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Transition matrices and Pieri-type rules for polysymmetric functions

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Abstract Asvin G and Andrew O'Desky recently introduced the graded algebra $\mathsf{P}\Lambda$ of polysymmetric functions as a generalization of the algebra Λ of symmetric functions. This article develops combinatorial formulas for some multiplication rules and transition matrix entries for $\mathsf{P}\Lambda$ that are analogous to well-known classical formulas for Λ. In more detail, we consider pure tensor bases $\{s_\tau^\otimes\}$, $\{p_\tau^\otimes\}$, and $\{m_\tau^\otimes\}$ for $\mathsf{P}\Lambda$ that arise as tensor products of the classical Schur basis, power-sum basis, and monomial basis for Λ. We find expansions in these bases of the non-pure bases $\{P_\delta\}$, $\{H_\delta\}$, $\{E_\delta^+\}$, and $\{E_\delta\}$ studied by Asvin G and O'Desky. The answers involve tableau-like structures generalizing semistandard tableaux, rim-hook tableaux, and the brick tabloids of Eğecioğlu and Remmel. These objects arise by iteration of new Pieri-type rules that give expansions of products such as $s_\sigma^\otimes H_\delta$, $p_\sigma^\otimes E_\delta$, etc.

1. Introduction

The ring Λ of symmetric functions is an object of great interest in modern algebraic combinatorics. Recently, Asvin G and Andrew O'Desky introduced a generalization $\mathsf{P}\Lambda$ called the ring of polysymmetric functions [4]. Our goal in this paper is to extend some of the rich combinatorial theory for symmetric functions to the new setting of polysymmetric functions. In particular, we develop combinatorial formulas for some multiplication rules and transition matrix entries for $\mathsf{P}\Lambda$ that are analogous to well-known classical formulas for Λ.

1.1. Review of symmetric functions.

We assume the reader has some prior familiarity with symmetric functions; background material may be found in texts such as [5, 7, 9]. We briefly recall some fundamental notation and terminology. An *integer partition of* n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of positive integers with sum n. We call λ_i the ith *part* of λ, and let $\ell(\lambda) = \ell$ be the number of nonzero parts of λ. We write $|\lambda| = n$ or $\text{area}(\lambda) = n$ or $\lambda \vdash n$ to mean that λ is an integer partition of n. We write $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \cdots)$ to indicate that λ is a partition with m_1 parts equal to 1, m_2 parts equal to 2, and so on. We denote the number of times i appears in λ by $m_i(\lambda)$. A *symmetric function over* $\mathbb{Q}$ is a formal power series of bounded degree in countably many variables with coefficients in $\mathbb{Q}$, say $f = f(\mathbf{x}) = f(x_1, x_2, \dots, x_m, \dots)$, that remains unchanged under any permutation of the variables x_i. Letting each variable x_i have degree 1, the set Λ^n of homogeneous

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symmetric functions of degree n is a vector space of dimension $p(n)$, the number of integer partitions of n. The set of all symmetric functions is a graded $\mathbb{Q}$-algebra $\Lambda = \bigoplus_{n \geqslant 0} \Lambda^n$.

Bases of the vector space Λ^n are naturally indexed by integer partitions of n. The *monomial symmetric function* $m_\lambda(\mathbf{x})$ is the formal sum of all distinct monomials obtained by permuting the subscripts in $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_\ell^{\lambda_\ell}$. The *complete symmetric function* $h_k(\mathbf{x})$ is the sum of all monomials $x_{i_1} x_{i_2} \cdots x_{i_k}$ where $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$. The *elementary symmetric function* $e_k(\mathbf{x})$ is the sum of all monomials $x_{i_1} x_{i_2} \cdots x_{i_k}$ where $1 \leqslant i_1 < i_2 < \cdots < i_k$. The *power-sum symmetric function* $p_k(\mathbf{x})$ is $x_1^k + x_2^k + \cdots + x_m^k + \cdots$. For any list of positive integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$, we define

$$h_\alpha(\mathbf{x}) = \prod_{i=1}^{s} h_{\alpha_i}(x), \quad e_\alpha(\mathbf{x}) = \prod_{i=1}^{s} e_{\alpha_i}(x), \quad p_\alpha(\mathbf{x}) = \prod_{i=1}^{s} p_{\alpha_i}(x).$$

The *Schur symmetric function* $s_\lambda(\mathbf{x})$ can be defined as $s_\lambda(\mathbf{x}) = \sum_\mu K_{\lambda,\mu} m_\mu(\mathbf{x})$, where the *Kostka number* $K_{\lambda,\mu}$ is the number of semistandard Young tableaux of shape λ and content μ. It is known that each of the sets $\{m_\lambda : \lambda \vdash n\}$, $\{h_\lambda : \lambda \vdash n\}$, $\{e_\lambda : \lambda \vdash n\}$, $\{p_\lambda : \lambda \vdash n\}$, and $\{s_\lambda : \lambda \vdash n\}$ is a basis of Λ^n. It follows that each of the sets $\{h_k : k \in \mathbb{Z}_{>0}\}$, $\{e_k : k \in \mathbb{Z}_{>0}\}$, and $\{p_k : k \in \mathbb{Z}_{>0}\}$ is algebraically independent over $\mathbb{Q}$. This leads to an abstract description of Λ as a polynomial ring $\Lambda = \mathbb{Q}[h_k : k > 0]$ in formal indeterminates h_k where $\deg(h_k) = k$. Similarly, we can think of Λ as a polynomial ring in the e_k or the p_k, where $\deg(e_k) = k = \deg(p_k)$.

Transition matrices between bases of Λ^n often exhibit interesting combinatorics [1, 3]. Given indexed bases $\{f_\lambda : \lambda \vdash n\}$ and $\{g_\lambda : \lambda \vdash n\}$ of Λ^n, the *transition matrix* $\mathcal{M}(f, g)$ is the unique matrix (with rows and columns indexed by partitions of n) such that

$$f_\mu = \sum_{\lambda \vdash n} \mathcal{M}(f, g)_{\lambda,\mu} g_\lambda. \tag{1}$$

For example, the definition of Schur functions (given above) states that $\mathcal{M}(s, m)_{\lambda,\mu}$ is the Kostka number $K_{\mu,\lambda}$. It is known that $\mathcal{M}(h, s)_{\lambda,\mu} = K_{\lambda,\mu}$, so that $\mathcal{M}(s, m)$ is the transpose of $\mathcal{M}(h, s)$. It is routine to check that matrix inversion switches the roles of the input basis and the output basis: $\mathcal{M}(g, f) = \mathcal{M}(f, g)^{-1}$. If $\{k_\lambda\}$ is another basis of Λ^n, then $\mathcal{M}(f, k)$ is the matrix product $\mathcal{M}(g, k)\mathcal{M}(f, g)$.

1.2. Polysymmetric functions. For each positive integer d, let $\Lambda_{(d)}$ be a copy of the ring Λ of symmetric functions where all degrees are multiplied by d. The $\mathbb{Q}$-algebra of *polysymmetric functions* may be defined abstractly as the tensor product

$$\mathsf{P}\Lambda = \Lambda_{(1)} \otimes \Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(d)} \otimes \cdots .$$

To get a more concrete description, we view $\Lambda_{(d)}$ as the ring of symmetric functions in a variable set $\mathbf{x}_{d*} = \{x_{d,1}, x_{d,2}, \ldots\}$, where $\deg(x_{d,i}) = d$ for all $i \geqslant 1$. Then $\mathsf{P}\Lambda$ appears as a particular subalgebra of the $\mathbb{Q}$-algebra $\mathbb{Q}[[\mathbf{x}_{**}]]$ of formal series of bounded degree in all the variables $x_{d,i}$ for $d, i \in \mathbb{Z}_{>0}$. A formal series $f = f(\mathbf{x}_{**})$ belongs to $\mathsf{P}\Lambda$ iff for each fixed d, f is unchanged by any permutation of the variables in $\mathbf{x}_{d*}$. An isomorphism between the abstract and concrete versions of $\mathsf{P}\Lambda$ is defined by sending the pure tensor $f_1 \otimes f_2 \otimes f_3 \otimes \cdots$ to the formal series $f_1(\mathbf{x}_{1*}) f_2(\mathbf{x}_{2*}) f_3(\mathbf{x}_{3*}) \cdots$. Like Λ, $\mathsf{P}\Lambda$ is a graded algebra: $\mathsf{P}\Lambda = \bigoplus_{n \geqslant 0} \mathsf{P}\Lambda^n$, where $\mathsf{P}\Lambda^n$ is the vector space of homogeneous polysymmetric functions of degree n.

Remark 1.1. Although Λ may be viewed algebraically as a polynomial ring $\mathbb{Q}[e_1, \ldots, e_k, \ldots]$ where $\deg(e_k) = k$, the combinatorics arising from bases of Λ^n is much richer than the combinatorics of an ordinary polynomial ring $\mathbb{Q}[x_1, \ldots, x_k, \ldots]$

where each x_k has degree 1. Similarly, the combinatorial features of $\mathsf{P\Lambda}$ are quite different from those of the regular tensor product of copies of Λ with no rescaling of degrees in each factor.

Remark 1.2. Polysymmetric functions have interesting applications to algebraic geometry and representation theory. In [4], Asvin G and O'Desky study motivic measures, configuration spaces, and a generalization of the Grothendieck ring of varieties. Polysymmetric functions emerge naturally from the plethystic exponential which plays a key role in this generalization. As mentioned in [4], $\mathsf{P\Lambda}$ is also related to the representation theory of the disconnected reductive wreath products $\mathrm{SL}_2 \wr S_n$. See Remark 1.5 for another connection between $\mathsf{P\Lambda}$ and representation theory.

Bases for $\mathsf{P\Lambda}^n$ are naturally indexed by *(splitting) types*, which we discuss next. A *block* is an ordered pair of positive integers (d, m), which we usually write as d^m. We say d^m has *degree* d, *multiplicity* m, and *weight* dm. We order blocks by writing $a^b \geqslant d^e$ to mean either $a > d$, or $a = d$ and $b \geqslant e$. A *type* of weight n is a weakly decreasing sequence of blocks $\tau = (d_1^{m_1}, d_2^{m_2}, \dots, d_s^{m_s})$ such that $d_1m_1 + d_2m_2 + \cdots + d_sm_s = n$. We write $|\tau| = n$ or $\tau \Vdash n$ to mean that τ is a type of weight n. We call s the *length of* τ and write $s = \ell(\tau)$. For fixed d, let $\tau|_d$ (sometimes abbreviated as τ_d) be the partition formed by taking the multiplicities of the blocks of τ of degree d. For example, $\tau = (3^4 3^4 3^2 2^3 2^2 2^1 2^1 1^4 1^3 1^3 1^1)$ is a type of weight 55 with $\tau|_3 = (4, 4, 2)$, $\tau|_2 = (3, 2, 1, 1)$, and $\tau|_1 = (4, 3, 3, 1)$. We may abbreviate any type τ by writing $\tau = (1^{\tau|_1} 2^{\tau|_2} 3^{\tau|_3} \cdots)$. The *sign* of type τ is $\mathrm{sgn}(\tau) = \prod_{i=1}^{k} (-1)^{m_i}$. The power of -1 in $\mathrm{sgn}(\tau)$ is $\sum_{i=1}^{k} m_i = \sum_{i=1}^{k} \mathrm{area}(\tau|_i)$.

Remark 1.3. Types of weight n encode the possible ways a polynomial $p(x) \in \mathbb{Q}[x]$ of degree n can split into irreducible factors. For example, $p = (x^2+1)^3(x^2-2)^3(x^2-3)(x-1)^2(x-2)^2$ has associated type $\tau = (2^3 2^3 2^1 1^2 1^2)$.

Suppose $\{f_\lambda\}$ is any fixed basis for Λ, where λ ranges over integer partitions, and $f_\lambda \in \Lambda^n$ whenever $\lambda \vdash n$. By the general theory of tensor products, it follows that the set of tensor products $f_{\lambda_{(1)}} \otimes f_{\lambda_{(2)}} \otimes f_{\lambda_{(3)}} \otimes \cdots$, where all but finitely many $f_{\lambda_{(d)}}$ are equal to 1, is a basis for the vector space $\mathsf{P\Lambda}$. We can identify the list $(\lambda_{(1)}, \lambda_{(2)}, \dots)$ with the type $\tau = (1^{\lambda_{(1)}} 2^{\lambda_{(2)}} \cdots)$. Define

$$f_\tau^{\otimes} = f_{\tau|_1} \otimes f_{\tau|_2} \otimes \cdots = \prod_{d \geqslant 1} f_{\tau|_d}(\mathbf{x}_{d*}).$$

This is a homogeneous element of $\mathsf{P\Lambda}$ of degree $\sum_{d\geqslant 1} d\,\mathrm{area}(\tau|_d) = |\tau|$. Letting τ range over all types, we get a basis $\{f_\tau^{\otimes}\}$ of $\mathsf{P\Lambda}$. For each $n \geqslant 0$, $\{f_\tau^{\otimes} : \tau \Vdash n\}$ is a basis of $\mathsf{P\Lambda}^n$. We call these bases of $\mathsf{P\Lambda}$ and $\mathsf{P\Lambda}^n$ the *pure tensor bases* associated with the given basis $\{f_\lambda\}$ of Λ.

Example 1.4. Given $\tau = (4^4 4^3 4^1 2^2 2^2 2^1 2^1 1^3 1^3 1^1 1^1) = (1^{3,3,1,1} 2^{2,2,1,1} 4^{4,3,1})$,

$$m_\tau^{\otimes} = m_{3311} \otimes m_{2211} \otimes 1 \otimes m_{431} \otimes 1 \otimes 1 \otimes \cdots = m_{3311}(\mathbf{x}_{1*}) m_{2211}(\mathbf{x}_{2*}) m_{431}(\mathbf{x}_{4*}).$$

Hereafter, we often omit trailing 1s in the tensor product presentation of a polysymmetric function.

Remark 1.5. Polysymmetric functions also appear in the recent work [8] in connection with the representation theory of the uniform block permutation (UBP) algebras. In the classical setting, the Frobenius characteristic map sends the irreducible character of the symmetric group indexed by a partition λ to the Schur symmetric function s_λ. In the setting of UBP algebras, the analogue of the Frobenius map sends an irreducible character of $\mathcal{U}_k$ to a tensor of the form $s_{\tau|_1}[h_1] \otimes s_{\tau|_2}[h_2] \otimes \cdots \otimes s_{\tau|_d}[h_d] \otimes \cdots$ inside a tensor product of copies of Λ, where each copy has its usual (unscaled) grading. For each $k > 1$, the map sending $f \in \Lambda$ to $f[h_k]$ is an injective algebra homomorphism that has the effect of rescaling the grading in the kth tensor factor by k. Thus, the image of the UBP Frobenius map is really another presentation of $\mathsf{P}\Lambda$. In our notation, the irreducible characters of $\mathcal{U}_k$ correspond to the elements $s^{\otimes}_{\tau}$ of the pure tensor Schur basis. This provides an extra motivation for finding $s^{\otimes}_{\tau}$-expansions of various polysymmetric functions. In particular, understanding the change-of-basis formulas between bases arising in algebraic geometry and the $s^{\otimes}$-basis can tell us whether those bases correspond to a representation of a UBP algebra.

1.3. The bases H, E^+, E, and P. The authors of [4] introduced four bases of $\mathsf{P}\Lambda$, denoted by $\{H_\tau\}$, $\{E^+_\tau\}$, $\{E_\tau\}$, and $\{P_\tau\}$, that are not pure tensor bases. These are polysymmetric analogues of the symmetric functions h_μ, e_μ, and p_μ, defined as follows. Order the subscripts of variables in $\mathbf{x}_{**}$ lexicographically: $(i,j) \leqslant (k,\ell)$ means $i < k$, or $i = k$ and $j \leqslant \ell$. For each positive integer d, define

$$H_d = \sum_{\substack{(i_1,j_1)\leqslant(i_2,j_2)\leqslant\cdots\leqslant(i_s,j_s)\\ i_1+i_2+\cdots+i_s=d}} x_{i_1,j_1}x_{i_2,j_2}\cdots x_{i_s,j_s}, \tag{2}$$

which is the sum of all distinct monomials of degree d. Define

$$E^+_d = \sum_{\substack{(i_1,j_1)<(i_2,j_2)<\cdots<(i_s,j_s)\\ i_1+i_2+\cdots+i_s=d}} x_{i_1,j_1}x_{i_2,j_2}\cdots x_{i_s,j_s}, \tag{3}$$

which is the sum of monomials of degree d where no variable x_{ij} appears more than once within any given monomial. Such monomials are called *square-free*. Define

$$E_d = \sum_{\substack{(i_1,j_1)<(i_2,j_2)<\cdots<(i_s,j_s)\\ i_1+i_2+\cdots+i_s=d}} (-1)^s x_{i_1,j_1}x_{i_2,j_2}\cdots x_{i_s,j_s}, \tag{4}$$

which is a signed variation of E^+_d. Define

$$P_d = \sum_{k|d} k \sum_{j\geqslant 1} x_{k,j}^{d/k}, \tag{5}$$

where "$\sum_{k|d}$" indicates a sum over positive divisors k of d. It is routine to check that H_d, E^+_d, E_d, and P_d all belong to $\mathsf{P}\Lambda^d$.

For any block d^m, define $H_{d^m} = H_d(\mathbf{x}^m_{**})$, which means that every variable x_{ij} appearing in every monomial of H_d gets replaced by x^m_{ij}. Similarly, define $E^+_{d^m} = E^+_d(\mathbf{x}^m_{**})$, $E_{d^m} = E_d(\mathbf{x}^m_{**})$, and $P_{d^m} = P_d(\mathbf{x}^m_{**})$. These objects are all in $\mathsf{P}\Lambda^{dm}$. Finally, for any ordered sequence of blocks $\delta = (d_1^{m_1}, d_2^{m_2}, \ldots, d_t^{m_t})$, define

$$H_\delta = \prod_{i=1}^t H_{d_i^{m_i}},\ E^+_\delta = \prod_{i=1}^t E^+_{d_i^{m_i}},\ E_\delta = \prod_{i=1}^t E_{d_i^{m_i}},\ P_\delta = \prod_{i=1}^t P_{d_i^{m_i}}.$$

In particular, this defines H_τ (etc.) when τ is a type. It is shown in [4] that each of the sets $\{H_\tau : \tau \Vdash n\}$, $\{E^+_\tau : \tau \Vdash n\}$, $\{E_\tau : \tau \Vdash n\}$, and $\{P_\tau : \tau \Vdash n\}$ is a linear basis of $\mathsf{P}\Lambda^n$. As in the case of Λ, this leads to an alternate algebraic characterization of $\mathsf{P}\Lambda$ as an abstract polynomial ring. Starting with formal indeterminates H_{d^m} for each block d^m, we can think of $\mathsf{P}\Lambda$ as $\mathbb{Q}[H_{d^m} : d, m > 0]$, where H_{d^m} has degree dm. Similarly, $\mathsf{P}\Lambda = \mathbb{Q}[E^+_{d^m} : d, m > 0] = \mathbb{Q}[E_{d^m} : d, m > 0] = \mathbb{Q}[P_{d^m} : d, m > 0]$.

1.4. Transition matrices for $\mathsf{P}\Lambda$. Our main goal in this paper is to develop the combinatorics of certain transition matrices between bases of $\mathsf{P}\Lambda^n$. We use notation analogous to the symmetric case. Given bases $\{F_\tau : \tau \Vdash n\}$ and $\{G_\tau : \tau \Vdash n\}$ of $\mathsf{P}\Lambda^n$, the *transition matrix* $\mathcal{M}(F,G)$ is the unique matrix (with rows and columns indexed by types of weight n) such that

$$F_\sigma = \sum_{\tau \Vdash n} \mathcal{M}(F,G)_{\tau,\sigma} G_\tau. \tag{6}$$

In the special case of pure tensor bases, we can immediately find transition matrices for $\mathsf{P}\Lambda$ if we know the corresponding transition matrices for Λ.

Proposition 1.6. *Let $\{f_\lambda\}$ and $\{g_\lambda\}$ be bases of Λ such that $f_\lambda, g_\lambda \in \Lambda^n$ whenever $\lambda \vdash n$. Let $F_\tau = f_\tau^\otimes$ and $G_\tau = g_\tau^\otimes$ be the corresponding pure tensor bases. For all types σ, τ,*

$$\mathcal{M}(F,G)_{\tau,\sigma} = \prod_{d \geqslant 1} \mathcal{M}(f,g)_{\tau|_d, \sigma|_d}.$$

Proof. For $\sigma \Vdash n$, we compute

$$\begin{aligned} F_\sigma = f_\sigma^\otimes &= \prod_{d\geqslant 1} f_{\sigma|_d}(\mathbf{x}_{d*}) = \prod_{d\geqslant 1} \sum_{\lambda_{(d)} \vdash \mathrm{area}(\sigma|_d)} \mathcal{M}(f,g)_{\lambda_{(d)},\sigma|_d} g_{\lambda_{(d)}}(\mathbf{x}_{d*}) \\ &= \sum_{\lambda_{(1)} \vdash \mathrm{area}(\sigma|_1)} \cdots \sum_{\lambda_{(d)} \vdash \mathrm{area}(\sigma|_d)} \cdots \prod_{d\geqslant 1} \mathcal{M}(f,g)_{\lambda_{(d)},\sigma|_d} g_{\lambda_{(d)}}(\mathbf{x}_{d*}) \\ &= \sum_{\tau \Vdash n} \left(\prod_{d \geqslant 1} \mathcal{M}(f,g)_{\tau|_d,\sigma|_d} \right) g_\tau^\otimes. \end{aligned}$$

where in the last step we set $\tau = (1^{\lambda_{(1)}} 2^{\lambda_{(2)}} \cdots d^{\lambda_{(d)}} \cdots)$. So the coefficient of G_τ in F_σ is $\prod_{d\geqslant 1} \mathcal{M}(f,g)_{\tau|_d,\sigma|_d}$, as needed. ☐

1.5. Main results. Transition matrices involving the bases H, E, E^+, and P are more subtle. In this paper, we find formulas for entries in the following transition matrices:

- $\mathcal{M}(P, s^\otimes)$, $\mathcal{M}(H, s^\otimes)$, $\mathcal{M}(E^+, s^\otimes)$, $\mathcal{M}(E, s^\otimes)$ (Section 2).
- $\mathcal{M}(P, p^\otimes)$, $\mathcal{M}(H, p^\otimes)$, $\mathcal{M}(E^+, p^\otimes)$, $\mathcal{M}(E, p^\otimes)$ (Section 3).
- $\mathcal{M}(P, m^\otimes)$, $\mathcal{M}(H, m^\otimes)$, $\mathcal{M}(E^+, m^\otimes)$, $\mathcal{M}(E, m^\otimes)$ (Section 4).

Our $s^\otimes$-expansions involve tableau-like structures that arise by iteration of certain rules analogous to the Pieri rules (giving the Schur expansions of $s_\mu h_k$ and $s_\mu e_k$) and the Murnaghan–Nakayama rule (giving the Schur expansion of $s_\mu p_k$). Letting $\delta = (d_1^{m_1}, d_2^{m_2}, \ldots, d_t^{m_t})$ be any ordered sequence of blocks, we prove Pieri-type rules for the $s^\otimes$-expansions of $s_\sigma^\otimes P_\delta$, $s_\sigma^\otimes H_\delta$, $s_\sigma^\otimes E_\delta^+$, and $s_\sigma^\otimes E_\delta$. Our $p^\otimes$-expansions have a more algebraic flavor and reveal some identities for $\mathsf{P}\Lambda$ analogous to corresponding power-sum identities for Λ. Our $m^\otimes$-expansions complement some comparable results in [4]. We give combinatorial descriptions of transition matrix entries using objects generalizing the brick tabloids studied by Eğecioğlu and Remmel [3]. We also prove Pieri-like rules for the $p^\otimes$-expansions of $p_\sigma^\otimes F_\delta$ and the $m^\otimes$-expansions of $m_\sigma^\otimes F_\delta$ where F is P, H, E^+, or E.

2. Expansions in the $s^\otimes$ basis

Recall that $\{s_\lambda\}$ is the Schur basis of Λ, and $\{s_\tau^\otimes\}$ is the associated pure tensor basis of $\mathsf{P}\Lambda$. This section provides combinatorial formulas for the coefficients in the $s^\otimes$-expansions of $s_\sigma^\otimes F$ where F is P_{d^m}, H_{d^m}, $E_{d^m}^+$, E_{d^m}, or any product of such factors.

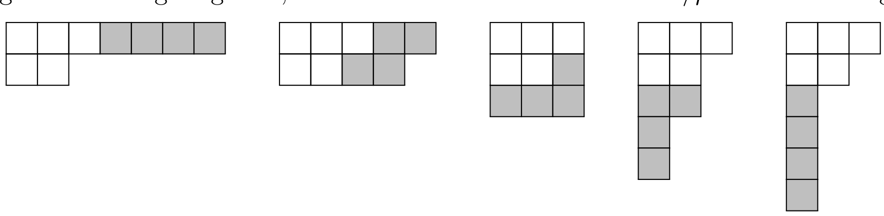
As special cases, we find the transition matrices $\mathcal{M}(P, s^{\otimes})$, $\mathcal{M}(H, s^{\otimes})$, $\mathcal{M}(E^{+}, s^{\otimes})$, and $\mathcal{M}(E, s^{\otimes})$.

2.1. Rule for $s^{\otimes}_{\sigma} P_{d^m}$. Before stating the rule for the $s^{\otimes}$-expansion of $s^{\otimes}_{\sigma} P_{d^m}$, we review the analogous classical rule for the Schur expansion of $s_\mu p_k$. Given an integer partition $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_s)$, the *diagram of* μ is the set $\mathrm{dg}(\mu) = \{(i, j) \in \mathbb{Z}^2 : 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant \mu_i\}$. We visualize the diagram of μ by drawing s rows of left-justified unit boxes with μ_i boxes in the ith row from the top. The *conjugate partition* μ' is the partition whose diagram is obtained from $\mathrm{dg}(\mu)$ by interchanging rows and columns. Given μ and another integer partition ν such that $\mathrm{dg}(\mu) \subseteq \mathrm{dg}(\nu)$, the *skew shape* ν/μ is the set difference $\mathrm{dg}(\nu) \setminus \mathrm{dg}(\mu)$. We visualize a skew shape as the collection of boxes in the diagram for ν that are outside the diagram for μ. A skew shape ν/μ is a *k-ribbon* (or a *k-rim-hook* or a *k-border strip*) if it consists of k boxes that can be labeled $b_1, \ldots, b_k$ so that, for $1 < i \leqslant k$, b_i is one unit left of b_{i-1} or one unit below b_{i-1}. Equivalently, this means that ν/μ is a connected strip of k boxes on the southeast border of $\mathrm{dg}(\nu)$ that contains no 2×2 square. The *sign* of a k-ribbon ν/μ that has boxes in r different rows is $\mathrm{sgn}(\nu/\mu) = (-1)^{r-1}$. The next result is often called the Murnaghan–Nakayama Rule, the Pieri Rule for Power-Sums, or the Slinky Rule.

Proposition 2.1 ([5, Theorem 10.46]). *For any integer partition μ and positive integer k,*

$$s_\mu p_k = \sum_{\nu:\ \nu/\mu \text{ is a } k\text{-ribbon}} \mathrm{sgn}(\nu/\mu) s_\nu.$$

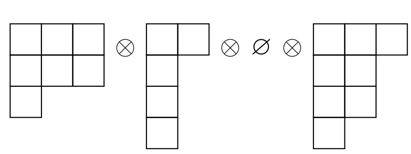
Example 2.2. We compute $s_{(3,2)} p_4 = s_{(7,2)} - s_{(5,4)} - s_{(3,3,3)} + s_{(3,2,2,1,1)} - s_{(3,2,1,1,1,1)}$ using the following diagrams, where the boxes in the 4-ribbon ν/μ are shaded in gray.



Turning to the polysymmetric case, let $\sigma = (1^{\sigma|_1} 2^{\sigma|_2} \cdots i^{\sigma|_i} \cdots)$ be a fixed type. The *tensor diagram of* σ is the formal symbol

$$\mathrm{dg}(\sigma) = \mathrm{dg}(\sigma|_1) \otimes \mathrm{dg}(\sigma|_2) \otimes \cdots \otimes \mathrm{dg}(\sigma|_i) \otimes \cdots.$$

We draw $\mathrm{dg}(\sigma)$ as a succession of partition diagrams joined by tensor signs; we draw $\varnothing$ in any position i where $\sigma|_i$ is the empty partition. For example, the diagram of $\sigma = (1^{3,3,1} 2^{2,1,1,1} 4^{3,2,2,1})$ is



The next theorem computes $s^{\otimes}_{\sigma} P_{d^m}$ by adding certain signed weighted ribbons to $\mathrm{dg}(\sigma)$ according to particular rules. If R is a ribbon added to the shape in position i of the tensor diagram, we let $\mathrm{wt}(R) = i$.

Theorem 2.3. *For any type σ and block d^m,*

$$s^{\otimes}_{\sigma} P_{d^m} = \sum_{\tau} \mathrm{sgn}(R)\, \mathrm{wt}(R) s^{\otimes}_{\tau},$$

where we sum over types τ that arise from σ by adding a (dm/k)-ribbon R to $\mathrm{dg}(\sigma|_k)$ *for some $k > 0$ that divides d.*

Proof. Combining (5) with the subsequent definition of P_{d^m}, we find

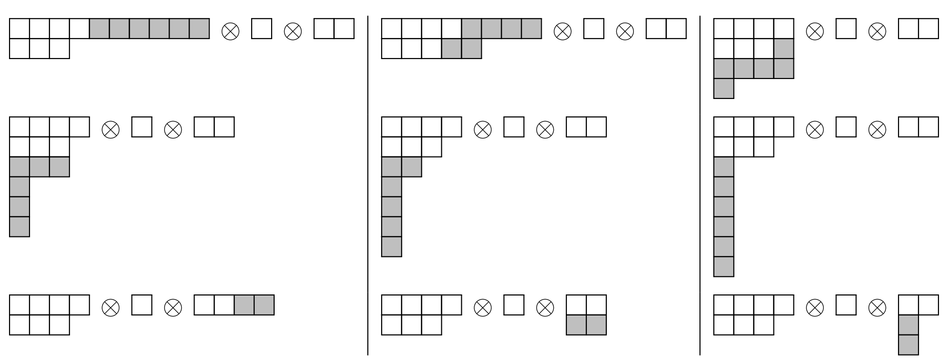
$$P_{d^m} = \sum_{k|d} k \sum_{j \geqslant 1} x_{k,j}^{dm/k} = \sum_{k|d} kp_{dm/k}(\mathbf{x}_{k*}) = \sum_{k|d} 1 \otimes \cdots \otimes 1 \otimes kp_{dm/k} \otimes 1 \otimes \cdots, \tag{7}$$

where $kp_{dm/k}$ occurs in the kth tensor factor. Multiplying $s_\sigma^\otimes$ by this expression, we get

$$s_\sigma^\otimes P_{d^m} = \sum_{k|d} s_{\sigma|_1} \otimes s_{\sigma|_2} \otimes \cdots \otimes s_{\sigma|_k} \cdot kp_{dm/k} \otimes s_{\sigma|_{k+1}} \otimes \cdots.$$

For a fixed choice of k dividing d, the classical Pieri rule replaces the factor $s_{\sigma|_k} p_{dm/k}$ by the sum of $\mathrm{sgn}(\nu/(\sigma|_k)) s_\nu$ over all ν such that $\nu/(\sigma|_k)$ is a (dm/k)-ribbon. We weight such a ribbon by k to account for the extra factor of k. Adding over all choices of k gives the formula in the theorem. □

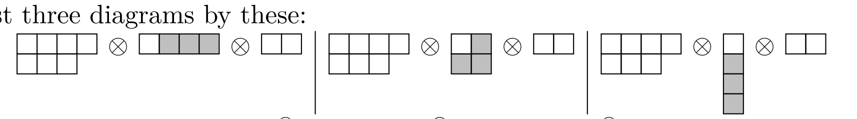
Example 2.4. Let $\sigma = (3^2 2^1 1^4 1^3)$. We compute $s_\sigma^\otimes P_{3^2}$ using the following nine diagrams, where the boxes in the newly added ribbons are shaded in gray.



The answer is

$$s^\otimes_{(3^2 2^1 1^{10,3})} - s^\otimes_{(3^2 2^1 1^{8,5})} + s^\otimes_{(3^2 2^1 1^{4,4,4,1})} - s^\otimes_{(3^2 2^1 1^{4,3,3,1,1,1})} + s^\otimes_{(3^2 2^1 1^{4,3,2,1,1,1,1})} - s^\otimes_{(3^2 2^1 1^{4,3,1,1,1,1,1,1})} + 3s^\otimes_{(3^4 2^1 1^{4,3})} + 3s^\otimes_{(3^{2,2} 2^1 1^{4,3})} - 3s^\otimes_{(3^{2,1,1} 2^1 1^{4,3})}.$$

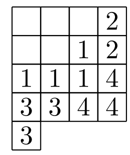
In contrast, when computing $s_\sigma^\otimes P_{2^3}$, we keep the first six diagrams but replace the last three diagrams by these:



The new terms are $+2s^\otimes_{(3^2 2^4 1^{4,3})} - 2s^\otimes_{(3^2 2^{2,2} 1^{4,3})} + 2s^\otimes_{(3^2 2^{1,1,1,1} 1^{4,3})}$.

2.2. Rule for $s_\sigma^\otimes P_\delta$ and $\mathcal{M}(P, s^\otimes)$. Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ be a list of positive integers. Iteration of Proposition 2.1 leads to the classical Schur expansion of $s_\mu p_\alpha$ in terms of rim hook tableaux, which we now describe. A *rim hook tableau (RHT)* of *shape* λ/μ and *content* α is a sequence of partitions $\mu = \nu^0, \nu^1, \nu^2, \ldots, \nu^s = \lambda$ such that ν^i/ν^{i-1} is an α_i-ribbon for $1 \leqslant i \leqslant s$. We visualize this skew RHT by drawing the skew shape λ/μ and filling the boxes in the ribbon ν^i/ν^{i-1} with the value i. The sign of the RHT is the product of the signs of all the ribbons appearing in it. The coefficient of s_λ in $s_\mu p_\alpha$ is the signed sum of all RHT of shape λ/μ and content α.

For example, here is one RHT that contributes $+1$ to the coefficient of $s_{(4,4,4,4,1)}$ in $s_{(3,2)}p_{(4,2,3,3)}$.

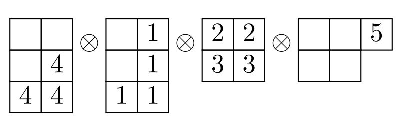


We get an analogous result for polysymmetric functions by iterating Theorem 2.3. Let $\delta = (d_1^{m_1}, \ldots, d_s^{m_s})$ be an ordered sequence of blocks. A *tensor rim hook tableau (TRHT)* of *shape* τ/σ and *content* δ is a sequence of types $\sigma = \tau^0, \tau^1, \tau^2, \ldots, \tau^s = \tau$ such that, for $1 \leqslant i \leqslant s$, τ^i arises from τ^{i-1} by adding a $d_i m_i/k_i$-ribbon R_i to $\mathrm{dg}(\tau^{i-1}|_{k_i})$ for some k_i dividing d_i. Let $\mathrm{TRHT}(\tau/\sigma, \delta)$ be the set of such objects. Write $\mathrm{TRHT}(\tau, \delta)$ when σ is the empty type. The sign (resp. weight) of a TRHT is the product of the signs (resp. weights) of all ribbons appearing in it. If the TRHT has r_k ribbons in the shape in tensor position k for each k, then the weight of the TRHT is $\prod_{k \geqslant 1} k^{r_k}$. As with RHT, we visualize a TRHT by filling all cells in ribbon R_i with the value i. This discussion proves the following theorem.

Theorem 2.5. *For any type σ and sequence $\delta = (d_1^{m_1}, \ldots, d_s^{m_s})$,*

$$s_\sigma^{\otimes} P_\delta = \sum_\tau \left[\sum_{T \in \mathrm{TRHT}(\tau/\sigma, \delta)} \mathrm{sgn}(T)\, \mathrm{wt}(T) \right] s_\tau^{\otimes}.$$

Example 2.6. The TRHT shown below contributes $(-1)^3 \cdot 2 \cdot 3^2 \cdot 4 = -72$ to the coefficient of $s^{\otimes}_{(1^{2,2,2}2^{2,2,2}3^{2,2}4^{3,2})}$ in $s^{\otimes}_{(1^{2,1}2^{1,1}4^{2,2})}P_{(4^2,3^2,6^1,3^1,4^1)}$.

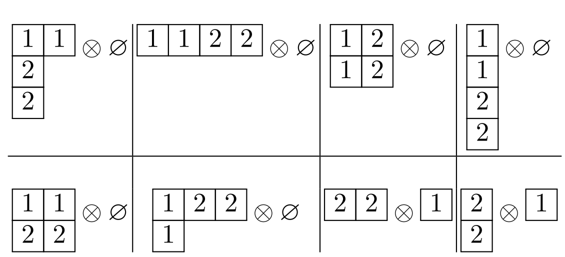


Starting with $s_0^{\otimes} = 1$ and multiplying by P_σ, we obtain the following transition matrix.

Corollary 2.7. *For all types $\sigma, \tau \Vdash n$, the coefficient of $s_\tau^{\otimes}$ in the $s^{\otimes}$-expansion of P_σ is*

$$\mathcal{M}(P, s^{\otimes})_{\tau,\sigma} = \sum_{T \in \mathrm{TRHT}(\tau,\sigma)} \mathrm{sgn}(T)\, \mathrm{wt}(T).$$

Example 2.8. We compute the $s^{\otimes}$-expansion of $P_{(2^1,1^2)}$. Creating the tensor rim hook tableaux according to the rules above, we get the following eight objects.



These give us the expansion

$$P_{(2^1 1^2)} = -s^{\otimes}_{(1^{2,1,1})} + s^{\otimes}_{(1^4)} + 2s^{\otimes}_{(1^{2,2})} + s^{\otimes}_{(1^{1,1,1,1})} - s^{\otimes}_{(1^{3,1})} + 2s^{\otimes}_{(1^2 2^1)} - 2s^{\otimes}_{(1^{1,1}2^1)}.$$

2.3. Rule for $s^{\otimes}_{\sigma} H_{d^r}$. In order to understand the effect of multiplying $s^{\otimes}_{\sigma}$ by H_{d^r}, we express H_{d^r} in terms of $h^{\otimes}$ and then use the plethystic Murnaghan–Nakayama Rule. We recall that *plethysm* is a binary operation, mapping an ordered pair (f, g) of symmetric functions to an output $f[g] \in \Lambda$, which satisfies the *Monomial Substitution Rule*: for any power-sum p_n and $f = f(\mathbf{x}) \in \Lambda$, $f[p_n] = f(\mathbf{x}^n)$. Plethysm appears in our discussion of $\mathsf{P}\Lambda$ since $H_{d^r}(\mathbf{x}_{**}) = H_d(\mathbf{x}^r_{**})$. We shall only need the Monomial Substitution Rule here, but readers interested in knowing more about plethysm may refer to [6]. Note that $f[p_n] = p_n[f]$ for all n.

Proposition 2.9. *For nonnegative integers d and r, the following expansions hold.*

$$\text{(a)}\qquad H_d = \sum_{\lambda\vdash d} h_{m_1(\lambda)}(\mathbf{x}_{1*})h_{m_2(\lambda)}(\mathbf{x}_{2*})\cdots h_{m_k(\lambda)}(\mathbf{x}_{k*})\cdots = \sum_{\lambda\vdash d} h_{m_1(\lambda)} \otimes h_{m_2(\lambda)} \otimes \ldots \otimes h_{m_k(\lambda)} \otimes \cdots .$$

$$\text{(b)}\qquad H_{d^r} = \sum_{\lambda\vdash d} h_{m_1(\lambda)}(\mathbf{x}^r_{1*})h_{m_2(\lambda)}(\mathbf{x}^r_{2*})\cdots h_{m_k(\lambda)}(\mathbf{x}^r_{k*})\cdots = \sum_{\lambda\vdash d} h_{m_1(\lambda)}[p_r] \otimes h_{m_2(\lambda)}[p_r] \otimes \cdots \otimes h_{m_k(\lambda)}[p_r] \otimes \cdots .$$

Proof. To prove part (a), consider the summand on the right side indexed by the partition $\lambda = (1^{m_1(\lambda)}2^{m_2(\lambda)}\ldots k^{m_k(\lambda)}\ldots)$. We know that for each i, any monomial that appears in $h_{m_i(\lambda)}(\mathbf{x}_{i*})$ is a product of $m_i(\lambda)$ variables chosen (with repetition allowed) from the variable set $\mathbf{x}_{i*}$. Thus, any monomial in $h_{m_1(\lambda)}(\mathbf{x}_{1*})h_{m_2(\lambda)}(\mathbf{x}_{2*})\ldots$ is a product of $m_k(\lambda)$ variables from $\mathbf{x}_{k*}$ (for each k) and has degree $\sum_{k\geqslant 1} km_k(\lambda) = |\lambda| = d$. This shows that each term in the sum on the right side of (a) appears in the expansion of H_d. To show that these are the only possible terms, we observe that any monomial f of degree d in variables $\{x_{ij}\}_{i,j\geqslant 1}$ can be expressed as a product $f_1(\mathbf{x}_{1*})f_2(\mathbf{x}_{2*})\cdots$ where each f_k is a monomial in the variables $\mathbf{x}_{k*}$ of degree d_k. Define $\lambda = (1^{d_1}2^{d_2}\cdots k^{d_k}\cdots)$. Then f appears as a monomial in the product $h_{m_1(\lambda)}(\mathbf{x}_{1*})h_{m_2(\lambda)}(\mathbf{x}_{2*})\cdots$ in the summand indexed by λ on the right side of (a).

Part (b) follows from the definition of H_{d^r}, part (a), and the Monomial Substitution Rule for plethysm. □

Example 2.10. The partitions of 4 are (1^4), $(1^2 2^1)$, $(1^1 3^1)$, (2^2), and (4^1). So $H_4 = h_4 \otimes 1 \otimes 1 \otimes 1 + h_2 \otimes h_1 \otimes 1 \otimes 1 + h_1 \otimes 1 \otimes h_1 \otimes 1 + 1 \otimes h_2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes h_1$.

To compute $s^{\otimes}_{\sigma} H_{d^r}$, we need to understand the combinatorial objects that appear in the Schur expansion of $s_\mu \cdot h_n[p_r]$. The formula appears in [2, pg. 29] and a combinatorial interpretation in terms of r-decomposable partitions was given by Wildon in [11]. We give a formula based on the notion of r^n-polyribbons following the description in Turek [10]. The notation r^n does not signify exponentiation or a block but is meant to evoke the n-fold iteration of the operation of adding an r-ribbon.

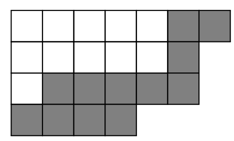
Here is the formal definition. Let γ/ρ be a k-ribbon. The *top row* of γ/ρ, denoted by $\top(\gamma/\rho)$, is the least row containing a cell of γ/ρ. A skew shape λ/μ is called an *r^n-polyribbon* if there exist partitions $\gamma_{(0)}, \gamma_{(1)}, \ldots, \gamma_{(n)}$ such that:

$$\mu = \gamma_{(0)} \subseteq \gamma_{(1)} \subseteq \cdots \subseteq \gamma_{(n)} = \lambda, \tag{8}$$

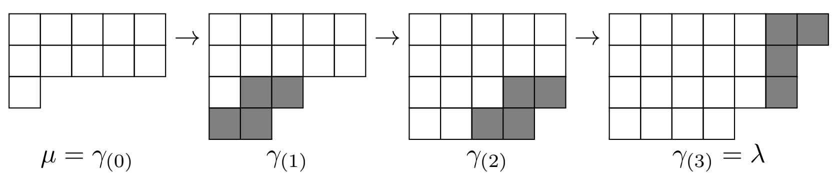
$\gamma_{(i)}/\gamma_{(i-1)}$ is an r-ribbon for $1 \leqslant i \leqslant n$, and $\top(\gamma_{(i)}/\gamma_{(i-1)}) \geqslant \top(\gamma_{(i+1)}/\gamma_{(i)})$ for $1 \leqslant i \leqslant n-1$. If λ/μ is an r^n-polyribbon, then (as is readily checked) only one list $\gamma_{(0)}, \ldots, \gamma_{(n)}$ satisfies the conditions stated here. Thus, we may define the *sign* of this r^n-polyribbon, written $\mathrm{sgn}_r(\lambda/\mu)$, to be $\prod_{i=1}^n \mathrm{sgn}(\gamma_{(i)}/\gamma_{(i-1)})$. If λ/μ is not an r^n-polyribbon for any n, then we set $\mathrm{sgn}_r(\lambda/\mu) = 0$.

Remark 2.11. The condition on top rows is equivalent to saying that the north-easternmost box of each inserted ribbon lies weakly north and strictly east of the northeasternmost box of the previously inserted ribbon.

Example 2.12. For $\mu = (5,5,1)$ and $\lambda = (7,6,6,4)$, λ/μ is a skew shape denoted by the gray cells in the figure below.



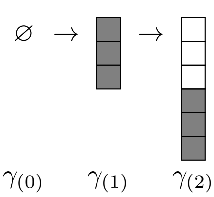
The skew shape λ/μ is a 4^3-polyribbon as it can be constructed by adding three 4-ribbons according to the aforementioned rules as shown here:



If we write t_i for $\top\left(\gamma_{(i)}/\gamma_{(i-1)}\right)$, then $t_1 = 3$, $t_2 = 3$, and $t_3 = 1$. This polyribbon has sign $\mathrm{sgn}_4(\lambda/\mu) = (-1)\cdot(-1)\cdot 1 = 1$.

Remark 2.13. The next examples illustrate some common pitfalls that may occur.

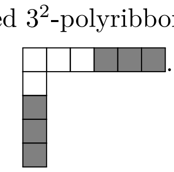
(a) The shape $(1,1,1,1,1,1)$ is *not* a 3^2-polyribbon as the only way to construct it is as follows:



Here $\top(\gamma_{(1)}/\gamma_{(0)}) = 1$, which is smaller than $\top(\gamma_{(2)}/\gamma_{(1)}) = 4$.

(b) The list of component ribbons of an r^n-polyribbon is unique when nonnegative integers r and n are fixed. For instance, $(3,3)$ is a 2^3-polyribbon constructed via $\varnothing \to (1,1) \to (2,2) \to (3,3)$. On the other hand, $(3,3)$ is a 3^2-polyribbon constructed via $\varnothing \to (2,1) \to (3,3)$; note that the alternate construction $\varnothing \to (3) \to (3,3)$ is invalid.

(c) An r^n-polyribbon may not be connected, in the sense that the skew shape might be the union of two subsets of boxes with no shared edges. For instance, $(6,1,1,1,1)/(3,1)$ is a disconnected 3^2-polyribbon, as one can see from this diagram:



(d) We use the phrase "adding an r^n-polyribbon to μ to give λ" to mean λ/μ is an r^n-polyribbon. If μ is given, we create a new r^n-polyribbon λ/μ by adding n r-ribbons moving northeast along the border of the growing shape. If instead λ/μ is given at the outset, we can test whether this shape is an r^n-polyribbon by trying to delete n r-ribbons moving southwest along the border as the shape λ shrinks to μ through intermediate partition shapes. For example, this test shows that $(2,2,2)$ is a 3^2-polyribbon but not a 2^3-polyribbon.

Here is the promised combinatorial description of the Schur expansion of $s_\mu \cdot h_n[p_r]$.

THEOREM 2.14 ([11, Equation (2)]). *Let μ be a partition and r, n be nonnegative integers. Then*

$$s_\mu \cdot h_n[p_r] = s_\mu \cdot p_r[h_n] = \sum_{\lambda} \operatorname{sgn}_r(\lambda/\mu) s_\lambda$$

where the sum is over all partitions λ obtained by adding an r^n-polyribbon to μ.

REMARK 2.15. In the case $n = 1$, $h_1[p_r] = p_r$, and the rule in the theorem reduces to the Slinky Rule stated in Proposition 2.1. In the case $r = 1$, $h_n[p_1] = h_n$, and the theorem reduces to the classical Pieri rule. This says that $s_\mu h_n = \sum_\nu s_\nu$ where we sum over partitions ν such that ν/μ is a horizontal n-strip, namely a collection of n boxes in distinct columns.

Applying Theorem 2.14 to the polysymmetric case leads to the following theorem.

THEOREM 2.16. *Let σ be any type and d^r be a block. Then*

$$s^{\otimes}_{\sigma} H_{d^r} = \sum_{\tau} \operatorname{sgn}^{\otimes}_r(\tau/\sigma) s^{\otimes}_{\tau},$$

where we sum over all types τ obtained from σ as follows: for some partition $\lambda \vdash d$, $\tau|_k$ is obtained by adding an $r^{m_k(\lambda)}$-polyribbon to $\sigma|_k$ for all $k \geqslant 1$; and $\operatorname{sgn}^{\otimes}_r(\tau/\sigma) = \prod_{k\geqslant 1} \operatorname{sgn}_r((\tau|_k)/(\sigma|_k))$.

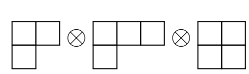
When τ is related to σ as described in this theorem, we say that τ/σ *is a d^r-tensor polyribbon.*

Proof. By Proposition 2.9(b),

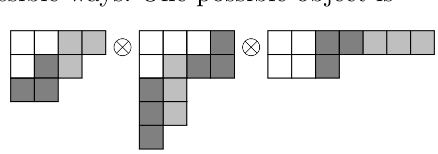
$$s^{\otimes}_{\sigma} H_{d^r} = \sum_{\lambda \vdash d} s_{\sigma|_1} \cdot h_{m_1(\lambda)}[p_r] \otimes s_{\sigma|_2} \cdot h_{m_2(\lambda)}[p_r] \otimes \cdots \otimes s_{\sigma|_k} \cdot h_{m_k(\lambda)}[p_r] \otimes \cdots .$$

The kth factor in the tensor product expands into $\sum_{\nu_{(k)}} \operatorname{sgn}_r(\nu_{(k)}/(\sigma|_k)) s_{\nu_{(k)}}$ where the sum is over all partitions $\nu_{(k)}$ obtained by adding an $r^{m_k(\lambda)}$-polyribbon to $\sigma|_k$. Using the distributive property of tensor products over addition gives the signed sum of $s^{\otimes}_{\tau}$ for the types τ described in the theorem. ☐

EXAMPLE 2.17. Let $\sigma = (3^2 3^2 2^3 2^1 1^2 1^1) = (1^{2,1} 2^{3,1} 3^{2,2})$, which has the tensor diagram shown here:



We describe one object in the expansion $s^{\otimes}_{\sigma} H_{14^3}$. First, we pick the partition $\lambda = (3, 3, 2, 2, 2, 1, 1) = (1^2 2^3 3^2)$ of 14. The theorem tells us to add a 3^2-polyribbon to the first diagram, a 3^3-polyribbon to the second diagram, and a 3^2-polyribbon to the third diagram in all possible ways. One possible object is



Here the gray cells show the added polyribbons, and the shading shows the constituent ribbons within each polyribbon. The sign of this object is $(-1 \cdot -1) \cdot (1 \cdot 1 \cdot -1) \cdot (-1 \cdot 1) = 1$, and the corresponding term is $+s^{\otimes}_{(3^7 3^3 2^4 2^4 2^2 2^2 2^1 1^4 1^3 1^2)}$.

2.4. Rule for $s^{\otimes}_{\sigma} H_{\delta}$ and $\mathcal{M}(H, s^{\otimes})$. We can iterate Theorem 2.16 to obtain the $s^{\otimes}$-expansions of $s^{\otimes}_{\sigma} H_{\delta}$ and H_{σ}. Let τ and σ be types. Let $\delta = (d_1^{r_1}, \dots, d_s^{r_s})$ be an ordered sequence of blocks. A *tensor polyribbon tableau (TPRT) T* of *shape τ/σ* and *content δ* is a sequence of types $\sigma = \tau_{(0)}, \tau_{(1)}, \dots, \tau_{(s)} = \tau$ such that, for all i between 1 and s, $\tau_{(i)}/\tau_{(i-1)}$ is a $d_i^{r_i}$-tensor polyribbon. Let $\mathrm{TPRT}(\tau/\sigma, \delta)$ be the set of such objects. We visualize T by drawing the tensor diagram of τ and filling all cells in $\mathrm{dg}(\tau_{(i)}) \smallsetminus \mathrm{dg}(\tau_{(i-1)})$ with the value i. The *sign* of T is $\mathrm{sgn}(T) = \prod_{i=1}^{s} \mathrm{sgn}^{\otimes}_{r_i}(\tau_{(i)}/\tau_{(i-1)})$.

Theorem 2.18. *Given a type σ and a sequence of blocks $\delta = (d_1^{r_1}, \dots, d_s^{r_s})$,*

$$s^{\otimes}_{\sigma} H_{\delta} = \sum_{\tau} \left[\sum_{T \in \mathrm{TPRT}(\tau/\sigma, \delta)} \mathrm{sgn}(T) \right] s^{\otimes}_{\tau}.$$

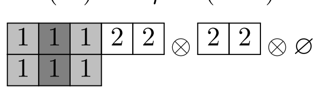
Proof. This follows by iterating Theorem 2.16 in the same way that Theorem 2.5 is deduced from Theorem 2.3. $\square$

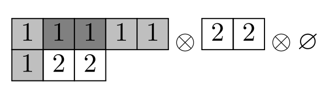
Corollary 2.19. *For all $\sigma, \tau \Vdash n$, the coefficient of $s^{\otimes}_{\tau}$ in the $s^{\otimes}$-expansion of H_{σ} is*

$$\mathcal{M}(H, s^{\otimes})_{\tau,\sigma} = \sum_{T \in \mathrm{TPRT}(\tau,\sigma)} \mathrm{sgn}(T).$$

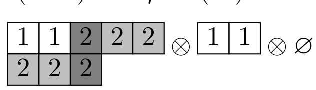
Example 2.20. We find the coefficient of $s^{\otimes}_{2^2 1^5 1^3}$ in the $s^{\otimes}$-expansion of $H_{3^2 3^2}$. Here, $d_1 = d_2 = 3$ and $r_1 = r_2 = 2$. We first pick $\lambda \vdash 3$ and add a $2^{m_k(\lambda)}$-polyribbon to an empty diagram in each position k. Then we pick $\mu \vdash 3$ and add a $2^{m_k(\mu)}$-polyribbon to the current diagram in each position k. We make such choices in all possible ways that lead to the target tensor diagram with $\mathrm{dg}(5,3)$ in position 1 and $\mathrm{dg}(2)$ in position 2. Since position 3 is empty, we cannot choose λ or μ to be (3^1).

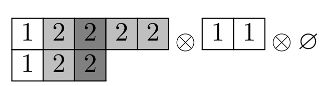
Choosing $\lambda = (1^3)$ and $\mu = (1^1 2^1)$ leads to these two TPRTs, both with sign -1:





Choosing $\lambda = (1^1 2^1)$ and $\mu = (1^3)$ leads to these two TPRTs, both with sign -1:





No other choice of λ, μ leads to the required tensor diagram. Thus the coefficient of $s^{\otimes}_{2^2 1^5 1^3}$ in $H_{3^2 3^2}$ is -4.

Remark 2.21. Let $\sigma = (1^{1,1,\dots,1}) \Vdash n$. The coefficient of $s^{\otimes}_{\tau}$ in the $s^{\otimes}$-expansion of H_{σ} is

$$\mathcal{M}(H, s^{\otimes})_{\tau,\sigma} = \begin{cases} f^{\lambda} & \text{if } \tau = (1^{\lambda}), \\ 0 & \text{otherwise,} \end{cases}$$

where f^{λ} is the number of standard Young tableaux of shape λ. This extends the analogous result for the symmetric function transition matrix $\mathcal{M}(h, s)_{\lambda, 1^n}$.

2.5. Rules for $s^{\otimes}_{\sigma} E^{+}_{d^m}$ and $s^{\otimes}_{\sigma} E_{d^m}$. The rules for E^+ and E follow from the rule for H. In this section, we make use of the involution ω on the algebra of symmetric functions. Under this map, $\omega(h_{\lambda}) = e_{\lambda}$, $\omega(s_{\mu}) = s_{\mu'}$, and $\omega(p_{\lambda}) = (-1)^{|\lambda| - \ell(\lambda)} p_{\lambda}$. For more information about this involution, refer to Section 9.20 of [5]. In this section, we use the following result.

Proposition 2.22 ([7, I.8, Ex 1(c)]). *Given nonnegative integers r and n,*

$$\omega(h_n[p_r]) = (-1)^{n(r-1)} e_n[p_r].$$

Using the proof technique from Proposition 2.9 and the idea of square-free monomials, we can find the $e^{\otimes}$-expansions of E_d^+ and E_d.

Proposition 2.23. *For nonnegative integers d and r, the following expansions of E^+ and E hold.*

$$\begin{aligned}
\text{(a)}\quad E_d^+ &= \sum_{\lambda\vdash d} e_{m_1(\lambda)}(\mathbf{x}_{1*})e_{m_2(\lambda)}(\mathbf{x}_{2*})\cdots e_{m_k(\lambda)}(\mathbf{x}_{k*})\cdots \\
&= \sum_{\lambda\vdash d} e_{m_1(\lambda)}\otimes e_{m_2(\lambda)}\otimes\cdots\otimes e_{m_k(\lambda)}\otimes\cdots . \\
\text{(b)}\quad E_d &= \sum_{\lambda\vdash d}(-1)^{\ell(\lambda)} e_{m_1(\lambda)}(\mathbf{x}_{1*})\cdots e_{m_k(\lambda)}(\mathbf{x}_{k*})\cdots \\
&= \sum_{\lambda\vdash d}(-1)^{\ell(\lambda)} e_{m_1(\lambda)}\otimes\cdots\otimes e_{m_k(\lambda)}\otimes\cdots . \\
\text{(c)}\quad E_{d^r}^+ &= \sum_{\lambda\vdash d} e_{m_1(\lambda)}(\mathbf{x}_{1*}^r)\cdots e_{m_k(\lambda)}(\mathbf{x}_{k*}^r)\cdots \\
&= \sum_{\lambda\vdash d} e_{m_1(\lambda)}[p_r]\otimes\cdots\otimes e_{m_k(\lambda)}[p_r]\otimes\cdots . \\
\text{(d)}\quad E_{d^r} &= \sum_{\lambda\vdash d}(-1)^{\ell(\lambda)} e_{m_1(\lambda)}(\mathbf{x}_{1*}^r)\cdots e_{m_k(\lambda)}(\mathbf{x}_{k*}^r)\cdots \\
&= \sum_{\lambda\vdash d}(-1)^{\ell(\lambda)} e_{m_1(\lambda)}[p_r]\otimes\cdots\otimes e_{m_k(\lambda)}[p_r]\otimes\cdots .
\end{aligned}$$

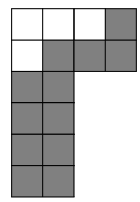
Proof. We prove (a) and (b), and the rest follows from the Monomial Substitution Rule. We proceed as in the proof of Proposition 2.9, but in this case each variable appears at most once. This gives us the expression for E_d^+. For E_d, the sign of a monomial f is given by $(-1)^{\mathrm{len}(f)}$, where $\mathrm{len}(f)$ is the number of indeterminates in f. Each e_k has exactly k indeterminates and thus has the sign $(-1)^k$. This shows that the sign for the monomial $e_{m_1(\lambda)}(\mathbf{x}_{1*})\cdots e_{m_k(\lambda)}(\mathbf{x}_{k*})\cdots$ is $(-1)^{m_1(\lambda)+\cdots+m_k(\lambda)+\cdots} = (-1)^{\ell(\lambda)}$. ∎

Before we present the analogue of Theorem 2.14 for multiplying a Schur function by $e_n[p_r]$, we introduce a notion dual to that of an r^n-polyribbon. For any skew shape λ/μ, let $\perp(\lambda/\mu)$ denote the least index of a column that contains a cell of λ/μ. A skew shape λ/μ is called an *$(r^n)'$-polyribbon* or a *dual r^n-polyribbon* if there exists a (necessarily unique) list of partitions $\gamma_{(0)},\gamma_{(1)},\ldots,\gamma_{(n)}$ such that $\mu=\gamma_{(0)}\subseteq\gamma_{(1)}\subseteq\cdots\subseteq\gamma_{(n)}=\lambda$, $\gamma_{(i)}/\gamma_{(i-1)}$ is an r-ribbon for $1\leqslant i\leqslant n$, and $\perp(\gamma_{(i)}/\gamma_{(i-1)})\geqslant\perp(\gamma_{(i+1)}/\gamma_{(i)})$ for $1\leqslant i\leqslant n-1$. Define the *sign* of an $(r^n)'$-polyribbon to be $\mathrm{sgn}'_r(\lambda/\mu)=\prod_{i=1}^{n}\mathrm{sgn}(\gamma_{(i)}/\gamma_{(i-1)})$.

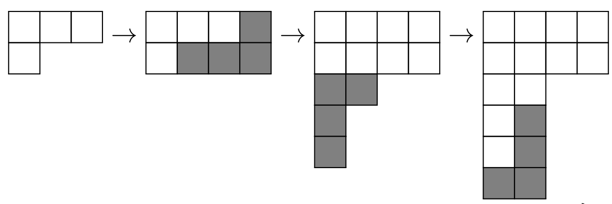
Remark 2.24. Equivalently, λ/μ is a dual r^n-polyribbon if we can go from $\mathrm{dg}(\mu)$ to $\mathrm{dg}(\lambda)$ by adding n r-ribbons in succession, where the southwesternmost box of each new r-ribbon lies strictly south and weakly west of the southwesternmost box of the previously added r-ribbon.

Remark 2.25. If λ/μ is an r^n-polyribbon, then λ'/μ' is an $(r^n)'$-polyribbon, and conversely.

Example 2.26. For $\mu = (3,1)$ and $\lambda = (4,4,2,2,2,2)$, λ/μ is the following skew shape:



The skew shape λ/μ is a dual 4^3-polyribbon since it can be constructed as follows:



$\mu = \gamma_{(0)}$ $\gamma_{(1)}$ $\gamma_{(2)}$ $\gamma_{(3)} = \lambda$

The values of $\perp\left(\gamma_{(i)}/\gamma_{(i-1)}\right)$ for $i = 1,2,3$ are 2, 1, and 1. This polyribbon has sign $\mathrm{sgn}'_4(\lambda/\mu) = (-1)\cdot 1 \cdot 1 = -1$.

Proposition 2.27. *Given a partition μ and nonnegative integers n and r,*

$$s_\mu \cdot e_n[p_r] = \sum_\lambda \mathrm{sgn}'_r(\lambda/\mu) s_\lambda,$$

where the sum is over all partitions λ obtained by adding a dual r^n-polyribbon to μ.

Proof. Recall from Theorem 2.14 that

$$s_\mu \cdot h_n[p_r] = \sum_\nu \mathrm{sgn}_r(\nu/\mu) s_\nu,$$

where the sum is over all partitions ν obtained by adding an r^n-polyribbon to μ. Acting on both sides by ω and then using Proposition 2.22 gives

$$s_{\mu'} \cdot (-1)^{n(r-1)} e_n[p_r] = \sum_\nu \mathrm{sgn}_r(\nu/\mu) s_{\nu'}.$$

Replacing μ' by μ gives

$$s_\mu \cdot e_n[p_r] = (-1)^{n(r-1)} \sum_\nu \mathrm{sgn}_r(\nu/\mu') s_{\nu'},$$

where the sum is over partitions ν obtained by adding an r^n-polyribbon to μ', the conjugate partition of μ. Equivalently, by Remark 2.25, ν' is obtained by adding the dual r^n-polyribbon ν'/μ to μ. Defining $\lambda = \nu'$, it suffices to show $\mathrm{sgn}'_r(\lambda/\mu) = (-1)^{n(r-1)}\,\mathrm{sgn}_r(\nu/\mu')$. If a skew shape α/β is a r-ribbon covering ℓ rows, then its sign is $(-1)^{\ell-1}$. The number of columns spanned by this ribbon is $r+1-\ell$ which determines the sign of α'/β', i.e. $\mathrm{sgn}(\alpha'/\beta') = (-1)^{r-\ell}$. Let the r^n-polyribbon ν/μ' have the decomposition $\gamma_{(0)}, \gamma_{(1)}, \ldots, \gamma_{(n)}$ as in eq. (8), where each $\gamma_{(i)}/\gamma_{(i-1)}$ covers ℓ_i rows and $r+1-\ell_i$ columns. This gives us

$$\begin{aligned}\mathrm{sgn}'_r(\lambda/\mu) &= (-1)^{(r-\ell_1)+(r-\ell_2)+\ldots+(r-\ell_n)}\\ &= (-1)^{nr}(-1)^{(\ell_1-1)+(\ell_2-1)+\cdots+(\ell_n-1)+n}\\ &= (-1)^{n(r-1)}\,\mathrm{sgn}_r(\nu/\mu').\end{aligned}$$

□

For types τ and σ, we say that τ/σ is a *dual d^r-tensor polyribbon* if, for some partition λ of d, each $\tau|_k$ is obtained from $\sigma|_k$ by adding a dual $r^{m_k(\lambda)}$-polyribbon. We call the partition λ the *associated partition of τ/σ*. In this situation, define $\operatorname{sgn}_r^+(\tau/\sigma) = \prod_{k=1}^{\infty} \operatorname{sgn}'_r((\tau|_k)/(\sigma|_k))$ and $\operatorname{sgn}_r^-(\tau/\sigma) = (-1)^{\ell(\lambda)} \prod_{k=1}^{\infty} \operatorname{sgn}'_r((\tau|_k)/(\sigma|_k))$, where λ is the associated partition of τ/σ. The extra power $(-1)^{\ell(\lambda)}$ is the total number of r-ribbons (within the various polyribbons) that are added to σ to reach τ.

Theorem 2.28. *Let σ be any type and d^r be a block. Then*

$$s_\sigma^\otimes E_{d^r}^+ = \sum_\tau \operatorname{sgn}_r^+(\tau/\sigma) s_\tau^\otimes \quad \text{and} \quad s_\sigma^\otimes E_{d^r} = \sum_\tau \operatorname{sgn}_r^-(\tau/\sigma) s_\tau^\otimes,$$

where the sums range over types τ such that τ/σ is a dual d^r-tensor polyribbon.

Proof. We prove it for the case of $E_{d^r}^+$, and the same proof works for E_{d^r} with an appropriate change of sign. From Proposition 2.23, we obtain

$$s_\sigma^\otimes \cdot E_{d^r}^+ = \sum_{\lambda \vdash d} s_{\sigma|_1} \cdot e_{m_1(\lambda)}[p_r] \otimes \cdots \otimes s_{\sigma|_k} \cdot e_{m_k(\lambda)}[p_r] \otimes \cdots .$$

Applying Proposition 2.27 to the above expression, the kth component of the tensor product expands to $\sum_\gamma \operatorname{sgn}'_r(\gamma/(\sigma|_k)) s_\gamma$, where the sum is over partitions γ that arise by adding a dual $r^{m_k(\lambda)}$-polyribbon to $\sigma|_k$. Using the distributive law gives us our result. □

2.6. Rules for $s_\sigma^\otimes E_\delta^+$, $s_\sigma^\otimes E_\delta$, $\mathcal{M}(E^+, s^\otimes)$, and $\mathcal{M}(E, s^\otimes)$.

To obtain the entries of the next transition matrices, we define a dual version of the tableaux in Section 2.4. Let τ and σ be types. Let $\delta = (d_1^{r_1}, \ldots, d_s^{r_s})$ be an ordered sequence of blocks. A *dual tensor polyribbon tableau* (*dual TPRT*) T of *shape* τ/σ and *content* δ is a sequence of types $\sigma = \tau_{(0)}, \tau_{(1)}, \ldots, \tau_{(s)} = \tau$ such that, for all i between 1 and s, $\tau_{(i)}/\tau_{(i-1)}$ is a dual $d_i^{r_i}$-tensor polyribbon. Let $\operatorname{TPRT}'(\tau/\sigma, \delta)$ be the set of such objects. We visualize T by drawing the tensor diagram of τ and filling all cells in $\operatorname{dg}(\tau_{(i)}) \smallsetminus \operatorname{dg}(\tau_{(i-1)})$ with the value i. Define the two corresponding signs associated with T to be $\operatorname{sgn}^+(T) = \prod_{i=1}^s \operatorname{sgn}_{r_i}^+(\tau_{(i)}/\tau_{(i-1)})$ and $\operatorname{sgn}^-(T) = \prod_{i=1}^s \operatorname{sgn}_{r_i}^-(\tau_{(i)}/\tau_{(i-1)})$.

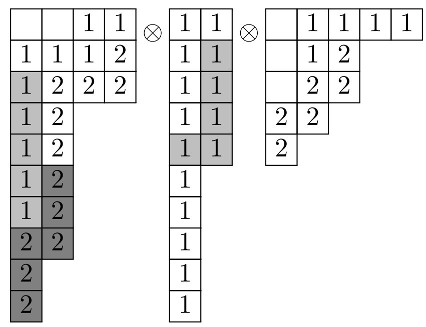
Theorem 2.29. *Given a type σ and a sequence of blocks $\delta = (d_1^{r_1}, \ldots, d_s^{r_s})$,*

$$s_\sigma^\otimes E_\delta^+ = \sum_\tau \left[\sum_{T \in \operatorname{TPRT}'(\tau/\sigma,\delta)} \operatorname{sgn}^+(T) \right] s_\tau^\otimes \quad \text{and}$$

$$s_\sigma^\otimes E_\delta = \sum_\tau \left[\sum_{T \in \operatorname{TPRT}'(\tau/\sigma,\delta)} \operatorname{sgn}^-(T) \right] s_\tau^\otimes.$$

Proof. These follow by iterating Theorem 2.28 in the same way that Theorem 2.5 is deduced from Theorem 2.3. □

Example 2.30. Let us construct one of the objects of shape $\tau = 1^{4^3 2^5 1^2} 2^{2^5 1^5} 3^{53^2 21}$ that appears in the $s^\otimes$-expansions of $s_\sigma^\otimes E_\delta$ and $s_\sigma^\otimes E_\delta^+$ for $\sigma = 1^2 3^{1,1,1}$ and $\delta = (11^5, 5^6)$. We first pick the partition $\lambda = 1^2 2^3 3^1 \vdash 11$. Starting with the tensor diagram of τ, we insert a dual 5^2-polyribbon in the first diagram, a dual 5^3-polyribbon in the second diagram, and a dual 5^1-polyribbon in the third diagram. We label the cells in these polyribbons by 1. Next we pick the partition $\mu = 1^2 3^1 \vdash 5$. We continue by adding a dual 6^2-polyribbon to the first diagram and a dual 6^1-polyribbon to the third diagram, with all new cells labeled by 2. Here is one possible object $T \in \mathrm{TPRT}(\tau/\sigma, \delta)$ arising from these insertions:



We compute $\mathrm{sgn}^+(T) = 1$ and $\mathrm{sgn}^-(T) = (-1)^{\ell(\lambda)+\ell(\mu)}\,\mathrm{sgn}^+(T) = (-1)^{6+3} = -1$.

Corollary 2.31. *For all types $\sigma, \tau \Vdash n$, the coefficients of $s^{\otimes}_{\tau}$ in the $s^{\otimes}$-expansions of E^+_{σ} and E_{σ} are*

$$\mathcal{M}(E^+, s^{\otimes})_{\tau,\sigma} = \sum_{T \in \mathrm{TPRT}'(\tau,\sigma)} \mathrm{sgn}^+(T) \quad \text{and} \quad \mathcal{M}(E, s^{\otimes})_{\tau,\sigma} = \sum_{T \in \mathrm{TPRT}'(\tau,\sigma)} \mathrm{sgn}^-(T).$$

3. Expansions in the $p^{\otimes}$ basis

3.1. Algebraic development of $p^{\otimes}$-expansions. Given integer partitions $\lambda = (1^{m_1(\lambda)}2^{m_2(\lambda)}\cdots)$ and $\mu = (1^{m_1(\mu)}2^{m_2(\mu)}\cdots)$, define their *union* to be $\lambda \cup \mu = (1^{m_1(\lambda)+m_1(\mu)}2^{m_2(\lambda)+m_2(\mu)}\cdots)$, which is the partition obtained by combining all the parts of λ and μ (with multiplicities) into a new weakly decreasing list. By definition of power-sums, we have $p_\lambda p_\mu = p_{\lambda\cup\mu}$. More generally, given integer partitions $\lambda^{(1)}, \ldots, \lambda^{(s)}$, $\prod_{i=1}^{s} p_{\lambda^{(i)}} = p_{\lambda^{(1)}\cup\cdots\cup\lambda^{(s)}}$.

Similar results hold for types and the $p^{\otimes}$-basis of $\mathsf{P}\Lambda$. For any types σ and ρ, let $\sigma\cup\rho$ be the type obtained by merging all the blocks in σ and ρ (with multiplicities) into a new list of blocks. Equivalently, using the union operation on integer partitions, we can define $\sigma\cup\rho$ by $(\sigma\cup\rho)|_k = \sigma|_k \cup \rho|_k$ for all $k \geqslant 1$. It follows from this definition that $p^{\otimes}_{\sigma\cup\rho} = p^{\otimes}_{\sigma} p^{\otimes}_{\rho}$. More generally, for all types $\tau^{(1)}, \ldots, \tau^{(s)}$,

$$\prod_{i=1}^{s} p^{\otimes}_{\tau^{(i)}} = \prod_{i=1}^{s} \bigotimes_{k\geqslant 1} p_{\tau^{(i)}|_k} = \bigotimes_{k\geqslant 1} p_{\tau^{(1)}|_k\cup\cdots\cup\tau^{(s)}|_k} = p^{\otimes}_{\tau^{(1)}\cup\cdots\cup\tau^{(s)}}. \tag{9}$$

Combining this formula with the distributive law, we get an algebraic prescription for the $p^{\otimes}$-expansion of a product $G_1G_2\cdots G_s$ assuming we already know the $p^{\otimes}$-expansions of each G_i. In particular, to get the transition matrices $\mathcal{M}(P, p^{\otimes})$, $\mathcal{M}(H, p^{\otimes})$, $\mathcal{M}(E^+, p^{\otimes})$, and $\mathcal{M}(E, p^{\otimes})$, it suffices to find the $p^{\otimes}$-expansions of P_{d^r}, H_{d^r}, $E^+_{d^r}$, and E_{d^r}.

Before presenting these expansions, we introduce some notation. For each integer partition λ, define $z_\lambda = \prod_{i\geqslant 1} i^{m_i(\lambda)} m_i(\lambda)!$. The factor z_λ appears when finding p-expansions of certain symmetric functions. In particular, $h_n = \sum_{\lambda\vdash n} \frac{p_\lambda}{z_\lambda}$ and $e_n = \sum_{\lambda\vdash n} (-1)^{n-\ell(\lambda)} \frac{p_\lambda}{z_\lambda}$ (see [5, §9.19]). The polysymmetric analog of z_λ is defined by $z^{\otimes}_{\tau} = \prod_{k\geqslant 1} z_{\tau|_k}$ for a type τ.

Example 3.1. For $\tau = (3^2 3^2 2^3 2^2 2^2 1^4 1^2)$, we have associated partitions $\tau|_1 = (4,2) = 4^1 2^1$, $\tau|_2 = (3,2,2) = 3^1 2^2$, and $\tau|_3 = (2,2) = 2^2$. We compute $z^{\otimes}_{\tau} = (4^1 1! 2^1 1!) \cdot (3^1 1! 2^2 2!) \cdot (2^2 2!) = 1536$.

For a type $\tau = (d_1^{m_1} d_2^{m_2} \dots d_s^{m_s})$ and an integer $r > 0$, define the type $\tau^r = (d_1^{rm_1} d_2^{rm_2} \dots d_s^{rm_s})$. Recall from §1.2 that $\operatorname{sgn}(\tau) = \prod_{i=1}^{s} (-1)^{m_i} = \prod_{k \geqslant 1} (-1)^{\operatorname{area}(\tau|_k)}$ and $\ell(\tau) = s = \sum_{k \geqslant 1} \ell(\tau|_k)$. The net exponent of -1 in $\operatorname{sgn}(\tau)$ is the number of blocks of τ with odd multiplicity, while the net exponent of -1 in $(-1)^{\ell(\tau)} \operatorname{sgn}(\tau)$ is the number of blocks of τ with even multiplicity.

Proposition 3.2. *For positive integers d and r, the following $p^{\otimes}$-expansions hold.*

(a) $P_{d^r} = \sum_{k|d} k\, p^{\otimes}_{k^{rd/k}}$.

(b) $H_{d^r} = \sum_{\tau \Vdash d} \dfrac{p^{\otimes}_{\tau^r}}{z^{\otimes}_{\tau}}$.

(c) $E^{+}_{d^r} = \sum_{\tau \Vdash d} (-1)^{\ell(\tau)} \operatorname{sgn}(\tau) \dfrac{p^{\otimes}_{\tau^r}}{z^{\otimes}_{\tau}}$.

(d) $E_{d^r} = \sum_{\tau \Vdash d} (-1)^{\ell(\tau)} \dfrac{p^{\otimes}_{\tau^r}}{z^{\otimes}_{\tau}}$.

Proof. Suppose we have found a required expansion when $r = 1$, say $F_d = \sum_{\tau} a_\tau p^{\otimes}_{\tau}$ where F is P or H or E^+ or E and $a_\tau \in \mathbb{Q}$. The plethysm property $p_m[p_r] = p_{rm}$ (for positive integers m, r) extends to $p_\lambda[p_r] = p_{r\lambda}$ (for a partition λ and integer r), where $r\lambda$ is λ with all parts scaled by r. Then the $p^{\otimes}$-expansion for general r is

$$F_{d^r} = \sum_{\tau} a_\tau p_{\tau|_1}[p_r] \otimes p_{\tau|_2}[p_r] \otimes \dots = \sum_{\tau} a_\tau p_{r\tau|_1} \otimes p_{r\tau|_2} \otimes \dots = \sum_{\tau} a_\tau p^{\otimes}_{\tau^r}. \tag{10}$$

(a) The sum $\sum_{j \geqslant 1} x_{k,j}^{d/k}$ is the power-sum symmetric function $p_{d/k}(\mathbf{x}_{k*}) = p^{\otimes}_{k^{d/k}}$. Thus, eq. (5) can be rephrased as $P_d = \sum_{k|d} k\, p^{\otimes}_{k^{d/k}}$. Part (a) now follows from (10).

(b) By Proposition 2.9, $H_d = \sum_{\lambda \vdash d} h_{m_1(\lambda)} \otimes h_{m_2(\lambda)} \otimes \cdots$. Using $h_n = \sum_{\mu \vdash n} \dfrac{p_\mu}{z_\mu}$ on each factor gives

$$H_d = \sum_{\lambda \vdash d} \sum_{\mu^{(1)} \vdash m_1(\lambda)} \sum_{\mu^{(2)} \vdash m_2(\lambda)} \cdots \sum_{\mu^{(d)} \vdash m_d(\lambda)} \frac{p_{\mu^{(1)}}}{z_{\mu^{(1)}}} \otimes \frac{p_{\mu^{(2)}}}{z_{\mu^{(2)}}} \otimes \cdots \otimes \frac{p_{\mu^{(d)}}}{z_{\mu^{(d)}}}. \tag{11}$$

The iterated sum here can be rewritten as a sum over types $\tau \Vdash d$ via the bijection sending $(\lambda, \mu^{(1)}, \mu^{(2)}, \dots, \mu^{(d)})$ to the type τ with $\tau|_k = \mu^{(k)}$ for all $k \geqslant 1$. We obtain $H_d = \sum_{\tau \Vdash d} \dfrac{p^{\otimes}_{\tau}}{z^{\otimes}_{\tau}}$. Part (b) now follows from (10). (c) The proof for E^+_d is like the proof for H_d, but with bookkeeping for signs. The kth tensor factor in (11) contributes the sign $(-1)^{\operatorname{area}(\mu^{(k)}) - \ell(\mu^{(k)})}$. Converting to a sum over τ as described above, the kth sign factor becomes $(-1)^{\operatorname{area}(\tau|_k) - \ell(\tau|_k)}$. Taking the product over $k \geqslant 1$ gives an overall sign of $\operatorname{sgn}(\tau)(-1)^{\ell(\tau)}$ for the coefficient of $p^{\otimes}_{\tau}$.

(d) For E_d, each summand on the right side of (11) now has the sign

$$(-1)^{\ell(\lambda)} \prod_{k \geqslant 1} (-1)^{m_k(\lambda)} \prod_{k \geqslant 1} (-1)^{\ell(\mu^{(k)})}.$$

But $\ell(\lambda) = \sum_{k \geqslant 1} m_k(\lambda)$, so that part of the sign disappears. We are left with a sign of $(-1)^{\ell(\tau)}$ for the coefficient of $p^{\otimes}_{\tau}$. □

Example 3.3. In this example, we illustrate Proposition 3.2 for the types 2^3 and 3^2. We compute:

$$P_{2^3} = p^{\otimes}_{1^6} + 2p^{\otimes}_{2^3}, \qquad P_{3^2} = p^{\otimes}_{1^6} + 3p^{\otimes}_{3^2},$$

$$H_{2^3} = \frac{p^{\otimes}_{1^6}}{2} + \frac{p^{\otimes}_{1^3 1^3}}{2} + p^{\otimes}_{2^3}, \qquad H_{3^2} = \frac{p^{\otimes}_{1^6}}{3} + \frac{p^{\otimes}_{1^4 1^2}}{2} + \frac{p^{\otimes}_{1^2 1^2 1^2}}{6} + p^{\otimes}_{2^2 1^2} + p^{\otimes}_{3^2},$$

$$E^{+}_{2^3} = -\frac{p^{\otimes}_{1^6}}{2} + \frac{p^{\otimes}_{1^3 1^3}}{2} + p^{\otimes}_{2^3}, \qquad E^{+}_{3^2} = \frac{p^{\otimes}_{1^6}}{3} - \frac{p^{\otimes}_{1^4 1^2}}{2} + \frac{p^{\otimes}_{1^2 1^2 1^2}}{6} + p^{\otimes}_{2^2 1^2} + p^{\otimes}_{3^2},$$

$$E_{2^3} = -\frac{p^{\otimes}_{1^6}}{2} + \frac{p^{\otimes}_{1^3 1^3}}{2} - p^{\otimes}_{2^3}, \qquad E_{3^2} = -\frac{p^{\otimes}_{1^6}}{3} + \frac{p^{\otimes}_{1^4 1^2}}{2} - \frac{p^{\otimes}_{1^2 1^2 1^2}}{6} + p^{\otimes}_{2^2 1^2} - p^{\otimes}_{3^2}.$$

For instance, we compute the coefficient of $p^{\otimes}_{1^4 1^2}$ in $E^{+}_{3^2}$ as follows. The type producing this term is $\tau = (1^2 1^1)$. Here, $\ell(\tau) = (-1)^2 = 1$, $\operatorname{sgn}(\tau) = (-1)^{2+1} = -1$, and $z^{\otimes}_{\tau} = z_{(2,1)} = 2$. So the required coefficient is $-1/2$.

Combining Proposition 3.2 with the remark following (9) leads to algebraic formulas for $p^{\otimes}$-expansions of various products of polysymmetric functions. In the following subsections, we supplement these algebraic formulas with combinatorial formulas that express the final answers in terms of tableau-like structures.

3.2. Rule for $p^{\otimes}_{\sigma} P_{\delta}$ and $\mathcal{M}(P, p^{\otimes})$.

Proposition 3.4. *For any type σ and block d^m,*

$$p^{\otimes}_{\sigma} P_{d^m} = \sum_{\tau} \operatorname{wt}(\sigma,\tau) p^{\otimes}_{\tau},$$

where we sum over all types τ that arise from σ by choosing a positive divisor k of d and inserting one new part of size dm/k into $\sigma|_k$; and $\operatorname{wt}(\sigma,\tau) = k$ for each such τ.

Proof. Recall from (7) that $P_{d^m} = \sum_{k|d} 1 \otimes \cdots \otimes 1 \otimes kp_{dm/k} \otimes 1 \otimes \cdots$, where $kp_{dm/k}$ occurs in position k. Multiplying $p^{\otimes}_{\sigma} = p_{\sigma|_1} \otimes p_{\sigma|_2} \otimes \cdots$ by this expression, we get

$$p^{\otimes}_{\sigma} P_{d^m} = \sum_{k|d} p_{\sigma|_1} \otimes \cdots \otimes p_{\sigma|_k} kp_{dm/k} \otimes p_{\sigma|_{k+1}} \otimes \cdots.$$

Multiplying $p_{\sigma|_k}$ by $p_{dm/k}$ produces $p_{\tau|_k}$ where τ is related to σ as described in the proposition. The resulting term $p^{\otimes}_{\tau}$ in the expansion has coefficient k. □

Fix a type $\sigma = (1^{\sigma|_1} 2^{\sigma|_2} \cdots)$ and an ordered sequence of blocks $\delta = (d_1^{m_1}, \ldots, d_s^{m_s})$. Iteration of the rule in Proposition 3.4 leads to the $p^{\otimes}$-expansion of $p^{\otimes}_{\sigma} P_{\delta}$. Starting with the tensor diagram of σ, we choose k_i dividing d_i (for $1 \leqslant i \leqslant s$) and add a new part (weighted by k_i) of size $d_i m_i / k_i$ to the current partition diagram in tensor position k_i. This produces the term $p^{\otimes}_{\tau}$ with the weight coefficient $\operatorname{wt}(\sigma,\tau) = k_i$. We get the required expansion by adding all such terms generated by making all possible choices of divisors $(k_1, \ldots, k_s)$.

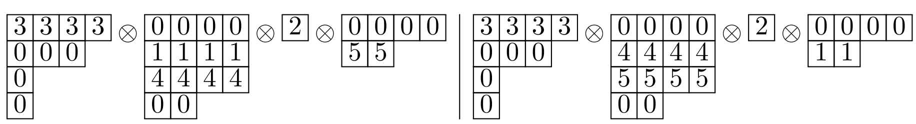
We now describe the answer in a different way, giving a combinatorial formula for the net coefficient of each $p^{\otimes}_{\tau}$ in the output. To do this, we define combinatorial structures (similar to TRHTs) that encode the required bookkeeping. We call these objects *increasing constant-row P-tableaux* (ICRPTs). Given σ and δ as above, let $\tau = (1^{\tau|_1} 2^{\tau|_2} \cdots)$ be a type such that for all k, r, $m_r(\tau|_k) \geqslant m_r(\sigma|_k)$. Intuitively, this condition means that the tensor diagram for τ arises from the tensor diagram for σ by adding new parts in various components. An ICRPT of *shape* τ and *extended content* $(\sigma; \delta)$ is a filling T of the cells in the tensor diagram of τ with integers $0, 1, \ldots, s$ satisfying these conditions:

- Each row of each $\tau|_k$ is constant (having the same value in each cell).

- For $1 \leqslant i \leqslant s$, exactly one row in the tensor diagram of τ contains the value i. If that row appears in $\tau|_k$ and has length r, then $rk = d_i m_i$.
- The cells containing 0 in T form a sub-tensor diagram that equals the tensor diagram of σ.
- For each r, k, the values in the rows of $\tau|_k$ of length r weakly increase reading down the first column.

The *weight* of the ICRPT T is $\mathrm{wt}(T) = \prod_{k \geqslant 1} k^{n_k(T)}$, where $n_k(T)$ is the number of rows in the diagram of $\tau|_k$ containing a nonzero value. Let $\mathrm{ICRPT}(\tau, (\sigma; \delta))$ be the set of fillings T satisfying these conditions. When σ is empty, we write $\mathrm{ICRPT}(\tau, \delta)$ for this set and call δ the *content* of T.

Example 3.5. For $\sigma = (1^{3,1,1}2^{4,2}4^4)$, $\tau = (1^{4,3,1,1}2^{4,4,4,2}3^1 4^{4,2})$, and $\delta = (4^2 3^1 4^1 2^4 4^2)$, the two objects in $\mathrm{ICRPT}(\tau, (\sigma; \delta))$ are shown here and explained below:



For each block $d_i^{m_i}$ of δ with $1 \leqslant i \leqslant 5$, denote by k_i the tensor factor where the brick labeled i is inserted and let r_i be the brick's length. To build the first object, choose $k_1 = 2$ with $r_1 = 4$, which gives us the brick of length 4 in the second factor. Then choose $k_2 = 3$ with $r_2 = 1$ thus inserting the box labeled 2 in the third tensor factor. We continue until choosing $k_5 = 4$, giving $r_5 = 2$, and we insert the brick labeled 5 of length 2 in the fourth tensor factor. Similarly, we construct the other diagram. Both objects have weight $1 \cdot 2 \cdot 2 \cdot 3 \cdot 4 = 48$ and thus the coefficient of $p_\tau^\otimes$ in $p_\sigma^\otimes P_\delta$ is 96. In general, the weight of $T \in \mathrm{ICRPT}(\tau, (\sigma; \delta))$ depends only on τ and σ, not δ.

Theorem 3.6. *For any type σ and sequence $\delta = (d_1^{m_1}, \ldots, d_s^{m_s})$,*

$$p_\sigma^\otimes P_\delta = \sum_\tau \left[\sum_{T \in \mathrm{ICRPT}(\tau, (\sigma; \delta))} \mathrm{wt}(T) \right] p_\tau^\otimes.$$

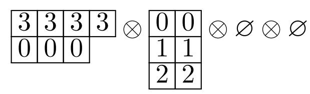
Proof. The entries in each ICRPT record the sequence of part additions caused by starting at $p_\sigma^\otimes$ and successively multiplying by $P_{d_1^{m_1}}, \ldots, P_{d_s^{m_s}}$ in accordance with Proposition 3.4. We start with the tensor diagram of σ, which is filled with 0s to indicate this is the initial shape. For $i = 1, 2, \ldots, s$, the unique row containing value i is the new row inserted into the tensor diagram due to the multiplication by $P_{d_i^{m_i}}$. This row must appear in tensor position k_i, for some k_i dividing d_i, and must have length $r = d_i m_i / k_i$. Each new row is inserted in the proper position within the k_ith diagram so that parts still appear in weakly decreasing order. If parts of length r already exist in the k_ith diagram, the new part is placed just below them. This is why values of T must increase as we scan down through equal-length parts in a given component of the tensor diagram. The net result of all the part additions is a term $p_\tau^\otimes$. Each new row added to the kth diagram multiplies this term by k, so the net coefficient of this term is $\mathrm{wt}(T)$. □

Corollary 3.7. *For all types $\sigma, \tau \Vdash n$, the coefficient of $p_\tau^\otimes$ in the $p^\otimes$-expansion of P_σ is*

$$\mathcal{M}(P, p^\otimes)_{\tau,\sigma} = \sum_{T \in \mathrm{ICRPT}(\tau, \sigma)} \mathrm{wt}(T).$$

Example 3.8. We find the $p^\otimes$-expansion of $p_{2^2 1^3}^\otimes P_{(2^2, 4^1, 2^2)}$. We compute one ICRPT step-by-step and present the rest in a figure. Here, $d_1^{m_1} = 2^2$, $d_2^{m_2} = 4^1$, and $d_3^{m_3} = 2^2$.

Choose $k_1 = 2$, $k_2 = 2$, and $k_3 = 1$. First, since $k_1 = 2$, we place a row of length $d_1m_1/k_1 = 2$ with cells labeled 1 in the second diagram. Second, since $k_2 = 2$, we place another row of length $d_2m_2/k_2 = 2$ with cells labeled 2 in the second diagram. Third, since $k_3 = 1$, we place a row of length $d_3m_3/k_3 = 4$ with cells labeled 3 in the first diagram. is added in the first tensor factor owing to the choice $k_3 = 1$. This gives the ICRPT



with weight $2 \cdot 2 \cdot 1 = 4$. Figure 1 shows all ICRPTs arising in Theorem 3.6 when $\sigma = 2^21^3$ and $\delta = (2^2, 4^1, 2^2)$. Below each ICRPT, we show the tuple (k_1, k_2, k_3) producing it and the weight of the ICRPT. Combining all of this, we find the $p^{\otimes}$-expansion of $p^{\otimes}_{1^32^2}P_{(2^2,4^1,2^2)}$ to be

$$p^{\otimes}_{1^4,4,4,3_{2^2}}+6p^{\otimes}_{1^4,4,3_{2^2,2}}+12p^{\otimes}_{1^4,3_{2^2,2,2}}+8p^{\otimes}_{1^3{2^2,2,2,2}}+4p^{\otimes}_{1^4,4,3_{2^2}4^1}+16p^{\otimes}_{1^4,3_{2^2,2}4^1}+16p^{\otimes}_{1^3{2^2,2,2}4^1}.$$

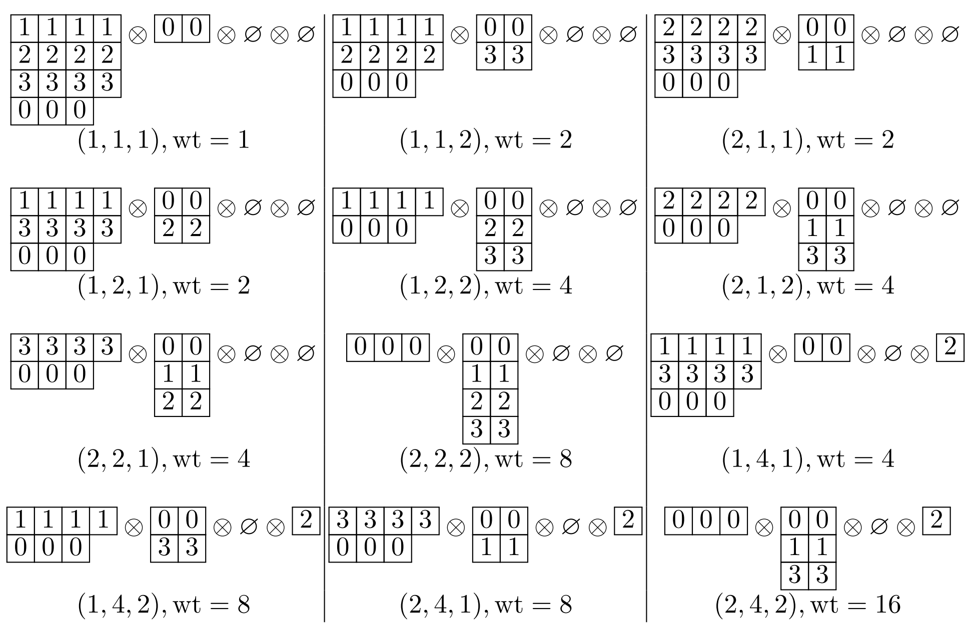


Figure 1. ICRPTs in Example 3.8.

3.3. Rule for $p^{\otimes}_{\sigma}H_{\delta}$ and $\mathcal{M}(H, p^{\otimes})$.

Proposition 3.9. *For any type σ and block d^m,*

$$p^{\otimes}_{\sigma}H_{d^m} = \sum_{\tau \Vdash d} \frac{1}{z^{\otimes}_{\tau}} p^{\otimes}_{\sigma \cup \tau^m}.$$

Proof. The formula follows immediately from Proposition 3.2(b), (9), and linearity. ☐

Here is a pictorial description of the rule in Proposition 3.9. To compute the $p^{\otimes}$-expansion of $p^{\otimes}_{\sigma}H_{d_m}$, start with the tensor diagram $\mathrm{dg}(\sigma)$. Choose any type $\tau \Vdash d$. For all $k \geqslant 1$, merge the partition diagrams $\mathrm{dg}(\sigma|_k)$ and $\mathrm{dg}(m\tau|_k)$ to get a new partition diagram in position k. Weight the new tensor diagram by $1/z^{\otimes}_{\tau} = \prod_{k\geqslant 1} z^{-1}_{\tau|_k}$. Add the resulting terms over all choices of the type τ.

Iteration of this rule leads to the $p^{\otimes}$-expansion of $p^{\otimes}_{\sigma}H_{\delta}$, where σ is a type and $\delta = (d_1^{m_1}, \ldots, d_s^{m_s})$ is a sequence of blocks. Define an *increasing constant-row H-tableau* (ICRHT) of shape τ and extended content $(\sigma;\delta)$ to be a filling T of the cells in the tensor diagram $\mathrm{dg}(\tau)$ with integers $0, 1, \ldots, s$ satisfying these conditions:

- Each row of each diagram $\mathrm{dg}(\tau|_k)$ is constant.
- The cells containing 0 in T form a sub-tensor diagram equal to $\mathrm{dg}(\sigma)$.
- For $1 \leqslant i \leqslant s$, the cells containing i in T form a sub-tensor diagram equal to $\mathrm{dg}(m_i\rho^{(i)})$ for some type $\rho^{(i)} \Vdash d_i$.
- For each r, k, the values in the rows of $\tau|_k$ of length r weakly increase reading down the first column.

Let $\mathrm{ICRHT}(\tau, (\sigma;\delta))$ be the set of all such objects. The *weight* of an object T in this set is $\prod_{i=1}^{s} 1/z^{\otimes}_{\rho^{(i)}}$. Define $\mathrm{sgn}^{+}(T) = \prod_{i=1}^{s}(-1)^{\ell(\rho^{(i)})}\,\mathrm{sgn}(\rho^{(i)})$ and $\mathrm{sgn}^{-}(T) = \prod_{i=1}^{s}(-1)^{\ell(\rho^{(i)})}$. The exponent of -1 in $\mathrm{sgn}^{-}(T)$ is the number of rows with positive labels in the tensor diagram of T. To compute $\mathrm{sgn}^{+}(T)$ from the tensor diagram we do the following: for every label $i > 0$, find the sub-tensor diagram formed by cells with label i, and divide the length of each row by m_i. Remove one cell from each row and call the total number of remaining cells c_i. Then $\mathrm{sgn}^{+}(T) = (-1)^{c_1+c_2+\cdots+c_s}$.

Theorem 3.10. *For any type σ and sequence $\delta = (d_1^{m_1}, \ldots, d_s^{m_s})$,*

$$p^{\otimes}_{\sigma}H_{\delta} = \sum_{\tau}\left[\sum_{T\in\mathrm{ICRHT}(\tau,(\sigma;\delta))}\mathrm{wt}(T)\right]p^{\otimes}_{\tau}.$$

Proof. Start with $p^{\otimes}_{\sigma}$, modeled by the tensor diagram $\mathrm{dg}(\sigma)$ with all cells containing 0. For $i = 1, 2, \ldots, s$, use Proposition 3.9 to modify the current diagram to enact multiplication by the next factor $H_{d_i^{m_i}}$. Do this by choosing a type $\rho^{(i)} \Vdash d_i$ and adding new parts given by $m_i\rho^{(i)}|_k$ to the kth diagram for all $k \geqslant 1$. Put i in all cells in these new parts to record which factor created them. As before, new parts of size r are placed immediately below existing parts of size r in each diagram. This explains the weakly increasing condition in the definition of ICRHTs. The factor $\mathrm{wt}(T)$ accounts for all the weights produced by each insertion step. Making these choices in all possible ways leads to the weighted set $\mathrm{ICRHT}(\tau, (\sigma;\delta))$ appearing in the theorem statement. ☐

Corollary 3.11. *For all types $\sigma, \tau \Vdash n$, the coefficient of $p^{\otimes}_{\tau}$ in the $p^{\otimes}$-expansion of H_{σ} is*

$$\mathcal{M}(H, p^{\otimes})_{\tau,\sigma} = \sum_{T\in\mathrm{ICRHT}(\tau,\sigma)}\mathrm{wt}(T).$$

Example 3.12. In this example, we compute the coefficient of $p^{\otimes}_{\tau}$ in the $p^{\otimes}$-expansion of H_{σ}, where $\tau = (3^{2,1}2^{2,2,1}1^{4})$ and $\sigma = (9^{1}6^{1}4^{1}2^{2})$. We construct the following six objects, each labeled by the tuple of types $(\rho^{(1)} \Vdash 9, \rho^{(2)} \Vdash 6, \rho^{(3)} \Vdash 4, \rho^{(4)} \Vdash 2)$ that produced it.

$T_1 = \begin{array}{\|c\|c\|c\|c\|}\hline 4&4&4&4\\\hline\end{array} \otimes \begin{array}{ll} 2&2\\ 3&3\\ 2& \end{array} \otimes \begin{array}{ll} 1&1\\ 1& \end{array}$ $((3^{2,1}),(2^{2,1}),(2^2),(1^2))$	$T_2 = \begin{array}{\|c\|c\|c\|c\|}\hline 3&3&3&3\\\hline\end{array} \otimes \begin{array}{ll} 2&2\\ 4&4\\ 2& \end{array} \otimes \begin{array}{ll} 1&1\\ 1& \end{array}$ $((3^{2,1}),(2^{2,1}),(1^4),(2^1))$
$T_3 = \begin{array}{\|c\|c\|c\|c\|}\hline 4&4&4&4\\\hline\end{array} \otimes \begin{array}{ll} 1&1\\ 3&3\\ 1& \end{array} \otimes \begin{array}{ll} 2&2\\ 1& \end{array}$ $((3^1 2^{2,1}),(3^2),(2^2),(1^2))$	$T_4 = \begin{array}{\|c\|c\|c\|c\|}\hline 3&3&3&3\\\hline\end{array} \otimes \begin{array}{ll} 1&1\\ 4&4\\ 1& \end{array} \otimes \begin{array}{ll} 2&2\\ 1& \end{array}$ $((3^1 2^{2,1}),(3^2),(1^4),(2^1))$
$T_5 = \begin{array}{\|c\|c\|c\|c\|}\hline 2&2&2&2\\\hline\end{array} \otimes \begin{array}{ll} 3&3\\ 4&4\\ 2& \end{array} \otimes \begin{array}{ll} 1&1\\ 1& \end{array}$ $((3^{2,1}),(2^1 1^4),(2^2),(2^1))$	$T_6 = \begin{array}{\|c\|c\|c\|c\|}\hline 1&1&1&1\\\hline\end{array} \otimes \begin{array}{ll} 3&3\\ 4&4\\ 1& \end{array} \otimes \begin{array}{ll} 2&2\\ 1& \end{array}$ $((3^1 2^1 1^4),(3^2),(2^2),(2^1))$

The weight of the first ICRHT is $\mathrm{wt}(T_1) = z_{(2,1)}^{-1} z_{(2,1)}^{-1} z_{(2)}^{-1} z_{(2)}^{-1} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$. Similarly, all six ICRHTs shown here have weight $\frac{1}{16}$. So $\mathcal{M}(H, p^{\otimes})_{\tau,\sigma} = \frac{3}{8}$.

Remark 3.13. In general, not all objects in $\mathrm{ICRHT}(\tau,\sigma)$ have the same weight. For example, let $\tau = (1^{1,1,1}2^{1,1,1})$ and $\sigma = (5^1 4^1)$. Two objects in $\mathrm{ICRHT}(\tau,\sigma)$ are $T' = \begin{array}{c} 1\\ 1\\ 1 \end{array} \otimes \begin{array}{c} 1\\ 2\\ 2 \end{array}$ and $T'' = \begin{array}{c} 1\\ 2\\ 2 \end{array} \otimes \begin{array}{c} 1\\ 1\\ 2 \end{array}$, arising from type choices $((1^{1,1,1}2^1),(2^{1,1}))$ for T' and $((1^1 2^{1,1}),(1^{1,1}2^1))$ for T''. We compute $\mathrm{wt}(T') = z_{(1^3)}^{-1} z_{(1)}^{-1} z_{(1^2)}^{-1} = 1/12$ and $\mathrm{wt}(T'') = z_{(1)}^{-1} z_{(1^2)}^{-1} z_{(1^2)}^{-1} z_{(1)}^{-1} = 1/4$.

3.4. Rule for $p_\sigma^{\otimes} E_\delta^+$, $p_\sigma^{\otimes} E_\delta$, $\mathcal{M}(E^+, p^{\otimes})$, and $\mathcal{M}(E, p^{\otimes})$. The next three results follow immediately by adapting the proofs in the previous subsection, keeping in mind Proposition 3.2(c) and (d).

Proposition 3.14. *For any type σ and block d^m,*

$$p_\sigma^{\otimes} E_{d^m}^+ = \sum_{\tau \Vdash d} \frac{(-1)^{\ell(\tau)}\operatorname{sgn}(\tau)}{z_\tau^{\otimes}} p_{\sigma\cup\tau^m}^{\otimes} \qquad \text{and} \qquad p_\sigma^{\otimes} E_{d^m} = \sum_{\tau \Vdash d} \frac{(-1)^{\ell(\tau)}}{z_\tau^{\otimes}} p_{\sigma\cup\tau^m}^{\otimes}.$$

Theorem 3.15. *For any type σ and sequence $\delta = (d_1^{m_1}, \ldots, d_s^{m_s})$,*

$$p_\sigma^{\otimes} E_\delta^+ = \sum_\tau \left[\sum_{T \in \mathrm{ICRHT}(\tau,(\sigma;\delta))} \operatorname{sgn}^+(T)\,\mathrm{wt}(T) \right] p_\tau^{\otimes};$$

$$p_\sigma^{\otimes} E_\delta = \sum_\tau \left[\sum_{T \in \mathrm{ICRHT}(\tau,(\sigma;\delta))} \operatorname{sgn}^-(T)\,\mathrm{wt}(T) \right] p_\tau^{\otimes}.$$

Corollary 3.16. *For all types $\sigma, \tau \Vdash n$, the coefficient of $p_\tau^{\otimes}$ in the $p^{\otimes}$-expansion of E_σ^+ is*

$$\mathcal{M}(E^+, p^{\otimes})_{\tau,\sigma} = \sum_{T \in \mathrm{ICRHT}(\tau,\sigma)} \operatorname{sgn}^+(T)\,\mathrm{wt}(T).$$

The coefficient of $p_\tau^{\otimes}$ in the $p^{\otimes}$-expansion of E_σ is

$$\mathcal{M}(E, p^{\otimes})_{\tau,\sigma} = \sum_{T \in \mathrm{ICRHT}(\tau,\sigma)} \operatorname{sgn}^-(T)\,\mathrm{wt}(T).$$

Example 3.17. We continue with Example 3.12 where $\tau = (3^{2,1}2^{2,2,1}1^4)$ and $\sigma = (9^1 6^1 4^1 2^2)$. For i between 1 and 6, $\operatorname{sgn}^-(T_i) = (-1)^6$ since there are 6 rows in $\mathrm{dg}(\tau)$, all filled with positive labels. So the coefficient of $p_\tau^{\otimes}$ in the $p^{\otimes}$-expansion of E_σ is

$\frac{3}{8}$. On the other hand, $\mathrm{sgn}^+(T_1) = \mathrm{sgn}^+(T_3) = (-1)^{10-6} = 1$, while $\mathrm{sgn}^+(T_2) = \mathrm{sgn}^+(T_4) = \mathrm{sgn}^+(T_5) = \mathrm{sgn}^+(T_6) = (-1)^{11-6} = -1$. So the coefficient of $p_\tau^\otimes$ in the $p^\otimes$-expansion of E_σ^+ is $-\frac{1}{8}$.

4. Expansions in the $m^\otimes$ basis

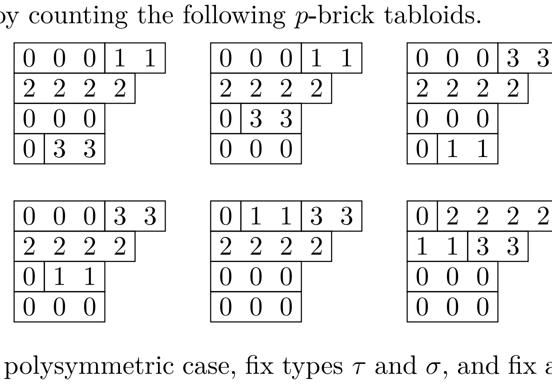
4.1. Rule for $m_\sigma^\otimes P_\delta$ and $\mathcal{M}(P, m^\otimes)$. Before stating our combinatorial rule for the $m^\otimes$-expansion of $m_\sigma P_\delta$, we describe an analogous rule (cf. [3]) for the monomial expansion of $m_\mu p_\alpha$, where $\mu = (\mu_1, \ldots, \mu_\ell)$ is an integer partition and $\alpha = (\alpha_1, \ldots, \alpha_s)$ is a sequence of positive integers. We create s horizontal bricks, namely, a brick containing α_1 boxes labeled 1, a brick containing α_2 boxes labeled 2, …, and a brick containing α_s boxes labeled s. We also create ℓ horizontal bricks of lengths $\mu_1, \ldots, \mu_\ell$ with all boxes in these bricks labeled 0. For a given partition λ, draw the diagram of λ and place these bricks in this diagram so that every box in the diagram is covered by exactly one brick, and the brick labels strictly increase reading left to right in each row. (Strict increase means that a row can contain at most one brick labeled 0.) Two bricks of the same length, with boxes labeled 0, are considered indistinguishable. Call such a configuration a *p-brick tabloid* of *shape* λ and *extended content* $(\mu; \alpha)$.

Proposition 4.1. *For any partitions λ, μ and list of positive integers α, the coefficient of m_λ in $m_\mu p_\alpha$ is the number of p-brick tabloids of shape λ and extended content $(\mu; \alpha)$.*

Proof. The coefficient of m_λ in the m-expansion of $m_\mu p_\alpha$ equals the coefficient of the particular monomial $\mathbf{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k} \cdots$ in the polynomial $m_\mu(\mathbf{x}) p_\alpha(\mathbf{x})$. The p-brick tabloids described in the proposition record all the ways the monomial $\mathbf{x}^\lambda$ can be generated by choosing particular monomials from each factor $m_\mu(\mathbf{x})$, $p_{\alpha_1}(\mathbf{x})$, …, $p_{\alpha_s}(\mathbf{x})$ and multiplying those monomials together in accordance with the distributive law.

In more detail, the placement of all the bricks labeled 0 in distinct rows $i_1, i_2, \ldots, i_\ell$ records a monomial $x_{i_1}^{\mu_1} x_{i_2}^{\mu_2} \cdots x_{i_\ell}^{\mu_\ell}$ coming from $m_\mu(\mathbf{x})$. The placement of the brick of length α_1 labeled 1 in some row j_1 records a monomial $x_{j_1}^{\alpha_1}$ coming from $p_{\alpha_1}(\mathbf{x})$. The placement of the brick of length α_2 labeled 2 in some row j_2 records a monomial $x_{j_2}^{\alpha_2}$ coming from $p_{\alpha_2}(\mathbf{x})$. And so on. Since the p-brick tabloid covers each cell in row k of the diagram of λ with exactly one brick, we see that the power of x_k in the generated monomial is λ_k for all k, as needed. Brick labels increase from left to right in each row since we place the bricks in the diagram in the same order that the choices of monomials are made from $m_\mu(\mathbf{x})$ (bricks labeled 0), $p_{\alpha_1}(\mathbf{x})$ (brick labeled 1), …, $p_{\alpha_s}(\mathbf{x})$ (brick labeled s). □

Example 4.2. Let $\mu = (3, 3, 1)$ and $\alpha = (2, 4, 2)$. We find the coefficient of $m_{(5,4,3,3)}$ in $m_\mu p_\alpha$ to be 6 by counting the following p-brick tabloids.



Turning to the polysymmetric case, fix types τ and σ, and fix an ordered sequence of blocks $\delta = (d_1^{e_1}, \ldots, d_s^{e_s})$. We seek the coefficient of $m_\tau^\otimes$ in the $m^\otimes$-expansion of

$m^{\otimes}_{\sigma} P_{\delta}$. We describe this coefficient as the weighted sum of *P-tensor brick tabloids* constructed as follows. We fill the tensor diagram of τ with certain horizontal bricks labeled $0, 1, 2, \dots, s$ so that every box is covered by exactly one brick. The brick labels in each row of each component diagram must strictly increase reading left to right. In each tensor component k, we use $\ell(\sigma|_k)$ bricks labeled 0, with lengths given by the parts of the partition $\sigma|_k$. Next, fix i between 1 and s. Recall from (7) that $P_{d_i^{e_i}} = \sum_{k_i|d_i} k_i p_{d_i e_i/k_i}(\mathbf{x}_{k_i *})$. When building a particular P-tensor brick tabloid, we may use exactly one brick labeled i, chosen as follows: pick a positive divisor k_i of d_i; make a brick labeled i containing $d_i e_i/k_i$ cells; and place that brick in the k_ith component diagram of $\mathrm{dg}(\tau)$. Every positively-labeled brick placed in component diagram k has a *weight* of k, while bricks labeled 0 have weight 1.

Any filling T of $\mathrm{dg}(\tau)$ satisfying all rules stated here is called a *P-tensor brick tabloid (PTBT)* of *shape* τ and *extended content* $(\sigma;\delta)$. Let $\mathrm{PTBT}(\tau,(\sigma;\delta))$ be the set of all such objects. When σ is empty, we write $\mathrm{PTBT}(\tau,\delta)$ and speak of PTBT of shape τ and content δ. The weight of a PTBT T, written $\mathrm{wt}(T)$, is the product of the weights of all the bricks in it. Equivalently, if component diagram k in T contains $n_k(T)$ bricks with positive labels, then $\mathrm{wt}(T) = \prod_{k\geqslant 1} k^{n_k(T)}$.

Theorem 4.3. *For any type σ and sequence of blocks δ,*

$$m^{\otimes}_{\sigma} P_{\delta} = \sum_{\tau} \left[\sum_{T \in \mathrm{PTBT}(\tau,(\sigma;\delta))} \mathrm{wt}(T) \right] m^{\otimes}_{\tau}.$$

Proof. We expand $m^{\otimes}_{\sigma}(\mathbf{x}_{**}) P_{\delta}(\mathbf{x}_{**})$ by choosing one monomial from each factor, multiplying those monomials, and adding over all possible choices of monomials. The weighted P-tensor brick tabloids in $\mathrm{PTBT}(\tau,(\sigma;\delta))$ record all possible ways the monomial $\mathbf{x}^{\tau} = \mathbf{x}_{1*}^{\tau|_1} \mathbf{x}_{2*}^{\tau|_2} \cdots \mathbf{x}_{k*}^{\tau|_k} \cdots$ can arise by such choices. The choice of a monomial from $m^{\otimes}_{\sigma}(\mathbf{x}_{**}) = \prod_{k\geqslant 1} m_{\sigma|_k}(\mathbf{x}_{k*})$ is recorded by the placement of all the bricks labeled 0. For $1 \leqslant i \leqslant s$, the choice of a monomial from $P_{d_i^{e_i}}(\mathbf{x}_{**})$ is recorded by the placement of the brick labeled i in some component diagram k_i, including the appropriate weight k_i. The monomial choices correspond bijectively to the objects in $\mathrm{PTBT}(\tau,(\sigma;\delta))$ as explained in the proof of Proposition 4.1. ☐

Corollary 4.4. *For all types $\sigma, \tau \Vdash n$, the coefficient of $m^{\otimes}_{\tau}$ in the $m^{\otimes}$-expansion of P_{σ} is*

$$\mathcal{M}(P, m^{\otimes})_{\tau,\sigma} = \sum_{T \in \mathrm{PTBT}(\tau,\sigma)} \mathrm{wt}(T).$$

Example 4.5. We compute the $m^{\otimes}$-expansion of $P_{2^2 2^2}$ by drawing the following PTBTs. Each PTBT T is labeled by the divisor pair (k_1, k_2) that produced it and its weight, namely $\mathrm{wt}(T) = k_1 k_2$.

[1 1 1 1] / [2 2 2 2] $\otimes \varnothing$	[2 2 2 2] / [1 1 1 1] $\otimes \varnothing$	[1 1 1 1 | 2 2 2 2] $\otimes \varnothing$	[1 1 1 1] $\otimes$ [2 2]
$(1,1), \mathrm{wt}=1$	$(1,1), \mathrm{wt}=1$	$(1,1), \mathrm{wt}=1$	$(1,2), \mathrm{wt}=2$
[2 2 2 2] $\otimes$ [1 1]	$\varnothing \otimes$ [1 1] / [2 2]	$\varnothing \otimes$ [2 2] / [1 1]	$\varnothing \otimes$ [1 1 | 2 2]
$(2,1), \mathrm{wt}=2$	$(2,2), \mathrm{wt}=4$	$(2,2), \mathrm{wt}=4$	$(2,2), \mathrm{wt}=4$

This gives $P_{2^2 2^2} = 2m^{\otimes}_{1^4 1^4} + m^{\otimes}_{1^8} + 4m^{\otimes}_{2^2 1^4} + 8m^{\otimes}_{2^2 2^2} + 4m^{\otimes}_{2^4}$.

4.2. Rule for $m^{\otimes}_{\sigma}H_{\delta}$ and $\mathcal{M}(H, m^{\otimes})$. In [4], the authors show that the coefficient of $m^{\otimes}_{\tau}$ in H_σ is the number of arrangements of one type into another. They write $a_{\tau,\sigma}$ for what we call $\mathcal{M}(H, m^{\otimes})_{\tau,\sigma}$, so $H_\sigma = \sum_{\tau \Vdash |\sigma|} a_{\tau,\sigma} m^{\otimes}_{\tau}$. Here we develop alternate combinatorial formulas for these coefficients based on tensor versions of brick tabloids, by extending classical results for the symmetric case (cf. [3]) to the polysymmetric case.

Let μ and λ be partitions, and let $\alpha = (\alpha_1, \ldots, \alpha_s)$ be a sequence of positive integers. Define an *h-brick tabloid of shape λ and extended content* $(\mu; \alpha)$ as follows. Construct α_i 1×1 bricks labeled i and $\ell(\mu)$ bricks labeled 0 of lengths $\mu_1, \mu_2, \ldots$. An h-brick tabloid is a non-overlapping cover of $\mathrm{dg}(\lambda)$ using these bricks such that each brick labeled 0 appears at most once in a row while brick labels weakly increase along rows.

Example 4.6. The next picture shows all h-brick tabloids of shape $(4, 4)$ with extended content $((2, 1); (2, 1, 2))$.

$$
\begin{array}{|cc|c|c|}\hline 0&0&1&1\\\hline 0&2&3&3\\\hline\end{array}\quad
\begin{array}{|cc|c|c|}\hline 0&0&1&3\\\hline 0&1&2&3\\\hline\end{array}\quad
\begin{array}{|cc|c|c|}\hline 0&0&3&3\\\hline 0&1&1&2\\\hline\end{array}\quad
\begin{array}{|cc|c|c|}\hline 0&0&2&3\\\hline 0&1&1&3\\\hline\end{array}\quad
\begin{array}{|cc|c|c|}\hline 0&0&1&2\\\hline 0&1&3&3\\\hline\end{array}
$$

$$
\begin{array}{|c|c|c|c|}\hline 0&2&3&3\\\hline 0&0&1&1\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 0&1&2&3\\\hline 0&0&1&3\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 0&1&1&2\\\hline 0&0&3&3\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 0&1&1&3\\\hline 0&0&2&3\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 0&1&3&3\\\hline 0&0&1&2\\\hline\end{array}
$$

There are 10 h-brick tabloids, and 10 is the coefficient of $m_{(4,4)}$ in the m-expansion of $m_{(2,1)}h_{(2,1,2)}$. This illustrates the result proved next.

Proposition 4.7. *Let λ, μ be partitions and $\alpha = (\alpha_1, \ldots, \alpha_s)$ be a sequence of positive integers. Then the coefficient of m_λ in $m_\mu h_\alpha$ is the number of h-brick tabloids of shape λ and extended content $(\mu; \alpha)$.*

Proof. As in the proof of Proposition 4.1, the coefficient of m_λ in the m-expansion of $m_\mu h_\alpha$ equals the coefficient of $\mathbf{x}^\lambda$ in $m_\mu(\mathbf{x})h_\alpha(\mathbf{x})$. In turn, this coefficient is the number of ordered factorizations of $\mathbf{x}^\lambda$ of the form $f_0 f_1 \cdots f_s$, where f_0 is a monomial in $m_\mu(\mathbf{x})$ and f_j is a monomial in $h_{\alpha_j}(\mathbf{x})$ for $j = 1, 2, \ldots, s$.

There is a bijection between the set of such factorizations and the set of h-brick tabloids described in the proposition. On one hand, given such an h-brick tabloid T, let $n_{ij}(T)$ be the number of cells in row i of T covered by a brick labeled j. Define $f_j = \prod_{i \geqslant 1} x_i^{n_{ij}(T)}$ for $j = 0, 1, 2, \ldots, s$. By the rules for the brick sizes, f_0 is one of the monomials in $m_\mu(\mathbf{x})$ and f_j is a monomial of degree α_j, which is one of the terms in $h_{\alpha_j}(\mathbf{x})$. Since every cell in $\mathrm{dg}(\lambda)$ is covered by exactly one brick, $f_0 f_1 \cdots f_s = \mathbf{x}^\lambda$ follows.

The inverse bijection acts as follows. Given an ordered factorization $f_0 f_1 \cdots f_s$ of x^λ, make the associated h-brick tabloid as follows. Write $f_j = \prod_{i \geqslant 1} x_i^{r_{ij}}$ for $j = 0, 1, 2, \ldots, s$. Since brick labels weakly increase in each row, with at most one brick labeled 0 in each row, there is exactly one way to cover $\mathrm{dg}(\lambda)$ with bricks such that the resulting tabloid has r_{ij} cells in row i covered by a brick labeled j for all i, j. $\square$

By putting $\mu = \varnothing$ and $\alpha = \nu$ (a partition) in Proposition 4.7, we can find the coefficient of m_λ in h_ν using objects of *shape* λ and *content* ν.

Example 4.8. The coefficient of $m_{(3,2)}$ in $h_{(2,2,1)}$ is 5, which is the number of h-brick tabloids of shape $(3, 2)$ and content $(2, 2, 1)$ shown below.

$$
\begin{array}{c|c|c|c|c}
\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array} &
\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2}\end{array} &
\begin{array}{|c|c|c|}\hline 2&2&3\\\hline 1&1\\\cline{1-2}\end{array} &
\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 1&3\\\cline{1-2}\end{array} &
\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 1&2\\\cline{1-2}\end{array} \\
x_1^2 \cdot x_1x_2 \cdot x_2 & x_1^2 \cdot x_2^2 \cdot x_1 & x_2^2 \cdot x_1^2 \cdot x_1 & x_1x_2 \cdot x_1^2 \cdot x_2 & x_1x_2 \cdot x_1x_2 \cdot x_1
\end{array}
$$

The ordered factorization under each h-brick tabloid is computed as in the proof: we have x_i appearing in the jth factor as many times as the label j appears in row i. For instance, for the leftmost h-brick tabloid, the first factor is x_1^2 as 1 appears twice in the first row. The second factor is x_1x_2 as 2 appears in the first and the second row. The third factor is x_2 because 3 appears once in row 2.

Remark 4.9. It is known that the coefficient of m_ν in h_λ and the coefficient of m_λ in h_ν are the same. This can be proved by a dual combinatorial construction illustrated in the next example, where the coefficient of m_λ in h_ν is found using objects of *shape* ν and *content* λ.

Example 4.10. The coefficient of $m_{(3,2)}$ in the expansion of $h_{(2,2,1)}$ is 5, which is the number of h-brick tabloids of shape $(2,2,1)$ and content $(3,2)$ shown below.

1 1	1 1	2 2	1 2	1 2
1 2	2 2	1 1	1 1	1 2
2	1	1	2	1
$x_1^2 \cdot x_1x_2 \cdot x_2$	$x_1^2 \cdot x_2^2 \cdot x_1$	$x_2^2 \cdot x_1^2 \cdot x_1$	$x_1x_2 \cdot x_1^2 \cdot x_2$	$x_1x_2 \cdot x_1x_2 \cdot x_1$

In this case, we convert h-brick tabloids to ordered factorizations as follows. For each brick labeled i in row j, we include a copy of x_i in the jth factor.

As seen in the last two examples, we have two bijections mapping h-brick tabloids to ordered factorizations. The first bijection forms the jth factor by recording the rows containing the bricks labeled j. The second bijection forms the jth factor by recording the brick labels in row j. By composing these maps, we get a bijective proof that $\mathcal{M}(h,m)_{\lambda,\nu} = \mathcal{M}(h,m)_{\nu,\lambda}$.

We now extend these results to the polysymmetric case. The objects here are versions of h-brick tabloids for tensor product diagrams. Let τ and σ be types and $\delta = (d_1^{r_1}, \ldots, d_s^{r_s})$ be a sequence of blocks. We define an *H-tensor brick tabloid (HTBT) of shape τ and extended content $(\sigma;\delta)$* as a filling of $\mathrm{dg}(\tau)$ built as follows. We first choose partitions $\lambda^{(i)}$ of d_i for $i = 1, 2, \ldots, s$. For each $k \geqslant 1$, we fill the kth component of $\mathrm{dg}(\tau)$ using these rules:

- Make $m_k(\lambda^{(i)})$ bricks labeled i, each of length r_i and height 1. Make $\ell(\sigma|_k)$ bricks labeled 0, each of height 1 and with lengths given by the parts of $\sigma|_k$.
- Cover $\mathrm{dg}(\tau|_k)$ with these bricks so that labels weakly increase in each row, and each row has at most one brick labeled 0.

Denote this set of objects by $\mathrm{HTBT}(\tau, (\sigma;\delta))$. This definition constructs objects similar to h-brick tabloids but with bricks scaled horizontally according to the multiplicity r_i of the block $d_i^{r_i}$. The degree d_i of the block determines the number of such bricks we make. More specifically, if the kth tensor diagram has $m_{k,i}$ bricks labeled i, then $\sum_{k\geqslant 1} k m_{k,i} = d_i$ for $i = 1, 2, \ldots, s$, where $m_{k,i} = m_k(\lambda^{(i)})$.

Theorem 4.11. *Let τ and σ be types and $\delta = (d_1^{r_1}, \ldots, d_s^{r_s})$ be a sequence of blocks. Then the coefficient of $m_\tau^\otimes$ in the $m^\otimes$-expansion of $m_\sigma^\otimes H_\delta$ is $|\mathrm{HTBT}(\tau, (\sigma;\delta))|$.*

Proof. Recall from Proposition 2.9(b) that $H_{d^r} = \sum_{\lambda\vdash d} \prod_{k\geqslant 1} h_{m_k(\lambda)}(\mathbf{x}_{k*}^r)$, so

$$m_\sigma^\otimes H_\delta = \sum_{\lambda^{(1)}\vdash d_1} \cdots \sum_{\lambda^{(s)}\vdash d_s} \prod_{k\geqslant 1} \left[m_{\sigma|_k}(\mathbf{x}_{k*}) \prod_{i=1}^{s} h_{m_k(\lambda^{(i)})}(\mathbf{x}_{k*}^{r_i}) \right]. \tag{12}$$

The kth component of the tensor diagram $\mathrm{dg}(\tau)$ is the partition $\tau|_k$. We fill this partition with bricks (using the rules above) to record all possible ways of getting the monomial $\mathbf{x}_{k*}^{\tau|_k}$ as part of the expression in (12). For a given choice of $\lambda^{(1)}, \ldots, \lambda^{(s)}$

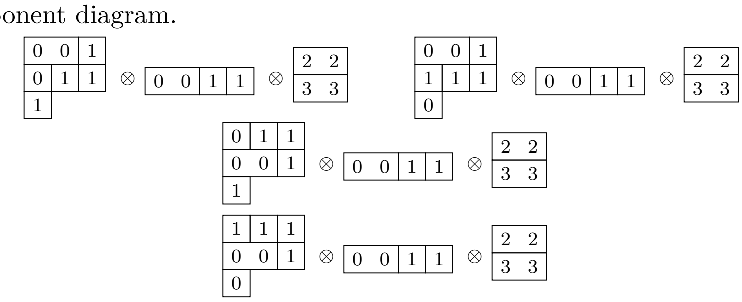
indexing the summands in (12) and for a given k, the part of the expression involving the variables $\mathbf{x}_{k*}$ is

$$m_{\sigma|_k}(\mathbf{x}_{k*})h_{m_k(\lambda^{(1)})}(\mathbf{x}_{k*}^{r_1})h_{m_k(\lambda^{(2)})}(\mathbf{x}_{k*}^{r_2})\dots h_{m_k(\lambda^{(s)})}(\mathbf{x}_{k*}^{r_s}).$$

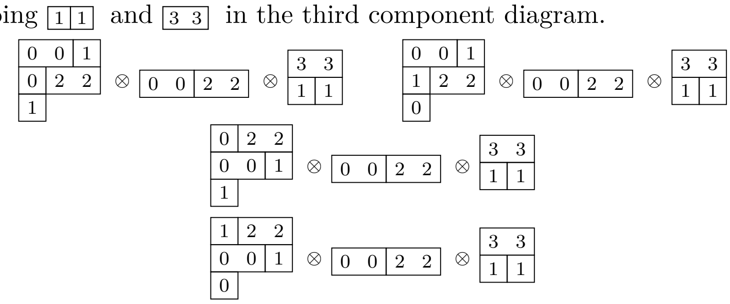
The result then follows as in the proof of Proposition 4.7, noting that raising the variables $\mathbf{x}_{k*}$ to the power r_i can be modeled by horizontally scaling 1×1 bricks to become bricks of length r_i. □

Example 4.12. Let $\tau = 3^2 3^2 2^4 1^3 1^3 1$, $\sigma = 2^2 1^2 1$ and $\delta = (8, 3^2, 3^2)$. We compute the coefficient of $m_\tau^\otimes$ in $m_\sigma^\otimes H_\delta$ to be 24 as follows.

(1) Corresponding to the choice of partitions $(2,2,1,1,1,1) \vdash 8$, $(3) \vdash 3$ and $(3) \vdash 3$, we get 8 objects in $\mathrm{HTBT}(\tau, (\sigma;\delta))$. We list 4 objects below, and the remaining 4 are obtained by swapping the $\boxed{2\ 2}$ and $\boxed{3\ 3}$ in the third component diagram.



(2) Now, we make a choice of partitions $(3,3,1,1) \vdash 8$, $(2,1) \vdash 3$ and $(3) \vdash 3$ which again gives us 8 objects. We list four objects and the rest can be obtained by swapping $\boxed{1|1}$ and $\boxed{3\ 3}$ in the third component diagram.



(3) For the choice of partitions $(3,3,1,1) \vdash 8$, $(3) \vdash 3$ and $(2,1) \vdash 3$, we construct 8 objects as in part (2) where the labels 2 and 3 are swapped.

(4) It is routine to check that the choices of partitions in (1), (2), and (3) are the only possibilities leading to collections of bricks that can fill $\mathrm{dg}(\tau)$ following the rules for HTBTs.

Corollary 4.13. *For all types $\tau, \sigma \Vdash n$, the coefficient of $m_\tau^\otimes$ in the $m^\otimes$-expansion of H_σ is*

$$a_{\tau\sigma} = \mathcal{M}(H, m^\otimes)_{\tau,\sigma} = |\,\mathrm{HTBT}(\tau,\sigma)|.$$

4.3. Rules for $m_\sigma^\otimes E_\delta^+$, $m_\sigma^\otimes E_\delta$, $\mathcal{M}(E^+, m^\otimes)$, and $\mathcal{M}(E, m^\otimes)$. In this section, we start by finding the m-expansion of the symmetric polynomial $m_\mu e_\alpha$. We then use similar ideas to obtain the $m^\otimes$-expansions of the polysymmetric functions E_σ and E_σ^+.

Define an *e-brick tabloid* to be an h-brick tabloid with the added condition that in each row, all bricks have distinct labels.

Example 4.14. The e-brick tabloids of shape $(4,4)$ with extended content given by $((2,1);(2,1,2))$ are

0	1	2	3		0	0	1	3
0	0	1	3		0	1	2	3

which are 2 of the 10 h-brick tabloids from Example 4.6.

Proposition 4.15. *Let λ, μ be partitions and $\alpha = (\alpha_1, \ldots, \alpha_s)$ be a sequence of positive integers. The coefficient of m_λ in $m_\mu e_\alpha$ is the number of e-brick tabloids of shape λ and extended content $(\mu; \alpha)$.*

Proof. We need to find the coefficient of $\mathbf{x}^\lambda$ in $m_\mu(\mathbf{x})e_\alpha(\mathbf{x})$. This is the number of ordered square-free factorizations of $\mathbf{x}^\lambda$, which have the form $\mathbf{x}^\lambda = f_0 f_1 \ldots f_s$ where f_0 is a monomial appearing in $m_\mu(\mathbf{x})$ and f_j is a monomial appearing in $e_{\alpha_j}(\mathbf{x})$ for $j = 1, 2, \ldots, s$. We proceed similarly to the proof of Proposition 4.7. For each $j > 0$, the condition that each row has at most one brick labeled j ensures that f_j is a square-free monomial of degree α_j and thus appears in $e_{\alpha_j}(\mathbf{x})$. The factor $f_0 = \prod_{i \geqslant 1} x_i^{r_i}$ is recorded in the brick tabloid by putting a brick of length r_i with label 0 in row i. For $j \geqslant 1$, if $f_j = x_{i_1} x_{i_2} \ldots x_{i_{\alpha_j}}$, then we put one brick labeled j in each row i_1, i_2, …, i_{α_j}. This gives us the e-brick tabloid recording the given square-free factorization of $\mathbf{x}^\lambda$. □

To get the analogue of Theorem 4.11, we define *E-tensor brick tabloids* (ETBTs) of shape τ and extended content $(\sigma; \delta)$, where τ and σ are types and $\delta = (d_1^{r_1}, \ldots, d_k^{r_k})$ is a sequence of blocks. To build such an ETBT, say T, first choose partitions $\lambda^{(i)}$ of d_i. For $k \geqslant 1$, the kth component of $\mathrm{dg}(\tau)$ is filled as follows.

- Make $m_k(\lambda^{(i)})$ bricks of length r_i and height 1, each with label i. Make $\ell(\sigma|_k)$ bricks of height 1 and label 0 with lengths corresponding to the parts of $\sigma|_k$.
- Cover $\mathrm{dg}(\tau|_k)$ with these bricks subject to the condition that brick labels increase strictly in each row.

Define the *sign* of the E-tensor brick tabloid thus constructed to be $\mathrm{sgn}(T) = \prod_{i=1}^{k} (-1)^{\ell(\lambda^{(i)})}$. Denote the set of such objects by $\mathrm{ETBT}(\tau, (\sigma; \delta))$. The power of -1 in $\mathrm{sgn}(T)$ is the total number of bricks in T with a positive label.

Theorem 4.16. *Let τ and σ be types and δ be a sequence of blocks.*
(a) The coefficient of $m_\tau^\otimes$ in the $m^\otimes$-expansion of $m_\sigma^\otimes E_\delta^+$ is $\sum_{T \in \mathrm{ETBT}(\tau, (\sigma;\delta))} 1 = |\mathrm{ETBT}(\tau, (\sigma; \delta))|$.
(b) The coefficient of $m_\tau^\otimes$ in the $m^\otimes$-expansion of $m_\sigma^\otimes E_\delta$ is $\sum_{T \in \mathrm{ETBT}(\tau, (\sigma;\delta))} \mathrm{sgn}(T)$.

Proof. We adapt the proof of Theorem 4.11. For (a), Equation (12) becomes

$$m_\sigma^\otimes E_\delta^+ = \sum_{\lambda^{(1)} \vdash d_1} \cdots \sum_{\lambda^{(s)} \vdash d_s} \prod_{k \geqslant 1} \left[m_{\sigma|_k}(\mathbf{x}_{k*}) \prod_{i=1}^{s} e_{m_k(\lambda^{(i)})}(\mathbf{x}_{k*}^{r_i}) \right]. \tag{13}$$

The part of this expression involving the variables $\mathbf{x}_{k*}$ is

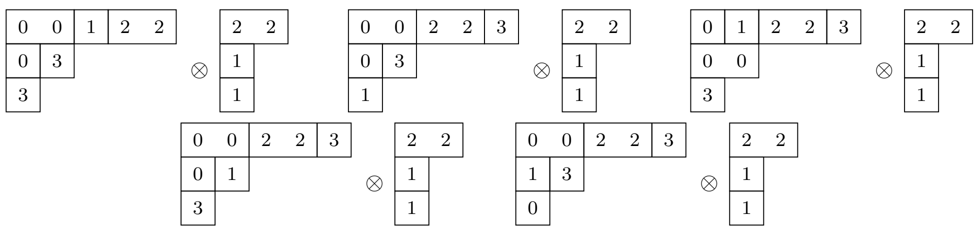
$$m_{\sigma|_k}(\mathbf{x}_{k*}) e_{m_k(\lambda^{(1)})}(\mathbf{x}_{k*}^{r_1}) e_{m_k(\lambda^{(2)})}(\mathbf{x}_{k*}^{r_2}) \ldots e_{m_k(\lambda^{(s)})}(\mathbf{x}_{k*}^{r_s}).$$

Choosing monomials from these factors corresponds to filling $\mathrm{dg}(\tau|_k)$ with bricks according to the rules in the definition of ETBTs. In particular, brick labels strictly increase in each row since the monomials in $e_m(\mathbf{x}_{k*})$ are square-free.

Part (b) is proved similarly, but now the right side of (13) includes the sign factor $\prod_{i=1}^{s} (-1)^{\ell(\lambda^{(i)})}$ for the summand indexed by $\lambda^{(1)}, \ldots, \lambda^{(s)}$. This sign equals $\mathrm{sgn}(T)$ for any ETBT T built from this choice of the partitions $\lambda^{(i)}$. □

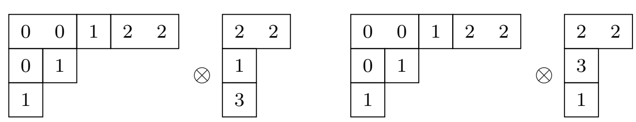
Example 4.17. Let $\sigma = 1^{(2,1)}$, $\tau = 2^{2,1,1}1^{5,2,1}$ and $\delta = (5^1, 3^2, 2^1)$. Via the objects below, we find that the coefficient of $m^{\otimes}_{\tau}$ in the $m^{\otimes}$-expansion of $m^{\otimes}_{\sigma}E^{+}_{\delta}$ is 7, while the coefficient of $m^{\otimes}_{\sigma}E_{\delta}$ is -7.

We first choose the partitions $(2,2,1) \vdash 5$, $(2,1) \vdash 3$ and $(1,1) \vdash 2$. Then we construct the five ETBTs shown below. Note that certain configurations that give valid HTBTs are not possible in the setting of ETBTs.



All these ETBTs have the same sign, namely $(-1)^{3+2+2} = -1$.

We now choose a different set of partitions $(2,1,1,1) \vdash 5$, $(2,1) \vdash 3$ and $(2) \vdash 2$. This gives us the two ETBTs shown below.



all with the sign $(-1)^{4+2+1} = -1$.

It is routine to check that no other choices of partitions lead to brick collections that can fill $\mathrm{dg}(\tau)$ following the rules for ETBTs.

Corollary 4.18.

(a) *For all* $\tau, \sigma \Vdash n$, *the coefficient of* $m^{\otimes}_{\tau}$ *in the* $m^{\otimes}$*-expansion of* E^{+}_{σ} *is* $\mathcal{M}(E^{+}, m^{\otimes})_{\tau,\sigma} = |\,\mathrm{ETBT}(\tau,\sigma)|$.

(b) *For all* $\tau, \sigma \Vdash n$, *the coefficient of* $m^{\otimes}_{\tau}$ *in the* $m^{\otimes}$*-expansion of* E_{σ} *is* $\mathcal{M}(E, m^{\otimes})_{\tau,\sigma} = \sum_{T \in \mathrm{ETBT}(\tau,(\sigma;\delta))} \mathrm{sgn}(T)$.

Remark 4.19. This paper is based on results contained in the first author's forthcoming Ph.D. thesis. That work will also discuss some additional transition matrices not covered here. The $h^{\otimes}$ and $e^{\otimes}$ expansions can be derived using abacus-based methods and the combinatorics of permutation statistics. The transition matrices between the non-classical polysymmetric bases P, E, and H can be derived via recursions similar to the symmetric case, leading to generalizations of the brick tabloids defined in [3].

Appendix A. Sample transition matrices

Below we give the transition matrices computed in this paper for bases of $\mathsf{P}\Lambda^4$. For example, the column marked 1^{22} in $\mathcal{M}(P, s^{\otimes})$ tells us that

$$P_{1^{22}} = 1s^{\otimes}_{1^4} - 1s^{\otimes}_{1^{31}} + 2s^{\otimes}_{1^{22}} - 1s^{\otimes}_{1^{211}} + 1s^{\otimes}_{1^{1111}}.$$

$$
\mathcal{M}(P,s^{\otimes}):\quad
\begin{array}{c|ccccccccccc|}
 & 1^{4} & 1^{31} & 1^{22} & 1^{211} & 1^{1111} & 2^{1}1^{2} & 2^{1}1^{11} & 3^{1}1^{1} & 2^{2} & 2^{11} & 4^{1} \\
1^{4} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1^{31} & -1 & 0 & -1 & 1 & 3 & -1 & 1 & 0 & -1 & -1 & -1 \\
1^{22} & 0 & -1 & 2 & 0 & 2 & 2 & 0 & -1 & 0 & 2 & 0 \\
1^{211} & 1 & 0 & -1 & -1 & 3 & -1 & -1 & 0 & 1 & -1 & 1 \\
1^{1111} & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\
2^{1}1^{2} & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 4 & 0 \\
2^{1}1^{11} & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & -4 & 0 \\
3^{1}1^{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
2^{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 \\
2^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & -2 \\
4^{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{array}
$$

$$
\mathcal{M}(H,s^{\otimes}):\quad
\begin{array}{c|ccccccccccc|}
 & 1^{4} & 1^{31} & 1^{22} & 1^{211} & 1^{1111} & 2^{1}1^{2} & 2^{1}1^{11} & 3^{1}1^{1} & 2^{2} & 2^{11} & 4^{1} \\
1^{4} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1^{31} & -1 & 0 & -1 & 1 & 3 & 0 & 2 & 1 & -1 & 1 & 0 \\
1^{22} & 0 & -1 & 2 & 0 & 2 & 1 & 1 & 0 & 1 & 1 & 0 \\
1^{211} & 1 & 0 & -1 & -1 & 3 & -1 & 1 & 0 & 0 & 0 & 0 \\
1^{1111} & -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2^{1}1^{2} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 \\
2^{1}1^{11} & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\
3^{1}1^{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
2^{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
2^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
4^{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$
\mathcal{M}(E^{+},s^{\otimes}):\quad
\begin{array}{c|ccccccccccc|}
 & 1^{4} & 1^{31} & 1^{22} & 1^{211} & 1^{1111} & 2^{1}1^{2} & 2^{1}1^{11} & 3^{1}1^{1} & 2^{2} & 2^{11} & 4^{1} \\
1^{4} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1^{31} & -1 & 0 & -1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\
1^{22} & 0 & -1 & 2 & 0 & 2 & -1 & 1 & 0 & 1 & 1 & 0 \\
1^{211} & 1 & 0 & -1 & -1 & 3 & 0 & 2 & 1 & -1 & 1 & 0 \\
1^{1111} & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
2^{1}1^{2} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
2^{1}1^{11} & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 2 & 1 \\
3^{1}1^{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
2^{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
2^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\
4^{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$
\mathcal{M}(E,s^{\otimes}):\quad
\begin{array}{c|ccccccccccc|}
 & 1^{4} & 1^{31} & 1^{22} & 1^{211} & 1^{1111} & 2^{1}1^{2} & 2^{1}1^{11} & 3^{1}1^{1} & 2^{2} & 2^{11} & 4^{1} \\
1^{4} & -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1^{31} & 1 & 0 & -1 & -1 & 3 & -1 & 1 & 0 & 0 & 0 & 0 \\
1^{22} & 0 & -1 & 2 & 0 & 2 & 1 & 1 & 0 & 1 & 1 & 0 \\
1^{211} & -1 & 0 & -1 & 1 & 3 & 0 & 2 & 1 & -1 & 1 & 0 \\
1^{1111} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2^{1}1^{2} & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\
2^{1}1^{11} & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -2 & -1 \\
3^{1}1^{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
2^{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
2^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
4^{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}
$$

$$\mathcal{M}(P,p^{\otimes}):\quad \begin{array}{c} \\ 1^4\\ 1^{31}\\ 1^{22}\\ 1^{211}\\ 1^{1111}\\ 2^11^2\\ 2^11^{11}\\ 3^11^1\\ 2^2\\ 2^{11}\\ 4^1 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1^4 & 1^{31} & 1^{22} & 1^{211} & 1^{1111} & 2^11^2 & 2^11^{11} & 3^11^1 & 2^2 & 2^{11} & 4^1 \end{array}\\ \left[\begin{array}{ccccccccccc} 1&0&0&0&0&0&0&0&1&0&1\\ 0&1&0&0&0&0&0&1&0&0&0\\ 0&0&1&0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&2&0&0&0&4&0\\ 0&0&0&0&0&0&2&0&0&0&0\\ 0&0&0&0&0&0&0&3&0&0&0\\ 0&0&0&0&0&0&0&0&2&0&2\\ 0&0&0&0&0&0&0&0&0&4&0\\ 0&0&0&0&0&0&0&0&0&0&4 \end{array}\right] \end{array}$$

$$\mathcal{M}(H,p^{\otimes}):\quad \begin{array}{c} \\ 1^4\\ 1^{31}\\ 1^{22}\\ 1^{211}\\ 1^{1111}\\ 2^11^2\\ 2^11^{11}\\ 3^11^1\\ 2^2\\ 2^{11}\\ 4^1 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1^4 & 1^{31} & 1^{22} & 1^{211} & 1^{1111} & 2^11^2 & 2^11^{11} & 3^11^1 & 2^2 & 2^{11} & 4^1 \end{array}\\ \left[\begin{array}{ccccccccccc} 1&0&0&0&0&0&0&0&\frac{1}{2}&0&\frac{1}{4}\\ 0&1&0&0&0&0&0&\frac{1}{3}&0&0&\frac{1}{3}\\ 0&0&1&0&0&\frac{1}{2}&0&0&\frac{1}{2}&\frac{1}{4}&\frac{1}{8}\\ 0&0&0&1&0&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&0&\frac{1}{2}&\frac{1}{4}\\ 0&0&0&0&1&0&\frac{1}{2}&\frac{1}{6}&0&\frac{1}{4}&\frac{1}{24}\\ 0&0&0&0&0&1&0&0&0&1&\frac{1}{2}\\ 0&0&0&0&0&0&1&1&0&1&\frac{1}{2}\\ 0&0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&1&0&\frac{1}{2}\\ 0&0&0&0&0&0&0&0&0&1&\frac{1}{2}\\ 0&0&0&0&0&0&0&0&0&0&1 \end{array}\right] \end{array}$$

$$\mathcal{M}(E^+,p^{\otimes}):\quad \begin{array}{c} \\ 1^4\\ 1^{31}\\ 1^{22}\\ 1^{211}\\ 1^{1111}\\ 2^11^2\\ 2^11^{11}\\ 3^11^1\\ 2^2\\ 2^{11}\\ 4^1 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1^4 & 1^{31} & 1^{22} & 1^{211} & 1^{1111} & 2^11^2 & 2^11^{11} & 3^11^1 & 2^2 & 2^{11} & 4^1 \end{array}\\ \left[\begin{array}{ccccccccccc} 1&0&0&0&0&0&0&0&-\frac{1}{2}&0&-\frac{1}{4}\\ 0&1&0&0&0&0&0&\frac{1}{3}&0&0&\frac{1}{3}\\ 0&0&1&0&0&-\frac{1}{2}&0&0&\frac{1}{2}&\frac{1}{4}&\frac{1}{8}\\ 0&0&0&1&0&\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&0&-\frac{1}{2}&-\frac{1}{4}\\ 0&0&0&0&1&0&\frac{1}{2}&\frac{1}{6}&0&\frac{1}{4}&\frac{1}{24}\\ 0&0&0&0&0&1&0&0&0&-1&-\frac{1}{2}\\ 0&0&0&0&0&0&1&1&0&1&\frac{1}{2}\\ 0&0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&1&0&-\frac{1}{2}\\ 0&0&0&0&0&0&0&0&0&1&\frac{1}{2}\\ 0&0&0&0&0&0&0&0&0&0&1 \end{array}\right] \end{array}$$

$$\mathcal{M}(E,p^{\otimes}):\quad \begin{array}{c} \\ 1^4\\ 1^{31}\\ 1^{22}\\ 1^{211}\\ 1^{1111}\\ 2^11^2\\ 2^11^{11}\\ 3^11^1\\ 2^2\\ 2^{11}\\ 4^1 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1^4 & 1^{31} & 1^{22} & 1^{211} & 1^{1111} & 2^11^2 & 2^11^{11} & 3^11^1 & 2^2 & 2^{11} & 4^1 \end{array}\\ \left[\begin{array}{ccccccccccc} -1&0&0&0&0&0&0&0&-\frac{1}{2}&0&-\frac{1}{4}\\ 0&1&0&0&0&0&0&\frac{1}{3}&0&0&\frac{1}{3}\\ 0&0&1&0&0&\frac{1}{2}&0&0&\frac{1}{2}&\frac{1}{4}&\frac{1}{8}\\ 0&0&0&-1&0&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&0&-\frac{1}{2}&-\frac{1}{4}\\ 0&0&0&0&1&0&\frac{1}{2}&\frac{1}{6}&0&\frac{1}{4}&\frac{1}{24}\\ 0&0&0&0&0&1&0&0&0&1&\frac{1}{2}\\ 0&0&0&0&0&0&-1&-1&0&-1&-\frac{1}{2}\\ 0&0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&-1&0&-\frac{1}{2}\\ 0&0&0&0&0&0&0&0&0&1&\frac{1}{2}\\ 0&0&0&0&0&0&0&0&0&0&-1 \end{array}\right] \end{array}$$

$\mathcal{M}(P, m^{\otimes})$:

	1^4	1^{31}	1^{22}	1^{211}	1^{1111}	$2^1 1^2$	$2^1 1^{11}$	$3^1 1^1$	2^2	2^{11}	4^1
1^4	1	1	1	1	1	1	1	1	1	1	1
1^{31}	0	1	0	2	4	0	2	1	0	0	0
1^{22}	0	0	2	2	6	2	2	0	0	2	0
1^{211}	0	0	0	2	12	0	2	0	0	0	0
1^{1111}	0	0	0	0	24	0	0	0	0	0	0
$2^1 1^2$	0	0	0	0	0	2	2	0	0	4	0
$2^1 1^{11}$	0	0	0	0	0	0	4	0	0	0	0
$3^1 1^1$	0	0	0	0	0	0	0	3	0	0	0
2^2	0	0	0	0	0	0	0	0	2	4	2
2^{11}	0	0	0	0	0	0	0	0	0	8	0
4^1	0	0	0	0	0	0	0	0	0	0	4

$\mathcal{M}(H, m^{\otimes})$:

	1^4	1^{31}	1^{22}	1^{211}	1^{1111}	$2^1 1^2$	$2^1 1^{11}$	$3^1 1^1$	2^2	2^{11}	4^1
1^4	1	1	1	1	1	1	1	1	1	1	1
1^{31}	0	1	0	2	4	1	3	2	0	2	1
1^{22}	0	0	2	2	6	2	4	2	1	3	1
1^{211}	0	0	0	2	12	1	7	3	0	4	1
1^{1111}	0	0	0	0	24	0	12	4	0	6	1
$2^1 1^2$	0	0	0	0	0	1	1	1	0	2	1
$2^1 1^{11}$	0	0	0	0	0	0	2	2	0	2	1
$3^1 1^1$	0	0	0	0	0	0	0	1	0	0	1
2^2	0	0	0	0	0	0	0	0	1	1	1
2^{11}	0	0	0	0	0	0	0	0	0	2	1
4^1	0	0	0	0	0	0	0	0	0	0	1

$\mathcal{M}(E^+, m^{\otimes})$:

	1^4	1^{31}	1^{22}	1^{211}	1^{1111}	$2^1 1^2$	$2^1 1^{11}$	$3^1 1^1$	2^2	2^{11}	4^1
1^4	1	1	1	1	1	0	0	0	0	0	0
1^{31}	0	1	0	2	4	1	1	0	0	0	0
1^{22}	0	0	2	2	6	0	2	0	1	1	0
1^{211}	0	0	0	2	12	1	5	1	0	2	0
1^{1111}	0	0	0	0	24	0	12	4	0	6	1
$2^1 1^2$	0	0	0	0	0	1	1	1	0	0	0
$2^1 1^{11}$	0	0	0	0	0	0	2	2	0	2	1
$3^1 1^1$	0	0	0	0	0	0	0	1	0	0	1
2^2	0	0	0	0	0	0	0	0	1	1	0
2^{11}	0	0	0	0	0	0	0	0	0	2	1
4^1	0	0	0	0	0	0	0	0	0	0	1

$\mathcal{M}(E, m^{\otimes})$:

	1^4	1^{31}	1^{22}	1^{211}	1^{1111}	$2^1 1^2$	$2^1 1^{11}$	$3^1 1^1$	2^2	2^{11}	4^1
1^4	−1	1	1	−1	1	0	0	0	0	0	0
1^{31}	0	1	0	−2	4	−1	1	0	0	0	0
1^{22}	0	0	2	−2	6	0	2	0	1	1	0
1^{211}	0	0	0	−2	12	−1	5	1	0	2	0
1^{1111}	0	0	0	0	24	0	12	4	0	6	1
$2^1 1^2$	0	0	0	0	0	1	−1	−1	0	0	0
$2^1 1^{11}$	0	0	0	0	0	0	−2	−2	0	−2	−1
$3^1 1^1$	0	0	0	0	0	0	0	1	0	0	1
2^2	0	0	0	0	0	0	0	0	−1	1	0
2^{11}	0	0	0	0	0	0	0	0	0	2	1
4^1	0	0	0	0	0	0	0	0	0	0	−1

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Aditya Khanna, Virginia Tech, Dept. of Mathematics, 225 Stanger St., 460 McBryde Hall, Blacksburg, VA 24061-0123 (USA)
E-mail : `adityakhanna@vt.edu`

Nicholas A. Loehr, Virginia Tech, Dept. of Mathematics, 225 Stanger St., 460 McBryde Hall, Blacksburg, VA 24061-0123 (USA)
E-mail : `nloehr@vt.edu`