ALGEBRAIC COMBINATORICS

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Volume 8, issue 3 (2025), p. 597-617.
https://doi.org/10.5802/alco.425

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

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Abstract A numerical semigroup is a cofinite subset of $\mathbb{Z}_{\geqslant 0}$ containing 0 and closed under addition. Each numerical semigroup S with smallest positive element m corresponds to an integer point in the Kunz cone $\mathcal{C}_m \subseteq \mathbb{R}^{m-1}$, and the face of $\mathcal{C}_m$ containing that integer point determines certain algebraic properties of S. In this paper, we introduce the Kunz fan, a pure, polyhedral cone complex comprised of a faithful projection of certain faces of $\mathcal{C}_m$. We characterize several aspects of the Kunz fan in terms of the combinatorics of Kunz nilsemigroups, which are known to index the faces of $\mathcal{C}_m$, and our results culminate in a method of "walking" the face lattice of the Kunz cone in a manner analogous to that of a Gröbner walk. We apply our results in several contexts, including a wealth of computational data obtained from the aforementioned "walks" and a proof of a recent conjecture concerning which numerical semigroups achieve the highest minimal presentation cardinality when one fixes the smallest positive element and the number of generators.

1. Introduction

A numerical semigroup is a cofinite subset $S \subseteq \mathbb{Z}_{\geqslant 0}$ containing 0 and closed under addition; see [1] for a thorough intro. Each numerical semigroup with smallest positive element m corresponds to an integer point in the Kunz cone $\mathcal{C}_m \subseteq \mathbb{R}^{m-1}$. Inspired by a construction of Kunz [19], this family of polyhedral cones has been of significant interest in the last decade. The original motivation for investigating these polyhedra was enumerative in nature (e.g., utilizing lattice point methods to enumerate numerical semigroups with a given number of gaps [27], or addressing some longstanding asymptotic questions [17]).

Much of the recent interest in the Kunz cone, however, has focused on the faces of $\mathcal{C}_m$. In this time, numerical semigroups S and T corresponding to points in the same face of $\mathcal{C}_m$ have been shown to share numerous algebraic properties, including embedding dimension (i.e. the number of minimal generators), Cohen-Macaulay type, and the symmetric property [5, 18]. The defining toric ideals of S and T have been shown to possess similar minimal binomial generators [13], and were recently shown to have identical Betti numbers up to reduction of graded degrees modulo m [4]. When S and T lie in certain popular families of numerical semigroups, such as those that are complete intersection, generated by (generalized) arithmetic sequences, or constructed via gluings, has also been shown to coincide [2].

At the heart of these shared properties is the Kunz nilsemigroup: a finite nilsemigroup associated to each face $F \subseteq \mathcal{C}_m$, derived from a portion of the divisibility poset

Manuscript received 22nd July 2024, revised 22nd February 2025, accepted 24th February 2025.
Keywords. numerical semigroup, poset, polyhedral cone.

of each numerical semigroup corresponding to a point in F [13, 18]. Kunz nilsemigroups provide a combinatorial framework for working with both the geometry of the faces of the Kunz cone [3] and the algebra of the numerical semigroups therein [9].

One application of the Kunz cone is a computational method of enumerating numerical semigroups with fixed m; this was utilized in [5] to make headway on Wilf's conjecture, one of the longest open problems in the numerical semigroups literature. Unfortunately, the number of faces of $\mathcal{C}_m$ grows quickly in m, so computing the full face lattice of $\mathcal{C}_m$ quickly becomes prohibitively difficult. If one is only interested in numerical semigroups with a relatively small number of generators (as is often the case), the relevant faces of $\mathcal{C}_m$ have small dimension and thus are far less numerous. However, since the cone $\mathcal{C}_m$ is defined by hyperplanes, existing computational methods make enumerating only lower-dimensional faces difficult.

The aim of the present manuscript is twofold: (i) to introduce a method of computing and enumerating, for fixed k, the faces of $\mathcal{C}_m$ in which the points corresponding to k-generated numerical semigroups reside; and (ii) to further develop the theoretical framework linking Kunz nilsemigroups and the geometry of the faces of $\mathcal{C}_m$ they index. To this end, we introduce the Kunz fan (Definition 3.1), a pure cone complex comprised of a faithful projection of such faces. We characterize several aspects of the Kunz fan, such as its boundary (Theorem 3.8 and Proposition 5.3), its chambers (Corollary 3.9), an H-description of its faces (Theorem 4.8), and chamber incidence (Theorem 5.5), in terms of the combinatorics of Kunz nilsemigroups.

On the computational front, our results yield a method of "walking" the face lattice of the Kunz cone (Section 5) in a manner analogous to that of a Gröbner walk [16] (this comparison is not just superficial; see Remark 5.7). Our algorithm represents a marked improvement over prior methods of enumerating the faces of $\mathcal{C}_m$: we computed all faces of every Kunz fan with $k = 3$ and $m \leqslant 42$ on our personal machines in under a day (we ran out of memory at $m = 43$), whereas computing the full face lattice for $m = 18$ for [5] took a cluster 3 weeks. This data is of particularly high interest in examining the relationship between a numerical semigroup's multiplicity, embedding dimension, and minimal presentation cardinality; indeed, these methods were utilized to obtain much of the table given in the introduction of [9], and several of the constructions given thereafter were identified from the extremal examples produced by those computations.

On the theoretical front, Section 6 includes several applications: we classify the Kunz nilsemigroups of 3-generated numerical semigroups; we identify a family of faces of $\mathcal{C}_m$ that yields an exponential lower bound on the number of rays of $\mathcal{C}_m$; we prove [9, Conjecture 7.3] concerning which numerical semigroups with fixed m and k achieve the highest minimal presentation cardinality; and we prove a result related to a longstanding open problem concerning Gröbner fans of toric ideals [30]. We also characterize the finite nilsemigroups that are Kunz (Theorem 4.9), answering a question posed in [18].

2. Background

In this section, we recall some necessary background on semigroups, polyhedral geometry, Kunz nilsemigroups, and the Kunz cone. For a more complete overview, see [1], [32], [9, Section 2] and [13, Section 3], respectively.

For a commutative semigroup $(N, +)$, an element $\infty \in N$ is *nil* if $a + \infty = \infty$ for all $a \in N$. We call an element $a \in N$ *nilpotent* if $na = \infty$ for some $n \in \mathbb{Z}_{\geqslant 1}$, and *partly cancellative* if $a + b = a + c \neq \infty$ implies $b = c$ for all $b, c \in N$. We say N is a *nilsemigroup* if N has an identity element and every non-identity element is nilpotent, and that N is *partly cancellative* if every non-nil element of N is partly cancellative.

Note that any nilsemigroup is *reduced*, meaning its only unit is the identity. We call any element of N that cannot be written as the sum of two non-identity elements an *atom* of N.

All semigroups N in this paper are assumed to be commutative, partly cancellative, finitely generated, and reduced. Under these assumptions, the atoms $n_0, \ldots, n_k \in N$ comprise the unique minimal generating set [25]; we denote this by $N = \langle n_0, \ldots, n_k \rangle$. A *factorization* of $n \in N$ is an expression

$$n = z_0 n_0 + \cdots + z_k n_k$$

where each $z_i \in \mathbb{Z}_{\geqslant 0}$. The *set of factorizations* of $n \in N$ is the set

$$\mathsf{Z}_N(n) = \{z \in \mathbb{Z}_{\geqslant 0}^{k+1} \mid n = z_0 n_0 + \cdots + z_k n_k\} \subset \mathbb{Z}_{\geqslant 0}^{k+1}.$$

The *factorization homomorphism* $\varphi_N : \mathbb{Z}_{\geqslant 0}^{k+1} \to N$ is the semigroup homomorphism

$$\varphi_N(z_0, \ldots, z_k) = z_0 n_0 + \cdots z_k n_k.$$

The *kernel* of φ_N is

$$\ker \varphi_N = \left\{(a, b) \in \mathbb{Z}_{\geqslant 0}^{k+1} \times \mathbb{Z}_{\geqslant 0}^{k+1} \mid \varphi_N(a) = \varphi_N(b)\right\}$$

which induces a *congruence relation* $\sim$ on $\mathbb{Z}_{\geqslant 0}^{k+1}$, setting $a \sim b$ whenever $(a, b) \in \ker \varphi_N$ (recall that a *congruence* is an equivalence relation such that $a \sim b$ implies $a + c \sim b + c$ for any $a, b, c \in \mathbb{Z}_{\geqslant 0}^{k+1}$). We call any such pair (a, b) or $a \sim b$ a *trade* of N. A set of trades ρ is said to *generate* $\sim$ if $\sim$ is the smallest congruence on $\mathbb{Z}_{\geqslant 0}^{k+1}$ containing ρ. A *presentation* of N is a set ρ of trades obtained by taking a generating set for $\sim$ and omitting any $a \sim b$ where $\varphi_N(a)$ is nil. A presentation ρ of $\sim$ is *minimal* if no proper subset of ρ is a presentation of $\sim$. It is known that any two minimal presentations of a finitely generated, partly cancellative semigroup have the same cardinality [13, 25].

The *support* of a factorization $z \in \mathbb{Z}_{\geqslant 0}^{k+1}$ is the set

$$\operatorname{supp}(z) = \{i \mid z_i > 0\}$$

of nonzero coordinates. For $Z \subseteq \mathbb{Z}_{\geqslant 0}^{k+1}$, define

$$\operatorname{supp}(Z) = \{i \mid z_i > 0 \text{ for some } z \in Z\},$$

and the *factorization graph* ∇_Z, whose vertices are elements of Z, and two factorizations $z, z' \in Z$ are connected by an edge if $\operatorname{supp}(z) \cap \operatorname{supp}(z') \neq \varnothing$. For $n \in N$, we write ∇_n for the factorization graph whose vertex set is $\mathsf{Z}_N(n)$. For each $i \in \operatorname{supp}(Z)$, let

$$Z - e_i = \{z - e_i \mid z \in Z,\, i \in \operatorname{supp}(z)\}.$$

Suppose N is a finite, partially cancellative nilsemigroup. An *outer Betti element* of N is a subset $B \subseteq \mathsf{Z}_N(\infty)$ such that

(i) for each $i \in \operatorname{supp}(B)$, $B - e_i = \mathsf{Z}_N(p)$ for some $p \in N \smallsetminus \{\infty\}$, and
(ii) the graph ∇_B is connected.

A *numerical semigroup* S is an additive subsemigroup of $(\mathbb{Z}_{\geqslant 0}, +)$ that is cofinite and contains 0. Numerical semigroups have a unique minimal generating set, the size of which we call the *embedding dimension*, and the smallest element of which we call the *multiplicity*. Letting m be an element of S, the Apéry set of S is the set

$$\operatorname{Ap}(S; m) = \{n \in S \mid n - m \notin S\}$$

containing the smallest element of S in each equivalence class modulo m. Let $\approx$ denote the congruence on S setting $a \approx b$ whenever $a = b$ or $a, b \notin \operatorname{Ap}(S; m)$. The quotient semigroup $S/{\approx}$ is a nilsemigroup with one non-nil element for each element of $\operatorname{Ap}(S; m)$. The *Kunz nilsemigroup* of S is given by $N = \mathbb{Z}_m \cup \{\infty\}$ as sets, and is obtained from $S/{\approx}$ by replacing each non-nil element by its equivalence class in $\mathbb{Z}_m$.

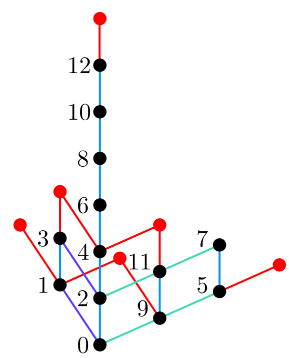

Figure 1. Kunz poset in Example 2.2 with outer Betti elements in red.

Theorem 2.1 ([13]). *If ρ is a minimal presentation of a numerical semigroup S, then $|\rho| = |\rho'| + \beta$, where ρ' is any minimal presentation for the Kunz nilsemigroup N of S, and β is the number of outer Betti elements of N.*

We briefly illustrate Theorem 2.1 and the definition preceding it in the following example. See [9, 13] for a more thorough introduction to outer Betti elements and their relationship to minimal presentations of numerical semigroups.

Example 2.2. Let $S = \langle 13, 53, 15, 35\rangle$. One can verify computationally [8] that

$$\mathrm{Ap}(S; 13) = \{0, 53, 15, 68, 30, 70, 45, 85, 60, 35, 75, 50, 90\},$$

where the elements are listed in order of their equivalence classes modulo $m = 13$. The partially ordered set (c) depicted in Figure 3 encodes the divisibility relations of the non-nil elements of Kunz nilsemigroup N of S. For instance, 3 precedes 1 in the diagram since $68 - 53 = 15 \in S$, but 3 does not precede 5 since $70 - 68 = 2 \notin S$. Moreover, $N = \langle 1, 2, 9\rangle$, as these are the elements covering 0.

Write $\mathrm{Ap}(S; 13) = \{0, a_1, \ldots, a_{12}\}$ with each $a_i \equiv i \bmod 13$. One can check that $\mathsf{Z}_N(11) = \{(0, 1, 1)\}$ since $a_{11} = 50 = 15 + 35 = a_2 + a_9$, and in fact this is the only factorization of a_{11}. Additionally, $(1, 2, 0) \in \mathsf{Z}_N(\infty)$ since $a_1 + 2a_2 = 83 > 70 = a_4$.

There are 6 outer Betti elements, each consisting of a single factorization from among

$$(2, 0, 0), \quad (1, 2, 0), \quad (0, 7, 0), \quad (1, 0, 1), \quad (0, 2, 1), \quad \text{and} \quad (0, 0, 3),$$

and each can be seen as a factorization in $\mathsf{Z}_N(\infty)$ that is minimal under the component-wise partial order. For instance,

$$\mathsf{Z}_N(11) = \{(0, 1, 1)\}, \qquad \mathsf{Z}_N(4) = \{(0, 2, 0)\}, \qquad \text{and} \qquad \{(0, 2, 1)\} \subseteq \mathsf{Z}_N(\infty),$$

imply $\{(0, 2, 1)\}$ is an outer Betti element of N.

A *rational polyhedral cone* $C \subseteq \mathbb{R}^d$ is the set of solutions to a finite set of linear inequalities with rational coefficients, i.e.

$$C = \{x \in \mathbb{R}^d \mid Ax \geqslant 0\}$$

for some rational matrix A. We say C is *strongly convex* (or *pointed*) if it does not contain any positive dimension linear subspace of $\mathbb{R}^d$, and the *dimension* of C is the vector space dimension $\dim C = \dim_{\mathbb{R}} \operatorname{span}_{\mathbb{R}} C$. If none of the rows of A can be omitted without changing C, we call A the *H-description* or *facet description* of C, and refer to each inequality therein as a *defining inequality* or *facet inequality* of P. If $\dim C = d$, then the facet description is unique up to the reordering and scaling of rows. Given a facet inequality $a_1x_1 + \cdots + a_dx_d \geqslant 0$ of C, the intersection of C with the hyperplane $a_1x_1 + \cdots + a_dx_d = 0$ is called a *facet* of C. Any intersection of facets is called a *face* of C; note that the condition of strong convexity is equivalent to $\{0\}$ being a face of C. Given a face $F \subseteq C$, the *relative interior* of F, denoted F°, is the set of points in F that do not lie in any proper faces of F. A finite collection $\mathcal{F}$ of polyhedral cones is called a *polyhedral fan* if

(i) for any $C \in \mathcal{F}$, every face of C is also in $\mathcal{F}$, and
(ii) the intersection of any $C, D \in \mathcal{F}$ is a face of both and lies in $\mathcal{F}$.

The elements of a fan $\mathcal{F}$ are often called its *faces*. A fan is *pure* if its maximal faces (under containment) have the same dimension, and in this case, we refer to the maximal faces as *chambers*.

For $m \geqslant 2$, the strongly convex cone $\mathcal{C}_m \subseteq \mathbb{R}^{m-1}_{\geqslant 0}$ with facet inequalities

$$x_i + x_j \geqslant x_{i+j} \qquad \text{for} \qquad i, j \in \mathbb{Z}_m \setminus \{0\} \qquad \text{with} \qquad i + j \neq 0$$

is called the *Kunz cone*. A point $z = (z_1, \ldots, z_{m-1}) \in \mathcal{C}_m \cap \mathbb{Z}^{m-1}$ is an *Apéry point* if each $z_i \equiv i \bmod m$. The following is the culmination of results from [13, 18, 19].

Theorem 2.3. *The Apéry points of $\mathcal{C}_m$ are in bijective correspondence with numerical semigroups containing m; more specifically, this bijection is given by $z \mapsto S$ where $\mathrm{Ap}(S; m) = \{0, z_1, \ldots, z_{m-1}\}$. Fix a face $F \subseteq \mathcal{C}_m$, the set*

$$H = \{0\} \cup \{i : x_i = 0 \text{ for all } x \in F\} \subseteq \mathbb{Z}_m$$

is a subgroup of $\mathbb{Z}_m$ (called the Kunz subgroup *of F). Define a binary operation $\oplus$ on $N = (\mathbb{Z}_m/H) \cup \{\infty\}$ so that ∞ is nil, and for any nonzero $a, b \in \mathbb{Z}_m$,*

$$\overline{a} \oplus \overline{b} = \begin{cases} \overline{a+b} & \text{if } x_a + x_b = x_{a+b} \text{ for all } x \in F; \\ \infty & \text{otherwise.} \end{cases}$$

Under the above definition, $(N, \oplus)$ is a partly cancellative nilsemigroup (called the Kunz nilsemigroup *of F), and if the Apéry point of a numerical semigroup S lies in F, then $H = \{0\}$ and the Kunz nilsemigroup of S equals the Kunz nilsemigroup of F.*

In view of the above theorem, we say a numerical semigroup S lies in the face $F \subseteq \mathcal{C}_m$, and write $S \in F$, if the Apéry point corresponding to S lies in F.

3. Kunz fans

Throughout this section, fix $A = \{p_1, \ldots, p_k\} \subseteq \mathbb{Z}_m \setminus \{0\}$ with $\gcd(A, m) = 1$, and let $p : \mathbb{R}^{m-1} \to \mathbb{R}^k$ denote the linear map that projects each point in $\mathbb{R}^{m-1}$ onto the coordinates whose indices lie in A, i.e.,

$$p(y) = (y_{p_1}, \ldots, y_{p_k}).$$

Definition 3.1. *Let $\mathcal{F}(m; A)$ denote the set of faces $F \subseteq \mathcal{C}_m$ for which each atom of the Kunz nilsemigroup N of F has a representative in A. The* Kunz fan *of A is the set*

$$\mathcal{G}(m; A) = \{p(F) : F \in \mathcal{F}(m; A)\}$$

of cones in $\mathbb{R}^k$ (we prove in Theorem 3.8 that $\mathcal{G}(m; A)$ is indeed a fan). Note that under this definition, $|A|$ may exceed the number of atoms of N, but this allowance is necessary to ensure $\mathcal{G}(m; A)$ is a fan; see Example 3.3 for a discussion of this subtlety.

One of the main results of this section is that $\mathcal{G}(m; A)$ is pure, and that the maximal faces of $\mathcal{G}(m; A)$ with respect to containment are precisely those whose corresponding nilsemigroup lies in the following family.

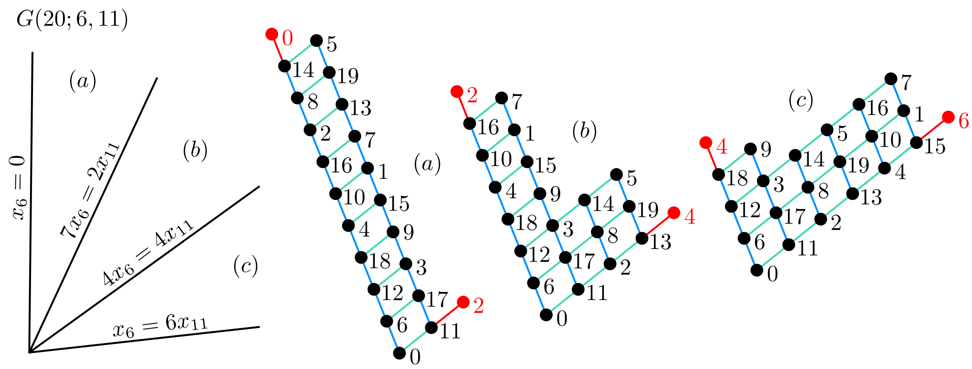

Figure 2. Posets from the chambers of the Kunz fan $\mathcal{G}(20;6,11)$.

Definition 3.2. *A finite, partly cancellative nilsemigroup N is a* staircase *nilsemigroup if every non-nil element uniquely factors as a sum of atoms, and a numerical semigroup S is called* staircase *if its Kunz nilsemigroup is staircase. As each outer Betti element of a staircase nilsemigroup N consists of a single factorization of ∞, when there can be no confusion we refer to each such factorization as an outer Betti element of N.*

A numerical semigroup S is staircase if and only if every element of $\mathrm{Ap}(S;m)$ factors uniquely. In this case, S is said to have *Apéry set of unique expression*; such semigroups have been studied in the literature [24] and were central to the constructions in [9].

The examples in this section examine two Kunz fans in detail, and reference results in this section and subsequent sections. We encourage the reader to peek ahead at these results while reading these examples, as they put the landscape of Sections 3, 4, and 5 in perspective.

Example 3.3. Let $m = 20$ and $A = \{6, 11\}$. The Kunz fan $\mathcal{G}(m;A)$ is depicted in Figure 2 alongside the staircase Kunz nilsemigroups corresponding to its chambers, with outer Betti elements depicted in red. Let N denote the Kunz nilsemigroup corresponding to chamber (b). As we will see in Proposition 3.4 below, any point (x_6, x_{11}) in the interior of chamber (b) must satisfy $7x_6 > 2x_{11}$, as $7 \cdot 6 \equiv 2 \bmod 20$ and $2 \cdot 11 \equiv 2 \bmod 20$, but $(0,2) \in \mathsf{Z}_N(2)$ while $\{(7,0)\}$ is an outer Betti element of N.

Let us examine the facets on the boundary of the fan $\mathcal{G}(m;A)$, as these each identify subtleties in Definition 3.1. One is defined by $x_6 \leqslant 6x_{11}$, which must be satisfied by every point in a face of $\mathcal{G}(m;A)$ by Proposition 3.4 since for any point $y \in \mathcal{C}_m$ that projects to a point (x_6, x_{11}) in a face of $\mathcal{G}(m;A)$, each coordinate y_i is a non-negative integral combination of x_6 and x_{11}. Note that the Kunz nilsemigroup N of this facet has $N = \langle 11 \rangle$, since $x_6 = 6x_{11}$ implies 6 is a multiple of 11 in N; this is why we do not require N to possess an atom for each element of A in Definition 3.1.

The other facet is defined by $x_6 \geqslant 0$; since $\gcd(6,20) > 1$ and $\gcd(6,11,20) = 1$, for each $x_{11} > 0$ one can locate points (x_6, x_{11}) in the interior of chamber (a) wherein x_6 is arbitrarily small. The Kunz subgroup H of this facet is nontrivial since $x_6 = 0$ for every point therein; this is why we only require each atom of the Kunz poset N to have a representative in A in Definition 3.1, rather than requiring A to equal the set of atoms of N.

Consider the piecewise linear map $q : \mathbb{R}^k_{\geqslant 0} \to \mathbb{R}^{m-1}_{\geqslant 0}$ given by $q(x) = y$, where

$$y_i = \min\{c \cdot x \mid c \in \mathbb{Z}^k_{\geqslant 0} \text{ with } c_1 p_1 + \cdots + c_k p_k \equiv i \bmod m\}.$$

Proposition 3.4. *Fix a face $F \in \mathcal{F}(m; A)$ with Kunz nilsemigroup N and Kunz subgroup H. The map p is injective on F, and if $x \in p(F)$, then $y = q(x)$ satisfies*

$$y_i = z_1 x_1 + \cdots + z_k x_k$$

for any factorization $z \in \mathsf{Z}_N(i)$. In particular, q restricts to a linear map on $p(F)$, and $q(p(y)) = y$ for every $y \in F$.

Proof. For each nonzero $i \in \mathbb{Z}_m$, Theorem 2.3 implies one of the following must hold: (i) $i \in H$ and $y_i = 0$; (ii) $i \in a + H$ for some $a \in A$ and $y_i = y_a$; or (iii) for any factorization $z \in \mathsf{Z}_N(i)$, we have

$$y_i = z_1 y_{p_1} + \cdots + z_k y_{p_k}$$

for all $y \in F$. As such, for any $y, y' \in F$, if $p(y) = p(y')$, then $y_a = y'_a$ for all $a \in A$, and thus $y = y'$. This proves p is injective on F.

Next, suppose $p(y) = x$, and let $y' = q(x)$. Fix i, and suppose that $c \in \mathbb{Z}^k_{\geqslant 0}$ satisfies $y'_i = c \cdot x$. We claim $c \in \mathsf{Z}_N(i)$. Indeed, if $i = 0$, then $y_i = 0$, and if $i \in a + H$ for some $a \in A$, then $y_i = y_a$. For all other cases, fix j with $c_j > 0$. This means $i + H$ covers $(i - p_j) + H$ in N, and by minimality of $c \cdot x$, $(c - e_j) \cdot x = y'_{i-p_j}$. By induction, we may assume $c - e_j \in \mathsf{Z}_N(j)$, and so we have $c = (c - e_j) + e_j \in \mathsf{Z}_N(i)$ since $y'_{i-p_j} + y_{p_j} = y'_i$. This proves the claim. By the preceding paragraph, we now have $y' = y$, and the remaining claims all immediately follow. □

It is not hard to show that $q(\mathbb{R}^k_{\geqslant 0}) \subseteq \mathcal{C}_m$, although the injectivity in Proposition 3.4 is lost if one considers input outside of the faces in $\mathcal{G}(m; A)$.

Remarks 3.5. Given a point $x \in \mathbb{R}^k_{\geqslant 0}$, one may compute $q(x)$ using the circle of lights algorithm [31], which is used to compute the Apéry set of a numerical semigroup from a given generating set. The version of the algorithm in [18, Algorithm 7.1] is faster, and as a byproduct computes the set of factorizations of each element of the Kunz nilsemigroup corresponding to the face of $\mathcal{C}_m$ containing $q(x)$.

We next consider the cone

$$C(m; A) = \{x \in \mathbb{R}^k_{\geqslant 0} : x_i \leqslant c \cdot x \text{ for all } c \in \mathbb{Z}^k_{\geqslant 0} \text{ with } c_1 p_1 + \cdots + c_k p_k \equiv p_i \bmod m\}.$$

Despite the fact that $C(m; A)$ is defined using an infinite collection of inequalities, only finitely many are necessary. Indeed, if $c \in \mathbb{Z}^k_{\geqslant 0}$ has some $c_j \geqslant m$, then

$$c \cdot x \geqslant (c - m e_j) \cdot x \qquad \text{for all} \qquad x \in \mathbb{R}^k_{\geqslant 0},$$

so in the definition of $C(m; A)$ one may restrict to c with coordinates in $[0, m-1]$. In particular, this means $C(m; A)$ is a rational polyhedral cone.

Example 3.6. Let $m = 13$ and $A = \{1, 2, 9\}$. The Kunz fan $\mathcal{G}(m; A)$ has five 3-dimensional chambers, the cross sections of which are depicted in Figure 3 alongside the Kunz nilsemigroups corresponding to each chamber. Any point in the interior of a chamber of $\mathcal{G}(m; A)$ must satisfy

$$x_1 < 3x_9, \qquad x_1 < 7x_2, \qquad x_2 < 2x_1, \qquad \text{and} \qquad x_9 < x_1 + 4x_2,$$

which arise from the "minimal" ways of expressing 1, 2, or 9 in $\mathbb{Z}_{13}$ as a sum of the other two and comprise the facets of $C(m; A)$.

Lemma 3.7. *For each $x \in C(m; A)^\circ$, the image $y = q(x)$ lies in a face $F \subseteq \mathcal{C}_m$ with trivial Kunz subgroup, and the Kunz nilsemigroup N of F has atom set A.*

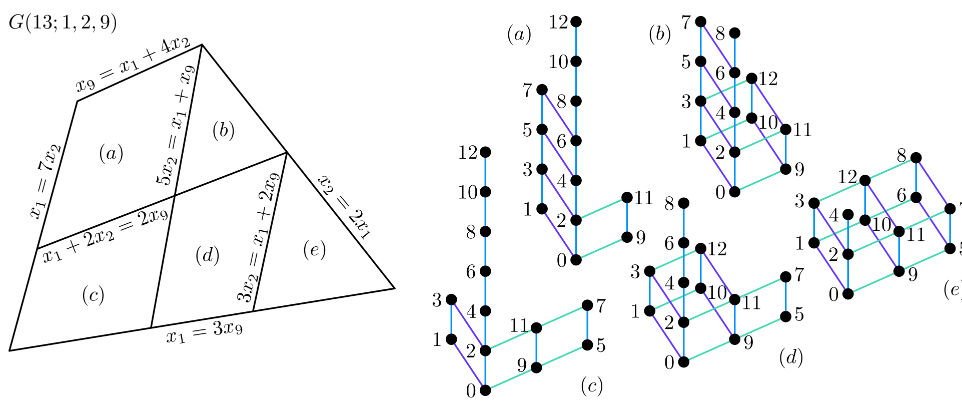

Figure 3. Posets from the chambers of the Kunz fan $\mathcal{G}(13;1,2,9)$.

Proof. Since $C(m;A) \subset \mathbb{R}^k_{\geqslant 0}$, each x_i is positive, so x has all positive coordinates as well. This ensures F has trivial Kunz subgroup. Moreover, the definition of q ensures N has atom set contained in A, but if some $p_i \in A$ were not an atom of N, then for any $z \in \mathsf{Z}_N(p_i)$, we would have

$$\begin{aligned} x_i \geqslant y_{p_i} &= z_1 y_{p_1} + \cdots + z_{i-1} y_{p_{i-1}} + z_{i+1} y_{p_{i+1}} + \cdots + z_k y_{p_k} \\ &= z_1 x_1 + \cdots + z_{i-1} x_{i-1} + z_{i+1} x_{i+1} + \cdots + z_k x_k, \end{aligned}$$

violating the assumption that x lies in the interior of $C(m;A)$. $\square$

Theorem 3.8. *The cones in $\mathcal{G}(m;A)$ form a polyhedral fan that is pure, and the union of the cones in $\mathcal{G}(m;A)$ equals $C(m;A)$.*

Proof. Cleary, if a face $F \subseteq \mathcal{C}_m$ lies in $\mathcal{F}(m;A)$, then all faces of F must as well, so every face of a cone in $\mathcal{G}(m;A)$ is a cone in $\mathcal{G}(m;A)$. Moreover, for any $F, F' \in \mathcal{F}(m;A)$, Proposition 3.4 implies $p(F) \cap p(F') = p(F \cap F')$, so the intersection of any two cones in $\mathcal{G}(m;A)$ is a cone in $\mathcal{G}(m;A)$. This verifies $\mathcal{G}(m;A)$ is a polyhedral fan.

Next, by Lemma 3.7, for each $x \in C(m;A)^\circ$, $y = q(x)$ lies in a face $F \in \mathcal{F}(m;A)$. As such, $p(y) = x$ by Proposition 3.4, which lies in $p(F) \in \mathcal{G}(m;A)$. This means the union of the cones in $\mathcal{G}(m;A)$ contains $C(m;A)^\circ$, and since rational polyhedral cones are topologically closed, the union must equal $C(m;A)$.

The final claim to prove is that every maximal cone in $\mathcal{G}(m;A)$ has dimension k. Indeed, since $\mathcal{G}(m;A)$ contains only finitely many cones, and their union equals the k-dimensional cone $C(m;A)$, the union of the k-dimensional cones in $\mathcal{G}(m;A)$ must also equal $C(m;A)$. This means every cone in $\mathcal{G}(m;A)$ is contained in a k-dimenisional cone in $\mathcal{G}(m;A)$. $\square$

Corollary 3.9. *A face $F \in \mathcal{F}(m;A)$ is a chamber if and only if its Kunz nilsemigroup is staircase.*

Proof. A maximal face of $\mathcal{F}(m;A)$ has dimension k by Theorem 3.8, and since the number of atoms of its Kunz nilsemigroup N is also k, [13, Theorem 4.3] implies N has no inner trades and therefore must be staircase. $\square$

4. Outer Betti elements and facets

Throughout this section, let $A = \{p_1, \ldots, p_k\} \subset \mathbb{Z}_m \setminus \{0\}$ with $\gcd(A, m) = 1$, and let N be a partly cancellative nilsemigroup with atom set A.

The main results of this section are Theorem 4.8, which gives an H-description of each face $F \subseteq \mathcal{C}_m$ in terms of its corresponding Kunz nilsemigroup, and Theorem 4.9, which characterizes the finite, partly cancellative nilsemigroups that are Kunz.

Definition 4.1. *A* modular[(1)] *nilsemigroup is a finite, partly cancellative nilsemigroup N together with a bijection $f : \mathbb{Z}_m \to N \setminus \{\infty\}$ such that for any $a, b \in \mathbb{Z}_m$, either $f(a)+f(b) = f(a+b)$ or $f(a)+f(b) = \infty$. For convenience, we write $a \in \mathbb{Z}_m$ in place of $f(a) \in N$, effectively viewing $N = \mathbb{Z}_m \cup \{\infty\}$ as sets. If $N = \langle f(p_1), \ldots, f(p_k)\rangle$ and $z \in \mathbb{Z}_{\geqslant 0}^k$, we write $\overline{z} = f(z_1p_1 + \cdots + z_kp_k)$, so if $z \in \mathsf{Z}_N(i)$ for $i \neq \infty$, then $\overline{z} = i$.*

A Betti equality *of a modular nilsemigroup N is an equation of the form*

$$c \cdot x = c_1x_1 + \cdots + c_kx_k = c_1'x_1 + \cdots c_k'x_k = c' \cdot x$$

where $c, c' \in \mathsf{Z}_N(p)$ for some $p \in N \setminus \{\infty\}$. Let H_N equal to subspace of $\mathbb{R}^k$ satisfying all such equalities for a given N (this coincides with the nullspace of the presentation matrix defined in [13, Section 4]*). A* Betti inequality *of N is an inequality of the form*

$$z \cdot x = z_1x_1 + \cdots + z_kx_k \geqslant a_1x_1 + \cdots + a_kx_k = a \cdot x$$

where $z \in \mathsf{Z}_N(\infty)$ is an outer Betti factorization and $a \in \mathsf{Z}_N(\overline{z})$. Let $F_N \subseteq H_N$ be the rational polyhedral cone containing points $x \in H_N$ that satisfy all Betti inequalities.

Let us see some examples of these definitions.

Example 4.2. Consider the Kunz nilsemigroup N depicted on the left in Figure 4. The trades $e_1 + e_3 \sim 2e_2$ and $e_7 + 3_8 \sim 2e_3$, occurring at $4, 6 \in N$ respectively, yield

$$(1, -2, 1, 0, 0) \cdot x = 0 \qquad \text{and} \qquad (0, 0, -2, 1, 1) \cdot x = 0$$

as Betti equalities, so $H_N \subseteq \mathbb{R}^5$ has $\dim H_N = 3$. Each factorization b from among

$$\begin{gathered}(1,0,0,0,1),\quad (0,1,0,1,0),\quad (-1,1,0,0,1),\quad (-1,0,1,1,0),\ (2,-1,0,0,0),\\ (0,-1,1,0,1),\ (1,1,-1,0,0),\ (0,-1,-1,2,0),\ (0,0,0,-1,2),\ (1,0,0,1,-1)\end{gathered}$$

yields a Betti inequality $b \cdot x \geqslant 0$, together defining a 3-dimensional cone $F_N \subseteq H_N$.

Example 4.3. Resume notation from Example 3.3. The interior of chamber (b) is defined by the inequalities $7x_6 > 2x_{11}$ and $4x_{11} > 4x_6$, which can be seen as consequences of the outer Betti elements $\{(7, 0)\}$ and $\{(0, 4)\}$, respectively. Note that N does have a third outer Betti element, namely $\{(3, 2)\}$, but the inequality $3x_6 + 2x_{11} \geqslant 0$ does not define a facet of chamber (b).

Example 4.4. The modular nilsemigroup N depicted in the middle in Figure 4 is not Kunz. Indeed, N has no inner Betti elements, so if it were Kunz, F_N would be a full dimensional cone inside $H_N = \mathbb{R}^4$ by Theorem 4.8. For a point $x \in F_N$, the inequalities

$$(-1, -1, 1, 1) \cdot x \geqslant 0, \qquad (1, -1, -1, 1) \cdot x \geqslant 0, \qquad \text{and} \qquad (0, 2, 0, -2) \cdot x \geqslant 0,$$

together imply $(0, 2, 0, -2) \cdot x = 0$. In other words, every point $x \in F_N$ lies in the hyperplane $x_2 = x_4$, i.e. F_N is not full dimensional in H_N.

In what follows, let

$$Z = \{z \in \mathsf{Z}_N(\infty) \mid z - e_i \notin \mathsf{Z}_N(\infty) \text{ for each } i \in \operatorname{supp}(z)\}$$

denote the set of minimal elements of $\mathsf{Z}_N(\infty)$ under the component-wise partial order.

Lemma 4.5. *Suppose N is a modular nilsemigroup and fix $x \in F_N^\circ$ and $y \in \mathsf{Z}_N(\infty)$. There exist $b, c \in \mathbb{Z}_{\geqslant 0}^k$ such that b is a factorization of an outer Betti element of N,*

$$y \cdot x = (b + c) \cdot x, \quad \text{and} \quad \overline{y} = \overline{b + c}.$$

(1) no relation to modular lattices

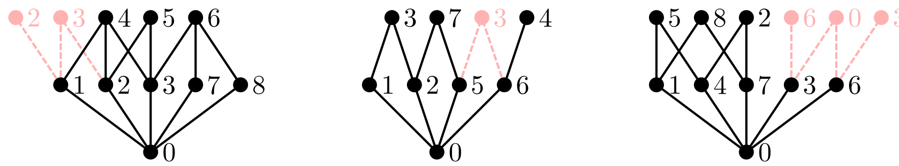

Figure 4. Kunz posets from Examples 4.2, 4.4, and 4.7

Proof. Choose $z \in Z$ so that $y = z + \ell$ for some $\ell \in \mathbb{Z}^k_{\geqslant 0}$. If z lies in an outer Betti element of N, then choosing $b = z$ and $c = \ell$ completes the proof. Otherwise, there exists $y' \in \mathsf{Z}_N(\infty) \setminus Z$ such that $y' \sim z$. Choose $z' \in Z$ so that $y' = z' + \ell'$ for some $\ell' \in \mathbb{Z}^k_{\geqslant 0} \setminus \{0\}$. If z' is a factorization of an outer Betti element of N, then choosing $b = z'$ and $c = \ell' + \ell$ completes the proof, since $y' \sim z$ ensures

$$(b+c)\cdot x = (y'+\ell)\cdot x = (z+\ell)\cdot x = y\cdot x \qquad \text{and} \qquad \overline{b+c} = \overline{y'+\ell} = \overline{z+\ell} = \overline{y}.$$

Otherwise, we may again repeat this process to obtain $z'' \in Z$. Notice that

$$(z\cdot x) - (z'\cdot x) = (y'\cdot x) - (z'\cdot x) = \ell'\cdot x \geqslant \min\{x_1, \ldots, x_k\},$$

so this process must eventually terminate in a suitable choice of b and c. □

Proposition 4.6. *Suppose N is modular, fix $p \in \mathbb{Z}_m \setminus \{\infty\}$, and fix $x \in F_N$ that satisfies all Betti inequalities strictly. If $z \in \mathsf{Z}_N(p)$ and $z' \in \mathbb{Z}^k_{\geqslant 0}$ with $\overline{z'} = p$, then*

$$z'\cdot x \geqslant z\cdot x,$$

with equality if and only if $z' \in \mathsf{Z}_N(p)$.

Proof. If $z' \in \mathsf{Z}_N(p)$, then the definition of F_N ensures $z\cdot x = z'\cdot x$, so it suffices to show that if $z' \in \mathsf{Z}_N(\infty)$ with $\overline{z'} = p$, we have $z'\cdot x > z\cdot x$. By Lemma 4.5, there is a factorization b of an outer Betti element and $c \in \mathbb{Z}^k_{\geqslant 0}$ such that

$$z'\cdot x = (b+c)\cdot x \qquad \text{and} \qquad \overline{b+c} = p.$$

Fixing $a \in \mathsf{Z}_N(\overline{b})$, the Betti inequality $z'\cdot x > (a+c)\cdot x$ must hold. If $a+c \in \mathsf{Z}_N(p)$, then $(a+c)\cdot x = z\cdot x$ and we are done. Otherwise, $(a+c) \in \mathsf{Z}_N(\infty)$, and we can again apply Lemma 4.5 to obtain a factorization b' of an outer Betti element, $c' \in \mathbb{Z}^k_{\geqslant 0}$, and $a' \in \mathsf{Z}_N(\overline{b_1})$ such that

$$z'\cdot x > (a+c)\cdot x = (b'+c')\cdot x > (a'+c')\cdot x \qquad \text{and} \qquad \overline{a+c} = \overline{a'+c'} = p.$$

As in the proof of Lemma 4.5, repeating this process eventually terminates in a factorization $a''+c'' \in \mathsf{Z}_N(p)$, at which point we obtain $z'\cdot x > (a''+c'')\cdot x = z\cdot x$. □

Example 4.7. The condition that x satisfies all Betti inequalities strictly cannot be omitted from Proposition 4.6. Indeed, consider the modular nilsemigroup N with $m = 9$ and atom set $A = \{1, 3, 4, 6, 7\}$ depicted on the right in Figure 4. The outer Betti elements of N have factorizations

$$\begin{array}{llllll} e_1 + e_3, & e_4 + e_3, & e_7 + e_3, & e_1 + e_6, & e_4 + e_6, & e_7 + e_6, \\ 2e_1, \quad 2e_4, & 2e_7, & 2e_3, & 2e_6, & e_3 + e_6, \quad \text{and} \quad e_1 + e_4 + e_7. \end{array}$$

One can check the point $(x_1, x_3, x_4, x_6, x_7) = (1, 3, 1, 4, 1)$ satisfies all Betti inequalities, but $z = e_6 \in \mathsf{Z}_N(1)$ and $z' = 2e_1 + e_4 \in \mathsf{Z}_N(\infty)$ have $z\cdot x = 4 > 3 = z'\cdot x$.

Theorem 4.8. *If $F \subseteq \mathcal{C}_m$ is a face with Kunz nilsemigroup N, then $p(F) = F_N$.*

Proof. Proposition 3.4 implies $p(F) \subseteq F_N$. Conversely, fix $x \in F_N$. Since $p(F) \subseteq F_N$ and both are rational polyhedral cones, it suffices to assume $x \in F_N^\circ$. Fix nonzero $p, q \in \mathbb{Z}_m$ so that $p + q \neq 0$, and fix factorizations $z_1 \in \mathsf{Z}_N(p)$ and $z_2 \in \mathsf{Z}_N(q)$. We must show for any $z \in \mathsf{Z}_N(p+q)$, we have $(z_1+z_2)\cdot x \geqslant z\cdot x$. If $z_1+z_2 \in \mathsf{Z}_N(p+q)$, then $(z_1 + z_2) \cdot x = z \cdot x$. Otherwise, $z_1 + z_2 \in Z_N(\infty)$, so $(z_1 + z_2) \cdot x \geqslant z \cdot x$ by Lemma 4.6. Thus $x \in p(F)$, thereby completing the proof. $\square$

Theorem 4.9. *A modular nilsemigroup N is Kunz if and only if there is a point $x \in F_N$ satisfying all Betti inequalities strictly.*

Proof. If N is a Kunz nilsemigroup, then there is some corresponding face $F \in \mathcal{C}_m$, and by Theorem 4.8, $p(F) = F_N$. Since $\dim p(F) = \dim F$ by Proposition 3.4, $p(F^\circ) = F_N^\circ$, so $p(x)$ satisfies all Betti inequalities strictly for any $x \in F^\circ$.

Conversely, suppose N is modular and there is some point x that satisfies all Betti inequalities of N strictly. By Proposition 4.6, $x \cdot e_i < z \cdot x$ for any $z \in \mathsf{Z}_N(\infty)$ with $\overline{z} = p_i$, so $x \in C(m; A)^\circ$. Since the interiors of the cones of $\mathcal{G}(m; A)$ partition $C(m; A)^\circ$ by Theorem 3.8, x lies in the interior of $p(F)$ for some face $F \subseteq \mathcal{C}_m$ whose Kunz nilsemigroup M has atom set A. From this, we can conclude $N = M$ since they have the same atom set and $Z_N(p) = Z_M(p)$ for any $p \in \mathbb{Z}_m$ by Proposition 4.6. $\square$

Example 4.10. Consider the nilsemigroup N from Example 4.4. Even though N is not Kunz, we have $F_N \in \mathcal{G}(8; 1, 2, 5, 6)$. Indeed, letting M denote the modular nilsemigroup obtained from N by adding $e_5 + e_6$ as a factorization of 3, then M is Kunz by Theorem 4.9, and one can check via computation that $F_N = F_M$.

Alternatively, if N is the nilsemigroup from Example 4.7 and $A = \{1, 3, 4, 6, 7\}$, then one can verify $F_N \notin \mathcal{G}(9; A)$ computationally. Indeed, $\mathcal{G}(9; A)$ has 6 chambers, all of which share a 3-dimensional face that is properly contained in F_N.

5. Walking the Kunz fan

Throughout this section, let $A = \{p_1, \ldots, p_k\} \subseteq \mathbb{Z}_m \setminus \{0\}$ with $\gcd(A, m) = 1$, and let N be a Kunz nilsemigroup with atom set A.

The goal of this section is to give an algorithm, outlined below, for computing the faces of $\mathcal{C}_m$ with atom set A by "walking" the chambers of $\mathcal{G}(m; A)$. Using results in this section, every step can be completed combinatorially (i.e., in terms of nilsemigroups, without relying on polyhedral computations) with the exception of step (3).

(1) Locate a point $x \in \mathbb{R}^k_{\geqslant 0}$ so that the face $F_N \in \mathcal{G}(m; A)$ with $x \in F_N$ has N staircase. This can be done by applying q to small random perturbation of the all-1's vector, since the all-1's vector is guaranteed to lie in some face of $\mathcal{G}(m; A)$, and in any open k-dimensional ball centered there, the set of points contained in non-maximal faces of $\mathcal{G}(m; A)$ has measure 0. Recall that N, along with its outer Betti elements, can be computed from x via the circle-of-lights algorithm, as in Remark 3.5.

(2) Use Theorem 4.8 to obtain an H-description of F_N with one inequality per outer Betti elements of N.

(3) Obtain an irredundant H-description of F_N by identifying which outer Betti elements of N yield supporting inequalities of F_N. We were not able to accomplish this without the use of a polyhedral computation (e.g. `Normaliz` [6]); Proposition 5.2 and Remark 5.6 identify sufficient and necessary conditions, respectively, but fall short of a full characterization.

(4) Determine which facets of F_N lie on the boundary of $C(m; A)$; we identify a combinatorial method for doing so in Proposition 5.3.

(5) For each remaining outer Betti element, determine the Kunz nilsemigroup N' for which $F_{N'}$ shares a facet with F_N; we identify a combinatorial method for doing so in Theorem 5.5.
(6) Using a graph traversal algorithm (e.g., depth-first search or breadth-first search), repeat steps (2) through (5) to compute all of the faces of $\mathcal{G}(m; A)$.

Prior efforts to enumerate the faces of $\mathcal{C}_m$ relied on computing the entire face lattice, or faces up to a particular codimension [5]. The above allows one to compute the faces containing numerical semigroup of small embedding dimension, which are of bounded dimension. For instance, when verifying [9, Conjecture 7.1], we computed all faces of every Kunz fan with $k = 3$ and $m \leqslant 42$ on our personal machines in under a day (we ran out of memory at $m = 43$), whereas computing the full face lattice for $m = 18$ for [5] took a cluster 3 weeks. The above is also prime for parallelization, since if one fixes m and k, each atom set A can be run independently.

The above algorithm turns out to coincide with the notion of a Gröbner walk [12]; Remark 5.7 makes this connection explicit.

Definition 5.1. *Fix a staircase Kunz nilsemigroup N. We say an outer Betti element $z \in \mathsf{Z}_N(\infty)$ is* irredundant *if its Betti inequality $z \cdot x \geqslant a \cdot x$ defines a facet of F_N, where $a \in \mathsf{Z}_N(\overline{z})$.*

We briefly identify some redundant outer Betti elements, for use in Section 6.

Proposition 5.2. *Let N be a staircase Kunz nilsemigroup.*

(a) *Any two outer Betti elements $b, b' \in \mathsf{Z}_N(\infty)$ with $\overline{b} = \overline{b}'$ have disjoint support.*
(b) *If $b \in \mathsf{Z}_N(\infty)$ is an outer Betti element and $\overline{b} = 0$, then b is irredundant if and only if* $\operatorname{supp}(b) = \{i\}$ *and $a_1, \ldots, a_k$ are distinct modulo* $\gcd(i, m)$.
(c) *There is at most one outer Betti element of N with full support, and such an outer Betti element is redundant if $|A| > 1$.*

Proof. Suppose $b_i > 0$ and $b'_i > 0$ for some i. Since b and b' are outer Betti elements, we have $b - e_i, b' - e_i \in Z_N(\overline{b} - p_i)$, and since N is staircase, $b - e_i = b' - e_i$ so $b = b'$.

Next, if $b_i > 0$ and $b_j > 0$, then b is redundant by the Farkas lemma, as its Betti inequality follows from $x_i \geqslant 0$ and $x_j \geqslant 0$. As such, suppose b_i is the only nonzero entry of b. Writing $d = \gcd(i, m)$ for the order of $i \in \mathbb{Z}_m$, [18, Corollary 3.7] and Proposition 3.4 imply $C(m; A) \cap \{x \in \mathbb{R}^d : x_i = 0\}$ projects faithfully onto $C(d; \overline{A} \smallsetminus \{\overline{0}\})$, where $\overline{A} \subseteq \mathbb{Z}_d$ is the set of residue classes of $a_1, \ldots, a_k$ in $\mathbb{Z}_d$. This cone has dimension $k - 1$ if and only if $a_1, \ldots, a_k$ are distinct modulo d, thereby proving the second claim.

Lastly, the final claim immediately follows from the first and second. $\square$

Proposition 5.3. *Suppose N is a staircase Kunz nilsemigroup and $b \in \mathsf{Z}_N(\infty)$ is an irredundant outer Betti element. The facet of F_N supported by the Betti inequality for b lies on the boundary of $C(m; A)$ if and only if either (i) $\overline{b} \in A$, or (ii) $\overline{b} = 0$.*

Proof. By definition, a facet inequality of $C(m; A)$ is either of the form $x_i \geqslant c \cdot x$ with

$$c_1 p_1 + \cdots + c_k p_k \equiv p_i \bmod m,$$

or of the form $x_i \geqslant 0$. As such, an outer Betti element whose Betti inequality is one of these two forms must fall into the claimed case (i) or (ii), respectively. $\square$

Example 5.4. Resume notation from Example 3.3. The boundaries between neighboring chambers are labeled in Figure 2 by an equality defining the linear subspace of $\mathbb{R}^2$ they lie in. By Theorem 5.5, each such equality induces a single trade in the Kunz nilsemigroup corresponding to that codimension 1 face. For instance, the ray between chambers (a) and (b) has corresponding Kunz nilsemigroup with the trade

$(7,0) \sim (0,2)$, since every point on that ray must satisfy $7x_6 \geqslant 2x_{11}$ from chamber (b) and $7x_6 \leqslant 2x_{11}$ from chamber (a). Additionally, the ray between chambers (b) and (c) has Kunz nilsemigroup with the trade $(4,0) \sim (0,4)$, and thus no numerical semigroups lie on that ray by [3, Corollary 3.16].

Theorem 5.5. *Suppose* $\dim F_N = k-1$ *and* F_N *is not on the boundary of* $C(m;A)$.

(a) *There is a unique inner Betti element* $p \in N$, *and* $\mathsf{Z}_N(p) = \{z, z'\}$.

(b) *For any non-nil* $q \in N$, *there exist unique factorizations* $w, w' \in \mathsf{Z}_N(q)$ *such that* $z \not\preceq w$ *and* $z' \not\preceq w'$, *where* $\preceq$ *denotes the component-wise partial order on* $\mathbb{Z}^k_{\geqslant 0}$. *Moreover,* $w \neq w'$ *if and only if* $p \preceq q$ *in* N.

(c) *For a staircase Kunz nilsemigroup* N' *with* $F_N \subseteq F_{N'}$, *either (i)* $z \not\preceq w$ *for all* $q \in N'$ *and* $w \in \mathsf{Z}_{N'}(q)$, *or (ii)* $z' \not\preceq w$ *for all* $q \in N'$ *and* $w \in \mathsf{Z}_{N'}(q)$.

Proof. Fix $c \in \mathbb{Z}^k$ with $\gcd(c) = 1$ and $H_N = \{x \in \mathbb{R}^k : c \cdot x = 0\}$. Write $c = c^+ - c^-$ where $c^+, c^- \in \mathbb{Z}^k_{\geqslant 0}$ have disjoint support, and let $d \in \mathbb{Z}_{\geqslant 1}$ be minimal with $\overline{dc^+} = \overline{dc^-}$. For any Betti element $p \in N$ and factorizations $z, z' \in \mathsf{Z}_N(p)$, we have $z - z' \in \mathbb{Z}c$. As such, $|\mathsf{Z}_N(p)| = 2$, as only one factorization of p can avoid the positive (or negative) support of c. In particular, $\mathsf{Z}_N(p) = \{z, z'\}$ such that $z \in d\mathbb{Z}_{\geqslant 1}c^+$ and $z' \in d\mathbb{Z}_{\geqslant 1}c^-$. To prove (a), it suffices to show $\overline{dc^+}$ is a Betti element of N. However, this follows from Proposition 3.4 and the fact that $\overline{dc^+} = \overline{dc^-} \neq 0$ by Propositions 5.2(c) and 5.3 since F_N is not on the boundary of $C(m;A)$.

Having proven part (a), let $p \in N$ denote the unique inner Betti element and write $\mathsf{Z}_N(p) = \{z, z'\}$. Part (b) then follows from part (a), since any two factorizations of a given non-nil element $q \in N$ differ by an integer multiple of $z - z'$, and Proposition 3.4 ensures that if a factorization is preceded by z or z', then performing the trade $z \sim z'$ or $z' \sim z$, respectively, results in another factorization of q. Lastly, for any $x \in F^\circ_{N'}$, the factorizations of any non-nil $q \in N$ are totally ordered by their dot product with x (in fact, they form an arithmetic sequence with step size $(z - z') \cdot x$), so since N' is staircase it must be as prescribed in part (c). ☐

Remarks 5.6. Theorem 5.5 identifies a necessary condition for an outer Betti element z of N to be irredundant. Indeed, one may naïvely apply steps (4) and (5) for z to obtain hypothetical factorizations of each element of $\mathbb{Z}_m$. If these do not form a staircase (e.g., some non-nil $p \in N$ and $z' \in \mathsf{Z}_N(p)$ has $z' - e_i \notin \mathsf{Z}_N(p - a_i)$ for some $z'_i > 0$), then z must be redundant. For example, in Example 3.6, the Kunz nilsemigroup N corresponding to the face in chamber (e) has outer Betti element $z = (1,2,0)$ with $\overline{z} = 5$, but applying Theorem 5.5 would move 5, 6, 7, and 8 to new locations in the Kunz poset, leave 6 "dangling" over the outer Betti element $(2,0,0)$. Geometrically, this amounts to crossing the hyperplane $x_1 + 2x_2 = 2x_9$ without first entering chamber (d).

Note that the condition identified in the previous paragraph is not sufficient. Indeed, consider the Kunz nilsemigroup N in Example 4.4, and let N' denote the nilsemigroup obtained from N by replacing $\mathsf{Z}_{N'}(3) = \{(0,0,1,1)\}$. One may readily check that N is Kunz and has an outer Betti element z above 1 and 2 with $\overline{z} = 3$, but applying Theorem 5.5 to z yields the modular nilsemigroup N, which is not Kunz.

Remarks 5.7. Under a different viewpoint, one may realize the fan $\mathcal{G}(m;A)$ as a portion of the Gröbner fan of a certain lattice ideal. We defer the reader to [29] for definitions of Gröbner bases and Gröbner fans, and [20] for definition of lattices and lattice ideals.

Given m and A, consider the lattice ideal

$$I_L = \langle x^a - x^b : a - b \in L \rangle \subseteq \Bbbk[x_1, \ldots, x_k]$$

for the rank k, index m lattice

$$L = \{(z_1, \dots, z_k) \in \mathbb{Z}^k : a_1 z_1 + \cdots + a_k z_k \in m\mathbb{Z}\}.$$

Each face in $\mathcal{G}(m; A)$ is a face of the Gröbner fan of I_L. Indeed, by [20, Corollary 7.29] and the discussion preceding it, the initial ideal corresponding to each chamber F of the Gröbner fan of I_L contains precisely the monomials whose exponent vectors are not "optimal" with respect to a vector $x \in F^\circ$; upon unraveling definitions, this result is encoded in Proposition 3.4.

Gröbner walks, as they are called [12], allow one to compute the Gröbner fan of a given polynomial ideal in a similar fashion to the algorithm at the start of this section. The manuscript [11] discusses a Gröbner walk for a general ideal I, and [16] concerns the special case where I is toric (i.e., I is the lattice ideal of a saturated lattice). Note that the lattice ideal I_L above is not toric, so the results in [16] cannot be directly applied in our setting, though some of our results have analogs in [16]. For instance, the necessary condition in Remark 5.6 is reminiscent of [16, Theorem 3.6]. Additionally, the lattice L defined above has the added property that each coset of L in $\mathbb{Z}^k$ naturally corresponds to an element of $\mathbb{Z}_m$, which provides the foundation of the nilsemigroup viewpoint used throughout this paper.

6. Examples and applications

6.1. Applications to open problems. We begin with a proof of [9, Conjecture 7.3]. Following the notation from [9], given a numerical semigroup S, we write $\eta(S) = |\rho|$, where ρ is any minimal presentation of S.

Definition 6.1. *Suppose N and N' are Kunz nilsemigroups. We say N is a* refinement *of N' if for each non-nil $i \in N$, we have $\mathsf{Z}_{N'}(i) \subseteq \mathsf{Z}_N(i)$.*

Theorem 6.2. *Any Kunz nilsemigroup N is a refinement of some staircase Kunz nilsemigroup N' with identical atom set. Moreover, if N and N' have β and β' outer Betti elements, respectively, and N has minimal presentation ρ, then $|\rho| + \beta \leqslant \beta'$. In particular, if S and S' are numerical semigroups whose Kunz nilsemigroups are N and N', respectively, then $\eta(S) \leqslant \eta(S')$.*

Proof. Let $m = |N|$, and let $F \subseteq \mathcal{C}_m$ denote the face whose Kunz nilsemigroup is N. By Theorem 3.8, $\mathcal{G}(m; A)$ is pure, so there exists $F' \subseteq \mathcal{C}_m$ with $F \subseteq F'$ whose Kunz nilsemigroup N' is staircase. This means N is a refinement of N', and moreover, if we write $\mathsf{Z}_{N'}(i) = \{z\}$, then for any $y \in (F')^\circ$, z is the element of $\mathsf{Z}_N(i)$ whose dot product with $p(y)$ is minimal.

Now, let B be an outer Betti element of N. Fix $y \in (F')^\circ$, and let $z \in B$ denote the element of B minimizing $z \cdot p(y)$. We claim $\{z\}$ is an outer Betti element of N'. Indeed, for each $i \in \operatorname{supp}(z)$, we have $(z - e_i) \in B - e_i = \mathsf{Z}_N(p)$ for some $p \in N$. By the minimality of z, $z - e_i$ has minimal dot product with $p(y)$ among elements of $\mathsf{Z}_N(p)$, so $\mathsf{Z}_{N'}(p) = \{z - e_i\}$. This proves the claim.

Next, fix a Betti element $i \in N$. Let $\mathsf{Z}_{N'}(i) = \{z\}$, and let $Z \subseteq \mathsf{Z}_N(i)$ denote a connected component of ∇_i not containing z. By an identical argument to the preceding paragraph, $\{z'\}$ is an outer Betti element of N' for some $z' \in Z$. Moreover, z' cannot lie in any outer Betti element of N since it lies in $\mathsf{Z}_N(i)$.

We now conclude the desired inequalities hold by Theorem 2.1. $\square$

Remarks 6.3. It remains an interesting open question to bound $\eta(S)$ in terms of $\mathsf{m}(S)$ and $\mathsf{e}(S)$. This was posed in [9, Conjecture 7.2], and recent progress and a survey can be found in [7, 22, 21]. Theorem 6.2 provides an avenue for further headway.

A numerical semigroup S with 4 generators can have $\eta(S)$ arbitrarily large; see [28] for examples and references. As such, if $k = 3$, there is no upper bound to the number of outer Betti elements a staircase Kunz nilsemigroup N can have. However, as Corollary 6.5 indicates, F_N can have at most 6 facets in this case.

Theorem 6.4. *If N is a staircase Kunz nilsemigroup, $k = 3$, and $z, z' \in \mathbb{Z}^3_{\geqslant 0}$ are outer Betti elements of N with* $\operatorname{supp}(z) = \operatorname{supp}(z') = \{1, 2\}$*, then z or z' is redundant.*

Proof. Let $z = (z_1, z_2, 0)$, $z' = (z_1', z_2', 0)$, $(0, 0, z_3) \in \mathsf{Z}_N(\overline{z})$, and $(0, 0, z_3') \in \mathsf{Z}_N(\overline{z'})$. After relabeling as necessary, we may assume

$$z_1 > z_1', \qquad z_2 < z_2', \qquad \text{and} \qquad z_3 > z_3'.$$

We will show that z' is redundant. Let $w = (z_1, z_2, -z_3)$, $w' = (z_1', z_2', -z_3')$, and

$$v = w' - w = (z_1' - z_1, z_2' - z_2, z_3 - z_3').$$

By the previous inequalities, we can decompose $v = v^+ - v^-$ where

$$v^+ = (0, z_2' - z_2, z_3 - z_3') \in \mathbb{Z}^3_{\geqslant 0} \qquad \text{and} \qquad v^- = (z_1 - z_1', 0, 0) \in \mathbb{Z}^3_{\geqslant 0}.$$

Now, since z is a factorization of a non-nil element of N, so is $z_1 e_1$, and thus

$$\mathsf{Z}_N(\overline{v^+}) = \{v^-\} \qquad \text{and} \qquad v^+ \in \mathsf{Z}_N(\infty)$$

since $\overline{v^+} = \overline{v^-}$. This means $v^+ = b + \ell$, where b is an outer Betti element and $\ell \in \mathbb{Z}^3_{\geqslant 0}$. Clearly $\operatorname{supp}(b) \subseteq \{2, 3\}$, and since $v_2^+ e_2$ and $v_3^+ e_3$ are factorizations of non-nil elements of N, we in fact have $\operatorname{supp}(b) = \{2, 3\}$. As such, $\mathsf{Z}_N(\overline{b}) = \{b_1 e_1\}$ for some $b_1 \geqslant 0$. Thus,

$$w' = w + v^+ - v^- = w + (-b_1, b_2, b_3) + \ell + (b_1 e_1 - v^-).$$

If $b_1 e_1 - v^- \in \mathbb{Z}^3_{\geqslant 0}$, then we are done by the Farkas lemma since $F_N \subseteq \mathbb{R}^3_{\geqslant 0}$. As such, suppose $z_1 - z_1' - b_1 > 0$. Computing equivalence classes modulo m, we have

$$(z_1 - z_1 - b_1)p_1 = \overline{v^-} - \overline{b} = \overline{v^+} - \overline{b} = \overline{\ell},$$

and since $(z_1 - z_1 - b_1)e_1$ is a factorization of a non-nil element of N, we have $\ell \in \mathsf{Z}_N(\infty)$. Proceeding as above, write $\ell = b' + \ell'$, where b' is an outer Betti element and $\ell' \in \mathbb{Z}^3_{\geqslant 0}$. By the same argument, $\operatorname{supp}(b') = \{2, 3\}$ and $\mathsf{Z}_N(\overline{b'}) = \{b_1' e_1\}$, so

$$w' = w + (-b_1, b_2, b_3) + (-b_1', b_2', b_3') + \ell' + (b_1 e_1 + b_1' e_1 - v^-).$$

We can continue applying this argument until the rightmost parenthetical lies in $\mathbb{Z}^3_{\geqslant 0}$, and this process will indeed terminate since $b_2, b_3, b_2', b_3', \ldots > 0$. ☐

Corollary 6.5. *If $k = 3$, then no two irredundant outer Betti elements of a staircase Kunz nilsemigroup N have identical support. In particular, F_N has at most 6 facets.*

Proof. Any outer Betti element b of N has nonempty support. If $\operatorname{supp}(b) = \{1, 2, 3\}$ then b is redundant by Proposition 5.2. Additionally, there are exactly three outer Betti elements of N with singleton support, none of which coincide, and the remaining cases are handled by Theorem 6.4 after appropriate permutation of indices. ☐

Remarks 6.6. Resuming notation and terminology from Remark 5.7, if $L \subseteq \mathbb{Z}^3$ is any lattice with $\mathbb{Z}^3/L \cong \mathbb{Z}_m$, then Corollary 6.5 implies each chamber of the Gröbner fan of the lattice ideal I_L has at most 6 facets. Indeed, choosing $(v_1, v_2, v_3) \in \mathbb{Z}^3$ whose image generates $\mathbb{Z}^3/L$, we have

$$L = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : v_1 x_1 + v_2 x_2 + v_3 x_3 \in m\mathbb{Z}\}.$$

As such, each chamber F of the Gröbner fan of I_L, there are two possibilities:

- $F \in \mathcal{G}(m; v_1, v_2, v_3)$, and thus has at most 6 facets by Corollary 6.5; or

- the initial ideal of I_L corresponding to F contains a variable, and thus has at most 4 minimal generators, since its staircase has at most one minimal generator with non-singleton support by Proposition 5.2.

According to [30, Conjecture 6.1], if L is a saturated d-dimensional lattice, then there exists a bound, in terms of d, on the number of facets of any chamber of the Gröbner fan of the toric ideal I_L. Though some progress has been made on this conjecture [15], a proof remains elusive, even in the case $d = 3$. Given the above conclusions, we pose the following generalization of this question.

Question 6.7. *Does there exist a function $\phi : \mathbb{Z} \to \mathbb{Z}$ such that for any d-dimensional lattice L, each chamber of the Gröbner fan of I_L has at most $\phi(d)$ facets?*

We state the following conjecture, which has been verified computations for $m \leqslant 42$, albeit with some reservation: [30, Conjecture 6.2] claimed the same was true for the chambers in the Gröbner fan of any toric ideal defined by a 3-dimensional lattice, but a counterexample was located soon thereafter [14].

Conjecture 6.8. *If $k = 3$ and N is staircase and Kunz, then F_N has at most 4 facets.*

6.2. Embedding Dimension 3. Throughout this subsection, let $k = 2$. In what follows, we characterize the faces of $\mathcal{C}_m$ containing embedding dimension 3 numerical semigroups by classifying the possible "shapes" of the Kunz nilsemigroup N of such a face. Any embedding dimension 3 numerical semigroup is either complete intersection with 2 minimal trades, or not complete intersection with 3 minimal trades; for more on this dichotomy, see [26, Chapter 10]. As we will show, the Kunz nilsemigroup N comes in 3 varieties; 2 in the former category, and 1 in the latter category.

We begin with the case where N is staircase. By Proposition 5.2, N has either 2 or 3 outer Betti elements since at most one can have full support. As such, the Kunz poset of N can have one of two staircase "shapes".

- If N has 2 outer Betti elements $(a, 0)$ and $(0, c)$, then $m = ac$ and its Kunz poset forms an $a \times c$ diamond. In this case, the *shape* of N is $(a, c) \in \mathbb{Z}^2_{\geqslant 2}$. Numerical semigroups with Kunz nilsemigroup N are complete intersection.
- If N has 3 outer Betti elements, then its Kunz poset forms a "V" with full support outer Betti element (a, c), and its other two outer Betti elements have the form $(a + b, 0)$ and $(0, c + d)$. In this case, we say N has *shape* $(a, b, c, d) \in \mathbb{Z}^4_{\geqslant 1}$, and
$$m = (a + b)(c + d) - bd.$$
Numerical semigroups with Kunz nilsemigroup N are not complete intersection, but are uniquely presented and possess 3 minimal trades [26, Chapter 10].

Example 6.9. The three posets from Figure 2 are all staircase posets, and their shapes are $(2, 0, 10, 0)$, $(2, 2, 3, 4)$ and $(2, 4, 3, 1)$, respectively.

The following result implies that there exists a Kunz nilsemigroup N with a given staircase shape if and only if there exists a "filling" with the elements of $\mathbb{Z}_m$. This amounts to choosing $p_1, p_2 \in \mathbb{Z}_m$ so as to "fill" the given staircase shape. One could even visualize the staircase of N as a Young tableaux, wherein each box (i, j) is filled with $p \in \mathbb{Z}_m$ if $\mathsf{Z}_N(p) = \{(i, j)\}$.

Proposition 6.10. *Any modular staircase nilsemigroup N with 2 atoms is Kunz.*

Proof. If N has shape (a, c), then p_1 has order a or p_2 has order c, as otherwise Bézout's identity yields a full support outer Betti element. Up to symmetry, assume

p_2 has order c. The Betti inequalities thus have the form

$$ax_1 \geqslant kx_2 \qquad \text{and} \qquad cx_2 \geqslant 0$$

where $\mathsf{Z}_N(ap_1) = \{(0,k)\}$. The point $(x_1,x_2) = (c,a)$ satisfies both strictly.

If N has shape (a,b,c,d), then the irredundant outer Betti inequalities are

$$(a+b)x_1 > k_2x_2, \qquad \text{and} \qquad (c+d)x_2 > k_1x_2,$$

for some $k_2 < c+d$ and $k_1 < a+b$, and the point $(x_1,x_2) = (c+d, a+b)$ satisfies both strictly. In either case, Theorem 4.9 completes the proof. $\square$

For any $(a,c) \in \mathbb{Z}_{\geqslant 2}$, there exists a Kunz nilsemigroup with shape (a,c); one may choose, for instance, $p_1 = 1$ and $p_2 = a$, as then each coset of the subgroup $\langle a\rangle \subseteq \mathbb{Z}_m$ forms a "row" of the staircase. More generally, up to symmetry, a choice of p_1 and p_2 fills the staircase shape if and only if p_2 with order c and p_1 generates $\mathbb{Z}_m/\langle a\rangle$.

Given a staircase shape (a,b,c,d), it follows from [10, Section 4] that a choice of p_1 and p_2 fills the staircase if and only the following 4 conditions hold:

$$(a+b)p_1 \equiv dp_2 \bmod m, \tag{1}$$
$$(c+d)p_2 \equiv bp_1 \bmod m, \tag{2}$$
$$ap_1 + cp_2 \equiv 0 \bmod m, \tag{3}$$
$$\gcd(p_1,p_2,m) = 1. \tag{4}$$

They cite the first author's PhD dissertation for a proof that such p_1, p_2 exist if and only if $\gcd(a,b,c,d) = 1$. We were not able to find a readable version of this, so we've elected to include a short proof for completeness.

Theorem 6.11. *There exists a Kunz nilsemigroup with 3 outer Betti elements and shape (a,b,c,d) if and only if $\gcd(a,b,c,d) = 1$.*

Proof. Suppose that $\gcd(a,b,c,d) = 1$, and let $f : \mathbb{Z}_m^2 \to \mathbb{Z}_m^2$ be the group homomorphism given by the matrix

$$\begin{pmatrix} a+b & -d \\ a & c \end{pmatrix}.$$

We claim that $\ker(f) \cong \mathbb{Z}_m$, and that any generator (p_1,p_2) of $\ker(f)$ satisfies (1)−(4). To see that $\ker(f) \cong \mathbb{Z}_m$, we use the fact that $\ker(f) \cong \ker(A)$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$$

is the Smith normal form [23] of f since $\gcd(a, a+b, -d, c) = 1$ and $(a+b)c + ad = m$. Clearly $\ker(A)$ is generated by $(0,1)$, and thus isomorphic to $\mathbb{Z}_m$.

Now, suppose (p_1,p_2) generate $\ker(f)$. By construction, both (1) and (3) are satisfied, and subtracting (1) from (3) yields (2). If $h_1 = \gcd(p_1,m)$ or $h_2 = \gcd(p_2,m)$ equal 1, then (4) is satisfied and we are done. Otherwise, suppose both $h_1, h_2 > 1$, and set $h = \gcd(h_1,h_2)$. Notice $\frac{m}{h_1}p_1 = \frac{m}{h_2}p_2 = 0$, but this means that $\frac{m}{h}(p_1,p_2) = (0,0)$. Since (p_1,p_2) generate $\ker(f)$, we must have $h = 1$, as desired.

Conversely, if $\gcd(a,b,c,d) = g > 1$, then the Smith normal form of f is

$$\begin{pmatrix} g & 0 \\ 0 & m \end{pmatrix},$$

so $\ker(f) \cong \mathbb{Z}_{m/g}$ and thus for any $(p_1,p_2) \in \ker f$, we have $\frac{m}{g}(p_1,p_2) = 0$. This means that p_1, p_2 fail to satisfy condition (4). $\square$

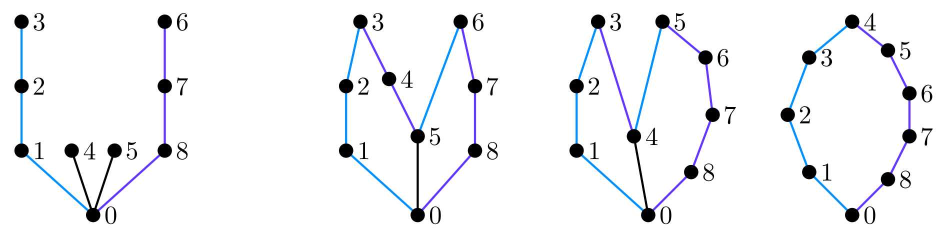

Figure 5. Cup poset for $d = 4$, alongside the posets of three of its rays.

This leaves the case where N is not staircase, i.e., $\dim F_N = 1$. By Theorem 3.8, this occurs when F_N is a ray on the shared boundary of two faces of $\mathcal{G}(m; A)$ of the form $F_{N'}$ with N' staircase. Corollary 6.12 below records all such rays, at which point Theorem 5.5 identifies the structure of N. Any numerical semigroup with Kunz nilsemigroup N is complete intersection since there is a trade involving 2 generators.

Corollary 6.12. *If N is staircase and has shape (a, c) and $p_2 = a$, then F_N has rays*

$$\vec{r}_1 = (1, 0) \qquad \textit{and} \qquad \vec{r}_2 = (k, c),$$

where $\mathsf{Z}_N(ap_1) = \{(0, k)\}$. If N has shape (a, b, c, d), then F_N has rays

$$\vec{r}_1 = (d, a + b) \qquad \textit{and} \qquad \vec{r}_2 = (c + d, b).$$

Proof. Apply Theorem 4.8 and the proofs of Proposition 6.10 and Theorem 6.11. ☐

The rays of the face $F \subseteq \mathcal{C}_m$ with $p(F) = F_N$ in the latter case of Corollary 6.12 are

$$\vec{r}_1 = (dx_1 + (a + b)y_1, \ldots, dx_{m-1} + (a + b)y_{m-1})$$
$$\vec{r}_2 = ((c + d)x_1 + by_1, \ldots, (c + d)x_{m-1} + by_{m-1})$$

where (x_p, y_p) is the unique factorization for each nonzero non-nil $p \in N$. A similar construction can be done in the former case in Corollary 6.12, though this case is also addressed in [2, Theorem 4.6].

6.3. Cup Posets. In [5], the authors compute the number of rays of $\mathcal{C}_m$ up to $m = 21$. This data suggests the number of rays of $\mathcal{C}_m$ grows exponentially in m. In this section, we explicitly construct a family of faces whose number of rays is exponential in m.

For this subsection, fix $d \geqslant 3$, let $m = 3(d - 1)$, and consider the modular nilsemigroup N with $A = \{1, d, d + 1, \ldots m - d, m - 1\}$ whose divisibility poset of non-nil elements has

$$1 \lessdot 2 \lessdot \cdots \lessdot d - 1 \qquad \text{and} \qquad m - 1 \lessdot m - 2 \lessdot \cdots \lessdot m - (d - 1)$$

as its cover relations (we call this a *cup poset*; $d = 4$ is depicted on the left in Figure 5). It is not hard to show $\dim F_N = d$ and the point $(1, d-1, \ldots, d-1, 1) \in F_N^\circ$ satisfies all Betti inequalities of N strictly. In particular, N is a Kunz nilsemigroup by Theorem 4.9.

Proposition 6.13. *We have $x \in F_N$ if and only if $y = q(x)$ satisfies*

$$dy_1 \geqslant y_d, \qquad dy_{m-1} \geqslant y_{m-d}, \qquad y_1 + y_k \geqslant y_{k+1}, \qquad \textit{and} \qquad y_{m-1} + y_k \geqslant y_{k-1}$$

for $d \leqslant k \leqslant m - d$.

Proof. Each inequality above is a Betti inequality of N, so by Theorem 4.8 it suffices to show the remaining outer Betti inequalities are redundant. Each such outer Betti inequality has the form $y_i + y_j \geqslant y_{i+j}$, where $d \leqslant i \leqslant j \leqslant m-d$ and $i+j \neq 0$. Notice

$$\begin{aligned} y_i + y_j &= y_i + y_j + (i-(d-1))y_{m-1} - (i-(d-1))y_{m-1} \\ &\geqslant (d-1)y_1 + y_j - (i-(d-1))y_{m-1} \\ &\geqslant \big((d-1) - (2(d-1)-j)\big)y_1 + (d-1)y_{m-1} - (i-(d-1))y_{m-1} \\ &= (j-(d-1))y_1 + (2(d-1)-i)y_{m-1} \end{aligned}$$

and

$$y_{i+j} = \begin{cases} (3(d-1)-(i+j))y_{m-1} & \text{if } i+j > m-d; \\ (i+j-3(d-1))y_1 & \text{if } i+j < d. \end{cases}$$

In either case the corresponding coefficient in $(j-(d-1))y_1 + (2(d-1)-i)y_{m-1}$ is larger than $y_{i,j}$, which means that $y_i + y_j \geqslant y_{i+j}$. $\square$

Proposition 6.14. *There is an invertible linear transformation H that sends F_N to the cone over a $(d-1)$-cube.*

Proof. For $x \in F_N$, the inequalities in Proposition 6.13 are

$$\begin{aligned} dx_1 - x_2 &\geqslant 0, & -(d-1)x_1 + x_2 + x_d &\geqslant 0, \\ x_1 + x_k - x_{k+1} &\geqslant 0, & -x_k + x_{k+1} + x_d &\geqslant 0, \\ x_1 + x_{d-1} - (d-1)x_d &\geqslant 0, & -x_{d-1} + dx_d &\geqslant 0, \end{aligned}$$

where $k \in \{2, \dots d-2\}$. Let $H_1, H_2 \in \mathbb{R}^{(d-1)\times d}$ denote the matrices corresponding to the first and second columns of inequalities, respectively. Every row of the matrix $J = H_1 + H_2$ equals $\mathbf{j} = e_1 + e_d$. Recall that the matrix defining the cone over the standard $(d-1)$-cube is

$$\begin{pmatrix} I_{d-1} & \mathbf{0} \\ -I_{d-1} & \mathbf{1} \end{pmatrix},$$

where $\mathbf{1}$ is the column vector of all 1's. Letting $H = \begin{pmatrix} H_1 \\ \mathbf{j} \end{pmatrix}$, one readily checks

$$\begin{pmatrix} I_{d-1} & \mathbf{0} \\ -I_{d-1} & \mathbf{1} \end{pmatrix} H = \begin{pmatrix} H_1 \\ -H_1 + J \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix},$$

which completes the proof. $\square$

Remarks 6.15. The Kunz posets of the rays of F_N have an interesting combinatorial structure. Picking a vertex of a cube (and thus a ray of F_N) is equivalent to making a binary choice for each pair of opposite faces [32, Chapter 7]. Following the map H from Proposition 6.14, each pair of opposing faces correspond to choosing either $x_i + x_1 = x_{i+1}$ or $x_i = x_{i+1} + x_{m-1}$ for each $i = d-1, \dots, 2(d-1)-1$, yielding the Kunz poset relation $i \prec i+1$ or $i+1 \prec i$ for each i. Three examples are depicted in Figure 5; we call these *mountain range posets.*

One may use the gluing constructions in [2, Section 6] to construct faces of $\mathcal{C}_m$ whose cross sections are simplicial. This raises the following.

Question 6.16. *Is there a family of faces of $\mathcal{C}_m$ whose cross sections are cones over cross polytopes?*

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Cole Brower, Mathematics Department, San Diego State University, San Diego, CA 92182
E-mail : `cole.brower@gmail.com`

Joseph McDonough, School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail : `mcdo1248@umn.edu`

Christopher O'Neill, Mathematics Department, San Diego State University, San Diego, CA 92182
E-mail : `cdoneill@sdsu.edu`