed. □

Remark 4.6. Now that we understand the orbit structure of G on Γ_2, it is trivial to determine a corresponding set of orbit representatives, viz. $\{12LL, 12RR, 12LR, 12RL\}$. For brevity we shall denote these orbits as LL, RR, LR, RL, respectively. We indicate the corresponding G-orbit partition of Γ_2 in Figure 7.

For $ijXY \in \Gamma_2$, we refer to iX and jY as its coordinates. In the result that follows, we demonstrate a manner in which the coordinates of the 12 Γ_2-neighbors of a fixed vertex in Γ_2 are balanced.

Lemma 4.7. *Let $ijXY$ be a fixed but arbitrary vertex in Γ_2 and consider collectively the* 24 *coordinates appearing among its* 12 *Γ_2-neighbors. Then the following hold:*

(1) *Each of iL, iR, jL, jR appears exactly once.*
(2) *Each of kL, kR appears exactly twice for each $k \neq i, j$.*
(3) *Each of L and R appears* 12 *times, i.e. the Γ_2-neighbors of a vertex in Γ_2 are L/R-balanced.*

Proof. As in Lemmas 3.2 and 4.5 we let $\{X, X^C\} = \{Y, Y^C\} = \{L, R\}$. We must show that each of iX, jY and iX^C, jY^C occurs exactly once as coordinates of the Γ_2-neighbors of $ijXY$. We treat the pair iX, jY first. Since $\lambda = 1$ and $ijXY$ is adjacent to $iX \in \Gamma_1$, there is a unique vertex in Γ_2 which is their common neighbor. Evidently this vertex is of the form $i\ell XW$ for some $W \in \{L, R\}$ with $\ell W \neq jY$. Thus iX occurs exactly once as a coordinate of a Γ_2-neighbor of $ijXY$ and by a symmetric argument the same holds true for jY.

We next treat the pair iX^C, jY^C. Observe that since $\lambda = 1$, we have that $ijXY$ and iX^C are nonadjacent. Since $\mu = 2$, the vertices $ijXY$ and iX^C must have a unique common neighbor $i\ell X^C W \in \Gamma_2$ with iX being their second common neighbor. Thus iX^C occurs exactly once as a coordinate of a Γ_2-neighbor of $ijXY$ with a similar result holding for jY^C. As $\{iX, iX^C\} = \{iL, iR\}$ and $\{jY, jY^C\} = \{jL, jR\}$, assertion (1) is proved.

Now let $k \neq i, j$ and $W \in \{L, R\}$. Since $\mu = 2$ and kW is nonadjacent to $ijXY$, there are exactly two vertices in Γ_2 that are common neighbors of kW and $ijXY$. Obviously, kW appears as a coordinate in each such neighbor, so twice in total. Thus assertion (2) is proved. Finally, observe that (3) follows directly from (1) and (2). □

As a consequence of Lemma 4.7(3), we have the following.

Lemma 4.8. *With notation as in Figure* 7*, $b_1 = b_{12} = b_2$, $b_{13} = b_{23}$, and $b_{14} = b_{24}$.*

Proof. We apply Lemma 4.7(3) to vertices in LL, RR, LR, RL in that order. Prefatory to this, note that the coordinates of vertices in LR and RL have a natural L/R-balance built into them. This means we may safely ignore edges that adjoin any vertex in Γ_2 to vertices in either of these two orbits when exploiting the property of balance.

Let u be a fixed vertex in LL. As u has valency b_1 in LL, it must have b_1 neighbors in RR in order to restore L/R-balance. This proves $b_{12} = b_1$. By a similar argument based on RR, we have $b_{12} = b_2$. Now let u be a vertex in LR. Clearly, every neighbor of u in LL must be reciprocated by a neighbor in RR to maintain balance. This proves $b_{13} = b_{23}$. Applying this latter argument to vertices in RL yields $b_{14} = b_{24}$. □

To further narrow down possible orbit valencies, we adopt the approach used in Section 3. That is to say, we analyze 2-paths between vertices from pairs of orbits. Let u be a fixed vertex in the orbit LL. We wish to count in two ways the cardinality of the set S given by

$$S = \{uvw : uvw \text{ is a 2-path with } w \in RR\}.$$

Observe that for each of the b_1 neighbors of u in RR there is a unique 2-path in S (since $\lambda = 1$), while for each of the $21 - b_1$ non-neighbors of u in RR there are two such paths (since $\mu = 2$). This gives $|S| = b_1 \cdot 1 + (21 - b_1) \cdot 2 = 42 - b_1$.

We next condition our count on the location of the intermediate vertex v. Here we rely heavily on Lemma 4.8.

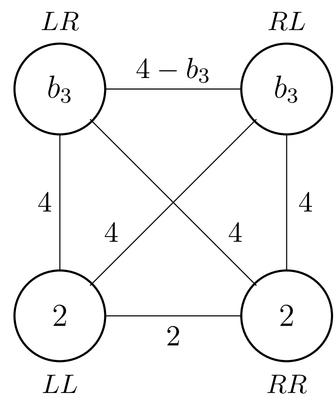

Figure 8. Narrowing down the valencies of a G-orbit partition of Γ_2

If v lies in LL, there are b_1 choices for v followed by b_1 independent choices for w. This gives a total of b_1^2 paths in S assuming the intermediate vertex v is in LL. Note that for v in RR the count is identical, i.e. there are b_1^2 paths in S assuming v is in RR.

Now suppose v is in the orbit LR. Since u has b_{13} neighbors in LR and each such vertex has b_{13} neighbors in RR, the total number of paths in this case is b_{13}^2. By a similar argument the number of paths with v in RL is b_{14}^2. Finally, we observe that there is no 2-path with intermediate vertex v in Γ_1. Indeed, this would require that $v = iL = jR$ for some i, j, which is absurd. Thus we obtain $|S| = 2b_1^2 + b_{13}^2 + b_{14}^2$. Equating the above two expressions for $|S|$ yields

$$42 - b_1 - 2b_1^2 = b_{13}^2 + b_{14}^2. \tag{3}$$

The only integral solution to equation 3 is $b_1 = 2$, $b_{13} = b_{14} = 4$ which, by virtue of Lemma 4.8, narrows down the orbit valencies to the ones indicated in Figure 8.

We are nearly at the point of determining the unique orbit structure of $G \cong Frob(21)$ acting on the second subconstituent Γ_2 of a putative Conway 99-graph Γ. To complete the process, we count in two ways the number of 2-paths originating at a fixed vertex u in LR and ending at some vertex in RL.

On one hand, there are $4-b_3$ edges from u to some vertex w in RL, and in each case there is a unique 2-path uvw since $\lambda = 1$. Similarly, for each of the $21-(4-b_3) = 17+b_3$ vertices in RL nonadjacent to v there are two distinct 2-paths of required form. This gives a total of $(4 - b_3) \cdot 1 + (17 + b_3) \cdot 2 = 38 + b_3$ such 2-paths.

We next focus our count on the location of the intermediate vertex v. If v is in either of LL or RR, there are $4 \cdot 4 = 16$ such 2-paths, while if v is in either of LR or RL there are $(4 - b_3)b_3 = 4b_3 - b_3^2$ such 2-paths. Lastly, we consider $v \in \Gamma_1$. Since u is in LR, it has coordinates $ijLR$ for some $i \neq j$ whence its two Γ_1-neighbors are iL and jR. Note that the 12 neighbors of iL in Γ_2 are of the form $ikLL$ and $ikLR$ $(= kiRL)$ where $k \neq i$. Thus six neighbors of iL lie in $LR \cup RL$. However, these neighbors are divided evenly between LR and RL, since $ikLR$ is in LR if and only if $ikRL$ is in RL. Thus iL has three neighbors in LR and by a symmetric argument this holds true for jR. This gives six more 2-paths starting at u and terminating at some vertex in RL. Thus the total number of such 2-paths is $2(16) + 2(4b_3 - b_3^2) + 6 = 38 + 8b_3 - 2b_3^2$ via this second count.

Equating the two counts yields $38 + b_3 = 38 + 8b_3 - 2b_3^2$ which simplifies to $2b_3^2 - 7b_3 = 0$. Clearly, the only integral solution is $b_3 = 0$ which gives us the following.

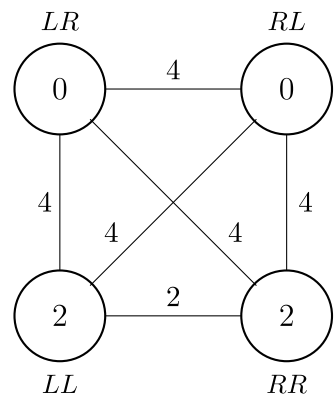

FIGURE 9. The unique G-orbit partition of Γ_2 where $G \cong Frob(21)$

PROPOSITION 4.9. *Assume $G \cong Frob(21)$. Then the only feasible G-orbit partition of Γ_2 is the one depicted in Figure 9.*

REMARK 4.10. Observe that each of $LR\cup\{x\}$ and $RL\cup\{x\}$ is a coclique of size 22. This is in fact the largest independence number allowable by the Hoffman Ratio Bound (aka Hoffman–Delsarte inequality), see [10]. Further note from Figure 9 that every vertex outside of $LR \cup \{x\}$ (resp. $RL \cup \{x\}$) has precisely four neighbors in $LR \cup \{x\}$ (resp. $RL \cup \{x\}$).

Let us call a 3-cycle $u_1u_2u_3$ *of type* (X_1Y_1, X_2Y_2, X_3Y_3) provided u_i is in the orbit X_iY_i where $X_i, Y_i \in \{L, R\}$ for $1 \leqslant i \leqslant 3$.

PROPOSITION 4.11. *Each of LL and RR consists of seven vertex-disjoint 3-cycles.*

Proof. By transitivity of G on its orbits, each orbit of 3-cycles in Γ_2 has size 21 with the exception being 3-cycles of type (LL, LL, LL) or (RR, RR, RR). Indeed, orbits of 3-cycles of these two types would have to be of size 7 since the 3-cycles remain internal to their respective orbits. As there are a total of 140 3-cycles in Γ_2 (cf. Lemma 2.4), their division into n_7 orbits of size 7 and n_{21} orbits of size 21 must satisfy $7n_7+21n_{21} = 140$. Moreover, $0 \leqslant n_7 \leqslant 2$ since each of LL and RR has internal valency 2. Since the only integral solution to the above is $n_7 = 2$ and $n_{21} = 6$, each of LL and RR must contain seven vertex-disjoint 3-cycles as claimed. □

Our goal is to prove the G-orbit partition depicted in Figure 9 cannot exist. To this end we consider the finer $\langle s\rangle$-orbit partition of Γ_2. (Recall that $G = \langle s, r\rangle$ with r as defined in Lemma 4.4.) Observe that each of LL, RR, LR, RL decomposes into three $\langle s\rangle$-orbits. We denote these latter 12 orbits by $(XY)_i$ where $XY \in \{LL, RR, LR, RL\}$ and $1 \leqslant i \leqslant 3$. Here, we are obviously requiring that $(XY)_i \subset XY$ for all i.

We will again apply the method we used in Section 3 to determine orbit valencies b_{ij} by counting 2-paths, only this time our focus will be on the aforementioned $\langle s\rangle$-orbit partition. Thus, we begin by counting in two ways the cardinality of the set

$$S = \{uvw : uvw \text{ is a 2-path with } w \in (XY)_i\},$$

where u is a fixed vertex in $(XY)_i$. For each of the b_i neighbors of u in $(XY)_i$ there is a unique 2-path from u to w (since $\lambda = 1$). Similarly, for each of the $6 - b_i$ non-neighbors of u in $(XY)_i$ there are two 2-paths from u to w (since $\mu = 2$). This gives $|S| = b_i \cdot 1 + (6 - b_i) \cdot 2 = 12 - b_i$.

We next condition our count on the location of the intermediate vertex v. For v in $(XY)_j$ there are $b_{ij}\,(b_{ij}-1)$ such 2-paths. Additionally, if $XY \in \{LL, RR\}$ then u has exactly two neighbors in Γ_1 each of which has a unique neighbor $w \in (XY)_i \setminus \{u\}$. In contrast, if $XY \in \{LR, RL\}$ there are no such neighbors in Γ_1. This gives $|S| = \sum_{j=1}^{12} b_{ij}\,(b_{ij}-1)+c$, where $c = 2$ if $XY \in \{LL, RR\}$ and $c = 0$ if u if $XY \in \{LR, RL\}$. Equating these two expressions for $|S|$, we obtain $12 - b_i = \sum_{j=1}^{12} b_{ij}\,(b_{ij}-1) + c$. But due to the fact that $\sum_{i=1}^{12} b_{ij} = 12$, this simplifies to

$$24 - c - \left(b_i^2 + b_i\right) = \sum_{j\neq i} b_{ij}^2 \tag{4}$$

We divide our analysis into cases based on an assumed value for b_i. Once a choice of b_i is made, we find all ways of expressing $24 - c - \left(b_i^2 + b_i\right)$ as a sum of eleven squares while maintaining the valency requirement $\sum_{i=1}^{12} b_{ij} = 12$.

Note that $b_i \leqslant 4$ since otherwise the value of $24 - c - \left(b_i^2 + b_i\right)$ would be negative. It is also clear by the Handshake Lemma that every internal valency b_i must be even because every $\langle s\rangle$-orbit is of size 7. In addition, we cannot have $b_i = 4$ since this we would lead to a violation of $\lambda = 1$. Thus $b_i \in \{0, 2\}$.

However, recall from the coarser G-orbit partition of Γ_2 that the internal valency of LR and RL is 0 while that of LL and RR is 2. By Proposition 4.11 the internal valencies of $(LL)_i$ and $(RR)_i$ must be zero since no 3-cycle in LL or RR can lie wholly within an $\langle s\rangle$-orbit. We conclude that the internal degree of every $\langle s\rangle$-orbit is zero.

We are now prepared to list all feasible solution to equation 4. Verification of the list is straightforward, so left to the reader.

Lemma 4.12. *Solutions for orbits of type $(LL)_i$ or $(RR)_i$ are listed as follows:*

(a) $\big(0, \big\{3, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0\big\}\big)$
(b) $\big(0, \big\{2, 2, 2, 2, 2, 1, 1, 0, 0, 0, 0\big\}\big)$

Solutions for orbits of type LR_i or RL_i are listed as follows:

(c) $\big(0, \big\{4, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0\big\}\big)$
(d) $\big(0, \big\{3, 3, 1, 1, 1, 1, 1, 1, 0, 0, 0\big\}\big)$
(e) $\big(0, \big\{3, 2, 2, 2, 1, 1, 1, 0, 0, 0, 0\big\}\big)$
(f) $\big(0, \big\{2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0\big\}\big)$

We can fairly quickly eliminate sequences (c) and (d) from this list. According to the G-orbit partition of Γ_2 depicted in Figure 9, the orbit LR has external orbit valency of 4 to each of LL, RR, RL. Assume now that sequence (c) corresponds to an $\langle s\rangle$-orbit inside LR, say $(LR)_1$. All three zeros in sequence (c) must be used in adjoining $(LR)_1$ to $(LR)_1, (LR)_2, (LR)_3$. This leaves the subsequence $\{4, 1, 1, 1, 1, 1, 1, 1, 1\}$ to account for all external orbit valencies of $(LR)_1$. But as there is no 3-element subset of this subsequence that sums to 4, there is no possible way to adjoin $(LR)_1$ to any $\langle s\rangle$-orbit outside of LR. (Of course the argument for RL is entirely symmetric.) This eliminates sequence (c) as a possible solution.

The argument for eliminating sequence (d) is similar. As above, all three zeros in sequence (d) are used to adjoin $(LR)_1$ to $(LR)_1, (LR)_2, (LR)_3$. This leaves the subsequence $\{3, 3, 1, 1, 1, 1, 1, 1\}$ to account for all external orbit valencies of $(LR)_1$. But as before, there is no 3-element subset of this subsequence that sums to 4. Hence, sequence (d) is ruled out as well.

For future reference, it is convenient to designate the surviving solution by type, e.g.

$$\text{I. } \big(0, \big\{3, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0\big\}\big) \quad \text{II. } \big(0, \big\{2, 2, 2, 2, 2, 1, 1, 0, 0, 0, 0\big\}\big)$$
$$\text{III. } \big(0, \big\{3, 2, 2, 2, 1, 1, 1, 0, 0, 0, 0\big\}\big) \quad \text{IV. } \big(0, \big\{2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0\big\}\big)$$

In what follows it behooves us to double count 2-paths of the form uvw where u is a fixed vertex in $(XY)_i$ and w is an arbitrary vertex in $(UV)_j$, $UV \neq XY$. For convenience we resort to the notation $\mathcal{O}_i$ ($1 \leqslant i \leqslant 12$) to represent the $\langle s \rangle$-orbits $(LL)_1, (LL)_2, (LL)_3, (RR)_1, (RR)_2, (RR)_3, (LR)_1, (LR)_2, (LR)_3, (RL)_1, (RL)_2, (RL)_3$ in that order.

For the first count, we observe that $u \in \mathcal{O}_i$ has b_{ij} neighbors in $\mathcal{O}_j$, and for each such neighbor $w \in \mathcal{O}_j$ there is a unique 2-path uvw (since $\lambda = 1$). Similarly, u has $7 - b_{ij}$ non-neighbors in $\mathcal{O}_j$, and for each such non-neighbor $w \in \mathcal{O}_j$ there are exactly two 2-paths of the form uvw (since $\mu = 2$). Thus, there are a total of $b_{ij} \cdot 1 + (7 - b_{ij}) \cdot 2 = 14 - b_{ij}$ 2-paths of required form.

For our second count, we focus on the location of the intermediate vertex v in each 2-path uvw. Here v can occur in any in the twelve orbits of Γ_2 as well as in Γ_1. In the case of orbits in Γ_2, there are b_{ik} choices for $v \in \mathcal{O}_k$ and for each such choice there are b_{kj} choices for $w \in \mathcal{O}_j$. This gives $b_{ik}b_{kj}$ 2-paths of this type. Finally, we denote by c_{ij} the number of 2-paths that have intermediate vertex v in Γ_1.

Equating these two counts yields

$$14 - b_{ij} = \sum_{k=1}^{12} b_{ik}b_{kj} + c_{ij}. \tag{5}$$

Let C be the 12×12 quotient matrix corresponding to the action of $\langle s \rangle \cong \mathbb{Z}_7$ restricted to Γ_2. Consistent with the above, we order the rows and columns of C by $(LL)_1$, $(LL)_2$, $(LL)_3$, $(RR)_1$, $(RR)_2$, $(RR)_3$, $(LR)_1$, $(LR)_2$, $(LR)_3$, $(RL)_1$, $(RL)_2$, (RL_3).

Proposition 4.13. *Under the assumption that $G \cong Frob(21)$, there is no $\langle s \rangle$-orbit LL_i or RR_j that has valency* 3.

Proof. By way of contradiction, suppose 3 occurs as a valency of $(LL)_i$ or $(RR)_j$ in the $\langle s \rangle$-orbit partition of Γ_2. This implies sequence I coincides with the valencies of such an orbit. Since the valency between the G-orbits LL and RR is 2, the valency 3 in sequence I must be used to adjoin to an $\langle s \rangle$-orbit within LR or RL. Since all $\langle s \rangle$-orbits of Γ_2 are of equal size, it follows that all external valencies are bidirectional. Thus without loss of generality, sequence III coincides with the valencies of the $\langle s \rangle$-orbit $(LR)_1$.

Our next observation depends solely on the assumption that $\langle s \rangle$ is a proper subgroup of $G = \langle s, r \rangle \cong Frob(21)$. Since r permutes the three $\langle s \rangle$-orbits within a given G-orbit, we have that $(XY)_1, (XY)_2, (XY)_3$ must share the same set of valencies for every $XY \in \{LL, RR, LR, RL\}$. Recalling that the G-orbits LL and LR are adjoined by a valency of 4, we may further assume that each $(LL)_i$ adjoins $(LR)_1, (LR)_2, (LR)_3$ with valencies $3, 1, 0$ in some order. This leads to the following 3×3 submatrix of C corresponding to the orbit valencies adjoining each of $(LL)_i$ to each of $(LR)_j$, where $1 \leqslant i, j \leqslant 3$.

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

We next consider the four 3×3 submatrices that occur as block diagonals of C. Since LR and RL are independent sets, the block corresponding to the $\langle s \rangle$-orbits within each of them is the 3×3 zero matrix. Next recall from Lemma 4.11 that each of LL and RR consists of seven disjoint 3-cycles. As the vertices of such 3-cycles must occupy different $\langle s \rangle$-orbits, we obtain the following 3×3 submatrix for the $\langle s \rangle$-orbits within LL, as well as for those within RR.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Finally, recall that in the G-orbit partition, the orbit LL is adjoined to RR and RL with orbit valencies of 2 and 4, respectively. This means we must arrange the unused portion of sequence I, namely $2,2,1,1,0,0$, into two groups of three that sum to 2 and 4. Here there are two possibilities: $\{\{1,1,0\},\{2,2,0\}\}$ and $\{\{2,0,0\},\{2,1,1\}\}$. We indicate below the two matrices that reflect these possibilities in addition to what we have deduced so far.

$$C_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 3 & 1 & 0 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 3 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 3 & 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & & & & & & \\ 1 & 0 & 1 & 1 & 0 & 1 & & & & & & \\ 1 & 1 & 0 & 1 & 1 & 0 & & & & & & \\ 3 & 1 & 0 & & & & 0 & 0 & 0 & & & \\ 1 & 0 & 3 & & & & 0 & 0 & 0 & & & \\ 0 & 3 & 1 & & & & 0 & 0 & 0 & & & \\ 0 & 2 & 2 & & & & & & & 0 & 0 & 0 \\ 2 & 0 & 2 & & & & & & & 0 & 0 & 0 \\ 2 & 2 & 0 & & & & & & & 0 & 0 & 0 \end{bmatrix} \quad C_2 = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 0 & 3 & 1 & 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 3 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 & 2 & 0 & 3 & 1 & 1 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 & 1 & & & & & & \\ 0 & 2 & 0 & 1 & 0 & 1 & & & & & & \\ 0 & 0 & 2 & 1 & 1 & 0 & & & & & & \\ 3 & 1 & 0 & & & & 0 & 0 & 0 & & & \\ 1 & 0 & 3 & & & & 0 & 0 & 0 & & & \\ 0 & 3 & 1 & & & & 0 & 0 & 0 & & & \\ 2 & 1 & 1 & & & & & & & 0 & 0 & 0 \\ 1 & 2 & 1 & & & & & & & 0 & 0 & 0 \\ 1 & 1 & 2 & & & & & & & 0 & 0 & 0 \end{bmatrix}$$

We first consider the matrix C_1. We claim this matrix cannot be completed to the desired matrix C. To prove this, we attempt to complete its fourth row, i.e. the row corresponding to the $\langle s\rangle$-orbit $(RR)_1$. Note that the value 1 already appears four times in this row, which dictates that the orbit valencies of $(RR)_1$ must coincide with sequence I. In particular, the six unfilled entries in this row must match up with those values in sequence I that have yet to be used, viz. $3,2,2,1,0,0$. As the external valency that adjoins RR to each of LR and RL is 4, the above six values must be partitioned into two groups of size 3 each summing to 4. Moreover, as the value 3 cannot occur twice in any column, the desired partition is $\{\{2,2,0\},\{3,1,0\}\}$.

We now apply equation 5 to the orbits $(LL)_2$ and $(RR)_1$. Note that in this case we have $b_{24} = 1$ and $c_{24} = 0$. Further note that the righthand side of equation 5 is simply the dot product of the second and fourth rows. For convenience, let us denote by a,b,c,d,e,f the six missing entries of row 4 in that order. Then the dot product in question is $2 + a + 3c + 2d + 2f$, an even integer due to $a, c \in \{0,2\}$. But this is a contradiction since the lefthand side of equation 5 is $14 - b_{24} = 13$ in this case.

We next turn our attention to the matrix C_2. Again, we claim there is no compatible way to complete the fourth row. The argument here is a bit more sophisticated than the one provided for C_1, but the basic ideas are the same. The only initial difference is that the roles of $\{2,2,0\}$ and $\{3,1,0\}$ have been reversed.

We first apply equation 5 to the first and fourth rows of C_2, noting that $b_{14} = 2$ and $c_{14} = 0$. Thus the lefthand side of equation 5 is $14 - 2 = 12$ while the righthand side is $3a + b + 2d + e + f$. Thus we obtain

$$12 = 3a + b + 2d + e + f = (a + b + c + d + e + f) + (2a - c + d) = 8 + 2a - c + d$$

which implies d is even since $c \in \{0,2\}$. Thus $d = 0$ because $d \in \{0,1,3\}$.

Our next step is to apply equation 5 to the second and fourth rows of C_2, noting that $b_{24} = 0$ and $c_{24} = 0$ in this case. Here equation 5 yields

$$14 = 4 + a + 3c + d + 2e + f = 4 + (a + b + c + d + e + f) + (2c - b + e) = 12 + (2c - b + e)$$

which implies e is even since $b \in \{0,2\}$. Thus $e = 0$ because $e \in \{0,1,3\}$. Together we now have $d = e = 0$ which contradicts the fact that $\{d,e,f\} = \{0,1,3\}$. Since neither C_1 nor C_2 can be completed to the desired quotient matrix C, it follows that 3 cannot occur as the valency of any orbit in an $\langle s\rangle$-orbit partition of Γ_2. $\square$

Proposition 4.14. *The G-orbit partition in Figure 9 cannot exist.*

Proof. We proceed as in the proof of Proposition 4.13 with the additional knowledge that the valencies of all $\langle s\rangle$-orbits $(LL)_i$ and $(RR)_j$ coincide with sequence II. From this, it is an easy matter to arrive at the partially completed matrix C_3 below.

$$C_3 = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 2 \\ 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 & 1 & 1 & & & & & & \\ 0 & 2 & 0 & 1 & 0 & 1 & & & & & & \\ 0 & 0 & 2 & 1 & 1 & 0 & & & & & & \\ 0 & 2 & 2 & & & & 0 & 0 & 0 & & & \\ 2 & 0 & 2 & & & & 0 & 0 & 0 & & & \\ 2 & 2 & 0 & & & & 0 & 0 & 0 & & & \\ 0 & 2 & 2 & & & & & & & 0 & 0 & 0 \\ 2 & 0 & 2 & & & & & & & 0 & 0 & 0 \\ 2 & 2 & 0 & & & & & & & 0 & 0 & 0 \end{bmatrix}$$

Just as in the proof of Proposition 4.13, we claim there is no possible way to complete the fourth row of C_3 in a manner that is compatible with rows $1, 2, 3$.

We first observe that the missing valencies of $(RR)_1$, denoted by a, b, c, d, e, f for notational convenience, must coincide with the remaining portion of sequence II, viz. $2, 2, 2, 2, 0, 0$. Applying equation 5 to the $\langle s\rangle$-orbits $(LL)_2$ and $(RR)_1$, we obtain $14 = 4 + 2a + 2c + 2d + 2f$ where we have used the fact that $b_{24} = c_{24} = 0$. However, this easily simplifies to $5 = a + c + d + f$, a contradiction since $a, c, d, f \in \{0, 2\}$. Together with Proposition 4.13, this proves there is no possible $\langle s\rangle$-orbit partition that fuses to the G-orbit partition in Figure 9. The result follows. $\square$

Corollary 4.15. *If 7 divides $|G|$ then $G \cong \mathbb{Z}_7$.*

Proof. The result follows at once from Corollary 3.13 and Propositions 4.9 and 4.14. $\square$

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