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Pattern bounds for principal specializations of β-Grothendieck polynomials

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Abstract There has been recent interest in lower bounds for the principal specializations of Schubert polynomials $\nu_w := \mathfrak{S}_w(1,\dots,1)$. We prove a conjecture of Yibo Gao in the setting of 1243-avoiding permutations that gives a lower bound for ν_w in terms of the permutation patterns contained in w. We extend this result to principal specializations of β-Grothendieck polynomials $\nu_w^{(\beta)} := \mathfrak{G}_w^{(\beta)}(1,\dots,1)$ by restricting to the class of vexillary 1243-avoiding permutations. Our methods are bijective, offering a combinatorial interpretation of the coefficients c_w and $c_w^{(\beta)}$ appearing in these conjectures.

1. Introduction

Schubert polynomials were introduced by Lascoux and Schützenberger [10] as canonical representatives for Schubert cycles in the cohomology ring of the complete flag variety $\mathcal{F}\ell(n)$, which can be identified with a quotient of the polynomial ring $\mathbb{Z}[x_1,\dots,x_n]$ by the ideal generated by symmetric polynomials with no constant term. In particular, given a permutation $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w(x)$ represents the cohomology class of the corresponding Schubert variety X_w.

There has been recent interest in establishing bounds for the *principal specialization* $\nu_w := \mathfrak{S}_w(1,\dots,1) \in \mathbb{Z}_{\geqslant 0}$, which is known to give the degree of the matrix Schubert variety corresponding to w [7]. The first result in this vein came from Weigandt [19] who showed that $\nu_w \geqslant 1 + p_{132}(w)$ where, in general, $p_u(w)$ represents the number of occurrences of the permutation pattern u in w (i.e. the number of subwords of w whose entries appear in the same relative order as u). This confirmed a conjecture of Stanley [17], who believed that $\nu_w = 2$ if and only if $p_{132}(w) = 1$.

Subsequently, Yibo Gao [4] improved this lower bound by showing that we can include the number of occurrences of 1432 patterns in w, that is, $\nu_w \geqslant 1 + p_{132}(w) + p_{1432}(w)$. In the same paper, Gao conjectured that there might be a way to include every permutation pattern in a lower bound for ν_w. Precisely, define the following sequence of coefficients c_w indexed by permutations w recursively:

(1) Set $c_\varnothing := 1$ (here $\varnothing$ is the unique permutation of size 0).

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(2) Assuming that c_u has been defined for all permutations u of size less than n, for $w \in S_n$ set
$$c_w := \nu_w - \sum_{u<w} c_u p_u(w)$$
where this sum is taken over all permutation patterns u properly contained in w (so $u \neq w$).

Conjecture 1.1 (Gao[(1)] [4]). *For all permutations w, $c_w \geqslant 0$.*

Note that if Conjecture 1.1 were true, then we would get a lower bound
$$\nu_w \geqslant \sum_{u<w} c_u p_u(w)$$
with equality exactly when $c_w = 0$. Since $c_\varnothing = c_{132} = c_{1432} = 1$ are the only nonzero values of c_w for permutations w of size $n \leqslant 4$, this would extend the previous bounds of Weigandt and Gao.

Conjecture 1.1 has been checked empirically to hold for all permutations $w \in S_n$ for $n \leqslant 8$ [4]. As more evidence, Mészáros and Tanjaya [15] showed that Conjecture 1.1 is true whenever w avoids both of the patterns 1432 and 1423.

In this paper, we prove Conjecture 1.1 for permutations w that do not contain the pattern 1243.

Theorem 1.2. *For all* 1243*-avoiding permutations w, $c_w \geqslant 0$.*

We briefly outline the proof of Theorem 1.2. It is known that ν_w enumerates a collection of diagrams corresponding to w called *reduced bumpless pipe dreams* [8]. To each of w's reduced BPDs, we assign both a subword v of w and another reduced BPD corresponding to the permutation pattern u that v is an occurrence of in w. Fixing v, this gives an injection from the reduced BPDs of w assigned to v into a certain subset of *minimal* reduced BPDs of u. This is bijective when w avoids the pattern 1243, in which case c_w will enumerate the set of minimal reduced bumpless pipe dreams for w and Theorem 1.2 will follow. In general, this yields an upper bound for the principal specialization ν_w in terms the minimal reduced BPDs of its permutation patterns (Corollary 3.5).

In parallel, we will consider the *β-Grothendieck polynomials* $\mathfrak{G}_w^{(\beta)}(x)$ of Fomin and Kirillov [2], which specialize to Schubert polynomials in the case $\beta = 0$. β-Grothendieck polynomials are known [6] to represent Schubert classes in the connective K-theory of the complete flag variety. We can set up the same machinery for principal specializations of β-Grothendieck polynomials $\nu_w^{(\beta)} := \mathfrak{G}_w^{(\beta)}(1,\dots,1)$, which in this case are polynomials in $\mathbb{Z}_{\geqslant 0}[\beta]$, by defining coefficients $c_w^{(\beta)} \in \mathbb{Z}[\beta]$ recursively as before:

(1) Set $c_\varnothing^{(\beta)} := 1$.

(2) Assuming that $c_u^{(\beta)}$ has been defined for all permutations u of size less than n, for $w \in S_n$ set
$$c_w^{(\beta)} := \nu_w^{(\beta)} - \sum_{u<w} c_u^{(\beta)} p_u(w).$$

Recall that a permutation is *vexillary* if it avoids the pattern 2143. The techniques used to prove Theorem 1.2 are also applicable in the Grothendieck setting by restricting our view to a smaller pattern-avoidance class of permutations.

Theorem 1.3. *For all vexillary* 1243*-avoiding permutations w, $c_w^{(\beta)} \in \mathbb{Z}_{\geqslant 0}[\beta]$.*

(1) Gao remarks that Conjecture 1.1 was also observed independently by Christian Gaetz.

The coefficients of $c_w^{(\beta)}$ have been verified by computer to be nonnegative for all permutations $w \in S_n$ of size $n \leqslant 9$. This, coupled with the establishment of Theorem 1.3, inspires us to make the following analogue of Conjecture 1.1.

Conjecture 1.4. *For all permutations w, $c_w^{(\beta)} \in \mathbb{Z}_{\geqslant 0}[\beta]$.*

We remark that ν_w and c_w are just the constant terms of $\nu_w^{(\beta)}$ and $c_w^{(\beta)}$, respectively, so Conjecture 1.4 is a strengthening of Conjecture 1.1.

2. Background & definitions

2.1. Pattern containment. Let S_n denote the set of permutations on the set $[n] := \{1, 2, \ldots, n\}$. We will write permutations as words in one-line notation. In other words, for $w \in S_n$ a permutation, we identify w with the word $w(1)w(2)\ldots w(n)$. Write $|w| := n$ for the size of w.

By a *subword* v of a permutation $w \in S_n$, we mean a map $v \colon [m] \to [n]$ obtained as a composition $[m] \to [n] \xrightarrow{w} [n]$ such that the first map is strictly increasing. As with permutations, we will write $v = v(1)v(2)\cdots v(m)$ and $|v| := m$ for the size of v. We use the relation $v \subseteq w$ to denote that v is a subword of w, whereas $v \subset w$ will denote the same with the additional constraint $v \neq w$. This notation is suggestive in that we can think of v as a subset of the inputs of w.

Let $w \in S_n$ be a permutation and let $v \subseteq w$ be a subword of size m. Define $\mathsf{perm}(v)$ to be the permutation $u \in S_m$ whose entries appear in the same relative order as the entries of v i.e. $v(i) \leqslant v(j)$ if and only if $u(i) \leqslant u(j)$ for all $i, j \in [m]$. As an example, the subword 253 of 12453 has $\mathsf{perm}(253) = 132$ since $2 < 3 < 5$.

Given two permutations $w \in S_n$ and $u \in S_m$, we define $p_u(w)$ to be the number of subwords $v \subseteq w$ satisfying $\mathsf{perm}(v) = u$. We say that w *avoids the pattern* u, or is u*-avoiding*, if $p_u(w) = 0$. Otherwise we say that w *contains the pattern* u.

Continuing our example, $p_{132}(12453) = 4$ (the occurrences are 143, 153, 243, and 253), so 12453 contains the pattern 132, even though 132 is not a *subword* of 12453. On the other hand, 12453 avoids the pattern 2143 since $p_{2143}(12453) = 0$.

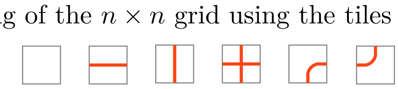
Permutations that are 2143-avoiding have a special name; they are known as *vexillary* permutations.

We will use the notation $u \leqslant w$ to denote that w contains the pattern u and the notation $u < w$ to additionally indicate that $u \neq w$.(2) It is important to make the distinction between the partial orders $\subseteq$, which is defined on *subwords* of w, and $\leqslant$, which is defined on *permutations* that appear as a pattern in w.

By convention, we define $[0]$ to be empty and let $\varnothing$ denote the unique permutation in S_0. Every permutation w contains $\varnothing$ as a pattern precisely once, that is $p_\varnothing(w) = 1$, and the occurrence of this pattern will also be denoted $\varnothing$.

We remark that the permutation statistics given by the pattern count functions p_u are interesting in their own right. As an example, $p_{21}(w)$ counts the number of *inversions* in w, which are pairs of indices (i, j) satisfying $i < j$ and $w(i) > w(j)$. This quantity is called the (*Coxeter*) *length* of w, which we denote by $\ell(w) := p_{21}(w)$.

2.2. Bumpless pipe dreams and alternating sign matrices. A *bumpless pipe dream* of size n is a filling of the $n \times n$ grid using the tiles



such that the resulting configuration is a network consisting of n "pipes," each entering from a column along the south edge of the grid and exiting through a row along the

(2)This is not to be confused with the *Bruhat order* on S_n, which is also often denoted by $\leqslant$.

east edge [8, 18]. In order from left-to-right, we will refer to these tiles as the *blank*, *horizontal*, *vertical*, *cross*, *r-elbow*, and *j-elbow*, respectively.

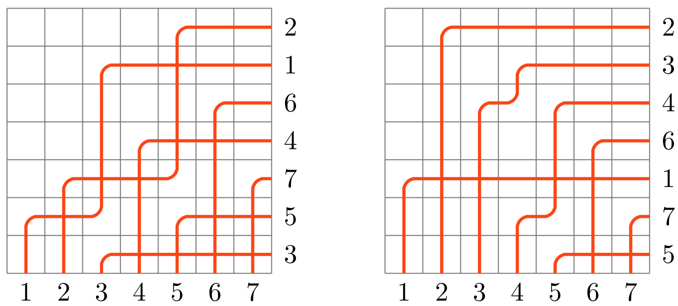


Figure 1. Two bumpless pipe dreams of size 7.

We let $\mathsf{BPD}(n)$ denote the collection of bumpless pipe dreams of size n. The standard convention is to plot each $B \in \mathsf{BPD}(n)$ using matrix coordinates. Write $B_{i,j}$ for the tile in B that is at position (i,j). We use the notation $B(\square)$ for the set of positions (i,j) such that $B_{i,j} = \square$ (and we make analogous definitions for $\boxminus$, $\boxbar$, $\boxplus$, $\ulcorner$, and $\lrcorner$).

We now make a brief digression to discuss a connection between bumpless pipe dreams and a related combinatorial object. An *alternating sign matrix* of size n is a square $n \times n$ matrix A consisting of the entries 0, $+1$, and -1 such that in each row and column of A, the entries sum to $+1$ with the nonzero entries alternating in sign. These conditions imply that each row and column of A has at least one nonzero entry with the initial and final nonzero entries always being $+1$'s. Let $\mathsf{ASM}(n)$ denote the set of alternating sign matrices of size n.

There is a bijection $\Phi\colon \mathsf{BPD}(n) \to \mathsf{ASM}(n)$ [18], where for B a bumpless pipe dream of size n, we get an $n \times n$ alternating sign matrix via the formula

$$\Phi(B)_{i,j} := \begin{cases} +1 & B_{i,j} = \ulcorner\,, \\ -1 & B_{i,j} = \lrcorner\,, \\ 0 & \text{else.} \end{cases}$$

This means that the elbows in each row and each column of B are alternating between $\ulcorner$ and $\lrcorner$ with the initial and final elbows always being $\ulcorner$'s. It is often helpful to have this alternative view of bumpless pipe dreams.

We now return to our introduction of bumpless pipe dreams. To each $B \in \mathsf{BPD}(n)$, we can assign a permutation $w(B) \in S_n$ by setting $w(B)(x) := y$ if and only if $y \to x$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 & +1 \\ +1 & 0 & -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 2. The alternating sign matrix $\Phi(B)$ corresponding to the first bumpless pipe dream B in Figure 1.

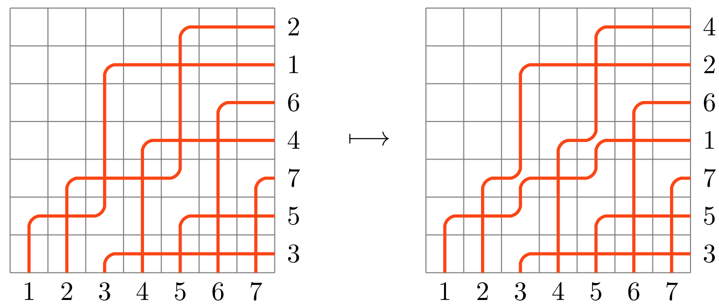
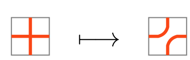


FIGURE 3. The resulting network of pipes $\mathsf{reduce}(B)$ obtained from the first bumpless pipe dream B in Figure 1.

is a pipe in B (that is, the pipe in B entering from column y exits in row x). We call $w(B)$ the *wiring* of the bumpless pipe dream B. The bumpless pipe dreams in Figure 1 have wirings 2164753 and 2346175, respectively. Write Wires(w) for set of bumpless pipe dreams that have wiring w.

There is an alternative way to interpret how the network of pipes in B behaves at crosses; namely, two pipes will cross only at the first $\boxplus$ they partake in, otherwise diverging away from each other at subsequent crosses. To make this precise, totally order $B(\boxplus)$ first by rows from bottom-to-top, then within each row from left-to-right. We now modify B as follows: for each $(i, j) \in B(\boxplus)$ in the prescribed order, check if the pipes passing through (i, j) also cross at another position southwest of (i, j). If they do, change the tile at (i, j) from a cross to a *bump*:(3)



Let $\mathsf{reduce}(B)$ denote the diagram obtained by this process. See Figure 3 for an example.

We get another permutation in S_n associated to B, denoted $\partial(B)$, by setting $\partial(B)(x) := y$ if and only if $y \to x$ is a pipe appearing in $\mathsf{reduce}(B)$. It turns out that this is the more natural permutation to attach to B, so we call $\partial(B)$ <u>the</u> *permutation* of B. Keeping with our example, the first bumpless pipe dream in Figure 1 has permutation 4261753, as shown in Figure 3. The second BPD in Figure 1 has no double crossings between its pipes and hence remains unchanged by this process, so it has permutation 2356175. Let $\mathsf{BPD}(w)$ denote the set of bumpless pipe dreams with permutation w.

We say that a bumpless pipe dream is *reduced* if no two of its pipes cross more than once. A bumpless pipe dream B is reduced if and only if $\mathsf{reduce}(B) = B$, and in this case the wiring of B and the permutation of B coincide. In Figure 1, the second BPD is reduced while the first is not. Write $\mathsf{BPD}_0(n)$ for the set of reduced bumpless pipe dreams of size n. For $w \in S_n$, we also write $\mathsf{BPD}_0(w) = \mathsf{BPD}(w) \cap \mathsf{BPD}_0(n)$ for the set of reduced BPDs with permutation w. Notice that equivalently $\mathsf{BPD}_0(w) = \mathsf{Wires}(w) \cap \mathsf{BPD}_0(n)$.

There is a unique bumpless pipe dream of size 0 which is reduced and has permutation $\varnothing$. We also use the symbol $\varnothing$ to denote this bumpless pipe dream.

2.3. GROTHENDIECK AND SCHUBERT POLYNOMIALS.

Throughout this subsection, fix a permutation $w \in S_n$.

(3)This eponymous tile, absent in the bumpless pipe dream model, is present in the older *pipe dream* model.

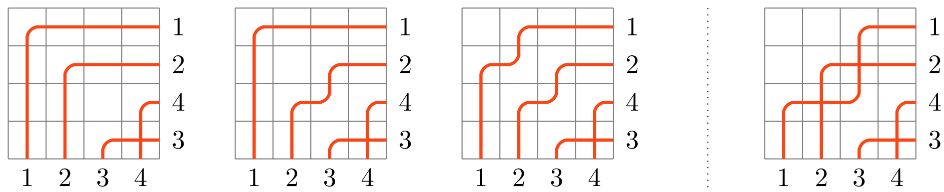


Figure 4. The four bumpless pipe dreams in $\mathsf{Wires}(1243)$. The three on the left are reduced and form $\mathsf{BPD}_0(1243) = \mathsf{BPD}(1243)$, while the one on the right is not.

The β-*Grothendieck polynomial* for w is a polynomial in $\mathbb{Z}_{\geqslant 0}[\beta][x_1, x_2, \dots]$ which we will define as the generating function

$$\mathfrak{G}_w^{(\beta)}(x_1, \dots, x_{n-1}) := \sum_{B \in \mathsf{BPD}(w)} \beta^{-\ell(w)} \Bigg(\prod_{(i,j) \in B(\square)} \beta x_i \Bigg) \Bigg(\prod_{(i,j) \in B(\ulcorner)} (1 + \beta x_i) \Bigg). \tag{1}$$

It is known that $|B(\square)| \geqslant \ell(w)$ with equality if and only if B is reduced [18], so $\mathfrak{G}_w^{(\beta)}(x)$ is indeed a polynomial in $\mathbb{Z}_{\geqslant 0}[\beta][x_1, x_2, \dots]$.

We compute the β-Grothendieck polynomial for $w = 1243$ as an example. The 3 bumpless pipe dreams for in $\mathsf{BPD}(1243)$ are illustrated in Figure 4. Hence $\mathfrak{G}_{1243}^{(\beta)}(x_1, x_2, x_3) = x_3 + x_2(1 + \beta x_3) + x_1(1 + \beta x_2)(1 + \beta x_3) = x_1 + x_2 + x_3 + \beta(x_1 x_2 + x_1 x_3 + x_2 x_3) + \beta^2 x_1 x_2 x_3$.

β-Grothendieck polynomials were originally introduced by Fomin and Kirillov [2] to study solutions to the Yang–Baxter equation. The above formulation comes from work of Lascoux [9] which was later translated by Weigandt [18] using the language of bumpless pipe dreams.

Of interest to us is the *principal specialization* of the β-Grothendieck polynomial $\nu_w^{(\beta)} := \mathfrak{G}_w^{(\beta)}(1, \dots, 1)$, which itself is a polynomial in $\mathbb{Z}_{\geqslant 0}[\beta]$. From the above, $\nu_w^{(\beta)}$ can be calculated as

$$\nu_w^{(\beta)} = \mathfrak{G}_w^{(\beta)}(1, \dots, 1) = \sum_{B \in \mathsf{BPD}(w)} \beta^{|B(\square)| - \ell(w)} (1 + \beta)^{|B(\ulcorner)|}.$$

Continuing our example above, we have $\nu_{1243}^{(\beta)} = 3 + 3\beta + \beta^2$. To simplify notation, define the β-*weight* of a bumpless pipe dream $B \in \mathsf{BPD}(w)$ to be the quantity $\mathrm{wt}^{(\beta)}(B) := \beta^{|B(\square)| - \ell(w)} (1 + \beta)^{|B(\ulcorner)|} \in \mathbb{Z}_{\geqslant 0}[\beta]$. Extend this weight to collections of bumpless pipe dreams $\mathcal{S} \subseteq \mathsf{BPD}(n)$ by setting $\mathrm{wt}^{(\beta)}(\mathcal{S}) := \sum_{B \in \mathcal{S}} \mathrm{wt}^{(\beta)}(B)$. Then $\nu_w^{(\beta)} = \mathrm{wt}^{(\beta)}(\mathsf{BPD}(w))$.

Let $B \in \mathsf{BPD}(w)$ be a bumpless pipe dream with permutation w. The term contributed by B in Equation 1 has a factor of β if and only if $|B(\square)| > \ell(w)$, which, as we remarked, occurs exactly when B is not reduced. The polynomial $\mathfrak{S}_w(x) := \mathfrak{G}_w^{(0)}(x)$ obtained by specializing $\beta = 0$ is thus a generating function over reduced bumpless pipe dreams:

$$\mathfrak{S}_w(x_1, \dots, x_{n-1}) = \sum_{B \in \mathsf{BPD}_0(w)} \prod_{(i,j) \in B(\square)} x_i.$$

This is the *Schubert polynomial* for w of Lascoux and Schützenberger [10, 8]. For example, the Schubert polynomial for $w = 1243$ is $\mathfrak{S}_{1243}(x_1, x_2, x_3) = x_1 + x_2 + x_3$, the β-degree 0 component of $\mathfrak{G}_{1243}^{(\beta)}$. Writing $\nu_w := \nu_w^{(0)}$, we see that the principal specialization of the Schubert polynomial $\mathfrak{S}_w(x)$ gives an enumeration

$$\nu_w = \mathfrak{S}_w(1, \dots, 1) = |\mathsf{BPD}_0(w)|.$$

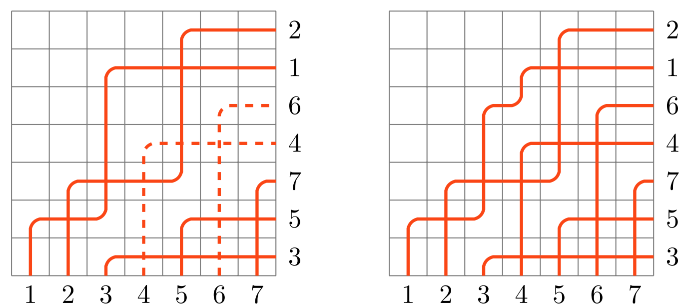


Figure 5. On the left, the first bumpless pipe dream from Figure 1 with its two removable pipes indicated via dashed lines. On the right, a similar BPD which is minimal.

We end this section with a few remarks. The first is that there are other specializations of β that appear in the literature. Taking $\beta = -1$ recovers the (ordinary) Grothendieck polynomials $\mathfrak{G}_w(x) = \mathfrak{G}_w^{(-1)}(x)$ of Lascoux and Schützenberger [11]. With $\beta = 1$, notice that the 1-weight of a bumpless pipe dream $B \in \mathsf{BPD}(n)$ is simply $\mathrm{wt}^{(1)}(B) = 2^{|B(\lrcorner)|}$. This weight is interesting when considering the bijection to alternating sign matrices: since the number of $\lrcorner$'s in B is the same as the number of -1's in $\Phi(B)$, $\mathrm{wt}^{(1)}(B)$ is the same as the weight of $\Phi(B)$ in what is known as the 2-*enumeration* of $\mathsf{ASM}(n)$ [1].

It should also be noted that the specializations $\nu_w^{(\beta)}$ have previously been studied for w vexillary, where a determinantal upper bound was established in terms of Delannoy paths on Young tableaux [16]. There is a q-analog of $\nu_w^{(\beta)}$ given by $\mathfrak{G}_w^{(\beta)}(1, q, q^2, \dots)$ where we instead send $x_i \mapsto q^{i-1}$. Fomin and Stanley [3] proved an elegant formula (originally conjectured by Macdonald [12]) which computes the Schubert q-analog principal specialization $\mathfrak{S}_w(1, q, q^2, \dots)$ as a weighted sum over the reduced words for w. This specializes to a formula for ν_w by taking $q = 1$. Marberg and Pawlowski [13] proved an analogous formula for the q-specializations of *back stable* Schubert and β-Grothendieck polynomials, which are power series variants of Schubert and β-Grothendieck polynomials that admit variables x_i with indices $i \leqslant 0$.

3. Main results

3.1. Removing pipes.

Definition 3.1. *Let $B \in \mathsf{BPD}(n)$.*

- *Say that a pipe $y \to x$ in B is hook-shaped if it only contains a single bend, which necessarily must occur at $(x, y) \in B(\ulcorner)$.*
- *Say that a pipe $y \to x$ in B is removable if it is hook-shaped, and (x, y) is the only $\ulcorner$ occurring in row x and in column y.*
- *Say that B is minimal if it contains no removable pipes.*

In the language of alternating sign matrices, as a pipe $y \to x$ is removable in B if and only if $\Phi(B)_{x,y} = +1$, and this is the only nonzero entry in row x and column y of $\Phi(B)$.

Fix a permutation $w \in S_n$. For $B \in \mathsf{Wires}(w)$ a BPD with (complete set of) removable pipes $y_1 \to x_1, \dots, y_r \to x_r$, we get a subword $v \subseteq w$ of size $n-r$ simply by ignoring the values $y_1, \dots, y_r$ in w. For the BPD on the left in Figure 5, this gives the subword 21753 of 2164753. Let $\mathsf{Wires}(w; v)$ denote the collection of all $B \in \mathsf{Wires}(w)$

that correspond to the subword $v \subseteq w$ in this way. This gives a partition

$$\mathsf{Wires}(w) = \bigsqcup_{v \subseteq w} \mathsf{Wires}(w; v).$$

In particular, $\mathsf{Wires}(w; w)$ is the set of minimal bumpless pipe dreams with wiring w.

Now, fix a subword $v \subseteq w$ of size m and write $u = \mathsf{perm}(v) \in S_m$. Let $y_1 \to x_1, \ldots, y_r \to x_r$ denote the removable pipes in any bumpless pipe dream in $\mathsf{Wires}(w; v)$. In other words, $r = n - m$, $y_1, \ldots, y_r$ are the entries in w not appearing in v, and $x_1, \ldots, x_r$ are the corresponding indices. Assume $y_1 < \cdots < y_r$. We also write $S = \{s_1 < \cdots < s_m\} = [n] \smallsetminus \{x_1, \ldots, x_r\}$ for the set of positions of v as a subword in w and $T = \{t_1 < \cdots < t_m\} = [n] \smallsetminus \{y_1, \ldots, y_r\}$ for the set of entries of v. Note that this means $w(s_i) = v(i) = t_{u(i)}$ for all $i \in [m]$.

Let $B \in \mathsf{Wires}(w; v)$. We now describe a procedure for removing the (suggestively named) removable pipes from B. This will produce a minimal pipe dream $\mathsf{minimize}(B) \in \mathsf{Wires}(u; u)$. The process is most easily described by viewing B as its corresponding alternating sign matrix $\Phi(B)$. In this setting, we simply take the $m \times m$ submatrix $\Phi(B)_{S,T}$ of $\Phi(B)$ on rows S and columns T. Since $\Phi(B)_{x_1,y_1}, \ldots, \Phi(B)_{x_r,y_r}$ are precisely the $+1$'s in $\Phi(B)$ with the property that they are the only nonzero entry in their row and column, the nonzero entries in each row of $\Phi(B)$ indexed by S occur only in the columns indexed by T, and vice versa, so the property of being an ASM is preserved.

We now describe the map $B \mapsto \mathsf{minimize}(B)$ using bumpless pipe dreams. We do this in three steps:

(1) Erase the pipes $y_1 \to x_1, \ldots, y_r \to x_r$ from B.
(2) Since each pipe $y_k \to x_k$ ($1 \leqslant k \leqslant r$) was removable, the remaining tiles in row x_k consist entirely of $\square$ and  tiles. We can now *contract* the rows $x_1, \ldots, x_r$ whilst maintaining the connectivity of the pipes that pass through them (i.e. delete the cells in these rows from B and then top-justify the resulting diagram).
(3) Likewise, the remaining tiles in each column y_k ($1 \leqslant k \leqslant r$) consist entirely of $\square$ and  tiles. We can now *contract* the columns $y_1, \ldots, y_r$ whilst maintaining the connectivity of the pipes that pass through them (i.e. delete the cells in these columns from B and then left-justify the resulting diagram).

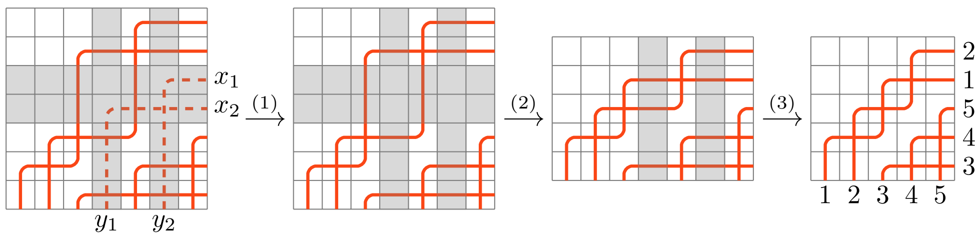
See Figure 6 for an example. Note that $\mathsf{minimize}(B)_{i,j} = B_{s_i,t_j}$ for all $i, j \in [m]$. Since all the wires remain connected, $\mathsf{minimize}(B)$ is a bumpless pipe dream which, in particular, is the bumpless pipe dream corresponding to the ASM $\Phi(B)_{S,T}$.

Lemma 3.2. *Let $w \in S_n$ be a permutation, and let $v \subseteq w$ be a subword with $u = \mathsf{perm}(v)$. If $B \in \mathsf{Wires}(w; v)$, then $\mathsf{minimize}(B) \in \mathsf{Wires}(u; u)$.*

Proof. Adopt the notation from above. We first show that $\mathsf{minimize}(B) \in \mathsf{Wires}(u)$. By construction, $j \to i$ is a pipe in $\mathsf{minimize}(B)$ if and only if $t_j \to s_i$ is a pipe in B. The pipes of this form in B are exactly the pipes $t_{u(i)} \to s_i$ (since $w(s_i) = t_{u(i)}$), so $u(i) \to i$ are exactly the pipes in $\mathsf{minimize}(B)$.

It remains to show that $\mathsf{minimize}(B)$ is minimal. This is clear on the ASM side: if (i, j) is a $+1$ in $\Phi(B)_{S,T}$, then (s_i, t_j) is a $+1$ in $\Phi(B)$ which is not among the $+1$'s corresponding to removable pipes: $(x_1, y_1), \ldots, (x_r, y_r)$. Hence there is another $+1$ in $\Phi(B)$ at some position $(s_{i'}, t_{j'})$ with $|\{i, j\} \cap \{i', j'\}| = 1$. Then $\Phi(B)_{S,T}$ has a $+1$ at (i', j'), so the $+1$ at (i, j) does not correspond to a removable pipe. Therefore $\mathsf{minimize}(B)$ is minimal. $\square$

Proposition 3.3. *Let $w \in S_n$ be a permutation, and let $v \subseteq w$ be a subword with $u = \mathsf{perm}(v)$. Then $\mathsf{minimize}\colon \mathsf{Wires}(w; v) \to \mathsf{Wires}(u; u)$ is a bijection.*



$$\begin{pmatrix} 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 & +1 \\ +1 & 0 & -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & +1 & 0 & -1 & +1 \\ +1 & 0 & -1 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 \end{pmatrix}$$

FIGURE 6. Top: minimize applied to the first bumpless pipe dream B in Figure 1 ($w = 2164753, v = 21753$). The final diagram $\mathsf{minimize}(B)$ is a minimal bumpless pipe dream with wiring $\mathsf{perm}(21753) = 21543$. Bottom: The corresponding map on ASMs.

Proof. Lemma 3.2 tells us that minimize is a well-defined map from $\mathsf{Wires}(w; v)$ to $\mathsf{Wires}(u; u)$. We shall show that minimize is a bijection by constructing an inverse map.

For $B \in \mathsf{Wires}(u; u)$, perform the following:

(1′) Expand out the columns in B by moving them to positions $t_1 < \cdots < t_m$ and then adding and tiles as appropriate in the gap positions between columns (i.e. in columns $y_1, \dots, y_r$) to preserve the linkage of pipes.

(2′) Expand out the rows in B by moving them to positions $s_1 < \cdots < s_m$ and then adding and tiles as appropriate in the gap positions between rows (i.e. in rows $x_1, \dots, x_r$) to preserve the linkage of pipes.

(3′) Add hook-shaped pipes $y_1 \to x_1, \dots, y_r \to x_r$ to B. We are able to do this since each row x_k only contains and tiles while each column y_k only contains and tiles ($1 \leqslant k \leqslant r$).

Let $\mathsf{expand}_v^w(B)$ denote the diagram obtained by this process. Since all pipes remain connected, $\mathsf{expand}_v^w(B)$ is a bumpless pipe dream of size n. Since $j \to i$ is a pipe in B if and only if $t_j \to s_i$ is a pipe in $\mathsf{expand}_v^w(B)$, and the pipes in B are exactly $u(i) \to i$ for $i \in [m]$, each $w(s_i) = t_{u(i)} \to s_i$ is a pipe in $\mathsf{expand}_v^w(B)$. The addition of the pipes $y_1 \to x_1, \dots, y_r \to x_r$ thus ensures that $\mathsf{expand}_v^w(B)$ has wiring w.

We need to show that $\mathsf{expand}_v^w(B) \in \mathsf{Wires}(w; v)$; that is, the removable pipes of $\mathsf{expand}_v^w(B)$ are precisely $y_1 \to x_1, \dots, y_r \to x_r$. This is again clear from the ASM perspective: the ASM corresponding to $\mathsf{expand}_v^w(B)$ is the $n \times n$ matrix A that has submatrix $A_{S,T} = \Phi(B)$, has $A_{x_k, y_k} = +1$ for all $k \in [r]$, and is zero elsewhere. It is immediate that each $y_k \to x_k$ is removable in $\mathsf{expand}_v^w(B)$, since each (x_k, y_k) is a $+1$ in A that is the only nonzero entry in its row and column.

We now look at the other $+1$'s in A, which are necessarily of the form (s_i, t_j) for some $i, j \in [m]$. Then (i, j) is a $+1$ in $\Phi(B)$, which is minimal, so there is another $+1$ in $\Phi(B)$ at a position (i', j') with $|\{i, j\} \cap \{i', j'\}| = 1$. Then $(s_{i'}, t_{j'})$ is a $+1$ in A, so the $+1$ at (s_i, t_j) cannot correspond to a removable pipe. We conclude that $\mathsf{expand}_v^w(B)$ is reduced.

It is clear that the procedure defining expand_v^w will undo the procedure defining $\mathsf{minimize}$, and vice versa. $\square$

We now examine how $\mathsf{minimize}$ acts on reduced bumpless pipe dreams. Write $\mathsf{BPD}_0(w;v) := \mathsf{Wires}(w;v) \cap \mathsf{BPD}_0(w)$. Since $\mathsf{BPD}_0(w) \subseteq \mathsf{Wires}(w)$, we get a decomposition

$$\mathsf{BPD}_0(w) = \bigsqcup_{v \subseteq w} \mathsf{BPD}_0(w;v).$$

Proposition 3.4. *Let $w \in S_n$ be a permutation, and let $v \subseteq w$ be a subword with $u = \mathsf{perm}(v)$. If $B \in \mathsf{BPD}_0(w;v)$, then $\mathsf{minimize}(B) \in \mathsf{BPD}_0(u;u)$.*

Proof. Consider two pipes $j \to i$ and $j' \to i'$ in $\mathsf{minimize}(B)$. Observe that $j \to i$ and $j' \to i'$ cross in $\mathsf{minimize}(B)$ at a position $(a,b) \in \mathsf{minimize}(B)(\boxplus)$ if and only if the corresponding pipes $t_j \to s_i$ and $t_{j'} \to s_{i'}$ in B cross at position $(s_a, t_b) \in B(\boxplus)$. This is since all $\boxplus$ tiles in B outside of the $S \times T$ subdiagram involve at least one of the pipes $y_k \to x_k$. Hence the number of times $j \to i$ and $j' \to i'$ cross in $\mathsf{minimize}(B)$ is equal to the number of times $t_j \to s_i$ and $t_{j'} \to s_{i'}$ cross in B, but this latter quantity is at most one since B is reduced. We conclude that $\mathsf{minimize}(B)$ is reduced. $\square$

The previous proposition gives us the inequality $|\mathsf{BPD}_0(w;v)| \leqslant |\mathsf{BPD}_0(u;u)|$, since $\mathsf{minimize}$ is injective on $\mathsf{Wires}(w;v)$. We will give a criterion on w for when $\mathsf{minimize}(\mathsf{BPD}_0(w;v)) = \mathsf{BPD}_0(u;u)$ in Proposition 3.9. For now, we have the following.

Corollary 3.5. *Let $w \in S_n$ be a permutation. There is an upper bound*

$$\nu_w = |\mathsf{BPD}_0(w)| \leqslant \sum_{u \leqslant w} |\mathsf{BPD}_0(u;u)| \cdot p_u(w).$$

Proof. We have

$$\begin{aligned} |\mathsf{BPD}_0(w)| &= \sum_{v \subseteq w} |\mathsf{BPD}_0(w;v)| \\ &\leqslant \sum_{v \subseteq w} |\mathsf{BPD}_0(\mathsf{perm}(v);\mathsf{perm}(v))| \\ &= \sum_{u \leqslant w} |\mathsf{BPD}_0(u;u)| \cdot p_u(w). \end{aligned}$$ $\square$

For $B \in \mathsf{BPD}(n)$, notice that none of the steps in obtaining $\mathsf{minimize}(B)$ from B change the number of $\lrcorner$ tiles appearing in the diagram, i.e. $\big|\mathsf{minimize}(B)(\lrcorner)\big| = \big|B(\lrcorner)\big|$. When $B \in \mathsf{BPD}_0(w)$ is reduced (so $|B(\square)| = \ell(w)$), we know that $\mathrm{wt}^{(\beta)}(B) = (1+\beta)^{|B(\lrcorner)|}$ only depends on $\big|B(\lrcorner)\big|$. Combining these with Proposition 3.4 gives the following result:

Proposition 3.6. *Let $B \in \mathsf{BPD}_0(n)$ be reduced. Then $\mathrm{wt}^{(\beta)}(B) = \mathrm{wt}^{(\beta)}(\mathsf{minimize}(B))$.*

Proof. Let $B \in \mathsf{BPD}_0(n)$ be a reduced bumpless pipe dream. Then $\mathsf{minimize}(B)$ is reduced by Proposition 3.4, so by the discussion above

$$\mathrm{wt}^{(\beta)}(B) = (1+\beta)^{|B(\lrcorner)|} = (1+\beta)^{|\mathsf{minimize}(B)(\lrcorner)|} = \mathrm{wt}^{(\beta)}(\mathsf{minimize}(B)).$$ $\square$

3.2. 1243-avoiding and vexillary permutations. We will now investigate properties of bumpless pipe dreams for permutations that avoid the patterns 1243 and 2143.

Proposition 3.7. *Let $B \in \mathsf{BPD}(n) \smallsetminus \mathsf{BPD}_0(n)$ be a bumpless pipe dream that is not reduced.*

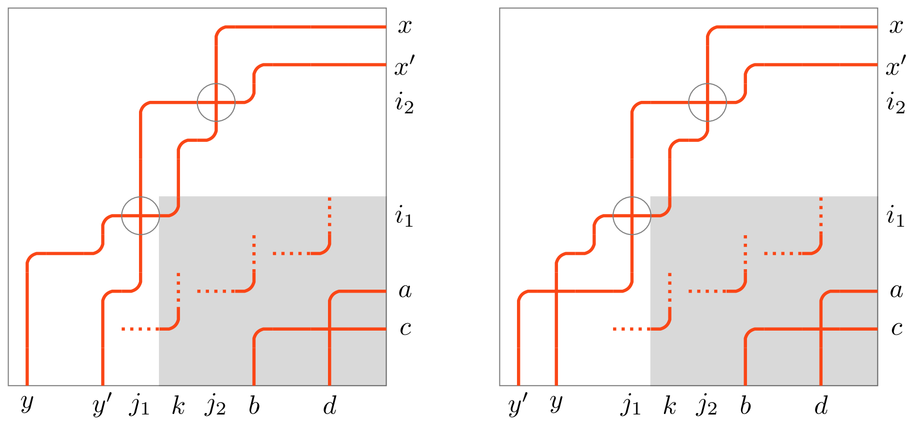


FIGURE 7. The two configurations of the pipes $y \to x$ and $y' \to x'$ in B as in the proof of Proposition 3.7 depending on if they cross an even or an odd number of times. The final two crosses shared by these pipes are circled. In each diagram, the shaded region consists of the positions weakly southeast of the at (i_1, k) and are populated with candidate maximal southeast 's (the partially drawn pipes). In the even configuration, we get the pattern 1243; in the odd configuration, we get the pattern 2143.

(a) *The wiring of B, $w(B)$, contains a* 1243 *pattern or a* 2143 *pattern. In particular, if B has two pipes that cross a nonzero even number of times, then $w(B)$ contains a* 1243 *pattern, and if B has two pipes that cross an odd number of times, then $w(B)$ contains a* 2143 *pattern.*

(b) *The permutation of B, $\partial(B)$, contains a* 2143 *pattern.*

Proof. Since B is not reduced, it has two pipes $y \to x$ and $y' \to x'$ $(x < x')$ that cross at least twice. Let (i_1, j_1) and (i_2, j_2) be the locations of the final two 's shared by $y \to x$ and $y' \to x'$ that we encounter when traversing $y \to x$ from south to east. Note that the at (i_1, j_1) must lie strictly southwest of the at (i_2, j_2), whereas both of these 's lie weakly southeast of both (x, y) and (x', y'). This means that $x < x' \leqslant i_2 < i_1$ and $\max\{y, y'\} \leqslant j_1 < j_2$. In order to exit in row $x < x'$, $y \to x$ must pass to underneath $y' \to x'$ horizontally through (i_1, j_1) and then emerge back above vertically through (i_2, j_2), at which point $y \to x$ will stay above $y' \to x'$ for the remainder of its path. This means that $y \to x$ must engage in a at some position (i_1, k) to the east of (i_1, j_1) (so $k > j_1$) in order to turn from east to north and eventually hit (i_2, j_2).

Of the 's in B that lie weakly southeast of (i_1, k) (including (i_1, k)), let (a, b) be the position of one chosen to be maximally southeast (so $a \geqslant i_1$, $b \geqslant k$, and $B_{a',b'} \neq$ for all $a' \geqslant a$, $b' \geqslant b$ with $(a', b') \neq (a, b)$). Observe that the pipe entering in column b must have its first at some position (c, b) strictly south of (a, b) (so $c > a$), at which point it cannot engage in a by the maximality of (a, b). This means that $b \to c$ is a hook-shaped pipe in B. Similarly, the pipe exiting in row a must have its final at some position (a, d) strictly to the east of (a, b) (so $d > b$), below which no can occur since (a, b) was chosen maximally, so $d \to a$ is also a hook-shaped pipe in B. See Figure 7 for a reference illustration.

We now have the sequences $x < x' \leqslant i_2 < i_1 \leqslant a < c$ and $\max\{y, y'\} \leqslant j_1 < k \leqslant b < d$. Since $y \to x$, $y' \to x'$, $b \to c$, and $d \to a$ are all pipes in B, $yy'db$ is a subword of $w(B)$ which is either an occurrence of the pattern 1243 (if $y < y'$) or an occurrence of the pattern 2143 (if $y > y'$). The first situation occurs precisely when $y \to x$ and $y' \to x'$ cross an *even* number of times, in which case $y \to x$ enters and exists on the outside of $y' \to x'$. When $y \to x$ and $y' \to x'$ cross an *odd* number of times, $y \to x$ still exits above $y' \to x'$ but instead enters underneath $y' \to x'$, so we get the second case. This completes (a).

To get (b), we need to see how the configuration of pipes in B changes as we pass to $\mathsf{reduce}(B)$. Observe that the hook-shaped pipes $b \to c$ and $d \to a$ lie to the southeast of the at (a, b). The southeast maximality of (a, b) tells us that there can be no 's underneath the hooks of either $b \to c$ and $d \to a$. In particular, $b \to c$ and $d \to a$ each cannot engage in double crosses with other pipes in B, so $b \to c$ and $d \to a$ are also pipes in $\mathsf{reduce}(B)$.

Since the tile at (i_1, j_1) is a $\boxplus$ in B, it is either a $\boxplus$ or a in $\mathsf{reduce}(B)$. Suppose that $t \to s$ and $t' \to s'$ ($s < s'$) are the pipes in $\mathsf{reduce}(B)$ meeting at (i_1, j_1). These pipes must cross in $\mathsf{reduce}(B)$; this is obvious if $\mathsf{reduce}(B)_{i_1,j_1} = \boxplus$. If instead $\mathsf{reduce}(B)_{i_1,j_1} =$, then these pipes must have crossed somewhere previously in $\mathsf{reduce}(B)$ as dictated by the construction of $\mathsf{reduce}(B)$. Since pipes in $\mathsf{reduce}(B)$ can cross at most once, $t \to s$ and $t' \to s'$ cross exactly once. Hence $t \to s$ enters $\mathsf{reduce}(B)$ to the right of $t' \to s'$ along the south edge and exits $\mathsf{reduce}(B)$ above $t' \to s'$ along the east edge. In particular, $t' < t$.

Since the pipes $t \to s$ and $t' \to s'$ visit the tile (i_1, j_1), we must have that $s' \leqslant i_1$ and $t \leqslant j_1$. Then $s < s' \leqslant i_1 \leqslant a < c$ and $t' < t \leqslant i_1 < k \leqslant b < d$. We conclude that $tt'bd$ is an occurrence of the pattern 2143 in $\partial(B)$, proving (b). □

Corollary 3.8. *Let $w \in S_n$ be a permutation.*

(a) *If w is vexillary* (2143*-avoiding), then* $\mathsf{BPD}(w) = \mathsf{BPD}_0(w)$.
(b) *If w is vexillary and* 1243*-avoiding, then* $\mathsf{Wires}(w) = \mathsf{BPD}(w) = \mathsf{BPD}_0(w)$.

Proof. In general, we have that $\mathsf{BPD}_0(w) \subseteq \mathsf{Wires}(w) \cap \mathsf{BPD}(w)$. If there is a nonreduced $B \in \mathsf{BPD}(w) \smallsetminus \mathsf{BPD}_0(w)$, then Proposition 3.7(b) says that w contains a 2143 pattern. This shows (a) and part of (b). Similarly, if there is $B \in \mathsf{Wires}(w) \smallsetminus \mathsf{BPD}_0(w)$ nonreduced, then w contains either 1243 or 2143 by Proposition 3.7(a). This gives the other part of (b). □

We remark that Corollary 3.8(a) was also proved by Weigandt [18].

3.3. Proofs of Theorems 1.2 and 1.3. We now move to prove our main results. Assume the notation from Section 3.1. In other words, for $w \in S_n$ a permutation and $v \subseteq w$ a subword with $u = \mathsf{perm}(v) \in S_m$, we write $y_1 \to x_1, \ldots, y_r \to x_r$ ($r = n - m$, $y_1 < \cdots < y_r$) for the removable pipes of any/all bumpless pipe dreams in $\mathsf{Wires}(w; v)$ and set $S = \{s_1 < \cdots < s_m\} = [n] \smallsetminus \{x_1, \ldots, x_r\}$, $T = \{t_1 < \cdots < t_m\} = [n] \smallsetminus \{y_1, \ldots, y_r\}$.

Proposition 3.9. *Let $w \in S_n$ be a permutation, and let $v \subseteq w$ be a subword with $u = \mathsf{perm}(v)$. If w is* 1243*-avoiding, then* $\mathsf{minimize}\colon \mathsf{BPD}_0(w; v) \to \mathsf{BPD}_0(u; u)$ *is a bijection.*

Proof. Recall that $\mathsf{minimize}\colon \mathsf{BPD}(w; v) \to \mathsf{BPD}(u; u)$ is a bijection (Proposition 3.3) and $\mathsf{minimize}(\mathsf{BPD}_0(w; v)) \subseteq \mathsf{BPD}_0(u; u)$ (Proposition 3.4). It then suffices to show that $\mathsf{expand}_v^w(\mathsf{BPD}_0(u; u)) \subseteq \mathsf{BPD}_0(w; v)$, where expand_v^w is the inverse map to $\mathsf{minimize}$ constructed in Proposition 3.3.

Let $B \in \mathsf{BPD}_0(u; u)$. Our goal is to show that $\mathsf{expand}_v^w(B)$ is reduced. To do so, we first note that there are two types of pipes in $\mathsf{expand}_v^w(B)$: pipes of the form $t_j \to s_i$ (indexed by pairs $i, j \in [m]$ where $j \to i$ is a pipe in B), and hook-shaped pipes of the form $y_k \to x_k$ (indexed by $k \in [r]$). Given two pipes in $\mathsf{expand}_v^w(B)$, we will show that these pipes cannot cross more than once in $\mathsf{expand}_v^w(B)$ by checking case-wise for each combination of the above two types.

Consider two pipes of first type, say $t_j \to s_i$ and $t_{j'} \to s_{i'}$. As in the proof of Proposition 3.4, the number of times these pipes cross in $\mathsf{expand}_v^w(B)$ is the same as the number of times the pipes $j \to i$ and $j' \to i'$ cross in B. Since B is reduced, it follows that $t_j \to s_i$ and $t_{j'} \to s_{i'}$ can cross at most once.

It is clear that two pipes of the second type (i.e. of the form $y_k \to x_k$) can cross at most once in $\mathsf{expand}_v^w(B)$ since pipes of this type are hook-shaped.

It remains to check that a pipe of the form $t_j \to s_i$ and a pipe of the form $y_k \to x_k$ cannot cross more than once in $\mathsf{expand}_v^w(B)$. Suppose otherwise. Since $y_k \to x_k$ is hook-shaped, it can only cross another given pipe at most twice, so $t_j \to s_i$ and $y_k \to x_k$ cross exactly twice. By Proposition 3.7(a), w must contain a 1243 pattern, but this contradicts our hypothesis on w. Thus $t_j \to s_i$ and $y_k \to x_k$ can cross at most once.

We have shown that every pair of pipes in $\mathsf{expand}_v^w(B)$ can cross at most once, so $\mathsf{expand}_v^w(B)$ is reduced. The proposition now follows. □

Proposition 3.10. *Let $w \in S_n$ be a permutation. If w is* 1243*-avoiding, then*

$$\nu_w = |\mathsf{BPD}_0(w)| = \sum_{u \leqslant w} |\mathsf{BPD}_0(u;u)| \cdot p_u(w).$$

Proof. Using Proposition 3.9, we can replace the inequality with an equality in the proof of Corollary 3.5. □

Proof of Theorem 1.2. Observe that $|\mathsf{BPD}_0(\varnothing;\varnothing)| = 1$ whereas

$$|\mathsf{BPD}_0(w;w)| = \nu_w - \sum_{u<w} |\mathsf{BPD}_0(u;u)| \cdot p_u(w)$$

whenever w is 1243-avoiding by Proposition 3.10. These are the same seed and recurrence satisfied by the coefficients c_w (which can be defined just on the class of 1243-avoiding permutations since pattern avoidance classes are closed under pattern containment), so $c_w = |\mathsf{BPD}_0(w;w)| \geqslant 0$ for every 1243-avoiding permutation w. □

In particular, we get a combinatorial description of the coefficients appearing in Conjecture 1.1 for 1243-avoiding permutations: c_w enumerates the set of minimal reduced bumpless pipe dreams with permutation w. See Figure 8 for an example of Proposition 3.10.

Proposition 3.11. *Let $w \in S_n$ be a permutation, and let $v \subseteq w$ be a subword with $u = \mathsf{perm}(v)$. If w is* 1243*-avoiding, then* $\mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(w;v)) = \mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(u;u))$.

Proof. A simple calculation shows

$$\begin{aligned}
\mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(w;v)) &= \sum_{B \in \mathsf{BPD}_0(w;v)} \mathrm{wt}^{(\beta)}(B) \\
\text{(Proposition 3.6)} \qquad &= \sum_{B \in \mathsf{BPD}_0(w;v)} \mathrm{wt}^{(\beta)}(\mathsf{minimize}(B)) \\
\text{(Proposition 3.9)} \qquad &= \sum_{B' \in \mathsf{BPD}_0(u;u)} \mathrm{wt}^{(\beta)}(B') \\
&= \mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(u;u)).
\end{aligned}$$

□

Proposition 3.12. *Let $w \in S_n$. If w is vexillary and* 1243*-avoiding, then*

$$\nu_w^{(\beta)} = \mathrm{wt}^{(\beta)}(\mathsf{BPD}(w)) = \sum_{u \leqslant w} \mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(u;u)) \cdot p_u(w).$$

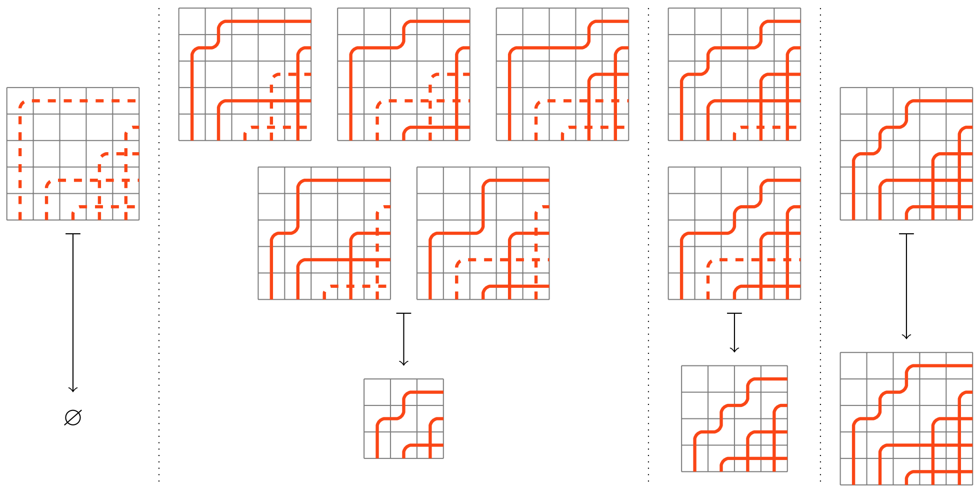


Figure 8. Illustration of Propositions 3.10 and 3.12 with $w = 15423$, which indeed avoids both 1243 and 2143. On top are the 9 bumpless pipe dreams in $\mathsf{BPD}_0(15423) = \mathsf{BPD}(15423)$, grouped based on their image under $\mathsf{minimize}$. The removable pipes in each BPD are indicated by dashed lines. On the bottom are the unique minimal reduced bumpless pipe dreams for the permutations $\varnothing$, 132, 1432, and 15423, respectively; these are the only patterns contained in 15423 that have any minimal reduced BPDs. We have $p_\varnothing(15423) = p_{15423}(15423) = 1$, $p_{132}(15423) = 5$, and $p_{1432}(15423) = 2$. Notice that the β-weight of each bumpless pipe dream is preserved under $\mathsf{minimize}$.

Proof. We have

$$\begin{aligned}
&\text{(Corollary 3.8(a))} && \mathrm{wt}^{(\beta)}(\mathsf{BPD}(w)) = \mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(w)) \\
& && = \sum_{v \subseteq w} \mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(w; v)) \\
&\text{(Proposition 3.11)} && = \sum_{v \subseteq w} \mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(\mathsf{perm}(v); \mathsf{perm}(v))) \\
& && = \sum_{u \leqslant w} \mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(u; u)) \cdot p_u(w). \qquad \square
\end{aligned}$$

Proof of Theorem 1.3. We have $\mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(\varnothing; \varnothing)) = 1$ and

$$\mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(w; w)) = \nu_w^{(\beta)} - \sum_{u < w} \mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(u; u)) \cdot p_u(w)$$

for vexillary 1243-avoiding w by Proposition 3.12. From this, it follows that $c_w^{(\beta)} = \mathrm{wt}^{(\beta)}(\mathsf{BPD}_0(w; w)) \in \mathbb{Z}_{\geqslant 0}[\beta]$ for all vexillary 1243-avoiding permutations w. Note that in this case, we also have $\mathsf{Wires}(w; w) = \mathsf{BPD}_0(w; w)$ by Corollary 3.8(b), so $c_w^{(\beta)} = \mathrm{wt}^{(\beta)}(\mathsf{Wires}(w; w))$ as well. $\square$

4. Closing remarks

4.1. Future directions. Theorem 1.2 establishes 1243-avoiding permutations the second known (nontrivial) pattern-avoidance class of permutations for which Conjecture 1.1 holds, the first being the avoidance class of 1432 and 1423 [15]. What is novel

n	max $\nu_w^{(1)}$	max $c_w^{(1)}$	$w \in S_n$
0	1	1	$\varnothing$
1	1	0	1
2	1	0	12 and 21
3	3	2	132
4	11	4	1432
5	71	44	12543 and 21543
6	1101	828	132654
7	38259	32160	1327654
8	1711251	1501128	13287654
9	190013835	177205856	143298765

FIGURE 9. The maximum values of $\nu_w^{(1)}$ and $c_w^{(1)}$ among all $w \in S_n$, as well as the $w \in S_n$ achieving these maxima.

about this development is that it is done bijectively through Propositions 3.9 and 3.10. In contrast, Conjecture 1.1 was shown to hold for permutations w avoiding 1432 and 1423 [15] indirectly by verifying the identity

$$\sum_{v \subseteq w} (-1)^{|w|-|v|} \nu_{\mathsf{perm}(v)} \geqslant 0.$$

With the observation $\nu_w = \sum_{v \subseteq w} c_{\mathsf{perm}(v)}$, we see by inclusion-exclusion that c_w is equal to the alternating sum on the left-hand side.

It is natural to ask if our methods can be used to show Conjecture 1.1 for other pattern-avoidance classes of permutations (including the avoidance class of 1432 and 1423). One prerequisite is narrowing our definition of minimal bumpless pipe dreams. To see why this is necessary, we examine the permutation 1243 itself. Recall that $\mathsf{BPD}_0(1243) = \mathsf{BPD}(1243)$ is given by the first three BPDs in Figure 4. Write B_1, B_2, and B_3 for these BPDs from left-to-right. 1243 has three subwords v for which $c_{\mathsf{perm}(v)} \neq 0$: 143 and 243 (occurrences of 132), and the empty subword $\varnothing$. Since $c_\varnothing = c_{132} = 1$, we would like $\mathsf{BPD}_0(1243; \varnothing)$, $\mathsf{BPD}_0(1243; 143)$, and $\mathsf{BPD}_0(1243; 243)$ to each contain a single BPD. This is the case for $\mathsf{BPD}_0(1243; \varnothing) = \{B_1\}$ and $\mathsf{BPD}_0(1243; 243) = \{B_2\}$, but not for $\mathsf{BPD}_0(1243; 143)$ which is empty. Instead $\mathsf{BPD}_0(1243; 1243) = \{B_3\}$ has one too many bumpless pipe dreams (since $c_{1243} = 0$). If we want our approach to work for 1243, B_3 should no longer be considered minimal and instead be reclassified as part of $\mathsf{BPD}_0(1243; 143)$. This entails generalizing the notion of removable pipes (e.g. pipe $2 \to 2$ in B_3 should be removable) and adapting $\mathsf{minimize}$ to account for these extra pipes.

4.2. LAYERED PERMUTATIONS. Of separate interest is a question regarding the maximum values attained by ν_w and c_w among $w \in S_n$ for a fixed size n. It has been conjectured by Merzon and Smirnov [14] that the permutations $w \in S_n$ maximizing ν_w are always *layered*, i.e. of the form $w = n_1 \dots 1\, n_2 \dots (n_1+1) \dots n_k \dots (n_{k-1}+1)$ for positive integers $1 \leqslant n_1 < n_2 < \cdots < n_k = n$, which has been confirmed for $n \leqslant 10$. Gao [4] found that the coefficients c_w are also maximized for layered $w \in S_n$, $n \leqslant 8$.

Continuing this trend, we have checked that for $n \leqslant 9$, the permutations $w \in S_n$ maximizing the $\beta = 1$ specializations $\nu_w^{(1)}$ and $c_w^{(1)}$ are all layered. Interestingly, the permutations maximizing $\nu_w^{(1)}$ are precisely those that maximize $c_w^{(1)}$. This was not the case with the sets of permutations maximizing ν_w and c_w, which differ for $n = 5$ [17, 4]. We have recorded our findings in the table in Figure 9.

The permutations listed in the table are in general not the same as the permutations maximizing ν_w and c_w, which can be found in [17] and [4], respectively.[(4)] In particular, they differ from the permutations maximizing ν_w for $n = 5$, $n = 6$, and $n = 9$ (and agree for other values $\leqslant 9$), and they differ from the permutations maximizing c_w for $n = 6$ and $n = 9$.

Except for $n = 2$ and $n = 5$, there is a unique permutation maximizing $\nu_w^{(1)}$ and $c_w^{(1)}$. In the cases where we do not have uniqueness, the polynomials $\nu_w^{(\beta)}$ and $c_w^{(\beta)}$ agree for both permutations attaining the maximum.

It may be interesting to see which permutations maximize $\nu_w^{(\beta)}$ and $c_w^{(\beta)}$ for other specializations of β. We remark that the $\beta = -1$ case is uninteresting since $\nu_w^{(-1)} = 1$ for all permutations w. This is because there is a unique bumpless pipe dream in $\mathsf{BPD}(w)$ without any $\ulcorner$'s, and this bumpless pipe dream contributes a weight of 1.

4.3. Formulae for skew sums. We finish with a simple observation about $\nu^{(\beta)}$ and $c^{(\beta)}$ for skew sums of permutations. Recall that the *skew sum* of two permutations $u \in S_m$ and $v \in S_n$ is the permutation $u \ominus v \in S_{m+n}$ given by $(u(1)+n)\dots(u(m)+n)\,v(1)\dots v(n)$. Note that $\ell(u \ominus v) = mn + \ell(u) + \ell(v)$.

Proposition 4.1. *Let $u \in S_m$, $v \in S_n$ be permutations. The diagrams in $\mathsf{BPD}(u \ominus v)$ are precisely those given in block form by*

$$B_u \ominus B_v = \begin{array}{|c|c|}\hline \square_{m\times n} & B_u \\ \hline B_v & \boxplus_{n\times m} \\ \hline \end{array}$$

where $B_u \in \mathsf{BPD}(u)$ and $B_v \in \mathsf{BPD}(v)$. Here $\square_{m\times n}$ is an $m \times n$ grid of $\square$ tiles, and $\boxplus_{n\times m}$ is an $n \times m$ grid of $\boxplus$ tiles.

We refer to $B_u \ominus B_v$ as the *skew sum* of B_u and B_v, which also makes sense to define for diagrams that are not bumpless pipe dreams. We remark that Proposition 4.1 is proven in [5] for reduced bumpless pipe dreams, with the generalization here essentially following the same argument. To help with our proof of Proposition 4.1, we first prove the following lemma:

Lemma 4.2. *Let $B_u \in \mathsf{BPD}(u)$ and $B_v \in \mathsf{BPD}(v)$. Then*

$$\mathsf{reduce}(B_u \ominus B_v) = \mathsf{reduce}(B_u) \ominus \mathsf{reduce}(B_v).$$

Proof. Write $B = B_u \ominus B_v$. Observe that pipes in B of the form $y \to x$ where $x \in [m]$ only pass through the regions B_u and $\boxplus_{n\times m}$. Similarly, the pipes of the form $y' \to x'$ where $x' \in [m+1, m+n] := \{m+1, \dots, m+n\}$ only pass through the regions B_v and $\boxplus_{n\times m}$. Thus each of the mn pairs of pipes $y \to x$ and $y' \to x'$ notated as above cross at one of the mn $\boxplus$'s in the region $\boxplus_{n\times m}$, and this is the only time this pair crosses in B. In particular, any pipes in B that cross more than once must do so internally to either of the regions B_u or B_v.

These observations remain unchanged no matter how many $\boxplus$'s in B_u or B_v we change into $\lrcorner\!\ulcorner$'s (though the specific pipes passing through each $\boxplus$ in $\boxplus_{n\times m}$ may change). Hence when passing to $\mathsf{reduce}(B)$, we resolve all of the double crossings in B_v while keeping the region $\boxplus_{n\times m}$ unchanged, then resolve the double crossings in B_u. The desired decomposition then follows. $\square$

(4) We checked for $n = 9$ that 109294 is the maximum value for c_w which is achieved by $w = 132987654$. This is the same permutation maximizing ν_w.

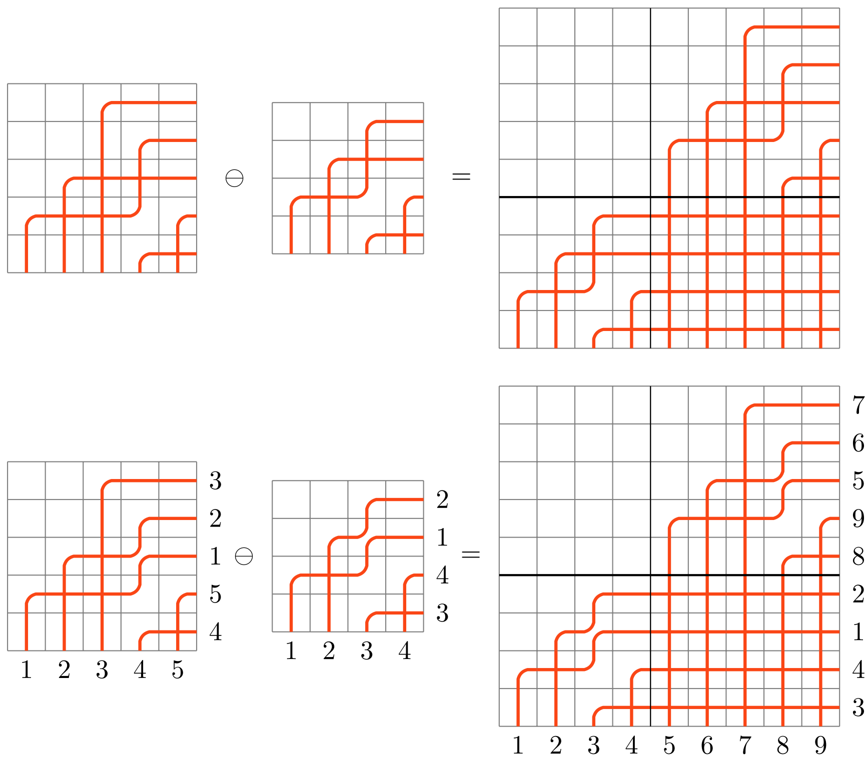


FIGURE 10. Top: The skew sum of two bumpless pipe dreams $B_u \in \mathsf{BPD}(2143)$ and $B_v \in \mathsf{BPD}(32154)$. Bottom: The skew sum of $\mathsf{reduce}(B_u)$ and $\mathsf{reduce}(B_v)$. Notice that $\mathsf{reduce}(B_u \ominus B_v) = \mathsf{reduce}(B_u) \ominus \mathsf{reduce}(B_v)$.

Proof of Proposition 4.1. Let $B \in \mathsf{BPD}(u \ominus v)$. If $y \to x$ is a pipe in $\mathsf{reduce}(B)$ with $x \in [m]$, then $y = u \ominus v(x) = u(x) + n \in [n+1, m+n]$, so $y \to x$ must stay within in the region $[m+n] \times [n+1, n+m]$. Likewise, if $y' \to x'$ is a pipe in $\mathsf{reduce}(B)$ with $x' \in [m+1, m+n]$, then $y' = u \ominus v(x') = v(x'-m) \in [n]$, so $y' \to x'$ must stay in the region $[m+1, m+n] \times [m+n]$. It follows that $\mathsf{reduce}(B)_{i,j} = \square$, and hence $B_{i,j} = \square$, for all $(i,j) \in [m] \times [n]$.

Furthermore, if $y \to x$ and $y' \to x'$ is a pair of pipes as above, then we saw $x < x'$ while $y > y'$, so $y \to x$ and $y' \to x'$ must cross in $\mathsf{reduce}(B)$. This can only happen in the region $[m+1, m+n] \times [n+1, m+n]$. Since there are mn such pairs of pipes, we must have that $\mathsf{reduce}(B)_{i,j} = \boxplus$, and hence $B_{i,j} = \boxplus$, for all $(i,j) \in [m+1, m+n] \times [n+1, m+n]$.

It follows that $B = B_1 \ominus B_2$ for some bumpless pipe dreams B_1 and B_2. By Lemma 4.2, $\mathsf{reduce}(B) = \mathsf{reduce}(B_1) \ominus \mathsf{reduce}(B_2)$. Since B has permutation $u \ominus v$, it follows that B_1 has permutation u and B_2 has permutation v. Hence $B_1 \in \mathsf{BPD}(u)$ and $B_2 \in \mathsf{BPD}(v)$.

Conversely, suppose that $B = B_u \ominus B_v$ for some $B_u \in \mathsf{BPD}(u)$ and $B_v \in \mathsf{BPD}(v)$. Then $\mathsf{reduce}(B) = \mathsf{reduce}(B_u) \ominus \mathsf{reduce}(B_v)$ by Lemma 4.2, from which it is clear that B has permutation $u \ominus v$. $\square$

Let $B = B_u \ominus B_v \in \mathsf{BPD}(u \ominus v)$. Notice that the contribution of B in Equation 1 for the β-Grothendieck polynomial $\mathfrak{G}^{(\beta)}_{u\ominus v}(x_1, \dots, x_{m+n-1})$ can be computed as

$$\begin{aligned}
\beta^{-\ell(u\ominus v)} \Bigg(\prod_{B(\square)} \beta x_i \Bigg) \Bigg(\prod_{B(\ulcorner)} (1 + \beta x_i) \Bigg) \\
&= \beta^{-mn-\ell(u)-\ell(v)} \Bigg(\prod_{\square_{m\times n}} \beta x_i \prod_{B_u(\square)} \beta x_i \prod_{B_v(\square)} \beta x_{m+i} \Bigg) \\
&\qquad \Bigg(\prod_{B_u(\ulcorner)} (1 + \beta x_i) \prod_{B_v(\ulcorner)} (1 + \beta x_{m+i}) \Bigg) \\
&= x_1^n x_2^n \cdots x_m^n \, \beta^{-\ell(u)} \Bigg(\prod_{B_u(\square)} \beta x_i \prod_{B_u(\ulcorner)} (1 + \beta x_i) \Bigg) \\
&\qquad \beta^{-\ell(v)} \Bigg(\prod_{B_v(\square)} \beta x_{m+i} \prod_{B_v(\ulcorner)} (1 + \beta x_{m+i}) \Bigg).
\end{aligned}$$

This is $x_1^n x_2^n \cdots x_m^n$ times the product of the contribution of B_u in $\mathfrak{G}^{(\beta)}_u(x_1, \dots, x_{m-1})$ and the contribution of B_v in $\mathfrak{G}^{(\beta)}_v(x_{m+1}, \dots, x_{m+n-1})$ with the indices shifted as indicated. Combining this with Proposition 4.1 gives the following multiplicative formulae for skew sums:

Proposition 4.3. *Let $u \in S_m$ and $v \in S_n$. Then*

$$\mathfrak{G}^{(\beta)}_{u\ominus v}(x_1, \dots, x_{m+n-1}) = x_1^n x_2^n \cdots x_m^n \mathfrak{G}^{(\beta)}_u(x_1, \dots, x_{m-1}) \mathfrak{G}^{(\beta)}_v(x_{m+1}, \dots, x_{m+n-1})$$

In particular, $\nu^{(\beta)}_{u\ominus v} = \nu^{(\beta)}_u \nu^{(\beta)}_v$ and $c^{(\beta)}_{u\ominus v} = c^{(\beta)}_u c^{(\beta)}_v$.

Proof. It remains to show that $c^{(\beta)}_{u\ominus v} = c^{(\beta)}_u c^{(\beta)}_v$. The subwords of $u \ominus v$ are themselves of the form $u' \ominus_n v'$ for subwords $u' \subseteq u$ and $v' \subseteq v$, where by $u' \ominus_n v'$ we mean the word $(u'(1)+n) \dots (u'(|u'|)+n)\ v'(1) \cdots v'(|v'|)$. It is clear that $\mathsf{perm}(u' \ominus_n v') = \mathsf{perm}(u') \ominus \mathsf{perm}(v')$. By applying inclusion-exclusion, we then get

$$\begin{aligned}
c^{(\beta)}_{u\ominus v} &= \sum_{u'\ominus_n v' \subseteq u\ominus v} (-1)^{|u\ominus v| - |u'\ominus v'|} \nu^{(\beta)}_{\mathsf{perm}(u'\ominus v')} \\
&= \sum_{u'\subseteq u} \sum_{v'\subseteq v} (-1)^{|u|+|v|-|u'|-|v'|} \nu^{(\beta)}_{\mathsf{perm}(u')\ominus\mathsf{perm}(v')} \\
&= \sum_{u'\subseteq u} \sum_{v'\subseteq v} (-1)^{|u|-|u'|} (-1)^{|v|-|v'|} \nu^{(\beta)}_{\mathsf{perm}(u')} \nu^{(\beta)}_{\mathsf{perm}(v')} \\
&= \Bigg(\sum_{u'\subseteq u} (-1)^{|u|-|u'|} \nu^{(\beta)}_{\mathsf{perm}(u')} \Bigg) \Bigg(\sum_{v'\subseteq v} (-1)^{|v|-|v'|} \nu^{(\beta)}_{\mathsf{perm}(v')} \Bigg) \\
&= c^{(\beta)}_u c^{(\beta)}_v. \qquad \square
\end{aligned}$$

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References

[1] Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp, *Alternating-sign matrices and domino tilings. I*, J. Algebraic Combin. **1** (1992), no. 2, 111–132.

[2] Sergey Fomin and Anatol N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, in Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique, DIMACS, Piscataway, NJ, sd, pp. 183–189.

[3] Sergey Fomin and Richard P Stanley, *Schubert polynomials and the nilCoxeter algebra*, Advances in Mathematics **103** (1994), no. 2, 196–207.

[4] Yibo Gao, *Principal specializations of Schubert polynomials and pattern containment*, European J. Combin. **94** (2021), article no. 103291 (12 pages).

[5] Zachary Hamaker, Oliver Pechenik, and Anna Weigandt, *Gröbner geometry of Schubert polynomials through ice*, Adv. Math. **398** (2022), article no. 108228 (29 pages).

[6] Thomas Hudson, *A Thom-Porteous formula for connective K-theory using algebraic cobordism*, J. K-Theory **14** (2014), no. 2, 343–369.

[7] Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, Ann. of Math. (2) **161** (2005), no. 3, 1245–1318.

[8] Thomas Lam, Seung Jin Lee, and Mark Shimozono, *Back stable Schubert calculus*, Compos. Math. **157** (2021), no. 5, 883–962.

[9] Alain Lascoux, *Chern and Yang through ice*, 2002, `https://www.math.uwaterloo.ca/~opecheni/lascoux-ice.pdf`.

[10] Alain Lascoux and Marcel-Paul Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 13, 447–450.

[11] Alain Lascoux and Marcel-Paul Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 11, 629–633.

[12] Ian G Macdonald, *Notes on Schubert polynomials*, Laboratorie de combinatoire et d'informatique mathématique (LACIM), Université du Québec à Montréal, 1991.

[13] Eric Marberg and Brendan Pawlowski, *Principal specializations of Schubert polynomials in classical types*, Algebr. Comb. **4** (2021), no. 2, 273–287.

[14] Grigory Merzon and Evgeny Smirnov, *Determinantal identities for flagged Schur and Schubert polynomials*, Eur. J. Math. **2** (2016), no. 1, 227–245.

[15] Karola Mészáros and Arthur Tanjaya, *Inclusion-exclusion on Schubert polynomials*, Algebr. Comb. **5** (2022), no. 2, 209–226.

[16] A. H. Morales, I. Pak, and G. Panova, *Hook formulas for skew shapes IV. Increasing tableaux and factorial Grothendieck polynomials*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **507** (2021), 59–98.

[17] Richard P Stanley, *Some Schubert shenanigans*, 2017, `https://arxiv.org/abs/1704.00851`.

[18] Anna Weigandt, *Bumpless pipe dreams and alternating sign matrices*, J. Combin. Theory Ser. A **182** (2021), article no. 105470 (52 pages).

[19] Anna E. Weigandt, *Schubert polynomials, 132-patterns, and Stanley's conjecture*, Algebr. Comb. **1** (2018), no. 4, 415–423.

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