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Abstract A cosmological polytope is defined for a given Feynman diagram, and its canonical form may be used to compute the contribution of the Feynman diagram to the wavefunction of certain cosmological models. Given a subdivision of a polytope, its canonical form is obtained as a sum of the canonical forms of the facets of the subdivision. In this paper, we identify such formulas for the canonical form via algebraic techniques. It is shown that the toric ideal of every cosmological polytope admits a Gröbner basis with a squarefree initial ideal, yielding a regular unimodular triangulation of the polytope. In specific instances, including trees and cycles, we recover graphical characterizations of the facets of such triangulations that may be used to compute the desired canonical form. For paths and cycles, these characterizations admit simple enumeration. Hence, we obtain formulas for the normalized volume of these polytopes, extending previous observations of Kühne and Monin.

1. Introduction

Arkani-Hamed, Benincasa and Postnikov [2] introduced the cosmological polytope $\mathcal{C}_G$ of an undirected, connected graph $G = (V, E)$, where V is the finite set of *vertices (or nodes)* of G and E is its finite collection of *edges*, i.e. pairs ij for some $i, j \in V$. When we would like to emphasize that V and E are, respectively, the vertex and edge set of G, we may write $V(G)$ and $E(G)$, respectively. We will use ij to denote an undirected edge between i and j, and (i, j) to denote a directed edge $i \to j$ when edge directions are needed.

We work in the finite real-Euclidean space $\mathbb{R}^{|V|+|E|}$ with standard basis vectors x_i and x_e for all $i \in V$, $e \in E$. The *cosmological polytope* $\mathcal{C}_G$ of G is

$$\mathcal{C}_G = \operatorname{conv}\{x_i + x_j - x_e, x_i - x_j + x_e, -x_i + x_j + x_e \ : \ e = ij \in E\},$$

where $\operatorname{conv}(S)$ denotes the convex hull of $S \subset \mathbb{R}^n$. This is a polytope of dimension $|V| + |E| - 1$. It is only required that the graph G is connected and undirected with a finite set of vertices and edges. For instance, G need not be simple. In [13], the authors work with a slight generalization of the definition of $\mathcal{C}_G$ that allows for G

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to be disconnected. For the purposes of this paper, however, we will consider only connected G.

In the physical context, the graph G can be interpreted as a Feynman diagram, in which case the cosmological polytope provides a geometric model for the computation of the contribution of the Feynman diagram represented by G to the so-called *wavefunction of the universe* [2]. Recent works study the physics of scattering amplitudes via a generalization of convex polytopes called *positive geometries* [1]. This connection arises via a unique differential form of the positive geometry that has only logarithmic singularities along its boundary. This form is termed its *canonical form* [1]. In the case of a cosmological polytope $\mathcal{C}_G$, the canonical form provides a formula for computing the contribution of the Feynman diagram G to certain wavefunctions of interest.

One way to compute the canonical form Ω_P of a polytope P is as a sum of the canonical forms of the facets $S_1, \dots, S_m$ of a subdivision of P [14], i.e.

$$\Omega_P = \Omega_{S_1} + \cdots + \Omega_{S_m}. \tag{1}$$

This technique has been applied successfully in several situations [5, 8, 9, 12, 17].

In the case of cosmological polytopes, it was observed in [2] for specific examples of G that special subdivisions of the cosmological polytope correspond to classical physical theories for the computation of the contribution of a Feynman diagram to a wavefunction. This observation suggests that subdivisions of the cosmological polytope may correspond to physical theories for the computation of wavefunctions, and hence motivates a search for nice subdivisions of cosmological polytopes that hold for any graph G. Such subdivisions have the potential to provide new physical theories for wavefunction computations. The investigation of subdivisions that hold for any graph G was left as future work in [2]. In this paper, we provide such subdivisions by way of algebraic techniques.

The cosmological polytope is a *lattice polytope*, i.e. the convex hull of a finite collection of points in $\mathbb{Z}^{|V|+|E|}$. From this perspective, it is perhaps most natural to investigate its subdivisions into *unimodular* simplices, i.e. lattice simplices of minimum Euclidean volume. Such subdivisions have the advantage that each summand in (1) has the relatively simple form of a rational function

$$\frac{\omega}{f_1 \cdots f_r}, \tag{2}$$

where $f_1, \dots, f_r$ are the facet-defining equations of the simplex and ω is a regular form on the associated positive geometry [3]. Hence, it is desirable to observe not only the existence of unimodular triangulations of $\mathcal{C}_G$ but also to provide a complete description of their facets.

In this paper, we initiate the study of this problem via algebraic techniques. We compute a family of Gröbner bases of the toric ideal for the cosmological polytope $\mathcal{C}_G$ for any connected, undirected graph G, each of which has a squarefree initial ideal. It is known that the initial terms of such a Gröbner basis correspond to the minimal non-faces of a regular unimodular triangulation of $\mathcal{C}_G$ [19]. Hence, we obtain the following main result:

Theorem A. *(Corollary 2.11) The cosmological polytope $\mathcal{C}_G$ of any undirected, connected graph G has a regular unimodular triangulation.*

An analogous result holds for another family of lattice polytopes associated to graphs; namely *symmetric edge polytopes* [11, 16]. Interestingly, these are related to cosmological polytopes, as the symmetric edge polytope of a graph appears as a projection of a facet of the cosmological polytope of the same graph. Specifically, the symmetric edge polytope of $G = (V, E)$ is recovered by intersecting $\mathcal{C}_G \subset \mathbb{R}^{|V| \cup |E|}$ with the hyperplane $\sum_{e \in E} x_e = 1$ and then projecting onto $\mathbb{R}^{|V|}$.

The identified Gröbner bases provide the minimal non-faces of triangulations, from which achieving a facet description is a non-trivial task. One of the contributions of this article is to obtain such a characterization for the cosmological polytope of notable families of graphs.

Theorem B. *For specific choices of a term order we obtain a facet description of a regular unimodular triangulation of the cosmological polytope $\mathcal{C}_G$, when G is:*

- *a path (Theorem* 3.2*);*
- *a cycle (Theorem* 4.1*);*
- *a tree (Theorem* 5.12*).*

In the case of paths and cycles these characterizations are of a relatively simple form that allows for enumeration. We thereby obtain formulas for the normalized volume of $\mathcal{C}_G$ in these two cases. While, for paths, we recover the formula identified in [13], for the cycle, the normalized volume of $\mathcal{C}_G$ was previously unknown. Indeed, our methods enable us to show the following simple formula:

Theorem C. *(Theorem* 4.2*) The cosmological polytope $\mathcal{C}_{C_n}$ of the n-cycle C_n has normalized volume*

$$\mathrm{Vol}(\mathcal{C}_{C_n}) = 4^n - 2^n.$$

While the normalized volume of these polytopes provides us with information on the number of summands in the formula (1) for computing $\Omega_{\mathcal{C}_G}$, the explicit description of the facets that we obtain for trees and cycles given in Theorems 3.2, 4.1, and 5.12 allows for the exact computation of this canonical form. Theorem A suggests that such characterizations should be feasible for more general graphs via further analysis of the Gröbner bases identified in this paper.

In Section 2 we define a family of term orders associated to the cosmological polytope $\mathcal{C}_G$, called *good* term orders (Definition 2.5) and give a Gröbner basis for the toric ideal of $\mathcal{C}_G$ with respect to a good term order. The initial terms of these Gröbner bases are all squarefree, and hence they index the minimal nonfaces of a regular unimodular triangulation of $\mathcal{C}_G$. In the subsequent sections we apply these results to derive explicit characterizations of the facets of regular unimodular triangulations of $\mathcal{C}_G$ arising from good term orders for special instances of G. In Section 3, we characterize the facets of this triangulation for a specific good term order when G is the path graph. In Section 4, we show that the techniques in Section 3 can be extended to yield an analogous characterization of the facets of a triangulation for the cycle. Finally, in Section 5, we extend the characterization of the facets of the triangulation for paths to general trees.

2. Gröbner bases for the toric ideal of $\mathcal{C}_G$

In this section, we describe a family of Gröbner bases for the toric ideal of a cosmological polytope with the property that the corresponding initial ideals are squarefree. First we observe that, other than the $3|E|$ many points which define $\mathcal{C}_G$, the standard basis vectors x_i and x_e are lattice points in $\mathcal{C}_G$. This follows from the convex combinations $x_i = \frac{1}{2}((x_i+x_i-x_e)+(x_i-x_j+x_e))$. It is not hard to see that no other lattice point is contained in $\mathcal{C}_G$. Indeed, any such point p would be an element of $\{-1,0,1\}^{V \cup E}$ with coordinates summing up to 1. The linear inequalities $x_i + x_j \leq 1$ and $x_i + x_e \leq 1$ hold for each of the defining points of $\mathcal{C}_G$, whenever i and j are not connected in a graph and $i \notin e$. This implies that either p has only one coordinate equal to 1 (i.e. it is a standard basis vector and a lattice point of $\mathcal{C}_G$) or it has exactly two coordinates equal to 1 which correspond either to pair of vertices joined by an edge, or a vertex and an edge containing it. In this case, since the inequality $x_i + x_j + \sum_{i,j\in e} x_e \leq 1$ holds for each of the defining points and for each pair of vertices i and j, p is one of the defining points.

For any undirected graph G with vertex set V and edge set E, we define a polynomial ring in $|V| + 4|E|$ variables, each corresponding to a lattice point of $\mathcal{C}_G$. More precisely, we introduce three families of variables:

- A variable z_k, for every $k \in V \cup E$. We refer to these as *z-variables.*
- Variables y_{ije} and y_{jie}, for every edge $e = ij \in E$. We refer to these as *y-variables.*
- A variable t_e for every edge $e \in E$. We refer to these as *t-variables.*

Let R_G be the polynomial ring in these $|V| + 4|E|$ many variables, with coefficients in a field K, and consider the surjective homomorphism of K-algebras defined by

$$\begin{aligned} \varphi_G : R_G &\to K[\mathbf{w}^p \ : \ p \in \mathcal{C}_G \cap \mathbb{Z}^{V\cup E}] \\ z_k &\mapsto w_k \\ y_{ije} &\mapsto w_i w_j^{-1} w_e \\ y_{jie} &\mapsto w_i^{-1} w_j w_e \\ t_e &\mapsto w_i w_j w_e^{-1}. \end{aligned}$$

The ideal $I_{\mathcal{C}_G} \coloneqq \ker(\varphi_G)$ is the *toric ideal* of $\mathcal{C}_G$. Observe that variables in R_G correspond to lattice points of $\mathcal{C}_G$. Moreover, it is well known that toric ideals are binomial, i.e. they have a set of generators in which every element is the difference of two monomials (see for instance [10, Theorem 3.2]). When the graph G is understood from the context, we may simply write φ for φ_G. We now define some distinguished binomials in $I_{\mathcal{C}_G}$, which will be elements of a Gröbner basis for this ideal.

Definition 2.1. *We define two types of pairs of directed subgraphs of G.*

(i) *Let P be a path in G with at least two edges, with edge set $E(P) = \{i_1i_2, i_2i_3, \dots, i_{k-1}i_k\}$. For any partition (P_1, P_2) of $E(P)$ into two nonempty blocks we consider $E_1 = \{i_j \to i_{j+1} \ : \ i_ji_{j+1} \in E(P_1)\}$, and $E_2 = \{i_{j+1} \to i_j \ : \ i_ji_{j+1} \in E(P_2)\}$. The pair (E_1, E_2) is called a* zig-zag pair *of G. Moreover, we define the* terminal vertices *of (E_1, E_2) to be $v_1 = i_k$ and $v_2 = i_1$.*

(ii) *Let C be a cycle in G, with $E(C) = \{i_1i_2, i_2i_3, \dots, i_{k-1}i_k, i_ki_1\}$. For any partition (C_1, C_2) of $E(C)$ into two blocks (with one possibly empty) we consider $E_1 = \{i_j \to i_{j+1} \ : \ i_ji_{j+1} \in E(C_1)\}$ and $E_2 = \{i_{j+1} \to i_j \ : \ i_ji_{j+1} \in E(C_2)\}$, where the indices are considered mod k. The pair (E_1, E_2) is called a* cyclic pair *of G.*

Definition 2.2. *For every zig-zag pair (E_1, E_2) we define the* zig-zag binomial

$$b_{E_1,E_2} = z_{v_1} \prod_{e=i\to j\in E_1} y_{ije} \prod_{e=i\to j\in E_2} z_e - z_{v_2} \prod_{e=i\to j\in E_2} y_{ije} \prod_{e=i\to j\in E_1} z_e \in I_{\mathcal{C}_G}.$$

For every cyclic pair (E_1, E_2) we define the cyclic binomial

$$b_{E_1,E_2} = \prod_{e=i\to j\in E_1} y_{ije} \prod_{e=i\to j\in E_2} z_e - \prod_{e=i\to j\in E_2} y_{ije} \prod_{e=i\to j\in E_1} z_e \in I_{\mathcal{C}_G}.$$

In the case that either $E_1 = \varnothing$ or $E_2 = \varnothing$, we call the resulting cyclic binomial a cycle binomial. *In particular, cycle binomials consist of one monomial containing only y-variables and one containing only z-variables.*

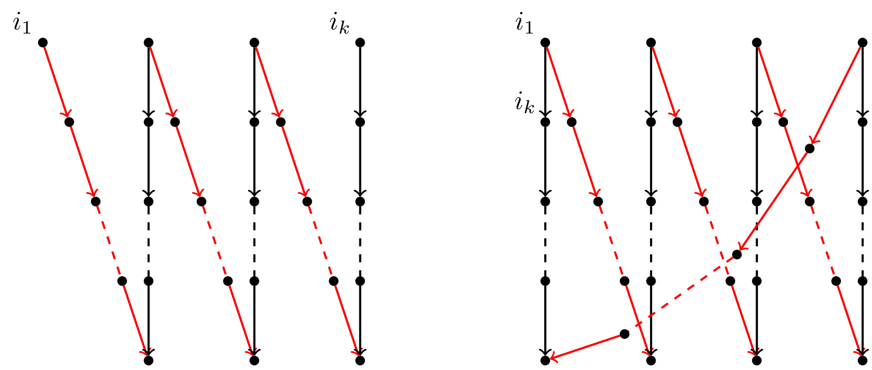


Figure 1. A zig-zag pair and a cyclic pair (E_1, E_2). The edges in E_1 are drawn in red.

Definition 2.3. *We define the following collection of binomials in $I_{\mathcal{C}_G}$.*

$$\begin{aligned} B_G = \{&\underline{y_{ije}y_{jie}} - z_e^2,\ \underline{y_{ije}t_e} - z_i^2,\ \underline{y_{jie}t_e} - z_j^2,\\ &\underline{y_{ije}z_j} - z_iz_e,\ \underline{y_{jie}z_i} - z_jz_e,\ \underline{t_ez_e} - z_iz_j\ :\ e = ij \in E(G)\}\cup\\ &\{b_{E_1,E_2}\ :\ (E_1,E_2) \text{ is a zig-zag pair or a cyclic pair of } G\}.\end{aligned}$$

We call the $6|E|$ many binomials contained in the first set the fundamental binomials*.*

Example 2.4. Let G be the complete graph on the vertices $V = \{1,2,3\}$. We observe that there are precisely 3 zig-zag pairs for G, namely $(\{(1\to2)\},\{(3\to2)\})$, $(\{(1\to3)\},\{(2\to3)\})$ and $(\{(2\to1)\},\{(3\to1)\})$. These give rise to the zig-zag binomials $z_3z_{23}y_{12(12)} - z_1z_{12}y_{32(23)}$, $z_2z_{23}y_{13(13)} - z_1z_{13}y_{23(23)}$ and $z_3z_{13}y_{21(12)} - z_2z_{12}y_{31(13)}$. Moreover, there are 8 ways to partition the 3 edges of G in two ordered blocks, which give rise to 8 cyclic binomials. Examples of these binomials are $y_{12(12)}y_{23(23)}z_{13} - y_{13(13)}z_{12}z_{23}$ and the cycle binomial $y_{12(12)}y_{23(23)}y_{31(13)} - z_{12}z_{23}z_{13}$.

Definition 2.5. *A term order on R_G is called a* good *term order if the leading terms of fundamental binomials in B_G are the elements underlined in Definition 2.3, and the leading terms of cycle binomials are the monomials containing only y-variables.*

We observe that good term orders exist. For instance we can consider any lexicographic term order for which y-variables and t-variables are larger than any z-variable. We will now show that, for any undirected, connected graph G, the set B_G is a Gröbner basis for $I_{\mathcal{C}_G}$ with respect to any good term order. To do so, we require a few lemmas.

Lemma 2.6. *Let $b = \mathbf{m}_1 - \mathbf{m}_2$ be a binomial in $I_{\mathcal{C}_G}$, and assume that no variable divides both $\mathbf{m}_1$ and $\mathbf{m}_2$. If $t_e|\mathbf{m}_1$, for some edge $e = ij$ of G, then $\mathbf{m}_1$ is divisible by the leading term of a fundamental binomial in B_G with respect to any good term order.*

Proof. Assume $t_e|\mathbf{m}_1$ and recall that $\varphi(t_e) = w_iw_jw_e^{-1}$. Since $b \in I_{\mathcal{C}_G}$, we have that $\varphi(\mathbf{m}_1) = \varphi(\mathbf{m}_2)$. In particular, the variable w_e either appears in the Laurent monomial $\varphi(\mathbf{m}_2)$ with a negative exponent, or it appears in $\varphi(\mathbf{m}_1/t_e)$ with positive exponent. The first case contradicts the fact that b is not the multiple of a variable: indeed, since t_e is the only variable for which the variable w_e appears in $\varphi(t_e)$ with a negative exponent, this would imply that $t_e|\mathbf{m}_2$.
In the second case we have that one of the variables v for which w_e appears in $\varphi(v)$ with a positive exponent must divide $\mathbf{m}_1$. These are either z_e, y_{jie} or y_{ije}. This

concludes the proof, since the monomials $t_e z_e$, $y_{jie} t_e$ and $y_{ije} t_e$ are leading terms of some binomial in B_G with respect to the chosen term order. □

We now associate to any binomial $\mathbf{m}_1 - \mathbf{m}_2$ a pair of directed graphs $(\overrightarrow{G_1}, \overrightarrow{G_2})$ of G in the following way: For any variable y_{ije} which divides $\mathbf{m}_1$ (respectively $\mathbf{m}_2$) the graph $\overrightarrow{G_1}$ (respectively $\overrightarrow{G_2}$) contains the vertices i and j and a number of directed edges from i to j equal to the degree of y_{ije} in $\mathbf{m}_1$ (respectively $\mathbf{m}_2$).

Definition 2.7. *For a directed graph $\overrightarrow{G}$ and a vertex $i \in V(\overrightarrow{G})$ we define $\deg_{\overrightarrow{G}}(i) = outdeg_{\overrightarrow{G}}(i) - indeg_{\overrightarrow{G}}(i)$, where $outdeg_{\overrightarrow{G}}(i) = |\{j \in V(\overrightarrow{G}) : (i,j) \in E(\overrightarrow{G})\}|$ and $indeg_{\overrightarrow{G}}(i) = |\{j \in V(\overrightarrow{G}) : (j,i) \in E(\overrightarrow{G})\}|$. If $\deg_{\overrightarrow{G}}(i) > 0$, we call i is* positive *vertex of $\overrightarrow{G}$. If $\deg_{\overrightarrow{G}}(i) < 0$, we call i a* negative *vertex of $\overrightarrow{G}$.*

Lemma 2.8. *Let $b = \mathbf{m}_1 - \mathbf{m}_2$ be a binomial in $I_{\mathcal{C}_G}$, and assume that no variable divides both $\mathbf{m}_1$ and $\mathbf{m}_2$. Let $(\overrightarrow{G_1}, \overrightarrow{G_2})$ be the associated pair of directed graphs. Assume that no leading term of a fundamental binomial in B_G with respect to a good term order divides $\mathbf{m}_1$ or $\mathbf{m}_2$. Then:*

1. *If $\deg_{\overrightarrow{G_1}}(i) < 0$, then $i \in V(\overrightarrow{G_1}) \cap V(\overrightarrow{G_2})$. Moreover, if $i \in V(\overrightarrow{G_1}) \cap V(\overrightarrow{G_2})$, then $\deg_{\overrightarrow{G_1}}(i) = \deg_{\overrightarrow{G_2}}(i)$.*
2. *If $i \in V(\overrightarrow{G_1}) \setminus V(\overrightarrow{G_2})$ and $\deg_{\overrightarrow{G_1}}(i) > 0$ ($i \in V(\overrightarrow{G_2}) \setminus V(\overrightarrow{G_1})$ and $\deg_{\overrightarrow{G_2}}(i) > 0$, respectively), then $z_i | \mathbf{m}_2$ ($z_i | \mathbf{m}_1$, respectively).*
3. *If $e \in E(\overrightarrow{G_1}) \setminus E(\overrightarrow{G_2})$ ($e \in E(\overrightarrow{G_2}) \setminus E(\overrightarrow{G_1})$ respectively), then $z_e | \mathbf{m}_2$ ($z_e | \mathbf{m}_1$ respectively).*

Proof. (1) If $\deg_{\overrightarrow{G_1}}(i) < 0$, then the degree of w_i in $\varphi(\mathbf{m}_1)$ is negative. Since $b \in I_{\mathcal{C}_G}$, the degree of w_i in $\varphi(\mathbf{m}_2)$ is also negative. As the only variables v such that w_i has negative exponent in $\varphi(v)$ are of the form y_{jie} for some vertex j and edge e, the claim follows. Since $i \in V(\overrightarrow{G_1})$ there is at least one edge incident to i in $\overrightarrow{G_1}$. Since the degree of w_i in $\varphi(\mathbf{m}_1)$ is negative, we have then that $\mathbf{m}_1$ is divisible by a variable of the form y_{jie}, for some vertex j and edge e. In particular, we conclude that z_i does not divide $\mathbf{m}_1$ as we assumed that $y_{jie} z_i$ does not divide $\mathbf{m}_1$. By symmetry z_i does not divide $\mathbf{m}_2$. It follows that $\deg_{\overrightarrow{G_1}}(i)$ equals the degree of the variable w_i in $\varphi(\mathbf{m}_1)$ and that $\deg_{\overrightarrow{G_2}}(i)$ equals the degree of the variable w_i in $\varphi(\mathbf{m}_2)$. Since $b \in I_{\mathcal{C}_G}$, we conclude that $\deg_{\overrightarrow{G_1}}(i) = \deg_{\overrightarrow{G_2}}(i)$.

(2) As in the previous case, the number $\deg_{\overrightarrow{G_1}}(i)$ is smaller or equal to the degrees of the variable w_i in $\varphi(\mathbf{m}_1)$ and $\varphi(\mathbf{m}_2)$. Since $i \notin V(\overrightarrow{G_2})$, the only variable in $\mathbf{m}_2$ which contributes to a positive degree in $\varphi(\mathbf{m}_2)$ is z_i.

(3) Again, since $b \in I_{\mathcal{C}_G}$, the degrees of w_e in $\varphi(\mathbf{m}_1)$ and $\varphi(\mathbf{m}_2)$ coincide. By Lemma 2.6 the variable t_e does not divide neither $\mathbf{m}_1$ nor $\mathbf{m}_2$, and this is the only variable v such that w_e has a negative degree in $\varphi(v)$. Hence w_e has positive degree in both $\varphi(\mathbf{m}_1)$ and $\varphi(\mathbf{m}_2)$. Since $e \notin E(\overrightarrow{G_2})$, the only variable which contributes to a positive degree of w_e in $\varphi(\mathbf{m}_2)$ is z_e. □

The following lemma collects some simple properties of directed acyclic graphs that will be of use.

Lemma 2.9. *Let H be a directed acyclic graph, with at least one edge and no isolated vertices. Then H has at least a positive and a negative vertex. Moreover, for every positive vertex $i \in V(H)$ there exists a negative vertex $j \in V(H)$ such that H contains* *a directed path from i to j, and for every negative vertex $j \in V(H)$ there exists a positive vertex $i \in V(H)$ such that H contains a directed path from i to j.*

Proof. Every directed acyclic graph with at least one edge has at least one sink and at least one source node. Since sinks are positive vertices and sources are negative vertices the first claim holds. The second claim follows from the fact that every vertex in the directed acyclic graph has at least one descendant that is a sink and every vertex has at least one source node as an ancestor. □

We are now ready to prove the main result of the section.

THEOREM 2.10. *The set B_G is a Gröbner basis of $I_{\mathcal{C}_G}$ with respect to every good term order.*

Proof. Recall that toric ideals can be generated by binomials which are the difference of two monomials [10, Theorem 3.2]. In order to show that B_G is a Gröbner basis of $I_{\mathcal{C}_G}$ it is then sufficient to show that for any binomial $b = \mathbf{m}_1 - \mathbf{m}_2$ be in $I_{\mathcal{C}_G}$ there exists a binomial $f \in B_G$ such that $\mathrm{lt}(f)|\mathbf{m}_1$ or $\mathrm{lt}(f)|\mathbf{m}_2$. This shows that any binomial in $I_{\mathcal{C}_G}$ can be reduced by an element of B_G. Since this reduction step produces another binomial and the sequence of reduction w.r.t. a term order terminates, it must terminate with the zero polynomial. In particular, all S-polynomials obtained from a generating set of binomials of $I_{\mathcal{C}_G}$ reduce to zero, which implies that B_G is a Gröbner basis. Since toric ideals are prime we can assume that no variable divides both $\mathbf{m}_1$ and $\mathbf{m}_2$. If the leading term of a fundamental binomial in B_G divides either $\mathbf{m}_1$ or $\mathbf{m}_2$, then we conclude.

Assume that no leading term of a fundamental binomial in B_G divides either $\mathbf{m}_1$ or $\mathbf{m}_2$. In particular, by Lemma 2.6, no variable of the form t_e divides either $\mathbf{m}_1$ or $\mathbf{m}_2$. Consider the pair $(\overrightarrow{G_1}, \overrightarrow{G_2})$ of directed subgraphs of G associated with $\mathbf{m}_1$ and $\mathbf{m}_2$.

If $\overrightarrow{G_1}$ ($\overrightarrow{G_2}$ respectively) has a directed cycle C, which is supported on a cycle of G, then by construction $\mathbf{m}_1$ ($\mathbf{m}_2$ respectively) is divisible by the monomial $\prod_{\vec{e}=(i,j)\in E(C)} y_{ije}$ which is the leading term of a cycle binomial by definition of good term order and so we conclude.
If $\overrightarrow{G_1}$ ($\overrightarrow{G_2}$ respectively) has a directed cycle C which is not supported on a cycle of G, then it must be a cycle on 2 vertices i and j. In this case the monomial $y_{ije}y_{jie}$, which is the leading term of a fundamental binomial, divides $\mathbf{m}_1$ ($\mathbf{m}_2$ respectively).

Assume that both $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ are directed acyclic. Since no variable divides both $\mathbf{m}_1$ and $\mathbf{m}_2$, $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ do not have any common directed edge, as those would correspond to y-variables which divide both $\mathbf{m}_1$ and $\mathbf{m}_2$.

Suppose there is a positive vertex i in $\overrightarrow{G_1}$ such that $i \in V(\overrightarrow{G_1}) \setminus V(\overrightarrow{G_2})$. Observe that by Lemma 2.8 (2) this implies that $z_i|\mathbf{m}_2$. We let $i_1 = i$ and j_1 be a negative vertex of $\overrightarrow{G_1}$ such that there is a directed path from i_1 to j_1. By Lemma 2.8 (1), j_1 is a negative vertex of $\overrightarrow{G_2}$ as well. By Lemma 2.9, there exists a positive vertex i_2 of $\overrightarrow{G_2}$ such that there is a directed path in $\overrightarrow{G_2}$ from i_2 to j_1. If $i_2 \in V(\overrightarrow{G_1})$, by Lemma 2.8 (1), we have $\deg_{\overrightarrow{G_1}}(i_2) = \deg_{\overrightarrow{G_2}}(i_2) > 0$. Otherwise we can find a path from i_2 to a negative vertex j_2, and we can keep looking for negative then iterate this procedure until one of the following possibilities occurs:

Case 1: $i_k \notin V(\overrightarrow{G_1})$. In this case, let E_1 be the union of the directed edges of the directed paths from i_t to j_t, for $t = 1, \dots, k-1$ and E_2 be the union of the directed edges of the directed paths from i_{t+1} to j_t, for $t = 1, \dots, k-1$. Hence (E_1, E_2) is a zig-zag pair. By definition of the graphs $(\overrightarrow{G_1}, \overrightarrow{G_2})$ we have that $\prod_{\vec{e}=(i,j)\in E_1} y_{ije}$ divides $\mathbf{m}_1$

and $\prod_{\vec{e}=(i,j)\in E_2} y_{ije}$ divides $\mathbf{m}_2$. Moreover, by Lemma 2.8 (2), we have that $z_{i_k}|\mathbf{m}_1$ and that $z_{i_1}|\mathbf{m}_2$. Finally, by Lemma 2.8 (3), $\prod_{\vec{e}=(i,j)\in E_2} z_e$ divides $\mathbf{m}_1$ and $\prod_{\vec{e}=(i,j)\in E_1} z_e$ divides $\mathbf{m}_2$. In particular, $\mathbf{m}_1$ and $\mathbf{m}_2$ are divisible by the two monomials of b_{E_1,E_2}, the binomial corresponding to the zig-zag pair (E_1, E_2).

Case 2: $i_k = i_\ell$, for some $\ell < k$. In this case, let E_1 be the union of the directed edges of the directed paths from i_t to j_t, for $t = \ell, \dots, k-1$ and E_2 be the union of the directed edges of the directed paths from i_{t+1} to j_t, for $t = \ell, \dots, k-1$ together with the directed edges from i_k to j_ℓ. The pair (E_1, E_2) is a cyclic pair. Again by definition of the graphs $(\overrightarrow{G_1}, \overrightarrow{G_2})$, we have that $\prod_{\vec{e}=(i,j)\in E_1} y_{ije}$ divides $\mathbf{m}_1$ and $\prod_{\vec{e}=(i,j)\in E_2} y_{ije}$ divides $\mathbf{m}_2$. Moreover, by Lemma 2.8 (3), $\prod_{\vec{e}=(i,j)\in E_2} z_e$ divides $\mathbf{m}_1$ and $\prod_{\vec{e}=(i,j)\in E_1} z_e$ divides $\mathbf{m}_2$. In particular, $\mathbf{m}_1$ and $\mathbf{m}_1$ are divisible by the two monomials of b_{E_1,E_2}, the binomial corresponding to the cyclic pair (E_1, E_2). This finishes Case 2.

If there is a positive vertex i in $\overrightarrow{G_2}$ such that $i \in V(\overrightarrow{G_2}) \setminus V(\overrightarrow{G_1})$, we can conclude by the same argument as above.

Suppose now that for all vertices i with $\deg_{\overrightarrow{G_1}}(i) > 0$ we have that $i \in V(\overrightarrow{G_2})$ and for all vertices i with $\deg_{\overrightarrow{G_2}}(i) > 0$ we have that $i \in V(\overrightarrow{G_1})$. We initialize i_1 to be any of the vertices with $\deg_{\overrightarrow{G_1}}(i) > 0$ and, as in the previous case, we start constructing disjoint directed paths from i_t to j_t in $\overrightarrow{G_1}$ and from i_{t+1} to j_t in $\overrightarrow{G_2}$. Since the graphs $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ are finite there exists k such that $i_k = i_\ell$ for some $\ell < k$. Let E_1 be the union of the directed edges of the directed paths from i_t to j_t, for $t = \ell, \dots, k-1$ and E_2 be the union of the directed edges of the directed paths from i_{t+1} to j_t, for $t = \ell, \dots, k-1$ together with the directed edges from i_k to j_ℓ. The pair (E_1, E_2) is a cyclic pair. Following verbatim Case 2 we obtain that $\mathbf{m}_1$ and $\mathbf{m}_2$ are divisible by the two monomials of b_{E_1,E_2}, the binomial corresponding to the cyclic pair (E_1, E_2). This completes the proof. □

2.1. Regular unimodular triangulations. Given a sufficiently generic vector $h \in \mathbb{R}^n$ and a finite collection of lattice points $\mathcal{A} = \{a_1, \dots, a_n\} \in \mathbb{Z}^d$, the vector h defines a *regular triangulation* $\mathcal{T}_h$ of $\mathcal{A}$ as follows: $\{a_{i_1}, \dots, a_{i_k}\} \in \mathcal{T}_h$ if there exists $c \in \mathbb{R}^d$ such that

$$\begin{aligned} \langle a_j, c\rangle &= h_j \quad \text{if } j \in \{i_1, \dots, i_k\}, \text{ and} \\ \langle a_j, c\rangle &< h_j \quad \text{if } j \notin \{i_1, \dots, i_k\} \end{aligned}$$

The triangulation $\mathcal{T}_h$ is called *unimodular* if the maximal dimensional faces (i.e. the *facets*) of $\mathcal{T}_h$ have smallest possible volume over all lattice simplices in the lattice spanned by $\mathcal{A}$. (When this lattice is $\mathbb{Z}^d$ this volume is $1/d!$.)

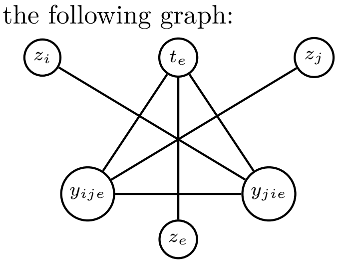
The *initial ideal* of a toric ideal $I_A \subset k[w_1, \dots, w_d]$ with respect to the term order $\prec$ is the ideal generated by the leading terms of all $f \in I_A$ with respect to $\prec$. By definition, the leading terms of the elements of a Gröbner basis of I_A with respect to $\prec$ forms a finite generating set for this initial ideal. In the ring R_G, the variables correspond to the lattice points in $\mathcal{C}_G \cap \mathbb{Z}^{|V|+|E|}$, and hence each element of this generating set indexes a subset of the lattice points in $\mathcal{C}_G$. When the leading terms of the polynomials in the Gröbner basis are all squarefree then these subsets form the set of minimal nonfaces of a regular unimodular triangulation of $\mathcal{C}_G$ (see [19, Theorem 8.3] or [7, Section 9.4]). Since by Theorem 2.10 the set B_G is a Gröbner basis of $I_{\mathcal{C}_G}$ with respect to any good term order with only squarefree leading terms, we obtain the following corollary.

Corollary 2.11. *Let G be any graph. The cosmological polytope $\mathcal{C}_G$ has a regular unimodular triangulation.*

Corollary 2.11 provides the existence of the desired subdivisions of $\mathcal{C}_G$ for any G. While the result is constructive, the presentation of the resulting triangulations is in the form of their minimal non-faces. In order to apply the formula in (1) to compute the canonical forms $\Omega_{\mathcal{C}_G}$, we require a description of the triangulations in terms of their facets. In the coming sections, we give such characterizations for families of G. To derive these results we will use some observations that can be seen to hold for all regular unimodular triangulations derived from good term orders for any graph G. In this subsection, we collect these results and the relevant notation that will be used throughout the remaining sections.

We start by introducing some notation. In the following, let us assume that we have a graph $G = (V, E)$ and a good term order. By Theorem 2.10 a Gröbner basis with squarefree initial ideal is given by the fundamental binomials, the zig-zag binomials and the cyclic binomials. Since the cosmological polytope $\mathcal{C}_G$ has dimension $|V| + |E| - 1$, the corresponding regular unimodular triangulation has facets given by all $(|V| + |E|)$-subsets of the variables $y_{ije}, y_{jie}, t_e, z_e, z_i$ that do not contain any leading term of the binomials in this Gröbner basis. Recall that every variable corresponds to a distinguished lattice point in $\mathcal{C}_G$ and as such to a vertex of the considered facet.

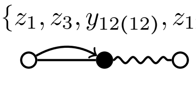
The fundamental binomials imply that certain 2-subsets of variables cannot be contained in the facets. These 2-subsets to be avoided for each edge $e = ij \in E$ correspond to the edges of the following graph:



To represent the facets of the triangulation, we introduce a symbol corresponding to each variable: Let $i \in V$ and $e = ij \in E$:

- the variable z_i is represented by the symbol $\circ$. The vertex i is instead represented by $\bullet$ if z_i is not present.
- the variable z_e is represented by the edge type $-$,
- the variable t_e is represented by the edge type $\sim\!\sim$,
- the variable y_{ije} is represented by the edge type $\to$ pointing from i to j, and
- the variable y_{jie} is represented by the edge type $\leftarrow$ pointing from j to i.

Given a subset S of the generators of R_G, we let G_S denote the graph drawn with the symbols above according to the elements in S. We also let $Z := \{z_i \ : \ i \in V\}$, $Z_S := S \cap Z$ and $\mathfrak{Z}_S := \{i \in V \ : \ z_i \in Z_S\}$. For example, if G is a path on 3 vertices, we represent the set of variables $S = \{z_1, z_3, y_{12(12)}, z_{12}, t_{23}\}$ via the graph G_S:



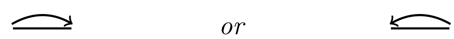
For this example, $Z = \{z_1, z_2, z_3\}$, $Z_S = \{z_1, z_3\}$ and $\mathfrak{Z}_S = \{1, 3\}$.

The fundamental binomials imply that if a subset of variables corresponds to a face of the triangulation, then its associated graph does not contain any of the following subgraphs, where the corresponding leading monomial (for the edge $e = ij$) is written below each subgraph:

(3)


We refer to these six subgraph as *fundamental obstructions*. The following lemma is immediate.

Lemma 2.12. *Let G be a simple, connected and undirected graph, and let $\mathcal{T}$ be a regular unimodular triangulation of $\mathcal{C}_G$ given by a good term order on R_G. If S is a facet of $\mathcal{T}$, then G_S contains only single edges and double edges. Moreover, any double edges are of the form, where the corresponding monomial is indicated for the edge $e = ij$ below:*



$y_{ije}z_e$ *or* $y_{jie}z_e$.

Given a subset S of the variables in R_G, we define the *support graph* of S (or G_S) to be the graph on vertex set V and edge set

$$\{e = ij \in E \ : \ S \cap \{t_e, z_e, y_{ije}, y_{jie}\} \neq \varnothing\}.$$

The following statement shows that the support graph of any facet is as large as possible.

Proposition 2.13. *Let $G = (V, E)$ be a connected, undirected graph and let $\mathcal{T}$ be a triangulation of $\mathcal{C}_G$ coming from a good term order. Let S be a subset of the variables of R_G. If S is a facet of $\mathcal{T}$, then the support graph G_S of S equals G. In particular, the support graph of S is connected.*

Proof. Let S be a facet of $\mathcal{T}$ and assume by contradiction that there is some edge $e = ij \in E$ such that $S \cap \{t_e, z_e, y_{ije}, y_{jie}\} = \varnothing$. We note that the variable t_e does not appear in any of the zig-zag binomials, the cyclic binomials nor the cyclic binomials. The only occurrence of t_e is in the leading term of the fundamental binomials $y_{ije}t_e - z_i^2$, $y_{jie}t_e - z_j^2$ and $t_e z_e - z_i z_j$. However, since by assumption, none of y_{ije}, y_{jie} and z_e is contained in S, it follows that $S \cup \{t_e\}$ is also a face of $\mathcal{T}$. This contradicts the fact that S is a facet. ☐

3. The cosmological polytope of the path

In this section, we give an explicit description of the regular unimodular triangulation corresponding to a Gröbner basis with respect to a good term order of the toric ideal for the cosmological polytope of the path with n edges I_n; that is, the graph with vertex set $V = [n+1]$ and edge set $E = \{ii+1 \ : \ i \in [n]\}$.

A combinatorial description of the facets of this triangulation is given that allows for enumeration of the facets. The resulting formula for the normalized volume of $\mathcal{C}_{I_n}$ agrees with the formula identified in [13]. The combinatorial description of the facets may also be used to compute the canonical form of the polytope in a novel way, which may suggest new physical theories for the computation of wavefunctions associated to such Feynman diagrams.

In the following, we use the variable order

$$\begin{aligned} &y_{12} > y_{23} > \cdots > y_{nn+1} > y_{n+1n} > \cdots > y_{32} > y_{21} > z_{12} > \cdots \\ &\cdots > z_{nn+1} > t_{12} > \cdots > t_{nn+1} > z_1 > \cdots > z_{n+1}, \end{aligned} \tag{4}$$

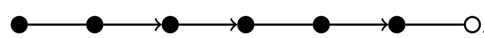
where for the edge $e = ii+1$, we write y_{ii+1} and y_{i+1i} for the variables y_{ii+1e} and y_{i+1ie}, respectively. It can be checked that the lexicographic term order, with respect to this ordering of the variables, on the monomials in R_{I_n} is a good term order according to Definition 2.5. Since the cosmological polytope $\mathcal{C}_G$ for a graph $G = (V, E)$ has dimension $|V| + |E| - 1$, the corresponding regular unimodular triangulation has facets given by all $(2n+1)$-subsets of the variables $y_{ije}, y_{jie}, t_e, z_e, z_i$ that do not contain the leading terms of the binomials in this Gröbner basis. Our goal is to characterize these subsets S in terms of the structure of their graphs G_S defined in Subsection 2.1.

By Proposition 2.13, we know that G_S is connected whenever S is a facet. We also know from Lemma 2.12 that all edges in G_S are either single or double edges and all double edges are of the form

 or 

Since the path graph contains no cycles, the Gröbner basis given in Theorem 2.10 for I_{I_n} contains only the fundamental binomials and zig-zag binomials. Now that we have specified a specific good term order on R_{I_n} we can also identify the subgraphs forbidden by the leading monomials of the zig-zag binomials.

Namely, if S is a facet of the triangulation then G_S does not contain any *partially directed paths to the right ending in a* $\circ$; that is, it does not contain any consecutive edges, undirected or directed to the right, where the last edge is undirected and ends in a $\circ$. For example, if S is a facet it cannot contain the subset of symbols R yielding the following graph:

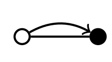
.

The following lemma collects some additional useful properties of G_S when S is a facet.

Lemma 3.1. *Let S be a subset of the variables generating the ring R_{I_n}. If S is a facet of the triangulation and $\mathfrak{Z}_S = \{i_1 < i_2 < \cdots < i_{n+1-k}\}$, then the induced subgraphs on $[i_1]$, $[i_j, i_{j+1}]$ for $1 \le j \le n-k$ and $[i_{n+1-k}, n+1]$ have exactly as many double edges as black nodes.*

Proof. Since S is a facet we know that $|S| = 2n+1$. We know also from Proposition 2.13 that the support graph of G_S is connected and equal to I_n. Hence, there is at least one edge in G_S for all n edges in I_n. Moreover, by Lemma 2.12 we know that G_S only contains single and double edges. Since $|Z_S| = n+1-k$ and $|S| = 2n+1$, it follows that G_S contains exactly k double edges.

Consider now the subgraph between $i < j$ of G_S where $z_i, z_j \in S$ and $z_\ell \notin S$ for all $i < \ell < j$. We claim that there are at most $j - i - 1$ double edges in this subgraph. To see this, suppose there are $j - i$ double edges instead. It follows that all edges in this subgraph are double and of the form specified in Lemma 2.12. Since the subgraph cannot include the fundamental obstruction $\circ \leftarrow$, it follows that the first pair of double edges is of the form



Since all remaining edges in the subgraph must also be double edges, and since these sets of doubles must each include the undirected edge $\ell\ell+1$ (for $i \le \ell \le j-1$), it follows that the subgraph contains a partially directed path to the right ending in $\circ$, which is a forbidden subgraph by the leading term of some zig-zag binomial. Hence, we have a contradiction.

It then follows from the Pigeonhole Principle that each subgraph of G_S given by a pair of nodes $i < j$ for $z_i, z_j \in S$ but $z_\ell \notin S$ for all $i < \ell < k$, or $z_i \in S$ but $z_\ell \notin S$ for all $\ell < i$, or $z_i \in S$ but $z_\ell \notin S$ for all $i < \ell$ contains exactly as many double edges as it does black nodes. This finishes the proof. □

Based on Lemma 3.1, it can be helpful to consider facets according to their intersection with the set $Z = \{z_i \ : \ i \in [n+1]\}$. If a facet S is such that $Z_S = S \cap Z =$

$\{z_{i_1}, \dots, z_{i_k}\}$, where $i_1 < i_2 < \cdots < i_k$, we can partition the graph G_S into the induced subgraphs on node sets $\{1, \dots, i_1\}$, $\{i_k, \dots, n+1\}$ and $\{i_j, i_j+1, \dots, i_{j+1}\}$ for all $j \in [k-1]$, and consider the possible placements of the appropriate number of edges in each induced subgraph so as to ensure that $|S| = 2n+1$. A rule for producing all such graphs in this way will yield a combinatorial description of the facets of the triangulation. The next theorem gives such a characterization of the graphs that correspond to facets of the triangulation. In the following we use $\leftrightarrow$ to denote that we are free to choose between either arrow (either $\leftarrow$ or $\rightarrow$).

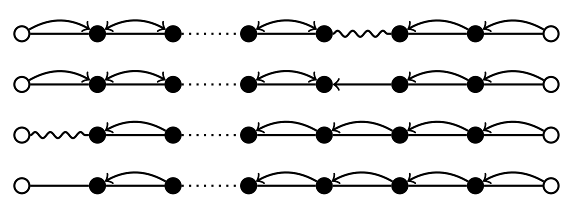
Theorem 3.2. *Let S be a subset of the generators of the ring R_{I_n} and let $Z_S = \{z_{i_1}, \dots, z_{i_k}\}$ where $i_1 < \cdots < i_k$ (and possibly, $Z_S = \varnothing$. Then S is a facet of the triangulation of $\mathcal{C}_{I_n}$ corresponding to the lexicographic order induced by* (4) *if and only if $Z_S \neq \varnothing$ and all three of the following hold:*

(1) *The induced subgraph of G_S on nodes $[i_1]$ is of the form*

.

That is, all edges are double with a $\leftarrow$.

(2) *For all $j \in [k-1]$, the induced subgraph of G_S on $\{i_j, i_j+1, \dots, i_{j+1}\}$ is of one of the following forms:*



That is, either (1) exactly one edge whose least vertex is a black node is either $\rightsquigarrow$ or $\leftarrow$, all edges to the right of this edge are double with a $\leftarrow$ and all edges to the left of this edge are double with either arrow ($\leftarrow$ or $\rightarrow$), except for the first edge which must have $\rightarrow$, or (2) the leftmost edge is either $-$ or $\rightsquigarrow$ and all edges to the right are double with a $\leftarrow$.

(3) *The induced subgraph of G_S on nodes $\{i_k, i_k+1, \dots, n+1\}$ is of the form*

.

That is, all edges are double with either arrow (either $\leftarrow$ or $\rightarrow$), except for the first edge which must have a $\rightarrow$.

Proof. We first observe that any set S such that G_S satisfies the listed properties, is a facet. Moreover, if S is a facet, then $|S| = 2n+1$. Since S can only contain double edges and I_n has n edges, it follows that $Z_S \neq \varnothing$.

Notice first that any choice of the edges for each of the possible subgraphs does not contain an induced subgraph excluded by the fundamental binomials. Furthermore, any partially directed path to the right is either interrupted by a single edge of the form $\leftarrow$ or $\rightsquigarrow$, or it terminates in a black node. Hence, such a G_S also does not contain any subgraph forbidden by the leading terms of the zig-zag binomials. Since there is exactly one double edge for every black node, it also follows that $|S| = 2n+1$. Since the dimension of $\mathcal{C}_{I_n}$ is $2n$, it follows that G_S is a facet of the triangulation.

Suppose now that S is a facet of the triangulation, and consider its associated graph G_S. Since S is a facet, we know $|S| = 2n+1$, and by Lemma 3.1 we also know that G_S is connected and any of the induced subgraphs on node sets $\{1, \dots, i_1\}$, $\{i_k, \dots, n+1\}$ and $\{i_j, i_j+1, \dots, i_{j+1}\}$ for $j \in [k-1]$ contains as many black nodes as it does double edges. It therefore suffices to show that these subgraphs of G_S are of one of the possible forms specified in the above list.

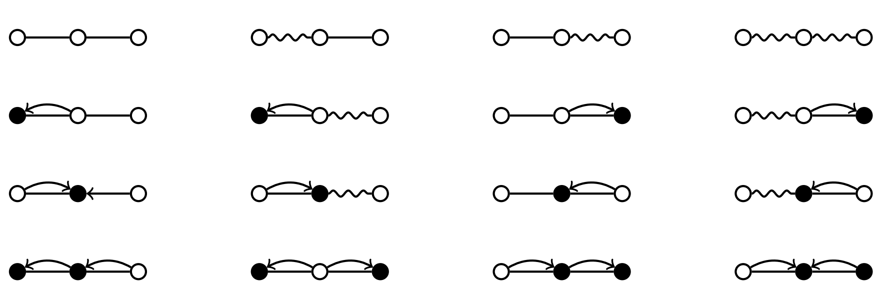
Consider first the induced subgraph of G_S on node set $[i_1]$. Since S is a facet, by Lemma 3.1, we know that every edge in this subgraph is a double edge, and hence of the form $\{(i,i+1), ii+1\}$ or $\{(i+1,i), ii+1\}$. Since S cannot contain the leading term of any fundamental binomial, it does not contain both z_{i_1} and $y_{i_1-1i_1}$. Hence, this subgraph must contain the double edge $\{(i_1, i_1-1), i_1-1i_1\}$. Similarly, since S cannot contain the leading term of any zig-zag binomial, this subgraph cannot contain any partially directed paths to the right. It follows that all double edges in this subgraph are of the form $\{(i+1,i), ii+1\}$. Hence, G_S fulfills the first criterion in the above list.

Similarly, for the induced graph of G_S on node set $\{i_j, i_j+1, \ldots, i_{j+1}\}$, we know that the graph must be connected and contain exactly $i_{j+1} - i_j - 1$ double edges by Lemma 3.1. Hence, there is exactly one single edge in the graph. Suppose that this edge is the leftmost edge (i.e. between i_j and i_j+1). In this case, the edge may be either $\rightsquigarrow$ or $-$, but not $\leftarrow$ or $\rightarrow$. To see that it cannot be $\leftarrow$, note that this would mean that the leading term of a fundamental binomial is contained in S. To see that it cannot be $\rightarrow$, note that, since all remaining edges in the subgraph must be doubled (and hence include a $-$), it would follow that S contains the leading term of a zig-zag binomial, which is a contradiction. In a similar fashion, all double edges must be of the form $\{(i+1,i), ii+1\}$. Otherwise S would contain the leading term of a zig-zag binomial.

Suppose now that the single edge in the subgraph is between $i_j + t$ and $i_j + t + 1$ for some $t > 1$. By the same argument as the previous case, all remaining edges must be double edges and all double edges to the right of $i_j + t + 1$ must be of the form $\{(i+1,i), ii+1\}$. We must also have that the double edge between i_j and i_j+1 is of the form $\{(i_j, i_j+1), i_j i_j+1\}$, since otherwise S would contain the leading term of a fundamental binomial. However, all double edges between $i_j + s$ and $i_j + s + 1$ for $1 \leq s < t$ can be of either form $\{(i+1,i), ii+1\}$ or $\{(i,i+1), ii+1\}$, since the single edge will interrupt any partially directed path to the right. Observe further that the single edge must be of the form $\leftarrow$ or $\rightsquigarrow$, since any other option would combine with the undirected edges and the directed edge between i_j and i_j+1 to yield a partially directed path to the right terminating in a $\circ$. It follows that if S is a facet, the corresponding induced subgraphs of G_S on the intervals $\{i_j, i_j+1, \ldots, i_{j+1}\}$ for all $j \in [k-1]$ are of the form in item (2) in the above list.

Finally, for the induced subgraph of G_S on node set $\{i_k, \ldots, n+1\}$, we know from Lemma 3.1 that all edges are double edges and hence of the form $\{(i+1,i), ii+1\}$ or $\{(i,i+1), ii+1\}$. To avoid a subgraph forbidden by a fundamental binomial, we must also have that the double edge between i_k and i_k+1 is of the form $\{(i,i+1), ii+1\}$. However, since the path does not contain any $\circ$ to the right of node i_k, we are free to choose the direction of the arrow in all remaining double edges. Hence, this subgraph is of the form given in item (3) in the above list, which completes the proof. □

Example 3.3. According to Theorem 3.2, the facets of the triangulation of $\mathcal{C}_{I_2}$ are given by the following sixteen graphs:



Each of these graphs encodes the collection of vertices of the corresponding facet in the triangulation of $\mathcal{C}_{I_2}$, from which we can recover the facet-defining equations of the simplex and thereby compute the canonical form Ω_{I_2}.

Since the triangulation is unimodular, it follows that the normalized volume of $\mathcal{C}_{I_n}$ is given by the sum over all graphs G_S that satisfy the properties listed in Lemma 3.2. Using the decomposition of these properties into subgraphs, we can recover the formula for the normalized volume of $\mathcal{C}_{I_n}$ given in [13].

Corollary 3.4. *The normalized volume of* $\mathcal{C}_{I_n}$ *is* 4^n.

Proof. We first deduce a formula for the normalized volume of $\mathcal{C}_{I_n}$ by enumerating the facets of the triangulation using Lemma 3.2. Then we show that this formula reduces to 4^n.

To enumerate the facets via Lemma 3.2, we first pick a subset $\{i_1, \dots, i_k\}$ of $[n+1]$ where we assume $i_1 < \dots < i_k$. Let S be a facet with $Z_S = \{z_{i_1}, \dots, z_{i_k}\}$. There is only one possible induced subgraph on node set $[i_1]$. The possible number of induced subgraphs of G_S on node set $\{i_k, i_k+1, \dots, n+1\}$ is the following

$$\begin{cases} 1, & \text{if } i_k = n+1, \\ 2^{n-i_k}, & \text{if } i_k < n+1. \end{cases}$$

Given an interval $\{i_j, i_j+1, \dots, i_{j+1}\}$, the possible induced subgraphs on this interval of G_S must contain a single edge between $i_j + s$ and $i_j + s + 1$ for exactly one $s \in \{0, 1, \dots, i_{j+1} - i_j - 1\}$. By Lemma 3.2 we always have two choices for this edge. Further, by the same lemma, when $s = 0$, there are exactly two possible subgraphs of G_S on this interval. For $s > 0$, there are 2^s choices (including the choice of edge type for the single edge). Hence, there are a total of

$$\begin{aligned} 2 + \sum_{s=1}^{i_{j+1}-i_j-1} 2^s &= 1 + \sum_{s=0}^{i_{j+1}-i_j-1} 2^s, \\ &= 1 + (2^{i_{j+1}-i_j} - 1), \\ &= 2^{i_{j+1}-i_j} \end{aligned}$$

possible subgraphs for this interval. The number of facets S of the triangulation with $Z_S = \{i_1, \dots, i_k\}$ is then equal to

$$\begin{cases} 1 \cdot \left(\prod_{j=1}^{k-1} 2^{i_{j+1}-i_j}\right) \cdot 1, & \text{if } i_k = n+1, \\ 1 \cdot \left(\prod_{j=1}^{k-1} 2^{i_{j+1}-i_j}\right) \cdot 2^{n-i_k}, & \text{if } i_k < n+1 \end{cases} = \begin{cases} 2^{i_k - i_1}, & \text{if } i_k = n+1, \\ 2^{n-i_1}, & \text{if } i_k < n+1. \end{cases}$$

Summing over all proper subsets of $[n+1]$, yields

$$\begin{aligned} \sum_{\varnothing \neq Z \in 2^{[n]}} 2^{n-\min(Z)} + \sum_{Z \in 2^{[n]}} 2^{n+1-\min(Z)} &= \sum_{\ell=1}^{n} 2^{n-\ell} \cdot 2^{n-\ell} + \sum_{\ell=1}^{n} 2^{n-\ell} \cdot 2^{n+1-\ell} + 1 \\ &= \sum_{\ell=0}^{n-1} 4^\ell + 2 \sum_{\ell=0}^{n-1} 4^\ell + 1 \\ &= 3 \sum_{\ell=0}^{n-1} 4^\ell + 1 \\ &= 3 \frac{4^n - 1}{4 - 1} + 1 = 4^n, \end{aligned}$$

which completes the proof. ☐

4. The cosmological polytope of the cycle

We now consider the cosmological polytope $\mathcal{C}_{C_n}$ associated with the n-cycle C_n, i.e. the graph with vertex set $V = [n]$ and edge set $E = \{ii+1 \ : \ i \in [n]\}$, where $i+1$ is considered modulo n. Via a mild extension of the observations made in Section 3, we can characterize the facets of a regular unimodular triangulation of $\mathcal{C}_{C_n}$ arising from a good term order. This yields a method for computing the canonical form Ω_{C_n}. Furthermore, we can enumerate these facets, yielding a closed formula for the normalized volume of $\mathcal{C}_{C_n}$, which was previously unknown.

We use the notation introduced in Section 3. In particular, we represent sets of variables S in R_{C_n} with the graphs G_S. For the edge $e = ii+1$ we also write y_{ii+1} and y_{i+1i} for the corresponding y-variables. Just as in Section 3, we consider the triangulation $\mathcal{T}$ of $\mathcal{C}_{C_n}$ induced by a lexicographic term order with respect to the following ordering of the variables

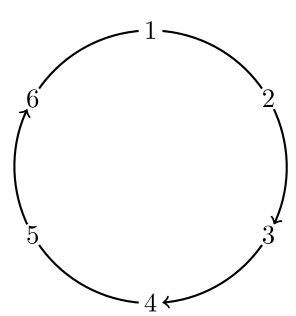
$$\begin{aligned}(5) \qquad & y_{12} > y_{23} > \cdots y_{n-1n} > y_{n1} > y_{1n} > y_{nn-1} > \cdots > y_{21} > z_{12} > \cdots \\ & \cdots > z_{n-1n} > t_{12} > \cdots > t_{n-1n} > z_1 > \cdots > z_n.\end{aligned}$$

This term order is seen to be a good term order (see Definition 2.5).

With respect to such a term order, the leading terms of zig-zag binomials correspond again to partially directed paths ending in a $\circ$, just as in Section 3. Note that this is indeed true even more general for facets of the cosmological polytope of any graph for any induced path (whose internal vertices have degree 2) with respect to any good term order for which variables corresponding to one direction of the path are greater than the ones for the other direction.

We must now also avoid the subgraphs corresponding to leading terms of cyclic binomials. For cycle binomials, this implies that we must avoid subgraphs that are directed cycles (both clockwise and counter-clockwise), such as

Here, we note that no vertices are chosen. For cyclic binomials that are not cycle binomials, since $y_{ii+1} > y_{j+1j}$ for every $i, j \in [n]$ (with addition taken modulo n), the leading terms will always correspond to partially directed cycles oriented clockwise, such as



Hence, we must avoid subgraphs that are partially directed cycles with a clockwise orientation. The following theorem provides a characterization of the facets of this triangulation in terms of these forbidden subgraphs.

Theorem 4.1. *Let S be a subset of the generators of the ring R_{C_n} and let $Z_S = \{z_{i_1}, \ldots, z_{i_k}\}$ where $i_1 < \cdots < i_k$. Then S is a facet of the triangulation of $\mathcal{C}_{C_n}$ corresponding to the lexicographic order induced by* (5) *if and only if all of the following hold:*

1. $Z_S \neq \varnothing$,
2. *the induced subgraph of G_S on $\{i_t, i_t+1, \ldots, i_{t+1}\}$ is of the form described in Theorem* 3.2 *(2) for all $t \in [k-1]$, and*
3. *the induced subgraph of G_S on $\{i_k, i_k+1 \mod n, \ldots, i_1\}$ is of the form described in Theorem* 3.2 *(2) where the right-most node is i_1.*

Proof. Suppose that S is a facet of $\mathcal{T}$. Then $|S| = 2n$. If $Z_S = \varnothing$, then by the forbidden subgraphs arising from the fundamental binomials, we know that every edge in G_S is a double edge consisting of an undirected edge together with a directed edge. This, however, would imply that G_S contains a subgraph corresponding to the leading term of a cyclic binomial, which is a contradiction. Hence, $Z_S \neq \varnothing$. The fact that conditions (2) and (3) hold follows from the specified variable ordering and the arguments given in the proof of Theorem 3.2.

Similarly, the converse follows from the arguments given in the proof of Theorem 3.2, with the additional observation that the specified paths between any two white nodes contains a single edge and these single edges prevent the existence of clockwise partially directed and directed cycles. □

Similar to the results in Section 3, we can use the characterization in Theorem 4.1 to enumerate the facets of the triangulation and derive a closed formula for the normalized volume of $\mathcal{C}_{C_n}$. The resulting formula appears as sequence A020522 in the Online Encyclopedia of Integers Sequences [18].

Theorem 4.2. *The cosmological polytope of the n-cycle C_n has normalized volume*

$$\mathrm{Vol}(\mathcal{C}_{C_n}) = 4^n - 2^n.$$

Proof. Let S be a facet of the triangulation $\mathcal{T}$ described above and $Z_S = \{z_{i_1}, \ldots, z_{i_k}\}$ where $i_1 < \cdots < i_k$. Then the induced subgraph of G_S on $\{i_\ell, \ldots, i_{\ell+1 \mod k}\}$ for $1 \leq \ell \leq k$ is of the form described in Theorem 3.2 (2). By the proof of Corollary 3.4 there are $2^{i_{\ell+1 \mod k} - i_\ell}$ possible subgraphs for this interval which gives $\prod_{\ell=1}^{k} 2^{i_{\ell+1 \mod k} - i_\ell} = 2^n$ possible graphs G_S with a prescribed set of white vertices. Varying the latter over all non-empty subsets of the vertices, we get a total of $(2^n-1)\dot{2}^n = 4^n - 2^n$ possible subgraphs G_S. It remains to verify that none of these subgraphs contains a clockwise partially oriented cycles or a completely oriented cycle. To see this, it suffices to note that if any induced subgraph on $\{i_\ell, \ldots, i_{\ell+1 \mod k}\}$ for $1 \leq \ell \leq k$ is of the first three types in Theorem 3.2 (2), then the unique single edge already prevents the existence of such a cycle. However, if all considered subgraphs are of the fourth type then none of the variables $y_{ii+1 \mod n}$ is present. Hence, G_S neither contains a clockwise partially oriented cycle nor a completely oriented cycle. This finishes the proof. □

5. The cosmological polytope of a tree

The description of the facets for a regular unimodular triangulation arising from a good term order for the path in Section 3 can be extended to any tree. To do so, we first specify a good term order associated to an arbitrary tree T on node set $[n+1]$ that generalizes the term order used in Section 3.

Fix a leaf node r of T and consider a planar embedding of T in which r is located at the origin and the vertices at distance d from r in T are located at (d, y) for some $y \in \mathbb{Z}$. We let $<_r$ denote the total order on the vertices of T given by reading the vertices from left-to-right, top-to-bottom, with the first vertex you read being the smallest in the order. A total ordering $\prec$ on the edges of T is analogously given by reading the edges from left-to-right, top-to-bottom.

Using these orderings, we can then define a total order $<$ of the variables y_{ij}, z_{ij}, t_{ij} and z_i such that

(1) If $(i,j),(s,t) \in E(\overrightarrow{T})$ and $(i,j) \prec (s,t)$, then
- $y_{ij} > y_{st}$,
- $y_{ts} > y_{ji}$,
- $z_{ij} > z_{st}$, and
- $t_{ij} > t_{st}$,

(2) If $(i,j),(s,t) \in E(\overrightarrow{T})$, then
- $y_{ij} > y_{ts}$,
- $y_{ij} > z_{st}$,
- $y_{ji} > z_{st}$,
- $y_{ij} > t_{st}$,
- $y_{ji} > t_{st}$,
- $y_{ij} > z_s$,
- $y_{ji} > z_s$,
- $z_{ij} > t_{st}$,
- $z_{ij} > z_s$,
- $t_{ij} > z_s$, and

(3) If $i <_r j$, then $z_i > z_j$.

This variable ordering is seen to generalize the variable ordering (4), and the associated lexicographic term order on the monomials in R_T is a good term order. Hence, by Corollary 2.11, we obtain a regular unimodular triangulation $\mathcal{T}$ of $\mathcal{C}_T$.

Note that the total ordering $<_r$ induces a directed version of T in which an edge between two adjacent nodes i, j in T is oriented as $i \to j$ if and only if $i <_r j$. A *floret* in this directed tree consists of a node i and all of its *children*, i.e. the nodes j such that $i \to j$ is an edge of the tree. Using this definition, we note the following property of the chosen edge ordering of T.

Lemma 5.1. *Suppose that $i <_r j$, and let $\pi = \{i_1 i_2, \ldots, i_{k-1} i_k\}$ be the unique path in T between $i_1 = i$ and $i_k = j$. Then $(i_2, i_1) \prec (i_{k-1}, i_k)$.*

Proof. Note first that since $i <_r j$, the distance from r to j in T is at least the distance from r to i in T. Suppose that these two distances are equal. Let $\alpha = \min_{<_r}\{i_1, \ldots, i_k\}$, and let π_1 and π_2 be the unique path between i_1 and α and i_k and α, respectively. We index π_1 as $\pi_1 = \{i_{0,1} i_{1,1}, i_{1,1} i_{2,1}, \ldots, i_{t-1,1} i_{t,1}\}$ and π_2 as $\pi_2 = \{i_{0,2} i_{1,2}, i_{1,2} i_{2,2}, \ldots, i_{t-1,2} i_{t,2}\}$ where $i_{0,1} = i_{0,2} = \alpha$, $i_{t,1} = i_1$ and $i_{t,2} = i_k$. Observe that π_1 and π_2 have the same length since i and j have the same distance from r.

We claim now that $i_{j,1} <_r i_{j,2}$ for all $j = 1, \ldots, t$. To see this, suppose for the sake of contradiction that there exists a j for which $i_{j,1} >_r i_{j,2}$. Then the floret for $i_{j,2}$ has its children ordered before that of $i_{j,1}$. This implies that $i_{j+1,2} >_r i_{j+1,1}$. Iterating this argument implies that $i = i_{t,1} >_r i_{t,2} = j$, which is a contradiction. Hence, $i_{j,1} <_r i_{j,2}$ for all $j = 1, \ldots, t$. It follows that

$$(i_2, i_1) = (i_{t-1,1}, i_{t,1}) \prec (i_{t-1,2}, i_{t,2}) = (i_{k-1}, i_k),$$

as desired.

Now suppose that the distance from r to j in T is strictly larger than the distance from r to i in T. Then π_1 contains only nodes of distance at most t from r and π_2 contains a node at distance $t+1$ from r, for minimally chosen t. By the chosen edge ordering, the edge in π_2 from the node at distance t to the node at distance $t+1$ is larger than all edges in π_1. This edge is also seen to be equal to, or smaller than (i_{k-1}, i_k), which completes the proof. □

For each edge $ij \in E(T)$ the fundamental binomials in B_T imply that if S is a subset of the generators of R_T corresponding to a face of $\mathcal{T}$, then the graph G_S does not contain any of the subgraphs listed in (3).

Similarly, the zig-zag binomials in B_T imply that the graph G_S must not contain certain subgraphs along paths if S is a face of $\mathcal{T}$. These subgraphs generalize the partially directed increasing paths from Section 3 and are defined as follows:

Let i_1, i_k be vertices of T such that $i_1 <_r i_k$, let $\pi = \{i_1i_2, i_2i_3, \ldots, i_{k-1}i_k\}$ be the unique path in T between i_1 and i_k, and let $\alpha = \min_{<_r}\{i_1, \ldots, i_k\}$. Further let π_1 and π_2 denote the subpaths of π between i_1 and α and i_k and α, respectively. Given the ordering of the variables above, the leading terms of any zig-zag binomial for a zig-zag pair on π have associated graphs being one of the following:

(1) Partially directed paths toward i_1 ending in $\circ$ that include a directed edge on π_1 pointing toward i_1.
(2) Partially directed paths toward i_k ending in $\circ$ that include an edge directed toward i_k on π_2, and
(3) Partially directed paths toward i_1 ending in $\circ$ with all edges on π_2 directed and all edges on π_1 undirected.

Observe that the paths in (2) include the paths excluded by the leading terms of zig-zag binomials for the path in Section 3 by taking $i_1 = \alpha$. To see that these three options contain all possible leading terms of zig-zag binomials, consider a zig-zag pair (E_1, E_2) on the path π, where E_1 are the edges directed toward i_k and E_2 are the edges directed toward i_1. If either E_1 contains edges on π_2 or E_2 contains edges on π_1, then under the given variable ordering one of the y-variables represented by these edges is the largest. Hence, under the given term order the leading term of the associated zig-zag pair is represented by a graph of type (1) or (2).

On the other hand, if E_1 is all the edges on π_1 and E_2 is all the edges on π_2, then by Lemma 5.1, the leading term is given by the y-variables in E_2. Hence, such a zig-zag pair is represented by the graphs in (3) listed above.

We call these paths *zig-zag obstructions*. Zig-zag obstructions of the form (i) for $i = 1, 2, 3$ are called *zig-zag obstructions of type i*.

Example 5.2. In Figure 2 we see four graphs. The first three each contain a zig-zag obstruction of type 1, 2, and 3, respectively, when considered from left-to-right. The second graph, which highlights in red a zig-zag obstruction of type 2 also contains three additional zig-zag obstructions (of the same type). These are given by replacing exactly the one of the directed edges with its undirected version, or alternatively considering the subgraph of the red edges where we forget the least of the two directed edges. The rightmost graph depicts a graph that contains no zig-zag obstructions.

For the chosen term order, we can further reduce the Gröbner basis identified for I_T. Consider paths of the form $\pi = \{i_1, \ldots, i_k\}$ in which $i_1 <_r i_2 <_r \cdots <_r i_k$. We define a *simple zig-zag pair of type 1* as zig-zag pair (E_1, E_2) where $E_1 = \{i_1 \to i_2\}$ and $E_2 = \{i_{t+1} \to i_t : t \in \{2, \ldots, k-1\}\}$. Consider also paths of the form $\pi = \{i_1, i_2, \ldots, i_k\}$ in which $i_j <_r i_1$ for all $j = 2, \ldots, k-1$ but $i_1 <_r i_k$. Let $\alpha = \min_{<_r}\{i_1, \ldots, i_k\}$, and take π_1 and π_2 as before. A *simple zig-zag pair of type 2* is a zig-zag pair (E_1, E_2) on this path where E_1 consists of all edges on π_1 oriented toward α and E_2 consists of all edges on π_2 oriented toward α.

Lemma 5.3. *Let T be a tree. The leading term of any zig-zag binomial under the lexicographic order on R_T corresponding to $<$ is divisible by the leading term of a simple zig-zag binomial.*

Proof. Under the given term order, the leading term of the zig-zag binomial for a simple zig-zag pair is graphically represented by a partially directed path from i_1 to i_k in which the first edge $i_1 \to i_2$ is directed toward i_k and all other edges are undirected, plus a symbol for the variable z_{i_1}. Given a zig-zag binomial whose leading term is

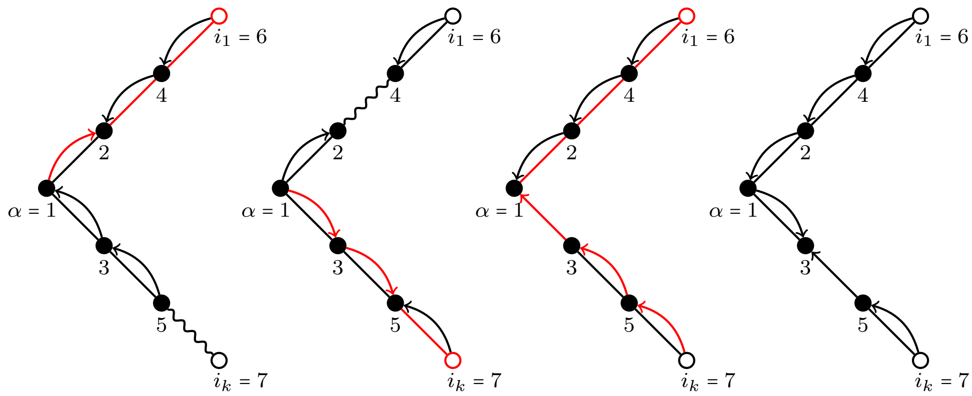


FIGURE 2. The first three graphs are, respectively from left-to-right, examples of zig-zag obstructions of type 1, 2 and 3 with the obstruction depicted in red. The rightmost graph is an example of a graph that contains no zig-zag obstructions. The order $<_r$ is the natural order on the vertex set.

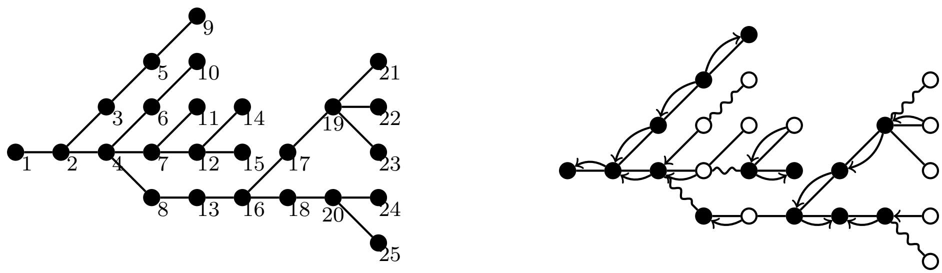
represented by a zig-zag obstruction of type 1, the associated path $\pi = \{i_1, \ldots, i_k\}$ is such that the subpath π_1 contains at least one directed edge pointing toward i_1. Pick the edge $i_s \leftarrow i_{s+1}$ of this form on π_1 with s minimal. The remaining edges between this edge and i_1 must be undirected, and hence the subpath on $\{i_1, \ldots, i_s\}$ is the graphical representation of the leading term of the zig-zag binomial of a simple zig-zag pair of type 1. Hence, the leading term of this zig-zag binomial is divisible by the leading term of a zig-zag binomial of a simple zig-zag pair. The same argument shows that all zig-zag binomials whose leading terms are represented by zig-zag obstructions of type 2 are also divisible by the leading term of some zig-zag binomial for a simple zig-zag pair of type 1.

For zig-zag binomials whose leading terms are represented by zig-zag obstructions of type 3, the subpath $\{i_1, \ldots, i_t = \alpha, i_{t+1}, \ldots, i_s\}$, where i_s is the first node on π_2 larger than i_1 under $<_r$, is the graphical representation of the leading term of a zig-zag binomial for a simple zig-zag pair of type 2. This follows from Lemma 5.1. Hence, all such zig-zag binomials also have leading terms divisible by the leading term of a zig-zag binomial for a simple zig-zag pair. This completes the proof. □

Let S be the subset of variables in the leading term of a zig-zag binomial for a simple zig-zag pair of type 1. We call the graph G_S a *simple zig-zag obstruction of type 1*. We similarly define *simple zig-zag obstructions of type 2*.

EXAMPLE 5.4. The zig-zag obstructions in the first and third graphs in Figure 2 are both simple (of type 1 and 2, respectively). On the other hand, the zig-zag obstruction in the second graph is not simple, but contains a simple zig-zag obstruction of type 2 as a subgraph. This obstruction is given by deleting the first of the two directed edges from the graph. Such subgraph inclusions correspond to the divisibility of leading terms as seen in the proof of Lemma 5.3.

Since T is a tree on vertex set $[n+1]$, the dimension of $\mathcal{C}_T$ is $|V| + |E| - 1 = 2n$. By Lemma 5.3, a characterization of the facets of the triangulation $\mathcal{T}$ of $\mathcal{C}_T$ given by the specified good term order consists of all subsets S of the variables generating R_T with $|S| = 2n + 1$ for which the graph G_S contains no fundamental obstructions and no simple zig-zag obstructions.



(a) A tree T. (b) The graph G_S.

Figure 3. A tree T and the graph G_S for a subset S of the variables in the ring R_T.

In the following, we say that two nodes i,j in a undirected graph $G=(V,E)$ are *connected given a subset* $C\subset V$ if there is a path $\pi=\{i_1i_2,\dots,i_{k-1}i_k\}$ in G such that $i_1=i$, $i_k=j$ and $i_2,\dots,i_{k-1}\notin C$. We say a subset of vertices B of G is *maximally connected given* C if all vertices in B are connected given C and there is no pair of vertices $i\in B$ and $j\notin B$ such that i and j are connected given C. For a tree T and subset S of variables in R_T, we let $\overline{G}_{S,1},\dots,\overline{G}_{S,M}$ denote the induced subgraphs of G_S on the maximally connected subsets of T given $\mathfrak{z}_S$. We call the collection of graphs $\overline{G}_{S,1},\dots,\overline{G}_{S,M}$ the *Z_S-components* of G_S.

Remark 5.5. One can intuitively think of the Z_S-components as the collection of subgraphs obtained in the following way: Delete all vertices from the set Z_S to obtain a collection of disjoint subgraphs of G. To each subgraph, add back in any adjacencies it had to elements in Z_S. For the special case that two nodes $i,j\in Z_S$ are adjacent in G, we include a graph in our Z_S-components that is exactly the edge between i and j.

Recall from Proposition 2.13 that the support graph of G_S for S a facet of the triangulation of $\mathcal{C}_T$ is the tree T. Hence, the support graph of each $\overline{G}_{S,j}$ is a subtree of T. In the following, we will want to refer to certain subgraphs of $\overline{G}_{S,j}$ that are induced subgraphs of $\overline{G}_{S,j}$ on the vertex set of the corresponding induced subgraphs of T. For instance, although a vertex i in $\overline{G}_{S,j}$ may have degree greater than 1, we will call it a *leaf node* of $\overline{G}_{S,j}$ if it is a leaf node in the support graph of $\overline{G}_{S,j}$. Similarly, we may refer to a subgraph of $\overline{G}_{S,j}$ as a *path* in $\overline{G}_{S,j}$ if it is the induced subgraph of $\overline{G}_{S,j}$ on the node set of a path in its support graph, despite the fact that it may include multiple edges between the same pair of vertices. This mild abuse of terminology should, however, be clear from context. As a first example, we call the graph $\overline{G}_{S,j}$ *Z_S-bounded* if all leaf nodes of $\overline{G}_{S,j}$ are in $\mathfrak{z}_S$. Otherwise, we call it *Z_S-unbounded*.

Example 5.6. Consider the tree T and the graph G_S depicted in Figure 3a and Figure 3b, respectively. For G_S, we have that

$$\mathfrak{z}_S=\{6,7,10,11,13,14,21,22,23,24,25\}.$$

The natural order on the vertex set is taken for $<_r$, where $r=1$, under which we see that G_S contains no fundamental obstructions and no simple zig-zag obstructions. From G_S we also see that $|S|=2n+1$, where $n=24$, and so it follows that S is a facet of the triangulation $\mathcal{T}$.

The graph G_S has the Z_S-components depicted in Figure 4, in which the two leftmost graphs are Z_S-unbounded and the remaining ones are Z_S-bounded.

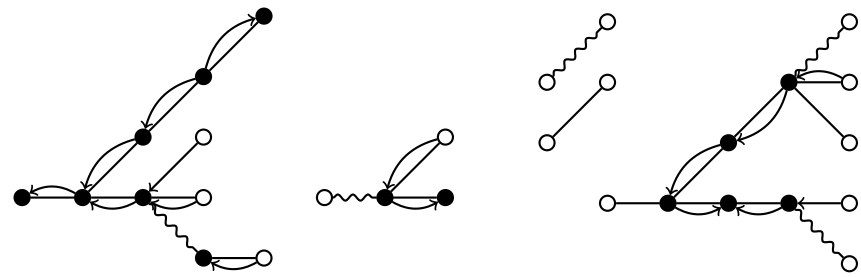


Figure 4. The Z_S-components of the graph G_S in Figure 3b.

To analyze the graphs $\overline{G}_{S,j}$ it will be helpful to have a notion of separation of vertices by edges. Given a graph G with vertex set V and edge set E, we say that the subset of nodes $A \subset V$ are *separated* by the subset of edges $B \subset E$ if for every pair of nodes $i, j \in A$ all paths in G from i to j include an edge in B. Note that the set B is a cut-set for which the associated cuts each contain a single vertex in A. The following lemma will be used.

Lemma 5.7. *Let T be a tree with set of leaf nodes A. If the set A is separated by B, then $|B| \geq |A| - 1$.*

Proof. Note that if B is a set of edges separating the leaf nodes A of the tree T then deleting the edges in B from T results in a forest with at least $|A|$ connected components such that no two leaves of T are in the same component. This is because removing a single edge from a forest always increases the number of connected components by exactly 1. Hence, to get $|A|$ connected components that each contain a unique leaf node we must remove $|A| - 1$ edges, which proves the desired lower bound. □

Lemma 5.8. *Let T be a tree with set of leaves A and let B be a set of edges in T that separate A. If $|B| = |A| - 1$, then for each edge $e \in B$ there exists a unique pair of vertices $i, j \in A$ such that B separates i and j but $B \setminus e$ does not.*

Proof. By Lemma 5.7 the set B is a minimal separating set for A. Hence, removing a single edge $e = st \in B$ from B connects at least one pair of leaves of T. Suppose we connect two pairs of leaves, say i, j and k, ℓ. Then, without loss of generality, i and k and j and ℓ are, respectively, on the same side of the single edge e, i.e. they are in the same connected component given by deleting the edge e. Say j and ℓ are in the component containing t and i and k are in the component containing s.

Since T is a tree there is a unique path between i and k, and this path must be the concatenation of the paths between i and s and k and s. Since this path does not contain e, it must contain another edge $e' \in B$. Without loss of generality, suppose e' lies on the path between i and s. This contradicts the assumption that removing e from the separating set connects i and j, which completes the proof. □

We say that an edge e *critically separates* a pair of nodes i and j in a graph $G = (V, E)$ with respect to $B \subset E$ if B separates i and j but $B \setminus e$ does not. Lemma 5.8 states that if G is a tree with set of leaf nodes A and B separates A with $|B| = |A| - 1$, then each edge in B critically separates a unique pair of leaf nodes in G. For a fixed tree T, we let $m_j \coloneqq |V(\overline{G}_{S,j}) \cap \mathfrak{Z}_S|$. The following generalizes Lemma 3.1 to arbitrary trees.

Lemma 5.9. *Let T be a tree on node set $[n+1]$ and let S be a facet of the triangulation of $\mathcal{C}_T$. If $\mathfrak{Z}_S = \{i_1 <_r \cdots <_r i_{n+1-k}\}$ it follows that*

(1) *G_S contains exactly k double edges,*

(2) $\overline{G}_{S,j}$ *contains exactly* $m_j - 1$ *single edges for all* $j \in [M]$.

Proof. We first show that (1) holds. Since S is a facet and $\dim(\mathcal{C}_T) = |V(T)|+|E(T)|-1 = 2n$, we have $|S| = 2n+1$. Suppose now that $|\mathfrak{Z}_S| = n-k+1$. We know that the support graph of G_S equals G by Proposition 2.13, and hence G_S contains at least n edges. Since $|\mathfrak{Z}_S| + n = 2n-k+1$ and $|S| = 2n+1$, and every edge in G_S is either a single edge or a double edge, it follows that G_S contains exactly k double edges.

It remains to see that (2) holds. Consider first the case when $\overline{G}_{S,j}$ is Z_S-bounded. Let $n_j = |V(\overline{G}_{S,j})|$. We claim that there are at most $n_j - m_j$ double edges in $\overline{G}_{S,j}$; equivalently, there are at least $m_j - 1$ single edges in $\overline{G}_{S,j}$. To see this, note that the single edges in $\overline{G}_{S,j}$ must separate all nodes in $V(\overline{G}_{S,j}) \cap \mathfrak{Z}_S$. That is, the edges in the support graph of $\overline{G}_{S,j}$ corresponding to the single edges in $\overline{G}_{S,j}$ must separate the leaf nodes of the support graph. Otherwise, there is at least one simple zig-zag obstruction in G_S, which would contradict S being a facet.

To see this claim, assume otherwise. Then there is a pair of vertices $i_1, i_k \in V(\overline{G}_{S,j}) \cap \mathfrak{Z}_S$ such that the unique path $\pi = \{i_1 i_2, \dots, i_{k-1} i_k\}$ in T between i_1 and i_k contains only double edges in $\overline{G}_{S,j}$. Suppose without loss of generality that $i_1 <_r i_k$. Since $i_1, i_k \in \mathfrak{Z}_S$, we know these nodes are $\circ$ nodes in G_S. Let $\alpha = \min_{<_r}\{i_1, \dots, i_k\}$, and let π_1 and π_2 denote the subpaths of π between i_1 and α and i_k and α, respectively. If there is a directed arrow pointing toward i_1 or i_k, respectively on π_1 or π_2 then G_S contains a zig-zag obstruction of type (1) or (2), and hence contains a simple zig-zag obstruction of type 1. This, would contradict S being a facet. Thus, all directed edges on π must be directed toward α. However, this implies that G_S contains a simple zig-zag obstruction of type 2, which is again a contradiction.

Thus, no such paths of double edges exist, and we conclude that the single edges must separate the nodes in $V(\overline{G}_{S,j}) \cap \mathfrak{Z}_S$. Since T is a tree, we require at least $m_j - 1$ such single edges in $\overline{G}_{S,j}$ by Lemma 5.7. Similarly, if $\overline{G}_{S,j}$ is Z_S-unbounded, we can consider the induced subgraph of $\overline{G}_{S,j}$ by all nodes on the unique paths in T between the vertices in $V(\overline{G}_{S,j}) \cap \mathfrak{Z}_S$, and the same argument applies as in the Z_S-bounded case. Hence, there are at least $m_j - 1$ single edges in $\overline{G}_{S,j}$ and at most $n_j - m_j$ double edges.

We now claim that there are exactly $m_j - 1$ single edges in $\overline{G}_{S,j}$. By our choice of Gröbner basis, the graph $\overline{G}_{S,j}$ also corresponds to a facet of the triangulation of the cosmological polytope of the support graph T' of $\overline{G}_{S,j}$. Since T' is a tree on n_j vertices, and since $|V(\overline{G}_{S,j}) \cap \mathfrak{Z}_S|$ contains m_j vertices, we know from (1) that $\overline{G}_{S,j}$ must exactly contain $n_j - m_j$ double edges. Equivalently, it must contain exactly $m_j - 1$ single edges, which completes the proof. □

In the proof of Lemma 5.9 we use the fact that the set of single edges in a Z_S-component $\overline{G}_{S,j}$ corresponds to a set of edges in the support graph of the component that separates its leaf nodes. In the following, we will simply say that a set of edges in a Z_S-component *separates* a set of nodes A if the corresponding edges in the support graph separate A.

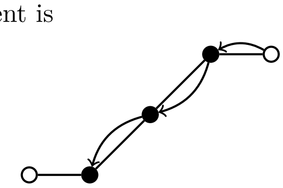
By definition, each of the Z_S-components $\overline{G}_{S,j}$ has a unique minimal node $r_{S,j}$ under the vertex ordering $<_r$. This node is the root of the induced subgraph of $\overrightarrow{T}$ on the vertex set $V(\overline{G}_{S,j})$. A root-to-leaf path in this subtree is a path connecting the root node $r_{S,j}$ to a leaf node i. We can consider the induced subgraph on the node set of such a path in the graph $\overline{G}_{S,j}$. For simplicity, we refer to such a subpath as a root-to-leaf path in $\overline{G}_{S,j}$, noting that it may contain multiple edges. When we refer to a leaf-to-root path we imagine reading such a root-to-leaf path backwards from the leaf to the root node.

For each node $i \in V(\overline{G}_{S,j}) \cap (\mathfrak{Z}_S \setminus \{r_{S,j}\})$ consider the first single edge encountered along the leaf-to-root path from i to $r_{S,j}$ in $\overline{G}_{S,j}$. By Lemma 5.8, this edge critically separates a unique pair of vertices $s, t \in V(\overline{G}_{S,j}) \cap \mathfrak{Z}_S$ (with respect to the set of single edges in $\overline{G}_{S,j}$). Moreover, it can be seen by applying a similar argument as in the proof of Lemma 5.8 that one of these two vertices must be i, say $s = i$. Let $\pi = \{i_1 i_2, \dots, i_{k-1} i_k\}$ denote the unique path in T between i and t, and let $\alpha = \min_{<_r}\{i_1, \dots, i_k\}$. We call the path between α and i the *threshold path for* i. If we take $i = i_k$ and $\alpha = i_t$ for some $t \in [k]$ in π, we see that $i_t <_r i_{t+1} <_r \cdots <_r i_k$; that is, the threshold path for i is a decreasing path from i to α under the ordering $<_r$. It follows that the threshold path for i always contains the unique single edge on π separating i and t as the leaf-to-root path used to define the threshold path is also decreasing.

If $i <_r t$ it follows that the threshold path is the path π_1, and if $t <_r i$ it is π_2. In the former case, we say the threshold path is *type 1*, and we say it is *type 2* in the latter case. A threshold path of type 1 is *blocking* if the portion of the path between i and the first single edge consists only of undirected edges paired with directed edges pointing away from i and the single edge is a directed edge pointing away from i or a $\rightsquigarrow$. A threshold path of type 2 is *blocking* if the portion of the path between i and the first single edge consists only of undirected edges paired with directed edges pointing away from i and one of the following holds:

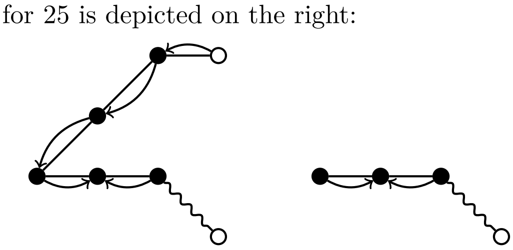
(1) The single edge is undirected and all directed edges on the threshold path point away from i, or
(2) the single edge is $\rightsquigarrow$, or
(3) the single edge is a directed edge pointing away from i and at least one directed edge on the portion of the path between α and this single edge points toward i.

Example 5.10. We consider some examples of threshold paths in the rightmost Z_S-component for the graph G_S in Example 5.6. For instance, the threshold path for the vertex 22 in this Z_S component is



The unique edge on the leaf-to-root path from 22 to $r = 1$ has first single edge the undirected edge between vertices 13 and 16 depicted here. This edge critically separates the two nodes $13, 22 \in \mathfrak{Z}_S$ (with respect to the set of all single edges in the Z_S-component), and hence the threshold path for 22 is the entire path in the Z_S-component between 22 to 13. Since $13 <_r 22$, this is a threshold path of type 2. We see that it is blocking since it satisfies (1).

As a second example, consider the leaf node 25 in the same Z_S-component. The first edge on its leaf-to-root path is a single edge. This edge critically separates nodes $22, 25 \in \mathfrak{Z}_S$. The path between these nodes is depicted on the left in the following, and the threshold path for 25 is depicted on the right:



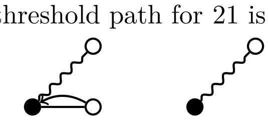
(a) A Z_S-component of the graph G_S in Figure 3b.

(b) The Z_S-component in Figure 5a with one edge direction reversed.

Figure 5. Examples and non-examples of partially directed branchings. An example of a partially directed branching appears in red.

Since $22 <_r 25$, the threshold path for 25 is also type 2, and we see that it is blocking by (2).

As a third, and final example, consider the root-to-leaf path from node 21 in the same Z_S-component. As the first edge on this path is a single edge which critically separates nodes 21 and 22, we have that the path between nodes 21 and 22 is that depicted on the left, and the threshold path for 21 is that depicted on the right:



Since $21 <_r 22$ this is a threshold path of type 1, which is seen to be blocking since the single edge is a $\rightsquigarrow$.

We will use the notion of blocking paths in our characterization of the facets of the triangulation of $\mathcal{C}_T$. We additionally require one more type of path. Given a node i of the graph T we say that a node j *covers* i if $i <_r j$ and no node along the unique path from j to r in T is larger than i. Let $\pi = \{i_1 i_2, \ldots, i_{k-1} i_k\}$ be the unique path in T between i and j and let $\alpha = \min_{<_r}\{i_1, \ldots, i_k\}$. A *partially directed branching* from j to i is a partial orientation of π such that all edges along the path between i and α are undirected and all edges along the path between j and α are directed toward α.

Example 5.11. We consider the Z_S-component depicted in Figure 5a for the graph G_S in Figure 3b. The node 23 is covered by nodes 24 and 25. We see from inspection of the graph that there are no partially directed branchings in this Z_S-component from 24 to 23 nor from 25 to 23. However, if we were to reverse the direction of the edge between nodes 16 and 18 we would then have a partially directed branching from 24 to 23, as depicted in red in Figure 5b.

The following theorem characterizes the facets of the triangulation of $\mathcal{C}_T$ for T a tree under the specified good term order.

Theorem 5.12. *Let S be a subset of generators of R_T where T is a tree on node set $[n+1]$. Let $\overline{G}_{S,1}, \ldots, \overline{G}_{S,M}$ be the Z_S-components of G_S. The set S is a facet of the triangulation $\mathcal{T}$ of $\mathcal{C}_T$ corresponding to a lexicographic order induced by the order $<$ if and only if the following hold:*

1. *G_S is connected,*
2. *G_S contains only single and double edges, where all double edges are of the form*

(3) *For all* $j \in [M]$

  (a) $\overline{G}_{S,j}$ *contains exactly* $|V(\overline{G}_{S,j}) \cap \mathfrak{Z}_S| - 1$ *single edges,*

  (b) *for all* $i \in V(\overline{G}_{S,j}) \cap (\mathfrak{Z}_S \smallsetminus \{r_{S,j}\})$ *the threshold path for* i *is blocking,*

  (c) *for all* $i \in V(\overline{G}_{S,j}) \cap (\mathfrak{Z}_S \smallsetminus \{r_{S,j}\})$ *there are no partially directed branchings to* i *from a node that covers* i*, and*

  (d) *if* $r_{S,j} \in \mathfrak{Z}_S$*, then the edges incident to it are either a single* $\rightsquigarrow$*, a single undirected edge, or an undirected edge together with an edge directed away from* $r_{S,j}$*.*

Proof. Assume first that S is a facet. By Proposition 2.13 we know that (1) is satisfied. By Lemma 2.12, (2) is satisfied. By Lemma 5.9 we know that (3)(a) is also satisfied.

To see that (3)(b) holds, consider a leaf-to-root path from $i \in V(\overline{G}_{S,j}) \cap (\mathfrak{Z}_S \smallsetminus \{r_{S,j}\})$ and the portion of this path up to and including its first single edge. We note that this is a subpath of the threshold path for i. Since S is a facet, all edges before the single edge are double, and by (2) they are of the specified form above. Since the path is leaf-to-root, then reading its vertices from the single edge out toward i is an increasing sequence of nodes under the ordering $<_r$. Hence, if any of these double edges contains an arrow pointing toward i, then G_S would contain a simple zig-zag obstruction of type 1, which is a contradiction to S being a facet.

Consider now the unique pair of vertices in $V(\overline{G}_{S,j}) \cap \mathfrak{Z}_S$ separated by the single edge on this path, one of which is i and the other of which we denote by t. Denote this path by π, and consider its associated α-value, and the two paths π_1 and π_2 between α and i and α and t. Here, we let π_1 denote the path between α and the least of the two vertices i and t under $<_r$, and π_2 denote the path between α and the largest of the two. Note that the single edge is always on the path π_1 or π_2 that contains i. Since the given single edge is the unique separator of i and t, we know that all other edges on this path are double and of the form specified in (2). If $i <_r t$ and the single edge is on π_1 (the path from the least of the two nodes i and t to α) then π_2 must consist of only double edge directed toward α. Otherwise, G_S would contain a simple zig-zag obstruction of type 1. Since all edges on π_1 are also double edges except for the single edge, G_S would contain a simple zig-zag obstruction of type 2 if the single edge were undirected. Since it would contain a simple zig-zag obstruction of type 1 if the edge were directed toward i, the only valid options are a $\rightsquigarrow$ or an edge directed toward α. Hence, the threshold path for i is of type 1 and blocking.

On the other hand, if $t <_r i$, then the unique single edge on the path between i and t would lie on π_2 (i.e. the path from the largest of the two nodes i and t to α). Hence, all edges on π_1 are double and directed toward α to avoid simple zig-zag obstructions of type 1. We note that the single edge cannot be a directed edge toward i as this would lead to a simple zig-zag obstruction of type 1 as well. If the edge is undirected, then G_S contains no simple zig-zag obstructions of type 1 only if all directed edges on π_2 point toward α. If the single edge is $\rightsquigarrow$, all directed edges between the single edge and i must point toward α. Finally, if the single edge is directed toward α there must be at least one directed edge between α and the single edge pointing toward i. Otherwise, G_S would contain a simple zig-zag obstruction of type 2. Hence, the threshold path must also be blocking in this case. Therefore, (3)(b) holds for S a facet.

To see that (3)(c) holds, note that if there was such a partially directed branching then G_S would necessarily contain a simple zig-zag obstruction of type 2, a contradiction to S being a facet.

To see that (3)(d) holds, note that any other choice of edge configuration at $r_{S,j}$ would imply that G_S contains a fundamental obstruction. Hence, the listed conditions are all satisfied if S is a facet of the triangulation.

Conversely, suppose that the listed conditions are satisfied by S. Let $|Z_S| = n+1-k$. We will show now that (3)(a) implies that G_S contains exactly k double edges. Since G_S is a connected graph on vertex set $[n+1]$, it contains at least n edges. If exactly k of these edges are double then $|S| = 2n+1$, which is the correct dimension for S to be a facet. To see this claim, assume (3)(a) holds, i.e. assume that each $\overline{G}_{S,j}$ contains exactly m_j-1 single edges, where we let $m_j = |V(\overline{G}_{S,j})\cap \mathfrak{Z}_S|$. We induct on the number $M \geq 1$ of Z_S-components in G_S. The result is seen to hold in the case that $M = 1$, as in this case $m_j - 1 = (n+1-k) - 1$. Suppose now that the result holds for G_S with at most $M - 1 \geq 0$ Z_S-components, and consider G_S with M Z_S-components. Since $M > 1$, there is at least one Z_S-component containing a sink node of $\overrightarrow{T}$ that does not contain the root node r. Suppose this component is $\overline{G}_{S,j}$ and that its support graph has n_j edges. Note that $r_{S,j}$ is necessarily a $\circ$ node, as $r_{S,j} \neq r$.

We then have that the subgraph of T given by deleting all edges of the support graph of $\overline{G}_{S,j}$ is a tree containing $n-n_j$ edges. Since $\overline{G}_{S,j}$ does not contain r and does contain a sink node of $\overrightarrow{T}$, deleting all vertices in $V(\overline{G}_{S,j})\smallsetminus\{r_{S,j}\}$ and all edges incident to these vertices from G_S, results in a graph $\widetilde{G}_{\tilde{S}}$ that contains $n+1-k-(m_j-1) = (n+1-k)-m_j+1$ $\circ$ nodes. By the inductive hypothesis, $\widetilde{G}_{\widetilde{S}}$ contains $(n+1-k)-m_j$ single edges. By assumption, $\overline{G}_{S,j}$ contains m_j-1 single edges. Hence, G_S contains $n-k$ single edges, or equivalently, k double edges.

Since $|S| = 2n+1$, which is the correct size for S to be a facet, it only remains to see that the set S contains no fundamental obstructions or simple zig-zag obstructions. The fact that G_S contains no fundamental obstructions follows from (2) together with (3)(d) and (3)(b) (when we consider the definition of blocking). The fact that G_S contains no simple zig-zag obstructions follows from (3)(b) and (3)(c), which completes the proof. □

Remark 5.13. The graphs characterized in Theorem 5.12 give an explicit vertex representation of the facets of a subdivision of $\mathcal{C}_T$. Since each facet is a simplex, this can be readily converted to a hyperplane representation which gives the denominator of the canonical form (2). Since each facet is further a unimodular simplex, the numerator of its canonical form is ±1 (see [14, Section 1]). Hence, Theorem 5.12 gives an algorithm for the computation of the canonical form of $\mathcal{C}_T$ for any tree T via (1).

6. Open problems

We conclude with a few problems of interest left open by the article. As we worked out in the case of cycles and trees, a combinatorial analysis of the Gröbner basis presented in Section 2 reveals an explicit facet description for the corresponding triangulation. Moreover, one of the features of having a regular unimodular triangulation is that the computation of the volume can be reduced to counting the facets. It would be interesting to push this understanding further.

Problem 6.1. Obtain a facet description for a regular unimodular triangulation of the cosmological polytope of more general families of graphs. Can the volume of the cosmological polytope of an arbitrary graph be expressed in terms of elementary graph invariants?

Since the initial release of this paper, additional progress has been made on Problem 6.1. In [15], a facet description of a good triangulation of $\mathcal{C}_G$ for G the complete bipartite graph $K_{2,m}$ was deduced and used to compute the normalized volume formula $\mathrm{Vol}(\mathcal{C}_{K_{2,m}}) = 12^m(1+(m/3))$. Facet descriptions for good triangulations and normalized volume formulas for G a multitree or multicycle were also given in the follow-up paper [6].

From a combinatorial viewpoint one is typically interested in finer invariants than the volume of a lattice polytope. One popular instance is the *h^*-polynomial*; that is, the univariate polynomial with integer coefficients which arises as the numerator of the *Ehrhart series* of a lattice polytope (see for instance [4, Chapter 3]). As an example, we observed experimentally that the h^*-polynomial of the cosmological polytope of a tree on $n+1$ vertices equals $h^*(t) = (1+3t)^n$. This has since been proven (and generalized) in the follow-up paper [6]. However, the following problem remains open for general graphs.

Problem 6.2. Find formulas for the h^*-polynomial of a cosmological polytope $\mathcal{C}_G$ in terms of graph invariants of G.

Finally, since, as we commented in the introduction, the computation of the canonical form of a polytope can be reduced to computing the canonical forms of the facets of any triangulation, we propose the following problem:

Problem 6.3. Describe (lattice) triangulations of cosmological polytopes with the minimum number of facets.

Corollary 3.4 shows that a unimodular triangulation of $\mathcal{C}_G$ for G a path graph with n edges has 4^n facets. However, this number of facets is far from minimal. As described in the following remark, for any tree G on n edges, there exists a triangulation of $\mathcal{C}_G$ having 2^{n-1} facets. Regarding Problem 6.3, it remains to show there is no (lattice) triangulation of $\mathcal{C}_G$ with fewer facets.

Remark 6.4. A tree G can be constructed iteratively, where at each step in the iterative construction we add a new leaf node j attached by an edge to a node i already in the tree. Letting G be a tree and $G+ij$ denoting the tree produced by such a leaf addition to G, we have that $\mathcal{C}_{G+ij} = \mathrm{conv}(\mathcal{C}_G \cup \{x_j+x_i-x_{ji}, -x_j+x_i+x_{ij}, x_j-x_i+x_{ij}\})$. The addition of only the first two of these additional three lattice points amounts to taking a bipyramid over $\mathcal{C}_G$. Then adding in the additional third lattice point realizes $\mathcal{C}_{G+ij}$ as a pyramid over this bipyramid (see [13, Proposition 4.1]). A triangulation $\mathcal{T}_G$ of $\mathcal{C}_G$ induces a canonical triangulation of the bipyramid with twice as many facets as $\mathcal{T}_G$, and this triangulation induces a canonical triangulation of the pyramid, also with twice as many facets as $\mathcal{T}_G$. Since the cosmological polytope of a single edge is a 2-dimensional simplex, it follows inductively that for G a tree on $n+1$ vertices, $\mathcal{C}_G$ admits a triangulation with 2^{n-1} facets.

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Martina Juhnke, Department of Mathematics, University of Osnabrück, Albrechtstraße 28a, 49076 Osnabrück, Germany
E-mail : `martina.juhnke@uni-osnabrueck.de`

Liam Solus, Department of Mathematics, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden
E-mail : `solus@kth.se`

Lorenzo Venturello, Department of Mathematics, Università di Pisa, L.go B. Pontecorvo, 5, 56127 Pisa, Italy
E-mail : `lorenzo.venturello@unipi.it`