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ABSTRACT In this paper, we study wall elements of rank 2 cluster scattering diagrams based on dilogarithm elements. We derive two major results. First, we give a method to calculate wall elements in lower degrees. By this method, we may see the explicit forms of wall elements including the Badlands, which is the complement of G-fan. In this paper, we write one up to 7 degrees. Also, by using this method, we derive some walls independent of their degrees. Second, we find a certain admissible form of them. In the proof of these facts, we introduce a matrix action on a structure group, which we call a similarity transformation, and we argue the relation between this action and ordered products.

1. INTRODUCTION

1.1. BACKGROUND. Cluster scattering diagrams (CSDs, for short) were introduced by [2]. They have great effects on cluster algebra theory, which was introduced by [1]. For example, the sign coherence of c-vectors and the Laurent positivity, both of which are important properties of cluster algebras, were shown by using CSDs. Roughly speaking, a CSD $\mathfrak{D}$ is a set of walls, and a wall contains a certain element of the structure group G of $\mathfrak{D}$, which is a non-abelian group. In particular, G has dilogarithm elements $\Psi[n]$, which are defined in Definition 3.7, and they play an important role in CSDs. In this paper, we concentrate on CSDs of rank 2. We write $\Psi[n]$ by $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ for $n = (a, b)$. Then, the consistency condition of a CSD of type (δ_1, δ_2), which is the most fundamental property of a CSD, has the following form, where $u_{(a,b)}(\delta_1, \delta_2)$ are some nonnegative rational numbers:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\delta_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\delta_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\delta_1} \left\{ \overrightarrow{\prod_{j;(a_j,b_j)\in\mathbb{Z}^2_{\geqslant 1}}} \begin{bmatrix} a_j \\ b_j \end{bmatrix}^{u_{(a_j,b_j)}(\delta_1,\delta_2)} \right\} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\delta_2}. \tag{1}$$

The right hand side is a product such that $\frac{a_j}{b_j} \geqslant \frac{a_i}{b_i}$ for any $j < i$. It is called the *strongly ordered product expressions* of $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]^{\delta_2} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^{\delta_1}$. Moreover, in [5], it is known that the above equality is obtained by applying the pentagon relation (possibly infinitely many times):

$$\begin{bmatrix} x \\ y \end{bmatrix}^{\gamma} \begin{bmatrix} z \\ w \end{bmatrix}^{\gamma} = \begin{bmatrix} z \\ w \end{bmatrix}^{\gamma} \begin{bmatrix} x+z \\ y+w \end{bmatrix}^{\gamma} \begin{bmatrix} x \\ y \end{bmatrix}^{\gamma}, \tag{2}$$

where $\gamma^{-1} = yz - xw$. The explicit value of $u_{(a,b)}(\delta_1, \delta_2)$ is well known when $\delta_1\delta_2 \leqslant 4$. When $\delta_1\delta_2 \leqslant 3$, a CSD is of finite type, and the product in (1) is finite [2, 5]. On the

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other hand, when $\delta_1\delta_2 = 4$, a CSD is of affine type, and the product in (1) is infinite [7, 6, 5, 4]. Also, the explicit forms of $u_{(a,b)}(\delta_1,\delta_2)$ are known in the case $\delta_1 = \delta_2$ and $a = b$ [8]. However, they are known only few cases. In particular, when $\delta_1\delta_2 \geqslant 5$, there is the region which is complement of the G-fan (that is so called *the Badlands*). The walls in the G-fan correspond to the cluster algebra theory, in particular, c-vectors and g-vectors [5, Thm. 6.13]. On the other hand, the structure in the Badland is hardly known. It is expected that every $u_{(a,b)}(\delta_1,\delta_2)$ is positive for such (a,b) belonging to the Badlands [2, 5].

1.2. Main results and ideas. In this paper, we treat $u_{(a,b)}(\delta_1,\delta_2)$ as a function of δ_1 and δ_2. The main purpose is to describe $u_{(a,b)}(\delta_1,\delta_2)$ explicitly. In order to emphasize that $u_{(a,b)}(\delta_1,\delta_2)$ is a function of δ_1 and δ_2, we write $\delta_1 = m$ and $\delta_2 = n$. Namely, we mainly consider $u_{(a,b)}(m,n)$ as a function of $(m,n) \in \mathbb{Z}^2_{\geqslant 0}$.

The main idea to obtain some results for CSDs is the similarity transformation, which is a group homomorphism defined by matrices $F \in \mathrm{Mat}_2(\mathbb{Z}_{\geqslant 0})$ with $|F| \neq 0$. By applying this action to dilogarithm elements, we have

$$F\begin{bmatrix} a \\ b \end{bmatrix} = \left[F\begin{pmatrix} a \\ b \end{pmatrix}\right]^{1/|F|}, \tag{3}$$

where $(a,b) \in \mathbb{Z}^2_{\geqslant 0}$. This action is compatible for the pentagon relation and ordered products. In particular, by applying this action to (1), we have

$$\begin{aligned} &\left(F\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^{\delta_2}\left(F\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^{\delta_1} \\ &= \left(F\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^{\delta_1}\left\{\prod^{\longrightarrow}_{j;(a_j,b_j)\in\mathbb{Z}^2_{\geqslant 1}}\left(F\begin{bmatrix} a_j \\ b_j \end{bmatrix}\right)^{u_{(a_j,b_j)}(\delta_1,\delta_2)}\right\}\left(F\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^{\delta_2}. \end{aligned} \tag{4}$$

This equality is the key to derive strongly ordered product expressions in lower degrees.

Let us see the main results. As the first result, we give a method to calculate $u_{(a,b)}(m,n)$ explicitly in order of $a+b$ (Method 5.7). More directly, we obtain the following recurrence relations:

Proposition 5.8. *Let $l \in \mathbb{Z}_{\geqslant 1}$, and let $(a,b) \in N^+$ with* $\deg(a,b) = l+1$. *Let $C_{(m,1)}$ and $C_{(m,n)}$ be the products which is defined by (*113*) and (*115*), respectively. The following two statements hold.*
(a) *By applying Algorithm* 5.5 *to $C_{(m,1)}$ repeatedly, we give a method to obtain the recurrence relation:*

$$u_{(a,b)}(m+1,1) = u_{(a,b)}(m,1) + p(m), \tag{5}$$

where $p(m)$ is some polynomial in m.
(b) *By applying Algorithm* 5.5 *to $C_{(m,n)}$ repeatedly, we obtain the recurrence relation:*

$$u_{(a,b)}(m,n+1) = u_{(a,b)}(m,n) + u_{(a,b)}(m,1) + p'(m,n), \tag{6}$$

where $p'(m,n)$ is some polynomial in m and n.

Moreover, $p(m)$ and $p'(m,n)$ are determined by the data of $u_{(x,y)}(m,n)$ with $\deg(x,y) \leqslant l$ *as functions of m and n.*

More strongly, we can show that $\gcd(a,b)p'(m,n)$ can be expressed as the following form:

$$\gcd(a,b)p'(m,n) = \sum_{\substack{0\leqslant k\leqslant A,\\ 0\leqslant l\leqslant B}} \alpha_{k,l}\binom{m}{k}\binom{n}{l} \quad (A,B,\alpha_{k,l} \in \mathbb{Z}_{\geqslant 0}). \tag{7}$$

In particular, polynomials of the above form are often used in this paper. We name them *polynomials in binomial coefficients* (*PBCs*, for short), and we derive some their properties in Section 2. By using this method up to $a+b \leqslant 5$, we obtain the following explicit forms:

$$\begin{aligned}
&\begin{bmatrix}0\\1\end{bmatrix}^{n}\begin{bmatrix}1\\0\end{bmatrix}^{m}\\
&\equiv \begin{bmatrix}1\\0\end{bmatrix}^{m}\begin{bmatrix}4\\1\end{bmatrix}^{\binom{m}{4}\binom{n}{1}}\begin{bmatrix}3\\1\end{bmatrix}^{\binom{m}{3}\binom{n}{1}}\begin{bmatrix}2\\1\end{bmatrix}^{\binom{m}{2}\binom{n}{1}}\\
&\times \begin{bmatrix}3\\2\end{bmatrix}^{2\binom{m}{2}\binom{n}{2}+\binom{m}{3}\binom{n}{1}+6\binom{m}{3}\binom{n}{2}}\begin{bmatrix}1\\1\end{bmatrix}^{\binom{m}{1}\binom{n}{1}}\\
&\times \begin{bmatrix}2\\2\end{bmatrix}^{2\binom{m}{2}\binom{n}{2}}\begin{bmatrix}2\\3\end{bmatrix}^{\binom{m}{1}\binom{n}{3}+2\binom{m}{2}\binom{n}{2}+6\binom{m}{2}\binom{n}{3}}\begin{bmatrix}1\\2\end{bmatrix}^{\binom{m}{1}\binom{n}{2}}\\
&\times \begin{bmatrix}1\\3\end{bmatrix}^{\binom{m}{1}\binom{n}{3}}\begin{bmatrix}1\\4\end{bmatrix}^{\binom{m}{1}\binom{n}{4}}\begin{bmatrix}0\\1\end{bmatrix}^{n} \qquad \bmod G^{>5}.
\end{aligned}$$

In principle, we may proceed to any order $a+b$. However, the calculation of this method becomes complicated rapidly when the order is larger. In Example 6.1, we write one up to $a+b \leqslant 7$.

As the second result, we give the following restriction for a possible form of $u_{(a,b)}(m,n)$.

Theorem 7.1. *Let a and b be positive integers. Then, we express*

$$u_{(a,b)}(m,n) = \gcd(a,b)^{-1} \sum_{\substack{1\leqslant i\leqslant a,\\ 1\leqslant j\leqslant b}} \alpha_{(a,b)}(i,j)\binom{m}{i}\binom{n}{j},$$

where $\alpha_{(a,b)}(i,j)$ are nonnegative integers independent of m and n.

Thus, for each (a,b), if we determine the ab factors $\alpha_{(a,b)}(i,j)$, $u_{(a,b)}(m,n)$ is completely determined. Moreover, by the following claim, the special values $u_{(a,b)}(k,l)$, where $1 \leqslant k \leqslant a$ and $1 \leqslant l \leqslant b$, suffice to determine $u_{(a,b)}(m,n)$ as a function of $(m,n) \in \mathbb{Z}^2_{\geqslant 0}$.

Proposition 8.3. *Let a and b be positive integers. Then, for any $1 \leqslant k \leqslant a$ and $1 \leqslant l \leqslant b$, it holds that*

$$\gcd(a,b)^{-1}\alpha_{(a,b)}(k,l) = \sum_{\substack{1\leqslant i\leqslant k,\\ 1\leqslant j\leqslant l}} (-1)^{i+j+k+l}\binom{k}{i}\binom{l}{j} u_{(a,b)}(i,j).$$

Last, we obtain the general formula of $u_{(a,2)}(m,n)$ as follows.

Theorem 8.9. *For any $a \in \mathbb{Z}_{>0}$, we have*

$$\begin{aligned}
&u_{(a,2)}(m,n)\\
&= \sum_{\frac{a}{2}<k\leqslant a} \left\lceil \frac{2k-a}{2} \right\rceil \binom{2k-a}{\lceil \frac{2k-a}{2} \rceil}\binom{k}{2k-a}\binom{m}{k}\binom{n}{2} \qquad (8)\\
&\quad + \sum_{\frac{a}{2}+1<k\leqslant a} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil \frac{2k-a-1}{2} \rceil} - 2^{2k-a-2} \right\} \binom{k}{2k-a}\binom{m}{k}\binom{n}{1}.
\end{aligned}$$

In the above relation, $\lceil x \rceil$ is the least integer greater than or equal to $x \in \mathbb{Q}$.

1.3. The structure of the paper. As we can see from Theorem 7.1, binomial coefficients play an important role in this paper. In Section 2, we show some equalities and properties which we use later.

In Section 3, we recall the definitions and notations of CSDs.

In Section 4, we introduce a similarity transformation, and consider the relation between a similarity transformation and ordering products.

In Section 5, we give the method to derive $u_{(a,b)}(m,n)$ explicitly.

The latter sections, we give some properties of $u_{(a,b)}(m,n)$. The most contents are independent from each other.

2. Polynomials in binomial coefficients

Binomial coefficients play an important role in exponents of ordering products. So, in this section, we prove some equalities to use later.

Definition 2.1. *Let $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geqslant 0}$. The binomial coefficients are defined by*

$$\binom{m}{k} = \begin{cases} \frac{m(m-1)(m-2)\cdots(m-k+1)}{k(k-1)\cdots 2\cdot 1} & k \geqslant 1, \\ 1 & k = 0. \end{cases} \tag{9}$$

We may also view them as polynomials in an indeterminate m. In this case, the degree of $\binom{m}{k} \in \mathbb{Q}[m]$ is k. Then, we call the following polynomial f in m and n a polynomial in binomial coefficients *(*PBC *for short).*

$$f(m,n) = \sum_{0\leqslant k,l} \alpha_{k,l}\binom{m}{k}\binom{n}{l} \in \mathbb{Q}[m,n] \quad (\alpha_{k,l} \in \mathbb{Z}). \tag{10}$$

Moreover, if $\alpha_{k,l} \geqslant 0$ for any k and l, we call f a nonnegative PBC*. For any nonnegative PBC f, if $f \neq 0$ as a polynomial, we call f a* positive PBC*.*

By definition, a sum of two nonnegative (resp. positive) PBCs is also a nonnegative (resp. positive) PBC. Also, the equalities $(m+1)\binom{m}{k} = (k+1)\binom{m+1}{k+1}$ and $(m-k)\binom{m}{k} = (k+1)\binom{m}{k+1}$ hold.

When m and n are viewed as indeterminates, the set $\left\{\binom{m}{k}\binom{n}{l}\right\}_{0\leqslant k,l}$ is a basis of $\mathbb{Q}[m,n]$. So, every polynomial $f(m,n) \in \mathbb{Q}[m,n]$ can be expressed as $f(m,n) = \sum_{0\leqslant k,l} \gamma_{k,l}\binom{m}{k}\binom{n}{l}$, where $\gamma_{k,l} \in \mathbb{Q}$. By the following claim, we may distinguish PBCs from mere polynomials.

Lemma 2.2. *Let $f(m,n)$ be a polynomial. Then, the following two conditions are equivalent.*

(a) *A polynomial $f(m,n)$ is a PBC.*
(b) *For any $(u,v) \in \mathbb{Z}_{\geqslant 0}$, $f(u,v) \in \mathbb{Z}$ holds.*

Proof. (a) $\Rightarrow$ (b) is immediately shown by $\binom{u}{k} \in \mathbb{Z}$ for any $u,k \in \mathbb{Z}_{\geqslant 0}$. We show $\neg$(a) $\Rightarrow$ $\neg$(b). Suppose that a polynomial $f(m,n) = \sum_{0\leqslant k,l} \alpha_{k,l}\binom{m}{k}\binom{n}{l}$ has non-integer coefficients $\alpha_{i,j} \notin \mathbb{Z}$. We chose (i_0,j_0) as $i+j$ is the smallest among such

(i,j). Consider $f(i_0,j_0)$. Then, we have

$$f(i_0,j_0) = \sum_{0\leqslant k,l} \alpha_{k,l}\binom{i_0}{k}\binom{j_0}{l}$$
$$= \sum_{\substack{0\leqslant k\leqslant i_0,\\ 0\leqslant l\leqslant j_0}} \alpha_{k,l}\binom{i_0}{k}\binom{j_0}{l} \quad \left(\binom{i_0}{k}=0 \text{ if } i_0<k\right) \tag{11}$$
$$= \alpha_{i_0,j_0} + \sum_{\substack{0\leqslant k\leqslant i_0,\\ 0\leqslant l\leqslant j_0,\\ (k,l)\neq(i_0,j_0)}} \alpha_{k,l}\binom{i_0}{k}\binom{j_0}{l}.$$

The second term on the RHS is an integer since $\alpha_{k,l}\in\mathbb{Z}$ for any (k,l) in the sum. By the assumption, the first term α_{i_0,j_0} is not an integer. Thus, $f(i_0,j_0)\notin\mathbb{Z}$ holds. ☐

For any $a,k\in\mathbb{Z}_{\geqslant 0}$, we can easily check the following equalities as polynomials in m and n (e.g. [9, Identity 1, Identity 11, Identity 57]).

$$\binom{m}{k}+\binom{m}{k+1}=\binom{m+1}{k+1}, \tag{12}$$

$$m\binom{m}{k}=(k+1)\binom{m}{k+1}+k\binom{m}{k}, \tag{13}$$

$$\binom{m+n}{a}=\sum_{\substack{0\leqslant a_1,a_2\\ a_1+a_2=a}}\binom{m}{a_1}\binom{n}{a_2}. \tag{14}$$

Moreover, the following equality holds when $m\in\mathbb{Z}_{\geqslant 1}$ and $k\in\mathbb{Z}_{\geqslant 0}$ (e.g. [9, Identity 58]).

$$\sum_{j=0}^{m-1}\binom{j}{k}=\binom{m}{k+1}. \tag{15}$$

The following lemma is a generalization of (13).

Lemma 2.3 (Product formula). *For any $s,r\in\mathbb{Z}_{\geqslant 0}$, the following equalities hold as the element of $\mathbb{Q}[m]$.*

$$\binom{m}{s}\binom{m}{r}=\sum_{0\leqslant k\leqslant s}\binom{s}{k}\binom{r+k}{s}\binom{m}{r+k},$$
$$=\sum_{\max(s,r)\leqslant k\leqslant s+r}\binom{s}{k-r}\binom{k}{s}\binom{m}{k}. \tag{16}$$

Moreover, for any $s,s',r,r'\in\mathbb{Z}_{\geqslant 0}$, $\left\{\binom{m}{s}\binom{n}{s'}\right\}\cdot\left\{\binom{m}{r}\binom{n}{r'}\right\}$ is a positive PBC in m and n.

Proof. The equality

$$\sum_{0\leqslant k\leqslant s}\binom{s}{k}\binom{r+k}{s}\binom{m}{r+k}=\sum_{\max(s,r)\leqslant k\leqslant s+r}\binom{s}{k-r}\binom{k}{s}\binom{m}{k} \tag{17}$$

can be shown by replacing k with $k-r$. We prove the first one by the induction on s. If $s=0$, the equality is obvious. (Both sides are equal to $\binom{m}{r}$.) We assume that

$\binom{m}{s}\binom{m}{r} = \sum_{0\leqslant k\leqslant s}\binom{s}{k}\binom{r+k}{s}\binom{m}{r+k}$ for some s. Then, by the inductive assumption, we have

$$\binom{m}{s+1}\binom{m}{r} = \frac{m-s}{s+1}\binom{m}{s}\binom{m}{r} = \frac{m-s}{s+1}\sum_{0\leqslant k\leqslant s}\binom{s}{k}\binom{r+k}{s}\binom{m}{r+k}. \tag{18}$$

By using (13), we have

$$m\binom{m}{r+k} = (r+k+1)\binom{m}{r+k+1} + (r+k)\binom{m}{r+k}. \tag{19}$$

Thus, (18) can be rearranged to the following form:

$$\begin{aligned}\binom{m}{s+1}\binom{m}{r} &= \frac{1}{s+1}\left\{\sum_{0\leqslant k\leqslant s}\binom{s}{k}\binom{r+k}{s}m\binom{m}{r+k}\right.\\ &\qquad\left. - \sum_{0\leqslant k\leqslant s} s\binom{s}{k}\binom{r+k}{s}\binom{m}{r+k}\right\}\\ &\overset{(19)}{=} \frac{1}{s+1}\left\{\sum_{0\leqslant k\leqslant s}\binom{s}{k}\binom{r+k}{s}(r+k+1)\binom{m}{r+k+1}\right.\\ &\qquad\left. + \sum_{0\leqslant k\leqslant s}(r+k-s)\binom{s}{k}\binom{r+k}{s}\binom{m}{r+k}\right\}.\end{aligned} \tag{20}$$

Now, the first term on the RHS can be written as follows:

$$\begin{aligned}&\sum_{0\leqslant k\leqslant s}\binom{s}{k}\binom{r+k}{s}(r+k+1)\binom{m}{r+k+1}\\ &= \sum_{1\leqslant k\leqslant s+1}\binom{s}{k-1}\binom{r+k-1}{s}(r+k)\binom{m}{r+k}\\ &= \binom{r+s}{s}(r+s+1)\binom{m}{r+s+1}\\ &\qquad + \sum_{1\leqslant k\leqslant s}\binom{s}{k-1}\binom{r+k-1}{s}(r+k)\binom{m}{r+k}.\end{aligned} \tag{21}$$

Since $(r+s+1)\binom{r+s}{s} = (s+1)\binom{r+s+1}{s+1}$ and $(r+k)\binom{r+k-1}{s} = (s+1)\binom{r+k}{s+1}$, we have

$$\begin{aligned}&\sum_{0\leqslant k\leqslant s}\binom{s}{k}\binom{r+k}{s}(r+k+1)\binom{m}{r+k+1}\\ &= (s+1)\binom{r+s+1}{s+1}\binom{m}{r+s+1}\\ &\qquad + \sum_{1\leqslant k\leqslant s}(s+1)\binom{s}{k-1}\binom{r+k}{s+1}\binom{m}{r+k}.\end{aligned} \tag{22}$$

Similarly, by using $(r+k-s)\binom{r+k}{s} = (s+1)\binom{r+k}{s+1}$, the second term on the RHS can be written as follows:

$$\begin{aligned}
&\sum_{0\leqslant k\leqslant s}(r+k-s)\binom{s}{k}\binom{r+k}{s}\binom{m}{r+k}\\
&= (s+1)\sum_{0\leqslant k\leqslant s}\binom{s}{k}\binom{r+k}{s+1}\binom{m}{r+k}\\
&= (s+1)\binom{r}{s+1}\binom{m}{r} + (s+1)\sum_{1\leqslant k\leqslant s}\binom{s}{k}\binom{r+k}{s+1}\binom{m}{r+k}.
\end{aligned}\tag{23}$$

Hence, putting these expressions to the last line of (20), we have

$$\begin{aligned}
&\binom{m}{s+1}\binom{m}{r}\\
&= \binom{r+s+1}{s+1}\binom{m}{r+s+1} + \binom{r}{s+1}\binom{m}{r}\\
&\qquad + \sum_{1\leqslant k\leqslant s}\left\{\binom{s}{k-1} + \binom{s}{k}\right\}\binom{r+k}{s+1}\binom{m}{r+k}\\
&\overset{(12)}{=} \binom{r+s+1}{s+1}\binom{m}{r+s+1} + \binom{r}{s+1}\binom{m}{r}\\
&\qquad + \sum_{1\leqslant k\leqslant s}\binom{s+1}{k}\binom{r+k}{s+1}\binom{m}{r+k}\\
&= \sum_{0\leqslant k\leqslant s+1}\binom{s+1}{k}\binom{r+k}{s+1}\binom{m}{r+k}.
\end{aligned}\tag{24}$$

The second statement follows from the following:

$$\begin{aligned}
&\left\{\binom{m}{s}\binom{n}{s'}\right\}\cdot\left\{\binom{m}{r}\binom{n}{r'}\right\} = \left\{\binom{m}{s}\binom{m}{r}\right\}\cdot\left\{\binom{n}{s'}\binom{n}{r'}\right\}\\
&= \sum_{\max(s,r)\leqslant k\leqslant s+r}\binom{s}{k-r}\binom{k}{s}\binom{m}{k}\sum_{\max(s',r')\leqslant l\leqslant s'+r'}\binom{s'}{k'-r'}\binom{l}{s'}\binom{n}{l}\\
&= \sum_{k,l}\left\{\binom{s}{k-r}\binom{k}{s}\binom{s'}{l-r'}\binom{l}{s'}\right\}\binom{m}{k}\binom{n}{l},
\end{aligned}\tag{25}$$

where $\binom{s}{k-r}\binom{k}{s}\binom{s'}{l-r'}\binom{l}{s'} \geqslant 0$. ☐

Corollary 2.4. *Let $f_1(m,n), f_2(m,n), \ldots, f_r(m,n) \in \mathbb{Q}[m,n]$ be positive PBCs. Then, $\prod_{j=1}^{r} f_j(m,n)$ is also a positive PBC.*

Proof. It suffices to show the case $r = 2$. Let $f(m,n) = \sum_{0\leqslant k,l}\alpha_{k,l}\binom{m}{k}\binom{n}{l}$ and $g(m,n) = \sum_{0\leqslant k',l'}\beta_{k',l'}\binom{m}{k'}\binom{n}{l'}$, where $\alpha_{k,l}, \beta_{k',l'} \in \mathbb{Z}_{\geqslant 0}$. Then,

$$f(m,n)g(m,n) = \sum_{0\leqslant k,k',l,l'}\alpha_{k,l}\beta_{k',l'}\binom{m}{k}\binom{n}{l}\binom{m}{k'}\binom{n}{l'}.\tag{26}$$

By Lemma 2.3, every $\binom{m}{k}\binom{n}{l}\binom{m}{k'}\binom{n}{l'}$ is a positive PBC. So, $f(m,n)g(m,n)$ is a positive PBC. ☐

For any PBC $f(m,n)$, we may consider a composition

$$\binom{f(m,n)}{k} = \frac{f(m,n)(f(m,n)-1)\cdots(f(m,n)-k+1)}{k\cdot(k-1)\cdots 2\cdot 1}. \tag{27}$$

Then, we have a following decomposition.

Lemma 2.5. *Let a be a nonnegative integer, and let $f(m,n)=\sum_{j=1}^{r} u_j\binom{m}{k_j}\binom{n}{l_j}$, with $u_j,k_j,l_j\in\mathbb{Z}_{\geqslant 0}$, be a positive PBC. Then, the following equality holds.*

$$\binom{f(m,n)}{a} = \sum_{\substack{0\leqslant a_j,\\ \sum_j a_j=a}} \prod_{j=1}^{r} \binom{u_j\binom{m}{k_j}\binom{n}{l_j}}{a_j}. \tag{28}$$

Proof. It is immediately shown by (14). □

Every factor of the above decomposition is also a positive PBC.

Lemma 2.6. *Let $a,s,t,u\in\mathbb{Z}_{\geqslant 0}$. Then, the following polynomial*

$$f(m,n)=\binom{u\binom{m}{s}\binom{n}{t}}{a} \tag{29}$$

is a nonnegative PBC in m and n.

Proof. If $u=0$, the claim is immediately shown by definition. Assume $u>0$. For any $p,q\in\mathbb{Z}$, we can easily check that $f(p,q)\in\mathbb{Z}$. Thus, by Lemma 2.2, $f(m,n)$ is a PBC. Now, we fix $u>0$ and $s,t\geqslant 0$. Since its degrees in m and n are sa and ta respectively, we may express

$$\binom{u\binom{m}{s}\binom{n}{t}}{a} = \sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \alpha^a_{k,l}\binom{m}{k}\binom{n}{l} \quad (\alpha^a_{k,l}\in\mathbb{Z}). \tag{30}$$

We show the following two claims.

(a) If $u\binom{k}{s}\binom{l}{t}<a$, then $\alpha^a_{k,l}=0$ holds.
(b) If $u\binom{k}{s}\binom{l}{t}\geqslant a$, then $\alpha^a_{k,l}>0$ holds.

(a) In this case, for any $k',l'\in\mathbb{Z}_{\geqslant 0}$ such that $k'\leqslant k$ and $l'\leqslant l$, we have $u\binom{k'}{s}\binom{l'}{t}\leqslant u\binom{k}{s}\binom{l}{t}<a$. It implies

$$\binom{u\binom{k'}{s}\binom{l'}{t}}{a}=0. \tag{31}$$

Thus, we have

$$0=\sum_{\substack{0\leqslant i\leqslant sa,\\ 0\leqslant j\leqslant ta}} \alpha^a_{i,j}\binom{k'}{i}\binom{l'}{j} = \sum_{\substack{0\leqslant i\leqslant k',\\ 0\leqslant j\leqslant l'}} \alpha^a_{i,j}\binom{k'}{i}\binom{l'}{j} \quad \left(\binom{k'}{i}=0 \text{ if } k'<i.\right) \tag{32}$$

for any $k'\leqslant k$ and $l'\leqslant l$. Considering $(k',l')=(0,0)$, we have $\alpha^a_{0,0}=0$. Next, considering $(k',l')=(1,0)$, we have $\alpha^a_{0,0}+\alpha^a_{1,0}=0$, and it implies $\alpha^a_{1,0}=0$. Repeating this process, we have $\alpha^a_{k,l}=0$.
(b) We show the claim by the induction on a. For $a=0$, we have

$$\binom{u\binom{m}{s}\binom{n}{t}}{0}=1. \tag{33}$$

Thus, $\alpha^0_{0,0} = 1 > 0$ holds. Suppose that the claim holds for some $a \geqslant 0$. We show that $\alpha^{a+1}_{k,l} > 0$ when $u\binom{k}{s}\binom{l}{t} \geqslant a+1$. We have the following equalities:

$$(34)\qquad \begin{aligned}(a+1)\binom{u\binom{m}{s}\binom{n}{t}}{a+1} &= \left(u\binom{m}{s}\binom{n}{t} - a\right)\binom{u\binom{m}{s}\binom{n}{t}}{a}\\ &= \left(u\binom{m}{s}\binom{n}{t} - a\right)\sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \alpha^a_{k,l}\binom{m}{k}\binom{n}{l}\\ &= u\sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \alpha^a_{k,l}\binom{m}{s}\binom{m}{k}\binom{n}{t}\binom{n}{l} - a\sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \alpha^a_{k,l}\binom{m}{k}\binom{n}{l}.\end{aligned}$$

By (16), the first term can be written as

$$(35)\qquad \begin{aligned}&\sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \alpha^a_{k,l}\left\{\binom{m}{s}\binom{m}{k}\right\}\left\{\binom{n}{t}\binom{n}{l}\right\}\\ &\overset{(16)}{=} \sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \alpha^a_{k,l}\left\{\sum_{i=0}^{s}\binom{s}{i}\binom{k+i}{s}\binom{m}{k+i}\right\}\left\{\sum_{j=0}^{t}\binom{t}{j}\binom{l+j}{t}\binom{n}{l+j}\right\}\\ &= \sum_{\substack{0\leqslant i\leqslant s,\\ 0\leqslant j\leqslant t}}\sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \alpha^a_{k,l}\binom{s}{i}\binom{k+i}{s}\binom{t}{j}\binom{l+j}{t}\binom{m}{k+i}\binom{n}{l+j}\\ &= \sum_{\substack{0\leqslant i\leqslant s,\\ 0\leqslant j\leqslant t}}\sum_{\substack{i\leqslant k\leqslant sa+i,\\ j\leqslant l\leqslant ta+j}} \alpha^a_{k-i,l-j}\binom{s}{i}\binom{k}{s}\binom{t}{j}\binom{l}{t}\binom{m}{k}\binom{n}{l}\\ &= \sum_{\substack{0\leqslant i\leqslant s,\\ 0\leqslant j\leqslant t}}\sum_{\substack{0\leqslant k\leqslant sa+i,\\ 0\leqslant l\leqslant ta+j}} \alpha^a_{k-i,l-j}\binom{s}{i}\binom{k}{s}\binom{t}{j}\binom{l}{t}\binom{m}{k}\binom{n}{l}.\end{aligned}$$

In the above last equality, we use $\binom{k}{s} = \binom{l}{t} = 0$ for any $k < i \leqslant s$ and $l < j \leqslant t$. We decompose the region of the latter sum as follows:

$$(36)\qquad \sum_{\substack{0\leqslant k\leqslant sa+i,\\ 0\leqslant l\leqslant ta+j}} = \sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} + \sum_{\substack{sa<k\leqslant sa+i,\\ \text{or } ta<l\leqslant ta+j}}.$$

Namely, we consider

$$(37)\qquad \begin{aligned}&\sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \alpha^a_{k,l}\left\{\binom{m}{s}\binom{m}{k}\right\}\left\{\binom{n}{t}\binom{n}{l}\right\}\\ &\overset{(35)}{=} \sum_{\substack{0\leqslant i\leqslant s,\\ 0\leqslant j\leqslant t}}\sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \alpha^a_{k-i,l-j}\binom{s}{i}\binom{k}{s}\binom{t}{j}\binom{l}{t}\binom{m}{k}\binom{n}{l}\\ &\qquad + \sum_{\substack{0\leqslant i\leqslant s,\\ 0\leqslant j\leqslant t}}\sum_{\substack{sa<k\leqslant sa+i,\\ \text{or } ta<l\leqslant ta+j}} \alpha^a_{k-i,l-j}\binom{s}{i}\binom{k}{s}\binom{t}{j}\binom{l}{t}\binom{m}{k}\binom{n}{l}.\end{aligned}$$

Then, in the first term, i, j, k, and l are independent. Thus, we can exchange the order of the sum. In the second term, we have

$$\sum_{\substack{0\leqslant i\leqslant s,\\ 0\leqslant j\leqslant t}} \sum_{\substack{sa<k\leqslant sa+i,\\ \text{or } ta<l\leqslant ta+j}} = \sum_{\substack{sa<k\leqslant s(a+1),\\ \text{or } ta<l\leqslant t(a+1)}} \sum_{\substack{k-sa\leqslant i\leqslant s,\\ l-ta\leqslant j\leqslant t}}. \tag{38}$$

Thus, we have

$$\begin{aligned}
&\sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \alpha^a_{k,l}\left\{\binom{m}{s}\binom{m}{k}\right\}\left\{\binom{n}{t}\binom{n}{l}\right\}\\
&= \sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \left\{\sum_{\substack{0\leqslant i\leqslant s,\\ 0\leqslant j\leqslant t}} \alpha^a_{k-i,l-j}\binom{s}{i}\binom{k}{s}\binom{t}{j}\binom{l}{t}\right\}\binom{m}{k}\binom{n}{l}\\
&\qquad + \sum_{\substack{sa<k\leqslant s(a+1),\\ \text{or } ta<l\leqslant t(a+1)}} \left\{\sum_{\substack{k-sa\leqslant i\leqslant s,\\ l-ta\leqslant j\leqslant t}} \alpha^a_{k-i,l-j}\binom{s}{i}\binom{k}{s}\binom{t}{j}\binom{l}{t}\right\}\binom{m}{k}\binom{n}{l}.
\end{aligned} \tag{39}$$

Putting the last expression to the last line of (34), we have

$$\begin{aligned}
&(a+1)\binom{u\binom{m}{s}\binom{n}{t}}{a+1}\\
&= \sum_{\substack{0\leqslant k\leqslant sa,\\ 0\leqslant l\leqslant ta}} \left\{u\sum_{\substack{0\leqslant i\leqslant s,\\ 0\leqslant j\leqslant t}} \alpha^a_{k-i,l-j}\binom{s}{i}\binom{k}{s}\binom{t}{j}\binom{l}{t} - a\alpha^a_{k,l}\right\}\binom{m}{k}\binom{n}{l}\\
&\qquad + \sum_{\substack{sa<k\leqslant s(a+1),\\ \text{or } ta<l\leqslant t(a+1)}} \left\{u\sum_{\substack{k-sa\leqslant i\leqslant s,\\ l-ta\leqslant j\leqslant t}} \alpha^a_{k-i,l-j}\binom{s}{i}\binom{k}{s}\binom{t}{j}\binom{l}{t}\right\}\binom{m}{k}\binom{n}{l}.
\end{aligned} \tag{40}$$

If $k > sa$ or $l > ta$, then we have

$$(a+1)\alpha^{a+1}_{k,l} = u\sum_{\substack{k-sa\leqslant i\leqslant s,\\ l-ta\leqslant j\leqslant t}} \alpha^a_{k-i,l-j}\binom{s}{i}\binom{k}{s}\binom{t}{j}\binom{l}{t} > 0. \tag{41}$$

This is because, for $i = k - sa$ and $j = l - ta$, $\alpha^a_{k-i,l-j} = \alpha^a_{sa,ta} > 0$. If $k \leqslant sa$ and $l \leqslant ta$, then we have

$$\begin{aligned}
(a+1)\alpha^{a+1}_{k,l} &= u\sum_{\substack{0\leqslant i\leqslant s,\\ 0\leqslant j\leqslant t}} \alpha^a_{k-i,l-j}\binom{s}{i}\binom{k}{s}\binom{t}{j}\binom{l}{t} - a\alpha^a_{k,l}\\
&= \alpha^a_{k,l}\left\{u\binom{k}{s}\binom{l}{t} - a\right\} + u\sum_{\substack{0\leqslant i\leqslant s,\\ 0\leqslant j\leqslant t,\\ (i,j)\neq(0,0)}} \alpha^a_{k-i,l-j}\binom{s}{i}\binom{k}{s}\binom{t}{j}\binom{l}{t}.
\end{aligned} \tag{42}$$

is positive since $u\binom{k}{s}\binom{l}{t} - a \geqslant 1$ and $\alpha^a_{k,l} > 0$. Hence, $\alpha^{a+1}_{k,l} > 0$ when $u\binom{k}{s}\binom{l}{t} \geqslant a+1$. This completes the proof. □

We have the main conclusion in this section.

Proposition 2.7. *If f, g, and h are positive PBCs, then $f(g(m,n), h(m,n))$ is also a positive PBC.*

Proof. Since the sum of positive PBCs is also a positive PBC, it suffices to show the case of $f(m,n) = \binom{m}{a}\binom{n}{b}$ for $a, b \in \mathbb{Z}_{\geqslant 0}$. In this case, we have $f(g(m,n), h(m,n)) =$

$\binom{g(m,n)}{a}\binom{h(m,n)}{b}$. Moreover, by Corollary 2.4, it suffices to show that $\binom{g(m,n)}{a}$ is a positive PBC. It is immediately shown by Lemma 2.5, Lemma 2.6, and Corollary 2.4. □

3. Dilogarithm elements and cluster scattering diagrams of rank 2

In this section, we summarize the definitions and properties of cluster scattering diagrams which were introduced by [2]. We concentrate on them of rank 2, and most notations mainly follow from [5].

3.1. Dilogarithm elements and ordered products.

Definition 3.1 (Fixed data and seed). *We define a* fixed data $\Gamma = (N, N^\circ, \{,\}, \delta_1, \delta_2)$ *and a* seed $\mathfrak{s} = (e_1, e_2)$ *as follows:*

- *A lattice $N \cong \mathbb{Z}^2$ with a skew-symmetric bilinear form $\{,\} : N \times N \to \mathbb{Q}$.*
- *Positive integers $\delta_1, \delta_2 \in \mathbb{Z}_{>0}$, and a basis (e_1, e_2) of N. They satisfy $\{\delta_i e_i, e_j\} \in \mathbb{Z}$ for $i, j = 1, 2$.*
- *A sublattice $N^\circ = \mathbb{Z}(\delta_1 e_1) \oplus \mathbb{Z}(\delta_2 e_2) \subset N$.*

For given fixed data Γ and seed $\mathfrak{s}$ as above, we have dual lattices $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and $M^\circ = \mathrm{Hom}_{\mathbb{Z}}(N^\circ, \mathbb{Z})$, and we define a real vector space $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. We regard

$$M \subset M^\circ \subset M_{\mathbb{R}}. \tag{43}$$

Also, we have the dual basis (e_1^*, e_2^*) of M. Let $f_i = e_i^*/\delta_i$. Then, (f_1, f_2) is a basis of M°. We define the canonical pairing

$$\begin{gathered}\langle\ ,\ \rangle : M_{\mathbb{R}} \times N \to \mathbb{R},\\ \left\langle \sum_{i=1}^{2} \alpha_i f_i, \sum_{j=1}^{2} \beta_j e_j \right\rangle = \sum_{i=1}^{2} \delta_i^{-1} \alpha_i \beta_i.\end{gathered} \tag{44}$$

Let

$$N^+ = \left\{ \sum_{i=1}^{2} a_i e_i \;\middle|\; a_i \in \mathbb{Z}_{\geqslant 0}, \sum_{i=1}^{2} a_i > 0 \right\}. \tag{45}$$

We define the degree function $\deg : N^+ \to \mathbb{Z}_{>0}$ as

$$\deg\left(\sum_{i=1}^{2} a_i e_i\right) = \sum_{i=1}^{2} a_i. \tag{46}$$

For any integer $l > 0$, we define the following sets.

$$\begin{gathered}(N^+)^{\leqslant l} = \{n \in N^+ \mid \deg(n) \leqslant l\}, \quad (N^+)^{>l} = \{n \in N^+ \mid \deg(n) > l\},\\ N^+_{\mathrm{pr}} = \{n \in N^+ \mid \text{For any } j \in \mathbb{Z}_{>1}, n/j \notin N^+\}.\end{gathered} \tag{47}$$

Definition 3.2 (Normalization factor). *Let $\mathfrak{s}$ be a seed for a fixed data Γ. For any $n \in N^+$, we define $\delta(n)$ as the smallest positive rational number such that $\delta(n)n \in N^\circ$, and we call it the* normalization factor *of n with respect to $(\Gamma, \mathfrak{s})$.*

Definition 3.3 (Structure group). *Let $\mathfrak{g}$ be an N^+*-graded Lie algebra *over $\mathbb{Q}$ with generators X_n $(n \in N^+)$ as follows:*

$$\mathfrak{g} = \bigoplus_{n \in N^+} \mathfrak{g}_n, \quad \mathfrak{g}_n = \mathbb{Q}X_n, \tag{48}$$

$$[X_n, X_{n'}] = \{n, n'\}\, X_{n+n'}. \tag{49}$$

For each integer $l \in \mathbb{Z}_{>0}$, we define an ideal $\mathfrak{g}^{>l}$ of $\mathfrak{g}$ as

$$\mathfrak{g}^{>l} = \bigoplus_{n \in (N^+)^{>l}} \mathfrak{g}_n, \tag{50}$$

and we define the quotient

$$\mathfrak{g}^{\leqslant l} = \mathfrak{g}/\mathfrak{g}^{>l}. \tag{51}$$

We define the group

$$G^{\leqslant l} = \left\{ \exp(X) \mid X \in \mathfrak{g}^{\leqslant l} \right\} \tag{52}$$

whose product is given by the Baker-Campbell-Hausdorff formula *(e.g.* [3, §V.5]*).*

$$\begin{aligned} &\exp(X)\exp(Y) \\ &= \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots \right). \end{aligned} \tag{53}$$

Since the canonical projection $\pi_{l',l} : \mathfrak{g}^{\leqslant l'} \to \mathfrak{g}^{\leqslant l}$ $(l' > l)$ induces the canonical projection $\pi_{l',l} : G^{\leqslant l'} \to G^{\leqslant l}$, we can consider the inverse limit of $\{\pi_{l+1,l}\}$, and we obtain a group

$$G = \lim_{\longleftarrow} G^{\leqslant l} \tag{54}$$

with the canonical projection $\pi_l : G \to G^{\leqslant l}$. This group G is called the structure group *corresponding to $(\Gamma, \mathfrak{s})$. We define $G^{>l} = \operatorname{Ker} \pi_l$.*

Definition 3.4. *For any $g, g' \in G$ and $l \in \mathbb{Z}_{\geqslant 1}$, we write $g \equiv g' \mod G^{>l}$ when $\pi_l(g) = \pi_l(g')$, and we say g is equal to g' in $G^{\leqslant l}$.*

By definition, for any $g, g' \in G$, $g = g$ is equivalent to $g \equiv g' \mod G^{>l}$ for any $l \in \mathbb{Z}_{\geqslant 1}$.

For any $g = \exp(X) \in G$ and $c \in \mathbb{Q}$, we define $g^c = \exp(cX)$. Then, $g^0 = \mathrm{id}$ and $g^c g^{c'} = g^{c+c'}$ hold.

Every structure group G is determined by a fixed data Γ and a seed $\mathfrak{s}$. However, there is some redundancy.

Definition 3.5 (Exchange matrix). *For any fixed data Γ and seed $\mathfrak{s}$, define a* exchange matrix *$B_{\Gamma,\mathfrak{s}}$ associated to Γ and $\mathfrak{s}$ by*

$$B_{\Gamma,\mathfrak{s}} = \begin{pmatrix} 0 & \{\delta_1 e_1, e_2\} \\ \{\delta_2 e_2, e_1\} & 0 \end{pmatrix}. \tag{55}$$

Proposition 3.6 ([5, Prop. 1.23]). *Let Γ and Γ' be fixed data, and let $\mathfrak{s}$ and $\mathfrak{s}'$ be seeds for Γ and Γ', respectively. Let G and G' be structure groups corresponding to $(\Gamma, \mathfrak{s})$ and $(\Gamma', \mathfrak{s}')$, respectively. If $B_{\Gamma,\mathfrak{s}} = B_{\Gamma',\mathfrak{s}'}$, then G and G' are isomorphic.*

If we focus on a structure group, it suffices to consider the case of $\{e_2, e_1\} = 1$. For any fixed data $\Gamma' = (N, (N^\circ)', \{,\}', \delta_1', \delta_2')$ and seed $\mathfrak{s} = (e_1, e_2)$ such that $\{e_2, e_1\}' > 0$ (if $\{e_2, e_1\}' < 0$, interchange e_1 and e_2), we define another fixed data $\Gamma = (N, N^\circ, \{,\}, \delta_1, \delta_2)$ by

$$\begin{aligned} &\delta_1 = -\{\delta_1' e_1, e_2\}', \quad \delta_2 = \{\delta_2' e_2, e_1\}', \quad \{e_2, e_1\} = 1, \\ &N^\circ = \{a(\delta_1 e_1) + b(\delta_2 e_2) \mid a, b \in \mathbb{Z}\} \subset N. \end{aligned} \tag{56}$$

Then, $\mathfrak{s}$ is also a seed for Γ. Moreover, because of Proposition 3.6, the structure group corresponding to $(\Gamma, \mathfrak{s})$ is isomorphic to the one corresponding to $(\Gamma', \mathfrak{s})$. From now on, we fix a seed (e_1, e_2) satisfying $\{e_2, e_1\} = 1$. Then, we may view $N = \mathbb{Z}^2$, $N^+ = \mathbb{Z}^2_{\geqslant 0} \backslash \{(0,0)\}$, and $M_{\mathbb{R}} = \mathbb{R}^2$ as follows:

$$\begin{aligned} &N \to \mathbb{Z}^2, && ae_1 + be_2 \mapsto (a, b), \\ &M_{\mathbb{R}} \to \mathbb{R}^2 && \alpha f_1 + \beta f_2 \mapsto (\alpha, \beta). \end{aligned} \tag{57}$$

In the above notations and our assumption, the skew-bilinear form $\{,\} : N \times N \to \mathbb{Q}$ may be viewed as

$$\{(a,b),(c,d)\} = bc - ad = -\begin{vmatrix} a & c \\ b & d \end{vmatrix}. \tag{58}$$

Moreover, for any $(a, b) \in N^+$, the normalization factor $\delta(a, b)$ only depends on the data of (δ_1, δ_2). So, we call it the normalization factor with respect to (δ_1, δ_2). In particular, when $(\delta_1, \delta_2) = (1, 1)$, we write it by $d(a, b)$. This is given by

$$d(a,b) = \frac{1}{\gcd(a,b)}. \tag{59}$$

Definition 3.7 (Dilogarithm element). *For each $n \in N^+$, we define the* dilogarithm element *for n as*

$$\Psi[n] = \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} X_{jn}\right) \in G. \tag{60}$$

Let $n \in N^+$. Then, the lowest term of $\Psi[n]$ is X_n. Thus, for any positive integer $l < \deg(n)$, it holds that $\pi_l(\Psi[n]) = \mathrm{id}$.

In this paper, the following relations may be viewed as fundamental relations.

Proposition 3.8 ([5, Prop. III.1.14]). *Let $n, n' \in N^+$. Then, the following relations hold.*

(a) *If $\{n', n\} = 0$, then for any $\gamma, \gamma' \in \mathbb{Q}$, it holds that*

$$\Psi[n']^{\gamma'} \Psi[n]^{\gamma} = \Psi[n]^{\gamma} \Psi[n']^{\gamma'}. \tag{61}$$

(b) (pentagon relation) *If $\{n', n\} = \gamma^{-1} \in \mathbb{Q} \backslash \{0\}$, it holds that*

$$\Psi[n']^{\gamma} \Psi[n]^{\gamma} = \Psi[n]^{\gamma} \Psi[n + n']^{\gamma} \Psi[n']^{\gamma}. \tag{62}$$

Also, by (53), the following relation is immediately shown for any $n, n' \in N^+$, $\gamma, \gamma'' \in \mathbb{Q}$, and $l < \deg(n + n')$.

$$\Psi[n']^{\gamma'} \Psi[n]^{\gamma} \equiv \Psi[n]^{\gamma} \Psi[n']^{\gamma'} \mod G^{>l}. \tag{63}$$

Now, we introduce the notation

$$\begin{bmatrix} a \\ b \end{bmatrix} = \Psi[(a,b)] \quad ((a,b) \in N^+). \tag{64}$$

Also, we define the degree $\deg\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ by $\deg\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = a + b$.

Definition 3.9. *We define the total order $\leqslant$ on N^+ as follows:*

$$\begin{aligned} (a,b) \leqslant (c,d) \Leftrightarrow &\{(a,b),(c,d)\} < 0, \\ &\text{or there exists } k \in \mathbb{Q}_{\geqslant 1} \text{ such that } (c,d) = k(a,b). \end{aligned} \tag{65}$$

Moreover, we also write it as $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \leqslant \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$.

Note that, if $(c,d)=k(a,b)$, then $\{(a,b),(c,d)\}=0$. Thus, if $(a,b)\leqslant(c,d)$, then $\{(a,b),(c,d)\}\leqslant 0$ holds.

Let J be an ordered countable set, and let

$$D=\left\{\left[\begin{smallmatrix}a_j\\ b_j\end{smallmatrix}\right]^{u_j}\mid j\in J,(a_j,b_j)\in N^+,u_j\in\mathbb{Q}\right\}$$

be a set. Suppose that for any $l\in\mathbb{Z}_{\geqslant 1}$, the set $J^{\leqslant l}=\{j\in J\mid \deg(a_j,b_j)\leqslant l\}$ is finite, and we write $J^{\leqslant l}=\{j_0^l<j_1^l<\cdots<j_s^l\}$. Then, we define

$$\prod_{j\in J}\begin{bmatrix}a_j\\ b_j\end{bmatrix}^{u_j}=\lim_{l\to\infty}\left(\begin{bmatrix}a_{j_0^l}\\ b_{j_0^l}\end{bmatrix}^{u_{j_0^l}}\begin{bmatrix}a_{j_1^l}\\ b_{j_1^l}\end{bmatrix}^{u_{j_1^l}}\cdots\begin{bmatrix}a_{j_s^l}\\ b_{j_s^l}\end{bmatrix}^{u_{j_s^l}}\right). \tag{66}$$

Definition 3.10. *Let J be an ordered countable set. Then, a product*

$$\prod_{j\in J}\begin{bmatrix}a_j\\ b_j\end{bmatrix}^{u_j}\quad\big((a_j,b_j)\in N^+,u_j\in\mathbb{Q}\big) \tag{67}$$

is said to be ordered *(resp.* anti-ordered*) if $\left[\begin{smallmatrix}a_i\\ b_i\end{smallmatrix}\right]\leqslant\left[\begin{smallmatrix}a_j\\ b_j\end{smallmatrix}\right]$ (resp. $\left[\begin{smallmatrix}a_i\\ b_i\end{smallmatrix}\right]\geqslant\left[\begin{smallmatrix}a_j\\ b_j\end{smallmatrix}\right]$) for any $i<j$ in J.*

In particular, if $(a_i,b_i)<(a_j,b_j)$ for any $i<j\in J$, we say that the product $\prod_{j\in J}\left[\begin{smallmatrix}a_j\\ b_j\end{smallmatrix}\right]^{u_j}$ is *strongly ordered*, and we write it by

$$\overrightarrow{\prod_{j\in J}}\begin{bmatrix}a_j\\ b_j\end{bmatrix}^{u_j}. \tag{68}$$

Every ordered product becomes the strongly ordered product by gathering same dilogarithm elements.

3.2. Cluster scattering diagrams.

We introduce the notation

$$\sigma(m)=\mathbb{R}_{\geqslant 0}m\quad(m\in M_{\mathbb{R}}\backslash\{0\}). \tag{69}$$

Definition 3.11 (Wall). *A* wall *$w=(\mathfrak{d},g)_n$ for a seed $\mathfrak{s}$ consists of the following:*

- *$n\in N^+_{\mathrm{pr}}$ (The* normal vector *of w).*
- *$\mathfrak{d}=\sigma(m)$ or $\mathfrak{d}=\sigma(m)\cup\sigma(-m)$, where $m\in M_{\mathbb{R}}\backslash\{0\}$ satisfies $\langle m,n\rangle=0$ (The* support *of w).*
- *$g\in G$ is expressed as the following form (The* wall element *of w).*

$$g=\exp\left(\sum_{j=1}^{\infty}c_jX_{jn}\right)\quad(c_j\in\mathbb{Q}). \tag{70}$$

It is known that the product of dilogarithm elements $\prod_{k=1}^{\infty}\Psi[kn]^{a_k}$ $(a_k\in\mathbb{Q})$ can be expressed as the form of (70) [5, Prop. 1.13].

Definition 3.12. *The group homomorphism $p^*:N\to M^\circ$ is defined by*

$$p^*(n)=\{\cdot,n\}. \tag{71}$$

Then, a wall $w=(\mathfrak{d},g)_n$ is incoming *(resp.* outgoing*) if $p^*(n)\in\mathfrak{d}$ (resp. $p^*(n)\notin\mathfrak{d}$).*

Definition 3.13 (Scattering diagram). *A* scattering diagram *$\mathfrak{D}=\{w_\lambda\}_{\lambda\in\Lambda}$ is a collection of walls such that the following conditions hold.*

- *The index set Λ is countable.*
- *For each integer $l\in\mathbb{Z}_{>0}$, there are only finitely many walls w_λ such that $\pi_l(g_\lambda)\neq\mathrm{id}$.*

We define the support *of $\mathfrak{D}$ by*

$$\mathrm{Supp}(\mathfrak{D})=\bigcup_{\lambda\in\Lambda}\mathfrak{d}_\lambda. \tag{72}$$

Definition 3.14 (Admissible curve). *Let $\mathfrak{D} = \{w_\lambda = (\mathfrak{d}_\lambda, g_\lambda)_{n_\lambda}\}_{\lambda\in\Lambda}$ be a scattering diagram. We say that a smooth curve $\gamma : [0,1] \to M_\mathbb{R}$ is* admissible *for $\mathfrak{D}$ if it satisfies the following conditions:*

- *$\gamma(0), \gamma(1) \notin \mathrm{Supp}(\mathfrak{D})$, and $\gamma(t) \neq 0$ for any t.*
- *If γ and $\mathfrak{d}_\lambda$ intersect, then γ intersects $\mathfrak{d}_\lambda$ transversally.*

Let γ be an admissible curve for $\mathfrak{D}$. For each positive integer l, there exist only finitely many walls $w_i = (\mathfrak{d}_i, g_i)_{n_i}$ $(i = 1, 2, \ldots, s)$ such that γ intersects $\mathfrak{d}_i$ and $\pi_l(g_i) \neq \mathrm{id}$. Let t_i be a real number such that $\gamma(t_i)$ be the intersection of γ and $\mathfrak{d}_i$, and assume

$$0 < t_1 \leqslant t_2 \leqslant \cdots \leqslant t_s < 1. \tag{73}$$

We define the intersection sign ϵ_i by

$$\epsilon_i = \begin{cases} 1 & \langle n_i, \gamma'(t_i)\rangle < 0, \\ -1 & \langle n_i, \gamma'(t_i)\rangle > 0. \end{cases} \tag{74}$$

Definition 3.15 (Path ordered product). *As the above notations, we define the* path ordered product $\mathfrak{p}_{\gamma,\mathfrak{D}}$ *as*

$$\mathfrak{p}_{\gamma,\mathfrak{D}} = \lim_{l\to\infty} (g_s^{\epsilon_s} \cdots g_1^{\epsilon_1}) \in G. \tag{75}$$

Definition 3.16 (Equivalence). *Let $\mathfrak{D}$ and $\mathfrak{D}'$ be scattering diagrams. We say that $\mathfrak{D}$ and $\mathfrak{D}'$ are* equivalent *if $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{\gamma,\mathfrak{D}'}$ for any admissible curve γ for both $\mathfrak{D}$ and $\mathfrak{D}'$.*

Definition 3.17 (Consistency). *Let $\mathfrak{D}$ be a scattering diagram. We say that $\mathfrak{D}$ is* consistent *if $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{\gamma',\mathfrak{D}}$ for any admissible curve γ and γ' with the same endpoints.*

Theorem 3.18 ([2, Thm. 1.21]). *For any fixed data Γ and seed $\mathfrak{s}$, there exists a consistent scattering diagram $\mathfrak{D}_{\Gamma,\mathfrak{s}}$ satisfying the following properties:*

- *Both $(e_1^\perp, \Psi[e_1]^{\delta_1})_{e_1}$ and $(e_2^\perp, \Psi[e_2]^{\delta_2})_{e_2}$ are incoming walls of $\mathfrak{D}_{\Gamma,\mathfrak{s}}$.*
- *Every wall except for $(e_1^\perp, \Psi[e_1]^{\delta_1})_{e_1}$ and $(e_2^\perp, \Psi[e_2]^{\delta_2})_{e_2}$ is outgoing.*

Moreover, a scattering diagram satisfying the above properties is unique up to the equivalence.

The above scattering diagram $\mathfrak{D}_{\Gamma,\mathfrak{s}}$ is called a *cluster scattering diagram* (*CSD* for short) for Γ and $\mathfrak{s}$. Recall the transformation (56) of Γ. Under our assumption $\{e_2, e_1\} = 1$, a CSD $\mathfrak{D}_{\Gamma,\mathfrak{s}}$ essentially depends on only positive integers δ_1 and δ_2. So, we write $\mathfrak{D}_{\Gamma,\mathfrak{s}}$ by $\mathfrak{D}_{\delta_1,\delta_2}$.

Let (f_1, f_2) be the dual basis of $(\delta_1 e_1, \delta_2, e_2)$, and let $\sigma = \{af_1 + bf_2 \in M_\mathbb{R} \mid a > 0, b < 0\}$. Then, the second condition in Theorem 3.18 implies that there are no outgoing walls in the region $M_\mathbb{R}\backslash\sigma$. Thus, we may consider the admissible curves $\mathfrak{p}_{\gamma_+,\mathfrak{D}_{\delta_1,\delta_2}}$ and $\mathfrak{p}_{\gamma_-,\mathfrak{D}_{\delta_1,\delta_2}}$ as Figure 1. Moreover, $\mathfrak{p}_{\gamma_+,\mathfrak{D}_{\delta_1,\delta_2}} = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^{\delta_2}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{\delta_1}$ holds. Thus, the consistency condition of a CSD is equivalent to $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^{\delta_2}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{\delta_1} = \mathfrak{p}_{\gamma_-,\mathfrak{D}_{\delta_1,\delta_2}}$.

For any $\delta_1, \delta_2 \in \mathbb{Z}_{>0}$, all wall elements in $\mathfrak{D}_{\delta_1,\delta_2}$ can be expressed as the form $\Psi[n]^{s\delta(n)}$, where $n \in N^+$ and $s \in \mathbb{Z}_{>0}$. The resulting CSD is called the *positive realization* [5]. The following theorem is a key for a positive realization.

Theorem 3.19 ([5, Prop. III.5.4]). *Let*

$$C^{\mathrm{in}} = \Psi[n'_k]^{s'_k\delta(n'_k)} \cdots \Psi[n'_1]^{s'_1\delta(n'_1)} \quad (n'_j \in N^+, s'_j \in \mathbb{Z}_{>0}) \tag{76}$$

be any finite anti-ordered product. Then, there exists an unique strongly ordered product

$$C^{\mathrm{out}} = \overrightarrow{\prod_j} \Psi[n_j]^{s_j\delta(n_j)} \quad (n_j \in N^+, s_j \in \mathbb{Z}_{>0}) \tag{77}$$

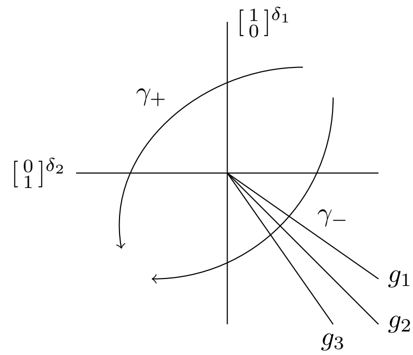

Figure 1. Cluster scattering diagram
$\mathfrak{p}_{\gamma_+,\mathfrak{D}} = [\begin{smallmatrix}0\\1\end{smallmatrix}]^{\delta_2}[\begin{smallmatrix}1\\0\end{smallmatrix}]^{\delta_1}$, $\mathfrak{p}_{\gamma_-,\mathfrak{D}} = [\begin{smallmatrix}1\\0\end{smallmatrix}]^{\delta_1} g_3 g_2 g_1 [\begin{smallmatrix}0\\1\end{smallmatrix}]^{\delta_2}$.
Moreover, g_i can be expressed as $\prod_{k=1}^{\infty} \Psi[kn_i]^{u_k}$, where $n_i \in N^+$ is the normal vector and $u_k \in \mathbb{Q}$.

which is equal to C^{in} as the element of G. Moreover, n_j satisfies $n'_1 \leqslant n_j \leqslant n'_k$.

In the above notations, we call C^{out} the *strongly ordered product expression* of C^{in}. In particular, for any $\delta_1, \delta_2 \in \mathbb{Z}^2_{>0}$, an anti-ordered product $[\begin{smallmatrix}0\\1\end{smallmatrix}]^{\delta_2}[\begin{smallmatrix}1\\0\end{smallmatrix}]^{\delta_1}$ is expressed as the (possibly infinite) strongly ordered product

$$\begin{bmatrix}0\\1\end{bmatrix}^{\delta_2}\begin{bmatrix}1\\0\end{bmatrix}^{\delta_1} = \overrightarrow{\prod_{j\in J}}\begin{bmatrix}a_j\\b_j\end{bmatrix}^{s_{(a_j,b_j)}\delta(a_j,b_j)}. \tag{78}$$

Then, the CSD $\mathfrak{D}_{\delta_1,\delta_2}$ is described as

$$\mathfrak{D}_{\delta_1,\delta_2} = \{\mathbf{w}_{e_1}, \mathbf{w}_{e_2}\} \cup \{\mathbf{w}_{(a,b)}\}_{(a,b)\in N^+_{\mathrm{pr}}\setminus\{(1,0),(0,1)\}},$$

where $\mathbf{w}_{e_1} = (e_1^{\perp}, [\begin{smallmatrix}1\\0\end{smallmatrix}]^{\delta_1})_{e_1}$, $\mathbf{w}_{e_2} = (e_2^{\perp}, [\begin{smallmatrix}0\\1\end{smallmatrix}]^{\delta_2})_{e_2}$, and for each pair $(a,b) \in N^+_{\mathrm{pr}}\setminus\{(1,0),(0,1)\}$, $\mathbf{w}_{(a,b)} = (\sigma(\delta_2 b, -\delta_1 a), g_{(a,b)})_{(a,b)}$ with

$$g_{(a,b)} = \prod_{k=1}^{\infty}\begin{bmatrix}ka\\kb\end{bmatrix}^{s_{(ka,kb)}\delta(ka,kb)}. \tag{79}$$

In this CSD, (78) coincides with the consistency condition. The main purpose of this paper is to describe these exponents $s_{(ka,kb)}\delta(ka,kb)$ explicitly.

The following property is known.

Lemma 3.20 ([5, Prop. III.1.24. (b)]). *Let δ_1 and δ_2 be positive integers. Then, the ordered product of $[\begin{smallmatrix}0\\1\end{smallmatrix}]^{\delta_2}[\begin{smallmatrix}1\\0\end{smallmatrix}]^{\delta_1}$ is expressed as*

$$\begin{bmatrix}1\\0\end{bmatrix}^{\delta_1}\begin{bmatrix}\delta_1\\1\end{bmatrix}^{\delta_2}\cdots\begin{bmatrix}1\\\delta_2\end{bmatrix}^{\delta_1}\begin{bmatrix}0\\1\end{bmatrix}^{\delta_2}. \tag{80}$$

4. Similarity transformations in the structure group

One of the most important properties in CSDs is the consistency condition (78), which is the relation in the structure group G. By Theorem 3.19, we may describe it by dilogarithm elements. In this section, we introduce an action on G, and we apply it for the consistency conditions.

We fix a seed (e_1, e_2) satisfying $\{e_2, e_1\} = 1$, and we view $N = \mathbb{Z}^2$ and $N^+ = \mathbb{Z}^2_{\geqslant 0}\setminus\{(0,0)\}$ as (57). For any $(\begin{smallmatrix}a\\b\end{smallmatrix}) \in N$ and matrix $F \in \mathrm{Mat}_2(\mathbb{Z})$, $F(\begin{smallmatrix}a\\b\end{smallmatrix}) \in N$ is defined by the usual matrix multiplication.

This definition and Proposition 4.2 are due to Peigen Cao.

Definition 4.1 (Similarity transformations). *Let* $F \in \mathrm{Mat}_2(\mathbb{Z}_{\geqslant 0})$ *with* $|F| \neq 0$*. Then, we define the linear action of* F *on* $\mathfrak{g}$ *by*

$$FX = \frac{1}{|F|} \sum_{n \in N^+} c_n X_{Fn} \quad \big(X = \sum_{n \in N^+} c_n X_n \in \mathfrak{g},\ c_n \in \mathbb{Q}\big). \tag{81}$$

Moreover, we define the action of F *on* G *by*

$$Fg = \exp(FX) \quad (g = \exp(X) \in G,\ X \in \mathfrak{g}). \tag{82}$$

We call it the similarity transformation *on* G *by* F*.*

Proposition 4.2. *Let* $F \in \mathrm{Mat}_2(\mathbb{Z}_{\geqslant 0})$ *with* $|F| \neq 0$*. Then, the following statements hold.*

(a) *For any* $n, n' \in N^+$*, it holds that*

$$\{Fn, Fn'\} = |F|\{n, n'\}. \tag{83}$$

(b) *For any* $X, Y \in \mathfrak{g}$*, it holds that*

$$F[X, Y] = [FX, FY]. \tag{84}$$

(c) *For any* $g, g' \in G$*, it holds that*

$$F(gg') = (Fg)(Fg'). \tag{85}$$

Namely, the similarity transformation by F *is a group homomorphism on* G*.*

Proof. (a) Let $n = (a, b)$ and $n' = (c, d)$. We have

$$\begin{aligned} &\left\{F\begin{pmatrix}a\\b\end{pmatrix}, F\begin{pmatrix}c\\d\end{pmatrix}\right\} \overset{(58)}{=} -\left|F\begin{pmatrix}a\\b\end{pmatrix}\ F\begin{pmatrix}c\\d\end{pmatrix}\right| = -|F|\begin{vmatrix}a & c\\ b & d\end{vmatrix} \\ &= |F|\left\{\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}c\\d\end{pmatrix}\right\}. \end{aligned} \tag{86}$$

(b) By the linearity of this action, it suffices to show that $F[X_n, X_{n'}] = [FX_n, FX_{n'}]$ for any $n, n' \in N^+$. We have

$$\begin{aligned} [FX_n, FX_{n'}] &\overset{(81)}{=} \left[\frac{1}{|F|}X_{Fn}, \frac{1}{|F|}X_{Fn'}\right] = \frac{1}{|F|^2}[X_{Fn}, X_{Fn'}] \\ &\overset{(49)}{=} \frac{1}{|F|^2}\{Fn, Fn'\}X_{F(n+n')} \overset{\text{(a)}}{=} \frac{1}{|F|}\{n, n'\}X_{F(n+n')} \\ &\overset{(81)}{=} F\left(\{n, n'\}X_{n+n'}\right) \overset{(49)}{=} F[X_n, X_{n'}]. \end{aligned} \tag{87}$$

(c) By (b), the action on $\mathfrak{g}$ preserves the Lie bracket. It implies that the action on G also preserves the product defined by the Backer-Campbell-Hausdorff formula. ☐

We apply this transformation for dilogarithm elements. For any $(a, b) \in N^+$, we have

$$\begin{aligned} F\begin{bmatrix}a\\b\end{bmatrix} &\overset{(60)}{=} F\exp\left(\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^2}X_{j\binom{a}{b}}\right) = \exp\left(\frac{1}{|F|}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^2}X_{jF\binom{a}{b}}\right) \\ &= \exp\left(\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^2}X_{jF\binom{a}{b}}\right)^{1/|F|} = \left[F\begin{pmatrix}a\\b\end{pmatrix}\right]^{1/|F|}. \end{aligned} \tag{88}$$

Next, we try to apply this transformation for ordered products.

Lemma 4.3. *Let* $(a,b),(c,d) \in N^+$*, and let* $F \in \mathrm{Mat}_2(\mathbb{Z}_{\geqslant 0})$ *with* $|F| > 0$*. Then, the following relations hold.*
(a).

$$\deg F \begin{pmatrix} a \\ b \end{pmatrix} \geqslant \deg \begin{pmatrix} a \\ b \end{pmatrix}. \tag{89}$$

Moreover, if $F \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *and* $a, b > 0$*, then*

$$\deg F \begin{pmatrix} a \\ b \end{pmatrix} > \deg \begin{pmatrix} a \\ b \end{pmatrix}. \tag{90}$$

(b). *If* $(c,d) > (a,b)$*, then*

$$F \begin{pmatrix} c \\ d \end{pmatrix} > F \begin{pmatrix} a \\ b \end{pmatrix}. \tag{91}$$

In the above lemma, the order in (b) is defined in Definition 3.9.

Proof. (a) Let $F = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$ with $\alpha, \beta, \gamma, \delta \geqslant 0$. Then, since $|F| \neq 0$, we have $\alpha + \beta, \gamma + \delta \geqslant 1$. Thus, we have

$$\begin{aligned} \deg F \begin{pmatrix} a \\ b \end{pmatrix} &= \deg \begin{pmatrix} a\alpha + b\gamma \\ a\beta + b\gamma \end{pmatrix} = a(\alpha+\beta) + b(\gamma+\delta) \\ &\geqslant a + b = \deg \begin{pmatrix} a \\ b \end{pmatrix}. \end{aligned} \tag{92}$$

Hence, the first statement holds. If $|F| \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, either $\alpha + \beta \geqslant 2$ or $\gamma + \delta \geqslant 2$ holds. Thus, by a similar argument to (92), we have $\deg F \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) > \deg \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$.
(b) The inequality $(c,d) > (a,b)$ implies either $\{\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right), \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\} > 0$ or $(c,d) = k(a,b)$ with $k > 1$. If $\{\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right), \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\} > 0$, then $\{F\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right), F\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\} = |F|\{\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right), \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\} > 0$. If $(c,d) = k(a,b)$, then $F\binom{c}{d} = k(F\binom{a}{b})$. Thus, $F\binom{c}{d} > F\binom{a}{b}$ holds. □

Let $m, n \in \mathbb{Z}_{\geqslant 1}$. Consider the equality

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}^m = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^m \left\{ \overrightarrow{\prod_{j \in J}} \begin{bmatrix} a_j \\ b_j \end{bmatrix}^{u_j} \right\} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^n. \tag{93}$$

By Theorem 3.19, the above equality exists. Let $F \in \mathrm{Mat}_2(\mathbb{Z}_{\geqslant 0})$ with $|F| > 0$, and we apply F to the above equality. Then, by Proposition 4.2 (c), we have

$$\left(F \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^n \left(F \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^m = \left(F \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^m \left\{ \overrightarrow{\prod_{j \in J}} \left(F \begin{bmatrix} a_j \\ b_j \end{bmatrix}\right)^{u_j} \right\} \left(F \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^n. \tag{94}$$

Moreover, by Lemma 4.3 (b), the RHS is strongly ordered. More strongly, we have the following statement.

Proposition 4.4. *Let* $F = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ *be a matrix with* $(a,b),(c,d) \in N^+$ *and* $|F| > 0$*. Assume* $F \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$*. Let* $m, n \in \mathbb{Z}_{>0}$*, and let* $C = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^m \{\overrightarrow{\prod}_{j \in J} \left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right]^{u_j}\} \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]^n$ *be the strongly ordered product expression of* $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]^n \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^m$*. Then, for any* $l \in \mathbb{Z}_{>0}$*, it holds that*

$$\begin{aligned} &\begin{bmatrix} c \\ d \end{bmatrix}^{n/|F|} \begin{bmatrix} a \\ b \end{bmatrix}^{m/|F|} \\ &\equiv \begin{bmatrix} a \\ b \end{bmatrix}^{m/|F|} \left\{ \overrightarrow{\prod_{\substack{j \in J, \\ \deg(a_j, b_j) \leqslant l}}} \left(F \begin{bmatrix} a_j \\ b_j \end{bmatrix}\right)^{u_j} \right\} \begin{bmatrix} c \\ d \end{bmatrix}^{n/|F|} \mod G^{>l+1}. \end{aligned} \tag{95}$$

Proposition 4.4 says that, if we find the strongly ordered expression of $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^n\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^m$ in $G^{\leqslant l}$, that is,

$$(96)\qquad \begin{bmatrix}0\\1\end{bmatrix}^n\begin{bmatrix}1\\0\end{bmatrix}^m \equiv \begin{bmatrix}1\\0\end{bmatrix}^m\left\{\overrightarrow{\prod_{\deg(a_j,b_j)\leqslant l}}\begin{bmatrix}a_j\\b_j\end{bmatrix}^{u_j}\right\}\begin{bmatrix}0\\1\end{bmatrix}^n \mod G^{>l},$$

then, for any anti-ordered product of the form $\left[\begin{smallmatrix}c\\d\end{smallmatrix}\right]^{n'/\gcd(a,b)}\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^{m'/\gcd(a,b)}$ $(m', n' \in \mathbb{Z}_{>0})$ except $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^{n'}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{m'}$, we can find the strongly ordered product expression (95) in $G^{\leqslant l+1}$. The difference between $G^{\leqslant l}$ and $G^{\leqslant l+1}$ is essential in this paper.

Proof. By (94), we have

$$(97)\qquad \begin{bmatrix}c\\d\end{bmatrix}^{n/|F|}\begin{bmatrix}a\\b\end{bmatrix}^{m/|F|} = \begin{bmatrix}a\\b\end{bmatrix}^{m/|F|}\left\{\overrightarrow{\prod_{j\in J}}\left(F\begin{bmatrix}a_j\\b_j\end{bmatrix}\right)^{u_j}\right\}\begin{bmatrix}c\\d\end{bmatrix}^{n/|F|}.$$

By Lemma 3.20, both a_j and b_j are positive integers for any $j \in J$. Moreover, by the assumptions, $F \neq \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ holds. Because of Lemma 4.3 (b), if $\deg(a_j, b_j) \geqslant l+1$, then $\deg F\left(\begin{smallmatrix}a_j\\b_j\end{smallmatrix}\right) \geqslant l+2$, namely, $\pi_{l+1}(F\left[\begin{smallmatrix}a_j\\b_j\end{smallmatrix}\right]) = \mathrm{id}$ holds. Thus, in $G^{\leqslant l+1}$, we may eliminate all factors $F\left[\begin{smallmatrix}a_j\\b_j\end{smallmatrix}\right]^{u_j}$ satisfying $\deg(a_j, b_j) \geqslant l+1$. Then, we have

$$(98)\qquad \prod_{j\in J}\left(F\begin{bmatrix}a_j\\b_j\end{bmatrix}\right)^{u_j} \equiv \prod_{\substack{j\in J,\\ \deg(a_j,b_j)\leqslant l}}\left(F\begin{bmatrix}a_j\\b_j\end{bmatrix}\right)^{u_j} \mod G^{>l+1}.$$

Putting the above expression to the (97), we have

$$(99)\qquad \begin{aligned}&\begin{bmatrix}c\\d\end{bmatrix}^{n/|F|}\begin{bmatrix}a\\b\end{bmatrix}^{m/|F|}\\ &\equiv \begin{bmatrix}a\\b\end{bmatrix}^{m/|F|}\left\{\overrightarrow{\prod_{\substack{j\in J,\\ \deg(a_j,b_j)\leqslant l}}}\left(F\begin{bmatrix}a_j\\b_j\end{bmatrix}\right)^{u_j}\right\}\begin{bmatrix}c\\d\end{bmatrix}^{n/|F|} \mod G^{>l+1}. \quad \square\end{aligned}$$

5. Calculation method and admissible forms of exponents

Let $(a, b) \in N^+$ and let $m, n \in \mathbb{Z}_{\geqslant 0}$. Then, we define the rational number $u_{(a,b)}(m, n)$ as the exponent of $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ in the strongly ordered product expression of $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^n\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^m$. Also, we define $\tilde{u}_{(a,b)}(m, n) = d(a, b)^{-1}u_{(a,b)}(m, n)$. Namely, $\tilde{u}_{(a,b)}(m, n)$ is defined by

$$(100)\qquad \begin{bmatrix}0\\1\end{bmatrix}^n\begin{bmatrix}1\\0\end{bmatrix}^m = \overrightarrow{\prod_{(a,b)\in N^+}}\begin{bmatrix}a\\b\end{bmatrix}^{d(a,b)\tilde{u}_{(a,b)}(m,n)}.$$

In this section, we introduce a method to calculate $u_{(a,b)}(m, n)$ as a function of m and n, and we show the following property based on this method.

Theorem 5.1. *For any $(a, b) \in N^+$, $\tilde{u}_{(a,b)}(m, n)$ is expressed as a nonnegative PBC in m and n.*

5.1. Calculation method.

By Theorem 3.19 for $\delta_1 = \delta_2 = 1$, for any $m, n \in \mathbb{Z}_{>0}$, $u_{(a,b)}(m, n) \in d(a, b)\mathbb{Z}_{\geqslant 0}$ holds, and it implies that $\tilde{u}_{(a,b)}(m, n) \in \mathbb{Z}_{\geqslant 0}$. Recall that $d(a, b) = 1/\gcd(a, b)$ is the normalization factor of (a, b) with respect to $(1, 1)$.

Example 5.2. Since the ordered product of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^m$ is itself, we have

$$(101)\qquad u_{(a,b)}(m, 0) = \begin{cases} m & (a, b) = (1, 0),\\ 0 & \text{otherwise.}\end{cases}$$

By Lemma 3.20, we have

$$u_{(a,0)}(m,n)=\begin{cases} m & a=1,\\ 0 & a\neq 1,\end{cases}\quad u_{(0,b)}(m,n)=\begin{cases} n & b=1,\\ 0 & b\neq 0.\end{cases} \tag{102}$$

Next, we find $u_{(1,1)}(m,n)$. By (63), the term $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ is commutative for every factor in $G^{\leqslant 2}$. Thus, we have

$$\begin{aligned}\begin{bmatrix}0\\1\end{bmatrix}^{n}\begin{bmatrix}1\\0\end{bmatrix}^{m} &= \begin{bmatrix}0\\1\end{bmatrix}^{n-1}\left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)\begin{bmatrix}1\\0\end{bmatrix}^{m-1} \overset{(62)}{=} \begin{bmatrix}0\\1\end{bmatrix}^{n-1}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{m-1}\\ &\overset{(62)}{=} \begin{bmatrix}0\\1\end{bmatrix}^{n-2}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{m-1}\\ &\overset{(63)}{\equiv} \begin{bmatrix}0\\1\end{bmatrix}^{n-2}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}^{2}\begin{bmatrix}0\\1\end{bmatrix}^{2}\begin{bmatrix}1\\0\end{bmatrix}^{m-1} \mod G^{>2}.\end{aligned} \tag{103}$$

By repeating this rearrangement until we obtain the strongly ordered product expression, the equality $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ is used mn times, and it implies that the factor $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ is produced mn times. Thus, we obtain

$$\begin{bmatrix}0\\1\end{bmatrix}^{n}\begin{bmatrix}1\\0\end{bmatrix}^{m} \equiv \begin{bmatrix}1\\0\end{bmatrix}^{m}\begin{bmatrix}1\\1\end{bmatrix}^{mn}\begin{bmatrix}0\\1\end{bmatrix}^{n}. \tag{104}$$

We have

$$u_{(1,1)}(m,n)=mn. \tag{105}$$

Note that Theorem 5.1 holds for any (a,b) with $a+b=1,2$.

Definition 5.3. *Let $C=\prod_{j\in J}\left[\begin{smallmatrix}a_j\\b_j\end{smallmatrix}\right]^{u_j}$ be a finite product. Then, we define the* stable part C^{stab} *and the* unstable part $\hat{C}$ *of C as follows:*

- *Let j_0 be the largest element in J such that there exists $k<j_0$ with $\left[\begin{smallmatrix}a_k\\b_k\end{smallmatrix}\right]\geqslant\left[\begin{smallmatrix}a_{j_0}\\b_{j_0}\end{smallmatrix}\right]$.*
- *Define $C^{\mathrm{stab}}=\prod_{j_0<j}\left[\begin{smallmatrix}a_j\\b_j\end{smallmatrix}\right]^{u_j}$, and $\hat{C}=\prod_{j\leqslant j_0}\left[\begin{smallmatrix}a_j\\b_j\end{smallmatrix}\right]^{u_j}$.*

If there are no such j_0, or equivalently, if C is strongly ordered, we define $C^{\mathrm{stab}}=C$ and $\hat{C}=\mathrm{id}$.

By definition, the following statements hold for any finite product C.

- $C=\hat{C}C^{\mathrm{stab}}$.
- The stable part C^{stab} is either a strongly ordered product or id.
- Let $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}z\\w\end{smallmatrix}\right]$ be dilogarithm elements appearing in C^{stab} and $\hat{C}$, respectively. Then, $\left[\begin{smallmatrix}z\\w\end{smallmatrix}\right]<\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ holds.

Let $\hat{C}'$ be the strongly ordered product expression of $\hat{C}$. Then, by Theorem 3.19, every dilogarithm element $\left[\begin{smallmatrix}z\\w\end{smallmatrix}\right]$ appearing in $\hat{C}'$ is smaller than all dilogarithm elements appearing in C^{stab}. Thus, $\hat{C}'C^{\mathrm{stab}}$ is the strongly ordered product expression of C.

In order to obtain the explicit forms of $u_{(a,b)}(m,n)$ as the function of m and n, we often consider the product

$$C=\prod_{j\in\bar{J}}\begin{bmatrix}a_j\\b_j\end{bmatrix}^{d(a_j,b_j)f_j(m,n)}, \tag{106}$$

where

- the index set $\bar{J}$ is finite.
- for each $j\in\bar{J}$, $f_j:\mathbb{Z}_{\geqslant 0}\times\mathbb{Z}_{\geqslant 0}\to\mathbb{Z}_{\geqslant 0}$ is a function.
- m and n are integer variables.

In this case, we view $\left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right]^{d(a_j,b_j)f_j(m,n)}$ as a factor of C for each $j \in \bar{J}$.

Lemma 5.4. *Let $l \in \mathbb{Z}_{\geqslant 1}$. Let C be a product with the above form, and let $\hat{C} = \prod_{j \in J} \left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right]^{d(a_j,b_j)f_j(m,n)}$ be the unstable part of C. Let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ be the greatest dilogarithm element appearing in $\hat{C}$. Namely, $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \geqslant \left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right]$ holds for any $j \in J$. Now, we assume the following conditions:*

a. $\deg\left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right] \leqslant l+1$ *for any* $j \in J$.
b. *If* $\deg\left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right] \leqslant l$*, then* $f_j(m,n)$ *can be expressed as a nonnegative PBC in* m *and* n.
c. $x \neq 0$ *or* $b_j \neq 0$ *for any* $\left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.

Let $\hat{J} = J \backslash \{j \in J \mid \left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\}$. Then, by applying Algorithm 5.5 below, we obtain the products

$$\hat{C}' = \left(\prod_{j \in J'} \begin{bmatrix} a'_j \\ b'_j \end{bmatrix}^{d(a'_j,b'_j)f'_j(m,n)} \right) \begin{bmatrix} x \\ y \end{bmatrix}^{d(x,y)g(m,n)} \tag{107}$$

and $C' = \hat{C}' C^{\mathrm{stab}}$ which satisfy the following conditions.

A. $\hat{C} \equiv \hat{C}' \mod G^{>l+1}$. *It implies that* $C \equiv C' \mod G^{>l+1}$.
B. $\deg\left[\begin{smallmatrix} a'_j \\ b'_j \end{smallmatrix}\right] \leqslant l+1$ *and* $\left[\begin{smallmatrix} a'_j \\ b'_j \end{smallmatrix}\right] < \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. *In particular, the stable part of* C' *includes* $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)g(m,n)} C^{\mathrm{stab}}$.
C. *The index set $\hat{J}$ can be embedded in J' as an ordered set, and it satisfies the following properties:*
   - *For any $j \in \hat{J} \subset J'$, it holds that* $\left[\begin{smallmatrix} a'_j \\ b'_j \end{smallmatrix}\right]^{d(a'_j,b'_j)f'_j(m,n)} = \left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right]^{d(a_j,b_j)f_j(m,n)}$.
   - *For any $j \in J' \backslash \hat{J}$, $f'_j(m,n)$ is expressed as a nonnegative PBC in m and n.*
D. *Every dilogarithm element* $\left[\begin{smallmatrix} a'_j \\ b'_j \end{smallmatrix}\right]$ *and the index set J' are independent of m and n.*
E. $g(m,n) = \sum_{\substack{j \in J, \\ \left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]}} f_j(m,n)$.

Algorithm 5.5.

Step 0. *Let $D = \hat{C}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)g(m,n)}$ with $g(m,n) = 0$.*

Step 1. *Let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)f(x,y)}$ be the second factor of D from the right hand side such that its dilogarithm element is $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. Namely, every factor $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]^{d(z,w)f'(m,n)}$ on the right side of this $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)f(x,y)}$ satisfies $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ except for the factor $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)g(m,n)}$ on the right end. Let $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^{d(a,b)f'(m,n)}$ be the right adjacent factor of $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)f(x,y)}$.*

Step 1.1. *If $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^{d(a,b)f'(m,n)} \neq \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)g(m,n)}$, let $F = \left(\begin{smallmatrix} a & x \\ b & y \end{smallmatrix}\right)$. By the assumptions, $(a,b) < (x,y)$, that is, $|F| = ay - bx \geqslant 0$ holds. Moreover, we have $F \neq I$ since $x \neq 0$ or $b \neq 0$. Proceed to (i) or (ii).*

(i). *If $|F| = 0$, applying (61), replace $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)}\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^{f'(m,n)}$ with $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^{d(a,b)f'(m,n)}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)f(m,n)}$. Back to Step 1.*

(ii). *If* $|F|>0$*, replace* $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]^{d(x,y)f(m,n)}\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^{d(a,b)f'(m,n)}$ *with*

$$\begin{bmatrix}a\\b\end{bmatrix}^{d(a,b)f'(m,n)}\left\{\overrightarrow{\prod_{\substack{p,q\in\mathbb{Z}_{\geqslant 1},\\ \deg(p,q)\leqslant l,\\ \deg F\left(\begin{smallmatrix}p\\q\end{smallmatrix}\right)\leqslant l+1}}}\left(F\begin{bmatrix}p\\q\end{bmatrix}\right)^{v_{(p,q)}(m,n)}\right\}\begin{bmatrix}x\\y\end{bmatrix}^{d(x,y)f(m,n)} \tag{108}$$

where

$$v_{(p,q)}(m,n)=u_{(p,q)}\left(d(a,b)|F|f'(m,n),d(x,y)|F|f(m,n)\right). \tag{109}$$

Back to Step 1.

Step 1.2. *If* $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^{d(a,b)f'(m,n)}=\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]^{d(x,y)g(m,n)}$*, replace* $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]^{d(x,y)f(m,n)}\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]^{d(x,y)g(m,n)}$ *with* $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]^{d(x,y)(g(m,n)+f(m,n))}$*, and we set* $g(m,n)$ *as* $f(m,n)+g(m,n)$*. If there exists a factor* $\left[\begin{smallmatrix}z\\w\end{smallmatrix}\right]^{d(z,w)h(z,w)}$ *such that* $\left[\begin{smallmatrix}z\\w\end{smallmatrix}\right]=\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ *and it is not at the right end, back to Step 1. Otherwise, proceed to Step 2.*

Step 2. *If every dilogarithm element appearing in* D *is not* $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ *except for the one at the right end, let* $\hat{C}'=D$*, and finish this algorithm.*

The replacement in Step 1.1 (ii) follows from the following relation and Proposition 4.4.

$$\begin{aligned}&\begin{bmatrix}0\\1\end{bmatrix}^{d(x,y)|F|f(m,n)}\begin{bmatrix}1\\0\end{bmatrix}^{d(a,b)|F|f'(m,n)}\\ \equiv\ &\begin{bmatrix}1\\0\end{bmatrix}^{d(a,b)|F|f'(m,n)}\left\{\overrightarrow{\prod_{\substack{p,q\in\mathbb{Z}_{\geqslant 1},\\ \deg(p,q)\leqslant l}}}\begin{bmatrix}p\\q\end{bmatrix}^{v_{(p,q)}(m,n)}\right\}\begin{bmatrix}0\\1\end{bmatrix}^{d(x,y)|F|f(m,n)}\\ &\hspace{20em}\mod G^{>l}.\end{aligned} \tag{110}$$

Roughly speaking, we change an anti-ordered pair $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]^{d(x,y)f(m,n)}\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^{d(a,b)f'(m,n)}$ to the strongly ordered product expression step by step, and we push $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]^{d(x,y)f(m,n)}$ out to the right end. By Proposition 4.4, this operation uses the information of the strongly ordered product expression of $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^n\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^m$ in $G^{\leqslant l}$. In particular, we do not use the data $u_{(p,q)}(m,n)$ for $p+q=l+1$.

Proposition 5.6. *Under the assumptions of Lemma* 5.4*, Algorithm* 5.5 *never fails, and finishes finitely many times.*

The number of $(x,y)\in N^+$ satisfying $\deg(x,y)\leqslant l+1$ is finite. Thus, by applying Algorithm 5.5 repeatedly, we obtain the strongly ordered product expression of C in $G^{\leqslant l+1}$ finitely many times. Moreover, by Lemma 5.4 C and D, the exponent of $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ in the strongly ordered product expression of C is

$$\sum_{j;\left[\begin{smallmatrix}a_j\\b_j\end{smallmatrix}\right]=\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\text{ in }C} d(a,b)f_j(m,n)+d(a,b)f(m,n) \tag{111}$$

for some nonnegative PBC $f(m,n)$.

Based on this algorithm, we give a method to calculate $u_{(a,b)}(m,n)$. We can see the example of this method in Section 6.

Method 5.7. *Let* $l\in\mathbb{Z}_{\geqslant 2}$*, and suppose that* $\tilde{u}_{(x,y)}(m,n)$ *is a nonnegative PBC for any* $(x,y)\in N^+$ *with* $\deg(x,y)\leqslant l$*. By applying Algorithm* 5.5 *to the following products* $C_{(m,1)}$ *and* $C_{(m,n)}$*, we may calculate* $u_{(a,b)}(m,n)$ *with* $\deg(a,b)=l+1$*.*

First, $C_{(m,1)}$ is defined as follows:

$$
\begin{aligned}
\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{m+1} &= \left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)\begin{bmatrix}1\\0\end{bmatrix}^{m} \\
&\overset{(62)}{=} \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{m}\right) \\
&\overset{(100)}{\equiv} \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{m}\left(\overrightarrow{\prod_{\substack{(x,y)\in N^+,\\ x+y\leqslant l+1,\\ x,y\geqslant 1,}}}\begin{bmatrix}x\\y\end{bmatrix}^{d(x,y)\tilde{u}_{(x,y)}(m,1)}\right)\begin{bmatrix}0\\1\end{bmatrix} \\
&\qquad \bmod G^{>l+1}.
\end{aligned}
\tag{112}
$$

Let

$$
C_{(m,1)} = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{m}\left(\overrightarrow{\prod_{\substack{(x,y)\in N^+,\\ x+y\leqslant l+1,\\ x,y\geqslant 1}}}\begin{bmatrix}x\\y\end{bmatrix}^{d(x,y)\tilde{u}_{(x,y)}(m,1)}\right). \tag{113}
$$

Next, $C_{(m,n)}$ is defined as follows:

$$
\begin{aligned}
\begin{bmatrix}0\\1\end{bmatrix}^{n+1}\begin{bmatrix}1\\0\end{bmatrix}^{m} &= \begin{bmatrix}0\\1\end{bmatrix}\left(\begin{bmatrix}0\\1\end{bmatrix}^{n}\begin{bmatrix}1\\0\end{bmatrix}^{m}\right) \\
&\overset{(100)}{\equiv} \left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{m}\right)\left(\overrightarrow{\prod_{\substack{(x,y)\in N^+,\\ x+y\leqslant l+1,\\ x,y\geqslant 1}}}\begin{bmatrix}x\\y\end{bmatrix}^{d(x,y)\tilde{u}_{(x,y)}(m,n)}\right)\begin{bmatrix}0\\1\end{bmatrix}^{n} \\
&\overset{(100)}{\equiv} \begin{bmatrix}1\\0\end{bmatrix}^{m}\left(\overrightarrow{\prod_{\substack{(z,w)\in N^+,\\ z+w\leqslant l+1\\ z,w\geqslant 1}}}\begin{bmatrix}z\\w\end{bmatrix}^{d(z,w)\tilde{u}_{(z,w)}(m,1)}\right)\begin{bmatrix}0\\1\end{bmatrix} \\
&\quad \times\left(\overrightarrow{\prod_{\substack{(x,y)\in N^+,\\ x+y\leqslant l+1\\ x,y\geqslant 1}}}\begin{bmatrix}x\\y\end{bmatrix}^{d(x,y)\tilde{u}_{(x,y)}(m,n)}\right)\begin{bmatrix}0\\1\end{bmatrix}^{n} \quad \bmod G^{>l+1}.
\end{aligned}
\tag{114}
$$

Then, $C_{(m,n)}$ is defined by

$$
\left(\overrightarrow{\prod_{\substack{(z,w)\in N^+,\\ z+w\leqslant l+1\\ z,w\geqslant 1}}}\begin{bmatrix}z\\w\end{bmatrix}^{d(z,w)\tilde{u}_{(z,w)}(m,1)}\right)\begin{bmatrix}0\\1\end{bmatrix}\left(\overrightarrow{\prod_{\substack{(x,y)\in N^+,\\ x+y\leqslant l+1\\ x,y\geqslant 1}}}\begin{bmatrix}x\\y\end{bmatrix}^{d(x,y)\tilde{u}_{(x,y)}(m,n)}\right)\begin{bmatrix}0\\1\end{bmatrix}^{n}. \tag{115}
$$

By Theorem 3.19 *and Theorem* 5.1*, $C_{(m,1)}$ and $C_{(m,n)}$ satisfy the assumptions of Lemma* 5.4*. Let $(a,b)\in N^+$ with $3\leqslant a+b=l+1$. Then, the factor $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^*$ in the initial $C_{(m,1)}$ is only $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^{d(a,b)\tilde{u}_{(a,b)}(m,1)}=\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^{u_{(a,b)}(m,n)}$, and the ones in the initial $C_{(m,n)}$ are only $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^{d(a,b)\tilde{u}_{(a,b)}(m,1)}=\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^{u_{(a,b)}(m,1)}$ and $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^{d(a,b)\tilde{u}_{(a,b)}(m,n)}=\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^{u_{(a,b)}(m,n)}$. The method to calculate $u_{(a,b)}(m,n)$ is as follows:*

1. *Apply Algorithm* 5.5 *to $C_{(m,1)}$ repeatedly until $C_{(m,1)}$ becomes the strongly ordered product.*

2. *After the operation 1, the exponent of* $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ *is* $u_{(a,b)}(m,1) + d(a,b)f(m)$ *for some nonnegative PBC* $f(m)$. *Thus, we obtain the relation*

$$\begin{aligned} & u_{(a,b)}(m+1,1) = u_{(a,b)}(m,1) + d(a,b)f(m) \\ \Leftrightarrow\ & u_{(a,b)}(m+1,1) - u_{(a,b)}(m,1) = d(a,b)f(m), \end{aligned} \tag{116}$$

and it implies that

$$\begin{aligned} u_{(a,b)}(m,1) &= u_{(a,b)}(0,1) + \sum_{j=0}^{m-1} \{u_{(a,b)}(j+1,1) - u_{(a,b)}(j,1)\} \\ &\overset{(116)}{=} u_{(a,b)}(0,1) + d(a,b) \sum_{j=0}^{m-1} f(j) \overset{(101)}{=} d(a,b) \sum_{j=0}^{m-1} f(j). \end{aligned} \tag{117}$$

Note that this $f(m)$ *is determined by the data of* $u_{(x,y)}(m,n)$ *for* $x+y \leqslant l$. *Thus, we may find the explicit form of* $u_{(a,b)}(m,1)$.

3. *Apply Algorithm* 5.5 *to* $C_{(m,n)}$ *repeatedly until* $C_{(m,n)}$ *becomes the strongly ordered product.*
4. *After the operation 3, the exponent of* $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ *is* $u_{(a,b)}(m,n) + u_{(a,b)}(m,1) + d(a,b)f'(m,n)$ *for some nonnegative PBC* $d(a,b)f'(m,n)$. *By a similar argument, we obtain the explicit form*

$$\begin{aligned} u_{(a,b)}(m,n) &= \sum_{j=0}^{n-1} u_{(a,b)}(m,1) + d(a,b) \sum_{j=0}^{n-1} f'(m,j) \\ &= u_{(a,b)}(m,1)n + d(a,b) \sum_{j=0}^{n-1} f'(m,j). \end{aligned} \tag{118}$$

To summarize, the following proposition holds.

Proposition 5.8. *Let* $l \in \mathbb{Z}_{\geqslant 1}$, *and let* $(a,b) \in N^+$ *with* $\deg(a,b) = l+1$. *Let* $C_{(m,1)}$ *and* $C_{(m,n)}$ *be the products which is defined by (*113*) and (*115*), respectively. The following two statements hold.*
(a) *By applying Algorithm* 5.5 *to* $C_{(m,1)}$ *repeatedly, we obtain the recurrence relation:*

$$u_{(a,b)}(m+1,1) = u_{(a,b)}(m,1) + d(a,b)f(m), \tag{119}$$

where $f(m)$ *is some nonnegative PBC in* m.
(b) *By applying Algorithm* 5.5 *to* $C_{(m,n)}$ *repeatedly, we obtain the recurrence relation:*

$$u_{(a,b)}(m,n+1) = u_{(a,b)}(m,n) + u_{(a,b)}(m,1) + d(a,b)f'(m,n), \tag{120}$$

where $f'(m,n)$ *is some nonnegative PBC in* m *and* n.

Moreover, $f(m)$ *and* $f'(m,n)$ *are determined by the data of* $u_{(x,y)}(m,n)$ *with* $\deg(x,y) \leqslant l$ *as functions of* m *and* n.

5.2. Proof of Theorem 5.1. Here, we prove Theorem 5.1, Lemma 5.4, and Proposition 5.6. We show these claims by the induction on the degree l. For any $l \in \mathbb{Z}_{\geqslant 1}$, we define the statements $(5.1)_l$, $(5.4)_l$, and $(5.6)_l$ as follows:

- $(5.1)_l$ Theorem 5.1 holds for $\deg(a,b) \leqslant l$.
- $(5.4)_l$ Lemma 5.4 holds for this l.
- $(5.6)_l$ Proposition 5.6 holds for this l.

The statement $(5.1)_1$ holds by (102). Thus, it suffices to show the following three statements:

$$\begin{gathered} (5.1)_l \Rightarrow (5.6)_l, \quad (5.1)_l, (5.6)_l \Rightarrow (5.4)_l, \\ (5.1)_l, (5.4)_l, (5.6)_l \Rightarrow (5.1)_{l+1}. \end{gathered}$$

Let us start the proof of $(5.1)_l \Rightarrow (5.6)_l$.

Proof of $(5.1)_l \Rightarrow (5.6)_l$. We prove that Algorithm 5.5 never fails. There are two claims in Step 1.1 (ii).

Since the replacement in Step 1.1 (ii) is derived from the equality (110), we should show that this equality holds for any $m, n \in \mathbb{Z}_{\geqslant 0}$. It suffices to show the following claim.

Claim 1. Both $d(a,b)|F|f'(m,n)$ and $d(x,y)|F|f(m,n)$ are nonnegative integers for any $m, n \in \mathbb{Z}_{\geqslant 0}$.

Second, we should show that the assumptions of this algorithm hold after any replacement. The assumption a and c are obvious. Thus, we should show the assumption b. If $\deg\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = l+1$ or $\deg\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = l+1$, the product (108) is $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^{d(x,y)f'(m,n)}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)f(m,n)}$. Thus, the assumption b holds. Hence, it suffices to show the following claim.

Claim 2. Assume $\deg(a,b), \deg(x,y) \leqslant l$. Consider the factor

$$\left(F\begin{bmatrix} p \\ q \end{bmatrix}\right)^{v_{(p,q)}(m,n)} = \left[F\begin{pmatrix} p \\ q \end{pmatrix}\right]^{v_{(p,q)}(m,n)/|F|}, \tag{121}$$

where $\deg(p,q) \leqslant l$, $F = \left(\begin{smallmatrix} a & x \\ b & y \end{smallmatrix}\right)$ and

$$v_{(p,q)}(m,n) = u_{(p,q)}\left(d(a,b)|F|f'(m,n), d(x,y)|F|f(m,n)\right). \tag{122}$$

Then, this exponent $\frac{v_{(p,q)}(m,n)}{|F|}$ can be expressed as $d(F\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right))h(m,n)$ for some nonnegative PBC $h(m,n)$.

Proof of Claim 1. We show $d(a,b)|F|f'(m,n) \in \mathbb{Z}_{\geqslant 0}$. By the assumptions, we have $f'(m,n) \in \mathbb{Z}$ for any $m,n \in \mathbb{Z}_{\geqslant 0}$. Now, we have the equality

$$d(a,b)|F| = \frac{a}{\gcd(a,b)}y - \frac{b}{\gcd(a,b)}x \in \mathbb{Z}_{\geqslant 0}. \tag{123}$$

Since $d(a,b)|F|, f'(m,n) \in \mathbb{Z}_{\geqslant 0}$, we have $d(a,b)|F|f'(m,n) \in \mathbb{Z}_{\geqslant 0}$ for any $m, n \in \mathbb{Z}_{\geqslant 0}$. ☐

Proof of Claim 2. We show that

$$\begin{aligned} &\frac{1}{d\left(F\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)\right)}\frac{v_{(p,q)}(m,n)}{|F|} \\ &= \frac{1}{d\left(F\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)\right)}\frac{d(p,q)\tilde{u}_{(p,q)}\left(d(a,b)|F|f'(m,n), d(x,y)|F|f(m,n)\right)}{|F|} \end{aligned} \tag{124}$$

is a nonnegative PBC. First, we show that (124) is expressed as $\sum_{0 \leqslant k,l} \gamma_{k,l}\binom{m}{k}\binom{n}{l}$ with $\gamma_{k,l} \in \mathbb{Q}_{\geqslant 0}$. By the assumption b and Claim 1, both $d(a,b)|F|f'(m,n)$ and $d(x,y)|F|f(m,n)$ are nonnegative PBCs. By the assumption $(5.1)_l$, $\tilde{u}_{(p,q)}(m,n)$ is expressed as a nonnegative PBC. Thus, by Proposition 2.7, we may express

$$\begin{aligned} &\tilde{u}_{(p,q)}\left(d(a,b)|F|f'(m,n), d(x,y)|F|f(m,n)\right) \\ &= \sum_{0 \leqslant k,l} \alpha_{k,l}\binom{m}{k}\binom{n}{l} \end{aligned} \tag{125}$$

for some nonnegative integers $\alpha_{k,l} \in \mathbb{Z}_{\geqslant 0}$. We have

$$\begin{aligned} &\frac{1}{d\left(F\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)\right)} \frac{v_{(p,q)}(m,n)}{|F|} \\ &= \frac{d(p,q)}{|F| d(F\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right))} \tilde{u}_{(p,q)}\left(d(a,b)|F|f'(m,n), d(x,y)|F|f(m,n)\right) \\ &= \frac{d(p,q)}{|F| d(F\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right))} \sum_{0 \leqslant k,l} \alpha_{k,l} \binom{m}{k}\binom{n}{l}. \end{aligned} \tag{126}$$

Let $\gamma_{k,l} = \frac{d(p,q)}{|F| d(F\binom{p}{q})} \alpha_{k,l} \in \mathbb{Q}_{\geqslant 0}$. Then $\frac{1}{d(F\binom{p}{q})} \frac{v_{(p,q)}(m,n)}{|F|} = \sum_{0 \leqslant k,l} \gamma_{k,l} \binom{m}{k}\binom{n}{l}$. Thus, it suffices to show that $\frac{1}{d(F\binom{p}{q})} \frac{v_{(p,q)}(m,n)}{|F|}$ is a PBC. Recall that $\frac{v_{(p,q)}(m,n)}{|F|}$ is the exponent of $\left[F\binom{p}{q}\right]$ in the strongly ordered product expression of (108). Consider Theorem 3.19 for $\delta_1 = \delta_2 = 1$. By the assumptions b and c, $C^{\text{in}} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)(m,n)} \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^{d(a,b)f'(m,n)}$ satisfies the assumption of Theorem 3.19. By (108) and Theorem 3.19,

$$\frac{v_{(p,q)}(m,n)}{|F|} \in d\left(F\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)\right)\mathbb{Z} \;\Leftrightarrow\; \frac{1}{d(F\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right))} \frac{v_{(p,q)}(m,n)}{|F|} \in \mathbb{Z} \tag{127}$$

holds for any $m, n \in \mathbb{Z}_{\geqslant 0}$. So, by Lemma 2.2, $\frac{1}{d(F\binom{p}{q})} \frac{v_{(p,q)}(m,n)}{|F|}$ is a PBC. This completes the proof. □

Next, we show that Algorithm 5.5 finishes in a finite number of steps. By applying Step 1.1 (i) and (ii), the number of factors on the right side of $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)f(m,n)}$ decreases by 1. Moreover, the product is always finite for each operation. Thus, this algorithm finishes finitely many times. □

Next, we show $(5.1)_l, (5.6)_l \Rightarrow (5.4)_l$.

Proof of $(5.1)_l, (5.6)_l \Rightarrow (5.4)_l$. The statements A, B, and D are shown by considering each step in Algorithm 5.5. Thus, we need to prove C and E.
C: In Step 1.1 (i) and Step 1.1 (ii), the anti-ordered pair $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)f(m,n)} \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^{d(a,b)f'(m,n)}$ is replaced with the strongly ordered product not changing $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^{d(a,b)f'(m,n)}$. Thus, every factor $\left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right]^{d(a_j,b_j)f_j(m,n)}$ $((a_j,b_j) \neq (x,y))$ in $\hat{C}$ exists in D. Moreover, for any factors $\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right]^{d(a_i,b_i)f_i(m,n)}$ and $\left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right]^{d(a_j,b_j)f_j(m,n)}$ such that $(a_i,b_i), (a_j,b_j) \neq (x,y)$, if $\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right]^{d(a_i,b_i)f_i(m,n)}$ appears before $\left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right]^{d(a_j,b_j)f_j(m,n)}$ in $\hat{C}$, then $\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right]^{d(a_i,b_i)f_i(m,n)}$ appears before $\left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right]^{d(a_j,b_j)f_j(m,n)}$ in D. Furthermore, this algorithm does not finish until all $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^*$ disappear in D except for the right end. Thus, the index set $\hat{J}$ can be embedded in J' preserving its order, and for any $j \in \hat{J}$, $\left[\begin{smallmatrix} a'_j \\ b'_j \end{smallmatrix}\right]^{d(a'_j,b'_j)f'_j(m,n)} = \left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right]^{d(a_j,b_j)f_j(m,n)}$ holds. Let $j \in J' \backslash \hat{J}$. Then, by the proof of Claim 2, $f'_j(m,n)$ is expressed as a nonnegative PBC.
E: In Step 1.1 (i) and (ii), every factor $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)f(m,n)}$ in D moves to the last $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^{d(x,y)g(m,n)}$ without changing its exponent, and new factors $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^*$ are not produced. Thus, E holds. □

Last, we prove $(5.1)_l, (5.4)_l, (5.6)_l \Rightarrow (5.1)_{l+1}$.

Proof of $(5.1)_l, (5.4)_l, (5.6)_l \Rightarrow (5.1)_{l+1}$. This is immediately shown by (117) and (118). Note that, for any PBC $f(m,n) = \sum_{0 \leqslant k,l} \alpha_{k,l} \binom{m}{k}\binom{n}{l}$ in m and n, $\sum_{j=0}^{n-1} f(m,j)$

is also expressed as a PBC as follows:

$$\sum_{j=0}^{n-1} f(m,j) = \sum_{0\leqslant k,l} \alpha_{k,l}\binom{m}{k}\sum_{j=0}^{n-1}\binom{j}{l} \overset{(15)}{=} \sum_{0\leqslant k,l}\alpha_{k,l}\binom{m}{k}\binom{n}{l+1}. \tag{128}$$

□

6. Examples in lower degrees

In Section 5, we introduced a method to calculate $u_{(a,b)}(m,n)$ (Method 5.7). In this section, we see some examples.

Example 6.1. By (105), we obtain

$$\begin{bmatrix}0\\1\end{bmatrix}^n\begin{bmatrix}1\\0\end{bmatrix}^m \equiv \begin{bmatrix}1\\0\end{bmatrix}^m\begin{bmatrix}1\\1\end{bmatrix}^{mn}\begin{bmatrix}0\\1\end{bmatrix}^n \mod G^{>2}. \tag{129}$$

For later, we write underlines on anti-ordered pairs where we apply the relation. First, consider $a+b=3$. Let $u_{m,n}=u_{(2,1)}(m,n)$ and $v_{m,n}=u_{(1,2)}(m,n)$, namely,

$$\begin{bmatrix}0\\1\end{bmatrix}^n\begin{bmatrix}1\\0\end{bmatrix}^m \equiv \begin{bmatrix}1\\0\end{bmatrix}^m\begin{bmatrix}2\\1\end{bmatrix}^{u_{m,n}}\begin{bmatrix}1\\1\end{bmatrix}^{mn}\begin{bmatrix}1\\2\end{bmatrix}^{v_{m,n}}\begin{bmatrix}0\\1\end{bmatrix}^n \mod G^{>3}. \tag{130}$$

Then, for any $m\in\mathbb{Z}_{\geqslant 0}$, we obtain

$$\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{m+1} \overset{(112)}{\equiv} \begin{bmatrix}1\\0\end{bmatrix}\underline{\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^m}\begin{bmatrix}2\\1\end{bmatrix}^{u_{m,1}}\begin{bmatrix}1\\1\end{bmatrix}^m\begin{bmatrix}1\\2\end{bmatrix}^{v_{m,1}}\begin{bmatrix}0\\1\end{bmatrix} \tag{131}$$

Now, we apply Step 1.1 (ii) to $[\begin{smallmatrix}1\\1\end{smallmatrix}][\begin{smallmatrix}1\\0\end{smallmatrix}]^m$. Namely, let $F=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, and we view

$$\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^m = F\left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^m\right). \tag{132}$$

Then, by using Proposition 4.4 and the result of (129), it holds that

$$\begin{aligned}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^m &\equiv F\left(\begin{bmatrix}1\\0\end{bmatrix}^m\begin{bmatrix}1\\1\end{bmatrix}^m\begin{bmatrix}0\\1\end{bmatrix}\right)\\ &\equiv \begin{bmatrix}1\\0\end{bmatrix}^m\begin{bmatrix}2\\1\end{bmatrix}^m\begin{bmatrix}1\\1\end{bmatrix} \mod G^{>3}.\end{aligned} \tag{133}$$

Putting this relation to the last line of (131), we have

$$\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{m+1} \equiv \begin{bmatrix}1\\0\end{bmatrix}^{m+1}\begin{bmatrix}2\\1\end{bmatrix}^m\underline{\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}^{u_{m,1}}}\begin{bmatrix}1\\1\end{bmatrix}^m\begin{bmatrix}1\\2\end{bmatrix}^{v_{m,1}}\begin{bmatrix}0\\1\end{bmatrix} \mod G^{>3}. \tag{134}$$

Next, we apply Step 1.1 (ii) to $[\begin{smallmatrix}1\\1\end{smallmatrix}][\begin{smallmatrix}2\\1\end{smallmatrix}]^{u_{m,1}}$. Since $\deg((1,1)+(2,1))=\deg(3,2)>3$, we have

$$\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}^{u_{m,1}} \equiv \begin{bmatrix}2\\1\end{bmatrix}^{u_{m,1}}\begin{bmatrix}1\\1\end{bmatrix} \mod G^{>3}. \tag{135}$$

Thus, we have

$$\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{m+1} \equiv \begin{bmatrix}1\\0\end{bmatrix}^{m+1}\begin{bmatrix}2\\1\end{bmatrix}^{m+u_{m,1}}\begin{bmatrix}1\\1\end{bmatrix}^{m+1}\begin{bmatrix}1\\2\end{bmatrix}^{v_{m,1}}\begin{bmatrix}0\\1\end{bmatrix} \mod G^{>3}. \tag{136}$$

The RHS is strongly ordered. So, we have $u_{m+1,1}=m+u_{m,1}$ and $v_{m+1,1}=v_{m,1}$. Moreover, we obtain

$$u_{m,1} = u_{0,1} + \sum_{k=0}^{m-1}(u_{k+1,1}-u_{k,1}) = \sum_{k=0}^{m-1}\binom{k}{1} \overset{(15)}{=} \binom{m}{2}, \tag{137}$$

$$v_{m,1} = v_{m-1,1} = \cdots = v_{0,1} = 0. \tag{138}$$

Next, by (114), we have

$$\begin{bmatrix}0\\1\end{bmatrix}^{n+1}\begin{bmatrix}1\\0\end{bmatrix}^{m}\equiv\begin{bmatrix}1\\0\end{bmatrix}^{m}\begin{bmatrix}2\\1\end{bmatrix}^{u_{m,1}}\begin{bmatrix}1\\1\end{bmatrix}^{m}\begin{bmatrix}1\\2\end{bmatrix}^{v_{m,1}}\underline{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}^{u_{m,n}}}\begin{bmatrix}1\\1\end{bmatrix}^{mn}\begin{bmatrix}1\\2\end{bmatrix}^{v_{m,n}}\begin{bmatrix}0\\1\end{bmatrix}^{n} \mod G^{>3}. \tag{139}$$

By applying Step 1.1 (ii) to $[\begin{smallmatrix}0\\1\end{smallmatrix}][\begin{smallmatrix}2\\1\end{smallmatrix}]^{u_{m,n}}$, we have

$$\begin{bmatrix}0\\1\end{bmatrix}^{n+1}\begin{bmatrix}1\\0\end{bmatrix}^{m}\equiv\begin{bmatrix}1\\0\end{bmatrix}^{m}\begin{bmatrix}2\\1\end{bmatrix}^{u_{m,1}}\begin{bmatrix}1\\1\end{bmatrix}^{m}\begin{bmatrix}1\\2\end{bmatrix}^{v_{m,1}}\begin{bmatrix}2\\1\end{bmatrix}^{u_{m,n}}\underline{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}^{mn}}\begin{bmatrix}1\\2\end{bmatrix}^{v_{m,n}}\begin{bmatrix}0\\1\end{bmatrix}^{n} \mod G^{>3}. \tag{140}$$

Next, apply Step 1.1 (ii) to $[\begin{smallmatrix}0\\1\end{smallmatrix}][\begin{smallmatrix}1\\1\end{smallmatrix}]^{mn}$. Let $F=(\begin{smallmatrix}1&0\\1&1\end{smallmatrix})$, and we view

$$\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}^{mn}=F\left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{mn}\right). \tag{141}$$

Thus, it holds that

$$\begin{aligned}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}^{mn}&\equiv F\left(\begin{bmatrix}1\\0\end{bmatrix}^{mn}\begin{bmatrix}1\\1\end{bmatrix}^{mn}\begin{bmatrix}0\\1\end{bmatrix}\right)\\&\equiv\begin{bmatrix}1\\1\end{bmatrix}^{mn}\begin{bmatrix}1\\2\end{bmatrix}^{mn}\begin{bmatrix}0\\1\end{bmatrix} \mod G^{>3}.\end{aligned} \tag{142}$$

Putting the last line to (140), we have

$$\begin{aligned}&\begin{bmatrix}0\\1\end{bmatrix}^{n+1}\begin{bmatrix}1\\0\end{bmatrix}^{m}\\&\equiv\begin{bmatrix}1\\0\end{bmatrix}^{m}\begin{bmatrix}2\\1\end{bmatrix}^{u_{m,1}}\begin{bmatrix}1\\1\end{bmatrix}^{m}\begin{bmatrix}1\\2\end{bmatrix}^{v_{m,1}}\begin{bmatrix}2\\1\end{bmatrix}^{u_{m,n}}\begin{bmatrix}1\\1\end{bmatrix}^{mn}\begin{bmatrix}1\\2\end{bmatrix}^{mn}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}^{v_{m,n}}\begin{bmatrix}0\\1\end{bmatrix}^{n}\\&\qquad \mod G^{>3}.\end{aligned} \tag{143}$$

Then, by a similar discussion of (135), we make this product strongly ordered by only exchange relations. Thus, we have

$$\begin{bmatrix}0\\1\end{bmatrix}^{n+1}\begin{bmatrix}1\\0\end{bmatrix}^{m}\equiv\begin{bmatrix}1\\0\end{bmatrix}^{m}\begin{bmatrix}2\\1\end{bmatrix}^{u_{m,1}+u_{m,n}}\begin{bmatrix}1\\1\end{bmatrix}^{m(n+1)}\begin{bmatrix}1\\2\end{bmatrix}^{mn+v_{m,n}}\begin{bmatrix}0\\1\end{bmatrix}^{n+1} \mod G^{>3}. \tag{144}$$

It implies that

$$u_{m,n}=\binom{m}{2}n, \tag{145}$$

$$v_{m,n}=m\binom{n}{2}, \tag{146}$$

and

$$\begin{bmatrix}0\\1\end{bmatrix}^{n}\begin{bmatrix}1\\0\end{bmatrix}^{m}\equiv\begin{bmatrix}1\\0\end{bmatrix}^{m}\begin{bmatrix}2\\1\end{bmatrix}^{\binom{m}{2}n}\begin{bmatrix}1\\1\end{bmatrix}^{mn}\begin{bmatrix}1\\2\end{bmatrix}^{m\binom{n}{2}}\begin{bmatrix}0\\1\end{bmatrix}^{n} \mod G^{>3}. \tag{147}$$

For any $a,b\in\mathbb{Z}_{>0}$, we can obtain $u_{(a,b)}(m,n)$ by the above method in principle. However, it becomes harder to complete this calculation when its degree $a+b$ is larger. The following relation is the result of $G^{\leqslant 7}$.

$$
\begin{aligned}
&\begin{bmatrix}0\\1\end{bmatrix}^{n}\begin{bmatrix}1\\0\end{bmatrix}^{m}\\
&\equiv \begin{bmatrix}1\\0\end{bmatrix}^{m}\begin{bmatrix}6\\1\end{bmatrix}^{\binom{m}{6}\binom{n}{1}}\begin{bmatrix}5\\1\end{bmatrix}^{\binom{m}{5}\binom{n}{1}}\begin{bmatrix}4\\1\end{bmatrix}^{\binom{m}{4}\binom{n}{1}}\begin{bmatrix}3\\1\end{bmatrix}^{\binom{m}{3}\binom{n}{1}}\\
&\times \begin{bmatrix}5\\2\end{bmatrix}^{3\binom{m}{3}\binom{n}{2}+4\binom{m}{4}\binom{n}{1}+24\binom{m}{4}\binom{n}{2}+7\binom{m}{5}\binom{n}{1}+30\binom{m}{5}\binom{n}{2}}\\
&\times \begin{bmatrix}2\\1\end{bmatrix}^{\binom{m}{2}\binom{n}{1}}\begin{bmatrix}4\\2\end{bmatrix}^{6\binom{m}{3}\binom{n}{2}+2\binom{m}{4}\binom{n}{1}+12\binom{m}{4}\binom{n}{2}}\begin{bmatrix}3\\2\end{bmatrix}^{2\binom{m}{2}\binom{n}{2}+\binom{m}{3}\binom{n}{1}+6\binom{m}{3}\binom{n}{2}}\\
&\times \begin{bmatrix}4\\3\end{bmatrix}^{2\binom{m}{2}\binom{n}{2}+8\binom{m}{2}\binom{n}{3}+30\binom{m}{3}\binom{n}{2}+72\binom{m}{3}\binom{n}{3}+\binom{m}{4}\binom{n}{1}+48\binom{m}{4}\binom{n}{2}+96\binom{m}{4}\binom{n}{3}}\\
&\times \begin{bmatrix}1\\1\end{bmatrix}^{\binom{m}{1}\binom{n}{1}}\begin{bmatrix}2\\2\end{bmatrix}^{2\binom{m}{2}\binom{n}{2}}\begin{bmatrix}3\\3\end{bmatrix}^{6\binom{m}{2}\binom{n}{3}+6\binom{m}{3}\binom{n}{2}+18\binom{m}{3}\binom{n}{3}}\\
&\times \begin{bmatrix}3\\4\end{bmatrix}^{\binom{m}{1}\binom{n}{4}+2\binom{m}{2}\binom{n}{2}+30\binom{m}{2}\binom{n}{3}+48\binom{m}{2}\binom{n}{4}+8\binom{m}{3}\binom{n}{2}+72\binom{m}{3}\binom{n}{3}+96\binom{m}{3}\binom{n}{4}}\\
&\times \begin{bmatrix}2\\3\end{bmatrix}^{\binom{m}{1}\binom{n}{3}+2\binom{m}{2}\binom{n}{2}+6\binom{m}{2}\binom{n}{3}}\begin{bmatrix}1\\2\end{bmatrix}^{\binom{m}{1}\binom{n}{2}}\begin{bmatrix}2\\4\end{bmatrix}^{2\binom{m}{1}\binom{n}{4}+6\binom{m}{2}\binom{n}{3}+12\binom{m}{2}\binom{n}{4}}\\
&\times \begin{bmatrix}2\\5\end{bmatrix}^{4\binom{m}{1}\binom{n}{4}+7\binom{m}{1}\binom{n}{5}+3\binom{m}{2}\binom{n}{3}+24\binom{m}{2}\binom{n}{4}+30\binom{m}{2}\binom{n}{5}}\\
&\times \begin{bmatrix}1\\3\end{bmatrix}^{\binom{m}{1}\binom{n}{3}}\begin{bmatrix}1\\4\end{bmatrix}^{\binom{m}{1}\binom{n}{4}}\begin{bmatrix}1\\5\end{bmatrix}^{\binom{m}{1}\binom{n}{5}}\begin{bmatrix}1\\6\end{bmatrix}^{\binom{m}{1}\binom{n}{6}}\begin{bmatrix}0\\1\end{bmatrix}^{n} \quad \bmod G^{>7}.
\end{aligned} \tag{148}
$$

REMARK 6.2. In [7, Theorem 5.2], when $\delta_1 = \delta_2 = \delta$, it was shown that there exists the wall $(\sigma(1,-1), f)_{(1,1)}$ in a CSD $\mathfrak{D}_{\delta,\delta}$, where

$$
\begin{aligned}
f &= \left(\sum_{k=0}^{\infty}\frac{1}{(\delta^2-2\delta)k+1}\binom{(\delta-1)^2k}{k}x^{k(-\delta,\delta)}\right)^{\delta}\\
&= \left(1+x^{(-\delta,\delta)}+(\delta-1)^2x^{2(-\delta,\delta)}+\frac{1}{2}(\delta-1)^2(3\delta^2-6\delta+2)x^{3(-\delta,\delta)}+\cdots\right)^{\delta}.
\end{aligned} \tag{149}
$$

This function $f \in \mathbb{Q}((x_1, x_2))$ corresponds to the element $g \in G$ by the group homomorphism induced by the following map [2, Lemma 1.3].

$$
\begin{aligned}
G &\to \mathbb{Q}((x_1,x_2)),\\
\Psi[n] &\mapsto (1+x^{p^*(n)})^{\delta(n)},
\end{aligned} \tag{150}
$$

where $\delta(n)$ is the normalization factor with respect to (δ, δ), that is, $\delta(a, b) = \delta/\gcd(a, b)$. Then, in (148), the wall element

$$
g = \begin{bmatrix}1\\1\end{bmatrix}^{\delta^2}\begin{bmatrix}2\\2\end{bmatrix}^{2\binom{\delta}{2}^2}\begin{bmatrix}3\\3\end{bmatrix}^{12\binom{\delta}{2}\binom{\delta}{3}+18\binom{\delta}{3}^2}\cdots \tag{151}
$$

corresponds to

$$\begin{aligned}(152)\qquad &(1+x^{(-\delta,\delta)})^{\delta}(1+x^{2(-\delta,\delta)})^{\frac{2}{\delta}\times 2\binom{\delta}{2}^2}(1+x^{3(-\delta,\delta)})^{\frac{3}{\delta}\times\left(12\binom{\delta}{2}\binom{\delta}{3}+18\binom{\delta}{3}^2\right)}\cdots\\ &=\left(1+x^{(-\delta,\delta)}+(\delta-1)^2x^{2(-\delta,\delta)}+\frac{1}{2}(\delta-1)^2(3\delta^2-6\delta+2)x^{3(-\delta,\delta)}+\cdots\right)^{\delta}.\end{aligned}$$

So, this result agrees with the result of (149) in the lower degrees.

7. Bounded property of PBCs for ordered products

In Section 5, we showed that every exponent $u_{(a,b)}(m,n)$ is essentially expressed as a nonnegative PBC in m and n. In this section, we give a property about its degree.

Recall that, by Theorem 5.1, we can express

$$(153)\qquad u_{(a,b)}(m,n)=d(a,b)\sum_{\substack{0\leqslant i\leqslant A,\\ 0\leqslant j\leqslant B}}\alpha_{(a,b)}(i,j)\binom{m}{i}\binom{n}{j}$$

for some $\alpha_{(a,b)}(i,j), A, B\in\mathbb{Z}_{\geqslant 0}$. More strongly, we have the following statement.

Theorem 7.1 (Bounded property). *Let a and b be positive integers. Then, we can express*

$$(154)\qquad u_{(a,b)}(m,n)=d(a,b)\sum_{\substack{1\leqslant i\leqslant a,\\ 1\leqslant j\leqslant b}}\alpha_{(a,b)}(i,j)\binom{m}{i}\binom{n}{j},$$

where $\alpha_{(a,b)}(i,j)$ are nonnegative integers independent of m and n.

Namely, (i,j) can be restricted as $1\leqslant i\leqslant a$ and $1\leqslant j\leqslant b$ in the sum of (153).

We show the following two claims.

Claim 1. For any $i=0,1,\dots,A$ and $j=0,1,\dots,B$, it holds that $\alpha_{(a,b)}(i,0)=\alpha_{(a,b)}(0,j)=0$.

Claim 2. For any $(k,l)\in\mathbb{Z}^2_{\geqslant 0}$ with $k>a$ or $l>b$, it holds that $\alpha_{(a,b)}(k,l)=0$.

Proof of Claim 1. By (101), we have

$$(155)\qquad u_{(a,b)}(A,0)=0.$$

By (153), we have

$$(156)\qquad u_{(a,b)}(A,0)=d(a,b)\sum_{0\leqslant i\leqslant A}\alpha_{(a,b)}(i,0)\binom{A}{i}=0.$$

Since $\alpha_{(a,b)}(i,0)\geqslant 0$ and $\binom{A}{i}>0$, we obtain $\alpha_{(a,b)}(i,0)=0$ for any $i=0,1,\dots,A$. Similarly, we have $\alpha_{(a,b)}(0,j)=0$ for any $j=0,1,\dots,B$. Thus, we have

$$(157)\qquad u_{(a,b)}(m,n)=d(a,b)\sum_{\substack{1\leqslant i\leqslant A,\\ 1\leqslant j\leqslant B}}\alpha_{(a,b)}(i,j)\binom{m}{i}\binom{n}{j}.\qquad\square$$

Before proving Claim 2, we show the following lemma.

Lemma 7.2. *Let $l\in\mathbb{Z}_{\geqslant 1}$, and let $C_{(m,n)}$ be the product which is defined by (115) in $G^{\leqslant l+1}$. Let $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]$ be a dilogarithm element with $x+y\leqslant l+1$, and let $\left[\begin{smallmatrix}z\\ w\end{smallmatrix}\right]$ be the greatest dilogarithm element such that $\left[\begin{smallmatrix}z\\ w\end{smallmatrix}\right]<\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]$ and $z+w\leqslant l+1$. Let D be the product which is obtained by applying Algorithm 5.5 to $C_{(m,n)}$ repeatedly until $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]^*$ is in the stable part. Then, the form of D is as follows:*

$$(158)\qquad D=\cdots\begin{bmatrix}z\\ w\end{bmatrix}^{u_{(z,w)}(m,n)}D^{\mathrm{stab}}.$$

In the above expression, $D^{\mathrm{stab}} = [\begin{smallmatrix} x \\ y \end{smallmatrix}]^* \cdots [\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}]^{n+1}$ is the stable part of D, which is defined in Definition 5.3.

Proof. Since $[\begin{smallmatrix} z \\ w \end{smallmatrix}]$ is the greatest element such that $[\begin{smallmatrix} z \\ w \end{smallmatrix}] < [\begin{smallmatrix} x \\ y \end{smallmatrix}]$ and $z + w \leqslant l + 1$, the form of initial $C_{(m,n)}$ is

$$\cdots \begin{bmatrix} z \\ w \end{bmatrix}^{u_{(z,w)}(m,n)} \begin{bmatrix} x \\ y \end{bmatrix}^{u_{(x,y)}(m,n)} \prod_{(x,y)<(p,q)} \begin{bmatrix} p \\ q \end{bmatrix}^{u_{(p,q)}(m,n)}. \tag{159}$$

In particular, every dilogarithm element on the right side of $[\begin{smallmatrix} z \\ w \end{smallmatrix}]^{u_{(z,w)}(m,n)}$ is greater than $[\begin{smallmatrix} z \\ w \end{smallmatrix}]$. When we apply Algorithm 5.5 repeatedly, every factor $[\begin{smallmatrix} u \\ v \end{smallmatrix}]^*$ appearing in the right side of $[\begin{smallmatrix} z \\ w \end{smallmatrix}]^{u_{(z,w)}(m,n)}$ satisfies $[\begin{smallmatrix} z \\ w \end{smallmatrix}] < [\begin{smallmatrix} u \\ v \end{smallmatrix}]$. Thus, if the form of D is $\cdots [\begin{smallmatrix} z \\ w \end{smallmatrix}]^{u_{(z,w)}(m,n)} \cdots [\begin{smallmatrix} u \\ v \end{smallmatrix}]^* \cdots D^{\mathrm{stab}}$, it holds that $[\begin{smallmatrix} z \\ w \end{smallmatrix}] < [\begin{smallmatrix} u \\ v \end{smallmatrix}] < [\begin{smallmatrix} x \\ y \end{smallmatrix}]$. (Note that D^{stab} has the factor $[\begin{smallmatrix} x \\ y \end{smallmatrix}]^*$.) It contradicts the assumptions of $[\begin{smallmatrix} z \\ w \end{smallmatrix}]$. □

Let us prove Claim 2. Let $f(m,n)$ be a polynomial or a PBC. We write the degree of $f(m,n)$ as a polynomial in n by $\deg_n(f(m,n))$.

Proof of Claim 2. Suppose that the claim does not hold; in other words, there exists $\alpha_{(a,b)}(k,l) > 0$ such that $k > a$ or $l > b$. Suppose $a + b$ is smallest among such (a,b), and there exists $l > b$ such that $\alpha_{(a,b)}(k,l) > 0$. (If there exists such $k > a$, we can do a similar argument.) Then, since

$$\begin{aligned} &u_{(a,b)}(m,n+1) - u_{(a,b)}(m,n) \\ &= d(a,b) \sum_{\substack{1\leqslant i\leqslant A,\\ 1\leqslant j\leqslant B}} \alpha_{(a,b)}(i,j) \binom{m}{i} \left\{ \binom{n+1}{j} - \binom{n}{j} \right\} \\ &\overset{(12)}{=} d(a,b) \sum_{\substack{1\leqslant i\leqslant A,\\ 1\leqslant j\leqslant B}} \alpha_{(a,b)}(i,j) \binom{m}{i} \binom{n}{j-1} \\ &= d(a,b) \alpha_{(a,b)}(k,l) \binom{m}{k} \binom{n}{l-1} \\ &\qquad + d(a,b) \sum_{\substack{1\leqslant i\leqslant A,\ 1\leqslant j\leqslant B,\\ (i,j)\neq(k,l)}} \alpha_{(a,b)}(i,j) \binom{m}{i} \binom{n}{j-1}, \end{aligned} \tag{160}$$

$u_{(a,b)}(m,n+1) - u_{(a,b)}(m,n)$ has a factor $\binom{m}{k}\binom{n}{l-1}$ with a positive coefficient. In particular, it holds that

$$\deg_n(u_{(a,b)}(m,n+1) - u_{(a,b)}(m,n)) \geqslant l - 1 \geqslant b. \tag{161}$$

Now, we apply Algorithm 5.5 to $C_{(m,n)}$ repeatedly. Then, by Proposition 5.8, we have a following relation:

$$\begin{aligned} u_{(a,b)}(m,n+1) = u_{(a,b)}(m,n) + u_{(a,b)}(m,1) \\ + \text{(factors produced in Algorithm 5.5)}. \end{aligned} \tag{162}$$

Because $\deg_n(u_{(a,b)}(m,n+1) - u_{(a,b)}(m,n)) \geqslant b$, there exists an anti-ordered pair $[\begin{smallmatrix} z \\ w \end{smallmatrix}]^g [\begin{smallmatrix} x \\ y \end{smallmatrix}]^f$ which produces $[\begin{smallmatrix} a \\ b \end{smallmatrix}]^h$ with $\deg_n(h) \geqslant b$. If $x + y = a + b$ or $z + w = a + b$, then this anti-ordered pair does not produce new factors. Assume $x+y, z+w < a+b$.

Let $F = \left(\begin{smallmatrix} x & z \\ y & w \end{smallmatrix}\right)$. Then, Step 1.1 (ii) in Algorithm 5.5 becomes

$$\begin{aligned}&\begin{bmatrix} z \\ w \end{bmatrix}^g \begin{bmatrix} x \\ y \end{bmatrix}^f \\ &\equiv \begin{bmatrix} x \\ y \end{bmatrix}^f \left\{ \overrightarrow{\prod_{\substack{(p,q)\in\mathbb{Z}^2_{\geqslant 1},\\ \deg(p,q)\leqslant a+b-1,\\ \deg F\binom{p}{q}\leqslant a+b}}} \left(F \begin{bmatrix} p \\ q \end{bmatrix} \right)^{u_{(p,q)}(|F|f,|F|g)} \right\} \begin{bmatrix} z \\ w \end{bmatrix}^g \mod G^{>a+b}.\end{aligned} \tag{163}$$

Since the above product on the RHS has a factor $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^h$, there exists $(p,q) \in \mathbb{Z}^2_{\geqslant 1}$ satisfying

$$F \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \iff \begin{cases} px + qz = a, \\ py + qw = b. \end{cases} \tag{164}$$

By using the above (p,q), the exponent of $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ is $h = u_{(p,q)}(|F|f, |F|g)/|F|$. Moreover, since $\deg_n(h) \geqslant b$, we have $\deg_n(u_{(p,q)}(|F|f, |F|g)) \geqslant b$. Since $\deg(p,q) < \deg(a,b)$ and the smallest assumption of (a,b), we have $\deg_m(u_{(p,q)}(m,n)) \leqslant p$ and $\deg_n(u_{(p,q)}(m,n)) \leqslant q$. Now, let $\deg_n(f) = t$ and $\deg_n(g) = t'$. Then, $\deg_n(u_{(p,q)}(|F|f, |F|g)) \leqslant tp + t'q$. Hence, we have

$$tp + t'q \geqslant b. \tag{165}$$

On the other hand, since $\deg_n(u_{(x,y)}(m,n)) \leqslant y$, we have $\deg_n(g) = t' \leqslant y$. Similarly, $\deg_n(f) = t \leqslant w$ holds. Moreover, since $p, q \geqslant 1$, we have

$$tp + t'q \leqslant yp + wq \overset{(164)}{=} b. \tag{166}$$

Combining the above two inequalities, we have $tp + t'q = b$, and it implies $t = y$ and $t' = w$. In particular, $\deg_n(g) = w$ holds. To summarize, $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^h$ is produced when $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]^g$ moves to the right hand side, and $\deg_n(g) \geqslant w$. We apply Algorithm 5.5 to $C_{(m,n)}$ until the greatest dilogarithm element appearing in its unstable part is $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$. Then, by Lemma 7.2, $C_{(m,n)}$ becomes

$$\cdots \begin{bmatrix} z \\ w \end{bmatrix}^g \cdots \begin{bmatrix} z \\ w \end{bmatrix}^{u_{(z,w)}(m,n)} \overrightarrow{\prod_{(u,v)>(z,w)}} \begin{bmatrix} u \\ v \end{bmatrix}^{u_{(u,v)}(m,n+1)}. \tag{167}$$

It implies that

$$u_{(z,w)}(m, n+1) = u_{(z,w)}(m,n) + g + \cdots. \tag{168}$$

Thus, $\deg_n(u_{(z,w)}(m,n)) > \deg_n(g) \geqslant w$, contradicting $\deg(z,w) < \deg(a,b)$. □

In Section 8.3, we can see some properties for these coefficients $\alpha_{(a,b)}(i,j)$.

8. Further results and examples

8.1. Inverse formula. Thanks to Theorem 7.1, $u_{(a,b)}(m,n)$ may be recovered from the special values $u_{(a,b)}(i,j)$ for $1 \leqslant i \leqslant a$ and $1 \leqslant j \leqslant b$.

Let $I = \{(i,j) \in \mathbb{Z} \mid 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b\}$, and let $u_{(a,b)}(m,n) = \sum_{\substack{1\leqslant k\leqslant a,\\ 1\leqslant l\leqslant b}} \alpha_{k,l}\binom{m}{k}\binom{n}{l}$. Namely, by using the notation of Theorem 7.1, we write $\alpha_{k,l} = d(a,b)\alpha_{(a,b)}(k,l)$.

First, we define the following three matrices:

$$\alpha = (\alpha_{i,j})_{(i,j)\in I} \in \mathrm{Mat}_{I\times 1}(\mathbb{Q}), \tag{169}$$

$$A = \left(\binom{i}{i'}\binom{j}{j'} \right)_{((i,j),(i',j'))\in I\times I}, \tag{170}$$

$$u = (u_{(a,b)}(i,j))_{(i,j)\in I} \in \mathrm{Mat}_{I\times 1}(\mathbb{Q}). \tag{171}$$

These three matrices have the relation

$$A\alpha = \left(\sum_{(i',j')\in I} \alpha_{i',j'} \binom{i}{i'}\binom{j}{j'} \right)_{(i,j)\in I} = (u_{(a,b)}(i,j))_{(i,j)\in I} = u. \tag{172}$$

This implies

$$\alpha = A^{-1}u. \tag{173}$$

Next, we obtain the inverse matrix A^{-1}. We define $P_a \in \mathrm{Mat}_a(\mathbb{Z})$ as follows:

$$P_a = \left(\binom{i}{j} \right)_{(i,j)}. \tag{174}$$

Similarly, we define $P_b \in \mathrm{Mat}_b(\mathbb{Z})$. Then, $A = P_a \otimes P_b$ holds, where $P_a \otimes P_b = \left(\binom{i}{i'}\binom{j}{j'}\right)_{((i,j),(i',j'))} \in \mathrm{Mat}_I(\mathbb{Z})$ is the tensor product. The inverse matrix of P_a is known as follows.

Lemma 8.1 (e.g. [9, Thm. 37]). *For any $a \in \mathbb{Z}_{>0}$, we have*

$$P_a^{-1} = \left((-1)^{i+j}\binom{i}{j} \right). \tag{175}$$

Thus, we obtain the inverse matrix A^{-1} as follows.

Lemma 8.2. *Let A be the matrix defined in (170). Then, it holds that*

$$A^{-1} = \left((-1)^{i+i'+j+j'} \binom{i}{i'}\binom{j}{j'} \right)_{((i,j),(i',j'))}. \tag{176}$$

Proof. It is immediately shown by

$$A^{-1} = (P_a \otimes P_b)^{-1} = P_a^{-1} \otimes P_b^{-1}. \tag{177}$$

□

By the above arguments, we may write the coefficients $\alpha_{k,l} = d(a,b)\alpha_{(a,b)}(k,l)$ by using special values $u_{(a,b)}(i,j)$.

Proposition 8.3 (*inverse formula*). *Let a and b be positive integers. Then, for any $1 \leqslant k \leqslant a$ and $1 \leqslant l \leqslant b$, it holds that*

$$d(a,b)\alpha_{(a,b)}(k,l) = \sum_{\substack{1\leqslant i\leqslant k,\\ 1\leqslant j\leqslant l}} (-1)^{i+j+k+l}\binom{k}{i}\binom{l}{j} u_{(a,b)}(i,j). \tag{178}$$

Proof. It is immediately shown by (173) and Lemma 8.2. □

Example 8.4. In [5, Algorithm 5.7], the method to find the exponent of $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ in the ordered product of $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]^n\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^m$ is known for certain m and n. For example, consider the exponent of $\left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right]$. Then, we can find the special values of $u_{(3,2)}(m,n)$ as follows:

$$\begin{gathered} u_{(3,2)}(1,1) = 0, u_{(3,2)}(1,2) = 0, u_{(3,2)}(2,1) = 0, \\ u_{(3,2)}(2,2) = 2, u_{(3,2)}(3,1) = 1, u_{(3,2)}(3,2) = 14. \end{gathered} \tag{179}$$

By Proposition 8.3, we have $u_{(3,2)}(m,n) = 2\binom{m}{2}\binom{n}{2} + \binom{m}{3}\binom{n}{1} + 6\binom{m}{3}\binom{n}{2}$.

8.2. Reciprocity.

Proposition 8.5 (reciprocity). *Let $(a,b)\in N^+$. Then, for any $m,n\in\mathbb{Z}_{\geqslant 0}$,*

$$u_{(a,b)}(m,n)=u_{(b,a)}(n,m) \tag{180}$$

holds.

Proof. By definition of $u_{(a,b)}(m,n)$, we have

$$\begin{bmatrix}0\\1\end{bmatrix}^n\begin{bmatrix}1\\0\end{bmatrix}^m=\overrightarrow{\prod_{(a,b)\in N^+}}\begin{bmatrix}a\\b\end{bmatrix}^{u_{(a,b)}(m,n)}. \tag{181}$$

Let $F=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then, by acting F, we have

$$\begin{bmatrix}1\\0\end{bmatrix}^{-n}\begin{bmatrix}0\\1\end{bmatrix}^{-m}=\prod_{(a,b)\in N^+}\begin{bmatrix}b\\a\end{bmatrix}^{-u_{(a,b)}(m,n)}. \tag{182}$$

Considering the inverse of the above relation, we have

$$\begin{bmatrix}0\\1\end{bmatrix}^{m}\begin{bmatrix}1\\0\end{bmatrix}^{n}=\prod_{(a,b)\in (N^+)^{\mathrm{op}}}\begin{bmatrix}b\\a\end{bmatrix}^{u_{(a,b)}(m,n)}. \tag{183}$$

The index set $(N^+)^{\mathrm{op}}$ is the opposite ordered set of N^+. The RHS is not the strongly ordered product because of the parallel dilogarithm elements. However, by using the relation (61), we may rearrange it to the strongly ordered product without changing these exponents. Thus, we have

$$\begin{bmatrix}0\\1\end{bmatrix}^{m}\begin{bmatrix}1\\0\end{bmatrix}^{n}=\overrightarrow{\prod_{(b,a)\in N^+}}\begin{bmatrix}b\\a\end{bmatrix}^{u_{(a,b)}(m,n)}.$$

It implies that

$$u_{(b,a)}(n,m)=u_{(a,b)}(m,n). \tag{184}$$

□

8.3. Properties of coefficients. By Theorem 7.1, we write

$$u_{(a,b)}(m,n)=d(a,b)\sum_{\substack{1\leqslant i\leqslant a,\\ 1\leqslant j\leqslant b}}\alpha_{(a,b)}(i,j)\binom{m}{i}\binom{n}{j}, \tag{185}$$

where $\alpha_{(a,b)}(i,j)\in\mathbb{Z}_{\geqslant 0}$. In the above expression, the upper bound is essential.

Proposition 8.6. *Let a and b be positive integers. Then, it holds that*

$$\alpha_{(a,b)}(a,b)>0. \tag{186}$$

Proof. We show the claim by the induction on $l=a+b$. If $a+b=2$, namely, if $a=b=1$, then, by (148), we have $u_{(1,1)}(m,n)=mn$. It indicates that $\alpha_{(1,1)}(1,1)=1$, and the claim holds. For some $l\geqslant 2$, suppose that the claim holds for any $(a,b)\in\mathbb{Z}^2_{\geqslant 1}$ with $a+b=l$. Let a and b be positive integers satisfying $a+b=l+1$. By Proposition 8.5, it suffices to show the claim when $a\leqslant b$. Then, since $a+b=l+1\geqslant 3$, it holds that $b\geqslant 2$. Consider the product $C_{(m,n)}$ which is defined in (115). By applying Algorithm 5.5 to $C_{(m,n)}$, there exists the following operation:

$$\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}a\\b-1\end{bmatrix}^{u_{(a,b-1)}(m,n)}\equiv\begin{bmatrix}a\\b-1\end{bmatrix}^{u_{(a,b-1)}(m,n)}\begin{bmatrix}a\\b\end{bmatrix}^{au_{(a,b-1)}(m,n)}\begin{bmatrix}0\\1\end{bmatrix} \mod G^{>a+b}. \tag{187}$$

The above relation follows from applying Proposition 4.4 to the following relation by $F = \left(\begin{smallmatrix} a & 0 \\ b-1 & 1 \end{smallmatrix}\right)$. Note that the factors $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ satisfying $\deg F\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \leqslant a+b$ are $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, and $u_{(1,1)}(au_{(a,b-1)}(m,n), a) = a^2 u_{(a,b-1)}(m,n)$.

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^a \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{au_{(a,b-1)}(m,n)} \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{au_{(a,b-1)}(m,n)} \cdots \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{a^2u_{(a,b-1)}(m,n)} \cdots \begin{bmatrix} 0 \\ 1 \end{bmatrix}^a \mod G^{>a+b-1}. \tag{188}$$

Hence, by Lemma 5.4 E, we have

$$u_{(a,b)}(m,n+1) = u_{(a,b)}(m,n) + au_{(a,b-1)}(m,n) + \cdots . \tag{189}$$

It holds that

$$\begin{aligned} u_{(a,b)}(m,n) &\overset{(118)}{=} \sum_{k=0}^{n-1} (au_{(a,b-1)}(m,k) + \cdots) \\ &= ad(a,b-1) \sum_{\substack{1\leqslant i\leqslant a,\\ 1\leqslant j\leqslant b-1}} \alpha_{(a,b-1)}(a,b-1) \binom{m}{i} \sum_{k=0}^{n-1} \binom{k}{j} + \cdots . \\ &\overset{(15)}{=} ad(a,b-1) \sum_{\substack{1\leqslant i\leqslant a,\\ 1\leqslant j\leqslant b-1}} \alpha_{(a,b-1)}(a,b-1) \binom{m}{i} \binom{n}{j+1} + \cdots . \end{aligned} \tag{190}$$

By focusing on the coefficient of $\binom{m}{a}\binom{n}{b}$, we have $d(a,b)\alpha_{(a,b)}(a,b) \geqslant ad(a,b-1)\alpha_{(a,b-1)}(a,b-1) > 0$. □

On the other hand, the lower bound is not known yet. However, a partial result can be derived.

Proposition 8.7. *Let $(a,b) \in N^+$ with $a > b \geqslant 1$. Then, for any positive integers $i < \frac{a}{b}$ and $j \leqslant b$, it holds that*

$$\alpha_{(a,b)}(i,j) = 0. \tag{191}$$

Proof. Let s be the largest integer such that $1 \leqslant s < \frac{a}{b}$. By Lemma 3.20, the strongly ordered product expression of $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]^b \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^s$ is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}^s \begin{bmatrix} s \\ 1 \end{bmatrix}^b \cdots \begin{bmatrix} 1 \\ b \end{bmatrix}^s \begin{bmatrix} 0 \\ 1 \end{bmatrix}^b . \tag{192}$$

Since $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] < \left[\begin{smallmatrix} s \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, we have $u_{(a,b)}(s,b) = 0$. By Theorem 7.1, it implies that

$$u_{(a,b)}(s,b) = d(a,b) \sum_{\substack{1\leqslant i\leqslant s,\\ 1\leqslant j\leqslant b}} \alpha_{(a,b)}(i,j) \binom{s}{i} \binom{b}{j} = 0. \tag{193}$$

Since $\binom{s}{i}\binom{b}{j} > 0$ and $\alpha_{(a,b)}(i,j) \geqslant 0$, we have $\alpha_{(a,b)}(i,j) = 0$ for any $i \leqslant s$ and $j \leqslant b$. This completes the proof. □

8.4. Special cases.

Proposition 8.8. *Let $a, b \in \mathbb{Z}_{\geqslant 0}$. Then, for any $m, n \in \mathbb{Z}_{\geqslant 0}$, we have*

$$u_{(a,1)}(m,n) = \binom{m}{a}\binom{n}{1}, \quad u_{(1,b)}(m,n) = \binom{m}{1}\binom{n}{b}. \tag{194}$$

Proof. By Theorem 7.1 and Proposition 8.7, $u_{(a,1)}(m,n)$ is expressed as

$$u_{(a,1)}(m,n) = d(a,1)\alpha_{(a,1)}(a,1)\binom{m}{a}\binom{n}{1} = \alpha_{(a,1)}(a,1)\binom{m}{a}\binom{n}{1}. \tag{195}$$

By Lemma 3.20, we have

$$u_{(a,1)}(a,1) = 1. \tag{196}$$

Thus, we obtain

$$\alpha_{(a,1)}(a,1)\binom{a}{a}\binom{1}{1} = \alpha_{(a,1)}(a,1) = 1, \tag{197}$$

and it implies that $u_{(a,1)}(m,n) = \binom{m}{a}\binom{n}{1}$. □

We may also find the exponent of $\left[\frac{a}{2}\right]$ explicitly. However, the proof is excessively long.

Theorem 8.9. *Let $a \in \mathbb{Z}_{\geqslant 0}$. Then, for any $m,n \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\begin{aligned} &u_{(a,2)}(m,n) \\ &= \sum_{\frac{a}{2}<k\leqslant a} \left\lceil \frac{2k-a}{2} \right\rceil \binom{2k-a}{\lceil \frac{2k-a}{2} \rceil}\binom{k}{2k-a}\binom{m}{k}\binom{n}{2} \\ &\qquad + \sum_{\frac{a}{2}+1<k\leqslant a} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil \frac{2k-a-1}{2} \rceil} - 2^{2k-a-2} \right\} \binom{k}{2k-a}\binom{m}{k}\binom{n}{1}. \end{aligned} \tag{198}$$

In the above relation, $\lceil x \rceil$ is the least integer more than or equal to $x \in \mathbb{Q}$. In (148), we can see the examples. Also, we can simplify the above formula as follows:

$$\begin{aligned} u_{(a,2)}(m,n) &= m\binom{m-1}{\lfloor \frac{a}{2} \rfloor}\binom{m-1}{\lceil \frac{a}{2}-1 \rceil}\binom{n}{2} + \frac{m}{2}\binom{m-1}{\lfloor \frac{a-1}{2} \rfloor}\binom{m-1}{\lceil \frac{a-1}{2} \rceil}\binom{n}{1} \\ &\qquad - \sum_{\frac{a}{2}<k\leqslant a} 2^{2k-a-2}\binom{k}{2k-a}\binom{m}{k}\binom{n}{1}. \end{aligned} \tag{199}$$

The proof of equivalence in these two expressions is given in Appendix B.

We can check that every coefficient in the relation (198) is nonzero. The proof of Theorem 8.9 is in Section 9.

9. Proof of Theorem 8.9

In this section, we express the exponent of $\left[\frac{a}{2}\right]$ explicitly. For the sake of simplicity, the proof of some equalities are given in Appendix A. For any $x \in \mathbb{Q}$, $\lfloor x \rfloor$ is the greatest integer less than or equal to x, and $\lceil x \rceil$ is the least integer more than or equal to x. First, we derive the recurrence relations enough to determine all $u_{(a,2)}(m,n)$ based on Method 5.7.

Proposition 9.1. (a). *Let $a \in \mathbb{Z}_{\geqslant 1}$. Then, for any $m,n \in \mathbb{Z}_{\geqslant 0}$, the following relation holds.*

$$\begin{aligned} &u_{(a,2)}(m,n+1) \\ &= u_{(a,2)}(m,n) + u_{(a,2)}(m,1) \\ &\qquad + \sum_{k=\lceil \frac{a}{2} \rceil}^{a} \left\{ \sum_{x=a-k}^{\lfloor \frac{a}{2} \rfloor} (a-2x)\binom{a-x}{k-x}\binom{k}{a-x} \right\} \binom{m}{k}\binom{n}{1}. \end{aligned} \tag{200}$$

(b). *Let $a \in \mathbb{Z}_{\geqslant 3}$. Then, for any $m \in \mathbb{Z}_{\geqslant 0}$, the following relation holds.*

$$\begin{aligned} &u_{(a,2)}(m+1,1) \\ &= u_{(a,2)}(m,1) + u_{(a-2,2)}(m,1) \\ &\quad + \sum_{k=\lceil \frac{a}{2} \rceil}^{a-1} \left\{ \sum_{x=a-k}^{\lfloor \frac{a}{2} \rfloor} (a-2x) \binom{a-x}{k-x+1} \binom{k}{a-x} \right\} \binom{m}{k}. \end{aligned} \tag{201}$$

Proof. (a). Let $C_{(m,n)}$ be the product which is defined in (115). Apply Algorithm 5.5 to $C_{(m,n)}$ repeatedly until it becomes strongly ordered. Suppose that an anti-ordered pair $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^g \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]^f$ produces a factor $\left[\begin{smallmatrix} a \\ 2 \end{smallmatrix}\right]^*$ in Step 1.1 (ii). Since every dilogarithm element $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$ appearing in the initial product $C_{(m,n)}$ satisfies $t \geqslant 1$, we have $y, w \geqslant 1$. By (108), there exists $(p,q) \in \mathbb{Z}^2_{\geqslant 1}$ satisfying

$$\begin{pmatrix} pz + qx \\ pw + qy \end{pmatrix} = \begin{pmatrix} a \\ 2 \end{pmatrix}. \tag{202}$$

Since $pw + qy = 2$ and $p, q, y, w \geqslant 1$, we have $p = q = y = w = 1$. Since $pz + qx = a$, we have $z = a - x$. By $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] > \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$, we have $xw - yz < 0$, and these imply that $2x - a < 0$. Thus, every anti-ordered pair $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^g \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]^f$ which produces the factor $\left[\begin{smallmatrix} a \\ 2 \end{smallmatrix}\right]^*$ has a following form:

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^g \begin{bmatrix} a-x \\ 1 \end{bmatrix}^f \quad \left(x < \frac{a}{2} \right). \tag{203}$$

Moreover, factors $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]^*$ ($x = 0, 1, 2, \dots$) are not produced when we apply Algorithm 5.5 to $C_{(m,n)}$. Thus, both $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]^g$ and $\left[\begin{smallmatrix} a-x \\ 1 \end{smallmatrix}\right]^f$ should be in the initial $C_{(m,n)}$. So, the anti-ordered pairs that produce $\left[\begin{smallmatrix} a \\ 2 \end{smallmatrix}\right]^*$ are only the following ones:

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^{u_{(x,1)}(m,1)} \begin{bmatrix} a-x \\ 1 \end{bmatrix}^{u_{(a-x,1)}(m,n)} \quad \left(x < \frac{a}{2} \right). \tag{204}$$

For any x, let $F = \left(\begin{smallmatrix} a-x & x \\ 1 & 1 \end{smallmatrix}\right)$. Then, the following relations hold by Proposition 4.4.

$$\begin{aligned} &\begin{bmatrix} x \\ 1 \end{bmatrix}^{u_{(x,1)}(m,1)} \begin{bmatrix} a-x \\ 1 \end{bmatrix}^{u_{(a-x,1)}(m,n)} \\ &= \left(F \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^{(a-2x)u_{(x,1)}(m,1)} \left(F \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^{(a-2x)u_{(a-x,1)}(m,n)} \\ &= \left(F \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{(a-2x)u_{(a-x,1)}(m,n)} \right) \cdots \\ &\qquad \times \left(F \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{u_{(1,1)}\left((a-2x)u_{(a-x,1)}(m,n), (a-2x)u_{(x,1)}(m,1)\right)} \\ &\qquad \times \cdots \left(F \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^{(a-2x)u_{(x,1)}(m,1)} \\ &= \begin{bmatrix} a-x \\ 1 \end{bmatrix}^{u_{(a-x,1)}(m,n)} \cdots \\ &\qquad \times \begin{bmatrix} a \\ 2 \end{bmatrix}^{\frac{1}{a-2x} u_{(1,1)}\left((a-2x)u_{(a-x,1)}(m,n), (a-2x)u_{(x,1)}(m,1)\right)} \\ &\qquad \times \cdots \begin{bmatrix} x \\ 1 \end{bmatrix}^{u_{(x,1)}(m,1)}. \end{aligned} \tag{205}$$

In the above relations, the third and the fourth products are strongly ordered. Moreover, because of $u_{(1,1)}(m,n)=mn$, we have

$$\begin{aligned}&\frac{1}{a-2x}u_{(1,1)}\big((a-2x)u_{(a-x,1)}(m,n),(a-2x)u_{(x,1)}(m,1)\big)\\&=(a-2x)u_{(a-x,1)}(m,n)u_{(x,1)}(m,1).\end{aligned}\tag{206}$$

By Proposition 8.8, it is

$$\begin{aligned}&(a-2x)\binom{m}{a-x}\binom{n}{1}\binom{m}{x}=(a-2x)\binom{m}{a-x}\binom{m}{x}\binom{n}{1}\\&\overset{(16)}{=}(a-2x)\sum_{k=a-x}^{a}\binom{a-x}{k-x}\binom{k}{a-x}\binom{m}{k}\binom{n}{1}.\end{aligned}\tag{207}$$

Thus, we have

$$\begin{aligned}&u_{(a,2)}(m,n+1)\\&=u_{(a,2)}(m,n)+u_{(a,2)}(m,1)\\&\qquad+\sum_{0\leqslant x<\frac{a}{2}}(a-2x)\sum_{k=a-x}^{a}\binom{a-x}{k-x}\binom{k}{a-x}\binom{m}{k}\binom{n}{1}\\&=u_{(a,2)}(m,n)+u_{(a,2)}(m,1)\\&\qquad+\sum_{0\leqslant x\leqslant\frac{a}{2}}\sum_{k=a-x}^{a}(a-2x)\binom{a-x}{k-x}\binom{k}{a-x}\binom{m}{k}\binom{n}{1}.\end{aligned}\tag{208}$$

Since

$$\begin{aligned}&\Big\{(x,k)\in\mathbb{Z}^2\mid 0\leqslant x\leqslant\Big\lfloor\frac{a}{2}\Big\rfloor,\ a-x\leqslant k\leqslant a\Big\}\\&=\Big\{(x,k)\in\mathbb{Z}^2\mid a-\Big\lfloor\frac{a}{2}\Big\rfloor\,(=\Big\lceil\frac{a}{2}\Big\rceil)\leqslant k\leqslant a,\ a-k\leqslant x\leqslant\Big\lfloor\frac{a}{2}\Big\rfloor\Big\},\end{aligned}\tag{209}$$

the equality (208) can be expressed as follows:

$$\begin{aligned}&u_{(a,2)}(m,n+1)\\&=u_{(a,2)}(m,n)+u_{(a,2)}(m,1)\\&\qquad+\sum_{k=\lceil\frac{a}{2}\rceil}^{a}\Bigg\{\sum_{x=a-k}^{\lfloor\frac{a}{2}\rfloor}(a-2x)\binom{a-x}{k-x}\binom{k}{a-x}\Bigg\}\binom{m}{k}\binom{n}{1}.\end{aligned}\tag{210}$$

(b). Let $C_{(m,1)}$ be the product which is defined in (113). Namely, consider

$$C_{(m,1)}=\begin{bmatrix}1\\0\end{bmatrix}\underline{\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^m}\Bigg(\overrightarrow{\prod_{\substack{x+y\leqslant a+2\\x,y\geqslant 1}}}\begin{bmatrix}x\\y\end{bmatrix}^{u_{(x,y)}(m,1)}\Bigg)\tag{211}$$

Let $F = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Then, by Proposition 4.4, we have

$$\begin{aligned}
\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^m &= \left(F\begin{bmatrix}0\\1\end{bmatrix}\right)\left(F\begin{bmatrix}1\\0\end{bmatrix}\right)^m \\
&\equiv \left(F\begin{bmatrix}1\\0\end{bmatrix}\right)^m \left\{\overrightarrow{\prod_{\substack{c+d\leqslant a+1,\\ c,d\geqslant 1}}}\left(F\begin{bmatrix}c\\d\end{bmatrix}\right)^{u_{(c,d)}(m,1)}\right\}\left(F\begin{bmatrix}0\\1\end{bmatrix}\right) \\
&= \begin{bmatrix}1\\0\end{bmatrix}^m \left(\overrightarrow{\prod_{\substack{c+d\leqslant a+1,\\ c,d\geqslant 1}}}\begin{bmatrix}c+d\\d\end{bmatrix}^{u_{(c,d)}(m,1)}\right)\begin{bmatrix}1\\1\end{bmatrix} \\
&\equiv \begin{bmatrix}1\\0\end{bmatrix}^m \left(\overrightarrow{\prod_{\substack{c+d\leqslant a+1,\\ c,d\geqslant 1,\\ z=c+d, w=d}}}\begin{bmatrix}z\\w\end{bmatrix}^{u_{(z-w,w)}(m,1)}\right)\begin{bmatrix}1\\1\end{bmatrix} \quad \bmod G^{>a+2}.
\end{aligned} \tag{212}$$

Putting the last expression to the RHS of (211), we have

$$\begin{aligned}
C_{(m,1)} \equiv\ & \begin{bmatrix}1\\0\end{bmatrix}^{m+1}\left(\overrightarrow{\prod}\begin{bmatrix}z\\w\end{bmatrix}^{u_{(z-w,w)}(m,1)}\right)\begin{bmatrix}1\\1\end{bmatrix} \\
& \times\left(\overrightarrow{\prod_{\substack{x+y\leqslant a+2\\ x,y\geqslant 1}}}\begin{bmatrix}x\\y\end{bmatrix}^{u_{(x,y)}(m,1)}\right).
\end{aligned} \tag{213}$$

Let

$$C' = \left(\overrightarrow{\prod}\begin{bmatrix}z\\w\end{bmatrix}^{u_{(z-w,w)}(m,1)}\right)\begin{bmatrix}1\\1\end{bmatrix}\left(\overrightarrow{\prod_{\substack{x+y\leqslant a+2\\ x,y\geqslant 1}}}\begin{bmatrix}x\\y\end{bmatrix}^{u_{(x,y)}(m,1)}\right). \tag{214}$$

Then, C' satisfies the following conditions:

- Every dilogarithm element $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ appearing in C' satisfies $y \geqslant 1$.
- The exponents of the factor $\left[\begin{smallmatrix}a\\2\end{smallmatrix}\right]^*$ are $u_{(a,2)}(m,1)$ and $u_{(a-2,2)}(m,1)$.

Thus, by a similar argument of (a), anti-ordered pairs producing $\left[\begin{smallmatrix}a\\2\end{smallmatrix}\right]^*$ are

$$\begin{bmatrix}x\\1\end{bmatrix}^{u_{(x-1,1)}(m,1)}\begin{bmatrix}a-x\\1\end{bmatrix}^{u_{(a-x,1)}(m,1)} \quad \left(1\leqslant x<\frac{a}{2}\right). \tag{215}$$

Moreover, for each $x = 1, 2, \cdots$, it produces $\left[\begin{smallmatrix}a\\2\end{smallmatrix}\right]^*$ whose exponent is

$$\begin{aligned}
(a-2x)u_{(x-1,1)}(m,1)u_{(a-x,1)}(m,1) &= (a-2x)\binom{m}{x-1}\binom{m}{a-x} \\
&\overset{(16)}{=} (a-2x)\sum_{k=a-x}^{a-1}\binom{a-x}{k-x+1}\binom{k}{a-x}\binom{m}{k}.
\end{aligned} \tag{216}$$

Thus, we have

$$\begin{aligned} &u_{(a,2)}(m+1,1)\\ &= u_{(a,2)}(m,1) + u_{(a-2,2)}(m,1)\\ &\qquad + \sum_{1\leqslant x<\frac{a}{2}} \sum_{k=a-x}^{a-1} (a-2x)\binom{a-x}{k-x+1}\binom{k}{a-x}\binom{m}{k}\\ &= u_{(a,2)}(m,1) + u_{(a-2,2)}(m,1)\\ &\qquad + \sum_{k=\lceil\frac{a}{2}\rceil}^{a-1} \left\{\sum_{x=a-k}^{\lfloor\frac{a}{2}\rfloor} (a-2x)\binom{a-x}{k-x+1}\binom{k}{a-x}\right\}\binom{m}{k}. \end{aligned} \tag{217}$$

This completes the proof. ☐

Next, we try to solve this recurrence relations. By Proposition 9.1 (a), we have

$$\begin{aligned} u_{(a,2)}(m,n) &= u_{(a,2)}(m,0) + \sum_{j=0}^{n-1}\left\{u_{(a,2)}(m,j+1) - u_{(a,2)}(m,j)\right\}\\ &\overset{(101)}{=} \sum_{j=0}^{n-1}\left\{u_{(a,2)}(m,j+1) - u_{(a,2)}(m,j)\right\}\\ &= \sum_{j=0}^{n-1} u_{(a,2)}(m,1)\\ &\qquad + \sum_{k=\lceil\frac{a}{2}\rceil}^{a}\left\{\sum_{x=a-k}^{\lfloor\frac{a}{2}\rfloor}(a-2x)\binom{a-x}{k-x}\binom{k}{a-x}\right\}\binom{m}{k}\sum_{j=1}^{n-1}\binom{j}{1}\\ &\overset{(15)}{=} u_{(a,2)}(m,1)\binom{n}{1}\\ &\qquad + \sum_{k=\lceil\frac{a}{2}\rceil}^{a}\left\{\sum_{x=a-k}^{\lfloor\frac{a}{2}\rfloor}(a-2x)\binom{a-x}{k-x}\binom{k}{a-x}\right\}\binom{m}{k}\binom{n}{2}. \end{aligned} \tag{218}$$

Similarly, by Proposition 9.1 (b), we have

$$\begin{aligned} &u_{(a,2)}(m,1)\\ &= \sum_{j=0}^{m-1} u_{(a-2,2)}(j,1)\\ &\qquad + \sum_{k=\lceil\frac{a}{2}\rceil}^{a-1}\left\{\sum_{x=a-k}^{\lfloor\frac{a}{2}\rfloor}(a-2x)\binom{a-x}{k-x+1}\binom{k}{a-x}\right\}\binom{m}{k+1}\\ &= \sum_{j=0}^{m-1} u_{(a-2,2)}(j,1) + \sum_{k=\lceil\frac{a}{2}\rceil+1}^{a}\left\{\sum_{x=a-k+1}^{\lfloor\frac{a}{2}\rfloor}(a-2x)\binom{a-x}{k-x}\binom{k-1}{a-x}\right\}\binom{m}{k}. \end{aligned} \tag{219}$$

These coefficients can be expressed concisely.

Lemma 9.2. (a). *For any $k=\lceil\frac{a}{2}\rceil,\dots,a$, the following equality holds.*

$$\sum_{x=a-k}^{\lfloor\frac{a}{2}\rfloor}(a-2x)\binom{a-x}{k-x}\binom{k}{a-x} = \left\lceil\frac{2k-a}{2}\right\rceil\binom{2k-a}{\lceil\frac{2k-a}{2}\rceil}\binom{k}{2k-a}. \tag{220}$$

(b). *For any* $k = \lceil \frac{a}{2} \rceil + 1, \dots, a$, *the following equality holds.*

$$\begin{aligned} &\sum_{x=a-k+1}^{\lfloor \frac{a}{2} \rfloor} (a-2x)\binom{a-x}{k-x}\binom{k-1}{a-x} \\ &= \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil \frac{2k-a-1}{2} \rceil} - 2^{2k-a-2} \right\} \binom{k-1}{2k-a-1}. \end{aligned} \tag{221}$$

The proof is given in Appendix. By using these equalities, (218) and (219) become as follows:

$$\begin{aligned} &u_{(a,2)}(m,n) \\ &= u_{(a,2)}(m,1)\binom{n}{1} \\ &\quad + \sum_{k=\lceil \frac{a}{2} \rceil}^{a} \left\{ \left\lceil \frac{2k-a}{2} \right\rceil \binom{2k-a}{\lceil \frac{2k-a}{2} \rceil}\binom{k}{2k-a} \right\} \binom{m}{k}\binom{n}{2}. \end{aligned} \tag{222}$$

$$\begin{aligned} &u_{(a,2)}(m,1) \\ &= \sum_{j=0}^{m-1} u_{(a-2,2)}(j,1) \\ &\quad + \sum_{k=\lceil \frac{a}{2} \rceil+1}^{a} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil \frac{2k-a-1}{2} \rceil} - 2^{2k-a-2} \right\} \binom{k-1}{2k-a-1}\binom{m}{k}. \end{aligned} \tag{223}$$

By using the above equalities, we show Theorem 8.9.

Proof of Theorem 8.9. By (222), it suffices to show that

$$u_{(a,2)}(m,1) = \sum_{\frac{a}{2}+1<k\leqslant a} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil \frac{2k-a-1}{2} \rceil} - 2^{2k-a-2} \right\} \binom{k}{2k-a}\binom{m}{k}. \tag{224}$$

We prove it by the induction on a. If $a = 1, 2$, then $u_{(a,2)}(m,1) = 0$. Thus, the statement holds. Let $a \geqslant 3$, and suppose that

$$\begin{aligned} &u_{(a-2,2)}(m,1) \\ &= \sum_{\frac{a}{2}<k\leqslant a-2} \left\{ \frac{2k-a+2}{2}\binom{2k-a+1}{\lceil \frac{2k-a+1}{2} \rceil} - 2^{2k-a} \right\} \binom{k}{2k-a+2}\binom{m}{k}. \end{aligned} \tag{225}$$

Then, by (223), we have

$$
\begin{aligned}
&u_{(a,2)}(m,1)\\
&= \sum_{j=0}^{m-1} \sum_{\frac{a}{2}<k\leqslant a-2} \left\{ \frac{2k-a+2}{2}\binom{2k-a+1}{\lceil\frac{2k-a+1}{2}\rceil} - 2^{2k-a} \right\} \binom{k}{2k-a+2}\binom{j}{k}\\
&\qquad + \sum_{k=\lceil\frac{a}{2}\rceil+1}^{a} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil} - 2^{2k-a-2} \right\} \binom{k-1}{2k-a-1}\binom{m}{k}\\
&\overset{(13)}{=} \sum_{\frac{a}{2}<k\leqslant a-2} \left\{ \frac{2k-a+2}{2}\binom{2k-a+1}{\lceil\frac{2k-a+1}{2}\rceil} - 2^{2k-a} \right\} \binom{k}{2k-a+2}\binom{m}{k+1}\\
&\qquad + \sum_{\frac{a}{2}+1<k\leqslant a} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil} - 2^{2k-a-2} \right\} \binom{k-1}{2k-a-1}\binom{m}{k}\\
&= \sum_{\frac{a}{2}+1<k\leqslant a-1} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil} - 2^{2k-a-2} \right\} \binom{k-1}{2k-a}\binom{m}{k}\\
&\qquad + \sum_{\frac{a}{2}+1<k\leqslant a} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil} - 2^{2k-a-2} \right\} \binom{k-1}{2k-a-1}\binom{m}{k}
\end{aligned}
\tag{226}
$$

Consider the first term. We can add the factor of $k=a$ since $\binom{k-1}{2k-a} = \binom{a-1}{a} = 0$. Thus, we have

$$
\begin{aligned}
&u_{(a,2)}(m,1)\\
&= \sum_{\frac{a}{2}+1<k\leqslant a} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil} - 2^{2k-a-2} \right\}\\
&\qquad \times \left\{ \binom{k-1}{2k-a} + \binom{k-1}{2k-a-1} \right\} \binom{m}{k}\\
&\overset{(12)}{=} \sum_{\frac{a}{2}+1<k\leqslant a} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil} - 2^{2k-a-2} \right\} \binom{k}{2k-a}\binom{m}{k}.
\end{aligned}
\tag{227}
$$

This completes the proof. □

Appendix A. Proof of Lemma 9.2

In this appendix, we prove Lemma 9.2.

Lemma A.1. *For any $u \in \mathbb{Z}_{\geqslant 0}$, the following relation holds.*

$$
\sum_{x=0}^{\lfloor\frac{u}{2}\rfloor} (u-2x)\binom{u}{x} = \left\lceil\frac{u}{2}\right\rceil \binom{u}{\lceil\frac{u}{2}\rceil}. \tag{228}
$$

Proof. We have

$$\begin{aligned}
&\sum_{x=0}^{\lfloor \frac{u}{2} \rfloor}(u-2x)\binom{u}{x} = \sum_{x=0}^{\lfloor \frac{u}{2} \rfloor}(u-x)\binom{u}{u-x} - \sum_{x=0}^{\lfloor \frac{u}{2} \rfloor} x\binom{u}{x} \\
&= \sum_{x=0}^{\lfloor \frac{u}{2} \rfloor} u\binom{u-1}{u-x-1} - \sum_{x=1}^{\lfloor \frac{u}{2} \rfloor} u\binom{u-1}{x-1} \\
&= u\left\{\sum_{x=0}^{\lfloor \frac{u}{2} \rfloor}\binom{u-1}{x} - \sum_{x=0}^{\lfloor \frac{u}{2} \rfloor - 1}\binom{u-1}{x}\right\} = u\binom{u-1}{\lfloor \frac{u}{2} \rfloor} \\
&= u\binom{u-1}{u-1-\lfloor \frac{u}{2} \rfloor} = u\binom{u-1}{\lceil \frac{u}{2} \rceil - 1} = \left\lceil \frac{u}{2} \right\rceil \binom{u}{\lceil \frac{u}{2} \rceil}. \quad \square
\end{aligned} \tag{229}$$

Lemma A.2. *For any $u \in \mathbb{Z}_{\geqslant 0}$, the following relation holds.*

$$\sum_{x=0}^{\lfloor \frac{u}{2} \rfloor}\binom{u}{x} = \begin{cases} 2^{u-1} + \frac{1}{2}\binom{u}{\frac{u}{2}} & u : \text{even}, \\ 2^{u-1} & u : \text{odd}. \end{cases} \tag{230}$$

Proof. Since $\sum_{x=0}^{\lfloor \frac{u}{2} \rfloor}\binom{u}{x} = \sum_{x=0}^{\lfloor \frac{u}{2} \rfloor}\binom{u}{u-x} = \sum_{x=u-\lfloor \frac{u}{2} \rfloor}^{u}\binom{u}{x}$, we have

$$\begin{aligned}
&2\sum_{x=0}^{\lfloor \frac{u}{2} \rfloor}\binom{u}{x} = \sum_{x=0}^{\lfloor \frac{u}{2} \rfloor}\binom{u}{x} + \sum_{x=u-\lfloor \frac{u}{2} \rfloor}^{u}\binom{u}{x} \\
&= \begin{cases} \sum_{x=0}^{u}\binom{u}{x} + \binom{u}{\frac{u}{2}} = 2^u + \binom{u}{\frac{u}{2}} & u\text{: even}, \\ \sum_{x=0}^{u}\binom{u}{x} = 2^u & u\text{: odd}. \end{cases}
\end{aligned} \tag{231}$$

So, Lemma A.2 holds. □

Proof of Lemma 9.2. (a). We can easily check $\binom{a-x}{k-x}\binom{k}{a-x} = \binom{k}{2k-a}\binom{2k-a}{k-x}$. Hence, we have

$$\begin{aligned}
&\sum_{x=a-k}^{\lfloor \frac{a}{2} \rfloor}(a-2x)\binom{a-x}{k-x}\binom{k}{a-x} = \binom{k}{2k-a}\sum_{x=a-k}^{\lfloor \frac{a}{2} \rfloor}(a-2x)\binom{2k-a}{k-x} \\
&= \binom{k}{2k-a}\sum_{x=0}^{\lfloor \frac{a}{2} \rfloor-(a-k)}(a-2(x+a-k))\binom{2k-a}{k-(x+a-k)} \\
&= \binom{k}{2k-a}\sum_{x=0}^{\lfloor \frac{2k-a}{2} \rfloor}((2k-a)-2x)\binom{2k-a}{x} \\
&= \left\lceil \frac{2k-a}{2} \right\rceil \binom{2k-a}{\lceil \frac{2k-a}{2} \rceil}\binom{k}{2k-a}. \quad \text{(Lemma A.1)}
\end{aligned} \tag{232}$$

(b). We can easily check $\binom{a-x}{k-x}\binom{k-1}{a-x} = \binom{k-1}{2k-a-1}\binom{2k-a-1}{k-a+x-1}$. Thus, we have

$$\begin{aligned}
&\sum_{x=a-k+1}^{\lfloor \frac{a}{2} \rfloor} (a-2x)\binom{a-x}{k-x}\binom{k-1}{a-x} \\
&= \binom{k-1}{2k-a-1} \sum_{x=a-k+1}^{\lfloor \frac{a}{2} \rfloor} (a-2x)\binom{2k-a-1}{k-a+x-1} \\
&= \binom{k-1}{2k-a-1} \sum_{x=0}^{\lfloor \frac{2k-a-2}{2} \rfloor} ((2k-a-2)-2x)\binom{2k-a-1}{x} \\
&= \binom{k-1}{2k-a-1} \left\{ \sum_{x=0}^{\lfloor \frac{2k-a-2}{2} \rfloor} ((2k-a-1)-2x)\binom{2k-a-1}{x} \right. \\
&\qquad \left. - \sum_{x=0}^{\lfloor \frac{2k-a-2}{2} \rfloor} \binom{2k-a-1}{x} \right\}.
\end{aligned} \tag{233}$$

(i) If a is odd, then $\lfloor \frac{2k-a-2}{2} \rfloor = \frac{2k-a-3}{2}$, $\lfloor \frac{2k-a-1}{2} \rfloor = \frac{2k-a-1}{2}$. Thus, we have

$$\begin{aligned}
&\sum_{x=0}^{\lfloor \frac{2k-a-2}{2} \rfloor} ((2k-a-1)-2x)\binom{2k-a-1}{x} \\
&= \sum_{x=0}^{\frac{2k-a-3}{2}} ((2k-a-1)-2x)\binom{2k-a-1}{x} \\
&= \sum_{x=0}^{\frac{2k-a-1}{2}} ((2k-a-1)-2x)\binom{2k-a-1}{x} \\
&= \frac{2k-a-1}{2}\binom{2k-a-1}{\frac{2k-a-1}{2}} \quad \text{(Lemma A.1)}.
\end{aligned} \tag{234}$$

Since $2k-a-1$ is even, we have

$$\begin{aligned}
&\sum_{x=0}^{\lfloor \frac{2k-a-2}{2} \rfloor} \binom{2k-a-1}{x} = \sum_{x=0}^{\frac{2k-a-3}{2}} \binom{2k-a-1}{x} \\
&= \sum_{x=0}^{\frac{2k-a-1}{2}} \binom{2k-a-1}{x} - \binom{2k-a-1}{\frac{2k-a-1}{2}} \\
&= 2^{2k-a-2} + \frac{1}{2}\binom{2k-a-1}{\frac{2k-a-1}{2}} - \binom{2k-a-1}{\frac{2k-a-1}{2}} \quad \text{(Lemma A.2)} \\
&= 2^{2k-a-2} - \frac{1}{2}\binom{2k-a-1}{\frac{2k-a-1}{2}}.
\end{aligned} \tag{235}$$

Hence, we have

$$\begin{aligned}
&\sum_{x=0}^{\lfloor\frac{2k-a-2}{2}\rfloor}((2k-a-1)-2x)\binom{2k-a-1}{x}-\sum_{x=0}^{\lfloor\frac{2k-a-2}{2}\rfloor}\binom{2k-a-1}{x}\\
&=\frac{2k-a-1}{2}\binom{2k-a-1}{\frac{2k-a-1}{2}}-\left(2^{2k-a-2}-\frac{1}{2}\binom{2k-a-1}{\frac{2k-a-1}{2}}\right)\\
&=\frac{2k-a}{2}\binom{2k-a-1}{\frac{2k-a-1}{2}}-2^{2k-a-2}.
\end{aligned}\tag{236}$$

Putting the last expression to (233), we have

$$\begin{aligned}
&\sum_{x=a-k+1}^{\lfloor\frac{a}{2}\rfloor}(a-2x)\binom{a-x}{k-x}\binom{k-1}{a-x}\\
&=\left\{\frac{2k-a}{2}\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil}-2^{2k-a-2}\right\}\binom{k-1}{2k-a-1}.
\end{aligned}\tag{237}$$

(ii) If a is even, then $\lfloor\frac{2k-a-2}{2}\rfloor=\lfloor\frac{2k-a-1}{2}\rfloor=\frac{2k-a-2}{2}$. Thus, we have

$$\begin{aligned}
&\sum_{x=0}^{\lfloor\frac{2k-a-2}{2}\rfloor}((2k-a-1)-2x)\binom{2k-a-1}{x}\\
&=\sum_{x=0}^{\lfloor\frac{2k-a-1}{2}\rfloor}((2k-a-1)-2x)\binom{2k-a-1}{x}\\
&=\frac{2k-a}{2}\binom{2k-a-1}{\frac{2k-a}{2}}.
\end{aligned}\tag{238}$$

Since $2k-a-1$ is odd, we have

$$\begin{aligned}
&\sum_{x=0}^{\lfloor\frac{2k-a-2}{2}\rfloor}\binom{2k-a-1}{x}=\sum_{x=0}^{\lfloor\frac{2k-a-1}{2}\rfloor}\binom{2k-a-1}{x}\\
&=2^{2k-a-2}.
\end{aligned}\tag{239}$$

Putting these expressions to (233), we obtain

$$\begin{aligned}
&\sum_{x=a-k+1}^{\lfloor\frac{a}{2}\rfloor}(a-2x)\binom{a-x}{k-x}\binom{k-1}{a-x}\\
&=\left\{\frac{2k-a}{2}\binom{2k-a-1}{\frac{2k-a}{2}}-2^{2k-a-2}\right\}\binom{k-1}{2k-a-1}\\
&=\left\{\frac{2k-a}{2}\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil}-2^{2k-a-2}\right\}\binom{k-1}{2k-a-1}.
\end{aligned}\tag{240}$$

□

Appendix B. The equivalence between (198) and (199)

First, we show the following lemma.

Lemma B.1. *Let $\alpha,\beta\in\mathbb{Z}_{\geqslant 0}$ with $\alpha\geqslant\beta$. Then, we have*

$$m\binom{m-1}{\alpha}\binom{m-1}{\beta}=\sum_{k=\alpha+1}^{\alpha+\beta+1}\binom{\alpha}{k-\beta-1}\binom{k-1}{\alpha}\binom{k}{k-1}\binom{m}{k}.\tag{241}$$

Proof. By Lemma 2.3, we have

$$\binom{m-1}{\alpha}\binom{m-1}{\beta}=\sum_{k=\alpha}^{\alpha+\beta}\binom{\alpha}{k-\beta}\binom{k}{\alpha}\binom{m-1}{k}. \tag{242}$$

By definition, we can check that $m\binom{m-1}{k}=(k+1)\binom{m}{k+1}$. Thus, we have

$$\begin{aligned}m\binom{m-1}{\alpha}\binom{m-1}{\beta}&=\sum_{k=\alpha}^{\alpha+\beta}\binom{\alpha}{k-\beta}\binom{k}{\alpha}(k+1)\binom{m}{k+1}\\&=\sum_{k=\alpha+1}^{\alpha+\beta+1}\binom{\alpha}{k-1-\beta}\binom{k-1}{\alpha}k\binom{m}{k}\\&=\sum_{k=\alpha+1}^{\alpha+\beta+1}\binom{\alpha}{k-1-\beta}\binom{k-1}{\alpha}\binom{k}{k-1}\binom{m}{k}.\end{aligned} \tag{243}$$

This completes the proof. □

By using this formula, we show the following two equalities.

Proposition B.2. *Let $a\in\mathbb{Z}_{\geqslant 0}$. Then, for any $m\in\mathbb{Z}_{\geqslant 0}$, we have*

$$\sum_{\frac{a}{2}<k\leqslant a}\left\lceil\frac{2k-a}{2}\right\rceil\binom{2k-a}{\lceil\frac{2k-a}{2}\rceil}\binom{k}{2k-a}\binom{m}{k}=m\binom{m-1}{\lfloor\frac{a}{2}\rfloor}\binom{m-1}{\lceil\frac{a}{2}-1\rceil}, \tag{244}$$

$$\sum_{\frac{a}{2}<k\leqslant a}\frac{2k-a}{2}\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil}\binom{k}{2k-a}\binom{m}{k}=\frac{m}{2}\binom{m-1}{\lfloor\frac{a-1}{2}\rfloor}\binom{m-1}{\lceil\frac{a-1}{2}\rceil}. \tag{245}$$

Proof. First, we show (244). By definition, we can check

$$\begin{aligned}&\left\lceil\frac{2k-a}{2}\right\rceil\binom{2k-a}{\lceil\frac{2k-a}{2}\rceil}\binom{k}{2k-a}=\binom{\lfloor\frac{a}{2}\rfloor}{k-1-\lceil\frac{a}{2}-1\rceil}\binom{k-1}{\lfloor\frac{a}{2}\rfloor}\binom{k}{k-1}\\&=\frac{k!}{(a-k)!(k-1-\lfloor\frac{a}{2}\rfloor)!(k-1-\lceil\frac{a}{2}-1\rceil)!}.\end{aligned} \tag{246}$$

In the above equalities, $k!=k(k-1)\cdots 2\cdot 1$ is the factorial of $k\in\mathbb{Z}_{\geqslant 0}$. Thus, by setting $\alpha=\lfloor\frac{a}{2}\rfloor$ and $\beta=\lceil\frac{a}{2}-1\rceil$ in (241), the equality (244) holds.

Next, we show (245). We have

$$\begin{aligned}&(2k-a)\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil}\binom{k}{2k-a}=\binom{\lceil\frac{a-1}{2}\rceil}{k-1-\lfloor\frac{a-1}{2}\rfloor}\binom{k-1}{\lceil\frac{a-1}{2}\rceil}\binom{k}{k-1}\\&=\frac{k!}{(a-k)!(k-1-\lceil\frac{a-1}{2}\rceil)!(k-1-\lfloor\frac{a-1}{2}\rfloor)}.\end{aligned} \tag{247}$$

Thus, by setting $\alpha=\lceil\frac{a-1}{2}\rceil$ and $\beta=\lfloor\frac{a-1}{2}\rfloor$ in (241), we have

$$\sum_{\frac{a}{2}<k\leqslant a}\frac{2k-a}{2}\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil}\binom{k}{2k-a}\binom{m}{k}=\frac{m}{2}\binom{m-1}{\lfloor\frac{a-1}{2}\rfloor}\binom{m-1}{\lceil\frac{a-1}{2}\rceil}. \tag{248}$$ □

Last, we show (199).

Proof of (199). By (244), the first term in the RHS of (198) is $m\binom{m-1}{\lfloor\frac{a}{2}\rfloor}\binom{m-1}{\lceil\frac{a}{2}-1\rceil}\binom{n}{2}$. Consider the second term of (198), that is,

$$\sum_{\frac{a}{2}+1<k\leqslant a}\left\{\frac{2k-a}{2}\binom{2k-a-1}{\lceil\frac{2k-a-1}{2}\rceil}-2^{2k-a-2}\right\}\binom{k}{2k-a}\binom{m}{k}\binom{n}{1}. \tag{249}$$

If $k = \lfloor \frac{a}{2} + 1 \rfloor$, or equivalently, if $2k - a = 1$ or 2, we can easily check that

$$\frac{2k-a}{2}\binom{2k-a-1}{\lceil \frac{2k-a-1}{2} \rceil} - 2^{2k-a-2} = 0. \tag{250}$$

Thus, we have

$$\begin{aligned}
&\sum_{\frac{a}{2}+1<k\leqslant a} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil \frac{2k-a-1}{2} \rceil} - 2^{2k-a-2} \right\} \binom{k}{2k-a}\binom{m}{k}\binom{n}{1} \\
&= \sum_{\frac{a}{2}<k\leqslant a} \left\{ \frac{2k-a}{2}\binom{2k-a-1}{\lceil \frac{2k-a-1}{2} \rceil} - 2^{2k-a-2} \right\} \binom{k}{2k-a}\binom{m}{k}\binom{n}{1} \\
&= \sum_{\frac{a}{2}<k\leqslant a} \frac{2k-a}{2}\binom{2k-a-1}{\lceil \frac{2k-a-1}{2} \rceil}\binom{k}{2k-a}\binom{m}{k}\binom{n}{1} \\
&\qquad - \sum_{\frac{a}{2}<k\leqslant a} 2^{2k-a-2}\binom{k}{2k-a}\binom{m}{k}\binom{n}{1} \\
&\overset{(245)}{=} \frac{m}{2}\binom{m-1}{\lfloor \frac{a-1}{2} \rfloor}\binom{m-1}{\lceil \frac{a-1}{2} \rceil} - \sum_{\frac{a}{2}<k\leqslant a} 2^{2k-a-2}\binom{k}{2k-a}\binom{m}{k}\binom{n}{1}.
\end{aligned} \tag{251}$$

Thus, (199) holds. ☐

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