ALGEBRAIC COMBINATORICS

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Volume 8, issue 4 (2025), p. 925-953.
https://doi.org/10.5802/alco.435

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

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ABSTRACT We study the geometry and topology of Δ-Springer varieties associated with two-row partitions. These varieties were introduced in recent work by Griffin–Levinson–Woo to give a geometric realization of a symmetric function appearing in the Delta conjecture by Haglund–Remmel–Wilson. We provide an explicit and combinatorial description of the irreducible components of the two-row Δ-Springer variety and compare it to the ordinary two-row Springer fiber as well as Kato's exotic Springer fiber corresponding to a one-row bipartition. In addition to that, we extend the action of the symmetric group on the homology of the two-row Δ-Springer variety to an action of a degenerate affine Hecke algebra and relate this action to a $\mathfrak{gl}_2$-tensor space.

1. Introduction

The Springer correspondence [35, 36] provides a powerful bridge between geometry (via Springer fibers) and algebra (via representations of Weyl groups) facilitating deep insights and results in both fields. In particular, it offers a geometric construction of the irreducible complex representations of Weyl groups. These representations are obtained in the top degree cohomology of Springer fibers, which are the fibers of the desingularization of the nilpotent cone.

In type A, nilpotent elements of the Lie algebra $\mathfrak{sl}_n$ are classified by their Jordan type, or equivalently, by a partition of n. Even in type A, the geometry of these fibers is not well understood. In the case of Springer fibers associated with two-row partitions, the situation is much nicer and has been studied extensively. For example, the irreducible components of two-row Springer fibers as well as their intersections are known to be smooth [18, 39]. The Springer fibers for which all irreducible components are smooth have been classified in [17]. In the two-row case, there also exists a nice diagrammatic description of the irreducible components in terms of cup diagrams [18, 39]. A homeomorphic topological model has been built for these varieties [26, 42] and the action of the symmetric group on the top degree cohomology has a skein theoretic interpretation [32, 31]. In types C and D, the geometry and topology of two-row Springer fibers have been studied in [12, 43, 40, 22]. As in type A, the irreducible components and their intersections are smooth and they admit explicit descriptions in terms of cup diagrams.

Manuscript received 16th July 2024, revised 20th April 2025, accepted 21st April 2025.

KEYWORDS. Springer theory, flag varieties, action on homology, degenerate affine Hecke algebra.

ACKNOWLEDGEMENTS. AL was partially supported by a PEPS JCJC grant from INSMI (CNRS). PV was supported by the Fonds de la Recherche Scientifique - FNRS under Grants no. MIS-F.4536.19 and CDR-J.0189.23. AW was supported by an AMS–Simons Grant for PUI Faculty.

ISSN: 2589-5486

There also exists another version of the Springer correspondence for the symplectic group [24] which is cleaner than the original Springer correspondence in type C. In fact, Kato's exotic Springer correspondence yields a bijection between orbits in the exotic nilpotent cone and irreducible representations of the Weyl group of type C. In contrast to that, the original Springer correspondence is more intricate outside of type A and requires extra data in terms of the component group. In the exotic case, the nilpotent orbits have been classified explicitly using bipartitions [1]. The irreducible components of exotic Springer fibers have been studied thoroughly in [29], as well as [33] in the specific case of one-row bipartitions. As for the ordinary two-row Springer fibers, the irreducible components of exotic Springer fibers for one-row bipartitions can be described using certain cup diagrams.

Even though the study of Springer fibers originated in the geometric representation theory of Weyl groups, many connections to representation theory, combinatorics, geometry, and topology have been established in recent years. These connections are already rich and interesting when one restricts to the two-row case. For example, the diagrammatics appearing in the study of two-row Springer fibers have an interpretation in terms of parabolic Kazhdan–Lusztig theory [18, 11, 10]. Furthermore, the cohomology of two-row Springer fibers in type A is related to Khovanov's arc algebra [25], which provides invariants of tangles, and thus an interesting connection to low-dimensional topology. In fact, it turns out that the cohomology ring of the Springer fiber is isomorphic to the center of the principal block of parabolic category $\mathcal{O}$, [3, 38]. Using a generalization of Khovanov's arc algebras, deep connections to the representation theory of Lie (super)algebras and (walled) Brauer algebras were established in work by Brundan–Stroppel [5, 4, 6, 7] and Ehrig–Stroppel [13, 14, 15, 16]. As evident from the above, two-row Springer fibers have proven to have important applications in the field of categorification and 2-representation theory.

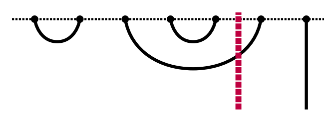
Motivation. The research that led to this paper originated in an attempt to understand [27] in terms of Springer theory, to define an arc algebra categorifying the Hecke algebra of type B with unequal parameters, or more precisely one of its quotients, the blob algebra of Martin–Saleur [28]. The representation theory of this algebra is governed by one-row bipartitions which naturally appear in Kato's exotic Springer correspondence. The exotic Springer fibers associated with one-row bipartitions share many geometric properties with the ordinary two-row Springer fibers in type A. Also note that the combinatorics of the blob algebra naturally appear when studying exotic Springer fibers for one-row bipartitions, [33].

Recently, yet another Springer-type variety, called a Δ-Springer variety, has been introduced in work of Griffin–Levinson–Woo, [20]. This variety gives a geometric interpretation of a ring generalizing both the cohomology ring of a Springer fiber in type A, and the Haglund–Rhoades–Shimozono ring, [21]. As remarked in [19], the Δ-Springer variety turns out to be a generalized Springer fiber in the sense of Borho–MacPherson, [2]. In the two-row case, the Δ-Springer variety is intimately related to the exotic Springer fiber (see Main Theorem B). We believe that it would be interesting to define an arc algebra whose center is isomorphic to the cohomology of a two-row Δ-Springer variety, and establish connections with Kazhdan–Lusztig theory, generalizing the rich picture known in type A.

What we do in this paper. In this paper, we specifically study Δ-Springer varieties associated with two-row partitions. We develop a diagrammatic combinatorics well suited for comparison with ordinary two-row Springer fibers as well as exotic Springer fibers associated with one-row bipartitions.

In order to define the Δ-Springer variety in the two-row case, we fix a two-row partition $(n-k,k)$ of n and an integer $0 \leqslant m \leqslant k$. Moreover, we fix a nilpotent endomorphism x of $\mathbb{C}^n$ of Jordan type $(n-k,k)$. The Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ consists of all partial flags in $\mathbb{C}^n$ of the form $\{0\} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-m} \subseteq \mathbb{C}^n$ satisfying $\dim(F_i) = i$, $xF_i \subseteq F_{i-1}$ for all $1 \leqslant i \leqslant n-m$, and $\operatorname{im} x^m \subseteq F_{n-m}$. We refer to Definition 2.3 for the general case.

The first step in our study is to provide an explicit description of the irreducible components of the two-row Δ-Springer fibers using the notion of a Δ-cup diagram. An example of a Δ-cup diagram is given by



This is a crossingless diagram consisting of cups and rays attached to finitely many vertices on a horizontal line. In addition to that, there exists a vertical dotted red line, a so-called cut line, dividing the diagram into a left and a right part. We only allow right endpoints of cups and rays to the right of the cut line. More details on Δ-cup diagrams can be found in Definition 2.7.

Section 3 is devoted to the study of the irreducible components of the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$, and we prove the following result.

Main Theorem A (Theorem 3.4). *There exists a bijection between the irreducible components of the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ and the Δ-cup diagrams on n points with k cups and m vertices to the right of the cut line. We give explicit relations describing all flags contained in the irreducible component associated with a given Δ-cup diagram. As a consequence, we show that each irreducible component is an iterated $\mathbb{P}^1$-bundle, and, in particular, it is smooth.*

To some extent, we like to think of the Δ-Springer varieties as an interpolation between Springer fibers in type A and exotic Springer fibers that have a type C flavor. Indeed, in the extremal case $m=0$, the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),0}$ is equal to the two-row Springer fiber associated with the partition $(n-k,k)$, and in the extremal case $m=k$, the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),k}$ is isomorphic to the exotic Springer fiber associated with the bipartition $((n-2k),(k))$. Using our explicit description of irreducible components, we relate in Section 4, for any value of $0 \leqslant m \leqslant k$, the two-row Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ to the exotic Springer fiber $\mathcal{B}^{\mathrm{e}}_{((n-m-k),(k))}$ associated with a one-row bipartition.

Main Theorem B (Theorem 4.4). *There exists an isomorphism of algebraic varieties from the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ to a closed subvariety of the exotic Springer fiber $\mathcal{B}^{\mathrm{e}}_{((n-m-k),(k))}$.*

We also give a negative answer to [20, Question 8.7] (see Example 4.1). The natural birational map from a union of irreducible components of the Springer fiber $\mathcal{B}_{(n-k,k)}$ to the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ described in [20, Remark 5.12] is not an isomorphism since it is not bijective.

The next step is a representation theoretic study of the homology of the two-row Δ-Springer variety. In [20], an action of a symmetric group is constructed on each degree of the homology of any Δ-Springer variety. Moreover, the top degree representation is identified with a Specht module associated with a skew partition. In Section 5, we construct a topological model for the Δ-Springer variety in the two-row case and use this model to identify the representation of the symmetric group S_{n-m} in every degree of its homology (not only the top degree).

Main Theorem C (Theorem 5.10). *For $0 \leqslant d \leqslant k$, the degree $2d$ of the homology of the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ is isomorphic, as an S_{n-m}-representation, to the Specht module associated with the skew partition $(n-d,d)/(m)$.*

Remark. We construct the action of the symmetric group on the homology of the Δ-Springer variety (or, more precisely, on its topology model) by embedding the homology into the homology of a product of 2-spheres. The symmetric group naturally acts on the product of 2-spheres by permuting spheres. We show that the action induced in homology restricts to an action on the homology of the Δ-Springer variety. This construction also provides a skein theoretic description of this representation, answering [20, Question 8.6].

The above result shows that each degree of the homology of the Δ-Springer variety is not always irreducible as a representation of the symmetric group S_{n-m}. In contrast to that, the representations of the symmetric group on the homology of ordinary two-row Springer fibers, as well as the representations of the Weyl group of type C on the homology of exotic Springer fibers associated with one-row bipartitions, are irreducible. In Section 6, we extend the action of the symmetric group on the homology of the Δ-Springer variety to an action of the degenerate affine Hecke algebra.

Main Theorem D (Theorem 6.4 and Theorem 6.19). *Each degree of the homology of a two-row Δ-Springer fiber is an irreducible representation of the degenerate affine Hecke algebra.*

In addition to the irreducibility of the representations in each degree, we find it surprising that the action of the degenerate affine Hecke algebra preserves the homological degree in the first place. In order to prove the above theorem, we identify the action on homology with the action of the degenerate affine Hecke algebra on a $\mathfrak{gl}_2$-tensor space, using a version of Schur–Weyl duality.

Conventions. In this paper, all varieties and vector spaces are defined over the field of complex numbers. If X is a topological space, we denote by $H_*(X)$ its singular homology with complex coefficients and by $H^*(X)$ its cohomology with complex coefficients. It follows from the universal coefficient theorem that homology and cohomology are dual to each other degreewise, that is $H_i(X) \cong \mathrm{Hom}(H^i(X), \mathbb{C}) \cong H^i(X)$ for all nonnegative integers i. Note that this duality is different from Poincaré duality. The duality implies that all results originally proved for cohomology remain true for homology, and vice versa, as long as they only depend on the vector space structure.

2. Δ-Springer varieties

Given an integer n and a sequence $(d_1, \dots, d_r)$, we will denote by $\mathcal{F}l_{(d_1,\dots,d_r)}(\mathbb{C}^n)$ the set of partial flags $F_\bullet = (F_i)_{1 \leqslant i \leqslant r}$ such that $\dim(F_i/F_{i-1}) = d_i$. Concerning flags, we will always use the convention that $F_0 = \{0\}$. We will write shortly $\mathcal{F}l(\mathbb{C}^n)$ for the set of complete flags in $\mathbb{C}^n$, that is the set $\mathcal{F}l_{(1^n)}(\mathbb{C}^n)$.

2.1. Springer fibers.

Springer fibers arise as fibers of the desingularization of the nilpotent cone (see [36]). The *Springer fiber* $\mathcal{B}_x$ associated with a nilpotent element $x \in \mathfrak{gl}_n(\mathbb{C})$ is the following subset of complete flags in $\mathbb{C}^n$:

$$\mathcal{B}_x = \{F_\bullet \in \mathcal{F}l(\mathbb{C}^n) \mid xF_i \subseteq F_{i-1} \text{ for } i \leqslant n\}.$$

This is a subvariety of the flag variety which, up to isomorphism, depends only on the orbit of x under the action of $GL_n(\mathbb{C})$ by conjugation. Given a partition λ of n, we will then usually write $\mathcal{B}_\lambda$ for the Springer fiber associated with a nilpotent element in $\mathfrak{gl}_n(\mathbb{C})$ of Jordan type λ.

The study of the cohomology of these Springer fibers is related to the representation theory of the symmetric group via the Springer correspondence. Given a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n, we denote by $n(\lambda)$ the integer $\sum_{i=1}^{r} \frac{\lambda_i(\lambda_i - 1)}{2}$.

Theorem 2.1 ([36, Section 1]). *The following hold:*

(1) *The Springer fiber $\mathcal{B}_\lambda$ is equidimensional of dimension $n(\lambda)$.*
(2) *There exists an irreducible action of the symmetric group S_n on $H^{2n(\lambda)}(\mathcal{B}_\lambda)$.*
(3) *The map $\lambda \mapsto H^{2n(\lambda)}(\mathcal{B}_\lambda)$ is a bijection between partitions of n and irreducible complex representations of S_n.*

2.2. Exotic Springer fibers. Exotic Springer fibers arise as fibers of the desingularization of the exotic nilpotent cone (see [24]). We first need to endow $\mathbb{C}^{2n}$ with a structure of a symplectic space. Fix a basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ of $\mathbb{C}^{2n}$ and define a symplectic form ω by $\omega(e_i, f_j) = -\omega(f_j, e_i) = \delta_{i+j,n+1}$. The action of $\mathrm{Sp}(\mathbb{C}^{2n}, \omega)$ on $\mathfrak{gl}_{2n}(\mathbb{C})$ by conjugation yields a decomposition $\mathfrak{gl}_{2n}(\mathbb{C}) = \mathfrak{sp}_{2n}(\mathbb{C}) \oplus \mathcal{S}(\mathbb{C}^{2n})$ of $\mathrm{Sp}(\mathbb{C}^{2n}, \omega)$-modules. Explicitly

$$\mathcal{S}(\mathbb{C}^{2n}) = \{x \in \mathfrak{gl}_{2n} \mid \forall v, w \in \mathbb{C}^{2n},\ \omega(xv, w) = \omega(v, xw)\}.$$

The exotic Springer fiber $\mathcal{B}^{\mathrm{e}}_{x,v}$ associated with a nilpotent element $x \in \mathcal{S}(\mathbb{C}^{2n})$ and a vector $v \in \mathbb{C}^{2n}$ is the following subset of flags in $\mathbb{C}^{2n}$:

$$\mathcal{B}^{\mathrm{e}}_{x,v} = \{F_\bullet \in \mathcal{F}l_{(1^n,n)}(\mathbb{C}^{2n}) \mid F_i \text{ is isotropic with respect to } \omega,\ xF_i \subseteq F_{i-1},\ v \in F_n\}.$$

Once again, up to isomorphism, this algebraic variety only depends on the orbit of the pair (x, v) under the action of $\mathrm{Sp}(\mathbb{C}^{2n}, \omega)$. These orbits were determined in [1] and are indexed by bipartitions of n. Therefore, given a bipartition (λ, μ) of n, we will write $\mathcal{B}^{\mathrm{e}}_{\lambda,\mu}$ for the exotic Springer fiber instead of $\mathcal{B}^{\mathrm{e}}_{x,v}$ if the pair (x, v) is in the orbit labeled by (λ, μ).

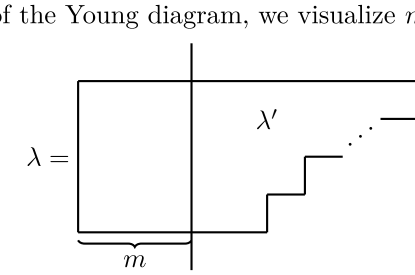
Similarly to the Springer correspondence in type A, we obtain a geometric construction of the irreducible complex representations of the Weyl group of type C.

Theorem 2.2 ([24]). *The following hold:*

(1) *The exotic Springer fiber $\mathcal{B}^{\mathrm{e}}_{\lambda,\mu}$ is equidimensional of dimension $n(\lambda) + |\mu|$.*
(2) *There exists an irreducible action of the Weyl group of type C_n on $H^{2(n(\lambda)+|\mu|)}(\mathcal{B}^{\mathrm{e}}_{\lambda,\mu})$.*
(3) *The map $(\lambda, \mu) \mapsto H^{2(n(\lambda)+|\mu|)}(\mathcal{B}^{\mathrm{e}}_{\lambda,\mu})$ is a bijection between bipartitions of n and irreducible complex representations of the Weyl group of type C_n.*

2.3. Definition of the Δ-Springer variety and basic results.

Let $\lambda = (\lambda_1, \dots, \lambda_s)$ be a partition of n, and let m be a positive integer such that $0 \leqslant m \leqslant \lambda_s$. In terms of the Young diagram, we visualize m as a cut line:



The part of the partition to the right of the cut line is denoted by λ'. Then $\lambda' = (\lambda_1 - m, \dots, \lambda_s - m)$ is a partition of $n' = n - ms$.

Definition 2.3. *Let $x \in \mathfrak{gl}_n(\mathbb{C})$ of Jordan type λ. We define the* Δ-Springer variety *by*

$$\mathcal{B}^{\Delta}_{\lambda,m} = \{F_\bullet \in \mathcal{F}l_{(1^{n'+m}, m(s-1))}(\mathbb{C}^n) \mid xF_i \subseteq F_{i-1} \text{ for } i \leqslant n' + m,\ \operatorname{im} x^m \subseteq F_{n'+m}\}.$$

Up to isomorphism, the algebraic variety $\mathcal{B}^{\Delta}_{\lambda,m}$ depends only on the Jordan type of the nilpotent element x, see [20, Lemma 3.4]. Note that we have $\mathcal{B}^{\Delta}_{\lambda,0} = \mathcal{B}_{\lambda}$.

Remark 2.4. The variety $\mathcal{B}^{\Delta}_{\lambda,m}$ is the variety denoted by $Y_{n'+m,\lambda',s}$ in [20]. This change of notation is justified by our comparison between Springer fibers, exotic Springer fibers and Δ-Springer varieties when λ is a two-row partition, see Section 4.

Let $\mathcal{T}_{\lambda,m}$ be the set of partial fillings of the Young diagram of λ with the labels $\{1, \ldots, n'+m\}$ (without repetition), such that the labels in each row are right justified and decrease from left to right, and the ith row contains at least λ'_i-many labels.

Let $\mathcal{P}(m, \lambda')$ be the set of all fillings of the Young diagram of λ' with the labels $\{1, \ldots, n' + m\}$, such that the labels decrease from left to right along each row and down each column. Since λ' is a partition of n', we do not use all the possible labels in such a filling.

The following proposition summarizes some of the results from [20].

Theorem 2.5. *The following hold.*

(1) *There exists an affine paving of $\mathcal{B}^{\Delta}_{\lambda,m}$ whose cells are in bijection with the set $\mathcal{T}_{\lambda,m}$.*
(2) *If $m = \lambda_s$, then there is a bijection between the irreducible components of $\mathcal{B}^{\Delta}_{\lambda,m}$ and the set $\mathcal{P}(m, \lambda')$. If $m < \lambda_s$, then there is a bijection between the irreducible components of $\mathcal{B}^{\Delta}_{\lambda,m}$ and those elements $S \in \mathcal{P}(m, \lambda')$ which satisfy the following condition: let $i_S \in \{1, \ldots, n'+m\}$ be the smallest number which does not appear in the filling S, then the bottom row of λ' is filled up by a subset of the numbers $\{1, \ldots, i_S - 1\}$.*
(3) *The variety $\mathcal{B}^{\Delta}_{\lambda,m}$ is equidimensional and its dimension equals*
$$n(\tilde{\lambda}') + m(s - 1),$$
where $\widetilde{\lambda'}$ is the conjugate of the partition λ'.

As for the usual and the exotic Springer fiber, there is an action of a Weyl group on the top degree cohomology of the Δ-Springer variety. We refer to [23] for the notion of the Specht module associated with a skew partition. The following is [20].

Theorem 2.6 ([20]). *There exists an action of the symmetric group $S_{n'+m}$ on $H^{2(n(\tilde{\lambda})+m(s-1))}(\mathcal{B}^{\Delta}_{\lambda,m})$ which is isomorphic to the Specht module associated with the skew partition $\lambda/(m^{s-1})$.*

In contrast with the (exotic) Springer correspondence, the top degree cohomology is not an irreducible representation of the symmetric group $S_{n'+m}$.

2.4. Special Case: two-row partitions.

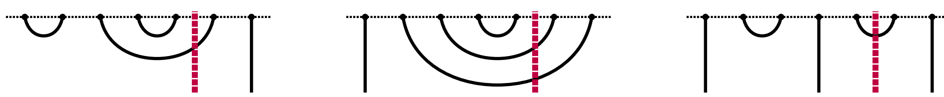
In this subsection, we restrict ourselves to the case of Δ-Springer varieties associated with two-row partitions. We fix $\lambda = (n - k, k)$ a two-row partition of n. In particular, we have $0 \leqslant k \leqslant \lfloor n/2 \rfloor$.

Definition 2.7. *Fix a horizontal line with $n = (n - m) + m$ vertices labeled by the numbers $1, \ldots, n$ in increasing order from left to right. A* cup diagram *is obtained by either connecting two vertices by a* cup*, or by attaching a vertical* ray *to a given vertex. We require that the resulting diagram is crossingless and that every vertex is connected to exactly one endpoint of a cup or ray. We use the notation i—j to indicate that vertices $i < j$ are connected by a cup. Moreover, we write $|$—i if vertex i is connected to a ray.*

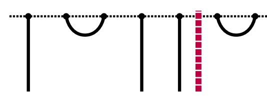
The set of all cup diagrams on n vertices with k cups such that the m rightmost vertices labeled by $\{n-m+1, \ldots, n\}$ are connected to rays or to right endpoints of cups only is denoted by $\mathbb{B}_{n-k,k,m}$. The diagrams in $\mathbb{B}_{n-k,k,m}$ are called Δ-cup diagrams*.*

Remark 2.8. Definition 2.7 makes sense for any nonnegative integers n, k, m such that $0 \leqslant k \leqslant \lfloor n/2 \rfloor$ and $0 \leqslant m \leqslant n$, but the Δ-Springer variety is only defined when $0 \leqslant m \leqslant k$. We will need the more general diagrams in the proof of Proposition 3.7.

Example 2.9. Below we give examples of Δ-cup diagrams for $n = 7$, $m = 2$. We use a vertical dashed line to indicate the m rightmost vertices.



The first two diagrams are elements of $\mathbb{B}_{4,3,2}$ and the last one an element of $\mathbb{B}_{5,2,2}$. Note that



is not a Δ-cup diagram (the second rightmost vertex is not connected to a ray nor is it the right endpoint of a cup).

Lemma 2.10. *Let $0 \leqslant m \leqslant k$. There exists a bijection between the irreducible components of $\mathcal{B}^{\Delta}_{(n-k,k),m}$ and the set $\mathbb{B}_{n-k,k,m}$.*

Proof. The irreducible components of $\mathcal{B}^{\Delta}_{(n-k,k),m}$ are indexed by a subset of $\mathcal{P}(m, \lambda')$ by Theorem 2.5. We remark that this subset of $\mathcal{P}(m, \lambda')$ is in bijection with the set of fillings of the skew Young diagram of $\lambda/(m)$ with the labels in $\{1, \ldots, n-m\}$. Such a bijection is obtained by first keeping the labels of λ' where they are and then completing the bottom row of $\lambda/(m)$ with the entries that are not in the filling of λ'.

We can now embed these fillings of $\lambda/(m)$ with the labels in $\{1, \ldots, n-m\}$ in the set of fillings of λ with the labels in $\{1, \ldots, n\}$ by simply adding the entries $n, \ldots, n-m+1$ into the first m boxes of the first row. The image of this embedding is exactly the set of fillings of λ with the labels in $\{1, \ldots, n\}$ which are decreasing along rows and columns and such that the entries $n, \ldots, n-m+1$ are in the first row (the decreasing condition forces these entries to be in the first m boxes).

Finally, to such a filling of λ we associate the unique element of $\mathbb{B}_{n-k,k,m}$ such that the vertices connected to the left endpoints of cups are the entries of the second row. Thus we obtain a bijection between the irreducible components of $\mathcal{B}^{\Delta}_{(n-k,k),m}$ and the set $\mathbb{B}_{n-k,k,m}$. ☐

Example 2.11. Let us take $\lambda = (3,3)$ and $m = 2$. The set of irreducible components is then parametrized by the following fillings of $\lambda' = (1,1)$ with entries in $\{1,2,3,4\}$

$$\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}, \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline\end{array}, \begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline\end{array}.$$

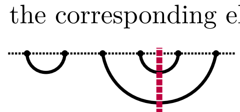
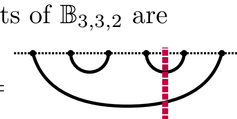
The skew tableaux of shape $\lambda/(m)$ and the tableaux of shape λ as obtained in the proof are

$$\begin{array}{ccc} & & \boxed{2} \\ \boxed{4} & \boxed{3} & \boxed{1} \end{array}, \begin{array}{ccc} & & \boxed{3} \\ \boxed{4} & \boxed{2} & \boxed{1} \end{array}, \begin{array}{ccc} & & \boxed{4} \\ \boxed{3} & \boxed{2} & \boxed{1} \end{array}$$

and

$$\begin{array}{|c|c|c|}\hline 6 & 5 & 2 \\ \hline 4 & 3 & 1 \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 6 & 5 & 3 \\ \hline 4 & 2 & 1 \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 3 & 2 & 1 \\ \hline\end{array}.$$

Finally, the corresponding elements of $\mathbb{B}_{3,3,2}$ are

$\mathbf{a} =$  $\mathbf{b} =$  $\mathbf{c} =$ 

3. Irreducible components for two-row Δ-Springer varieties

As in the previous section, let $\lambda = (n-k, k)$ be a two-row partition of n and $0 \leqslant m \leqslant k$. We also define $\lambda' = (n-k-m, k-m)$ which is a two-row partition of $n' = n - 2m$.

3.1. Embedding the Δ-Springer variety into the Cautis–Kamnitzer variety. Fix a large integer $N > 0$ (see Remark 3.1 for details on what is considered "large") and let $z\colon \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ be a nilpotent linear endomorphism with two Jordan blocks of the same size. In particular, there exists a Jordan basis

$$e_1 \leftarrow e_2 \leftarrow \dots \leftarrow e_N \qquad\qquad f_1 \leftarrow f_2 \leftarrow \dots \leftarrow f_N \tag{1}$$

of $\mathbb{C}^{2N}$ where the action of z is indicated by the arrows (the vectors e_1 and f_1 are sent to zero).

In [8, §2], Cautis–Kamnitzer define a smooth, projective variety given by

$$Y_n := \{F_\bullet \in \mathcal{F}l_{(1^n, 2N-n)}(\mathbb{C}^{2N}) \mid zF_i \subseteq F_{i-1}\}, \tag{2}$$

which will play an important role for our results.

Remark 3.1. Note that the inclusions $zF_i \subseteq F_{i-1}$ imply that

$$F_n \subseteq z^{-1}F_{n-1} \subseteq \dots \subseteq z^{-n}(0) = \operatorname{span}(e_1, \dots, e_n, f_1, \dots, f_n).$$

Hence, the variety Y_n does not depend on the choice of N as long as $N \geqslant n$. In particular, we can always assume (by making N larger, if necessary) that all the vector spaces of a flag in Y_n are contained in the image of z.

Define $E_{n-k,k} \subseteq \mathbb{C}^{2N}$ to be the subspace spanned by

$$e_1, \dots, e_{n-k}, f_1, \dots, f_k.$$

Then we can view the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ as a subvariety of Y_{n-m} via the following identification

$$\mathcal{B}^{\Delta}_{(n-k,k),m} \cong \{F_\bullet \in Y_{n-m} \mid z^m(E_{n-k,k}) \subseteq F_{n-m} \subseteq E_{n-k,k}\}. \tag{3}$$

By the z-invariance of the flags, the following observation is immediate:

Lemma 3.2. *We have*

$$\mathcal{B}^{\Delta}_{(n-k,k),m} \cong \{F_\bullet \in Y_{n-m} \mid F_{n-m} \subseteq E_{n-k,k}, e_{n-k-m}, f_{k-m} \in F_{n-m}\}.$$

3.2. Explicit description of the irreducible components. For the remainder of this article, we will write $\mathcal{B}^{\Delta}_{(n-k,k),m}$ to denote the embedded Δ-Springer variety via the identification (3). The following subvarieties will describe irreducible components of the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$.

Definition 3.3. *Let $\mathbf{a} \in \mathbb{B}_{n-k,k,m}$. We define $K_{\mathbf{a}} \subseteq Y_{n-m}$ to be the subvariety of Y_{n-m} consisting of all flags $(F_1, \dots, F_{n-m})$ satisfying the following conditions imposed by the Δ-cup diagram $\mathbf{a}$:*

(i) *If i—j, $i, j \in \{1, \dots, n-m\}$, then*
$$F_j = z^{-\frac{1}{2}(j-i+1)}F_{i-1}.$$
(ii) *If $|$—i, $i \in \{1, \dots, n-m\}$, then*
$$F_i = F_{i-1} + \operatorname{span}\Big(e_{\frac{1}{2}\left(i+\rho_{\mathbf{a}}(i)\right)}\Big).$$
Here, $\rho_{\mathbf{a}}(i)$ is the number of rays to the left of vertex i (including the vertex i) in $\mathbf{a}$.

There are no relations for a vector space indexed by a vertex connected to a cup whose right endpoint is connected to a vertex in $\{n-m+1, \dots, n\}$.

Theorem 3.4. *Let $0 \leqslant m \leqslant k$. The following statements hold:*

(a) *The subvariety $K_{\mathbf{a}} \subseteq Y_{n-m}$ is an irreducible component of the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m} \subseteq Y_{n-m}$.*

(b) *The irreducible component $K_{\mathbf{a}} \subseteq \mathcal{B}^{\Delta}_{(n-k,k),m}$ is a k-fold iterated fiber bundle over $\mathbb{P}^1$: there exist spaces $K_{\mathbf{a}} = X_1, X_2, \dots, X_k, X_{k+1} = \mathrm{pt}$ together with maps $p_1, p_2, \dots, p_k$ such that $p_j \colon X_j \to \mathbb{P}^1$ is a fiber bundle with typical fiber X_{j+1}. In particular, the irreducible component $K_{\mathbf{a}}$ is smooth.*

(c) *The map $\mathbf{a} \mapsto K_{\mathbf{a}}$ defines a bijection between the Δ-cup diagrams in $\mathbb{B}_{n-k,k,m}$ and the irreducible components of $\mathcal{B}^{\Delta}_{(n-k,k),m}$.*

Remark 3.5. We could replace z by the restriction z_λ of z to $E_{n-k,k}$ in Definition 3.3, which is justified by Proposition 3.7. Hence, the description of the irreducible components of the Δ-Springer variety in Theorem 3.4(a) also makes sense without the embedding into Y_{n-m}.

For the proof of Theorem 3.4 we consider the subvariety $X^i_{n-m} \subseteq Y_{n-m}$, $1 \leqslant i < n-m$, defined by

$$X^i_{n-m} := \{F_\bullet \in Y_{n-m} \mid F_{i+1} = z^{-1}F_{i-1}\}, \tag{4}$$

and the surjective morphism of varieties $q^i_{n-m} \colon X^i_{n-m} \twoheadrightarrow Y_{n-m-2}$ given by

$$(F_1, \dots, F_{n-m}) \mapsto (F_1, \dots, F_{i-1}, zF_{i+2}, \dots, zF_{n-m}), \tag{5}$$

see also [8, §2].

Lemma 3.6. *Let $\mathbf{a} \in \mathbb{B}_{n-k,k,m}$ be a cup diagram with a cup connecting vertices i and $i+1$ and let $\tilde{\mathbf{a}} \in \mathbb{B}_{n-k-1,k-1,m}$ be the cup diagram obtained by deleting this cup. Then we have $K_{\mathbf{a}} = (q^i_{n-m})^{-1}(K_{\tilde{\mathbf{a}}})$.*

Proof. We have to show that a flag $(F_1, \dots, F_{n-m}) \in Y_{n-m}$ satisfies the conditions (i), (ii) from Definition 3.3 with respect to the Δ-cup diagram $\mathbf{a}$ if and only if

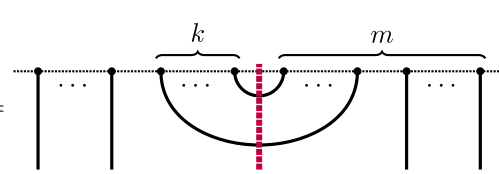
$$q^i_{n-m}(F_1, \dots, F_{n-m}) = (F_1, \dots, F_{i-1}, zF_{i+2}, \dots, zF_{n-m}) \in Y_{n-m-2}$$

satisfies these conditions with respect to $\tilde{\mathbf{a}}$. For a proof, we refer to [33, Lemma 17]. □

Proposition 3.7. *Let $\mathbf{a} \in \mathbb{B}_{n-k,k,m}$ and $F_\bullet \in K_{\mathbf{a}}$. Then $F_{n-m} \subseteq E_{n-k,k}$ and e_{n-k-m}, f_{k-m} belong to F_{n-m}. In particular, if $0 \leqslant m \leqslant k$, we have $K_{\mathbf{a}} \subseteq \mathcal{B}^{\Delta}_{(n-k,k),m}$.*

Proof. We use the same argument as in the proof of [33, Proposition 18]: we proceed by induction on the number of cups in $\mathbf{a}$ with both endpoints on the left of the cut line.

If there is no cup with both endpoints on the left of the cut line, then $m \geqslant k$ and

$\mathbf{a} =$ 

By convention, $f_{k-m} = 0$ since $k \leqslant m$, so that $f_{k-m} \in F_{n-m}$. By definition of $K_{\mathbf{a}}$, the flag $F_\bullet$ satisfies $F_i = \mathrm{span}(e_1, \dots, e_i)$ for $1 \leqslant i \leqslant n-m-k$. Therefore, we have that $e_{n-k-m} \in F_{n-m}$ since $e_{n-k-m} \in F_{n-k-m}$ and $F_{n-k-m} \subseteq F_{n-m}$. Finally, $F_{n-m} \subseteq z^{-k}F_{n-k-m}$ is included in $\mathrm{span}(e_1, \dots, e_{n-m}, f_1, \dots, f_k) = E_{n-k,k}$.

Now, suppose that there is a cup in $\mathbf{a}$ with both endpoints on the left of the cut line. Fix then $1 \leqslant i < n-m$ such that the vertices i and $i+1$ are joined by a cup in $\mathbf{a}$. We consider $\tilde{\mathbf{a}} \in \mathbb{B}_{n-k-1,k-1,m}$ the diagram obtained by removing this cup.

Then $q^i_{n-m}(F_\bullet) = (F_1,\dots,F_{i-1},zF_{i+2},\dots,zF_{n-m}) \in K_{\tilde{\mathbf{a}}}$ and so, by induction, we have that $zF_{n-m}\subseteq\operatorname{span}(e_1,\dots,e_{n-k-1},f_1,\dots,f_{k-1})$ and hence that $F_{n-m}\subseteq\operatorname{span}(e_1,\dots,e_{n-k},f_1,\dots,f_k)=E_{n-k,k}$.

We finally show that e_{n-k-m} and f_{k-m} are in F_{n-m}. By induction hypothesis, we also have $e_{n-k-m-1} \in zF_{n-m-1}$. There then exists $v \in F_{n-m-1}$ such that $z(v) = e_{n-k-m-1}$. Any such v is of the form $v = e_{n-k-m} + \alpha e_1 + \beta f_1$. Since i and $i+1$ are joined by a cup, we have $F_{i+1} = z^{-1}F_{i-1} \supseteq z^{-1}\{0\} = \operatorname{span}(e_1, f_1)$. Therefore the vectors e_1 and f_1 belong to F_{n-m} and $e_{n-k-m} = v - \alpha e_1 - \beta f_1 \in F_{n-m}$. One shows similarly that $f_{k-m} \in F_{n-m}$.

If $0 \leqslant m \leqslant k$, then Lemma 3.2 implies that $K_{\mathbf{a}} \subset \mathcal{B}^{\Delta}_{(n-k,k),m}$. □

Proof of Theorem 3.4. We first note that $K_{\mathbf{a}}$ is an k-fold iterated fiber bundle over $\mathbb{P}^1$. The proof is the same as for the irreducible components of two-row Springer fibers because the defining relations (i) and (ii) of $K_{\mathbf{a}}$ in Definition 3.3 are the same as for two-row Springer fibers. However, for the reader's convenience, we briefly recall the argument. We refer to [18, Proposition 5.1] and [34, Section 8] for additional details. Let $i_1 < i_2 < \ldots < i_k$ denote the vertices connected to a left endpoint of a cup in $\mathbf{a}$. Note that the space F_{i_1-1} is the same for every flag $(F_1,\dots,F_{n-m}) \in K_{\mathbf{a}}$ because each vertex strictly to the left of i_1 is connected to a ray. Hence, by successively applying relation (ii) in Definition 3.3, we see that F_{i_1-1} is uniquely determined. As a result, we can consider the fiber bundle

$$p_1\colon K_{\mathbf{a}} \to \mathbb{P}(z^{-1}F_{i_1-1}/F_{i_1-1}) \cong \mathbb{P}^1\,,\ (F_1,\dots,F_{n-m}) \mapsto F_{i_1}/F_{i_1-1}.$$

Its typical fiber is denoted X_2 and consists of all flags $(F_1,\dots,F_{n-m}) \in K_{\mathbf{a}}$ with F_{i_1} (and F_j, if (i_1+1) and j are connected by a cup) fixed. Now we repeat the above construction replacing X_1 by X_2 and the vertex i_1 by i_2, and continue until we have exhausted all of the vertices $i_1 < \cdots < i_k$.

In order to prove parts (a) and (b) of Theorem 3.4 we first note that $K_{\mathbf{a}}$ is smooth since it is a k-fold iterated fiber bundle over $\mathbb{P}^1$. This shows that $K_{\mathbf{a}}$ is irreducible since it is as well connected as an iterated fiber bundle over $\mathbb{P}^1$. By Proposition 3.7 the variety $K_{\mathbf{a}}$ is contained in $\mathcal{B}^{\Delta}_{(n-k,k),m}$. Finally, the dimension of $K_{\mathbf{a}}$ equals k which is the dimension of $\mathcal{B}^{\Delta}_{(n-k,k),m}$ by Theorem 2.5. Hence, $K_{\mathbf{a}}$ is an irreducible component of the (embedded) Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$. In particular, parts (a) and (b) of Theorem 3.4 are now clear from the above.

By Lemma 2.10 we know that the cup diagrams in $\mathbb{B}_{n-k,k,m}$ are in bijective correspondence with the irreducible components of $\mathcal{B}^{\Delta}_{(n-k,k),m}$. Since the irreducible components $K_{\mathbf{a}}$ are different for different $\mathbf{a} \in \mathbb{B}_{n-k,k,m}$, we see that the map $\mathbf{a} \mapsto K_{\mathbf{a}}$ explicitly realizes this bijection which proves part (c) of the theorem. □

Example 3.8. Let us continue Example 2.11. The irreducible components are then given by

- $K_{\mathbf{a}} = \{F_1 \subset \operatorname{span}(e_1, f_1) \subset F_2 \subset F_3\}$,
- $K_{\mathbf{b}} = \{F_1 \subset F_2 \subset z^{-1}(F_1) \subset F_3\}$,
- $K_{\mathbf{c}} = \{F_1 \subset F_2 \subset F_3 \subset z^{-1}(F_2)\}$.

3.3. $\mathbb{C}^*$-action and generalized components. In [20], an affine paving of the Δ-Springer variety is constructed in order to compute the cohomology ring of the variety. We make this explicit in the two-row case using a $\mathbb{C}^*$-action.

Let $\mathbb{C}^*$ act on $\mathbb{C}^{2N}$ by

$$t \cdot e_i = t^{-1} e_i \quad \text{and} \quad t \cdot f_i = t f_i.$$

This action restricts to $E_{n-k,k}$ and induces a $\mathbb{C}^*$-action on the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$.

Definition 3.9. *A* combinatorial weight of type $(n-k,k)$ *is a sequence in* $\{\wedge,\vee\}^n$ *containing* $n-k$ *ups (*$\wedge$*) and* k *downs (*$\vee$*).*

Such sequence is called a Δ-weight of type $(n-k,k,m)$ *if there is no* $\vee$ *to the left of any* $\wedge$ *among the last* m *symbols.*

When writing weights we use a $|$ to indicate the m rightmost entries and we suppress the unnecessary commas.

Example 3.10. Let $n=5$ and $k=2$ and $m=2$. Then $\vee\wedge\vee|\wedge\wedge$ is a Δ-weight of type $(3,2,2)$ and $\wedge\vee\wedge|\vee\wedge$ is not.

Proposition 3.11. *There is a bijection between* Δ*-weights of type* $(n-k,k,m)$ *and the fixed points under the action of* $\mathbb{C}^*$ *on* $\mathcal{B}^{\Delta}_{(n-k,k),m}$*.*

Explicitly, given a Δ*-weight* $\alpha=(\alpha_1,\ldots,\alpha_n)$ *of type* $(n-k,k,m)$*, the corresponding fixed point is the flag* $F^{\alpha}_{\bullet}$ *with ith subspace given by*

$$F^{\alpha}_i = \operatorname{span}(e_1,\ldots,e_{\#\{\wedge\text{'s weakly to the left of } i\}}, f_1,\ldots,f_{\#\{\vee\text{'s weakly to the left of } i\}}).$$

Proof. It is clear that the flag $F^{\alpha}_{\bullet}$ is fixed by the $\mathbb{C}^*$-action since each space of the flag is generated by weight vectors with respect to $\mathbb{C}^*$-action.

Conversely, let $F_{\bullet}=(F_1,\ldots,F_{n-m})\in\mathcal{B}^{\Delta}_{(n-k,k),m}$ fixed under the action of $\mathbb{C}^*$. Using induction on l, one see that the vector space F_l must be spanned by $e_1,\ldots,e_p,f_1,\ldots,f_q$ for some $0\leqslant p\leqslant n-k$ and $0\leqslant q\leqslant k$ with $p+q=l$. By counting occurrences of e's and f's, we recover the entries $\alpha_1,\ldots,\alpha_{n-m}$ of the Δ-weight corresponding to the fixed flag $F_{\bullet}$. Let r be the number of $\vee$ among these $\alpha_1,\ldots,\alpha_{n-m}$. Then the condition $z^m(E_{n-k,k})\subseteq F_{n-m}$ forces to have $0\leqslant r\leqslant k-m$. We then complete the weight $\alpha_1,\ldots,\alpha_{n-m}$ by $r+m-k$ $\wedge$ followed by $k-r$ $\vee$ in order to obtain the Δ-weight of type $(n-k,k,m)$ corresponding to $F_{\bullet}$. ☐

We define the attracting cell of the fixed point $F^{\alpha}_{\bullet}$ by

$$K_{\alpha}=\{F_{\bullet}\in\mathcal{B}^{\Delta}_{(n-k,k),m}\mid \lim_{t\to+\infty} t\cdot F_{\bullet}=F^{\alpha}_{\bullet}\}.$$

To test whether a flag $F_{\bullet}$ lies inside K_{α}, we follow [39, Section 2.2]. We define P to be the subspace of $E_{n-k,k}$ spanned by the e_i's and Q to be the subspace of $E_{n-k,k}$ spanned by the f_i's. Given a flag $F_{\bullet}$ in $\mathcal{B}^{\Delta}_{(n-k,k),m}$, we associate a new flag $F^{\text{ass}}_{\bullet}$ by setting $F^{\text{ass}}_i=P_i+Q_i\subset P\oplus Q$, where $P_i=F_i\cap P$ and Q_i is the image of F_i by the projection onto Q along P. It is then clear that this flag is stable under the $\mathbb{C}^*$-action. The following is similar to [39, Proposition 14].

Lemma 3.12. *Let* α *be a* Δ*-weight of type* $(n-k,k,m)$ *and* $F_{\bullet}\in\mathcal{B}^{\Delta}_{(n-k,k),m}$*. The flag* $F_{\bullet}$ *is in the attracting cell* K_{α} *if and only* $F^{\text{ass}}_{\bullet}=F^{\alpha}_{\bullet}$*.*

Lemma 3.12 implies that the cell K_{α} is an affine variety. We can also explicitly describe these attracting cells. Let α be a Δ-weight and construct a Δ-cup diagram $C(\alpha)$ as follows. First successively connect neighboring pairs $\vee\wedge$ by a cup, ignoring symbols that are already connected. When there are no more neighboring pairs $\vee\wedge$ among the remaining symbols, then connect all remaining symbols to ray.

Theorem 3.13.

(1) *There is a bijection between the* Δ*-weights of type* $(n-k,k,m)$ *and the cells of an affine paving of* $\mathcal{B}^{\Delta}_{(n-k,k),m}$*.*

(2) *The attracting cell* K_{α} *consists of all flags* $F_{\bullet}\in\mathcal{B}^{\Delta}_{(n-k,k),m}$ *satisfying the following conditions:*

(i) $F_j = z^{-\frac{j-i+1}{2}} F_{i-1}$ *if* $1 \leqslant i < j \leqslant n-m$ *and* i—j *in* $C(\alpha)$,
(ii) $F_i = F_i^{\alpha}$ *if* $|$—i *in* $C(\alpha)$,
(iii) $F_{i-1} \cap P = F_i \cap P$ *if* i *is the left endpoint of a cup in* $C(\alpha)$.

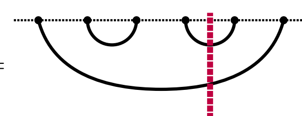
Proof. The first item follows from Proposition 3.11. Concerning the second item, we can use the same arguments as in [33, Theorem 36]; working with a Δ-Springer variety does not affect the proof. □

Remark 3.14. We also obtain a description of the closures of the cells K_α by removing the condition (iii) in the description of K_α in the above theorem.

In addition, if $C(\alpha)$ contains k cups, then it is clear that the closure of K_α is the irreducible component associated with the Δ-cup diagram $C(\alpha)$.

Example 3.15. Let $\lambda = (3,3)$ and $m = 2$.

- for $\alpha = \vee \vee \wedge \vee | \wedge \wedge$ we get

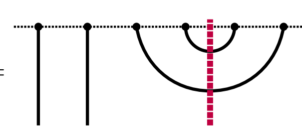
$C(\alpha) =$ 

and $F_\bullet^\alpha = (\mathrm{span}(f_1), \mathrm{span}(f_1, f_2), \mathrm{span}(e_1, f_1, f_2), \mathrm{span}(e_1, f_1, f_2, f_3))$. Then the closure of the attracting cell consists of all flags $F_\bullet \in \mathcal{B}^{\Delta}_{(3,3),2}$ such that

$$F_1 \subseteq F_2 \subseteq z^{-1}(F_1) \subseteq F_4 \subseteq \mathbb{C}^6.$$

This is in fact the irreducible component of $\mathcal{B}^{\Delta}_{(3,3),2}$ labeled by the diagram $C(\alpha)$.

- for $\alpha = \wedge \vee \vee \vee | \wedge \wedge$, we get

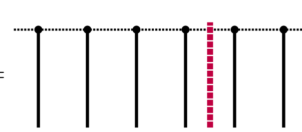
$C(\alpha) =$ 

and $F_\bullet^\alpha = (\mathrm{span}(e_1), \mathrm{span}(e_1, f_1), \mathrm{span}(e_1, f_1, f_2), \mathrm{span}(e_1, f_1, f_2, f_3))$. Then the closure of the attracting cell consists of all flags $F_\bullet \in \mathcal{B}^{\Delta}_{(3,3),2}$ such that

$$\mathrm{span}(e_1) \subseteq \mathrm{span}(e_1, f_1) \subseteq F_3 \subseteq F_4 \subseteq \mathbb{C}^6.$$

This attracting cell is not an irreducible component of $\mathcal{B}^{\Delta}_{(3,3),2}$.

- for $\alpha = \wedge \wedge \wedge \vee | \vee \vee$, we get

$C(\alpha) =$ 

and $F_\bullet^\alpha = (\mathrm{span}(e_1), \mathrm{span}(e_1, e_2), \mathrm{span}(e_1, e_2, e_3), \mathrm{span}(e_1, e_2, e_3, f_1))$. Then the closure of the attracting cell consists only of the flag $F_\bullet^\alpha$

As a corollary to Theorem 3.13, we obtain a diagrammatic description of the homology of the Δ-Springer variety.

Corollary 3.16. *The homology* $H_*(\mathcal{B}^{\Delta}_{(n-k,k),m})$ *has a basis indexed by the* Δ*-weights of type* $(n-k,k,m)$*. Moreover, the homological degree of an element of this basis is given by twice the number of cups in* $C(\alpha)$*.*

4. Comparison with Springer fibers and exotic Springer fibers

We still work with a Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ for a two-row partition $(n-k,k)$ and $0 \leqslant m \leqslant k$. We compare this variety with the two-row Springer fiber $\mathcal{B}_{(n-k,k)}$ and with the exotic Springer fiber $\mathcal{B}^{\mathrm{e}}_{((n-m-k),(k))}$.

4.1. Comparison with the two-row Springer fiber.

In this subsection, we give a negative answer to [20, Question 8.7]. We already have remarked that if $m = 0$ then the Δ-Springer variety $\mathcal{B}^{\Delta}_{\lambda,0}$ is equal to the Springer fiber $\mathcal{B}_{\lambda}$. Consider $\pi\colon Y_n \to Y_{n-m}$ the morphism of algebraic varieties which forgets the last m subspaces of a flag:

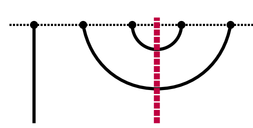
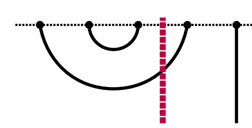
$$\pi(F_1, \ldots, F_n) = (F_1, \ldots, F_{n-m}).$$

Forgetting the cut line defines an injection $\iota\colon \mathbb{B}_{n-k,k,m} \to \mathbb{B}_{n-k,k,0}$. The set $\mathbb{B}_{n-k,k,m}$ indexes the irreducible components of the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ and the set $\mathbb{B}_{n-k,k,0}$ indexes the irreducible components of the Springer fiber $\mathcal{B}_{(n-k,k)}$. Therefore, if $\mathbf{a} \in \mathbb{B}_{n-k,k,m}$, then $K_{\iota(\mathbf{a})}$ is an irreducible component of $\mathcal{B}_{(n-k,k)}$. In [20, Remark 5.12], the authors showed that π induces a birational morphism form $\bigcup_{\mathbf{a}\in\mathbb{B}_{n-k,k,m}} K_{\iota(\mathbf{a})}$ to $\mathcal{B}^{\Delta}_{(n-k,k),m}$, and asked whether this map is an isomorphism. We now give an example that answers this question negatively.

Example 4.1. Let $\lambda = (3,2)$ and $m = 2$. The corresponding Δ-Springer variety $\mathcal{B}^{\Delta}_{(3,2),2}$ is the union of three irreducible components. Consider the flag

$$F_\bullet = (F_1, F_2, F_3) = (\mathrm{span}(e_1), \mathrm{span}(e_1, e_2), \mathrm{span}(e_1, e_2, f_1)).$$

This flag lies in the irreducible components corresponding to the following diagrams:

 and 

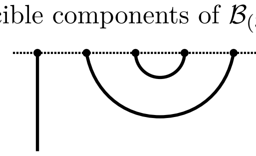
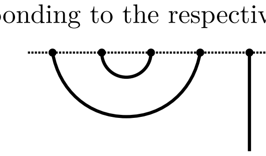
Now, consider the two different flags

$$(F_1, F_2, F_3, \mathrm{span}(e_1, e_2, e_3, f_1), \mathrm{span}(e_1, e_2, e_3, f_1, f_2))$$

and

$$(F_1, F_2, F_3, \mathrm{span}(e_1, e_2, f_1, f_2), \mathrm{span}(e_1, e_2, e_3, f_1, f_2)).$$

in $\mathcal{B}_{(3,2)}$. By definition, the image of both of these flags under π is $F_\bullet$. They belong to the irreducible components of $\mathcal{B}_{(3,2)}$ corresponding to the respective diagrams

 and 

which are both in $\mathbb{B}_{3,2,2}$. Therefore, the restriction of π to $\bigcup_{\mathbf{a}\in\mathbb{B}_{3,2,2}} K_{\iota(\mathbf{a})}$ is not an isomorphism onto the Δ-Springer variety $\mathcal{B}^{\Delta}_{(3,2),2}$ since it is not bijective. Note that we cannot remove more irreducible components of the Springer fiber $\mathcal{B}_{(3,2)}$ to make the map injective. Consider the three flags

$$\begin{array}{lll}
(\mathrm{span}(e_1), & \mathrm{span}(e_1, e_2), & \mathrm{span}(e_1, e_2, e_3)),\\
(\mathrm{span}(e_1 + f_1), & \mathrm{span}(e_1 + f_1, e_2 + f_2), & \mathrm{span}(e_1, f_2, e_2 + f_2)),\\
(\mathrm{span}(e_1 + f_1), & \mathrm{span}(e_1, f_1), & \mathrm{span}(e_1, f_1, e_2 - f_2)).
\end{array}$$

in $\mathcal{B}^{\Delta}_{(3,2),2}$. For each of these flags, one can check that they are in the image of a unique irreducible component of $\mathcal{B}_{(3,2)}$. In particular, throwing out an additional irreducible component of $\mathcal{B}_{(3,2)}$ would not yield a surjection onto $\mathcal{B}^{\Delta}_{(3,2),2}$.

4.2. Comparison with the exotic Springer fiber for a one-row bipartition. We now compare a two-row Δ-Springer variety with an exotic Springer fiber associated with a one-row bipartition. Before doing so, let us recall the description of [33] of the irreducible components of exotic Springer fibers for one-row bipartitions.

Firstly, [33, Proposition 14] shows that we can ignore the symplectic structure in the case of exotic Springer fibers for one-row bipartitions. If the bipartition is $((p),(q-p))$ then

$$\mathcal{B}^{\mathrm{e}}_{((p),(q-p))} \simeq \{F_\bullet \in Y_q \mid e_p \in F_q\}. \tag{6}$$

As for the Δ-Springer variety, the notation $\mathcal{B}^{\mathrm{e}}_{((p),(q-p))}$ will refer to the exotic Springer fiber embedded in Y_q. We first deal with the extremal case $m=k$.

Proposition 4.2. *The Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),k}$ and the exotic Springer fiber $\mathcal{B}^{\mathrm{e}}_{((n-2k),(k))}$ are isomorphic.*

Proof. Using an inductive argument as in [33, Proposition 18], one can prove that if a flag $F_\bullet \in \mathcal{B}^{\mathrm{e}}_{((p),(q-p))}$ then $F_q \subseteq \operatorname{span}(e_1,\dots,e_q,f_1,\dots,f_{q-p})$. Therefore $\mathcal{B}^{\Delta}_{(n-k,k),k}$ and $\mathcal{B}^{\mathrm{e}}_{((n-2k),(k))}$ are isomorphic to subvarieties of $\{F_\bullet \in Y_{n-k} \mid F_{n-k} \subseteq E_{n-k,k}\}$, see (3) and (6). By Lemma 3.2, we have

$$\mathcal{B}^{\Delta}_{(n-k,k),k} \cong \{F_\bullet \in Y_{n-k} \mid F_{n-k} \subseteq E_{n-k,k}, e_{n-2k}, f_0 \in F_{n-k}\}.$$

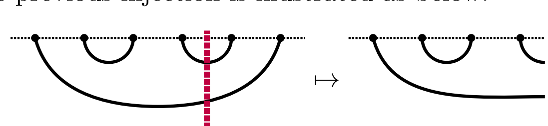
Since $f_0 = 0$, comparing with (6) shows that both varieties are equal as subvarieties of Y_{n-k}. □

We turn back to the general case of $0 \leqslant m \leqslant k$. Elements of the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ are described by flags in Y_{n-m} and so are the elements of the exotic Springer fiber $\mathcal{B}^{\mathrm{e}}_{((n-m-k),(k))}$.

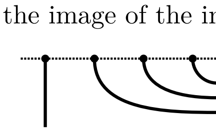
We quickly recall the diagrammatics describing the irreducible components of the exotic Springer fiber associated with the bipartition $((n-m-k),(k))$, see [33] for more details. These irreducible components are indexed by one-boundary diagrams on $n-m$ points which are endpoints of rays, cups or half-cups: cups connect two points, and both rays and half-cups connect only one point. Doing so, we require that the diagram is crossingless. We will denote by $\mathbb{B}_{((n-m-k),(k))}$ this set of diagrams with a total number of cups and half-cups equal to k.

There is a map $\mathbb{B}_{n-k,k,m}$ into $\mathbb{B}_{((n-m-k),(k))}$ by deleting the part of the diagram right of the cut line. Since there is no cup among the last m points of an element of $\mathbb{B}_{n-k,k,m}$, this map is an injection: we can reconstruct the initial diagram by completing the half-cups and then complete the remaining points with rays.

Example 4.3. The previous injection is illustrated as below:



The following diagram is not in the image of the injection $\mathbb{B}_{3,3,2} \to \mathbb{B}_{((1),(3))}$:



Indeed, three points are needed to complete the half-cups and only two points are allowed on the right of the cut line of an element of $\mathbb{B}_{3,3,2}$.

It easy to see that the image of this map is the subset of $\mathbb{B}_{((n-m-k),(k))}$ with at most m half-cups. Using this identification, given $\mathbf{a} \in \mathbb{B}_{n-k,k,m}$, we will denote by $K^{\mathrm{e}}_{\mathbf{a}}$ the corresponding irreducible component of the exotic Springer fiber $\mathcal{B}^{\mathrm{e}}_{((n-m-k),(k))}$.

Theorem 4.4. *The (embedded) Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ is equal to the closed subvariety $\bigcup_{\mathbf{a}\in\mathbb{B}_{n-k,k,m}} K^{\mathrm{e}}_{\mathbf{a}}$ of the (embedded) exotic Springer fiber $\mathcal{B}^{\mathrm{e}}_{((n-m-k),(k))}$.*

Proof. Since $e_{n-k-m} \in z^m(E_{n-k,k})$, the result follows from [33, Proposition 14] and the description of irreducible components of the exotic Springer fiber in [33, Theorem 15]. ☐

5. A topological model and the action of the symmetric group

Recall that we have fixed a two-row partition $(n-k, k)$ and $0 \leqslant m \leqslant k$. Using Theorem 4.4 and the topological model for the exotic Springer fiber of [33], we obtain a topological model of the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$. We then deduce a skein theoretic description of the action of the symmetric group S_{n-m} on the homology of the Δ-Springer variety.

5.1. A topological model. Let $\mathbb{S}^2 \subseteq \mathbb{R}^3$ be the two dimensional standard unit sphere with north pole $p = (0,0,1)$. Given a Δ-cup diagram $\mathbf{a} \in \mathbb{B}_{n-k,k,m}$, define

$$S_{\mathbf{a}} = \left\{(x_1,\dots,x_{n-m}) \in \left(\mathbb{S}^2\right)^{n-m} \mid x_j = -x_i \text{ if } i \text{—} j, \text{ and } x_i = p \text{ if } \vdash\!\text{—} i\right\}.$$

Note that $S_{\mathbf{a}}$ is homeomorphic to a product of 2-spheres. Each left endpoint of a cup in the diagram $\mathbf{a} \in \mathbb{B}_{n-k,k,m}$ contributes exactly one sphere.

Definition 5.1. *The* topological Δ-Springer variety $\mathcal{S}^{\Delta}_{(n-k,k),m}$ *is defined as the union*

$$\mathcal{S}^{\Delta}_{(n-k,k),m} := \bigcup_{\mathbf{a}\in\mathbb{B}_{n-k,k,m}} S_{\mathbf{a}} \subseteq \left(\mathbb{S}^2\right)^{n-m}.$$

The above definition of $S_{\mathbf{a}}$ does not use the part of the diagram right of the cut line, and makes sense for any $\mathbf{a} \in \mathbb{B}_{((n-m-k),(k))}$.

Proposition 5.2. *There exists a homeomorphism between the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ and the topological Δ-Springer variety $\mathcal{S}^{\Delta}_{(n-k,k),m}$ such that the irreducible component $K_{\mathbf{a}}$ of $\mathcal{B}^{\Delta}_{(n-k,k),m}$ is sent to $S_{\mathbf{a}}$.*

Proof. In [33], a homeomorphism between the exotic Springer variety $\mathcal{B}^{\mathrm{e}}_{((n-m-k),(k))}$ and

$$\mathcal{S}^{\mathrm{e}}_{((n-m-k),(k))} := \bigcup_{\mathbf{a}\in\mathbb{B}_{((n-m-k),(k))}} S_{\mathbf{a}} \subseteq \left(\mathbb{S}^2\right)^{n-m}$$

is constructed and the irreducible component $K^{\mathrm{e}}_{\mathbf{a}}$ corresponds to $S_{\mathbf{a}}$. Therefore, using Theorem 4.4, the restriction of the homeomorphism $\mathcal{B}^{\mathrm{e}}_{((n-m-k),(k))} \simeq \mathcal{S}^{\mathrm{e}}_{((n-m-k),(k))}$ to the closed subvarieties provides a homeomorphism $\mathcal{B}^{\Delta}_{(n-k,k),m} \simeq \mathcal{S}^{\Delta}_{(n-k,k),m}$. The irreducible component $K_{\mathbf{a}}$ of $\mathcal{B}^{\Delta}_{(n-k,k),m}$ is then sent to $S_{\mathbf{a}}$. ☐

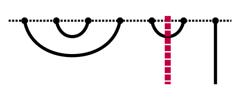
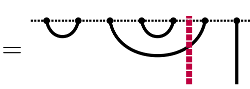
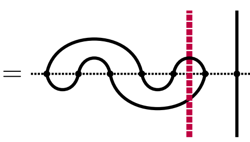
Exactly as in [33, Section 4.3], we describe the pairwise intersection of irreducible components in terms of circle diagrams.

For $\mathbf{a} \in \mathbb{B}_{n-k,k,m}$, denote by $\overline{\mathbf{a}}$ the reflection of $\mathbf{a}$ along a horizontal line. For $\mathbf{a}$ and $\mathbf{b}$ two Δ-cup diagrams in $\mathbb{B}_{n-k,k,m}$, we call the concatenation $\overline{\mathbf{a}}\mathbf{b}$ a *circle diagram.* Such a diagram only contains circles and (open) lines. In a circle diagram, a line is called *non-propagating* if it is entirely on the left of the cut line and its endpoints are on the same side of the horizontal line. Otherwise, a line is called *propagating.*

Proposition 5.3. *Let $\mathbf{a},\mathbf{b}\in\mathbb{B}_{n-k,k,m}$. The intersection $S_{\mathbf{a}}\cap S_{\mathbf{b}}$ is nonempty if and only if all the lines in the circle diagram $\overline{\mathbf{a}}\mathbf{b}$ are propagating. Moreover, $S_{\mathbf{a}}\cap S_{\mathbf{b}}$ is homeomorphic to $(\mathbb{S}^2)^{\ell}$, where ℓ counts the number of circles plus the number of open lines with both endpoints on the cut line, in the diagram obtained from $\overline{\mathbf{ab}}$ by erasing everything on the right of the cut line.*

Proof. This follows from [33, Theorem 29]. □

Example 5.4. Below we give examples of cup diagrams $\mathbf{a}$, $\mathbf{b}$ and a circle diagram $\overline{\mathbf{a}}\mathbf{b}$.

$\mathbf{a}=$  $\mathbf{b}=$  $\overline{\mathbf{a}}\mathbf{b}=$ 

Therefore the intersection $S_{\mathbf{a}}\cap S_{\mathbf{b}}$ is a 2-sphere.

5.2. Diagrammatic description of the homology. The topological model constructed in the previous section naturally embeds in $(\mathbb{S}^2)^{n-m}$ and this inclusion induces a map

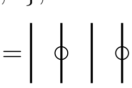
$$H_*(\mathcal{S}^{\Delta}_{(n-k,k),m})\hookrightarrow H_*((\mathbb{S}^2)^{n-m}). \tag{7}$$

We describe explicitly this map using line diagrams and show that this map is injective and the natural action of the symmetric group S_{n-m} on $H_*((\mathbb{S}^2)^{n-m})$ stabilizes the subspace $H_*(\mathcal{S}^{\Delta}_{\lambda,m})$.

The decomposition of the sphere $\mathbb{S}^2$ as $\{p\}\cup(\mathbb{S}^2\smallsetminus\{p\})$ defines a CW-structure on $\mathbb{S}^2$ with one 0-cell and one 2-cell. We fix this CW-structure and then equip $(\mathbb{S}^2)^{n-m}$ with the Cartesian product CW-structure. The cells of this CW-structure are indexed by the subsets U of $\{1,\dots,n-m\}$ by mapping U to the cell C_U which is defined by choosing the 0-cell in the i-th component if $i\notin U$ and the 2-cell if $i\in U$. We will denote by l_U the homology class of the cell C_U in $H_*((\mathbb{S}^2)^{n-m})$ so that $(l_U)_U$ is a basis of $H_*((\mathbb{S}^2)^{n-m})$, the degree of l_U is given by twice the cardinality of U.

Following [33, Section 6.1], we will write the elements l_U as line diagrams: we attach $n-m$ vertical lines, which are decorated by (empty) dots on the lines with endpoints not in U.

Example 5.5. If $U=\{1,3\}\subseteq\{1,2,3,4\}$, the corresponding line diagram is

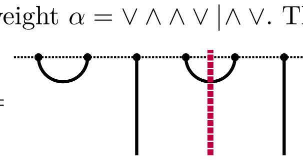
$l_U=$  .

Given a Δ-weight α of type $(n-k,k,m)$, we define an element $L_\alpha\in H_*((\mathbb{S}^2)^{n-m})$ as follows. Denote by $\mathcal{U}_\alpha$ the set of all $U\subseteq\{1,\dots,n-m\}$ containing one endpoint of each cup of $C(\alpha)$ left of the cut line and the left endpoints of cups which cross the cut line. We set

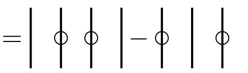
$$L_\alpha=\sum_{U\in\mathcal{U}_\alpha}(-1)^{\Lambda_\alpha(U)}l_U,$$

where $\Lambda_\alpha(U)$ is the number of right endpoints of cups of $C(\alpha)$ in U. Note that L_α is in degree $2d$, where d is the number of cups in $C(\alpha)$.

Example 5.6. Consider the Δ-weight $\alpha=\vee\wedge\wedge\vee|\wedge\vee$. The associated diagram is

$C(\alpha)=$ 

We then have

$$L_\alpha =$$ ,

the first term corresponding to $U = \{1, 4\}$ and the second one to $U = \{2, 4\}$.

Proposition 5.7. *The inclusion $\mathcal{S}^{\Delta}_{(n-k,k),m} \hookrightarrow (\mathbb{S}^2)^{n-m}$ induces an embedding*

$$H_*(\mathcal{S}^{\Delta}_{(n-k,k),m}) \hookrightarrow H_*((\mathbb{S}^2)^{n-m})$$

which sends the homology class of K_α to L_α.

Proof. The proof is similar to [33, Proposition 57]. In order to sketch the idea, consider the commutative diagram

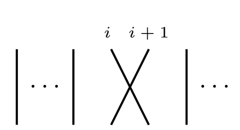
$$\bigoplus_{\mathbf{a} \in \mathbb{B}_{n-k,k,m}} H_*(S_{\mathbf{a}}) \longrightarrow H_*(\mathcal{S}^{\Delta}_{(n-k,k),m}) \longrightarrow H_*((\mathbb{S}^2)^{n-m}).$$

All maps are induced by the natural inclusions. Since $K_\alpha \subset S_{\mathbf{a}}$ for some $\mathbf{a} \in \mathbb{B}_{n-k,k,m}$, one can apply the upper map to the homology class of K_α and check that its image in $H_*((\mathbb{S}^2)^{n-m})$ is the line diagram sum L_α. The left horizontal map sends the homology class of K_α in $H_*(S_{\mathbf{a}})$ to the homology class of K_α in $H_*(\mathcal{S}^{\Delta}_{(n-k,k),m})$. Hence, by the commutativity of the diagram, the homology class of K_α in $H_*(\mathcal{S}^{\Delta}_{(n-k,k),m})$ gets sent to L_α. Since the homology classes of the K_α form a basis of $H_*(\mathcal{S}^{\Delta}_{(n-k,k),m})$, and the line diagram sums L_α, where α varies over all Δ-weights of type $(n-k, k, m)$, can be shown to be linearly independent, it follows that the map $H_*(\mathcal{S}^{\Delta}_{(n-k,k),m}) \hookrightarrow H_*((\mathbb{S}^2)^{n-m})$ is injective. $\square$

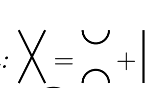
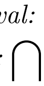
5.3. Action of the symmetric group. The symmetric group S_{n-m} acts naturally on $(\mathbb{S}^2)^{n-m}$ by permutation. This induces an action at the level of the homology, which is again described by permutations of the lines in a line diagram. From now on, we identify the homology $H_*(\mathcal{S}^{\Delta}_{(n-k,k),m})$ with its image in $H_*((\mathbb{S}^2)^{n-m})$.

We give a skein theoretic description of the action of the symmetric group, thereby answering [20, Question 8.6]. We write $s_i \in S_{n-m}$ for the transposition $(i, i+1)$. To state the explicit action, we will identify the basis element L_α of the homology corresponding to the Δ-weight α with the corresponding Δ-cup diagram $C(\alpha)$.

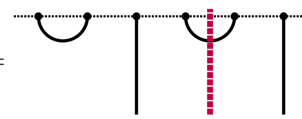
Proposition 5.8. *The subspace $H_*(\mathcal{S}^{\Delta}_{(n-k,k),m})$ is stable under the action of the symmetric group. Moreover, the action of s_i on L_α is obtained by stacking the diagram*



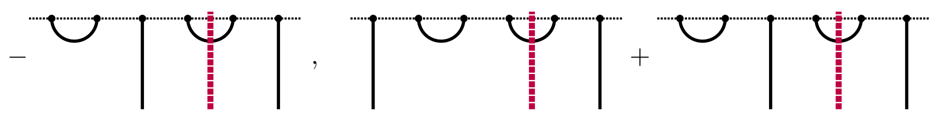
on top of $C(\alpha)$ and using the following relations

- *isotopies,*
- *skein relation:* ,
- *bubble removal:*  $= -2$,
- *cap removal:*  $= 0$,
- *if these operations create diagrams which are not allowed (*i.e. *with an entire cup right of the cut line), then these diagrams are set to zero.*

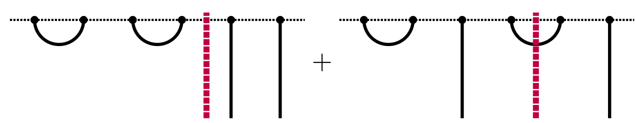
Example 5.9. Consider the weight α of Example 5.6, with corresponding Δ-cup diagram

$C(\alpha) =$ 

Then the actions of s_1, s_2 and s_3 on $C(\alpha)$ are respectively given by



and



Proof of Proposition 5.8. This is a direct computation, along the lines of [31]. □

As already shown in [20] for the top degree, the representations obtained in each degree are not simple. In the following, if μ is a skew partition of $n-m$, we denote by V_μ the corresponding skew Specht module which is a complex representation of S_{n-m}. If μ is a partition of $n-m$, then it is the usual irreducible Specht module.

Theorem 5.10. *Let $0 \leqslant d \leqslant k$. As S_{n-m} representations, we have the following isomorphisms:*

$$H_{2d(\mathcal{S}^{\Delta}_{(n-k,k),m})} \simeq \bigoplus_{j=\max(0,2d+m-n)}^{\min(m,d)} V_{(n-m-d+j,d-j)} \simeq V_{(n-d,d)/(m)}.$$

Proof. For $0 \leqslant j \leqslant \min(d,m)$, we consider the subspace W_j of $H_{2d(\mathcal{S}^{\Delta}_{(n-k,k),m})}$ spanned by the vectors L_α such that $C(\alpha)$ has at most j cups through the cut line.

Let us first notice that $W_{\min(d,m)} = H_{2d(\mathcal{S}^{\Delta}_{(n-k,k),m})}$. Indeed, if $L_\alpha \in H_{2d(\mathcal{S}^{\Delta}_{(n-k,k),m})}$ then the diagram $C(\alpha)$ has d cups and since there are m points on the right of the cut line, there are at most m cups of $C(\alpha)$ through the cut line.

We also have $W_j = \{0\}$ if $j < m-n+2d$. Suppose that $m-n+2d > 0$, otherwise there is nothing to prove. If C is a cup diagram on n points with d cups, then there are $n-2d$ rays on this diagram. Denote by r the number of rays on the right of the cut line. This implies that $m-r$ cups must pass through the cut line. But since $r \leqslant n-2d$, we obtain that C must have at least $2d+m-n$ cups through the cut line.

From the description of the action of S_{n-m} in Proposition 5.8, it is clear that W_j is stable under the action of S_{n-m}. Therefore, we obtain a filtration

$$\{0\} \subset W_{\max(0,2d+m-n)} \subset W_{\max(0,2d+m-n)+1} \subset \cdots \subset W_{\min(d,m)} = H_{2d(\mathcal{S}^{\Delta}_{(n-k,k),m})}$$

by S_{n-m} invariant subspaces. We claim that $W_j/W_{j-1} \simeq V_{(n-m-d+j,d-j)}$, which will prove the first isomorphism of the proposition.

Indeed, the quotient space W_j/W_{j-1} has a basis given by Δ-cup diagrams on n points with d cups and exactly j cups through the cut line. Forgetting the right part of such a diagram and replacing the j half-cups created this way by rays, we obtain a cup diagram on $n-m$ points with $d-j$ cups, and all such diagrams can be obtained by this process.

We thus obtain an isomorphism of vector spaces between W_j/W_{j-1} and $H_{2(d-j)(\mathcal{S}_{(n-m-d+j,d-j)})}$, which is easily checked to be S_{n-m}-equivariant thanks to

Proposition 5.8. Here, $\mathcal{S}_{(n-m-d+j,d-j)}$ is the topological model for the Springer fiber for the partition $(n-m-d+j, d-j)$, see [31]. Therefore, since $H_{2(d-j)(\mathcal{S}_{(n-m-d+j,d-j)})}$ is isomorphic to $V_{(n-m-d+j,d-j)}$ as a representation of S_{n-m}, we have proven our claim.

Concerning the isomorphism with the skew Specht module $V_{(n-d,d)/(m)}$, we use that the multiplicity of V_μ in $V_{(n-d,d)/(m)}$ as an S_{n-m}-representation is equal to the multiplicity of $V_{(n-d,d)}$ in $\mathrm{Ind}_{S_{n-m}\times S_m}^{S_n}(V_\mu \otimes V_{(m)})$ as S_n-representations [23, Section 3]. From Pieri's formula, we deduce that, as an S_{n-m}-representation, we have $V_{(n-d,d)/(m)} \simeq \bigoplus_\mu V_\mu$, the direct sum being on partitions μ obtained from $(n-d, d)$ by removing m boxes, no two in the same column. We easily check that we obtain the expected direct sum. □

Note that the isomorphism $H_*(\mathcal{B}^\Delta_{(n-k,k),m}) \simeq H_*(\mathcal{S}^\Delta_{(n-k,k),m})$ intertwines the S_{n-m}-action since the first Chern class of the dual ith line bundle maps to the hyperplane class of the ith copy of $\mathbb{P}^1 \simeq \mathbb{S}^2$, see [8, Theorem 2.1].

6. Action of the degenerate affine Hecke algebra

We enhance the action of the symmetric group S_{n-m} on the cohomology of the Δ-Springer variety to an action of the degenerate affine Hecke algebra. Each degree of the cohomology will be irreducible for the action of this algebra.

6.1. Degenerate affine Hecke algebra and action on the homology of $(\mathbb{S}^2)^{n-m}$.

We start by defining the degenerate affine Hecke algebra.

Definition 6.1. *The degenerate affine Hecke algebra H_{n-m} is the $\mathbb{C}$-algebra with generators $\sigma_1, \dots, \sigma_{n-m-1}$, $x_1, \dots, x_{n-m}$ and relations*

$$\begin{aligned}
\sigma_i^2 &= 1 && \text{for } 1 \leqslant i < n-m\\
\sigma_i\sigma_j &= \sigma_j\sigma_i && \text{if } |i-j| > 1,\\
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } 1 \leqslant i < n-m-1,\\
x_ix_j &= x_jx_i && \text{for } 1 \leqslant i,j \leqslant n-m,\\
\sigma_ix_j &= x_j\sigma_i && \text{if } j \neq i, i+1,\\
x_{i+1}\sigma_i - \sigma_ix_i &= 1 && \text{for } 1 \leqslant i < n-m.
\end{aligned}$$

There is a well-known polynomial action of H_{n-m} on $\mathbb{C}[X_1, \dots, X_{n-m}]$ given by the so-called Dunkl operators. This action preserves the degree, and the following definition is the restriction of the action of Dunkl operators to the subspace $\mathrm{span}(X_{i_1} \cdots X_{i_k} \mid k \in \mathbb{N}, 1 \leqslant i_1 < \cdots < i_k \leqslant n-m)$, written in the language of line diagrams.

Definition 6.2. *Let $\boldsymbol{\xi} = (\xi_0, \dots, \xi_{n-m}) \in \mathbb{C}^{n-m+1}$. Given a line diagram l_U, with U a subset of $\{1, \dots, n-m\}$ of cardinality d, and $1 \leqslant i \leqslant n-m$, we define*

$$\mathcal{D}_i^{(\boldsymbol{\xi})}(l_U) = \begin{cases} (\xi_d + i - n + m - |\{1 \leqslant j < i \mid j \notin U\}|)l_U - \displaystyle\sum_{\substack{j=i+1\\ j\notin U}}^{n-m} l_{U\smallsetminus\{i\}\cup\{j\}} & \text{if } i \in U,\\ (i - n + m + |\{i < j \leqslant n-m \mid j \in U\}|)\, l_U + \displaystyle\sum_{\substack{j=1\\ j\in U}}^{i-1} l_{U\smallsetminus\{j\}\cup\{i\}} & \text{if } i \notin U. \end{cases}$$

We will call $\boldsymbol{\xi}$ the parameters of the action. Since this action arises from Dunkl operators, this defines an action of the degenerate affine Hecke algebra on $H_*((\mathbb{S}^2)^{n-m})$, see [9].

Proposition 6.3. *The assignment $\sigma_i \mapsto s_i$ and $x_i \mapsto \mathcal{D}_i^{(\boldsymbol{\xi})}$ is a well-defined action of the degenerate affine Hecke algebra H_{n-m} on $H_*((\mathbb{S}^2)^{n-m})$.*

6.2. Restriction to the homology of the Δ-Springer variety. It turns out that the cohomology of the Δ-Springer variety, viewed as a subspace of the cohomology of the product of $n-m$ spheres, is stable under the action of the degenerate affine Hecke algebra for specific values of the parameter $\boldsymbol{\xi}$.

Theorem 6.4. *Let $\boldsymbol{\xi} = (\xi_0, \ldots, \xi_{n-m})$ with $\xi_d = n+1-d$. Then the subspace $H_*(\mathcal{B}^{\Delta}_{(n-k,k),m})$ of $H_*((\mathbb{S}^2)^{n-m})$ is stable under the action of H_{n-m}.*

We will prove the theorem as follows: since the degenerate affine Hecke algebra H_{n-m} is generated by the symmetric group and x_{n-m}, it suffices to show that the subspace $H_*(\mathcal{B}^{\Delta}_{(n-k,k),m})$ is stable under the action of the symmetric group and under the action of x_{n-m}. The action of the symmetric group is given in Proposition 5.8, therefore it suffices to consider the action of x_{n-m}. We give explicit formulas in terms of line diagrams, which have a skein theoretic interpretation akin to the action of the symmetric group.

Let α be a Δ-weight of type $(n-k,k,m)$ and suppose that the corresponding Δ-cup diagram $C(\alpha)$ has d cups. In other words, the element L_α corresponding to α is in $H_{2d}(\mathcal{B}^{\Delta}_{(n-k,k),m})$. We discuss three cases, depending whether $n-m$ is the endpoint of a ray, the right endpoint of a cup or the left endpoint of a cup in $C(\alpha)$.

Lemma 6.5. *Suppose that $n-m$ is the endpoint of a ray in $C(\alpha)$. Then $x_{n-m} \cdot L_\alpha = 0$.*

Proof. By definition, we have

$$L_\alpha = \sum_{U \in \mathcal{U}_\alpha} (-1)^{\Lambda_\alpha(U)} l_U.$$

Since we have supposed that $n-m$ is the endpoint of a ray in $C(\alpha)$, we know that $n-m \notin U$ for all $U \in \mathcal{U}_\alpha$. Therefore, the definition of the action of the degenerate affine Hecke algebra via Dunkl operators implies that

$$x_{n-m} \cdot L_\alpha = \sum_{U \in \mathcal{U}_\alpha} (-1)^{\Lambda_\alpha(U)} \sum_{i \in U} l_{U \smallsetminus \{i\} \cup \{n-m\}} = \sum_{\substack{U \in \mathcal{U}_\alpha \\ i \in U}} (-1)^{\Lambda_\alpha(U)} l_{U \smallsetminus \{i\} \cup \{n-m\}}.$$

We consider the bijection ψ on the set $\{(U,i) \mid U \in \mathcal{U}_\alpha, i \in U\}$ given by $\psi(U,i) = (U \smallsetminus \{i\} \cup \{j\}, j)$ where j is such that i and j are connected by a cup in $C(\alpha)$. This is well defined: there exists no cup that crosses the cut line because $n-m$ is the endpoint of a ray. Moreover, it is clear that $\Lambda_\alpha(U \smallsetminus \{i\} \cup \{j\}) = \Lambda_\alpha(U) \pm 1$ if $i \in U$ and j is connected to i by a cup in $C(\alpha)$.

Therefore, using the bijection ψ, we obtain that

$$\begin{aligned} x_{n-m} \cdot L_\alpha &= \sum_{\substack{U \in \mathcal{U}_\alpha \\ i \in U}} (-1)^{\Lambda_\alpha(U)} l_{U \smallsetminus \{i\} \cup \{n-m\}} \\ &= -\sum_{\substack{U \in \mathcal{U}_\alpha \\ i \in U}} (-1)^{\Lambda_\alpha(U)} l_{U \smallsetminus \{i\} \cup \{n-m\}} = -x_{n-m} \cdot L_\alpha, \end{aligned}$$

which implies that $x_{n-m} \cdot L_\alpha = 0$. □

Remark 6.6. The proof of Lemma 6.5 (as well as the proof of Lemma 6.9 below) uses a well-known argument from algebraic combinatorics. As in [37, Section 2.6], we show the vanishing of some terms by constructing sign-reversing involutions on the summands.

Lemma 6.7. *Suppose that $n-m$ is the left endpoint of a cup in $C(\alpha)$. Then*

$$x_{n-m} \cdot L_\alpha = (m+1)L_\alpha.$$

Proof. Since $n-m$ is the left endpoint of a cup in $C(\alpha)$, the corresponding right endpoint is on the right of the cut line. Therefore $n-m \in U$ for all $U \in \mathcal{U}_\alpha$ and the definition of the action of the degenerate affine Hecke algebra in terms of Dunkl operators gives

$$\begin{aligned} x_{n-m} \cdot l_U &= (\xi_d - |\{1 \leqslant i < n-m \mid i \notin U\}|)\, l_U \\ &= (n+1-d-(n-m-d))l_U = (m+1)l_U \end{aligned}$$

for all $U \in \mathcal{U}_\alpha$. This implies that $x_{n-m} \cdot L_\alpha = (m+1)L_\alpha$. □

In order to state the last case, we need to introduce some notation. Suppose that $n-m$ is the right endpoint of a cup in $C(\alpha)$. Denote by $a_1 < \cdots < a_r$ the left endpoints of cups in $C(\alpha)$ with right endpoints among the last m points, *i.e.* right of the cut line. We also let a_{r+1} be the left endpoint of the cup of $C(\alpha)$ with right endpoint $n-m$.

Therefore, the weight α satisfies the following:

- $\alpha_j = \wedge$ for $n-m+1 \leqslant j \leqslant n-m+r$ and $\alpha_j = \vee$ for $n-m+r+1 \leqslant j \leqslant n$,
- $\alpha_j = \vee$ for $j \in \{a_1, \dots, a_{r+1}\}$ and $\alpha_{n-m} = \wedge$.

For $1 \leqslant i \leqslant r$ define a Δ-weight α^i of type $(n-k, k, m)$ by:

- $\alpha^i_{n-m} = \vee$ and $\alpha^i_{a_{i+1}} = \wedge$,
- $\alpha^i_j = \alpha_j$ for $j \neq a_{i+1}, n-m$.

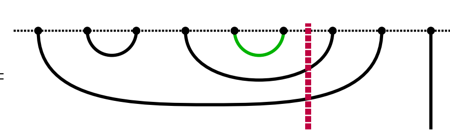
In terms of cup diagrams, the diagram $C(\alpha^i)$ is obtained from $C(\alpha)$ by moving the cup joining a_{r+1} and $n-m$ to a cup joining a_i and a_{i+1}, and by moving the left endpoints of the $r-i+1$ rightmost cups crossing the cut line.

If the number of right endpoints of cups crossing the cut line in $C(\alpha)$ is smaller than m, that is $r < m$, we also define a weight α^0 by:

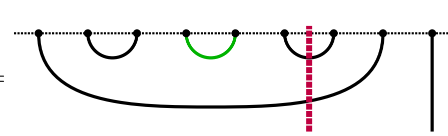
- $\alpha^0_{n-m} = \vee$ and $\alpha^0_{n-m+r+1} = \wedge$,
- $\alpha^0_j = \alpha_j$ for $j \neq a_1, n-m$.

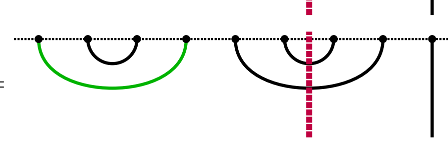
In terms of cup diagram, all the points $a_2, \dots, a_{r+1}$ and $n-m$ are left endpoints of a cup and a_1 is the endpoint of a ray.

Example 6.8. Let us take $\alpha = \vee\vee\wedge\vee\vee\wedge|\wedge\wedge\vee$. The associated cup diagram is then

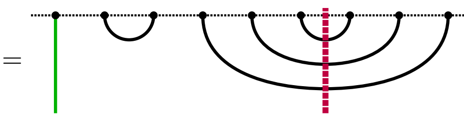
$C(\alpha) =$ 

We have $r = 2$ and $a_1 = 1$, $a_2 = 4$ and $a_3 = 5$. Therefore, the weights α^2, α^1 and α^0 and their respective associated cup diagram are given by

$\alpha^2 = \vee\vee\wedge\vee\wedge\vee|\wedge\wedge\vee$, $C(\alpha^2) =$ 

$\alpha^1 = \vee\vee\wedge\wedge\vee\vee|\wedge\wedge\vee$, $C(\alpha^1) =$ 

$\alpha^0 = \vee \vee \wedge \vee \vee \vee | \wedge \wedge \wedge,$ $\qquad C(\alpha^0) =$



We have drawn in green the cup that "moves along the diagrams when going from α to α^i".

Lemma 6.9. *Suppose that $n-m$ is the right endpoint of a cup in $C(\alpha)$, and keep the notations from above. Then*

$$x_{n-m} \cdot L_\alpha = -\sum_{i=0}^{r} (m-r+i) L_{\alpha^i}. \tag{8}$$

Remark 6.10. In (8), the term L_{α^0} is not defined if $m = r$. Nonetheless, in the case $m = r$, this term appears with a coefficient 0 and it is unnecessary to distinguish the cases $m = r$ and $m \neq r$.

Example 6.11. In the above example, we obtain $x_6 \cdot L_\alpha = -3L_{\alpha^2} - 2L_{\alpha^1} - L_{\alpha^0}$.

Proof of Lemma 6.9. First, we compute the action of x_{n-m} on L_α following the definition in terms of Dunkl operators:

$$x_{n-m} \cdot L_\alpha = \sum_{\substack{U \in \mathcal{U}_\alpha \\ n-m \in U}} (-1)^{\Lambda_\alpha(U)} (m+1) l_U + \sum_{\substack{U \in \mathcal{U}_\alpha \\ n-m \notin U}} (-1)^{\Lambda_\alpha(U)} \sum_{i \in U} l_{U \smallsetminus \{i\} \cup \{n-m\}},$$

where we have used that $\xi_d = n+1-d$ and $|\{1 \leqslant i < n-m \mid i \notin U\}| = n-m-d$ for $U \in \mathcal{U}_\alpha$ with $n-m \notin U$.

In order to compare with the right hand side of the equality to prove, we introduce some subsets of $\{1, \dots, n-m\}$. For $1 \leqslant i \leqslant r$ we define

$$\mathcal{U}_i = \{U \in \mathcal{U}_{\alpha^i} \mid a_i \in U\} = \{U \in \mathcal{U}_{\alpha^i} \mid a_{i+1} \notin U\},$$

and also $\mathcal{U}_0 = \mathcal{U}_{\alpha^0}$ (if α^0 is not defined, then we set $\mathcal{U}_0 = \varnothing$). It is then straightforward to check that, for all $1 \leqslant i \leqslant r$, we have $\mathcal{U}_{\alpha^i} = \mathcal{U}_{i-1} \cup \mathcal{U}_i$, and that these unions are disjoint. Moreover, an element $U \in \mathcal{U}_i$ is as well in $\mathcal{U}_{\alpha^{i+1}}$ and we have $\Lambda_{\alpha^{i+1}}(U) = \Lambda_{\alpha^i}(U) + 1$. Therefore

$$\begin{aligned}
\sum_{i=0}^{r} (m-r+i) L_{\alpha^i} &= \sum_{i=1}^{r} (m-r+i) \sum_{U \in \mathcal{U}_{i-1}} (-1)^{\Lambda_{\alpha^i}(U)} l_U + \sum_{i=0}^{r} (m-r+i) \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_{\alpha^i}(U)} l_U \\
&= -\sum_{i=0}^{r-1} (m-r+i+1) \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_{\alpha^i}(U)} l_U + \sum_{i=0}^{r} (m-r+i) \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_{\alpha^i}(U)} l_U \\
&= m \sum_{U \in \mathcal{U}_r} (-1)^{\Lambda_{\alpha^r}(U)} l_U - \sum_{i=0}^{r-1} \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_{\alpha^i}(U)} l_U \\
&= (m+1) \sum_{U \in \mathcal{U}_r} (-1)^{\Lambda_{\alpha^r}(U)} l_U - \sum_{i=0}^{r} \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_{\alpha^i}(U)} l_U.
\end{aligned}$$

But we have $\mathcal{U}_r = \{U \in \mathcal{U}_\alpha \mid n-m \in U\}$ and if $U \in \mathcal{U}_r$ then $\Lambda_\alpha(U) = \Lambda_{\alpha^r}(U) + 1$. It remains to prove the following equality:

$$\sum_{\substack{U \in \mathcal{U}_\alpha \\ n-m \notin U}} (-1)^{\Lambda_\alpha(U)} \sum_{i \in U} l_{U \smallsetminus \{i\} \cup \{n-m\}} = \sum_{i=0}^{r} \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_{\alpha^i}(U)} l_U.$$

We simplify the left hand side as follows. Suppose that i and j are connected by a cup in $C(\alpha)$. Then we have a bijection

$$\{U \in \mathcal{U}_\alpha \mid n-m \notin U \text{ and } i \in U\} \to \{U \in \mathcal{U}_\alpha \mid n-m \notin U \text{ and } j \in U\}$$

given by $U \mapsto U \smallsetminus \{i\} \cup \{j\}$, and $\Lambda_\alpha(U \smallsetminus \{i\} \cup \{j\}) = \Lambda_\alpha(U) \pm 1$. Therefore

$$\sum_{\substack{U\in\mathcal{U}_\alpha\\ n-m\notin U}} (-1)^{\Lambda_\alpha(U)} \sum_{i\in U} l_{U\smallsetminus\{i\}\cup\{n-m\}} = \sum_{\substack{U\in\mathcal{U}_\alpha\\ n-m\notin U}} (-1)^{\Lambda_\alpha(U)} \sum_{\substack{i=1\\ a_i\in U}}^{r+1} l_{U\smallsetminus\{a_i\}\cup\{n-m\}},$$

since we can cancel the other terms by pairs that carry opposite signs. Finally, for all $1 \leqslant i \leqslant r+1$, we have a bijection between $\{U \in \mathcal{U}_\alpha \mid n-m \notin U \text{ and } a_i \in U\}$ and $\mathcal{U}_{i-1}$ given by $U \mapsto U \smallsetminus \{a_i\} \cup \{n-m\}$ which moreover satisfies $\Lambda_\alpha(U) = \Lambda_{\alpha^i}(U \smallsetminus \{a_i\} \cup \{n-m\})$. Therefore

$$\sum_{\substack{U\in\mathcal{U}_\alpha\\ n-m\notin U}} (-1)^{\Lambda_\alpha(U)} \sum_{\substack{i=1\\ a_i\in U}}^{r+1} l_{U\smallsetminus\{a_i\}\cup\{n-m\}} = \sum_{i=0}^{r} \sum_{U\in\mathcal{U}_i} (-1)^{\Lambda_{\alpha^i}(U)} l_U$$

and the proof is complete. ☐

Proof of Theorem 6.4. The theorem follows from Proposition 5.8, Lemma 6.5, Lemma 6.7 and Lemma 6.9, together with the fact that the degenerate affine Hecke algebra is generated by the symmetric group and by x_{n-m}. ☐

6.3. Comparison with a tensor space. The goal is to give a proof of the irreducibility of the action of H_{n-m} on the cohomology of the Δ-Springer variety, using Schur–Weyl duality. We consider the Lie algebra $\mathfrak{gl}_2$ with basis given by (h_1, h_2, e, f) where

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We denote by Ω its Casimir element, that is $\Omega = h_1 \otimes h_1 + h_2 \otimes h_2 + e \otimes f + f \otimes e$. We denote by $\Omega_{i,j} \in \mathfrak{gl}_2^{\otimes n}$, for $1 \leqslant i < j \leqslant n$, the Casimir element for the ith and jth factor.

We also consider the natural representation V with basis (v_+, v_-) with the following $\mathfrak{gl}_2$-action:

$$\begin{aligned} &h_1 \cdot v_+ = v_+, && h_2 \cdot v_+ = 0, && e \cdot v_+ = 0, && f \cdot v_+ = v_-, \\ &h_1 \cdot v_- = 0, && h_2 \cdot v_- = v_-, && e \cdot v_- = v_+, && f \cdot v_- = 0. \end{aligned}$$

The m-th symmetric power S^mV of the natural representation has $(v_0, \ldots, v_m)$ as a basis and the following $\mathfrak{gl}_2$-action:

$$h_1 \cdot v_i = (m-i)v_i, \qquad h_2 \cdot v_i = iv_i \qquad e \cdot v_i = iv_{i-1}, \qquad f \cdot v_i = (m-i)v_{i+1}.$$

The main player will be the tensor space $V^{\otimes(n-m)} \otimes S^mV$, which is naturally a $\mathfrak{gl}_2$-module. The following theorem has been proved in [41, Proposition 7.1].

Theorem 6.12. (1) *The mapping $s_i \mapsto \Omega_{i,i+1}$ and $x_i \mapsto -\sum_{i<j\leqslant n-m+1} \Omega_{i,j} + m\,\mathrm{Id}$ defines an action of H_{n-m} on $V^{\otimes(n-m)} \otimes S^mV$, which commutes with the action of $\mathfrak{gl}_2$.*

(2) *The maps $H_{n-m} \to \mathrm{End}_{\mathfrak{gl}_2}(V^{\otimes(n-m)} \otimes S^mV)$ and $U(\mathfrak{gl}_2) \to \mathrm{End}_{H_{n-m}}(V^{\otimes(n-m)} \otimes S^mV)$ are surjective.*

(3) *As an $(U(\mathfrak{gl}_2), H_{n-m})$-bimodule, we have*

$$V^{\otimes(n-m)} \otimes S^mV \simeq \bigoplus_{d=0}^{\min(n-m,\lfloor n/2\rfloor)} V_{(n-d,d)} \boxtimes L_d,$$

Where $V_{(n-d,d)}$ is the simple highest $\mathfrak{gl}_2$-weight module of highest weight $(n-d,d)$ and L_d is a simple H_{n-m}-module.

This theorem provides a construction of certain simple H_{n-m}-modules, which will be compared to the modules obtained via the cohomology of the Δ-Springer variety.

Remark 6.13. In [41], the action of x_i is slightly different. We recover ours after applying the automorphism $\sigma_i \mapsto \sigma_{n-m-i}$, $x_i \mapsto -x_{n-m+1-i}$ of H_{n-m} and shifting the action of all the x_i by a common constant.

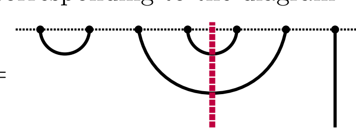
The simple module L_d is a calibrated simple module, in the sense of [30], and is then indexed by a placed skew shape. It can be checked that the corresponding skew partition is $(n-d,d)/(m)$.

First, to each $\mathbf{a} \in \mathbb{B}_{n-d,d,m}$ we associate a highest weight vector $p_{\mathbf{a}}$ of weight $(n-d,d)$ in $V^{\otimes(n-m)} \otimes S^m V$. We convert each cup left of the cut line into the vector $v_- \otimes v_+ - v_+ \otimes v_-$, each ray left of the cut line into v_+, and the remaining r cups with endpoints on both sides of the cut line into the vector

$$w_r = \sum_{I \subset \{1,\dots,r\}} (-1)^{|I|} v_I \otimes v_{|I|},$$

where $v_I \in V^{\otimes r}$ has its i-th tensorand equal to v_+ if $i \in I$ and to v_- otherwise. We then place these vectors in the appropriate position in $V^{\otimes(n-m)} \otimes S^m V$: the $n-m$ endpoints left of the cut line correspond to the various tensorands V and the m endpoints right of the cut line correspond to the tensorand $S^m V$. In this process, we completely ignore the rays right of the cut line.

Example 6.14. The vector corresponding to the diagram

$\mathbf{a} =$ 

is

$$(v_- \otimes v_+ - v_+ \otimes v_-) \otimes (v_- \otimes v_- \otimes v_0 - v_+ \otimes v_- \otimes v_1 - v_- \otimes v_+ \otimes v_1 + v_+ \otimes v_+ \otimes v_2).$$

Lemma 6.15. *Let $\mathbf{a} \in \mathbb{B}_{n-d,d,m}$. Then $p_{\mathbf{a}}$ is a highest weight vector of weight $(n-d,d)$.*

Proof. Since permuting the first $n-m$ tensorands of $V^{\otimes(n-m)} \otimes S^m V$ commutes with the action of $\mathfrak{gl}_2$, it suffices to check that $v_- \otimes v_+ - v_+ \otimes v_-$, v_+ and w_r are highest weight vectors, which is immediate. The weights of $v_- \otimes v_+ - v_+ \otimes v_-$, v_+ and w_r are respectively $(1,1)$, $(1,0)$ and (m,r), the weight of $p_{\mathbf{a}}$ is then $(n-d,d)$. □

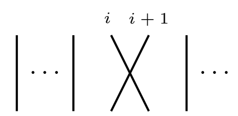
We also notice that $p_{\mathbf{a}} = v_{\alpha(\mathbf{a})} +$higher terms, where $\alpha(\mathbf{a})$ is the Δ-weight obtained by orienting the cups in $\mathbf{a}$ counterclockwise and the rays upward, and $v_{\alpha(\mathbf{a})}$ is obtained by replacing $\vee$ by v_- and $\wedge$ by v_+ and then tensoring by v_0. Here, the higher terms are with respect to $v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_{n-m}} \otimes v_i < v_{\varepsilon'_1} \otimes \cdots \otimes v_{\varepsilon'_{n-m}} \otimes v_{i'}$ if, for the first index such that $\varepsilon_i \neq \varepsilon'_i$, we have $\varepsilon_i < \varepsilon'_i$, with the convention $- < +$. Therefore, the family $\{p_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{B}_{n-d,d,m}\}$ is free.

Lemma 6.16. *The set $\{p_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{B}_{n-d,d,m}\}$ is a basis of the highest weight vectors of weight $(n-d,d)$ in $V^{\otimes(n-m)} \otimes S^m V$.*

Proof. Since the family is linearly independent, it remains to show that we have the correct numbers of such vectors. But, using the branching rules for $\mathfrak{gl}_2$-modules, we see that the number of summands of $V^{\otimes(n-m)} \otimes S^m V$ isomorphic to $V_{(n-d,d)}$ is the number of standard tableaux of skew shape $(n-d,d)/(m)$, which is also the cardinality of $\mathbb{B}_{n-d,d,m}$. □

As a corollary, we have a basis of the H_{n-m}-module L_d, and we now compute the action of the generators of the degenerate affine Hecke algebra to compare with the representation obtained on the cohomology of the Δ-Springer variety.

Proposition 6.17. *Let* $\mathbf{a} \in \mathbb{B}_{n-d,d,m}$. *The action of* s_i *on* $p_{\mathbf{a}}$ *is obtained by stacking the diagram*



on top of $\mathbf{a}$ *and using the same relations as in Proposition 5.8.*

Proof. This follows from standard calculations. For example, the vector w_r is clearly invariant by permutation of any two consecutive tensorands among the first r, which corresponds to the last point of Proposition 5.8. □

It remains to compute the action of the polynomial generator of the degenerate affine Hecke algebra. We compute the action of x_{n-m} by distinguishing three cases.

Proposition 6.18. *Let* $\mathbf{a} \in \mathbb{B}_{n-d,d,m}$. *We have:*

$$x_{n-m} \cdot p_{\mathbf{a}} = \begin{cases} 0 & \text{if } n-m \text{ is the endpoint of a ray,} \\ (m+1)p_{\mathbf{a}} & \text{if } n-m \text{ is the left endpoint of a cup,} \\ -\sum_{i=0}^{r}(m-r+i)p_{\mathbf{a}^i} & \text{if } n-m \text{ is the right endpoint of a cup,} \end{cases}$$

where, in the third case, the integer r *and the diagrams* $\mathbf{a}^i$ *are defined similarly as in the previous section.*

Proof. We first suppose that $n-m$ is the endpoint of a ray in $\mathbf{a}$. Therefore, there are only rays right of the cut line, and $p_{\mathbf{a}} = v \otimes v_+ \otimes v_0$ for some $v \in V^{\otimes(n-m-1)}$. Since $\Omega \cdot (v_+ \otimes v_0) = mv_+ \otimes v_0$, we find that $x_{n-m} \cdot p_{\mathbf{a}} = 0$.

Now, we suppose that $n-m$ is the left endpoint of a cup. The corresponding right endpoint is then right of the cut line. Using the action of the symmetric group, it then suffices to compute the action of $\Omega_{r,r+1}$ on the vector w_r. Given $I \subset \{1,\ldots,r\}$, we have

$$\Omega_{r,r+1} \cdot (v_I \otimes v_{|I|}) = \begin{cases} (m-|I|)v_I \otimes v_{|I|} + |I|v_{I\setminus\{n-m\}} \otimes v_{|I|-1} & \text{if } n-m \in I, \\ |I|v_I \otimes v_{|I|} + (m-|I|)v_{I\cup\{n-m\}} \otimes v_{|I|+1} & \text{if } n-m \notin I. \end{cases}$$

This implies that $\Omega_{r,r+1} \cdot w_r = -w_r$ and that $x_{n-m} \cdot p_{\mathbf{a}} = (m+1)p_{\mathbf{a}}$.

Finally, suppose that $n-m$ is the right endpoint of a cup. Once again, using the action of the symmetric group, it suffices to compute the action of $\Omega_{r+2,r+3}$ on the vector

$$\sum_{I\subset\{1,\ldots,r\}} (-1)^{|I|} v_I \otimes (v_- \otimes v_+ - v_+ \otimes v_-) \otimes v_{|I|}.$$

For $1 \leqslant i \leqslant r$, denote by $w_{r,i,i+1}$, the vector in $V^{\otimes(r+2)} \otimes S^m V$ given by inserting into w_r the vector $v_- \otimes v_+ - v_+ \otimes v_i$ at the tensorands i and $i+1$. We have

$$\Omega_{r+2,r+3} \cdot w_{r,r+1,r+2} = \sum_{I\subset\{1,\ldots,r\}} (-1)^{|I|} v_I \otimes \big(v_- \otimes ((m-|I|)v_+ \otimes v_{|I|} + |I|v_- \otimes v_{|I|-1}) - v_+ \otimes (|I|v_- \otimes v_{|I|} + (m-|I|)v_+ \otimes v_{|I|+1})\big).$$

We then find that $\Omega_{r+2,r+3} \cdot w_{r,r+1,r+2} + w_{r,r+1,r+2}$ is a linear combination of terms of the form $v_J \otimes v_{|J|-1}$, with $J \subset \{1,\ldots,r+2\}$, with coefficients $(-1)^{|J|-1}(m-|J|+2)$

if $r+2 \in J$ and $(-1)^{|J|}|J|$ if $r+2 \notin J$. We fix $J \subset \{1,\dots,r+2\}$ and compute the coefficient of $v_J \otimes v_{|J|}$ in the sum

$$\sum_{i=1}^{r+1}(m-r+i)w_{r,i,i+1} + (m-r)v_+ \otimes w_{r+1}. \tag{9}$$

We will distinguish four cases, whether 1 and $r+2$ are in J or not. We will treat only the case $1, r+2 \in J$, the other cases being similar. The set J correspond to a word $+^{\alpha_1} -^{\beta_1} \cdots -^{\beta_k} +^{\alpha_{k+1}}$, with $\alpha_i, \beta_i > 0$. Then, for each subword $-+$ in the position $i, i+1$, $v_J \otimes v_{|J|-1}$ appears in $w_{r,i,i+1}$ with a coefficient $(-1)^{|J|-1}$ and for each subword $+-$ in the position $i, i+1$, $v_J \otimes v_{|J|-1}$ appears in $w_{r,i,i+1}$ with a coefficient $-(-1)^{|J|-1}$. Therefore, the total coefficient of $v_J \otimes v_{|J|-1}$ in the sum (9) is

$$(-1)^{|J|-1}\left(\sum_{j=1}^{k}(m-r+\sum_{s=1}^{j}(\alpha_s+\beta_s)) - \sum_{j=1}^{k}(m-r+\sum_{s=1}^{j-1}(\alpha_s+\beta_s)+\alpha_j) + (m-r)\right)$$
$$= (-1)^{|J|-1}(m-r+\sum_{j=1}^{k}\beta_k).$$

But $\sum_{j=1}^{k}\beta_k = (r+2)-|J|$ and we find that the coefficient is $(-1)^{|J|-1}(m-|J|+2)$ as expected. We finally obtain that

$$\begin{aligned}\Omega_{r+2,r+3}\cdot w_{r,r+1,r+2} &= -w_{r,r+1,r+2} + \sum_{i=1}^{r+1}(m-r+i)w_{r,i,i+1} + (m-r)v_+ \otimes w_{r+1} \\ &= m w_{r,r+1,r+2} + \sum_{i=1}^{r}(m-r+i)w_{r,i,i+1} + (m-r)v_+ \otimes w_{r+1},\end{aligned}$$

which concludes the proof. ☐

Theorem 6.19. *The H_{n-m}-module $H_{2d(\mathcal{S}^{\Delta}_{(n-k,k),m})}$ is irreducible.*

Proof. The map $L_\alpha \mapsto p_{\mathbf{a}(\alpha)}$ is an isomorphism of vector spaces since it sends a basis of $H_{2d(\mathcal{S}^{\Delta}_{(n-k,k),m})}$ onto a basis of L_d. Furthermore, the comparison of the action of H_{n-m} on both sides shows that this isomorphism is H_{n-m}-equivariant, see Lemma 6.5, Lemma 6.7 and Lemma 6.9 for one side, and Proposition 6.18 for the other. Since L_d is irreducible, so is $H_{2d}(\mathcal{B}^{\Delta}_{(n-k,k),m})$. ☐

6.4. Action in the extremal cases.

6.4.1. *Springer fiber of type A.* In this section, we assume that $m=0$. In this extremal case, the Δ-Springer variety is isomorphic to the Springer fiber associated with the two-row partition $(n-k,k)$. There is a well-known action of the symmetric group S_n on the (co)homology of the Springer fiber, which has a skein theoretic interpretation [31]. We recover this action if we restrict the action of the degenerate affine Hecke algebra H_n to the symmetric group S_n.

Proposition 6.20. *Suppose that $m=0$. Then the generator x_n acts by 0.*

Proof. Since $m=0$, there are no points on the right of the cut line, and n cannot be the left endpoint of a cup. The result then follows from Lemma 6.5 and Lemma 6.9. ☐

The action of the generator x_i is then given by the Jucys–Murphy element $\sum_{j=i+1}^{n}(i\ j)$. The action of H_n is entirely recovered from the action of the (group algebra of the) symmetric group S_n.

6.4.2. *Exotic Springer fiber.* In the extremal case $m = k$, the Δ-Springer variety is isomorphic to the exotic Springer fiber associated with the one-row bipartition $(n-2k, k)$. There is a well-known action of the Weyl group W_{n-k} of type C_{n-k} on the (co)homology of the exotic Springer fiber, which can be easily described [33].

Recall that in Subsection 6.1 we have constructed an action of the degenerate affine Hecke algebra H_{n-m} on the homology of the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$. We now describe this action using W_{n-m}, which we consider as the group of signed permutations on the set $\{-(n-m), \dots, -1, 1, \dots, n-m\}$.

For $1 \leqslant i < n-m$, we denote by s_i the permutation $(i\ i+1)(-i\ \ -(i+1))$ and by s_i' the permutation $(i\ -i)$. The group W_{n-m} acts on $H_{2d}((\mathbb{S}^2)^{n-m})$: the element s_i acts by usual permutations and s_i' acts on a line diagram $l_U \in H_{2d}((\mathbb{S}^2)^{n-m})$ by

$$s_i' \cdot l_U = \begin{cases} l_U & \text{if } i \notin U, \\ -l_U & \text{if } i \in U. \end{cases}$$

For $1 \leqslant i \leqslant n-m$, we define the Jucys–Murphy elements J_i and $\tilde{J}_i$ by

$$J_i = \sum_{j=1}^{i-1} (j\ \ i) \quad \text{and} \quad \tilde{J}_i = \sum_{j=i+1}^{n-m} (i\ \ j).$$

Proposition 6.21. *The action of x_i on $H_{2d}((\mathbb{S}^2)^{n-m})$ is equal to the action of*

$$\frac{1-s_i'}{2}(a_d - n + m + 1 + J_i) - \frac{1+s_i'}{2}\tilde{J}_i.$$

Proof. Let $U \subset \{1, \dots, n-m\}$. We have

$$J_i \cdot l_U = \begin{cases} |\{1 \leqslant j < i \mid j \in U\}| l_U + \displaystyle\sum_{\substack{j=1 \\ j \notin U}}^{i-1} l_{U \smallsetminus \{i\} \cup \{j\}} & \text{if } i \in U, \\ |\{1 \leqslant j < i \mid j \notin U\}| l_U + \displaystyle\sum_{\substack{j=1 \\ j \in U}}^{i-1} l_{U \smallsetminus \{j\} \cup \{i\}} & \text{if } i \notin U, \end{cases}$$

and

$$\tilde{J}_i \cdot l_U = \begin{cases} |\{i < j \leqslant n-m \mid j \in U\}| l_U + \displaystyle\sum_{\substack{j=i+1 \\ j \notin U}}^{n-m} l_{U \smallsetminus \{i\} \cup \{j\}} & \text{if } i \in U, \\ |\{i < j \leqslant n-m \mid j \notin U\}| l_U + \displaystyle\sum_{\substack{j=i+1 \\ j \in U}}^{n-m} l_{U \smallsetminus \{j\} \cup \{i\}} & \text{if } i \notin U. \end{cases}$$

The proof is complete once we notice that $\frac{1-s_i'}{2} \cdot l_U = \delta_{i \in U} l_U$ and $\frac{1+s_i'}{2} \cdot l_U = \delta_{i \notin U} l_U$. □

Nonetheless, the homology of the Δ-Springer variety $\mathcal{B}^{\Delta}_{(n-k,k),m}$ is not stable by the action of W_{n-m}, but only under the action of the degenerate affine Hecke algebra. In the extremal case $m = k$, the homology of the Δ-Springer variety is equal to the homology of $(\mathbb{S}^2)^{n-k}$ in degree $0, 2, \dots, 2k$ and is $\{0\}$ in all other degrees. Therefore, the description of the action of W_{n-k} given in [33] is equivalent to the action of the degenerate affine Hecke algebra H_{n-k}: since both actions are irreducible, the Jacobson density theorem implies that the two actions generate $\mathrm{End}(H_{2d}((\mathbb{S}^2)^{n-k}))$.

Acknowledgements. The authors would like to thank Sean T. Griffin for helpful conversations and clarifications on [20]. They also want to thank Alexis Langlois-Rémillard, Grégoire Naisse, Loïc Poulain d'Andecy, Simon Riche, Toshiaki Shoji, and Catharina Stroppel for discussions around this work, and the anonymous referees for their detailed comments.

References

[1] Pramod N. Achar and Anthony Henderson, *Orbit closures in the enhanced nilpotent cone*, Adv. Math. **219** (2008), no. 1, 27–62.

[2] Walter Borho and Robert MacPherson, *Partial resolutions of nilpotent varieties*, in Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101-102, Soc. Math. France, Paris, 1983, pp. 23–74.

[3] Jonathan Brundan, *Symmetric functions, parabolic category $\mathcal{O}$, and the Springer fiber*, Duke Math. J. **143** (2008), no. 1, 41–79.

[4] Jonathan Brundan and Catharina Stroppel, *Highest weight categories arising from Khovanov's diagram algebra. II. Koszulity*, Transform. Groups **15** (2010), no. 1, 1–45.

[5] Jonathan Brundan and Catharina Stroppel, *Highest weight categories arising from Khovanov's diagram algebra I: cellularity*, Mosc. Math. J. **11** (2011), no. 4, 685–722, 821–822.

[6] Jonathan Brundan and Catharina Stroppel, *Highest weight categories arising from Khovanov's diagram algebra III: category $\mathcal{O}$*, Represent. Theory **15** (2011), 170–243.

[7] Jonathan Brundan and Catharina Stroppel, *Highest weight categories arising from Khovanov's diagram algebra IV: the general linear supergroup*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 2, 373–419.

[8] Sabin Cautis and Joel Kamnitzer, *Knot homology via derived categories of coherent sheaves. I. The $\mathfrak{sl}(2)$-case*, Duke Math. J. **142** (2008), no. 3, 511–588.

[9] Ivan Cherednik, *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series, vol. 319, Cambridge University Press, Cambridge, 2005.

[10] Anton Cox and Maud De Visscher, *Diagrammatic Kazhdan-Lusztig theory for the (walled) Brauer algebra*, J. Algebra **340** (2011), 151–181.

[11] Anton Cox, Maud De Visscher, Stephen Doty, and Paul Martin, *On the blocks of the walled Brauer algebra*, J. Algebra **320** (2008), no. 1, 169–212.

[12] Michael Ehrig and Catharina Stroppel, *2-row Springer fibres and Khovanov diagram algebras for type D*, Canad. J. Math. **68** (2016), no. 6, 1285–1333.

[13] Michael Ehrig and Catharina Stroppel, *Diagrammatic description for the categories of perverse sheaves on isotropic Grassmannians*, Selecta Math. (N.S.) **22** (2016), no. 3, 1455–1536.

[14] Michael Ehrig and Catharina Stroppel, *Koszul gradings on Brauer algebras*, Int. Math. Res. Not. IMRN (2016), no. 13, 3970–4011.

[15] Michael Ehrig and Catharina Stroppel, *On the category of finite-dimensional representations of $\mathrm{OSp}(r|2n)$: Part I*, in Representation theory—current trends and perspectives, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2017, pp. 109–170.

[16] Michael Ehrig and Catharina Stroppel, *Deligne categories and representations of $\mathrm{OSp}(r|2n)$*, preprint, 2021, `http://www.math.uni-bonn.de/ag/stroppel/ospii.pdf`.

[17] Lucas Fresse and Anna Melnikov, *On the singularity of the irreducible components of a Springer fiber in $\mathfrak{sl}_n$*, Selecta Math. (N.S.) **16** (2010), no. 3, 393–418.

[18] Francis Y. C. Fung, *On the topology of components of some Springer fibers and their relation to Kazhdan-Lusztig theory*, Adv. Math. **178** (2003), no. 2, 244–276.

[19] Maria Gillespie and Sean T. Griffin, *Cocharge and skewing formulas for Δ-Springer modules and the Delta conjecture*, Int. Math. Res. Not. IMRN (2024), no. 14, 10895–10917.

[20] Sean T. Griffin, Jake Levinson, and Alexander Woo, *Springer fibers and the Delta conjecture at $t = 0$*, Adv. Math. **439** (2024), article no. 109491 (53 pages).

[21] James Haglund, Brendon Rhoades, and Mark Shimozono, *Ordered set partitions, generalized coinvariant algebras, and the Delta conjecture*, Adv. Math. **329** (2018), 851–915.

[22] Mee Seong Im, Chun-Ju Lai, and Arik Wilbert, *Irreducible components of two-row Springer fibers for all classical types*, Proc. Amer. Math. Soc. **150** (2022), no. 6, 2415–2432.

[23] Gordon D. James and Michael H. Peel, *Specht series for skew representations of symmetric groups*, J. Algebra **56** (1979), no. 2, 343–364.

[24] Syu Kato, *An exotic Springer correspondence for symplectic groups*, 2006, `https://arxiv.org/abs/math/0607478`.

[25] Mikhail Khovanov, *A functor-valued invariant of tangles*, Algebr. Geom. Topol. **2** (2002), 665–741.

[26] Mikhail Khovanov, *Crossingless matchings and the cohomology of (n,n) Springer varieties*, Commun. Contemp. Math. **6** (2004), no. 4, 561–577.

[27] Abel Lacabanne, Grégoire Naisse, and Pedro Vaz, *Tensor product categorifications, Verma modules and the blob 2-category*, Quantum Topol. **12** (2021), no. 4, 705–812.

[28] Paul Martin and Hubert Saleur, *The blob algebra and the periodic Temperley-Lieb algebra*, Lett. Math. Phys. **30** (1994), no. 3, 189–206.

[29] Vinoth Nandakumar, Daniele Rosso, and Neil Saunders, *Irreducible components of exotic Springer fibres*, J. Lond. Math. Soc. (2) **98** (2018), no. 3, 609–637.

[30] Arun Ram, *Skew shape representations are irreducible*, in Combinatorial and geometric representation theory (Seoul, 2001), Contemp. Math., vol. 325, Amer. Math. Soc., Providence, RI, 2003, pp. 161–189.

[31] Heather M. Russell, *A topological construction for all two-row Springer varieties*, Pacific J. Math. **253** (2011), no. 1, 221–255.

[32] Heather M. Russell and Julianna S. Tymoczko, *Springer representations on the Khovanov Springer varieties*, Math. Proc. Cambridge Philos. Soc. **151** (2011), no. 1, 59–81.

[33] Neil Saunders and Arik Wilbert, *Exotic Springer fibers for orbits corresponding to one-row bipartitions*, Transform. Groups **27** (2022), no. 3, 1111–1147.

[34] Gisa Schäfer, *A graphical calculus for 2-block Spaltenstein varieties*, Glasg. Math. J. **54** (2012), no. 2, 449–477.

[35] Tonny A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173–207.

[36] Tonny A. Springer, *A construction of representations of Weyl groups*, Invent. Math. **44** (1978), no. 3, 279–293.

[37] Richard P. Stanley, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.

[38] Catharina Stroppel, *Parabolic category $\mathcal{O}$, perverse sheaves on Grassmannians, Springer fibres and Khovanov homology*, Compos. Math. **145** (2009), no. 4, 954–992.

[39] Catharina Stroppel and Ben Webster, *2-block Springer fibers: convolution algebras and coherent sheaves*, Comment. Math. Helv. **87** (2012), no. 2, 477–520.

[40] Catharina Stroppel and Arik Wilbert, *Two-block Springer fibers of types C and D: a diagrammatic approach to Springer theory*, Math. Z. **292** (2019), no. 3-4, 1387–1430.

[41] Takeshi Suzuki, *Representations of degenerate affine Hecke algebra and $\mathfrak{gl}_n$*, in Combinatorial methods in representation theory (Kyoto, 1998), Adv. Stud. Pure Math., vol. 28, Kinokuniya, Tokyo, 2000, pp. 343–372.

[42] Stephan M. Wehrli, *A remark on the topology of (n,n) Springer varieties.*, 2009, `https://arxiv.org/abs/0908.2185`.

[43] Arik Wilbert, *Topology of two-row Springer fibers for the even orthogonal and symplectic group*, Trans. Amer. Math. Soc. **370** (2018), no. 4, 2707–2737.

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