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Weights on homogeneous coherent configurations

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Abstract D. G. Higman generalized the notion of a coherent configuration and defined a weight. In this article, we will modify the definition and investigate weights on coherent configurations. If our weights are on a thin homogeneous coherent configuration, that is essentially a finite group, then there is a natural correspondence between the set of equivalence classes of weights and the 2-cohomology group of the group. We also give a construction of weights as a generalization of Higman's method using monomial representations of finite groups.

1. Introduction

In [3], D. G. Higman established basic theory of coherent configurations, and in [4], he generalized the notion of a coherent configuration and defined a weight. Typical examples of coherent configurations are defined using permutation representations of finite groups, and weights are defined using monomial representations [2]. However, after [4], weights have been little studied. In this article, we will modify the definition and investigate weights on coherent configurations. Especially, we will consider weights on homogeneous coherent configurations. Homogeneous coherent configurations are just (not necessarily commutative) association schemes in [8]. In Higman's definition, a weight is a generalization of a coherent configuration, but we will consider a weight on a coherent configuration.

In Section 2, we will give definitions. Also we will define equivalence of weights, which is not considered in [4]. A finite group G can be considered as a homogeneous coherent configuration. In Section 3, we consider this case and show that there is a natural correspondence between the set of equivalence classes of weights and the 2-cohomology group of the group G. In Section 4, we generalize the method by Higman using monomial representations of finite groups.

Let X be a finite set. We denote by $M_X(\mathbb{C})$ the matrix algebra both rows and columns of whose matrices are indexed by the set X. For $a_x \in \mathbb{C}$ $(x \in X)$, $\mathrm{diag}(a_x \mid x \in X) \in M_X(\mathbb{C})$ is the diagonal matrix with the (x,x)-entry a_x. For $c \subset X \times X$, $A_c \in M_X(\mathbb{C})$ is defined by $(A_c)_{x,y} = 1$ if $(x,y) \in c$ and 0 otherwise. For $c \subset X \times X$, $c^* = \{(y,x) \mid (x,y) \in c\}$. Obviously, $A_{c^*} = A_c^T$, the transposed matrix of A_c.

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2. Coherent configurations and weights

2.1. Coherent configurations. Let X be a finite set. We consider the following conditions.

(C1) C is a partition of $X \times X$. Namely, C is a collection of non-empty subsets of $X \times X$, $X \times X = \bigcup_{c \in C} c$, and $c \cap c' = \varnothing$ for $c \neq c' \in C$.
(C2) If $c \in C$, then $c^* \in C$.
(C3) There is a subset Δ of C such that $\bigcup_{d \in \Delta} d = \{(x,x) \mid x \in X\}$.
(C4) $\mathbb{C}C = \bigoplus_{c \in C} \mathbb{C}A_c$ is an algebra.

The pair $\mathfrak{X} = (X, C)$ is called a *configuration* if (C1) holds. The configuration $\mathfrak{X}$ is said to be *precoherent* if (C1), (C2) and (C3) hold. The configuration $\mathfrak{X}$ is said to be *coherent* if (C1), (C2), (C3) and (C4) hold. The algebra $\mathbb{C}C$ is called the *adjacency algebra* of $\mathfrak{X}$. When Δ is a singleton, a coherent configuration $\mathfrak{X}$ is said to be *homogeneous*. Homogeneous coherent configurations are just (non-commutative) *association schemes* in [8].

Example 2.1. [Thin homogeneous coherent configurations] Let G be a finite group. For $g \in G$, define $c_g = \{(x,y) \in G \times G \mid xg = y\}$. Then $\mathfrak{X}(G) = (G, \{c_g \mid g \in G\})$ is a homogeneous coherent configuration. In this case, we say that $\mathfrak{X}(G)$ is *thin*.

Example 2.2. [Schurian homogeneous coherent configuration, centralizer algebra] Let G be a finite transitive permutation group on X, and let T be the permutation representation of G related to X. Set $V = \{A \in M_X(\mathbb{C}) \mid AT(g) = T(g)A \text{ for all } g \in G\}$, the *centralizer algebra*. Then V has a basis consisting of 01-matrices (G-orbits on $X \times X$). We can define a homogeneous coherent configuration. In this case, we say that the homogeneous coherent configuration is *schurian*.

2.2. Weights. Let $\mathfrak{X} = (X, C)$ be a coherent configuration. For $W \in M_X(\mathbb{C})$, we define the *support* of W by

$$\mathrm{spt}(W) = \{(x,y) \in X \times X \mid W_{xy} \neq 0\}.$$

For $c \in C$, we set

$$A_c^W = A_c \circ W,$$

where $\circ$ is the entry-wise product (Hadamard product). We consider the following conditions.

(W1) $\mathrm{spt}(W) = \bigcup_{d \in D} d$ for some subset D of C, and, if $d \subset \mathrm{spt}(W)$, then $d^* \subset \mathrm{spt}(W)$.
(W2) $W_{xx} \neq 0$ for all $x \in X$.
(W3) $\mathbb{C}^W C = \bigoplus_{c \in C} \mathbb{C}A_c^W$ is an algebra.
(W4) W is hermitian, $||W_{xy}|| \in \{0,1\}$ for all $x, y \in X$, and $W_{xx} = 1$ for all $x \in X$.

We call W a *weight* on $\mathfrak{X}$ if (W1), (W2) and (W3) hold. We call W an *H-weight* on $\mathfrak{X}$ if (W1), (W2), (W3) and (W4) hold ("H-" is due to Higman).

Remark 2.3. In [4], Higman called W a *weight* if $\mathfrak{X}$ is a precoherent configuration and (W1) and (W4) hold, and a *coherent weight* if W is a weight and (W3) holds (the condition (W2) automatically holds by (W4)). Thus, in Higman's sense, weights are not necessarily on coherent configurations, and $\mathbb{C}C$ is not necessarily an algebra. In our definition, weights are on coherent configurations and we require that $\mathbb{C}C$ is also an algebra.

Example 2.4. The "all one" matrix W is a weight on any coherent configuration, and called the *standard weight*. In this case, $\mathbb{C}^W C = \mathbb{C}C$. The identity matrix W is a weight on any coherent configuration, and called the *trivial weight*.

Example 2.5. Let G be a finite group, H a subgroup of G. The induced representation from the trivial representation of H to G is the transitive permutation representation and defines a homogeneous coherent configuration $\mathfrak{X}(G,H)$ as in Example 2.2. Let φ be a linear character of H, and let U_φ be the representation of H affording φ. Consider the monomial representation $U_\varphi^{\uparrow G}$ and set $V_\varphi = \{A \in M_{|G:H|}(\mathbb{C}) \mid AU_\varphi^{\uparrow G}(g) = U_\varphi^{\uparrow G}(g)A \text{ for all } g \in G\}$. Then V_φ determines an H-weight on $\mathfrak{X}(G,H)$ [2].

2.3. Equivalence. We define an equivalence of weights on a coherent configuration. Let W and W' be weights on a coherent configuration $\mathfrak{X} = (X, C)$. Set $\mathrm{Aut}(\mathfrak{X}) = \{\sigma \in \mathrm{Sym}(X) \mid c^\sigma = c \text{ for all } c \in C\}$, where $c^\sigma = \{(x^\sigma, y^\sigma) \mid (x,y) \in c\}$ and $\mathrm{Sym}(X)$ is the symmetric group on X. Let $P_\sigma \in M_X(\mathbb{C})$ be the permutation matrix corresponding to $\sigma \in \mathrm{Sym}(X)$. We say that W and W' are *equivalent* and write $W \sim W'$ if there exist $a_x \in \mathbb{C}^\times$ $(x \in X)$, $\gamma(c) \in \mathbb{C}^\times$ $(c \in C)$ and $\sigma \in \mathrm{Aut}(\mathfrak{X})$ such that

$$W' = \mathrm{diag}(a_x \mid x \in X)^{-1} P_\sigma^{-1} \sum_{c \in C} \gamma(c) A_c^W P_\sigma \,\mathrm{diag}(a_x \mid x \in X).$$

We remark that $W = \sum_{c \in C} A_c^W$. We say that two H-weights W and W' are *H-equivalent* and write $W \sim_H W'$ if $W \sim W'$ for $||a_x|| = ||\gamma(c)|| = 1$ and $\gamma(c^*) = \gamma(c)^{-1}$ $(x \in X,\ c \in C)$.

3. Weights on a finite group

In this section, we consider weights and H-weights on a thin homogeneous coherent configuration, a finite group (Example 2.1). We will show that weights are essentially factor sets, in this case.

We recall the basic theory of 2-cohomology of a finite group with reference to [7]. Let G be a finite group. A function $\alpha : G \times G \to \mathbb{C}^\times$ is called a *factor set* or a 2-*cocycle* if

$$(*) \qquad \alpha(g,h)\alpha(gh,k) = \alpha(g,hk)\alpha(h,k)$$

hold for all $g, h, k \in G$. It is possible to change $\mathbb{C}^\times$ to an abelian group M and consider the action of G on M, but we will consider only the trivial action on $\mathbb{C}^\times$. If we consider a vector space $\bigoplus_{g \in G} \mathbb{C}v_g$, where $\{v_g \mid g \in G\}$ is a formal basis of the space, with the multiplication $v_g v_h = \alpha(g,h)v_{gh}$, then the condition $(*)$ is equivalent to the associativity of the multiplication. Thus, for a factor set α, we can define the *generalized group algebra* (also known as *twisted group algebra*) $\mathbb{C}^{(\alpha)}G = \bigoplus_{g \in G} \mathbb{C}v_g$. The set $Z^2(G, \mathbb{C}^\times)$ of all factor sets is an abelian group by $(\alpha\beta)(g,h) = \alpha(g,h)\beta(g,h)$. The change of basis $v_g \mapsto \gamma(g)v_g$ yields the change of the factor set $\alpha(g,h) \mapsto \gamma(g)\gamma(h)\gamma(gh)^{-1}\alpha(g,h)$. We say that factor sets α and β are *equivalent* and write $\alpha \sim \beta$ if there exists a map $\gamma : G \to \mathbb{C}^\times$ such that $\beta(g,h) = \gamma(g)\gamma(h)\gamma(gh)^{-1}\alpha(g,h)$. A factor set α is called a 2-*coboundary* if there exists a map $\gamma : G \to \mathbb{C}^\times$ such that $\alpha(g,h) = \gamma(g)\gamma(h)\gamma(gh)^{-1}$. The set $B^2(G, \mathbb{C}^\times)$ of all 2-coboundaries is a subgroup of $Z^2(G, \mathbb{C}^\times)$. The factor group $H^2(G, \mathbb{C}^\times) = Z^2(G, \mathbb{C}^\times)/B^2(G, \mathbb{C}^\times)$ is called the 2-*cohomology group* of G. An element of $H^2(G, \mathbb{C}^\times)$ is an equivalence class of factor sets.

Proposition 3.1 ([7, II. Section 7.2, Theorem 7.3, III. Lemma 5.4]). *For a factor set α of a finite group G, there exists a factor set β equivalent to α satisfying the following conditions.*

(1) $\beta(g,1) = \beta(1,g) = 1$, $\beta(g, g^{-1}) = \beta(g^{-1}, g) = 1$ *for all* $g \in G$.
(2) $\beta(g,h)^{2|G|} = 1$ *for all* $g, h \in G$.

Now we consider weights on a thin homogeneous coherent configuration. The next lemma is clear by definition.

Lemma 3.2. *Let W be a weight on a thin homogeneous coherent configuration $\mathfrak{X}(G)$ defined by a finite group G. Then* $\mathrm{spt}(W) = \bigcup_{k \in K} c_k$ *for some subgroup K of G, where* $c_k = \{(x,y) \in G \times G \mid xk = y\}$.

We say that W has "no zero entry", if $\mathrm{spt}(W) = G$. Clearly, if $W \sim W'$ and W has no zero entry, then so does W'. The following theorems are the main results in this section.

Theorem 3.3. *Let $\mathfrak{X}(G)$ be a thin homogeneous coherent configuration defined by a finite group G. Then there exist natural bijections between the following sets:*

(1) $\mathfrak{W}$ *: the set of $\sim$ equivalence classes of weights on $\mathfrak{X}(G)$ having no zero entry,*
(2) $\mathfrak{W}_H$ *: the set of $\sim_H$ equivalence classes of H-weights on $\mathfrak{X}(G)$ having no zero entry,*
(3) $H^2(G, \mathbb{C}^\times)$ *: the 2-cohomology group of G.*

Theorem 3.4. *Let $\mathfrak{X}(G)$ be a thin homogeneous coherent configuration defined by a finite group G. Then there exist natural bijections between the following sets:*

(1) *the set of $\sim$ equivalence classes of weights on $\mathfrak{X}(G)$,*
(2) *the set of $\sim_H$ equivalence classes of H-weights on $\mathfrak{X}(G)$,*
(3) $\bigcup_K H^2(K, \mathbb{C}^\times)$*, where K runs over all subgroups of G.*

To prove the above theorems, we need some lemmas. For a moment, we suppose that W is a weight on $\mathfrak{X}(G)$ having no zero entry. Set $c_g = \{(x,y) \in G \times G \mid xg = y\}$ and $A_g = A_{c_g}$, the adjacency matrix. By

$$A_g^W A_h^W = \alpha_W(g,h) A_{gh}^W$$

for $g, h \in G$, we obtain a factor set α_W. For a factor set α, we can define $W_\alpha \in M_G(\mathbb{C})$ by

$$(W_\alpha)_{xy} = \alpha(x, x^{-1}y).$$

Lemma 3.5. *For a factor set α, W_α is a weight and $\alpha_{W_\alpha} = \alpha$.*

Proof. By definition,

$$\begin{aligned}(A_g^{W_\alpha} A_h^{W_\alpha})_{xy} &= \sum_{z \in G} \delta_{xg,z} \alpha(x, x^{-1}z) \delta_{zh,y} \alpha(z, z^{-1}y) = \delta_{xgh,y} \alpha(x,g) \alpha(xg,h) \\ &= \delta_{xgh,y} \alpha(x, gh) \alpha(g,h), \\ (A_{gh}^{W_\alpha})_{xy} &= \delta_{xgh,y} \alpha(x, x^{-1}y) = \delta_{xgh,y} \alpha(x, gh).\end{aligned}$$

Thus $A_g^{W_\alpha} A_h^{W_\alpha} = \alpha(g,h) A_{gh}^{W_\alpha}$ and the result holds. □

The definition of W_α is quite natural, because it is based on the regular representation of the generalized group algebra $\mathbb{C}^{(\alpha)}G$. The matrix W_α is also known as a *cocyclic matrix* and considered in several papers, for example in [1, 6].

Lemma 3.6. *Let W be a weight on a thin homogeneous coherent configuration $\mathfrak{X}(G)$ defined by a finite group G. Then*

$$\alpha_W(g,h) = \frac{W_{x,xg} W_{xg,xgh}}{W_{x,xgh}}$$

for all $g, h, x \in G$. In particular, if W is an H-weight, then $||\alpha_W(g,h)|| = 1$ for all $g, h \in G$.

Proof. The result follows from

$$\begin{aligned}(A_g^W A_h^W)_{xy} &= \sum_{z\in G} \delta_{xg,z} W_{x,z} \delta_{zh,y} W_{z,y} \\ &= \delta_{xgh,y} W_{x,xg} W_{xg,xgh}, \\ (A_g^W A_h^W)_{xy} &= \alpha_W(g,h)(A_{gh}^W)_{x,y} = \alpha_W(g,h)\delta_{xgh,y} W_{x,xgh}\end{aligned}$$

for $g,h,x,y \in G$. □

For a factor set α, a weight W, and an H-weight W, we denote the $\sim$ equivalence class containing α by $[\alpha]$, the $\sim$ equivalence class containing W by $[W]$, and the $\sim_H$ equivalence class containing W by $[W]_H$, respectively. We define

$$\Phi : H^2(G, \mathbb{C}^\times) \to \mathfrak{W}, \quad \Phi([\alpha]) = [W_\alpha]$$

and show that Φ is a bijection.

Lemma 3.7. *For a weight W, $W \sim W_{\alpha_W}$.*

Proof. Set

$$W' = \operatorname{diag}((W_{1g})^{-1} \mid g \in G)^{-1} W \operatorname{diag}((W_{1g})^{-1} \mid g \in G).$$

Then $\alpha_W = \alpha_{W'}$, $W \sim W'$, and $W'_{1g} = W'_{11}$ ($g \in G$). By setting $x = 1$ in Lemma 3.6, we have $\alpha_W(g,h) = W'_{g,gh}$ and thus $W' = W_{\alpha_W}$. □

Lemma 3.8. *For factor sets α and β, if $\alpha \sim \beta$, then $W_\alpha \sim W_\beta$. (Namely, Φ is well-defined.)*

Proof. Suppose $\beta(g,h) = \gamma(g)\gamma(h)\gamma(gh)^{-1}\alpha(g,h)$. We have $W_\alpha = \sum_{g\in G} A_g^{W_\alpha}$ and set $W' = \sum_{g\in G} \gamma(g) A_g^{W_\alpha}$. Then $W' \sim W_\alpha$ and W' is a weight with the factor set β. By Lemma 3.7, $W' \sim W_\beta$ and $W_\alpha \sim W_\beta$. □

Lemma 3.9. *The map Φ is surjective.*

Proof. This is clear by Lemma 3.7. □

Lemma 3.10. *For factor sets α and β, if $W_\alpha \sim W_\beta$, then $\alpha \sim \beta$. (This means that Φ is injective.)*

Proof. For $W_\alpha = \sum_{g\in G} A_g^{W_\alpha}$ and $\sigma \in \operatorname{Aut}(\mathfrak{X})$, we can write

$$W_\beta = \operatorname{diag}(a_g \mid g \in G)^{-1} P_\sigma^{-1} \sum_{g\in G} \gamma(g) A_g^{W_\alpha} P_\sigma \operatorname{diag}(a_g \mid g \in G).$$

Thus the factor set obtained by W_β is $\gamma(g)\gamma(h)\gamma(gh)^{-1}\alpha(g,h)$. We have $\alpha \sim \beta$. □

By Lemmas 3.8, 3.9, 3.10, $\Phi : H^2(G, \mathbb{C}^\times) \to \mathfrak{W}$ is bijective.

We consider

$$\Psi : \mathfrak{W}_H \to \mathfrak{W}, \quad \Psi([W]_H) = [W]$$

and show that Ψ is a bijection. By definition, it is clear that Ψ is well-defined, namely, if $W \sim_H W'$, then $W \sim W'$.

Lemma 3.11. *If a factor set α satisfies the conditions (1) and (2) in Proposition* 3.1*, then W_α is an H-weight.*

Proof. Suppose that α satisfies (1) and (2). Recall that $(W_\alpha)_{xy} = \alpha(x, x^{-1}y)$. Since $\alpha(x,1) = 1$, $(W_\alpha)_{xx} = 1$ for $x \in G$. For all $x,y \in G$, $||(W_\alpha)_{xy}|| = ||\alpha(x,x^{-1}y)|| = 1$ hold. By $\alpha(x,x^{-1}) = 1$, $A_{x^{-1}}^{W_\alpha} = (A_x^{W_\alpha})^{-1} = (A_x^{W_\alpha})^*$. Thus $W_\alpha = \sum_{x\in G} A_x^{W_\alpha}$ is hermitian. Now W_α is an H-weight. □

Lemma 3.12. *Ψ is surjective.*

Proof. Let W be a weight. By Lemma 3.7, $W \sim W_{\alpha_W}$. By Proposition 3.1, there exists a factor set $\beta \sim \alpha_W$ which satisfies the conditions (1) and (2) in Proposition 3.1. By Lemmas 3.8, 3.11, W_β is an H-weight and $W_{\alpha_W} \sim W_\beta$. Now $\Psi([W_\beta]_H) = [W_\beta] = [W]$ and Ψ is surjective. □

Lemma 3.13. *For H-weights W and W', if $W \sim W'$, then $W \sim_H W'$. (This means that Ψ is injective.)*

Proof. Suppose that W and W' are H-weights and set

$$W' = \operatorname{diag}(a_g \mid g \in G)^{-1} P_\sigma^{-1} \sum_{g \in G} \gamma(g) A_g^W P_\sigma \operatorname{diag}(a_g \mid g \in G).$$

We have $\alpha_{W'}(g,h) = \gamma(g)\gamma(h)\gamma(gh)^{-1}\alpha_W(g,h)$ and $||\alpha_W(g,h)|| = ||\alpha_{W'}(g,h)|| = 1$ for all $g, h \in G$. Choose $g_0 \in G$ such that $||\gamma(g_0)||$ is maximal. By $\alpha_{W'}(g_0,g_0) = \gamma(g_0)^2\gamma(g_0^2)^{-1}\alpha(g_0,g_0)$, we have $||\gamma(g_0)||^2 = ||\gamma(g_0^2)|| \leqslant ||\gamma(g_0)||$ and this shows $||\gamma(g_0)|| \leqslant 1$. Conversely, choose $g_1 \in G$ such that $||\gamma(g_1)||$ is minimal. Then we have $||\gamma(g_1)|| \geqslant 1$. This shows that $||\gamma(g)|| = 1$ for all $g \in G$.

The absolute values of entries of $P_\sigma^{-1} \sum_{g \in G} \gamma(g) A_g^W P_\sigma$ are 1. For all $g, h \in G$, we can see that $||a_g^{-1} a_h|| = 1$ and this shows $||a_g|| = 1$ for all $g \in G$. Now $W \sim_H W'$. □

Proof of Theorem 3.3. By Lemmas 3.8, 3.9, 3.10, $\Phi : H^2(G, \mathbb{C}^\times) \to \mathfrak{W}$ is bijective. By Lemmas 3.12, 3.13, $\Psi : \mathfrak{W}_H \to \mathfrak{W}$ is bijective. □

We consider weights having zero entries.

Proof of Theorem 3.4. Let W be a weight on $\mathfrak{X}(G)$ which can have zero entries. The support of W determines a subgroup K of G. We can write W as a block diagonal matrix, and all blocks are weights on K of the same factor set of K. We remark that the support is invariant under equivalence of weights. Conversely, for a factor set α of a subgroup K, we define a block diagonal matrix W with all diagonal blocks W_α. Then W is a weight on $\mathfrak{X}(G)$ with support K. Now the result holds. □

Remark 3.14. By the above arguments, we can see that every weight W on a finite group is equivalent to an H-weight W'. If the factor set of W is α and we choose β as in Proposition 3.1, then $W \sim W_\beta$, W_β is an H-weight, and every entry of W_β is 0 or a root of unity.

4. A construction

In this section, we will generalize the result by Higman [2] to construct weights on homogeneous coherent configurations. Higman used monomial representations of finite groups. We will define monomial representations of homogeneous coherent configurations, and construct weights on homogeneous coherent configurations. As we mentioned, homogeneous coherent configurations are association schemes in the sense in [8]. We will use terminologies in [8, 9].

We summarize the theory of homogeneous coherent configurations (association schemes) and their representations with reference to [3, 8].

Let $\mathfrak{X} = (X, C)$ be a homogeneous coherent configuration. The adjacency algebra $\mathbb{C}C$ is known to be semisimple. Thus we can write $\mathbb{C}C \cong \bigoplus_{i=1}^r M_{n_i}(\mathbb{C})$, and character theory works well. By $\operatorname{Irr}(C)$, we denote the set of all irreducible characters of $\mathbb{C}C$. Naturally $\mathbb{C}C$ acts on $\mathbb{C}X$, and we call $\mathbb{C}X$ the *standard module*. The *standard character* is also defined. The multiplicity of $\chi \in \operatorname{Irr}(C)$ in the standard character is called the *multiplicity* of χ and denoted by m_χ. There is a natural $\mathbb{C}C$-monomorphism from the regular module $\mathbb{C}C$ to the standard module $\mathbb{C}X$, and thus $\chi(1) \leqslant m_\chi$ holds for

$\chi \in \operatorname{Irr}(C)$. Let e_χ be the central primitive idempotent corresponding to $\chi \in \operatorname{Irr}(C)$. Then the rank of e_χ as an element in $M_X(\mathbb{C})$ is $m_\chi\chi(1)$.

A subset D of C is called a *closed subset* of C (or $\mathfrak{X}$) if $\mathbb{C}D = \bigoplus_{d\in D}\mathbb{C}A_d$ is a subalgebra of $\mathbb{C}C$ (see [9, Lemma 2.1.6]). Suppose that D is a closed subset. For $x \in X$, set $xD = \{y \in X \mid (x,y) \in \bigcup_{d\in D} d\}$. We have a partition $X = x_1D \cup \cdots \cup x_mD$. Then (x_iD, D_{x_iD}) is also a homogeneous coherent configuration, called a *sub homogeneous coherent configuration*, where $D_{x_iD} = \{d \cap (x_iD \times x_iD) \mid d \in D\}$. We remark that these sub homogeneous coherent configurations (x_iD, D_{x_iD}), $i = 1, \ldots, m$ are not necessarily isomorphic, but their adjacency algebras are isomorphic to $\mathbb{C}D$. Thus we can identify $\operatorname{Irr}(D_{x_iD})$ and write $\operatorname{Irr}(D)$. Set $X/D = \{x_1D, \ldots, x_mD\}$. Define $c^D = \{(x_iD, x_jD) \mid c \cap (x_iD \times x_jD) \neq \varnothing\}$ for $c \in C$ and $C/\!/D = \{c^D \mid c \in C\}$. Then $(X/D, C/\!/D)$ is a homogeneous coherent configuration, called the *factor homogeneous coherent configuration*. We remark that $C/\!/D$ defines a partition of C.

Now we define monomial representations (characters) of homogeneous coherent configurations. Let D be a closed subset of $\mathfrak{X} = (X, C)$, and let $\varphi \in \operatorname{Irr}(D)$ be of multiplicity one. The induced character $\varphi^{\uparrow C}$ is called a *monomial character* of $\mathfrak{X}$. Let $e_i \in M_{x_iD}(\mathbb{C})$ be the central primitive idempotent corresponding to $\varphi \in \operatorname{Irr}(D)$. Since $m_\varphi = 1$, the rank of e_i is 1. By [5, Theorem 2.8], φ is essentially a character of a cyclic group, namely, there exists a closed subset K of C such that the factors of its sub homogeneous coherent configurations are isomorphic cyclic groups. Thus we may assume that all e_i are same matrices e. Now we can set

$$e_\varphi = \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix} \in \mathbb{C}D \subset \mathbb{C}C \subset M_X(\mathbb{C}),$$

the primitive idempotent in $\mathbb{C}D$ corresponding to $\varphi \in \operatorname{Irr}(D)$. The right $\mathbb{C}D$-module $e_\varphi\mathbb{C}D$ affords the character φ, and so the induced module, the module of the monomial representation, is

$$e_\varphi\mathbb{C}D_{\mathbb{C}D} \otimes \mathbb{C}C \cong e_\varphi\mathbb{C}C.$$

We consider the endomorphism algebra $\operatorname{End}_{\mathbb{C}C}(e_\varphi\mathbb{C}C) \cong e_\varphi\mathbb{C}Ce_\varphi$. We set $|x_iD| = \ell$. Then the rank of $eM_\ell(\mathbb{C})e = 1$ and so $eM_\ell(\mathbb{C})e = \mathbb{C}e$. Thus

$$\begin{aligned} e_\varphi\mathbb{C}Ce_\varphi \subset e_\varphi M_X(\mathbb{C})e_\varphi &\cong \left\{ \begin{pmatrix} a_{11}e & \ldots & a_{1m}e \\ & \ldots & \\ a_{m1}e & \ldots & a_{mm}e \end{pmatrix} \,\middle|\, a_{ij} \in \mathbb{C} \quad (1 \leqslant i,j \leqslant m) \right\} \\ &\cong M_m(\mathbb{C}) \cong M_{X/D}(\mathbb{C}). \end{aligned}$$

This defines an injective algebra homomorphism $\Gamma : e_\varphi\mathbb{C}Ce_\varphi \to M_{X/D}(\mathbb{C})$.

We will choose representatives of c^D.

Lemma 4.1. *Suppose $e_\varphi A_c e_\varphi \neq 0$. Then, for $c' \in C$ with $c^D = c'^D$, there exists $\mu \in \mathbb{C}$ such that $e_\varphi A_{c'} e_\varphi = \mu e_\varphi A_c e_\varphi$.*

Proof. Write

$$A_c = \begin{pmatrix} (A_c)_{11} & \ldots & (A_c)_{1m} \\ & \ldots & \\ (A_c)_{m1} & \ldots & (A_c)_{mm} \end{pmatrix}, \quad A_{c'} = \begin{pmatrix} (A_{c'})_{11} & \ldots & (A_{c'})_{1m} \\ & \ldots & \\ (A_{c'})_{m1} & \ldots & (A_{c'})_{mm} \end{pmatrix}.$$

Then we can write

$$e_\varphi A_c e_\varphi = \begin{pmatrix} e(A_c)_{11}e & \ldots & e(A_c)_{1m}e \\ & \ldots & \\ e(A_c)_{m1}e & \ldots & e(A_c)_{mm}e \end{pmatrix} = \begin{pmatrix} a_{11}e & \ldots & a_{1m}e \\ & \ldots & \\ a_{m1}e & \ldots & a_{mm}e \end{pmatrix},$$

$$e_\varphi A_{c'} e_\varphi = \begin{pmatrix} e(A_{c'})_{11}e & \dots & e(A_{c'})_{1m}e \\ & \dots & \\ e(A_{c'})_{m1}e & \dots & e(A_{c'})_{mm}e \end{pmatrix} = \begin{pmatrix} b_{11}e & \dots & b_{1m}e \\ & \dots & \\ b_{m1}e & \dots & b_{mm}e \end{pmatrix}$$

for some $a_{ij}, b_{ij} \in \mathbb{C}$. Suppose $a_{st} \neq 0$. Set $L = a_{st}^{-1} b_{st} e_\varphi A_c e_\varphi - e_\varphi A_{c'} e_\varphi$. Then the (s,t)-part of L is 0. We remark that

$$(x_i D, x_j d) \in c^D \Longleftrightarrow (A_c)_{ij} \neq 0,$$
$$(x_i D, x_j d) \notin c^D \Longrightarrow e(A_c)_{ij} e = 0,\ a_{ij} = 0.$$

We put $U = \{c_1 \in C \mid c_1^D = c^D\}$. Since $L \in \mathbb{C}C$, we can write $L = \sum_{c_1 \in U} \mu(c_1) A_{c_1}$ for some $\mu(c_1) \in \mathbb{C}$. By the definition of c^D, every A_{c_1} $(c_1 \in U)$ has non-zero entries in the (s,t)-part, and thus $\mu(c_1) = 0$ for all $c_1 \in U$. Now $e_\varphi A_{c'} e_\varphi = a_{st}^{-1} b_{st} e_\varphi A_c e_\varphi$. □

Choose $c_\lambda \in C$ $(\lambda \in \Lambda)$ such that $C/\!/D = \{c_\lambda^D \mid \lambda \in \Lambda\}$, $c_\lambda^D \neq c_{\lambda'}^D$ if $\lambda \neq \lambda'$, and $e_\varphi A_{c_\lambda} e_\varphi \neq 0$ if such c_λ exists. Then $\{e_\varphi A_{c_\lambda} e_\varphi \mid \lambda \in \Lambda,\ e_\varphi A_{c_\lambda} e_\varphi \neq 0\}$ is a basis of $e_\varphi \mathbb{C}C e_\varphi$. We put

$$W = \sum_{\lambda \in \Lambda} \Gamma(e_\varphi A_{c_\lambda} e_\varphi)$$

and show that W is a weight on $(X/D, C/\!/D)$. We remark that c_λ is not unique to c_λ^D, but $\Gamma(e_\varphi A_{c_\lambda} e_\varphi)$ is unique up to scalar multiple by Lemma 4.1 and thus W is unique up to equivalence of weights.

Theorem 4.2. *Let $\mathfrak{X} = (X, C)$ be a homogeneous coherent configuration, and D a closed subset of C. Let φ be an irreducible character of D of multiplicity one. Then W defined above is a weight on the factor homogeneous coherent configuration $(X/D, C/\!/D)$ and $\mathbb{C}^W(C/\!/D) \cong e_\varphi \mathbb{C}C e_\varphi$.*

Proof. It is easy to see that $\mathrm{spt}(W) = \bigcup_c c^D$, where c runs over $\{c \in C \mid e_\varphi A_c e_\varphi \neq 0\}$. Suppose $c^D \subset \mathrm{spt}(W)$. We may assume $e_\varphi A_c e_\varphi \neq 0$. We remark that e is hermitian, because e is essentially a central primitive idempotent corresponding to a linear character of a finite group. Thus e_φ is also hermitian. We have $e_\varphi A_{c^*} e_\varphi = (e_\varphi A_c e_\varphi)^* \neq 0$, and so $(c^D)^* \subset \mathrm{spt}(W)$. The condition (W1) holds.

(W2) is clear. (W3) is also clear since $\mathbb{C}^W(C/\!/D)$ is the image of the algebra homomorphism Γ. □

Remark 4.3. In the above definition of W, we may suppose the representatives $\{c_\lambda\}$ are closed under the transposition. Then, by $e_\varphi A_{c^*} e_\varphi = (e_\varphi A_c e_\varphi)^*$, W is hermitian.

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