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Crystals for set-valued decomposition tableaux

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ABSTRACT We describe two crystal structures on set-valued decomposition tableaux. These provide the first examples of interesting "K-theoretic" crystals on shifted tableaux. Our first crystal is modeled on a similar construction of Monical, Pechenik, and Scrimshaw for semistandard (unshifted) set-valued tableaux. Our second crystal is adapted from the "square root" operators introduced by Yu on the same set. Neither of our shifted crystals is normal, but we conjecture that our second construction is connected with a unique highest weight element. These results lead to partial progress on a conjectural formula of Cho–Ikeda for K-theoretic Schur P-functions. We also study a new category of "square root crystals" that includes our second construction and Yu's set-valued tableau crystals as examples. We observe that Buch's formula for the coefficients expanding products of symmetric Grothendieck functions has a simple description in terms of the tensor product for this category.

1. INTRODUCTION

Work by Grantcharov et al. [8, 9, 10] defines a family of *crystals* for the queer Lie superalgebra $\mathfrak{q}_n$. Concretely, these objects consist of certain directed acyclic graphs with labeled weights and weighted vertices. There is a natural tensor product for $\mathfrak{q}_n$-crystals, as well as a *standard* $\mathfrak{q}_n$*-crystal*. The connected crystals that appear in tensor powers of the standard crystal, as well as arbitrary disjoint unions of such crystals, are called *normal*.

It is shown in [8] that every connected normal $\mathfrak{q}_n$-crystal has a unique highest weight element, whose weight is a strict partition λ with at most n parts, and that any two such crystals with the same highest weight are isomorphic. Conversely, the authors in [8] provide a simple, explicit construction of a connected normal $\mathfrak{q}_n$-crystal with any feasible highest weight λ: this is given by introducing natural crystal operators on all *decomposition tableaux* of shape λ.

A *decomposition tableau* is a certain filling of the shifted shape of a strict partition λ by positive integers. In such tableaux, each row must be a *hook word*, meaning a weakly decreasing sequence followed by a (possibly empty) strictly increasing sequence. Additionally, consecutive rows in a decomposition tableau must avoid the

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patterns represented in French notation as

$$\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \cdots & b \\ \hline a & \cdots & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|}\cline{2-5} \multicolumn{1}{c|}{} & \cdots & c & \cdots & b \\ \hline & \cdots & a & \cdots & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \cdots & x \\ \hline y & \cdots & z \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|c|c|c|c|}\cline{2-5} \multicolumn{1}{c|}{} & \cdots & & \cdots & x \\ \hline & \cdots & y & \cdots & z \\ \hline \end{array}$$

for all $a \leqslant b \leqslant c$ and $x < y < z$. See Section 2 for the precise definitions.

Our purpose here is to describe new crystal structures on a "set-valued" generalization of decomposition tableaux that was first considered in unpublished work of Cho and Ikeda [14]. Several authors (for example, [13, 26, 29, 33]) have recently studied crystal structures on unshifted set-valued tableaux. The characters of these crystals give K-theoretic symmetric functions of independent interest. It has been an open problem to extend such constructions to shifted tableaux.

Addressing this problem, we first show that the family of *set-valued decomposition tableaux* $\mathsf{SetDecTab}_n(\lambda)$ has an abstract $\mathfrak{q}_n$-crystal structure. The elements of $\mathsf{SetDecTab}_n(\lambda)$ are the set-valued shifted tableaux (that is, fillings of a shifted shape by nonempty subsets of $\{1, 2, \dots, n\}$) whose distributions all belong to $\mathsf{DecTab}_n(\lambda)$, the set of decomposition tableaux of shape λ. By a *distribution* of a set-valued tableau, we mean any tableau of the same shape formed by replacing each set-valued entry by one of its elements. For example, we can draw an element of $\mathsf{SetDecTab}_4(\lambda)$ for $\lambda = (3, 2)$ in French notation as

$$\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 12 & 3 \\ \hline 4 & 3 & 234 \\ \hline \end{array}$$

since it holds that

$$\left\{ \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 3 & 2 \\ \hline \end{array}, \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 2 & 3 \\ \hline 4 & 3 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 2 & 3 \\ \hline 4 & 3 & 2 \\ \hline \end{array}, \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 3 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 2 & 3 \\ \hline 4 & 3 & 4 \\ \hline \end{array} \right\} \subset \mathsf{DecTab}_4(\lambda).$$

Our $\mathfrak{q}_n$-crystal structure on $\mathsf{SetDecTab}_n(\lambda)$ is formally similar to the one in [26] for unshifted set-valued tableaux. The details of its construction appear in Section 3.2.

Cho and Ikeda [14] have conjectured that the weight generating function for set-valued decomposition tableaux recovers the *K-theoretic Schur P-function* GP_λ introduced in [15] (see Conjecture 3.2). By interpreting this generating function as the character of our $\mathfrak{q}_n$-crystal $\mathsf{SetDecTab}_n(\lambda)$, we are able to prove that it is at least equal to GP_λ plus a (possibly infinite) linear combination of GP_μ's with $|\mu| > |\lambda|$ (see Corollary 3.11).

Our first crystal structure on $\mathsf{SetDecTab}_n(\lambda)$, like the one in [26], is typically disconnected. Yu [33] introduced a modified set of "square root" crystal operators on set-valued words that square to a seminormal crystal structure. These operators determine a connected crystal when restricted to semistandard set-valued tableaux of a fixed shape (identified with their column reading words). It turns out that Yu's crystal operators also make sense as operators on set-valued decomposition tableaux (now using the reverse row reading word), and provide an alternate "square root" crystal structure on these objects (see Theorem 4.18). We conjecture that the resulting crystal, when equipped with additional "square root" queer crystals operators, is actually connected (see Conjecture 4.24).

We also discuss an axiomatic framework to contain there constructions. Specifically, we introduce definitions of abstract *$\sqrt{\mathfrak{gl}_n}$- and $\sqrt{\mathfrak{q}_n}$-crystals*, which include Yu's crystal of semistandard set-valued tableaux and our second crystal of set-valued decomposition tableaux as special cases. These "crystals" possess a natural tensor product and a plausible choice of a "standard object," which give rise to families of *normal $\sqrt{\mathfrak{gl}_n}$- and $\sqrt{\mathfrak{q}_n}$-crystals*. The crystals do not generally have highest weight properties that are as nice as in the classical case, but they nevertheless display several interesting features (see, for example, Theorem 4.28, Theorem 4.30, and Proposition 4.40). We discuss these nuances in Section 4.4 and present several related conjectures in Section 4.5.

We close this introduction with a brief comparison of our crystal structures on $\mathsf{SetDecTab}_n(\lambda)$. Ultimately, these are both based on analogues of the *signature rule* that governs tensor products of Kashiwara crystals. The positivity conjectures in Section 4.5 suggest that our second construction may be more fundamental, but the crystal operators for our first construction are easier to compute. In any event, our proofs that the relevant crystals are well-defined in each case rely on the same set of technical lemmas (see Section 3.3). For this reason, it makes sense to present both constructions in one place, although *a priori* they may not appear directly related to each other.

2. Preliminaries

This section contains some background material on shifted tableaux and crystals from [3, 8, 19]. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{P} = \{1, 2, 3, \dots\}$. Fix a nonnegative integer $n \in \mathbb{N}$ and let $[n] = \{1, 2, \dots, n\}$.

2.1. Tableaux. A *partition* is a weakly decreasing sequence of integers $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots > 0)$ with finite sum. The nonzero numbers λ_i are the *parts* of λ; if these are all distinct then λ is *strict*. Let $\ell(\lambda)$ be the number of nonzero parts of λ. The *diagram* of λ is the set

$$\mathsf{D}_\lambda := \{(i, j) : i \in [\ell(\lambda)] \text{ and } j \in [\lambda_i]\}.$$

When $\lambda = (\lambda_1 > \lambda_2 > \dots > 0)$ is a strict partition, its *shifted diagram* is

$$\mathsf{SD}_\lambda := \{(i, i + j - 1) : i \in [\ell(\lambda)] \text{ and } j \in [\lambda_i]\} = \{(i, i + j - 1) : (i, j) \in \mathsf{D}_\lambda\}.$$

For the moment, we define a *tableau* of shape λ to be any map $\mathsf{D}_\lambda \to \{1, 2, 3, \dots\}$. We often view this object as a filling of the positions in D_λ, drawn as boxes, by positive integers. Likewise, a *shifted tableau* of shape λ is a map $\mathsf{SD}_\lambda \to \{1' < 1 < 2' < 2 < \dots\}$. Here, one can view the *primed numbers* $1' < 2' < 3' < \dots$ either as formal symbols or as the half-integers $i' := i - \frac{1}{2}$.

If T is a (shifted) tableau, then we write $(i, j) \in T$ to indicate that (i, j) belongs to the domain of T and we let T_{ij} denote the value assigned to this position. We draw tableaux in French notation, so that row indices increase bottom to top and column indices increase left to right. For example,

$$R = \begin{array}{|c|c|c|c} \hline 3 & 3 & 7 & \multicolumn{1}{|c}{} \\ \hline 1 & 2 & 2 & \multicolumn{1}{|c|}{6} \\ \hline \end{array}, \quad S = \begin{array}{c|c|c|c|} \cline{2-4} & 3 & 5 & 7 \\ \hline \multicolumn{1}{|c|}{1} & 2 & 4 & 6 \\ \hline \end{array}, \quad \text{and} \quad T = \begin{array}{c|c|c|c|} \cline{2-4} & 2' & 2 & 4' \\ \hline \multicolumn{1}{|c|}{1'} & 1 & 1 & 4' \\ \hline \end{array} \tag{2.1}$$

are all tableaux of shape $\lambda = (4, 3)$, with $R_{23} = 7$, $S_{23} = 5$, and $T_{23} = 2$. The tableau R is unshifted while S and T are shifted. The *(main) diagonal* of a shifted tableau is the set of boxes (i, j) in its domain with $i = j$.

A (shifted or unshifted) tableau is *semistandard* if its rows and columns are weakly increasing, such that no primed entry is repeated in any row and no unprimed entry is repeated in any column. The examples in (2.1) are all semistandard. We write $\mathsf{Tab}(\lambda)$ for the set of all semistandard tableaux of shape λ, $\mathsf{ShTab}^+(\lambda)$ for the set of all semistandard shifted tableaux of shape λ, and $\mathsf{ShTab}(\lambda)$ for the subset of elements in $\mathsf{ShTab}^+(\lambda)$ with no primed entries on the diagonal.

Notation 2.1. Throughout, if $\mathsf{FAMILY}(\lambda)$ is any family of tableaux associated to a partition λ, then we write $\mathsf{FAMILY}_n(\lambda)$ for the subset of $T \in \mathsf{FAMILY}(\lambda)$ with all entries at most n.

If T is a (shifted) tableau, then set $x^T := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ where a_k is the number of times k or k' appears in T. The *Schur function* of an arbitrary partition λ is

$$s_\lambda := \textstyle\sum_{T \in \mathsf{Tab}(\lambda)} x^T. \tag{2.2}$$

Similarly, the *Schur P- and Q-functions* of a strict partition λ are

$$P_\lambda := \textstyle\sum_{T \in \mathsf{ShTab}(\lambda)} x^T \quad\text{and}\quad Q_\lambda := \sum_{T \in \mathsf{ShTab}^+(\lambda)} x^T = 2^{\ell(\lambda)} P_\lambda. \tag{2.3}$$

These power series all belong to the ring of bounded degree symmetric functions Sym. The Schur functions $\{s_\lambda\}$ are a $\mathbb{Z}$-basis for Sym, while $\{P_\lambda\}$ and $\{Q_\lambda\}$ (as λ varies over all strict partitions) are $\mathbb{Z}$-bases for two distinct subrings of Sym [19, Chapter III, §8].

The Schur P-functions have a second tableau generating function formula. A *hook word* is a finite sequence of positive integers $w = w_1 w_2 \cdots w_n$ such that $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_m < w_{m+1} < w_{m+2} < \cdots < w_n$ for some $m \in [n]$. Given such a hook word, let $w\!\downarrow\, := w_1 w_2 \cdots w_m$ denote the *decreasing part* and let $w\!\uparrow\, := w_{m+1} w_{m+2} \cdots w_n$ denote the *increasing part*.

A *decomposition tableau* of strict partition shape λ is a map $T : \mathsf{SD}_\lambda \to \{1,2,3,\dots\}$ such that if ρ_i denotes row i of T, then (1) each ρ_i is a hook word and (2) ρ_i is a hook subword of maximal length in $\rho_{i+1}\rho_i$ for each $i \in [\ell(\lambda)-1]$. This definition follows [8] but differs from [4, 5, 31], which uses the opposite weak/strict inequality convention for hook words.

Let $\mathsf{DecTab}(\lambda)$ be the set of all decomposition tableaux of shape λ. Then we have

$$P_\lambda = \textstyle\sum_{T \in \mathsf{DecTab}(\lambda)} x^T \tag{2.4}$$

by [31, Thm. 2.17] and [5, Thm. 3.9].

Example 2.2. We have $\begin{array}{|c|c|c|}\hline & 1 & 1 \\ \hline 2 & 2 & 1 \\ \hline\end{array} \in \mathsf{DecTab}_3((3,2))$ but $\begin{array}{|c|c|c|}\hline & 1 & 1 \\ \hline 2 & 2 & 3 \\ \hline\end{array} \notin \mathsf{DecTab}_3((3,2))$. The second example is not a decomposition tableau because its row reading word $\rho_2\rho_1 = 11223$ contains the hook subword 1123 which is longer than $\rho_1 = 223$.

It is useful to reformulate the maximal hook subword condition as follows:

Lemma 2.3 ([8, Prop. 2.3]). *Let T be shifted tableau of shape λ whose rows are each hook words. Then T is a decomposition tableaux if and only if none of the following conditions holds for any $i \in [\ell(\lambda)-1]$ and $j,k \in [\lambda_{i+1}]$:*

(a) *$T_{i,i} \leqslant T_{i+1,i+k}$ or $T_{i,i+j} \leqslant T_{i+1,i+k} \leqslant T_{i+1,i+j}$ when $j<k$,*
(b) *$T_{i+1,i+k} < T_{i,i} < T_{i,i+k}$ or $T_{i+1,i+k} < T_{i,i+j} < T_{i,i+k}$ when $j<k$.*

That is, we forbid rows i and $i+1$ of T from having configurations of entries

$$\begin{array}{|c|c|c|}\hline & \cdots & b \\ \hline a & \cdots & \\ \hline\end{array}\,,\quad \begin{array}{|c|c|c|c|c|}\hline & \cdots & c & \cdots & b \\ \hline & \cdots & a & \cdots & \\ \hline\end{array}\,,\quad \begin{array}{|c|c|c|}\hline & \cdots & x \\ \hline y & \cdots & z \\ \hline\end{array}\,,\quad \text{or}\quad \begin{array}{|c|c|c|c|c|}\hline & \cdots & & \cdots & x \\ \hline & \cdots & y & \cdots & z \\ \hline\end{array}$$

with $a \leqslant b \leqslant c$ and $x < y < z$. Here, the leftmost boxes are on the main diagonal and the ellipses "$\cdots$" indicate sequences of zero or more columns.

2.2. Abstract crystals. Let $\mathcal{B}$ be a set with maps $\mathrm{wt} : \mathcal{B} \to \mathbb{Z}^n$ and $e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ for $i \in [n-1]$, where $0 \notin \mathcal{B}$. Write $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbb{Z}^n$ for the standard basis.

Definition 2.4. *The set $\mathcal{B}$ is a $\mathfrak{gl}_n$-crystal if for all $i \in [n-1]$ and $b,c \in \mathcal{B}$ it holds that $e_i(b) = c$ if and only if $f_i(c) = b$, in which case $\mathrm{wt}(c) = \mathrm{wt}(b) + \mathbf{e}_i - \mathbf{e}_{i+1}$.*

Assume $\mathcal{B}$ is a $\mathfrak{gl}_n$-crystal. Then the maps e_i and f_i encode a directed graph with vertex set $\mathcal{B}$, to be called the *crystal graph*, with an edge $b \xrightarrow{i} c$ if and only if $f_i(b) = c$. Define the *string lengths* $\varepsilon_i, \varphi_i : \mathcal{B} \to \{0,1,2,\dots\} \sqcup \{\infty\}$ by

$$\varepsilon_i(b) := \sup\left\{k \geqslant 0 : e_i^k(b) \neq 0\right\} \text{ and } \varphi_i(b) := \sup\left\{k \geqslant 0 : f_i^k(b) \neq 0\right\}. \tag{2.5}$$

DEFINITION 2.5. *The* $\mathfrak{gl}_n$*-crystal* $\mathcal{B}$ *is* seminormal *if the string lengths always take finite values such that* $\varphi_i(b) - \varepsilon_i(b) = \mathrm{wt}(b)_i - \mathrm{wt}(b)_{i+1}$ *for all* $b \in \mathcal{B}$.

If $\mathcal{B}$ is finite then its *character* is

$$\mathrm{ch}(\mathcal{B}) := \textstyle\sum_{b\in\mathcal{B}} x^{\mathrm{wt}(b)}$$

where $x^{\mathrm{wt}(b)} := \prod_{i\in[n]} x_i^{\mathrm{wt}(b)_i}$. The character is symmetric in $x_1, x_2, \dots, x_n$ if $\mathcal{B}$ is seminormal [3, §2.6].

We refer to wt as the *weight map*, to each e_i as a *raising operator*, and to each f_i as a *lowering operator*. Each connected component of the crystal graph of $\mathcal{B}$ may be viewed as a $\mathfrak{gl}_n$-crystal by restricting the weight map and crystal operators; these objects are called *full subcrystals*. For $\mathfrak{gl}_n$-crystals (and for the $\mathfrak{q}_n$-crystals defined in Section 2.3), an *isomorphism* means a weight-preserving map that defines a graph isomorphism of the corresponding crystal graphs.

EXAMPLE 2.6. The *standard* $\mathfrak{gl}_n$*-crystal* $\mathbb{B}_n = \left\{\boxed{i} : i \in [n]\right\}$ has crystal graph

$$\boxed{1} \xrightarrow{\;1\;} \boxed{2} \xrightarrow{\;2\;} \boxed{3} \xrightarrow{\;3\;} \cdots \xrightarrow{\;n-1\;} \boxed{n} \qquad \text{with } \mathrm{wt}(\boxed{i}) := \mathbf{e}_i.$$

Its character is $\mathrm{ch}(\mathbb{B}_n) = x_1 + x_2 + x_3 + \cdots + x_n$.

If $\mathcal{B}$ and $\mathcal{C}$ are $\mathfrak{gl}$-crystals then the set $\mathcal{B}\otimes\mathcal{C} := \{b\otimes c : b \in \mathcal{B},\ c \in \mathcal{C}\}$ of formal tensors has a unique $\mathfrak{gl}_n$-crystal structure (which is seminormal if $\mathcal{B}$ and $\mathcal{C}$ are seminormal) in which $\mathrm{wt}(b \otimes c) := \mathrm{wt}(b) + \mathrm{wt}(c)$ and

$$\begin{aligned} e_i(b\otimes c) &:= \begin{cases} b\otimes e_i(c) & \text{if } \varepsilon_i(b) \leqslant \varphi_i(c) \\ e_i(b)\otimes c & \text{if } \varepsilon_i(b) > \varphi_i(c) \end{cases} \\ f_i(b\otimes c) &:= \begin{cases} b\otimes f_i(c) & \text{if } \varepsilon_i(b) < \varphi_i(c) \\ f_i(b)\otimes c & \text{if } \varepsilon_i(b) \geqslant \varphi_i(c) \end{cases} \end{aligned} \tag{2.6}$$

for $i \in [n-1]$, where we set $b \otimes 0 = 0 \otimes c = 0$ [3, §2.3]. This follows the "anti-Kashiwara convention," which reverses the tensor product order in [9, 8]. The natural map $\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}) \to (\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}$ is a crystal isomorphism, so we can dispense with parentheses in iterated tensor products.

A $\mathfrak{gl}_n$-crystal is *normal* if each of its full subcrystals is isomorphic to a full subcrystal of $\mathbb{B}_n^{\otimes m}$ for some $m \in \mathbb{N}$. Such crystals have particularly nice properties, such as Schur positive characters; see [3, Thms. 3.2 and 8.6] and [32].

A *word* is a finite sequence of positive integers. Let $\mathsf{Words}_n(m)$ be the set of m-letter words $w = w_1w_2\cdots w_m$ with $w_i \in [n]$. We identify each word $w \in \mathsf{Words}_n(m)$ with the tensor $w_1 \otimes w_2 \otimes \cdots \otimes w_m$ in order to view $\mathsf{Words}_n(m)$ as a crystal isomorphic to $\mathbb{B}_n^{\otimes m}$. This allows us to evaluate $\mathrm{wt}(w)$, $e_i(w)$, and $f_i(w)$ for $i \in [n-1]$ using the definition of the $\mathfrak{gl}_n$-crystal $\mathbb{B}_n^{\otimes m}$. The weight of w under this convention is the usual integer vector whose ith component is the number of letters equal to i.

REMARK 2.7. The following *signature rule* [3, §2.4]) can be used to compute $e_i(w)$ and $f_i(w)$. Suppose $w = w_1w_2\cdots w_m \in \mathsf{Words}_n(m)$ is a word and $i \in [n-1]$. Mark each entry $w_k = i$ by a right parenthesis ")" and each entry $w_j = i+1$ by a left parenthesis "(". The *i-unpaired indices* in w are the indices $j \in [m]$ with $w_j \in \{i, i+1\}$ that are not the positions of matching parentheses.

- If no i-unpaired index j of w has $w_j = i+1$ then $e_i(w) = 0$.
  Otherwise, if j is the smallest such index, then $e_i(w) = w_1\cdots(w_j-1)\cdots w_m$.
- If no i-unpaired index j of w has $w_j = i$ then $f_i(w) = 0$.
  Otherwise, if j is the largest such index, then $f_i(w) = w_1\cdots(w_j+1)\cdots w_m$.

2.3. Queer crystals. Grantcharov et al. developed a theory of crystals for the *queer Lie superalgebra* $\mathfrak{q}_n$ in [9, 8, 10], which we review here.

Assume $n \geqslant 2$ and let $\mathcal{B}$ be a $\mathfrak{gl}_n$-crystal with additional maps $e_{\overline{1}}, f_{\overline{1}} : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$. Define $\varepsilon_{\overline{1}}, \varphi_{\overline{1}} : \mathcal{B} \to \mathbb{N} \sqcup \{\infty\}$ as in (2.5) with $i = \overline{1}$. We say that one map $\phi : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ *preserves* another map $\eta : \mathcal{B} \to \mathcal{X}$ if $\eta(\phi(b)) = \eta(b)$ whenever $\phi(b) \neq 0$.

Definition 2.8. *The $\mathfrak{gl}_n$-crystal $\mathcal{B}$ is a $\mathfrak{q}_n$-crystal if both of the following hold:*

(a) *$e_{\overline{1}}$, $f_{\overline{1}}$ commute with e_i, f_i while preserving ε_i, φ_i for all $3 \leqslant i \leqslant n-1$,*
(b) *if $b, c \in \mathcal{B}$ then $e_{\overline{1}}(b) = c$ if and only if $f_{\overline{1}}(c) = b$, in which case*

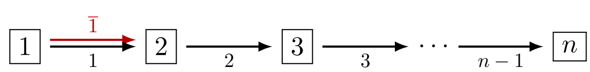
$$\mathrm{wt}(c) = \mathrm{wt}(b) + \boldsymbol{e}_1 - \boldsymbol{e}_2.$$

Assume $\mathcal{B}$ is a $\mathfrak{q}_n$-crystal. The corresponding *$\mathfrak{q}_n$-crystal graph* has vertex set $\mathcal{B}$ and edges $b \xrightarrow{i} c$ whenever $f_i(b) = c$ for any $i \in \{\overline{1}, 1, 2, \dots, n-1\}$.

Definition 2.9. *A $\mathfrak{q}_n$-crystal $\mathcal{B}$ is seminormal if it is seminormal as a $\mathfrak{gl}_n$-crystal and for all $b \in \mathcal{B}$ one has $\mathrm{wt}(b) \in \mathbb{N}^n$ and $\varphi_{\overline{1}}(b) + \varepsilon_{\overline{1}}(b) \leqslant 1$, with strict inequality if and only if $\mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0$.*

If $\mathcal{B}$ is a finite seminormal $\mathfrak{q}_n$-crystal then $\mathrm{ch}(B)$ is a $\mathbb{Z}$-linear combination of Schur P-polynomials $P_\lambda(x_1, x_2, \dots, x_n)$ by [20, Prop. 2.5].

Example 2.10. The *standard $\mathfrak{q}_n$-crystal* $\mathbb{B}_n = \left\{ \boxed{i} : i \in [n] \right\}$ has crystal graph



with $\mathrm{wt}(\boxed{i}) := \mathbf{e}_i$.

Suppose $\mathcal{B}$ and $\mathcal{C}$ are $\mathfrak{q}_n$-crystals. The set $\mathcal{B} \otimes \mathcal{C}$ already has a $\mathfrak{gl}_n$-crystal structure. There is a unique way of viewing this object as a $\mathfrak{q}_n$-crystal with

$$e_{\overline{1}}(b \otimes c) := \begin{cases} b \otimes e_{\overline{1}}(c) & \text{if } e_{\overline{1}}(b) = f_{\overline{1}}(b) = 0 \\ e_{\overline{1}}(b) \otimes c & \text{otherwise} \end{cases}$$
$$f_{\overline{1}}(b \otimes c) := \begin{cases} b \otimes f_{\overline{1}}(c) & \text{if } e_{\overline{1}}(b) = f_{\overline{1}}(b) = 0 \\ f_{\overline{1}}(b) \otimes c & \text{otherwise} \end{cases} \tag{2.7}$$

where it is again understood that $b \otimes 0 = 0 \otimes c = 0$ [8, Thm. 1.8]. The natural map $\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}) \to (\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}$ is again an isomorphism, and if $\mathcal{B}$ and $\mathcal{C}$ are seminormal then so is $\mathcal{B} \otimes \mathcal{C}$.

A $\mathfrak{q}_n$-crystal is *normal* if each of its full subcrystals is isomorphic to a full subcrystal of $\mathbb{B}_n^{\otimes m}$ for some $m \in \mathbb{N}$. As in the $\mathfrak{gl}_n$-case, these crystals have many nice properties, such as Schur P-positive characters; see [8, Thm. 2.5 and Cor. 4.6].

Remark 2.11. The formulas below for the action of $e_{\overline{1}}$ and $f_{\overline{1}}$ on $w = w_1 w_2 \cdots w_m \in \mathsf{Words}_n(m) \cong \mathbb{B}_n^{\otimes m}$ are easy to check from (2.7) by induction on m; see [7]:

- If w has no 2's or if a 1 appears before the first 2, then $e_{\overline{1}}(w) = 0$. Otherwise, $e_{\overline{1}}(w)$ is formed from w by changing the first 2 to 1.
- If w has no 1's or if a 2 appears before the first 1, then $f_{\overline{1}}(w) = 0$. Otherwise, $f_{\overline{1}}(w)$ is formed from w by changing the first 1 to 2.

The operators e_i and f_i for $i \in [n-1]$ act on w exactly as in the $\mathfrak{gl}_n$-case.

The *row reading word* of a shifted tableau T is the word $\mathsf{row}(T)$ formed by reading the rows from left to right, but starting with last row. The *reverse row reading word* of T is the reversal of $\mathsf{row}(T)$; we denote this by $\mathsf{revrow}(T)$.

Example 2.12. We have $\mathsf{row}\left(\begin{array}{|c|c|c|} \hline & 2 & 1 \\ \hline 2 & 2 & 3 \\ \hline \end{array} \right) = 21223$ and $\mathsf{revrow}\left(\begin{array}{|c|c|c|} \hline & 2 & 1 \\ \hline 2 & 2 & 3 \\ \hline \end{array} \right) = 32212.$

A *crystal embedding* is a weight-preserving injective map $\phi : \mathcal{B} \to \mathcal{C}$ between crystals that commutes with all crystal operators, in the sense that $\phi(e_i(b)) = e_i(\phi(b))$ and $\phi(f_i(b)) = f_i(\phi(b))$ for all $b \in \mathcal{B}$ when we set $\phi(0) = 0$. An example of a crystal embedding is provided by the following theorem from [8].

Theorem 2.13 ([8, Thm. 2.5(a)]). *Let λ be a strict partition of m with at most n parts. There is a unique $\mathfrak{q}_n$-crystal structure on $\mathit{DecTab}_n(\lambda)$ that makes*

$$\mathsf{revrow} : \mathit{DecTab}_n(\lambda) \to \mathsf{Words}_n(m)$$

into a $\mathfrak{q}_n$-crystal embedding. The weight map for this crystal has $x^{\mathrm{wt}(T)} = x^T$.

Since $\mathsf{Words}_n(m) \cong \mathbb{B}_n^{\otimes m}$, the $\mathfrak{q}_n$-crystal $\mathsf{DecTab}_n(\lambda)$ is automatically normal.

3. First construction

This section contains our first main result, which describes a "set-valued" analogue of the crystal on decomposition tableaux just introduced in Theorem 2.13. We review some preliminaries on *set-valued tableaux* in Section 3.1 before defining the relevant crystal operators.

3.1. Set-valued tableaux. A *set-valued tableaux* of partition shape λ is a filling T of the diagram D_λ by finite, nonempty sets of $\mathbb{P} = \{1, 2, 3, \dots\}$. A *set-valued shifted tableaux* of strict partition shape λ is likewise a filling T of SD_λ by finite, nonempty sets of $\{1' < 1 < 2' < 2 < \dots\}$. A *distribution* of a (shifted) set-valued tableau is a tableau of the same shape formed by replacing every set-valued entry by one of its elements. For example,

$$T = \begin{array}{|c|c|c|} \hline \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 1'3 & 234 \\ \hline \end{array}$$

has 6 distributions given by

$$\begin{array}{|c|c|c|} \hline \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 1' & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 1' & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 1' & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 3 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 3 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 3 & 4 \\ \hline \end{array}.$$

A set-valued (shifted) tableau is *semistandard* if its distributions are all semistandard according to the definition in Section 2.1.

Let $\mathsf{SetTab}(\lambda)$ be the set of all semistandard set-valued (unshifted) tableaux of shape λ. Then let $\mathsf{SetShTab}^+(\lambda)$ be the set of all semistandard set-valued shifted tableaux of shape λ, and define $\mathsf{SetShTab}(\lambda)$ to be the subset of such tableaux with no primed numbers appearing in any set-valued entries on the main diagonal. Define $\mathsf{SetTab}_n(\lambda)$, $\mathsf{SetShTab}^+_n(\lambda)$, and $\mathsf{SetShTab}_n(\lambda)$ using our convention in Section 2.1. The last three sets are nonempty if and only if $\ell(\lambda) \leqslant n$.

Given any set-valued tableau T, let

$$\mathrm{wt}(T) := (a_1, a_2, a_3, \dots) \quad \text{and} \quad x^T := \textstyle\prod_{k\in\mathbb{P}} x_k^{a_k} = x^{\mathrm{wt}(T)}$$

where a_k is how many times k or k' appears in T. The *symmetric Grothendieck function* of λ is

$$G_\lambda = \textstyle\sum_{T\in\mathsf{SetTab}(\lambda)} x^T. \tag{3.1}$$

Similarly, Ikeda and Naruse's *K-theoretic Schur P- and Q-functions* of λ are

$$GP_\lambda = \textstyle\sum_{T\in\mathsf{SetShTab}(\lambda)} x^T \quad \text{and} \quad GQ_\lambda = \textstyle\sum_{T\in\mathsf{SetShTab}^+(\lambda)} x^T. \tag{3.2}$$

Often in the literature the definitions of these power series involve a bookkeeping parameter β. Here, for simplicity, we have set $\beta = 1$. The more general definition is recovered by making the variable substitutions $x_i \mapsto \beta x_i$ and then dividing by $\beta^{|\lambda|}$.

Remark 3.1. Work of Buch [1] shows that each G_λ is symmetric with lowest degree term given by the Schur function s_λ. Similarly, GP_λ and GQ_λ are both Schur positive symmetric functions, though of unbounded degree [24, Thms. 3.27 and 3.40].

Specializations of GP_λ and GQ_λ give equivariant K-theory representatives for Schubert varieties in the maximal isotropic Grassmannians of orthogonal and symplectic types [15, Cor. 8.1]. These symmetric functions have another geometric interpretation as the stable limits of K-theory representatives for certain orbit closures in the complete flag variety [22, 23] as well as remarkable positivity properties [6, 12, 18, 30] (see [21, §4.6] for a survey of results and open problems). They also have other formulas besides the tableau generating functions given above; see [15, 16, 27].

Following [14], we define a *set-valued decomposition tableau* of strict partition shape λ to be a set-valued shifted tableau whose distributions are each decomposition tableaux of shape λ. Let $\mathsf{SetDecTab}(\lambda)$ be the set of all such tableaux and let $\mathsf{SetDecTab}_n(\lambda)$ be the subset with all entries at most n. A large amount of computation supports the following conjecture:

Conjecture 3.2 (Cho–Ikeda [14]). *It holds that* $GP_\lambda = \sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$.

Currently we do not know of any analogous conjectural formula for GQ_λ involving a set-valued analogue of decomposition tableaux. It would be interesting to find such a formula.

3.2. A seminormal crystal. Crystals for $\mathfrak{gl}_n$ have been identified on $\mathsf{SetTab}_n(\lambda)$ in [26] and [33] (see also [13, 29]). Both of these prior constructions have analogues for set-valued decomposition tableaux, which can be used to derive a weaker form of Conjecture 3.2. The simpler of these crystals is described in this section.

Fix a strict partition λ and $T \in \mathsf{SetDecTab}(\lambda)$. The *reverse row reading word* of T is the word $\mathsf{revrow}(T)$ formed by iterating over the boxes of T in the reverse row reading word order (starting with the last box of the first row and proceeding row by row, reading each row right to left), and listing the entries of each box in decreasing order. Define $\mathrm{wt}(T) = \mathrm{wt}(\mathsf{revrow}(T))$. For example, if

$T =$

	1	3
4	123	234

then $\mathsf{revrow}(T) = 432321431$ and $\mathrm{wt}(T) = (2, 2, 3, 2)$.

Fix $i \in \mathbb{P}$. Consider the word formed by replacing each i in $\mathsf{revrow}(T)$ by a ")" and each $i+1$ by a "(". We define a letter in $\mathsf{revrow}(T)$ to be *i-unpaired* if it is equal to i or $i+1$ but does not belong to a matching pair of parentheses in this modified word.

Definition 3.3. *Given* $i \in \mathbb{P}$ *and* $T \in \mathsf{SetDecTab}(\lambda)$, *form* $e_i(T) \in \mathsf{SetDecTab}(\lambda) \sqcup \{0\}$ *as follows:*

- *Define* $e_i(T) = 0$ *if there are no i-unpaired letters in* $\mathsf{revrow}(T)$ *equal to* $i+1$.
- *Otherwise, suppose the first i-unpaired $i+1$ in* $\mathsf{revrow}(T)$ *is in box* (x, y).
  (a) *Form* $e_i(T)$ *from* T *by changing the* $i+1$ *in box* (x, y) *to* i *if this yields a set-valued decomposition tableau. For example,*

$e_2 :$

	1	2
3	13	123

$\mapsto$

	1	2
3	12	123

.

  (b) *Otherwise, some box* (a, b) *preceding* (x, y) *in the reverse row reading word order has* $\{i, i+1\} \subseteq T_{ab}$. *If* (a, b) *is the last such box, then form*

$e_i(T)$ by removing $i+1$ from T_{ab} and adding i to T_{xy}. The box (a,b) either has $a = x$ and $b > y$, as in the example

$$e_3 : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 4 & 1 & 34 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 34 & 1 & 3 \\ \hline \end{array},$$

or has $a = x-1$ and $b < y$, as in the example

$$e_2 : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 123 & 234 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 23 \\ \hline 4 & 12 & 234 \\ \hline \end{array}.$$

Definition 3.4. *Given $i \in \mathbb{P}$ and $T \in \mathsf{SetDecTab}(\lambda)$, form $f_i(T) \in \mathsf{SetDecTab}(\lambda) \sqcup \{0\}$ as follows:*

- *Define $f_i(T) = 0$ if there are no i-unpaired letters in $\mathsf{revrow}(T)$ equal to i.*
- *Otherwise, suppose the last i-unpaired i in $\mathsf{revrow}(T)$ is in box (x,y) of T.*
  (a) *Form $f_i(T)$ from T by changing the i in box (x,y) to $i+1$ if this yields a set-valued decomposition tableau. For example,*

$$f_2 : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 3 & 12 & 123 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 3 & 13 & 123 \\ \hline \end{array}.$$

  (b) *Otherwise, some box (a,b) following (x,y) in the reverse row reading word order has $\{i, i+1\} \subseteq T_{ab}$. If (a,b) is the first such box, then form $f_i(T)$ by removing i from T_{ab} and adding $i+1$ to T_{xy}. The box (a,b) either has $a = x$ and $b < y$, as in the example*

$$f_3 : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 34 & 1 & 3 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 4 & 1 & 34 \\ \hline \end{array},$$

or has $a = x+1$ and $b > y$, as in the example

$$f_2 : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 23 \\ \hline 4 & 12 & 234 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 123 & 234 \\ \hline \end{array}.$$

It is not obvious that the operators e_i and f_i just given are well-defined, as one needs to verify the existence of the box (a,b) used in part (b) of each definition. This will be justified later in Lemmas 3.12 and 3.13.

A *multiset-valued decomposition tableau* is defined in the same way as a set-valued decomposition tableau, only we allow entries to be multisets. Let $\mathsf{SetDecTab}^*(\lambda)$ be the set of multiset-valued decomposition tableaux of shape λ with the property that only the number 1 appears more than once in any given box.

Definition 3.5. *Form $e^*_{\overline{1}}(T)$ and $f^*_{\overline{1}}(T)$ from $T \in \mathsf{SetDecTab}^*(\lambda)$ as follows:*

(a) *If $2 \notin \mathsf{revrow}(T)$ or the first $2 \in \mathsf{revrow}(T)$ is after a 1, then $e^*_{\overline{1}}(T) = 0$. Otherwise, change the first 2 in $\mathsf{revrow}(T)$ to 1. For example,*

$$e^*_{\overline{1}} : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 11 & 2 \\ \hline 3 & 13 & 123 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 11 & 2 \\ \hline 3 & 13 & 113 \\ \hline \end{array} \quad \textit{and} \quad e^*_{\overline{1}} : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 11 & 2 \\ \hline 3 & 13 & 113 \\ \hline \end{array} \mapsto 0.$$

(b) *If $1 \notin \mathsf{revrow}(T)$ or the first $1 \in \mathsf{revrow}(T)$ is after a 2, then $f^*_{\overline{1}}(T) = 0$. Otherwise, change the first 1 in $\mathsf{revrow}(T)$ to 2. For example,*

$$f^*_{\overline{1}} : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 111 & 2 \\ \hline 3 & 13 & \color{red}{13} \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 111 & 2 \\ \hline 3 & 13 & 23 \\ \hline \end{array} \quad \textit{and} \quad f^*_{\overline{1}} : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 3 & 13 & 123 \\ \hline \end{array} \mapsto 0.$$

Proposition 3.6. *The operations $e^*_{\overline{1}}$ and $f^*_{\overline{1}}$ define maps*

$$\mathsf{SetDecTab}^*(\lambda) \to \mathsf{SetDecTab}^*(\lambda) \sqcup \{0\},$$

and for all $T, U \in \mathsf{SetDecTab}^*(\lambda)$ it holds that $e^*_{\overline{1}}(T) = U$ if and only if $T = f^*_{\overline{1}}(U)$.

Proof. The claim that $e^*_{\overline{1}}(T) = U$ if and only if $T = f^*_{\overline{1}}(U)$ is clear from the definitions. It remains to check the less evident property that $e^*_{\overline{1}}(T)$ and $f^*_{\overline{1}}(T)$ both belong to $\mathsf{SetDecTab}^*(\lambda) \sqcup \{0\}$.

For this, choose $T \in \mathsf{SetDecTab}^*(\lambda)$ with $e^*_{\overline{1}}(T) \neq 0$. Consider the box of T containing the first 2 in the reverse row reading word order. On every distribution of T containing the 2 in this box, the crystal operator $e_{\overline{1}}$ from Theorem 2.13 acts by changing this 2 to 1. It follows that $e^*_{\overline{1}}(T) \in \mathsf{SetDecTab}^*(\lambda)$. Likewise, if $f^*_{\overline{1}}(T) \neq 0$ and we consider the box of T containing the first 1 in the reverse row reading word order, then $f_{\overline{1}}$ acts on every distribution of T containing the 1 in this box by changing this 1 to 2, so $f^*_{\overline{1}}(T) \in \mathsf{SetDecTab}^*(\lambda)$. ☐

Definition 3.7. *Define $f_{\overline{1}} : \mathsf{SetDecTab}(\lambda) \to \mathsf{SetDecTab}(\lambda) \sqcup \{0\}$ to be the restriction of $f^*_{\overline{1}}$. Then let $e_{\overline{1}}$ be the map $\mathsf{SetDecTab}(\lambda) \to \mathsf{SetDecTab}(\lambda) \sqcup \{0\}$ with $e_{\overline{1}}(T) = e^*_{\overline{1}}(T)$ if this is not a multiset-valued tableau, and with $e_{\overline{1}}(T) = 0$ otherwise.*

We observe that $e_{\overline{1}}$ acts as zero on set-valued decomposition tableaux for which the first box in the reverse row reading word order containing 1 or 2 contains both 1 and 2; notice that $f_{\overline{1}}$ also has this property. On all other tableaux $T \in \mathsf{SetDecTab}(\lambda)$ we have $e_{\overline{1}}(T) = e^*_{\overline{1}}(T)$.

Theorem 3.8. *Let λ be a strict partition with at most n parts, so that $\mathsf{SetDecTab}_n(\lambda)$ is nonempty. Then for the operators $e_{\overline{1}}, e_1, e_2, \dots, e_{n-1}$ and $f_{\overline{1}}, f_1, f_2, \dots, f_{n-1}$ given above, $\mathsf{SetDecTab}_n(\lambda)$ is a $\mathfrak{q}_n$-crystal that is seminormal as a $\mathfrak{gl}_n$-crystal.*

We defer the proof to Section 3.3 and discuss some applications here.

Remark 3.9. The crystal $\mathsf{SetDecTab}_n(\lambda)$ is not typically $\mathfrak{q}_n$-seminormal as it can occur that $\varphi_{\overline{1}}(T) + \varepsilon_{\overline{1}}(T) = 0$ outside the case when $\mathrm{wt}(T)_1 = \mathrm{wt}(T)_2 = 0$. More surprisingly, $\mathsf{SetDecTab}_n(\lambda)$ is not always a normal $\mathfrak{gl}_n$-crystal; see Figure 1. Thus we cannot deduce that $\sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$ is Schur positive from our crystal structure, although this is expected in view of Conjecture 3.2 as GP_λ is Schur positive [24, Thm. 3.27]. Even in special cases when the $\mathfrak{q}_n$-crystal $\mathsf{SetDecTab}_n(\lambda)$ is normal, it is often quite disconnected. See Figures 2 and 3 for examples.

We can at least use Theorem 3.8 to show that $\sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$ is symmetric, and a little more. A polynomial or formal power series f in an ordered sequence of variables $\{x_i\}$ satisfies the *K-theoretic Q-cancelation property* if for all variable indices $i < j$ the substitution

$$f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{j-1}, \tfrac{-t}{1+t}, x_{j+1}, \dots)$$

does not depend on t. When f is symmetric, to check this property it suffices to take $i = 1$ and $j = 2$.

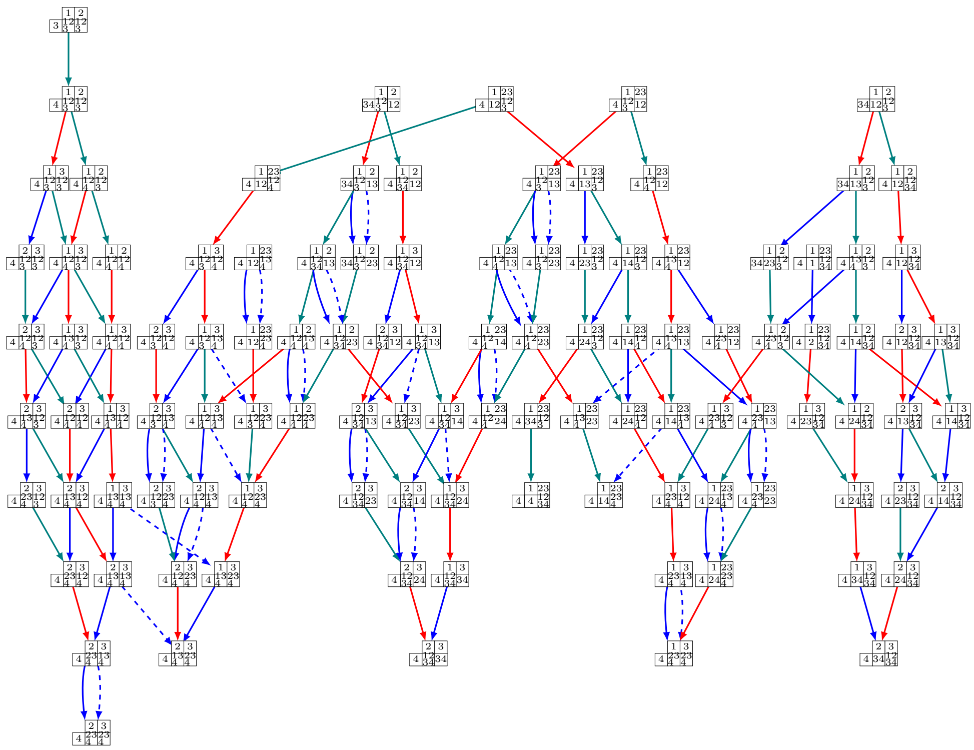


Figure 1. A connected component of the $\mathfrak{q}_4$-crystal of set-valued decomposition tableaux $\mathsf{SetDecTab}_4(\lambda)$ for $\lambda = (3,2)$ that is not normal as a $\mathfrak{gl}_n$-crystal. Solid blue, red, and green arrows indicate $\xrightarrow{1}$, $\xrightarrow{2}$, and $\xrightarrow{3}$ edges, while dotted blue arrows indicate $\xrightarrow{\overline{1}}$ edges.

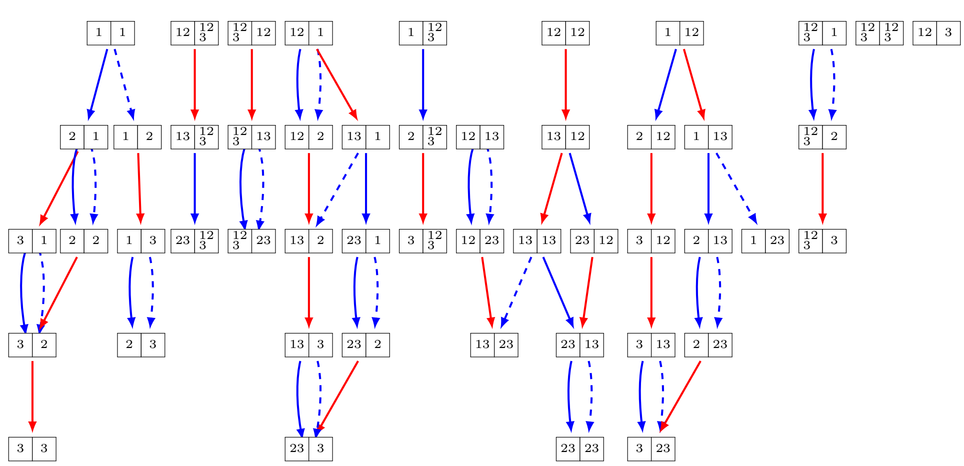


Figure 2. Crystal graph of the $\mathfrak{q}_3$-crystal $\mathsf{SetDecTab}_3(\lambda)$ for $\lambda = (2)$. Here, solid blue and red arrows respectively indicate $\xrightarrow{1}$ and $\xrightarrow{2}$ edges while dashed blue arrows indicate $\xrightarrow{\overline{1}}$ edges.

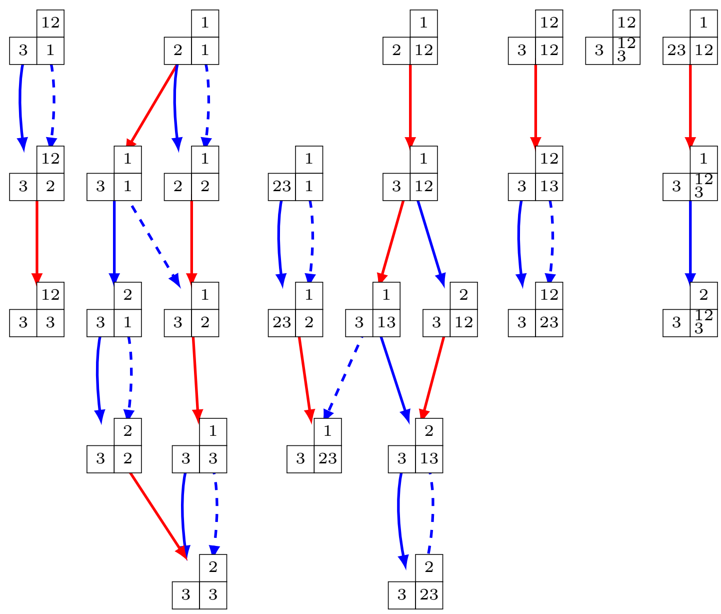


Figure 3. Crystal graph of the $\mathfrak{q}_3$-crystal $\mathsf{SetDecTab}_3(\lambda)$ for $\lambda = (2,1)$. Here, solid blue and red arrows respectively indicate $\xrightarrow{1}$ and $\xrightarrow{2}$ edges while dashed blue arrows indicate $\xrightarrow{\overline{1}}$ edges.

The symmetric functions in $\mathbb{Z}[\![x_1, x_2, \dots]\!]$ with the K-theoretic Q-cancelation property are the ones that may be (uniquely) expressed as a possibly infinite $\mathbb{Z}$-linear combinations of GP-functions [15, Prop. 3.4]. In finitely many variables, the polynomials $GP_\lambda(x_1, x_2, \dots, x_n)$, with λ ranging over all strict partitions having at most n parts, form a $\mathbb{Z}$-basis for the subring of symmetric elements in $\mathbb{Z}[x_1, x_2, \dots, x_n]$ with the K-theoretic Q-cancelation property [15, Thm. 3.1].

Proposition 3.10. *The power series* $\sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$ *is a symmetric function with the K-theoretic Q-cancelation property and lowest degree term* P_λ.

Proof. Let $GP^{[\mathsf{dec}]}_\lambda := \sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$. This power series is symmetric since each polynomial $GP^{[\mathsf{dec}]}_\lambda(x_1, x_2, \dots, x_n)$ is the character of a seminormal $\mathfrak{gl}_n$-crystal, namely $\mathsf{SetDecTab}_n(\lambda)$. The lowest degree term of $GP^{[\mathsf{dec}]}_\lambda$ is evidently $P_\lambda = \sum_{T \in \mathsf{DecTab}(\lambda)} x^T$.

To check the K-theoretic Q-cancelation property, observe that

$$GP^{[\mathsf{dec}]}_\lambda\left(\tfrac{x_1}{1-x_1}, -x_2, x_3, x_4, \dots\right) = \sum_{T \in \mathsf{SetDecTab}^*(\lambda)} (-1)^{c_2(T)} x^T$$

where $c_i(T)$ denotes the number of times that i appears in T. We have $e^*_{\overline{1}}(T) = f^*_{\overline{1}}(T) = 0$ if and only if $c_1(T) = c_2(T) = 0$. If $e^*_{\overline{1}}(T) \neq 0$ then $c_2(T) = c_2(e^*_{\overline{1}}(T)) + 1$ and $x^T = x^{e^*_{\overline{1}}(T)} x_2/x_1$, in which case $\left((-1)^{c_2(T)} x^T + (-1)^{c_2(e^*_{\overline{1}}(T))} x^{e^*_{\overline{1}}(T)}\right)\big|_{x_1=x_2} = 0$. Therefore

$$GP^{[\mathsf{dec}]}_\lambda\left(\tfrac{u}{1-u}, -u, x_3, x_4, \dots\right) = \sum_{\substack{T \in \mathsf{SetDecTab}^*(\lambda) \\ c_1(T)=c_2(T)=0}} x^T = GP^{[\mathsf{dec}]}_\lambda(x_3, x_4, \dots).$$

Setting $u = \frac{t}{1+t}$ turns this into the K-theoretic Q-cancelation property. ☐

As a corollary of the preceding result, we get this weaker form of Conjecture 3.2:

Corollary 3.11. *If λ is a strict partition then $\sum_{T\in\mathsf{SetDecTab}(\lambda)} x^T$ is equal to GP_λ plus a possibly infinite $\mathbb{Z}$-linear combination of power series GP_μ with $|\mu| > |\lambda|$.*

Proof. As $GP_\lambda^{[\mathsf{dec}]}$ has the K-theoretic cancellation property by Proposition 3.10, it is a formal $\mathbb{Z}$-linear combination of GP-functions. Therefore, we can write $GP_\lambda^{[\mathsf{dec}]} = \sum_\mu a_\mu GP_\mu$ for some integers a_μ indexed by strict partitions μ.

We now consider the lowest degree terms of both sides of this equation. Both $GP_\lambda^{[\mathsf{dec}]}$ and GP_λ have lowest degree term P_λ. Therefore, the lowest degree term of the left side is P_λ while the lowest degree term of the right side is $\sum_{|\mu|=k} a_\mu P_\mu$ where $k \in \mathbb{N}$ is minimal with $a_\mu \neq 0$. Since Schur P-functions are linearly independent and homogeneous with $\deg P_\lambda = |\lambda|$, these terms can only be equal if we have $k = |\lambda|$, $a_\lambda = 1$, and $a_\mu = 0$ for all μ with $|\mu| < k$ or with $|\mu| = k$ and $\mu \neq \lambda$. ☐

3.3. Proof of Theorem 3.8. Fix a strict partition λ. We derive Theorem 3.8 after proving a pair of technical lemmas.

Lemma 3.12. *Fix $i \in [n-1]$ and $T \in \mathsf{SetDecTab}_n(\lambda)$. Suppose there are i-unpaired letters equal to $i+1$ in $\mathsf{revrow}(T)$. Let (x, y) be the box of T containing the first such $i+1$. Assume that changing this $i+1$ to i does not yield a set-valued decomposition tableau. Then:*

(1) *None of the following occurs:*
   (a) *$x = y$ and some $z > y$ has $i \in T_{x+1,z}$.*
   (b) *$x < y$ and some $z > y$ has $\max(T_{x+1,y}) \geqslant i \in T_{x+1,z}$.*
   (c) *$1 < x < y$ and some $1 < z < y$ has $\min(T_{x-1,z}) \leqslant i \in T_{xz}$.*

Equivalently, T does not contain any of the following configurations:

	$\cdots$	i
T_{xy}	$\cdots$	

,

$\geqslant i$	$\cdots$	i
T_{xy}	$\cdots$	

, *or*

i	$\cdots$	T_{xy}
$\leqslant i$	$\cdots$	

.

(2) *A box of T preceding (x, y) in the reverse row reading word order contains both i and $i+1$. Let (a, b) be the last such box. Then $a \in \{x-1, x\}$ and one of the following holds:*
   (a) *$a = x$, $b = y+1$, and $\min(T_{x,y+2}) \leqslant i$, so T has the configuration*

$\cdots$	$i+1 \in T_{xy}$	$i, i+1 \in T_{ab}$	$\leqslant i$	$\cdots$

.

   (b) *$a = x$, $b > y$, and $\min(T_{x+1,b}) < i$, so T has the configuration*

	$\cdots$		$\cdots$	$< i$
	$\cdots$	$i+1 \in T_{xy}$	$\cdots$	$i, i+1 \in T_{ab}$

.

(c) $a = x-1$, $b < y$, *and* $\max(T_{ay}) > i+1$, *so* T *has the configuration*

	$\cdots$		$\cdots$	$i+1$ $\in T_{xy}$
	$\cdots$	$i, i+1$ $\in T_{ab}$	$\cdots$	$> i+1$

In addition, no box of T *between* (a,b) *and* (x,y) *in the reverse row reading word order contains* i *or* $i+1$.

(3) *If* (a,b) *is the box defined in part (2), then removing* $i{+}1$ *from* T_{ab} *and adding* i *to* T_{xy} *yields a set-valued decomposition tableau.*

Proof. Suppose the configuration in (1)(a) occurs. By Lemma 2.3(a), row $x{+}1$ cannot contain an $i+1$. But then the $i+1$ in T_{xx} would have paired with the i in $T_{x+1,z}$ in $\mathsf{revrow}(T)$, contradict to the fact that this $i+1$ is i-unpaired. Thus, the configuration in (1)(a) must not occur.

Let $\mathrm{distr}_{i,max}(T)$ be the distribution of T formed by choosing i or $i+1$ in every box that contains $\{i, i+1\}$ appearing before or after (x,y) in $\mathsf{revrow}(T)$ respectively, choosing either i or $i+1$ in boxes that contain i or $i+1$, respectively, but not both, and choosing the largest entry in all other boxes. The $i+1$ in box (x,y) remains the first i-unpaired letter in $\mathsf{revrow}(T)$. Also, the tableaux $\mathrm{distr}_{i,max}(T)$, $e_j(\mathrm{distr}_{i,max}(T))$ and $f_j(\mathrm{distr}_{i,max}(T))$ must be valid decomposition tableaux.

Now suppose the configuration in (1)(b) occurs. Then $\mathrm{distr}_{i,max}(T)$ selects $i+1$ from T_{xy}, i from $T_{x+1,z}$ and either $\max(T_{x+1,y})$ or $i+1$ or i from $T_{x+1,y}$. In any case, $e_i(\mathrm{distr}_i(T))$ is formed from $\mathrm{distr}_i(T)$ by changing the $i+1$ in box (x,y) to i. Now boxes $(x{+}1, y)$, $(x{+}1, z)$ and (x,y) form a forbidden pattern in Lemma 2.3(a), which is impossible.

Suppose the configuration in (1)(c) occurs. By the hook word condition in row x, no $i{+}1$ can be found between T_{xz} and T_{xy}. However, the $i{+}1 \in T_{xy}$ would be paired with $i \in T_{xz}$ in $\mathsf{revrow}(T)$, contradict to the assumption that $i{+}1 \in T_{xy}$ is not paired.

We conclude that none of the configurations listed in part (1) occurs.

For part (2), we observe that changing $i+1$ to i in T_{xy} fails to produce a set-valued decomposition tableau when either (i) the row x is not a hook word for some distribution of T, or (ii) the row x is part of a forbidden pattern from Lemma 2.3, which may involve either row $x+1$ or $x-1$.

For case (i), there are two possibilities to consider: (α) the $i{+}1$ in box (x,y) belongs to the decreasing part of row x in a particular distribution of T, and there exists an $i+1$ in box $(x, y+1)$ or (β) the $i+1$ in box (x,y) is part of the increasing part of row x in a distribution of T, and there exists an i in box $(x, y-1)$ that is also part of the increasing part of row x in the same distribution.

Assume that possibility (α) occurs. Then there is also an i in box $(x, y+1)$ as otherwise the $i+1$ in box $(x, y+1)$ would be an earlier unpaired $i+1$. It also must hold that $\min(T_{x,y+2}) \leqslant i+1$ because changing $i+1 \in T_{xy}$ to i violates the hook word condition. Finally, we cannot have $\min(T_{x,y+2}) = i+1$ because then this $i+1$ would be unpaired in $\mathsf{revrow}(T)$ while preceding the $i+1 \in T_{xy}$. Thus we actually have $\min(T_{x,y+2}) \leqslant i$ and T is as described in case (2)(a).

On the other hand, case (β) cannot occur. If case (β) occur, then $i+1$ would also be present in box $(x, y-1)$; otherwise, the i in box $(x, y-1)$ would be paired with the $i+1$ in box (x,y). However, this would lead to the contradiction that an element in box $(x, y-2)$ is smaller than i, implying that there exists a distribution that has consecutive entries $a, i+1, i+1$ with $a < i$ in row x, which violates the condition of each row being a hook word in a decomposition tableau.

Now we examine case (ii). In view of part (1), when changing $i+1$ to i in T_{xy}, only the forbidden patterns described in Lemma 2.3(b) remain as possibilities. These patterns occur if and only if one of the following configurations occur:

	$\cdots$		$\cdots$	$< i$
	$\cdots$	$i+1 \in T_{xy}$	$\cdots$	$i+1$

or

	$\cdots$		$\cdots$	$i+1 \in T_{xy}$
	$\cdots$	$i+1$	$\cdots$	$> i+1$

.

Define (c,d) to be the box containing the red $i+1$. We now show that T_{cd} must contain both i and $i+1$. Observe that T_{cd} must contain $i+1$ to ensure that some forbidden pattern arises when changing the $i+1$ in T_{xy} to i. Suppose T_{cd} does not contain i. Since T_{xy} represents the first occurrence of an i-unpaired $i+1$ in $\mathsf{revrow}(T)$, the $i+1$ in box T_{cd} must be paired with an i in a box T_{xf} where $y < f < d$ for case (2)(b), or in a box T_{ef} where $e = x$, $f > y$, or $e = x-1$, $f < d$ for case (2)(c). However, in case (2)(b), the distribution with entries $\min(T_{x+1,b}) < i$, T_{xf}, and T_{cd} forms the same type of forbidden pattern right from the start. This is illustrated by the picture below

	$\cdots$		$\cdots$		$\cdots$	$< i$
	$\cdots$	$i+1 \in T_{xy}$	$\cdots$	$i \in T_{xf}$	$\cdots$	$i+1 \in T_{cd}$

where the forbidden pattern is highlighted in red.

In the subcase for (2)(c) where i is present in $T_{x-1,f}$ with $f < d$, we consider two slightly different situations: one where $d = y-1$ and another where $d < y-1$. In both situations, we have $\max(T_{xd}) < i+1$, since if $\max(T_{xd}) \geqslant i+1$, it would lead to a forbidden pattern with boxes T_{xd}, T_{xy}, and $T_{cd} = T_{x-1,d}$, as described in Lemma 2.3(a).

Now, we focus on the situation where $d = y-1$. In this case, $\max(T_{xd})$ cannot be i, because if it were, the $i+1$ in T_{xy} would be paired with this i. However, in this situation, the boxes $T_{x,y-1}$, $T_{x-1,f}$, and $T_{x-1,y-1}$ form a forbidden pattern as described in Lemma 2.3(b). The situation is illustrated below:

	$\cdots$		$\cdots$	$< i$	$i+1 \in T_{xy}$	$\cdots$
	$\cdots$	i	$\cdots$	$i+1 \in T_{cd}$	$> i+1$	$\cdots$

.

If $d < y-1$, then $\max(T_{xd})$ still cannot be equal to i. If we assume that $\max(T_{xd}) = i$, then there would be an $i+1$ in a box T_{xg} where $d < g < y$. However, it is impossible to have a subword $i, i+1, i+1$ in the boxes T_{xd}, T_{xg}, T_{xy} because the $i+1 \in T_{xg}$ would lie in both the increasing and decreasing parts of a hook word in a specific distribution of T. Therefore, $\max(T_{xd})$ must be strictly less than i, but then the boxes T_{xd}, $T_{x-1,f}$, and $T_{x-1,d}$ form a forbidden pattern described in

Lemma 2.3(b). The situation is illustrated below:

	$\cdots$		$\cdots$	$< i$	$\cdots$	$i+1$ $\in T_{xy}$	$\cdots$
	$\cdots$	i	$\cdots$	$i+1$ $\in T_{cd}$	$\cdots$	$> i+1$	$\cdots$

.

Finally, in the subcase for (2)(c) where i is present in T_{xf} with $f > y$, note that our assumptions imply that the $i+1$ in T_{xy} must be part of the decreasing part of row x in any distribution where i is in T_{xf}. Consequently, we have $\max(T_{c+1,d}) \geqslant i+1$. However, this leads to a forbidden pattern of type (a) in Lemma 2.3 with the boxes $T_{c+1,d}$, T_{xy}, and T_{cd}. The situation is illustrated below:

	$\cdots$	$\geqslant i+1$	$\cdots$	$i+1$ $\in T_{xy}$	$\cdots$	i	$\cdots$
	$\cdots$	$i, i+1$ $\in T_{cd}$	$\cdots$	$> i+1$	$\cdots$		$\cdots$

.

Therefore, we can conclude that the box T_{cd} must contain both i and $i+1$.

We will now show that if there exists a box containing both i and $i+1$ located strictly between the box marked (c,d) and (x,y) in the reverse row reading order, for cases (2)(b) and (2)(c), then this box must also yield a forbidden pattern in cases (2)(b) or (2)(c). Consequently, without loss of generality, we can deduce that box (c,d) coincides with (a,b), which is defined as the last such box preceding (x,y) in the reverse row reading order containing both i and $i+1$.

In fact, since $i+1 \in T_{cd}$, the same argument as above demonstrates that no box in T located strictly between (c,d) and (x,y) in the reverse row reading word order can contain i. This also implies that no box in T located strictly between (c,d) and (x,y) in the reverse row reading word order can contain $i+1$, since if it did, this $i+1$ would be an earlier i-unpaired letter in $\mathsf{revrow}(T)$. Hence, we can conclude that (2) is proven by asserting $(a,b) = (c,d)$.

Finally, part (3) can be concluded in two steps. First, removing $i+1$ from T_{ab} will certainly result in a set-valued decomposition tableau. This is because removing $i+1$ does not introduce new forbidden patterns, and we are only eliminating a subset of the distributions that form decomposition tableaux in T. Second, adding i to T_{xy} in case (2)(a) will not violate the hook word condition, since $\max(T_{ab}) = i+1$ but $i+1$ is already removed from T_{ab}.

Moreover, we observe that adding i to T_{xy} creates new forbidden patterns in the cases (2)(b) and (2)(c) only under the occurrence of one of the following two cases:

	$\cdots$		$\cdots$	$< i$	$< i$
	$\cdots$	$i, i+1$ $\in T_{xy}$	$\cdots$	$i \in T_{ab}$	$i+1$

or

	$\cdots$				$\cdots$	$i, i+1$ $\in T_{xy}$
	$\cdots$	$i \in T_{ab}$	$i+1$	$> i+1$	$\cdots$	$\gg i+1$

.

Here we use the notation "$\gg i+1$" to mean that the maximum entry in that box is greater than the maximum entry in the box labeled $> i+1$. However, these cases lead to forbidden patterns in distributions of T right from the start, as highlighted in red in the above tableaux. Therefore, we can conclude that removing $i+1$ from T_{ab} and adding i to T_{xy} indeed results in a set-valued decomposition tableau. □

Our second technical lemma is the f-version of Lemma 3.12.

Lemma 3.13. *Fix $i \in [n-1]$ and $T \in \mathsf{SetDecTab}_n(\lambda)$. Suppose there are i-unpaired letters equal to i in $\mathsf{revrow}(T)$. Let (x, y) be the box of T containing the last such i. Assume that changing this i to $i+1$ does not yield a set-valued decomposition tableau. Then:*

(1) *None of the following occurs:*
  (a) $1 < x \leqslant y$ *and* $i+1 \in T_{x-1,x-1}$.
  (b) $1 < x \leqslant y$ *and some* $z > y$ *has* $\min(T_{x-1,y}) \leqslant i+1 \in T_{xz}$.
  (c) $1 < x < y$ *and some* $1 < z < y$ *has* $\max(T_{xz}) \geqslant i+1 \in T_{x-1,z}$.
  Equivalently, T does not contain any of the following configurations:

	$\cdots$	T_{xy}
$i+1$	$\cdots$	

,

T_{xy}	$\cdots$	$i+1$
$\leqslant i+1$	$\cdots$	

, *or*

$\geqslant i+1$	$\cdots$	T_{xy}
$i+1$	$\cdots$	

.

(2) *A box of T following (x, y) in the reverse row reading word order contains both i and $i+1$. Let (a, b) be the first such box. Then $a \in \{x, x+1\}$ and one of the following holds:*
  (a) $a = x$, $b = y-1$, *and* $\min(T_{x,y+1}) \leqslant i$, *so T has the configuration*

$\cdots$	$i, i+1$ $\in T_{ab}$	i $\in T_{xy}$	$\leqslant i$	$\cdots$

.

  (b) $a = x$, $b < y$, *and* $\min(T_{x+1,y}) < i$, *so T has the configuration*

	$\cdots$		$\cdots$	$< i$
	$\cdots$	$i, i+1$ $\in T_{ab}$	$\cdots$	i $\in T_{xy}$

.

  (c) $a = x+1$, $b > y$, *and* $\max(T_{xb}) > i+1$, *so T has the configuration*

	$\cdots$		$\cdots$	$i, i+1$ $\in T_{ab}$
	$\cdots$	i $\in T_{xy}$	$\cdots$	$> i+1$

.

In addition, no box of T between (x, y) and (a, b) in the reverse row reading word order contains i or $i+1$.

(3) *If (a, b) is the box defined in part (2), then removing i from T_{ab} and adding $i+1$ to T_{xy} yields a set-valued decomposition tableau.*

Proof. Suppose the configuration in (1)(a) occurs. By Lemma 2.3(a), row x cannot contain an $i+1$. But then the i in $T_{x,y}$ would have paired with the $i+1$ in $T_{x-1,x-1}$ in $\mathsf{revrow}(T)$, contradict to the fact that this i is i-unpaired. Thus, the configuration in (1)(a) must not occur.

Suppose the configuration in (1)(b) occurs. By the hook word condition in row x, no $i+1$ can be found between T_{xy} and T_{xz}. However, the $i \in T_{xy}$ would be paired with $i+1 \in T_{xz}$ in $\mathsf{revrow}(T)$, contradict to the assumption that $i \in T_{xy}$ is not paired.

Define $\mathrm{distr}_{i,max}(T)$ in the same way as in Lemma 3.12, but now using the definition of (x, y) as the box of T containing the last i-unpaired letter equal to i in $\mathsf{revrow}(T)$.

Now suppose the configuration in (1)(c) occurs. Then $\mathrm{distr}_{i,max}(T)$ selects i from T_{xy}, $i+1$ from $T_{x-1,z}$ and either $\max(T_{x+1,y})$ or $i+1$ from $T_{x,z}$. In any case, $f_i(\mathrm{distr}_i(T))$ is formed from $\mathrm{distr}_i(T)$ by changing the i in box (x, y) to $i+1$. Now boxes $(x-1, z)$, (x, y) and (x, z) form a forbidden pattern in Lemma 2.3(a), which is impossible.

We conclude that none of the configurations listed in part (1) occurs.

For part (2), note that changing i to $i+1$ in T_{xy} fails to produce a set-valued decomposition tableau when either (i) the row x is not a hook word for some distribution of T, or (ii) the row x is part of a forbidden pattern as described in Lemma 2.3, which may involve either row $x+1$ or $x-1$.

For case (i), there are two possibilities to consider: (α) the i in box (x, y) belongs to the decreasing part of row x in a particular distribution T, and there exists an i in box $(x, y-1)$ or (β) the i in box (x, y) is part of the increasing part of row x in a distribution T, and there exists an $i+1$ in box $(x, y+1)$.

Assume that possibility (α) occurs. Then there is also an $i+1$ in box $(x, y-1)$ as otherwise the i in box $(x, y+1)$ would be a later unpaired i. Moreover, it must hold that $\min(T_{x,y+1}) \leqslant i+1$ because changing $i \in T_{xy}$ to $i+1$ violates the hook word condition. Finally, we cannot have $\min(T_{x,y+2}) = i+1$ as then the $i \in T_{xy}$ would be paired with $i+1 \in T_{x,y+1}$ in $\mathsf{revrow}(T)$. Thus we actually have $\min(T_{x,y+2}) \leqslant i$ and T is as described in case (2)(a).

On the other hand, possibility (β) cannot actually arise. If such a scenario were to occur, it would mean that i would also be present in box $(x, y+1)$; otherwise, the i in box (x, y) would be paired with the $i+1$ in box $(x, y+1)$. However, this would lead to the contradiction that an element in box $(x, y-1)$ is smaller than i, implying that there exists a distribution that has consecutive entries a, i, i with $a < i$ in row x, which violates the condition of each row being a hook word in a decomposition tableau.

Now we examine case (ii). In view of part (1), when changing i to $i+1$ in T_{xy}, only the forbidden patterns described in Lemma 2.3(b) remain as possibilities. These patterns occur if and only if one of the following configurations occur:

	$\cdots$		$\cdots$	$< i$
	$\cdots$	i	$\cdots$	i $\in T_{xy}$

or

	$\cdots$		$\cdots$	i
	$\cdots$	i $\in T_{xy}$	$\cdots$	$> i+1$

.

Define (c, d) to be the box containing the red i. Next we show that T_{cd} must contain both i and $i+1$. Observe that T_{cd} must contain i to ensure that some forbidden pattern arises when changing the i in T_{xy} to $i+1$. Suppose T_{cd} does not contain $i+1$. Since T_{xy} represents the last occurrence of an i-unpaired i in $\mathsf{revrow}(T)$, the i in box T_{cd} must be paired with an $i+1$ in a box T_{xf} where $d < f < y$ for case (2)(b), or in a box T_{ef} where $e = x$, $f < y$, or $e = x+1$, $f > d$ for case (2)(c). However, in case (2)(b), the distribution with entries $i \in T_{cd}$, $x+1 \in T_{xf}$, and $i \in T_{xy}$ violates the condition of being a hook word for each row in T right from the start. This is illustrated in the picture below:

	$\cdots$	$i \in T_{cd}$	$\cdots$	$i+1$ $\in T_{xf}$	$\cdots$	$i \in T_{xy}$

.

In the subcase for (2)(c) where $i+1$ is present in $T_{x+1,f}$ with $f > d$, notice that $\max(T_{xf}) > \max(T_{xd})$, since $\max(T_{xd}) > i+1 > i \in T_{xy}$. However, the entries $i+1$, $\max(T_{xd})$ and $\max(T_{xf})$ in boxes $T_{x+1,f}$, T_{xd} and T_{xf} respectively form a forbidden pattern of type (b) in Lemma 2.3. The situation is illustrated below:

	$\cdots$		$\cdots$	$i \in T_{cd}$	$\cdots$	$i+1$ $\in T_{x+1,f}$
	$\cdots$	$i \in T_{xy}$	$\cdots$	$> i+1$	$\cdots$	$\gg i+1$

,

where the notation "$\gg i+1$" again means that the maximum entry in that box is greater than the maximum entry in the box labeled $> i+1$.

Finally, in the subcase for (2)(c) where $i+1$ is present in T_{xf} with $f < y$, the entries i, $i+1$, and $\max(T_{xd})$ in boxes T_{cd}, T_{xf} and T_{xd} respectively form a forbidden pattern of type (b) in Lemma 2.3 The situation is illustrated below:

	$\cdots$		$\cdots$		$\cdots$	$i \in T_{cd}$	$\cdots$
	$\cdots$	$i+1$ $\in T_{xf}$	$\cdots$	T_{xy}	$\cdots$	$> i+1$	$\cdots$

.

Therefore, we conclude that T_{cd} must contain both i and $i+1$.

We will now show that if there exists a box containing both i and $i+1$ located strictly between the box marked (x, y) and (c, d) in the reverse row reading order, for cases (2)(b) and (2)(c), then this box must also yield a forbidden pattern in cases (2)(b) or (2)(c). Consequently, without loss of generality, we can deduce that box (c, d) coincides with (a, b), which is defined as the first box after (x, y) in the reverse row reading order containing both i and $i+1$.

In fact, since $i \in T_{cd}$, the same argument as above demonstrates that no box in T located strictly between (x, y) and (c, d) in the reverse row reading word order can contain $i+1$. This also implies that no box in T located strictly between (x, y) and (c, d) in the reverse row reading word order can contain i, since if it did, this i would be a later i-unpaired letter in $\mathsf{revrow}(T)$. Hence, we can conclude that (2) is proven by asserting $(a, b) = (c, d)$.

Finally, part (3) can be concluded in two steps. First, removing i from T_{ab} will certainly result in a set-valued decomposition tableau. Second, adding $i+1$ to T_{xy} in case (2)(a) will not violate the hook word condition, since $\min(T_{ab}) \geqslant i$ but i is already removed from T_{ab}.

Moreover, we observe that adding $i+1$ to T_{xy} creates new forbidden patterns of the form (2)(b) and (2)(c) only when one of the following cases occurs:

	$\cdots$		$\cdots$	$< i$	$\cdots$	i
	$\cdots$	$i+1$ $\in T_{ab}$	$\cdots$	$i, i+1$ $\in T_{xy}$	$\cdots$	$> i+1$

or

	$\cdots$		$\cdots$	i	$\cdots$	$i+1$ $\in T_{ab}$
	$\cdots$	$i, i+1$ $\in T_{xy}$	$\cdots$	$> i+1$	$\cdots$	$\gg i+1$

.

However, these cases lead to forbidden patterns in distributions of T right from the start, as highlighted in red in the above tableaux. Therefore, we can conclude that removing i from T_{ab} and adding $i+1$ to T_{xy} indeed results in a set-valued decomposition tableau. □

We can now supply the main proof of this section.

Proof of Theorem 3.8. It is clear from Lemmas 3.12 and 3.13—in particular, the last part of each statement—that if $i \in \{\overline{1}, 1, 2, \dots, n-1\}$ then our definitions of e_i and f_i give well-defined maps $\mathsf{SetDecTab}_n(\lambda) \to \mathsf{SetDecTab}_n(\lambda) \sqcup \{0\}$.

It is also evident that when they do not act as a zero, e_i and f_i add or subtract $\mathbf{e}_i - \mathbf{e}_{i+1}$ to the weight, while $e_{\overline{1}}$ and $f_{\overline{1}}$ add or subtract $\mathbf{e}_1 - \mathbf{e}_2$. For $i \in [n-1]$ and $T \in \mathsf{SetDecTab}_n(\lambda)$, we have $\varphi_i(T) - \varepsilon_i(T) = \mathrm{wt}(T)_i - \mathrm{wt}(T)_{i+1}$ since $\varphi_i(T)$ and $\varepsilon_i(T)$ are the respective numbers of i-unpaired letters in $\mathsf{revrow}(T)$ equal to i and $i+1$.

The remarks before Theorem 3.8 show that if $T, U \in \mathsf{SetDecTab}_n(\lambda)$ then $e_{\overline{1}}(T) = U$ if and only if $T = f_{\overline{1}}(U)$. The only thing remaining is the analogous property for e_i and f_i when $i \in [n-1]$, which we can check using Lemmas 3.12 and 3.13.

Suppose $e_i(T) \neq 0$. If $e_i(T)$ is formed from T by changing the first i-unpaired $i+1$ letter in $\mathsf{revrow}(T)$ to i, then this newly created i will be the last i-unpaired i in $\mathsf{revrow}(e_i(T))$ and so f_i will act to change it back to $i+1$, giving $f_i(e_i(T)) = T$.

If instead $e_i(T)$ is formed by removing $i+1$ from T_{ab} and adding i to T_{xy} using the notation in Lemma 3.12, then the remaining $i \in T_{ab}$ will become the last i-unpaired i in $\mathsf{revrow}(e_i(T))$. Inspecting cases (a), (b), and (c) in part (2) of Lemma 3.12 shows that changing this i to $i+1$ will not produce a set-valued decomposition tableau. We are therefore in the situation of Lemma 3.13, from which it follows again that $f_i(e_i(T)) = T$. The argument to show that $e_i(f_i(T)) = T$ when $f_i(T) \neq 0$ is similar, just swapping the roles of Lemmas 3.12 and 3.13. □

4. Second construction

This section describes another crystal structure on set-valued decomposition tableaux, which is based on a construction of Yu [33] for unshifted tableaux. Unlike our results in the previous section, we can place this crystal in a more general abstract framework, involving objects that we call *square root crystals*. We will see that a natural tensor product for square root crystals motivates the crystal operator definitions in [33], and also provides a novel interpretation of the nonnegative integer coefficients appearing in the product expansion $G_\lambda G_\mu = \sum_\nu c_{\lambda\mu}^\nu G_\nu$.

4.1. Abstract square root crystals. Let n be any positive integer. Suppose $\mathcal{B}$ is a nonempty set with maps $\mathrm{wt} : \mathcal{B} \to \mathbb{Z}^n$ and $e'_i, f'_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ for $i \in [n-1]$, where $0 \notin \mathcal{B}$. Define $\varepsilon'_i, \varphi'_i : \mathcal{B} \to \{0,1,2,\dots\} \sqcup \{\infty\}$ by

$$\varepsilon'_i(b) := \sup\left\{k \geqslant 0 : (e'_i)^k(b) \neq 0\right\} \text{ and } \varphi'_i(b) := \sup\left\{k \geqslant 0 : (f'_i)^k(b) \neq 0\right\}. \tag{4.1}$$

Definition 4.1. *The set $\mathcal{B}$ is a $\sqrt{\mathfrak{gl}_n}$-crystal if for all $i \in [n-1]$ and $b, c \in \mathcal{B}$ both of the following conditions hold:*

(a) $\varepsilon'_i(b) + \varphi'_i(b) \in 2\mathbb{N}$ *is even with* $\frac{\varphi'_i(b)-\varepsilon'_i(b)}{2} = \mathrm{wt}(b)_i - \mathrm{wt}(b)_{i+1}$, *and*
(b) $e'_i(b) = c$ *if and only if* $b = f'_i(c)$, *in which case*

$$\mathrm{wt}(c) - \mathrm{wt}(b) = \begin{cases} \boldsymbol{e}_i & \text{if } \varepsilon'_i(b) \text{ is even} \\ -\boldsymbol{e}_{i+1} & \text{if } \varepsilon'_i(b) \text{ is odd.} \end{cases}$$

The "queer" extension of this definition goes as follows. Suppose $\mathcal{B}$ is a $\sqrt{\mathfrak{gl}_n}$-crystal with extra maps $e'_{\overline{1}}, f'_{\overline{1}} : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$. Define $\varepsilon'_{\overline{1}}$ and $\varphi'_{\overline{1}}$ as in (4.1) with $i = \overline{1}$.

Definition 4.2. *When $n \geqslant 2$, we define $\mathcal{B}$ to be a $\sqrt{\mathfrak{q}_n}$-crystal if $e'_{\overline{1}}$ and $f'_{\overline{1}}$ commute with e'_i and f'_i while preserving ε'_i and φ'_i for all $3 \leqslant i \leqslant n-1$, such that for all $b, c \in \mathcal{B}$ both of the following conditions hold:*

(a) $\mathrm{wt}(b) \in \mathbb{N}^n$ *and* $\varepsilon'_{\overline{1}}(b) + \varphi'_{\overline{1}}(b) = \begin{cases} 0 & \text{if } \mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0 \\ 2 & \text{otherwise;} \end{cases}$
(b) $e'_{\overline{1}}(b) = c$ *if and only if* $b = f'_{\overline{1}}(c)$, *in which case*

$$\mathrm{wt}(c) - \mathrm{wt}(b) = \begin{cases} \boldsymbol{e}_1 & \text{if } \varepsilon'_{\overline{1}}(b) = 2 \\ -\boldsymbol{e}_2 & \text{if } \varepsilon'_{\overline{1}}(b) = 1. \end{cases}$$

When $n = 1$, we define a $\sqrt{\mathfrak{q}_1}$-crystal to be the same thing as a $\sqrt{\mathfrak{gl}_1}$-crystal.

We associate to each $\sqrt{\mathfrak{gl}_n}$- and $\sqrt{\mathfrak{q}_n}$-crystal a *crystal graph* in the usual way, with arrows $b \xrightarrow{i} c$ whenever $f'_i(b) = c \neq 0$. Unlike the classical case, these edge-labeled directed graphs are no longer acyclic in general. Two $\sqrt{\mathfrak{gl}_n}$- or $\sqrt{\mathfrak{q}_n}$-crystals are *isomorphic* if there is a weight-preserving graph isomorphism between their crystal graphs.

Example 4.3. Let $\mathbb{S}_n$ be the set of nonempty subsets of $[n]$. For $i \in [n-1]$ and $S \in \mathbb{S}_n$, define

$$e'_i(S) := \begin{cases} S \sqcup \{i\} & \text{if } S \cap \{i, i+1\} = \{i+1\}, \\ S \setminus \{i+1\} & \text{if } S \cap \{i, i+1\} = \{i, i+1\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$f'_i(S) := \begin{cases} S \sqcup \{i+1\} & \text{if } S \cap \{i, i+1\} = \{i\}, \\ S \setminus \{i\} & \text{if } S \cap \{i, i+1\} = \{i, i+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Also let $\mathrm{wt}(S) = \sum_{i \in S} \mathbf{e}_i \in \mathbb{N}^n$. Relative to these maps, the set $\mathbb{S}_n$ is a $\sqrt{\mathfrak{gl}_n}$–crystal, which we refer to as the *standard crystal*. Its crystal graph for some small values of n is shown in Figure 4. When $n \geqslant 2$, define $e'_{\overline{1}}(S) = e'_1(S)$ and $f'_{\overline{1}}(S) = f'_1(S)$. Then $\mathbb{S}_n$ is also a $\sqrt{\mathfrak{q}_n}$-crystal.

We define the *character* of a finite $\sqrt{\mathfrak{gl}_n}$- or $\sqrt{\mathfrak{q}_n}$-crystal in the same way as for ordinary crystals. The following result is immediate from the definitions, and motivates our terminology:

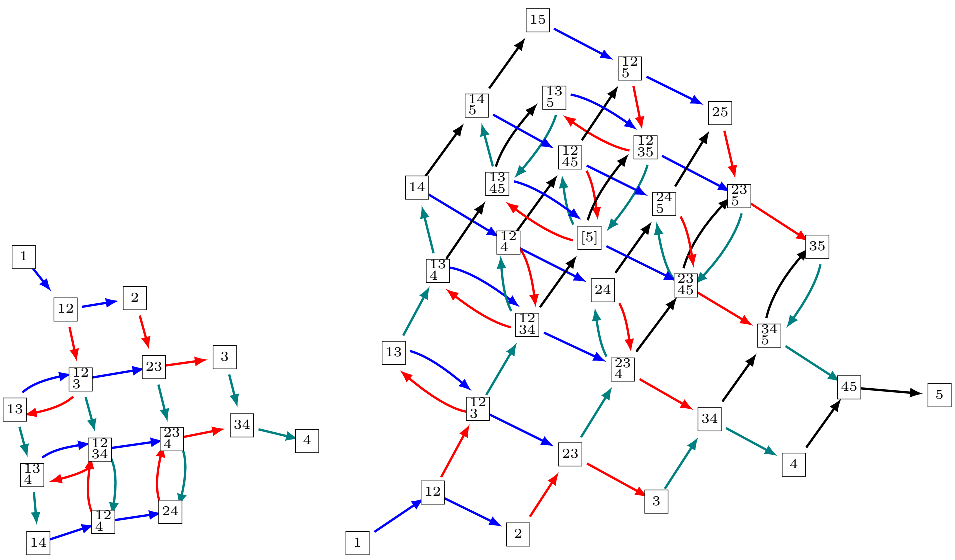


Figure 4. Crystal graphs of the standard $\sqrt{\mathfrak{gl}_4}$ and $\sqrt{\mathfrak{gl}_5}$-crystals. We have omitted the edge labels, but have drawn all arrows with a given label using the same color. The omitted edge labels can be inferred by comparing the weights of each arrow's source and target.

Proposition 4.4. *Any $\sqrt{\mathfrak{gl}_n}$-crystal (respectively, $\sqrt{\mathfrak{q}_n}$-crystal) is a $\mathfrak{gl}_n$-crystal (resp., $\mathfrak{q}_n$-crystal) relative to the operators $(e'_i)^2$ and $(f'_i)^2$, setting $e'_i(0) = f'_i(0) = 0$. This object is seminormal as a $\mathfrak{gl}_n$-crystal and so, when finite, its character is a symmetric polynomial.*

The "square" of a $\sqrt{\mathfrak{q}_n}$-crystal is not necessarily seminormal as a $\mathfrak{q}_n$-crystal since one can have $e_{\overline{1}}(b) = f_{\overline{1}}(b) = 0$ without having $\mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0$. The $\mathfrak{gl}_n$- and $\mathfrak{q}_n$-crystals derived from $\mathbb{S}_n$ via Proposition 4.4 are the standard crystals $\mathbb{B}_n$ from Examples 2.6 and 2.10.

4.2. Square root crystals of words. Given $m \in \mathbb{N}$, let $\mathsf{SetWords}_n(m)$ be the set of m-element sequences $S = (S_1, S_2, \dots, S_m)$ where each $\varnothing \subsetneq S_i \subseteq [n]$, or equivalently $S_i \in \mathbb{S}_n$. We refer to elements of $\mathsf{SetWords}_n(m)$ as *set-valued words*.

In [33], Yu identifies a remarkable $\sqrt{\mathfrak{gl}_n}$-crystal on $\mathsf{SetWords}_n(m)$. We review this structure below, and indicate how it can be extended to a $\sqrt{\mathfrak{q}_n}$-crystal. In the next section, we show how this construction leads to our second crystal structure on set-valued decomposition tableaux.

For the rest of this part, fix a set-valued word $S = (S_1, S_2, \dots, S_m)$. All of the following definitions were introduced in [33] in a slightly more restricted context.

Definition 4.5. *For each $i \in \mathbb{P}$, the* i-word *of S is the following word composed of "(", ") ", and "− " characters concatenated together. Read through the set-valued entries of S from left to right. For each entry containing i but not $i+1$, we write the single character ") ". For each entry containing $i+1$ but not i, we write the single character "(". Finally, for each entry containing both i and $i+1$, we write the three characters ") − (".*

The only difference between this definition and [33, Def. 4.1] is that Yu requires S to be (the column reading word of) a semistandard set-valued tableau, rather than any set-valued word.

Example 4.6. We represent the set-valued word $S = (S_1, S_2, \ldots, S_m)$ as the one-column tableau

$$S = \begin{array}{|c|}\hline S_1 \\ \hline S_2 \\ \hline \vdots \\ \hline S_m \\ \hline \end{array},$$

with commas omitted in each entry. Using this convention, the set-valued word

$$S = \begin{array}{|c|}\hline 23 \\ \hline 2 \\ \hline 23 \\ \hline 34 \\ \hline 1 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \in \mathsf{SetWords}_4(7)$$

has 1-word "((()()", 2-word ")−())−(()", and 3-word ")))−(".

Yu [33] divides the characters in the i-word into equivalence classes in the following way. Ignore the "−" symbols and pair the parentheses "(" with ")" in the usual way. Then we require:

- If a left parenthesis "(" is paired with a right parenthesis ")", then these two characters and everything between them is in the same equivalence class.
- For each ")−(", these three characters are in the same equivalence class.

Each of the resulting equivalence classes is a contiguous sub-word.

Example 4.7. If the i-word is "))−(())−())−(()−(" then its distinct equivalence classes are ")" and ")−(())−()" and ")−(" and "()−(".

Observe that any unpaired ")" in the i-word must be the first character in its class, and any unpaired "(" must be the last character in its class. As in [33], we distinguish between classes in the i-word using the terminology below:

(a) A class is a *null form* if it has no unpaired "(" or ")" characters.
For example: "(()−())−()".

(b) A class is a *left form* if it has no unpaired ")" characters but ends with an unpaired "(". This class is either "(" or "(u)−(" for some word u.
For example: "(())−(".

(c) A class is a *right form* if it has no unpaired "(" characters but starts with an unpaired ")". This class is either ")" or ")−(u)" for some word u.
For example: ")−()−()".

(d) A class is a *combined form* if it starts with an unpaired ")" and ends with an unpaired "(". This class is either ")−(" or ")−(u)−(" for some word u.
For example: ")−()−(())−(".

If we ignore the null forms, then in any i-word we have zero or more right forms, followed by at most one combined form, followed by zero or more left forms.

We may now define $\sqrt{\mathfrak{q}_n}$-crystal operators on $\mathsf{SetWords}_n(m)$. The operators indexed by $i \in [n-1]$ are identical to ones considered in [33, §4.1], with the same caveats as after Definition 4.5. Recall that we let $S = (S_1, S_2, \ldots, S_m)$ be an arbitrary element of $\mathsf{SetWords}_n(m)$.

Definition 4.8. *For $i \in [n-1]$, construct $e'_i(S) \in \mathsf{SetWords}_n(m) \sqcup \{0\}$ as follows:*

- *If the i-word of S has a combined form, then we find the entry in S that corresponds to "$) - ($" at the end of this combined form. We remove $i+1$ from this entry to obtain $e'_i(S)$.*
- *Otherwise, if the i-word of S has no left forms, then we set $e'_i(S) = 0$.*
- *Otherwise, find the first left form in the i-word of S, and then find the entry in S that corresponds to "$($" at the start of this left form. Add i to this entry to obtain $e'_i(S)$.*

Definition 4.9. *For $i \in [n-1]$, construct $f'_i(S) \in \mathsf{SetWords}_n(m) \sqcup \{0\}$ as follows:*

- *If the i-word of S has a combined form, then we find the entry in S that corresponds to "$) - ($" at the beginning of this combined form. We remove i from this entry to obtain $f'_i(S)$.*
- *Otherwise, if the i-word of S has no right forms, then we set $f'_i(S) = 0$.*
- *Otherwise, find the last right form in the i-word of S, and then find the entry in S that corresponds to "$)$" at the end of this right form. Add $i+1$ to this entry to obtain $f'_i(S)$.*

Example 4.10. The f'_2 operator has this effect on these set-valued words:

$$\begin{array}{|c|}\hline 23\\\hline 2\\\hline 2\\\hline 34\\\hline 1\\\hline 2\\\hline 1\\\hline\end{array} \xrightarrow{f'_2} \begin{array}{|c|}\hline 23\\\hline 2\\\hline 23\\\hline 34\\\hline 1\\\hline 2\\\hline 1\\\hline\end{array} \xrightarrow{f'_2} \begin{array}{|c|}\hline 23\\\hline 2\\\hline 3\\\hline 34\\\hline 1\\\hline 2\\\hline 1\\\hline\end{array} \xrightarrow{f'_2} \begin{array}{|c|}\hline 23\\\hline 23\\\hline 3\\\hline 34\\\hline 1\\\hline 2\\\hline 1\\\hline\end{array} \xrightarrow{f'_2} \begin{array}{|c|}\hline 3\\\hline 23\\\hline 3\\\hline 34\\\hline 1\\\hline 2\\\hline 1\\\hline\end{array} \xrightarrow{f'_2} 0.$$

We include one lemma before proceeding. Given $S = (S_1, S_2, \dots, S_m) \in \mathsf{SetWords}_n(m)$, let $w(S)$ be the word formed by replacing each S_i by its entries listed in increasing order. For example, if $S = (\{4,5\}, \{3\}, \{3,6\}, \{1,2\}, \{2\}, \{2,3\})$ then $w(S) = 4533612223$. Following [1], we say that a word $w_1 w_2 \cdots w_l$ is a *reverse lattice word* if $\mathrm{wt}(w_i w_{i+1} w_{i+2} \cdots w_l)$ is a partition for each $i \in [l]$.

Lemma 4.11. *A set-valued word $S \in \mathsf{SetWords}_n(m)$ has $e'_i(S) = 0$ for all $i \in [n-1]$ if and only if $w(S)$ is a reverse lattice word.*

Proof. We have $e'_i(S) \neq 0$ if and only if there is at least one unpaired "(" character in the i-word of S. This occurs if and only if there is at least one i-unpaired letter (in the sense of Remark 2.7) equal to $i+1$ in $w(S)$.

Write $w(S) = w_1 w_2 \cdots w_l$. For this word to have an i-unpaired letter $w_j = i+1$, we must have $\mathrm{wt}(w_{j+1} w_{j+2} \cdots w_l)_i \leqslant \mathrm{wt}(w_{j+1} w_{j+2} \cdots w_l)_{i+1}$, but then $\mathrm{wt}(w_j w_{j+1} \cdots w_l)_i < \mathrm{wt}(w_j w_{j+1} \cdots w_l)_{i+1}$ so $\mathrm{wt}(w_j w_{j+1} w_{j+2} \cdots w_l)$ is not a partition. Thus, if $w(S)$ is a reverse lattice word then we must have $e'_i(S) = 0$ for all $i \in [n-1]$.

Conversely, if $w(S)$ is not a reverse lattice word, then for some $i \in [n-1]$, there must exist $j \in [l]$ with $\mathrm{wt}(w_j w_{j+1} \cdots w_l)_i < \mathrm{wt}(w_j w_{j+1} \cdots w_l)_{i+1}$. If j is chosen to be maximal after fixing i, then $w_j = i+1$ (as otherwise the strict inequality would still hold after incrementing j by one), and this letter is i-unpaired in $w(S)$ (since if it were paired with $w_k = i$ for some $k > j$ then the strict inequality would still hold after replacing j by $k+1$). Thus, if $w(S)$ is not a reverse lattice word, then it has an i-unpaired letter equal to $i+1$ for some $i \in [n-1]$, so we have $e'_i(S) \neq 0$. ☐

The following more straightforward operators are new. Assume $n \geqslant 2$ and $S \in \mathsf{SetWords}_n(m)$.

Definition 4.12. *Construct* $e'_{\overline{1}}(S) \in \mathsf{SetWords}_n(m) \sqcup \{0\}$ *as follows. If there are no entries in* S *containing* 1 *or* 2*, then* $e'_{\overline{1}}(S) = 0$*. Otherwise, suppose* $i \in [m]$ *is minimal with* $\{1, 2\} \cap S_i \neq \varnothing$.

- *If* $1 \notin S_i$ *and* $2 \in S_i$ *then add* 1 *to* S_i *to obtain* $e'_{\overline{1}}(S)$.
- *If* $1 \in S_i$ *and* $2 \in S_i$ *then remove* 2 *from* S_i *to obtain* $e'_{\overline{1}}(S)$.
- *If* $1 \in S_i$ *and* $2 \notin S_i$ *then* $e'_{\overline{1}}(S) = 0$.

Definition 4.13. *Construct* $f'_{\overline{1}}(S) \in \mathsf{SetWords}_n(m) \sqcup \{0\}$ *as follows. If there are no entries in* S *containing* 1 *or* 2*, then* $f'_{\overline{1}}(S) = 0$*. Otherwise, suppose* $i \in [m]$ *is minimal with* $\{1, 2\} \cap S_i \neq \varnothing$.

- *If* $1 \in S_i$ *and* $2 \notin S_i$ *then add* 2 *to* S_i *to obtain* $f'_{\overline{1}}(S)$.
- *If* $1 \in S_i$ *and* $2 \in S_i$ *then remove* 1 *from* S_i *to obtain* $f'_{\overline{1}}(S)$.
- *If* $1 \notin S_i$ *and* $2 \in S_i$ *then* $f'_{\overline{1}}(S) = 0$.

Example 4.14. The $f'_{\overline{1}}$ operator has this effect on these set-valued words:

$$\begin{array}{|c|}\hline 3 \\ \hline 13 \\ \hline 3 \\ \hline 34 \\ \hline \end{array} \xrightarrow{f'_{\overline{1}}} \begin{array}{|c|}\hline 3 \\ \hline 123 \\ \hline 3 \\ \hline 34 \\ \hline \end{array} \xrightarrow{f'_{\overline{1}}} \begin{array}{|c|}\hline 3 \\ \hline 23 \\ \hline 3 \\ \hline 34 \\ \hline \end{array} \xrightarrow{f'_{\overline{1}}} 0.$$

Theorem 4.15. *For the operators* e'_i *and* f'_i *given above,* $\mathsf{SetWords}_n(m)$ *is a* $\sqrt{\mathfrak{q}_n}$*-crystal.*

The claim that $\mathsf{SetWords}_n(m)$ is a $\sqrt{\mathfrak{gl}_n}$-crystal essentially follows from [33, §4]; the results there technically only consider set-valued words that arise as column reading words of semistandard set-valued tableaux, but this property is not used in the relevant proofs. Once we know that the set $\mathsf{SetWords}_n(m)$ is a $\sqrt{\mathfrak{gl}_n}$-crystal, very little needs to be checked to deduce that it is also a $\sqrt{\mathfrak{q}_n}$-crystal. We will give a self-contained alternate proof of Theorem 4.15 in Section 4.4.

Remark 4.16. Observe that $\mathsf{SetWords}_n(1) \cong \mathbb{S}_n$ as $\sqrt{\mathfrak{q}_n}$-crystals if $\mathbb{S}_n$ is defined as in Example 4.3. Also notice that the $\mathfrak{gl}_n$- or $\mathfrak{q}_n$-crystal afforded by "squaring" $\mathsf{SetWords}_n(m)$ in the sense of Proposition 4.4 contains the normal object $\mathsf{Words}_n(m)$ as a union of full subcrystals, once we identify each ordinary word $w_1 w_2 \cdots w_m$ with the set-valued word $(\{w_1\}, \{w_2\}, \ldots, \{w_m\})$. Most other full subcrystals of this crystal are not normal as either $\mathfrak{gl}_n$-crystals or $\mathfrak{q}_n$-crystals.

4.3. Square root crystals of tableaux. Suppose T is a (shifted or unshifted) set-valued tableau with m boxes and all entries contained in $[n]$. Define the *set-valued row reading word* of T to be the sequence $\mathsf{row}_{SV}(T) \in \mathsf{SetWords}_n(m)$ formed by listing the set-valued entries of T in the usual row reading order. Define $\mathsf{revrow}_{SV}(T) \in \mathsf{SetWords}_n(m)$ to be the same sequence read in the opposite order. Finally, form the *set-valued column reading word* $\mathsf{col}_{SV}(T) \in \mathsf{SetWords}_n(m)$ by reading the entries of T down each column in French notation, iterating over columns from left to right. For example, we have

$$\mathsf{row}_{SV}\left(\begin{array}{|c|c|c|}\hline 45 & & \\ \hline 3 & 36 & \\ \hline 12 & 2 & 23 \\ \hline\end{array}\right) = (\{4,5\},\{3\},\{3,6\},\{1,2\},\{2\},\{2,3\}),$$

$$\mathsf{revrow}_{SV}\left(\begin{array}{cccc} & & 1 & \\ & 2 & 1 & \\ 34 & 2 & 2 & 23 \end{array}\right) = (\{2,3\},\{2\},\{2\},\{3,4\},\{1\},\{2\},\{1\}),$$

$$\mathsf{col}_{SV}\left(\begin{array}{|c|c|c|}\hline 45 & & \\ \hline 3 & 36 & \\ \hline 12 & 2 & 23 \\ \hline\end{array}\right) = (\{4,5\},\{3\},\{1,2\},\{3,6\},\{2\},\{2,3\}),$$

An *embedding* of $\sqrt{\mathfrak{gl}_n}$- or $\sqrt{\mathfrak{q}_n}$-crystals is defined in the same way as for ordinary crystals (see Theorem 2.13): this means a weight-preserving injective map $\phi : \mathcal{B} \to \mathcal{C}$ that commutes with all crystal operators, in the sense that $\phi(e'_i(b)) = e'_i(\phi(b))$ and $\phi(f'_i(b)) = f'_i(\phi(b))$ for all $b \in \mathcal{B}$ and all relevant indices i when we set $\phi(0) = e'_i(0) = f'_i(0) = 0$.

Recall that $\mathsf{SetTab}_n(\lambda)$ is the set of semistandard set-valued tableaux of shape λ whose entries are all subsets of $[n]$. The following is equivalent to [33, Lems. 4.8 and 4.9]. An example of the resulting crystal is shown in [33, Fig. 4.1].

Theorem 4.17 ([33]). *Let λ be a partition. Then there is a unique $\sqrt{\mathfrak{gl}_n}$-crystal structure on $\mathsf{SetTab}_n(\lambda)$ for which $\mathsf{col}_{SV} : \mathsf{SetTab}_n(\lambda) \to \mathsf{SetWords}_n(m)$ is a $\sqrt{\mathfrak{gl}_n}$-crystal embedding.*

Our main new result is a decomposition tableau analogue of the previous theorem:

Theorem 4.18. *Let λ be a strict partition. Then there is a unique $\sqrt{\mathfrak{q}_n}$-crystal structure on $\mathsf{SetDecTab}_n(\lambda)$ for which $\mathsf{revrow}_{SV} : \mathsf{SetDecTab}_n(\lambda) \to \mathsf{SetWords}_n(m)$ is a $\sqrt{\mathfrak{q}_n}$-crystal embedding.*

The hardest parts of the proof of this theorem will turn out to follow directly from the technical lemmas in Section 3.3. Before explaining this argument, we discuss some examples.

Example 4.19. Let us write e'_i and f'_i for the unique maps $\mathsf{SetDecTab}_n(\lambda) \to \mathsf{SetDecTab}_n(\lambda) \sqcup \{0\}$ with $\mathsf{revrow}_{SV} \circ e'_i = e'_i \circ \mathsf{revrow}_{SV}$ and $\mathsf{revrow}_{SV} \circ f'_i = f'_i \circ \mathsf{revrow}_{SV}$, which exist by Theorem 4.18. Then f'_2 has this effect on these set-valued decomposition tableaux:

$$\begin{array}{cccc} & & 1 & \\ & 2 & 1 & \\ 34 & 2 & 2 & 23 \end{array} \xrightarrow{f'_2} \begin{array}{cccc} & & 1 & \\ & 2 & 1 & \\ 34 & 23 & 2 & 23 \end{array} \xrightarrow{f'_2} \begin{array}{cccc} & & 1 & \\ & 2 & 1 & \\ 34 & 3 & 2 & 23 \end{array} \xrightarrow{f'_2} \begin{array}{cccc} & & 1 & \\ & 2 & 1 & \\ 34 & 3 & 23 & 23 \end{array} \xrightarrow{f'_2} \begin{array}{cccc} & & 1 & \\ & 2 & 1 & \\ 34 & 3 & 23 & 3 \end{array} \xrightarrow{f'_2} 0.$$

Likewise, the operator f'_1 acts as follows:

$$\begin{array}{cccc} & & 1 & \\ & 2 & 1 & \\ 34 & 3 & 13 & 3 \end{array} \xrightarrow{f'_1} \begin{array}{cccc} & & 1 & \\ & 2 & 1 & \\ 34 & 3 & 123 & 3 \end{array} \xrightarrow{f'_1} \begin{array}{cccc} & & 1 & \\ & 2 & 1 & \\ 34 & 3 & 23 & 3 \end{array} \xrightarrow{f'_1} 0.$$

Figures 5 and 7 show two $\sqrt{\mathfrak{q}_n}$-crystal graphs for $\mathsf{SetDecTab}_n(\lambda)$, while Figures 6 and 8 show the $\mathfrak{q}_n$-crystals obtained by squaring the crystal operators e'_i and f'_i as in Proposition 4.4. The latter objects are usually not isomorphic to the $\mathfrak{q}_n$-crystals defined in Section 3, although they have the same elements and weights.

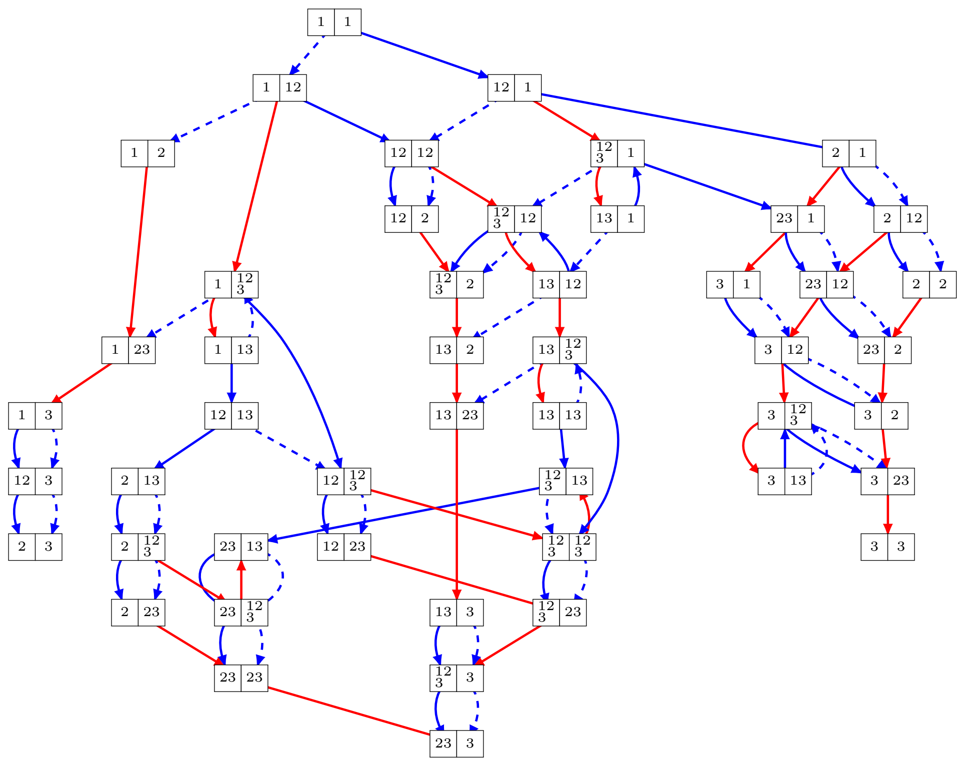


FIGURE 5. Crystal graph of the $\sqrt{\mathfrak{q}_3}$-crystal $\mathsf{SetDecTab}_3(\lambda)$ for $\lambda = (2)$. Here, solid blue and red arrows respectively indicate $\xrightarrow{f'_1}$ and $\xrightarrow{f'_2}$ edges while dashed blue arrows indicate $\xrightarrow{f'_{\overline{1}}}$ edges.

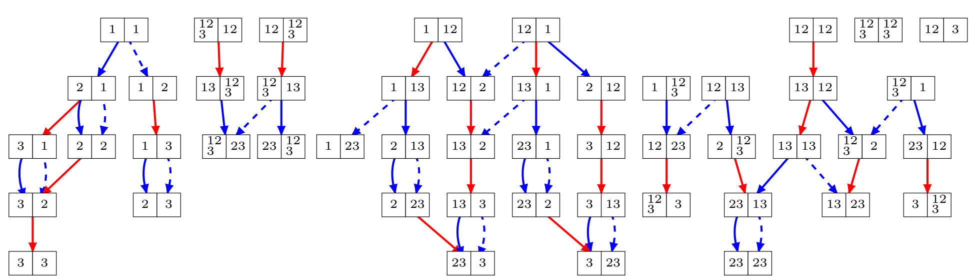


FIGURE 6. The $\mathfrak{q}_3$-crystal obtained by squaring the crystal operators in Figure 5; compare with Figure 2. Solid blue and red arrows are $\xrightarrow{1}$ and $\xrightarrow{2}$ edges. Dashed blue arrows are $\xrightarrow{\overline{1}}$ edges. Notice that $\mathsf{DecTab}_3(\lambda)$ for $\lambda = (2)$ is a connected component of this graph.

REMARK 4.20. In view of Remark 4.16, it always holds that the respective subsets of "non-set-valued" tableaux in $\mathsf{SetTab}_n(\lambda)$ and $\mathsf{SetDecTab}_n(\lambda)$ (whose entries are all singleton sets) are full subcrystals of the $\mathfrak{gl}_n$- and $\mathfrak{q}_n$-crystals derived from Theorems 4.17 and 4.18 via Proposition 4.4.

Proof of Theorem 4.18. Fix $T \in \mathsf{SetDecTab}_n(\lambda)$. When $e'_i(\mathsf{revrow}_{SV}(T))$ is nonzero, there is certainly a unique set-valued tableau U of shape λ with $\mathsf{revrow}_{SV}(U) =$

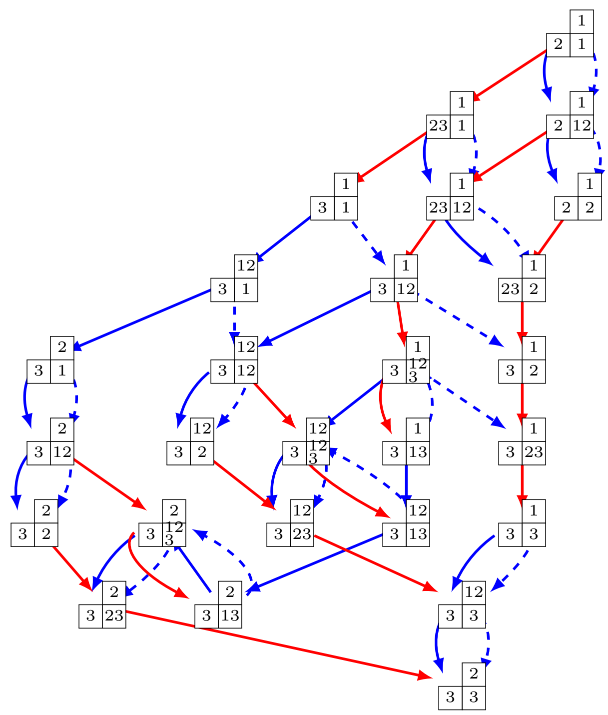


Figure 7. Crystal graph of the $\sqrt{\mathfrak{q}_3}$-crystal $\mathsf{SetDecTab}_3(\lambda)$ for $\lambda = (2,1)$. Here, solid blue and red arrows respectively indicate $\xrightarrow{f'_1}$ and $\xrightarrow{f'_2}$ edges while dashed blue arrows indicate $\xrightarrow{f'_{\overline{1}}}$ edges.

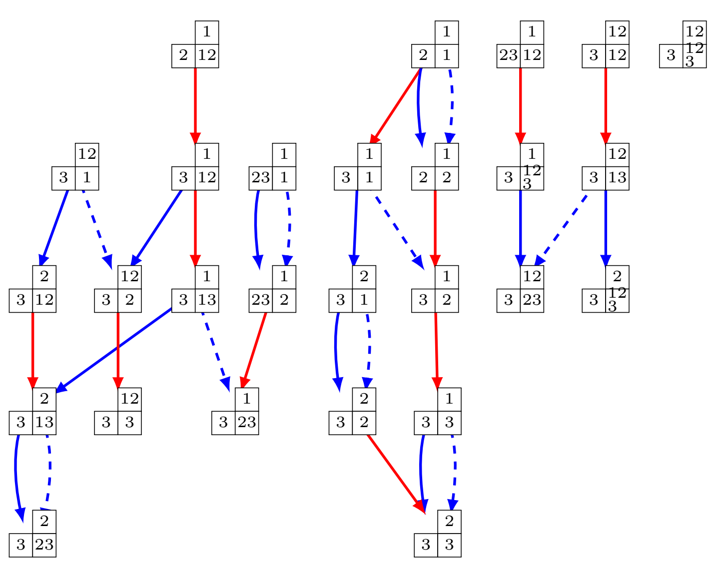


Figure 8. The $\mathfrak{q}_3$-crystal obtained by squaring the crystal operators in Figure 7; compare with Figure 3. Solid blue and red arrows are $\xrightarrow{1}$ and $\xrightarrow{2}$ edges. Dashed blue arrows are $\xrightarrow{\overline{1}}$ edges. Notice that $\mathsf{DecTab}_3(\lambda)$ for $\lambda = (2,1)$ is a connected component of this graph.

$e'_i(\mathsf{revrow}_{SV}(T))$, although it is not obvious that this is a decomposition tableau. Set $e'_i(T) := U$ when $e'_i(\mathsf{revrow}_{SV}(T)) \neq 0$ and define $e'_i(T) := 0$ otherwise. Form $f'_i(T)$ analogously. The most difficult thing to check is that these operators e'_i and f'_i really do give maps $\mathsf{SetDecTab}_n(\lambda) \to \mathsf{SetDecTab}_n(\lambda) \sqcup \{0\}$.

For this, first suppose $i \in [n-1]$. Observe that e'_i or f'_i act on T by either removing a letter from some box or adding a letter to some box, when they do not send $T \mapsto 0$. In the cases when e'_i or f'_i remove a letter from some box, it is clear $e'_i(T)$ and $f'_i(T)$ belong to $\mathsf{SetDecTab}_n(\lambda)$. On the other hand, if the i-word of $\mathsf{revrow}_{SV}(T)$ has no combined forms and at least one left form, then the number that corresponds to the "(" at the start of the first left form is also the first i-unpaired letter equal to $i+1$ in $\mathsf{revrow}(T)$ in the sense of Lemma 3.12. Changing this $i+1$ to i must yield an element of $\mathsf{SetDecTab}_n(\lambda)$ since otherwise parts (2) and (3) of Lemma 3.12 would imply that the "(" character being considered is not the start of a left form in the linked i-word. Therefore adding i to the box of $i+1$ also gives an element of $\mathsf{SetDecTab}_n(\lambda)$.

Similarly, if the i-word of $\mathsf{revrow}_{SV}(T)$ has no combined forms and at least one right form, then the number that corresponds to the ")" at the start of the last right form is also the last i-unpaired letter equal to i in $\mathsf{revrow}(T)$ in the sense of Lemma 3.13. Changing this i to $i+1$ must yield an element of $\mathsf{SetDecTab}_n(\lambda)$ since otherwise parts (2) and (3) of Lemma 3.13 would imply that the ")" character being considered is not the start of a right form in the i-word. Therefore adding $i+1$ to the box of i also gives an element of $\mathsf{SetDecTab}_n(\lambda)$.

One can either deduce from Theorem 4.15 that $e'_i(T) = U \neq 0$ if and only if $T = f'_i(U)$, or derive this fact repeating the argument in the proof of [33, Lem. 4.9]. Since $\varepsilon'_i(T)$ is twice the number of left forms in the i-word of $\mathsf{revrow}_{SV}(T)$ plus one if there is a combined form, and $\varphi'_i(T)$ is twice the number of right forms in the i-word of $\mathsf{revrow}_{SV}(T)$ plus one if there is a combined form, the sum $\varepsilon'_i(T) + \varphi'_i(T)$ is even. Moreover, $\frac{\varphi'_i(T)-\varepsilon'_i(T)}{2}$ is exactly the difference between the number of unpaired letters equal to i and $i+1$ in the i-word of $\mathsf{revrow}_{SV}(T)$, which is just $\mathrm{wt}(T)_i - \mathrm{wt}(T)_{i+1}$. Finally, the weight identity in part (b) of Definition 4.1 is immediate.

This shows that $\mathsf{SetDecTab}_n(\lambda)$ is a $\sqrt{\mathfrak{gl}_n}$-crystal. We move on to the conditions in Definition 4.2, assuming $n \geqslant 2$. If $e'_{\overline{1}}$ or $f'_{\overline{1}}$ acts on T by removing a 2 or 1 from some box then clearly $e'_{\overline{1}}(T)$ and $f'_{\overline{1}}(T)$ still belong to $\mathsf{SetDecTab}(\lambda)$. Suppose (i,j) is the first box of T in the reverse row reading word containing 1 or 2. If we have $1 \notin T_{ij}$ and $2 \in T_{ij}$, then $e_{\overline{1}}$ acts on every distribution of T containing $1 \in T_{ij}$ by changing this 1 to 2; therefore adding 2 to T_{ij} gives an element of $\mathsf{SetDecTab}_n(\lambda)$. Likewise, if we have $1 \in T_{ij}$ and $2 \notin T_{ij}$, then $f_{\overline{1}}$ acts on every distribution of T containing $2 \in T_{ij}$ by changing this 2 to 1; therefore adding 1 to T_{ij} again gives an element of $\mathsf{SetDecTab}_n(\lambda)$. The remaining things to check in Definition 4.2 are straightforward. ☐

It follows from [33, Cor. 5.2] that if λ is a partition with at most n parts then the $\sqrt{\mathfrak{gl}_n}$-crystal $\mathsf{SetTab}_n(\lambda)$ is connected (in the sense that its crystal graph has only one weakly connected component). More strongly, Yu's result show that $\mathsf{SetTab}_n(\lambda)$ has a unique highest weight element (namely, the tableau of shape λ with the set $\{i\}$ in every entry in row i) that is sent to zero by all e'_i operators. We predict that a similar phenomenon holds for the $\sqrt{\mathfrak{q}_n}$-crystal $\mathsf{SetDecTab}_n(\lambda)$:

CONJECTURE 4.21. *For each strict partition λ, the $\sqrt{\mathfrak{q}_n}$-crystal $\mathsf{SetDecTab}_n(\lambda)$ is connected.*

This is supported by all examples that we can compute. These examples also exhibit a more technical highest weight property, which we now explain.

Let λ be a strict partition with $\ell(\lambda) = k$ nonzero parts. The *first border strip* of the shifted diagram SD_λ is the minimal set of positions S such that

(a) one has $(1, \lambda_1) \in S$ and
(b) if $(i, j) \in S$ and $i \neq j$, then either $(i+1, j) \in S$, or $(i, j-1) \in S$ when $(i+1, j) \notin \mathsf{SD}_\lambda$.

Let $\mathsf{SD}^{(1)}_\lambda$ be the first border strip of SD_λ. The set difference $\mathsf{SD}_\lambda - \mathsf{SD}^{(1)}_\lambda$ is either empty when $k = 1$ or equal to SD_μ for a strict partition μ with $\ell(\mu) = k - 1$. For $i \in [k-1]$ let $\mathsf{SD}^{(i+1)}_\lambda$ be the first border strip of $\mathsf{SD}_\lambda - (\mathsf{SD}^{(1)}_\lambda \sqcup \cdots \sqcup \mathsf{SD}^{(i)}_\lambda)$. Finally, let $T^{\mathsf{highest}}_\lambda \in \mathsf{SetDecTab}_n(\lambda)$ be the set-valued decomposition tableau of shape λ whose entries in each border strip $\mathsf{SD}^{(i)}_\lambda$ are all $\{i\}$.

Example 4.22. If $\lambda = (6, 4, 2, 1)$ then $T^{\mathsf{highest}}_\lambda =$

			1		
		2	1		
	3	2	1	1	
4	3	2	2	1	1

.

Assume $\mathcal{B}$ is a $\sqrt{\mathfrak{q}_n}$-crystal. For any given index i, an *i-string* in $\mathcal{B}$ is a connected component in the subgraph of the crystal graph of $\mathcal{B}$ retaining only the $\xrightarrow{i}$ arrows. Let $\sigma'_i : \mathcal{B} \to \mathcal{B}$ be the involution that reverses each i-string.

Next, define $e'_{\overline{i}} : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ for indices $2 \leqslant i < n$ by

$$e'_{\overline{i}} := (\sigma'_{i-1}\sigma'_i) \cdots (\sigma'_2\sigma'_3)(\sigma'_1\sigma'_2) e'_{\overline{1}} (\sigma'_2\sigma'_1)(\sigma'_3\sigma'_2) \cdots (\sigma'_i\sigma'_{i-1}), \tag{4.2}$$

using the convention that $\sigma'_i(0) = 0$. Finally, let us say that an element $b \in \mathcal{B}$ is *$\sqrt{\mathfrak{q}_n}$-highest weight* if $e'_i(b) = e'_{\overline{i}}(b) = 0$ for all $i \in [n-1]$.

This construction is motivated by the definition of a *highest weight* element in a $\mathfrak{q}_n$-crystal [8, Def. 1.12], which is specified in the same way via the ordinary $\mathfrak{q}_n$-crystal operators. It is known from [8, Thm. 2.5] that $T^{\mathsf{highest}}_\lambda$ is the unique highest weight element of the $\mathfrak{q}_n$-crystal $\mathsf{DecTab}_n(\lambda)$ from Theorem 2.13. This implies the following non-obvious fact:

Proposition 4.23. *The tableau $T^{\mathsf{highest}}_\lambda$ is a $\sqrt{\mathfrak{q}_n}$-highest weight element of the crystal* $\mathsf{SetDecTab}_n(\lambda)$.

Proof. Let σ_i be the operator that reverses all i-strings in the $\mathfrak{q}_n$-crystal $\mathsf{DecTab}_n(\lambda)$, and identify $\mathsf{DecTab}_n(\lambda)$ with a subset of $\mathsf{SetDecTab}_n(\lambda)$. Recall from Remark 4.20 that the crystal operators e_i and f_i on $\mathsf{DecTab}_n(\lambda)$ are related to e'_i and f'_i by the relations $e_i = e'^2_i$ and $f_i = f'^2_i$.

Fix $T \in \mathsf{DecTab}_n(\lambda)$ and $i \in [n-1]$. Then each i-word of $\mathsf{revrow}_{SV}(T)$ has no combined forms, so if $e'_i(T)$ is nonzero then the i-word of $\mathsf{revrow}_{SV}(e'_i(T))$ does have a combined form, as does the i-word of $\mathsf{revrow}_{SV}(f'_i(T))$ if $f'_i(T) \neq 0$. It follows that for either operator $\phi \in \{e'_i, f'_i\}$, we have $\phi(T) \neq 0$ if and only if $\phi^2(T) \neq 0$, in which case $\phi^2(T)$ is also in $\mathsf{DecTab}_n(\lambda)$. This implies that the i-string through T in $\mathsf{SetDecTab}_n(\lambda)$ contains an odd number of vertices, which alternate between elements of $\mathsf{DecTab}_n(\lambda)$ and $\mathsf{SetDecTab}_n(\lambda) \setminus \mathsf{DecTab}_n(\lambda)$, and we have $\sigma_i(T) = \sigma'_i(T) \in \mathsf{DecTab}_n(\lambda)$.

It is also true for $T \in \mathsf{DecTab}_n(\lambda)$ that $e'_{\overline{1}}(T) \neq 0$ if and only if $e_{\overline{1}}(T) = e'^2_{\overline{1}}(T) \neq 0$, as both situations occur precisely when the reverse row reading word of T contains a 2 before any 1. We conclude that $T \in \mathsf{DecTab}_n(\lambda)$ satisfies $e'_i(T) = e'_{\overline{i}}(T) = 0$ for all $i \in [n-1]$ if and only if $e_i(T) = e_{\overline{i}}(T) = 0$ for all $i \in [n-1]$, where

$$e_{\overline{i}} := (\sigma_{i-1}\sigma_i) \cdots (\sigma_2\sigma_3)(\sigma_1\sigma_2) e_{\overline{1}} (\sigma_2\sigma_1)(\sigma_3\sigma_2) \cdots (\sigma_i\sigma_{i-1}).$$

As already mentioned, results in [8] imply that $T^{\mathsf{highest}}_\lambda$ satisfies the second set of conditions. ☐

The following stronger version of the previous result holds in all of our examples. Note that as $\sqrt{\mathfrak{q}_n}$-crystal graphs may have cycles, this property does not immediately imply Conjecture 4.21.

Conjecture 4.24. *The tableau* $T_\lambda^{\mathsf{highest}}$ *is the unique* $\sqrt{\mathfrak{q}_n}$*-highest weight element of the crystal* $\mathsf{SetDecTab}_n(\lambda)$*.*

We have checked Conjectures 4.21 and 4.24 by computer when either $n = 3$ and $|\lambda| \leqslant 20$, $n = 4$ and $|\lambda| \leqslant 10$, $n = 5$ and $|\lambda| \leqslant 6$, $n = 6$ and $|\lambda| \leqslant 5$, or $n \in \{7, 8\}$ and $|\lambda| \leqslant 3$. Beyond this, we can verify both conjectures in the following special cases:

Proposition 4.25. *Conjectures* 4.21 *and* 4.24 *hold if* $n \leqslant 2$ *or* $\ell(\lambda) \leqslant 1$*.*

Proof. We will prove the stronger statement that if $n \leqslant 2$ or $\ell(\lambda) \leqslant 1$, then for each $T \in \mathsf{SetDecTab}_n(\lambda)$ there are indices $i_1, i_2, \dots, i_q \in \{\overline{1}, 1, 2, \dots, n-1\}$ with $e'_{i_1} e'_{i_2} \cdots e'_{i_q}(T) = T_\lambda^{\mathsf{highest}}$.

If $n = 1$ or $\ell(\lambda) = 0$ then $T_\lambda^{\mathsf{highest}}$ is the unique element of $\mathsf{SetDecTab}_n(\lambda)$, so the claim is immediate. Choose any $T \in \mathsf{SetDecTab}_n(\lambda)$. Next assume $n = 2$. In view of Lemma 2.3, T has at most two rows and all boxes in its second row must contain the set $\{1\}$. Suppose there are k boxes in the second row. Then the first row T read left to right must consist of k or more boxes containing $\{2\}$, optionally followed by a box containing $\{1, 2\}$, then zero or more boxes containing $\{1\}$, optionally followed by a final box containing $\{2\}$ or $\{1, 2\}$.

We now explain how to apply a sequence to raising operators to turn T into $T_\lambda^{\mathsf{highest}}$. If the last box in the first row of T contains $\{1, 2\}$ (respectively, $\{2\}$) then we apply $e'_{\overline{1}}$ once (respectively, twice) to change this box's entry to $\{1\}$. Next, if there is a box in the middle of the first row containing $\{1, 2\}$, then the 1-word of $\mathsf{revrow}_{SV}(T)$ will have a combined form and we can apply e'_1 to change the middle box's entry to $\{1\}$. After applying these operators, the first row of T consists of some number l boxes containing $\{2\}$, with $l \geqslant k$, followed by zero or more boxes containing $\{1\}$. Applying $(e'_1)^{2(l-k)}$ turns T into $T_\lambda^{\mathsf{highest}}$ as needed.

Finally suppose $n > 2$ but $\ell(\lambda) = 1$, so that $\lambda = (m)$ for some $m \in \mathbb{P}$. We prove our claim by induction on the set of entries $\mathcal{E} = \mathcal{E}(T) := \bigcup_{(i,j)\in T} T_{ij}$ appearing in T, using graded lexicographic order. If $\mathcal{E} = \{1\}$, then $T = T_\lambda^{\mathsf{highest}}$. Suppose $\mathcal{E} \neq \{1\}$ but $i := \min(\mathcal{E} \setminus \{1\})$ is greater than 2. If the number i appears exactly k times in T, then applying $(e'_{i-1})^{2k}$ to T will change all of these i entries to $i - 1$, and the set $\mathcal{E}$ will become $(\mathcal{E} \setminus \{i\}) \sqcup \{i - 1\}$. Then by induction we can find a sequence of raising operators that will turn $(e'_{i-1})^{2k}(T)$ into $T_\lambda^{\mathsf{highest}}$.

Assume that 2 appears in some entry of T. Since T is a set-valued decomposition tableaux, the entries of T containing 1 or 2 must occur in a contiguous sequence of boxes, say in columns $a + 1, a + 2, \dots, b$ where $0 \leqslant a < b$. If we consider just these boxes and remove all numbers greater than 2, then we obtain a set-valued decomposition tableau $U \in \mathsf{SetDecTab}_2((b - a))$ that is not equal to $T_{(b-a)}^{\mathsf{highest}}$. The argument given in the $n = 2$ case above shows that there is a sequence of indices $i_1, i_2, \dots, i_q \in \{\overline{1}, 1\}$ with $e'_{i_1} e'_{i_2} \cdots e'_{i_q}(U) = T_{(b-a)}^{\mathsf{highest}}$. Clearly for these indices we also have $\mathcal{E}(e'_{i_1} e'_{i_2} \cdots e'_{i_q}(T)) = (\mathcal{E} \setminus \{2\}) \cup \{1\}$, and so by induction we can find additional raising operators that will turn $e'_{i_1} e'_{i_2} \cdots e'_{i_q}(T)$ into $T_\lambda^{\mathsf{highest}}$ as needed. □

Yu [33] also demonstrates a close relationship between his $\sqrt{\mathfrak{gl}_n}$-crystals $\mathsf{SetTab}_n(\lambda)$ and *Lascoux polynomials*. It would be interesting to know if there is a similar relationship between the $\sqrt{\mathfrak{q}_n}$-crystals $\mathsf{SetDecTab}_n(\lambda)$ and the *P-Lascoux polynomials* introduced in [24].

4.4. Tensor products. This section investigates a tensor product for $\sqrt{\mathfrak{gl}_n}$- and $\sqrt{\mathfrak{q}_n}$-crystals. This can be defined by reusing the same tensor product rules as for $\mathfrak{gl}_n$- and $\mathfrak{q}_n$-crystals. The resulting construction is algebraically very natural, although it is no longer motivated by the representation theory of an associated quantum group.

Theorem 4.26. *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are $\sqrt{\mathfrak{gl}_n}$-crystals. Then the set $\mathcal{B}\otimes\mathcal{C} = \{b\otimes c : b \in \mathcal{B},\ c \in \mathcal{C}\}$ has a unique $\sqrt{\mathfrak{gl}_n}$-crystal structure in which* $\mathrm{wt}(b\otimes c) = \mathrm{wt}(b) + \mathrm{wt}(c)$ *and*

$$e'_i(b\otimes c) = \begin{cases} b\otimes e'_i(c) & \text{if } \varepsilon'_i(b) \leqslant \varphi'_i(c), \\ e'_i(b)\otimes c & \text{if } \varepsilon'_i(b) > \varphi'_i(c), \end{cases}$$

$$f'_i(b\otimes c) = \begin{cases} b\otimes f'_i(c) & \text{if } \varepsilon'_i(b) < \varphi'_i(c), \\ f'_i(b)\otimes c & \text{if } \varepsilon'_i(b) \geqslant \varphi'_i(c). \end{cases}$$

If $\mathcal{B}$ and $\mathcal{C}$ are $\sqrt{\mathfrak{q}_n}$-crystals, then the $\sqrt{\mathfrak{gl}_n}$-crystal $\mathcal{B}\otimes\mathcal{C}$ has a unique $\sqrt{\mathfrak{q}_n}$-crystal structure with

$$e'_{\overline{1}}(b\otimes c) = \begin{cases} b\otimes e'_{\overline{1}}(c) & \text{if } e'_{\overline{1}}(b) = f'_{\overline{1}}(b) = 0, \\ e'_{\overline{1}}(b)\otimes c & \text{otherwise}, \end{cases}$$

$$f'_{\overline{1}}(b\otimes c) = \begin{cases} b\otimes f'_{\overline{1}}(c) & \text{if } e'_{\overline{1}}(b) = f'_{\overline{1}}(b) = 0, \\ f'_{\overline{1}}(b)\otimes c & \text{otherwise}. \end{cases}$$

Finally, the natural maps $\mathcal{B}\otimes(\mathcal{C}\otimes\mathcal{D}) \to (\mathcal{B}\otimes\mathcal{C})\otimes\mathcal{D}$ are $\sqrt{\mathfrak{gl}_n}$- and $\sqrt{\mathfrak{q}_n}$-crystal isomorphisms.

Proof. Fix $i \in [n-1]$, $b \in \mathcal{B}$, and $c \in \mathcal{C}$. Then

$$\varepsilon'_i(b\otimes c) = \max\{0, \varepsilon'_i(b) - \varphi'_i(c)\} + \varepsilon'_i(c)$$

and

$$\varphi'_i(b\otimes c) = \max\{0, \varphi'_i(c) - \varepsilon'_i(b)\} + \varphi'_i(b)$$

so we have

$$\varepsilon'_i(b\otimes c) + \varphi'_i(b\otimes c) \equiv \varepsilon'_i(b) + \varphi'_i(b) + \varepsilon'_i(c) + \varphi'_i(c) \equiv 0 \pmod 2$$

as well as

$$\tfrac{\varphi'_i(b\otimes c) - \varepsilon'_i(b\otimes c)}{2} = \tfrac{\varphi'_i(b) - \varepsilon'_i(b)}{2} + \tfrac{\varphi'_i(c) - \varepsilon'_i(c)}{2} = \mathrm{wt}(b\otimes c)_i - \mathrm{wt}(b\otimes c)_{i+1}.$$

It is clear that e'_i and f'_i act as inverse operators when they are nonzero. Suppose $e'_i(b\otimes c) \neq 0$. If $\varepsilon'_i(b) \leqslant \varphi'_i(c)$ then $\mathrm{wt}(e'_i(b\otimes c)) - \mathrm{wt}(b\otimes c) = \mathrm{wt}(e'_i(c)) - \mathrm{wt}(c)$ and $\varepsilon'_i(b\otimes c) = \varepsilon'_i(c)$. If $\varepsilon'_i(b) > \varphi'_i(c)$ then $\mathrm{wt}(e'_i(b\otimes c)) - \mathrm{wt}(b\otimes c) = \mathrm{wt}(e'_i(b)) - \mathrm{wt}(b)$ and

$$\varepsilon'_i(b\otimes c) = \varepsilon'_i(b) - \varphi'_i(c) + \varepsilon'_i(c) \equiv \varepsilon'_i(b) + \varphi'_i(c) + \varepsilon'_i(c) \equiv \varepsilon'_i(b) \pmod 2.$$

Therefore, property (b) in Definition 4.1 for $\mathcal{B}$ and $\mathcal{C}$ implies the same property for $\mathcal{B}\otimes\mathcal{C}$. We conclude that $\mathcal{B}\otimes\mathcal{C}$ is a $\sqrt{\mathfrak{gl}_n}$-crystal.

The conditions we need to check to show that $\mathcal{B}\otimes\mathcal{C}$ is a $\sqrt{\mathfrak{q}_n}$-crystal when $\mathcal{B}$ and $\mathcal{C}$ are $\sqrt{\mathfrak{q}_n}$-crystals are all straightforward. Notice in particular that the first two components of the weight of $b\otimes c$ are both zero if and only if the same is true of both b and c, as we require all weights to be in $\mathbb{N}^n$.

Finally, the argument to show that the natural map $\mathcal{B}\otimes(\mathcal{C}\otimes\mathcal{D}) \to (\mathcal{B}\otimes\mathcal{C})\otimes\mathcal{D}$ is a $\sqrt{\mathfrak{gl}_n}$- or $\sqrt{\mathfrak{q}_n}$-crystal isomorphism is the same as the proof of the associativity of the tensor product for ordinary $\mathfrak{gl}_n$- or $\mathfrak{q}_n$-crystals; see [3, §2.3]. □

Remark 4.27. The tensor products for $\mathfrak{gl}_n$- and $\sqrt{\mathfrak{gl}_n}$-crystals (respectively, $\mathfrak{q}_n$- and $\sqrt{\mathfrak{q}_n}$-crystals) do not commute with the "squaring" operation in Proposition 4.4: if $\mathcal{B}^{(2)}$ denotes the crystal obtained from $\mathcal{B}$ via that result, then one can have $\mathcal{B}^{(2)} \otimes \mathcal{C}^{(2)} \not\cong (\mathcal{B}\otimes\mathcal{C})^{(2)}$. It does hold as usual that $\mathrm{ch}(\mathcal{B}\otimes\mathcal{C}) = \mathrm{ch}(\mathcal{B})\mathrm{ch}(\mathcal{C})$ when $\mathcal{B}$ and $\mathcal{C}$ are both finite, however.

Fix $m \in \mathbb{N}$ and recall the definition of $\mathbb{S}_n$ from Example 4.3. There is an obvious weight-preserving bijection $\mathbb{S}_n^{\otimes m} \to \mathsf{SetWords}_n(m)$ given by identifying tensors with tuples, which sends

$$S_1 \otimes S_2 \otimes \cdots \otimes S_m \mapsto (S_1, S_2, \ldots, S_m) \in \mathsf{SetWords}_n(m). \tag{4.3}$$

The following theorem shows that the $\sqrt{\mathfrak{q}_n}$-crystal structure on $\mathsf{SetWords}_n(m)$, which may have seemed unmotivated in Section 4.2, is in fact the one induced by this bijection. This statement implies Theorem 4.15, and so provides an alternate proof that result.

Theorem 4.28. *The bijection* (4.3) *is an isomorphism of* $\sqrt{\mathfrak{q}_n}$*-crystals*

$$\mathbb{S}_n^{\otimes m} \xrightarrow{\sim} \mathsf{SetWords}_n(m).$$

Proof. Fix $m \geqslant 2$ and $T = (T_1, T_2, \ldots, T_m) \in \mathsf{SetWords}_n(m)$. Let $U = (T_2, \ldots, T_m)$. Recall that $\mathbb{S}_n \cong \mathsf{SetWords}_n(1)$ as $\sqrt{\mathfrak{q}_n}$-crystals. To show that the map $\mathbb{S}_n^{\otimes m} \to \mathsf{SetWords}_n(m)$ is an isomorphism of $\sqrt{\mathfrak{gl}_n}$-crystals, it suffices by induction on m to check for each $i \in [n-1]$ that

$$e'_i(T) = \begin{cases} (T_1, e'_i(U)) & \text{if } \varepsilon'_i(T_1) \leqslant \varphi'_i(U) \\ (e'_i(T_1), U) & \text{if } \varepsilon'_i(T_1) > \varphi'_i(U) \end{cases}$$

and

$$f'_i(T) = \begin{cases} (T_1, f'_i(U)) & \text{if } \varepsilon'_i(T_1) < \varphi'_i(U) \\ (f'_i(T_1), U) & \text{if } \varepsilon'_i(T_1) \geqslant \varphi'_i(U), \end{cases}$$

where we set

$$(T_1, e'_i(U)) := \begin{cases} 0 & \text{if } e'_i(U) = 0 \\ (T_1, V_2, \ldots, V_m) & \text{if } e'_i(U) = (V_2, \ldots, V_m) \neq 0, \end{cases}$$

and interpret $(e'_i(T_1), U)$, $(T_1, f'_i(U))$, and $(f'_i(T_1), U)$ similarly. We will just prove the formula for $e'_i(T)$ since the argument for $f'_i(T)$ is not much different.

Recall that $\varphi'_i(U)$ is twice the number of right forms in the i-word of U plus one if there is a combined form. On the other hand, $\varepsilon'_i(T_1)$ is 2 if $T_1 \cap \{i, i+1\} = \{i+1\}$, 1 if $T_1 \cap \{i, i+1\} = \{i, i+1\}$, and 0 otherwise. The three possibilities here combine with the two in our desired formula for $e'_i(T)$ to give six different cases:

(1) Suppose $\varepsilon'_i(T_1) = 2 \leqslant \varphi'_i(U)$. Then the i-word of U must have at least one right form, and the first such form will merge with the "(" contributed by T_1 to create a null form in the i-word of T. In this case, the i-word of T will have a combined form (respectively, a left form) if and only if the i-word of U does as well, and the entries that correspond to ") – (" at the end of the combined forms (respectively, "(" at the start of the first left forms) in T and U coincide, so $e'_i(T) = (T_1, e'_i(U))$.

(2) Suppose $\varepsilon'_i(T_1) = 1 \leqslant \varphi'_i(U)$. Then the i-word of U must have a right or combined form. If there is a right form, then it will merge with the ") – (" contributed by T_1 to create the first right form in the i-word of T, and it follows exactly as in case (1) that $e'_i(T) = (T_1, e'_i(U))$. If there are no right forms, then the unique combined form in the i-word of U will merge with the ") – (" contributed by T_1 to create a larger combined form in the i-word

of T. When this happens, the entries that correspond to ") – (" at the end of the combined forms in T and U coincide, so again $e'_i(T) = (T_1, e'_i(U))$.

(3) Suppose $\varepsilon'_i(T_1) = 0 \leqslant \varphi'_i(U)$. Then the i-word of T is either identical to the i-word of U, or is given by appending the i-word of U after the single right form ")". Either way, the same reasoning as in case (1) shows that $e'_i(T) = (T_1, e'_i(U))$.

(4) Suppose $\varepsilon'_i(T_1) = 2 > \varphi'_i(U)$. Then the i-word of U must have no right forms. If the i-word of U has a combined form, then it will merge with the "(" contributed by T_1 to create the first left form in the i-word of T, which will have no combined forms. If there is no combined form in the i-word of U then the "(" contributed by T_1 will become on its own the first left in the i-word of T. In both situations, the i-word of T has no combined forms and T_1 is the entry that corresponds to the "(" at the start of the first left form, so $e'_i(T) = (e'_i(T_1), U)$.

(5) Suppose $\varepsilon'_i(T_1) = 1 > \varphi'_i(U)$. Then the i-word of U must have no right or combined forms, so the "(" contributed by T_1 will become the first left in the i-word of T, which will also have no combined forms. Then, it follows exactly as in case (4) that $e'_i(T) = (e'_i(T_1), U)$.

(6) The final case $\varepsilon'_i(T_1) = 0 > \varphi'_i(U)$ is impossible since $\varphi'(U) \in \mathbb{N}$.

This cases analysis shows that $e'_i(T)$ does have the desired formula in the equation displayed above. We conclude that $\mathbb{S}_n^{\otimes m} \to \mathsf{SetWords}_n(m)$ is an isomorphism of $\sqrt{\mathfrak{gl}_n}$-crystals.

To upgrade this to a $\sqrt{\mathfrak{q}_n}$-isomorphism, it suffices (again by induction on m) just to show that

$$e'_{\overline{1}}(T) = \begin{cases} (T_1, e'_{\overline{1}}(U)) & \text{if } 1,2 \notin T_1 \\ (e'_{\overline{1}}(T_1), U) & \text{otherwise} \end{cases} \quad \text{and} \quad f'_{\overline{1}}(T) = \begin{cases} (T_1, f'_{\overline{1}}(U)) & \text{if } 1,2 \notin T_1 \\ (f'_{\overline{1}}(T_1), U) & \text{otherwise.} \end{cases}$$

These formulas are immediate from the relevant Definitions 4.12 and 4.13. □

Corollary 4.29. *If $p, q \in \mathbb{N}$ then $\mathsf{SetWords}_n(p) \otimes \mathsf{SetWords}_n(q) \cong \mathsf{SetWords}_n(p+q)$.*

We mention an interesting $\sqrt{\mathfrak{gl}_n}$-crystal theoretic interpretation of the coefficients in the symmetric Grothendieck expansion of the product $G_\lambda G_\mu = \sum_\nu c^\nu_{\lambda\mu} G_\nu$. The coefficients $c^\nu_{\lambda\mu}$ in this expansion were first shown to be nonnegative integers by Buch [1]. Using Lemma 4.11 and Corollary 4.29, we can reformulate Buch's description of these numbers as follows:

Theorem 4.30 (Buch [1]). *If λ and μ are partitions then*

$$G_\lambda(x_1, x_2, \dots, x_n) G_\mu(x_1, x_2, \dots, x_n) = \sum_{\substack{b \in \mathsf{SetTab}_n(\lambda) \otimes \mathsf{SetTab}_n(\mu) \\ e'_i(b) = 0 \ \forall i \in [n-1]}} G_{\mathrm{wt}(b)}(x_1, x_2, \dots, x_n).$$

Proof. We attribute this theorem to Buch as it is almost immediate from [1, Thm. 5.4]. That result is equivalent to the statement that $G_\lambda(x_1, \dots, x_n) G_\mu(x_1, \dots, x_n) = \sum G_{\mathrm{wt}(T \otimes U)}(x_1, \dots, x_n)$ where the sum is over tensors $T \otimes U \in \mathsf{SetTab}_n(\lambda) \otimes \mathsf{SetTab}_n(\mu)$ such that $w(\mathsf{col}_{SV}(T)\mathsf{col}_{SV}(U))$ is a reverse lattice word, with $w(T)$ defined as in Lemma 4.11. By Corollary 4.29, we can embed

$$\mathsf{SetTab}_n(\lambda) \otimes \mathsf{SetTab}_n(\mu) \subseteq \mathsf{SetWords}_n(|\lambda|) \otimes \mathsf{SetWords}_n(|\mu|) \cong \mathsf{SetWords}_n(|\lambda| + |\mu|)$$

and then identify $T \otimes U$ with the concatenation $\mathsf{col}_{SV}(T)\mathsf{col}_{SV}(U)$, so it follows from Lemma 4.11 that $w(\mathsf{col}_{SV}(T)\mathsf{col}_{SV}(U))$ is a reverse lattice word if and only if $e'_i(T \otimes U) = 0$ for all $i \in [n-1]$. □

The preceding result does not lift to crystals, as the full subcrystal of $T \in \mathsf{SetTab}_n(\lambda) \otimes \mathsf{SetTab}_n(\mu)$ with $e'_i(T) = 0$ for all $i \in [n-1]$ is not always isomorphic to $\mathsf{SetTab}_n(\mathrm{wt}(T))$.

Remark 4.31. If λ and μ are strict partitions then $GP_\lambda(x_1, \dots, x_n)GP_\mu(x_1, \dots, x_n)$ is an $\mathbb{N}$-linear combination GP-functions $GP_\nu(x_1, \dots, x_n)$ [6, 30]. However, the coefficients in this expansion appear to be overcounted by the $\sqrt{\mathfrak{q}_n}$-highest weights in $\mathsf{SetDecTab}_n(\lambda) \otimes \mathsf{SetDecTab}_n(\mu)$. To be precise, in all examples we can compute, if $a^\nu_{\lambda\mu} \in \mathbb{N}$ are the numbers with

$$GP_\lambda(x_1, \dots, x_n)GP_\mu(x_1, \dots, x_n) = \sum_\nu a^\nu_{\lambda\mu} GP_\nu(x_1, \dots, x_n)$$

then

$$0 \leqslant a^\nu_{\lambda\mu} \leqslant \left|\left\{b \in \mathsf{SetDecTab}_n(\lambda) \otimes \mathsf{SetDecTab}_n(\mu) : e'_i(b) = e'_{\bar{i}}(b) = 0\ \forall i \in [n-1]\right\}\right|.$$

We use the tensor products in Theorem 4.26 to introduce families of *normal $\sqrt{\mathfrak{gl}_n}$-crystals* and *normal $\sqrt{\mathfrak{q}_n}$-crystals*: we define these to be the smallest full subcategories of crystals that contain all tensor powers of the standard object $\mathbb{S}_n$ and that are closed under disjoint unions and restriction to any connected component of the crystal graph. Theorems 4.17, 4.18, and 4.28 imply that:

Corollary 4.32. *The objects* $\mathsf{SetWords}_n(m)$ *and* $\mathsf{SetDecTab}_n(\lambda)$ *are normal as both* $\sqrt{\mathfrak{gl}_n}$- *and* $\sqrt{\mathfrak{q}_n}$-*crystals, while* $\mathsf{SetTab}_n(\lambda)$ *is a normal* $\sqrt{\mathfrak{gl}_n}$-*crystal.*

From Lemma 4.11, we also know that:

Corollary 4.33. *In a normal* $\sqrt{\mathfrak{gl}_n}$-*crystal, the weight of any highest weight element (that is, an element* b *with* $e'_1(b) = e'_2(b) = \dots = e'_{n-1}(b) = 0$*) is a partition with at most* n *nonzero parts.*

The categories of normal $\sqrt{\mathfrak{gl}_n}$- and $\sqrt{\mathfrak{q}_n}$-crystals are automatically closed under tensor products and disjoint unions. However, they are not so well-behaved as their classical counterparts. Every connected normal $\mathfrak{gl}_n$- or $\mathfrak{q}_n$-crystal has a unique highest weight element, whose weight uniquely determines the crystal's isomorphism class, and all finite normal $\mathfrak{gl}_n$- or $\mathfrak{q}_n$-crystals with the same character are isomorphic (see [25, §1.2]). By contrast:

- There are connected normal $\sqrt{\mathfrak{gl}_n}$- and $\sqrt{\mathfrak{q}_n}$-crystals with multiple highest weight elements. For example, there is a full $\sqrt{\mathfrak{gl}_3}$-subcrystal of $\mathsf{SetWords}_3(4)$ with two highest weight elements

  $$(\{1,3\},\{1\},\{1,2\},\{1\}) \quad \text{and} \quad (\{1,2,3\},\{1\},\{1,2\},\{1\})$$

  whose character is $G_{(4,1,1)}(x_1,x_2,x_3) + G_{(4,2,1)}(x_1,x_2,x_3)$, and there is a full $\sqrt{\mathfrak{q}_n}$-subcrystal of $\mathsf{SetWords}_3(5)$ with two highest weight elements

  $$(\{1\},\{2\},\{1\},\{1,2\},\{1\}) \quad \text{and} \quad (\{1\},\{2,3\},\{1\},\{1,2\},\{1\})$$

  whose character is $GP_{(4,2)}(x_1,x_2,x_3)$.
- Among the connected normal $\sqrt{\mathfrak{gl}_n}$- and $\sqrt{\mathfrak{q}_n}$-crystals that do have unique highest weight elements, there exist non-isomorphic objects with the same highest weight. For example,

  $$(\{1\},\{2\},\{1\}) \in \mathsf{SetWords}_3(3)$$

  is the unique highest weight element of a full $\sqrt{\mathfrak{gl}_3}$-subcrystal that is not isomorphic to $\mathsf{SetTab}_3((2,1))$, although both have character $G_{(2,1)}(x_1,x_2,x_3)$. Similarly,

  $$(\{1\},\{2\},\{1\},\{1\}) \in \mathsf{SetWords}_3(4)$$

is the unique highest weight element of a full $\sqrt{\mathfrak{q}_3}$-subcrystal that is not isomorphic to $\mathsf{SetDecTab}_3((3,1))$, although both have character $GP_{(3,1)}(x_1,x_2,x_3)$.

- We expect that the normal crystals $\mathsf{SetTab}_n(\lambda)$ and $\mathsf{SetDecTab}_n(\lambda)$ have unique highest weight elements of weight λ; this is known for $\mathsf{SetTab}_n(\lambda)$ but only conjectural for $\mathsf{SetDecTab}_n(\lambda)$. One might hope that taking the isomorphism classes of just these set-valued tableau crystals would give good substitutes for the "too large" categories of normal $\sqrt{\mathfrak{gl}_n}$- and $\sqrt{\mathfrak{q}_n}$-crystals. However, neither of the resulting subcategories is closed under tensor products. For example,

$$\mathsf{SetTab}_3((1)) \otimes \mathsf{SetTab}_3((2))$$

and

$$\mathsf{SetDecTab}_3((1)) \otimes \mathsf{SetDecTab}_3((2))$$

each have a full subcrystal of unique highest weight $(3,1)$. The subcrystal of the first object is not isomorphic to $\mathsf{SetTab}_3((3,1))$ as a $\sqrt{\mathfrak{gl}_3}$-crystal, and the subcrystal of the second object is not isomorphic to $\mathsf{SetDecTab}_3((3,1))$ as a $\sqrt{\mathfrak{q}_3}$-crystal.

Despite these observations, normal $\sqrt{\mathfrak{gl}_n}$- and normal $\sqrt{\mathfrak{q}_n}$-crystals do appear to have some interesting properties, and there seems to be a close connection between these crystal categories and the rings of symmetric functions generated by $\{G_\lambda\}$ and $\{GP_\lambda\}$. This speculative relationship is indicated by the conjectures explained in the next section.

4.5. Conjectures.

We use this final section to present several conjectures about normal $\sqrt{\mathfrak{gl}_n}$- and $\sqrt{\mathfrak{q}_n}$-crystals that would be interesting to explore in future work.

First, we say that an element of $\mathbb{Z}[x_1,\dots,x_n]$ is *G-positive* (respectively, *GP-positive*) if it is an $\mathbb{N}$-linear combination of polynomials of the form $G_\lambda(x_1,\dots,x_n)$ (respectively, $GP_\lambda(x_1,\dots,x_n)$).

Example 4.34. The character of a tensor power of $\mathbb{S}_n$ is both G- and GP-positive as

$$\mathrm{ch}(\mathbb{S}_n^{\otimes m}) = \mathrm{ch}(\mathbb{S}_n)^m = G_{(1)}(x_1,x_2,\dots,x_n)^m = GP_{(1)}(x_1,x_2,\dots,x_n)^m$$

and the Pieri rules in [1, 2] expand the last expression as a sum of G- or GP-functions.

Although it is known that $\mathrm{ch}(\mathsf{SetTab}_n(\lambda)) = G_\lambda(x_1,\dots,x_n)$ and conjectured that $\mathrm{ch}(\mathsf{SetDecTab}_n(\lambda)) = GP_\lambda(x_1,\dots,x_n)$, the characters of other connected normal $\sqrt{\mathfrak{gl}_n}$- and $\sqrt{\mathfrak{q}_n}$-crystals are not single G- or GP-polynomials. However, the following holds in all examples we can compute:

Conjecture 4.35. *The character of any finite normal $\sqrt{\mathfrak{gl}_n}$-crystal is G-positive.*

Conjecture 4.36. *The character of any finite normal $\sqrt{\mathfrak{q}_n}$-crystal is GP-positive.*

A more precise version of the first conjecture, generalizing Theorem 4.30, also seems to hold:

Conjecture 4.37. *If $\mathcal{B}$ is a finite normal $\sqrt{\mathfrak{gl}_n}$-crystal then*

$$\mathrm{ch}(\mathcal{B}) = \sum_b G_{\mathrm{wt}(b)}(x_1,\dots,x_n)$$

where the sum is over all $b \in \mathcal{B}$ with $e_i'(b) = 0$ for all $i \in [n-1]$.

We have checked these conjectures by computer for all subcrystals of $\mathbb{S}_n^{\otimes m} \cong \mathsf{SetWords}_n(m)$ when $n = 2$ and $m \leqslant 12$, $n = 3$ and $m \leqslant 7$, $n = 4$ and $m \leqslant 5$, $n = 5$ and $m \leqslant 4$, or $n = 6$ and $m \leqslant 3$. One might try to prove them by identifying set-valued analogues of *RSK insertion* or Haiman's *shifted mixed insertion* [11], perhaps extending the *uncrowding algorithms* in [1, 28].

As in Remark 4.31, it is not clear how to convert Conjecture 4.37 to a formula for the GP-expansion of the character of a finite normal $\sqrt{\mathfrak{q}_n}$-crystal. Like Theorem 4.30, the character formula in Conjecture 4.37 does not lift to an isomorphism of $\sqrt{\mathfrak{gl}_n}$-crystals.

We highlight the special case of Conjecture 4.37 with $\mathcal{B} = \mathsf{SetDecTab}_n(\lambda)$. This asserts that the $\sqrt{\mathfrak{gl}_n}$-highest weight elements in $\mathsf{SetDecTab}_n(\lambda)$ encode the G-expansion of its character, which via Conjecture 3.2 is expected to be $GP_\lambda(x_1, x_2, \ldots, x_n)$. The coefficients in the G-expansion of GP_λ are already known to be nonnegative integers by [24, Thm. 3.27].

Conjecture 4.38. *If λ is a strict partition then*

$$GP_\lambda(x_1, x_2, \ldots, x_n) = \sum_{\substack{T \in \mathsf{SetDecTab}_n(\lambda) \\ e'_i(T) = 0 \ \forall i \in [n-1]}} G_{\mathrm{wt}(T)}(x_1, x_2, \ldots, x_n).$$

We have checked this conjecture when $n = 2$ and $|\lambda| \leqslant 30$, $n = 3$ and $|\lambda| \leqslant 12$, $n = 4$ and $|\lambda| \leqslant 8$, $n = 5$ and $|\lambda| \leqslant 6$, or $n = 6$ and $|\lambda| \leqslant 4$.

Recall the definition of $w(T)$ for $T \in \mathsf{SetWords}_n(m)$ from Lemma 4.11. Via that result and Theorem 4.18, Conjecture 4.38 is equivalent to the following statement:

Conjecture 4.39. *If λ is a strict partition then $GP_\lambda = \sum_\nu d_{\lambda\mu} G_\mu$ where the sum is over strict partitions μ and $d_{\lambda\mu}$ is the number of set-valued decomposition tableaux T of shape λ and weight $\mathrm{wt}(T) = \mu$ such that $w(\mathsf{revrow}_{SV}(T))$ is a reverse lattice word.*

The conjectures above are plausible in view of the following observations. We have seen that the character of any finite $\sqrt{\mathfrak{gl}_n}$-crystal is symmetric. It is well-known that any symmetric polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ is a $\mathbb{Z}$-linear combination of symmetric Grothendieck polynomials. In addition:

Proposition 4.40. *The character of any finite normal $\sqrt{\mathfrak{q}_n}$-crystal is symmetric with the K-theoretic Q-cancelation property, so is a $\mathbb{Z}$-linear combination of GP-polynomials $GP_\lambda(x_1, \ldots, x_n)$.*

Proof. We prove this when $\mathcal{B}$ is a connected normal $\sqrt{\mathfrak{gl}_n}$-crystal. We may assume that $\mathcal{B} \subseteq \mathsf{SetWords}_n(m)$ for some m. We already know that $\mathrm{ch}(\mathcal{B})$ is symmetric by Proposition 4.4 so we just need to verify that it has the K-theoretic Q-cancelation property.

Let $\mathcal{B}^*$ be the set of *multiset-valued words* $S = (S_1, \ldots, S_m)$ with the following properties: (a) in each multiset S_i only the number 1 may be repeated, and (b) when the repeated 1's are replaced by a single copy, one gets a set-valued word $\overline{S} \in \mathcal{B}$. Then

$$\mathrm{ch}(\mathcal{B})(\tfrac{x_1}{1-x_1}, -x_2, x_3, \ldots, x_n) = \sum_{S \in \mathcal{B}^*} (-1)^{c_2(S)} x_1^{c_1(S)} x_2^{c_2(S)} x_3^{c_3(S)} \cdots x_n^{c_n(S)}$$

where $c_j(S)$ denotes the multiplicity of j in the multiset union $S_1 \cup S_2 \cup \cdots \cup S_m$.

As in the proof of Proposition 3.10, to show that $\mathrm{ch}(\mathcal{B})$ has the K-theoretic Q-cancelation property, we must demonstrate that the right side of the previous equation is independent of x_1 when we set $x_1 = x_2$. For this, it is enough to produce an involution of $\phi : \mathcal{B}^* \to \mathcal{B}^*$ such that $\phi(S) = S$ when $c_1(S) = c_2(S) = 0$ and such that for all other $S \in \mathcal{B}^*$ we have

- $c_2(\phi(S)) = c_2(S) \pm 1$,
- $c_1(\phi(S)) + c_2(\phi(S)) = c_1(S) + c_2(S)$, and
- $c_j(\phi(S)) = c_j(S)$ for $2 < j < n$.

Here is such an involution. Suppose $i \in [m]$ is minimal with $1 \in S_i$ or $2 \in S_i$. If there is no such index then set $\phi(S) := S$. Otherwise, form $\phi(S)$ from S by modifying S_i as follows. If there is already a (unique) copy of $2 \in S_i$, then remove this 2 and add an extra copy of 1. If $2 \notin S_i$, then instead replace one copy of $1 \in S_i$ by 2.

Clearly $\phi(S)$ has the desired properties and $\phi(\phi(S)) = S$. It only remains to justify why $\phi(S) \in \mathcal{B}^*$. Recall the definition of $\overline{S} \in \mathcal{B}$ from the first paragraph of this proof. It is enough to check that $\overline{\phi(S)} \in \mathcal{B}$. If $2 \in S_i$ and $1 \in S_i$, then $\overline{\phi(S)} = e'_{\overline{1}}(\overline{S})$, which certainly belongs to $\mathcal{B}$. Likewise, if $2 \in S_i$ and $1 \notin S_i$, then $\overline{\phi(S)} = e'_{\overline{1}}(e'_{\overline{1}}(\overline{S})) \in \mathcal{B}$. Finally, if $2 \notin S_i$ then $\overline{\phi(S)} = f'_{\overline{1}}(\overline{S})$ when $c_1(S) > 1$ and $\overline{\phi(S)} = f'_{\overline{1}}(f'_{\overline{1}}(\overline{S}))$ when $c_1(S) = 1$, and in both cases $\overline{\phi(S)} \in \mathcal{B}$ as needed. □

Finally, we recall from [3, Ex. 5.2] that if $\mathcal{B}$ is a $\mathfrak{gl}_n$-crystal then its *Lusztig dual* $\mathcal{B}^\vee$ is the $\mathfrak{gl}_n$-crystal on the same underlying set with weight map

$$\mathrm{wt}^\vee(b) := (\mathrm{wt}(b)_n, \dots, \mathrm{wt}(b)_2, \mathrm{wt}(b)_1)$$

and crystal operators $e_i^\vee := f_{n-i}$ and $f_i^\vee := e_{n-i}$. We define the *Lusztig dual* of a $\sqrt{\mathfrak{gl}_n}$-crystal in the same way, setting $e_i'^\vee := f'_{n-i}$ and $f_i'^\vee := e'_{n-i}$.

It easy to check that the operators on $\mathcal{B}^\vee$ satisfy the relevant $\mathfrak{gl}_n$- or $\sqrt{\mathfrak{gl}_n}$-crystal axioms, and that the map $b \otimes c \mapsto c \otimes b$ is a crystal isomorphism $(\mathcal{B} \otimes \mathcal{C})^\vee \cong \mathcal{C}^\vee \otimes \mathcal{B}^\vee$. Moreover, one has $\mathbb{B}_n \cong \mathbb{B}_n^\vee$ as $\mathfrak{gl}_n$-crystals and $\mathbb{S}_n \cong \mathbb{S}_n^\vee$ as $\sqrt{\mathfrak{gl}_n}$-crystals. It follows as a consequence that:

Proposition 4.41. *If $\mathcal{B}$ is a connected normal $\mathfrak{gl}_n$- or $\sqrt{\mathfrak{gl}_n}$-crystal, then so is $\mathcal{B}^\vee$.*

Any connected normal $\mathfrak{gl}_n$-crystal is isomorphic to its Lusztig dual, which has the same unique highest weight. This property does not hold for all connected normal $\sqrt{\mathfrak{gl}_n}$-crystals, but does apply in the following cases.

A partition λ is a *rectangle* if its nonzero parts all have the same size, and a *fat hook* if it has exactly two distinct part sizes, so that $|\{\lambda_1, \lambda_2, \dots\} \setminus \{0\}| = 2$.

Proposition 4.42. *Suppose λ is a partition with $\ell(\lambda) \leqslant n$. Then $\mathsf{SetTab}_n(\lambda) \cong \mathsf{SetTab}_n(\lambda)^\vee$ as $\sqrt{\mathfrak{gl}_n}$-crystals if λ is a rectangle, or if λ is a fat hook with $\ell(\lambda) = n$.*

Proof. First assume that λ is a rectangle. Given $T \in \mathsf{SetTab}_n(\lambda)$ form $T^\vee$ by rotating the tableau 180° and then applying the substitution $S \mapsto n+1-S$ to each set-valued entry. The resulting map $T \mapsto T^\vee$ is an involution of $\mathsf{SetTab}_n(\lambda)$. (When T is not set-valued, this operation is the classical *Sch́utzenberger involution* [17, §2.4].) The set-valued column reading word of $T^\vee$ is formed by reversing $\mathsf{col}_{SV}(T)$ and then applying the substitution $S \mapsto n+1-S$ to each entry. Once we make this observation, checking that $T \mapsto T^\vee$ is an isomorphism $\mathsf{SetTab}_n(\lambda) \cong \mathsf{SetTab}_n(\lambda)^\vee$ is a straightforward exercise.

Now suppose that λ is a fat hook with $\ell(\lambda) = n$. Let $m \geqslant 1$ be the number of columns in the diagram of λ with n boxes. Then each $T \in \mathsf{SetTab}_n(\lambda)$, being semistandard, must have $T_{ij} = \{i\}$ for all $(i,j) \in [n] \times [m]$, and removing these boxes yields an arbitrary element $U \in \mathsf{SetTab}_n(\mu)$ where μ is the rectangular partition $(\lambda_1 - m, \lambda_2 - m, \dots, \lambda_n - m)$. Define $T^\vee$ from T by leaving the first m columns unchanged and replacing U by $U^\vee$ as given in the previous paragraph. It is again a simple exericise to check that $T \mapsto T^\vee$ is a crystal isomorphism $\mathsf{SetTab}_n(\lambda) \cong \mathsf{SetTab}_n(\lambda)^\vee$. □

Our computations suggest that the converse to the previous result holds.

CONJECTURE 4.43. *Suppose λ is a partition with $\ell(\lambda) \leqslant n$. Then* $\mathsf{SetTab}_n(\lambda) \cong \mathsf{SetTab}_n(\lambda)^\vee$ *as $\sqrt{\mathfrak{gl}_n}$-crystals only if λ is a rectangle, or λ is a fat hook with $\ell(\lambda) = n$.*

We require λ to have $\ell(\lambda) \leqslant n$ in these statements since otherwise $\mathsf{SetTab}_n(\lambda)$ is empty. We have checked by computer that this conjecture holds at least for $n \leqslant 5$ and $|\lambda| \leqslant 12$.

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REFERENCES

[1] Anders Skovsted Buch, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78.
[2] Anders Skovsted Buch and Vijay Ravikumar, *Pieri rules for the K-theory of cominuscule Grassmannians*, J. Reine Angew. Math. **668** (2012), 109–132.
[3] Daniel Bump and Anne Schilling, *Crystal bases: Representations and combinatorics*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.
[4] Soojin Cho, *A new Littlewood-Richardson rule for Schur P-functions*, Trans. Amer. Math. Soc. **365** (2013), no. 2, 939–972.
[5] Seung-Il Choi, Sun-Young Nam, and Young-Tak Oh, *Bijections among combinatorial models for shifted Littlewood-Richardson coefficients*, J. Combin. Theory Ser. A **128** (2014), 56–83.
[6] Edward Clifford, Hugh Thomas, and Alexander Yong, *K-theoretic Schubert calculus for* $\mathrm{OG}(n, 2n+1)$ *and jeu de taquin for shifted increasing tableaux*, J. Reine Angew. Math. **690** (2014), 51–63.
[7] Maria Gillespie, Graham Hawkes, Wencin Poh, and Anne Schilling, *Characterization of queer supercrystals*, J. Combin. Theory Ser. A **173** (2020), article no. 105235 (53 pages).
[8] Dimitar Grantcharov, Ji Hye Jung, Seok-Jin Kang, Masaki Kashiwara, and Myungho Kim, *Crystal bases for the quantum queer superalgebra and semistandard decomposition tableaux*, Trans. Amer. Math. Soc. **366** (2014), no. 1, 457–489.
[9] Dimitar Grantcharov, Ji Hye Jung, Seok-Jin Kang, Masaki Kashiwara, and Myungho Kim, *Crystal bases for the quantum queer superalgebra*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 7, 1593–1627.
[10] Dimitar Grantcharov, Ji Hye Jung, Seok-Jin Kang, and Myungho Kim, *Highest weight modules over quantum queer superalgebra* $U_q(\mathfrak{q}(n))$, Comm. Math. Phys. **296** (2010), no. 3, 827–860.
[11] Mark D. Haiman, *On mixed insertion, symmetry, and shifted Young tableaux*, J. Combin. Theory Ser. A **50** (1989), no. 2, 196–225.
[12] Zachary Hamaker, Adam Keilthy, Rebecca Patrias, Lillian Webster, Yinuo Zhang, and Shuqi Zhou, *Shifted Hecke insertion and the K-theory of* $OG(n, 2n+1)$, J. Combin. Theory Ser. A **151** (2017), 207–240.
[13] Graham Hawkes and Travis Scrimshaw, *Crystal structures for canonical Grothendieck functions*, Algebr. Comb. **3** (2020), no. 3, 727–755.
[14] Takeshi Ikeda, Personal communication.
[15] Takeshi Ikeda and Hiroshi Naruse, *K-theoretic analogues of factorial Schur P- and Q-functions*, Adv. Math. **243** (2013), 22–66.
[16] Shinsuke Iwao, *Neutral-fermionic presentation of the K-theoretic Q-function*, J. Algebraic Combin. **55** (2022), no. 2, 629–662.
[17] Cristian Lenart, *On the combinatorics of crystal graphs. I. Lusztig's involution*, Adv. Math. **211** (2007), no. 1, 204–243.
[18] Joel Brewster Lewis and Eric Marberg, *Combinatorial formulas for shifted dual stable Grothendieck polynomials*, Sém. Lothar. Combin. **89B** (2023), article no. 22 (12 pages).
[19] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995.
[20] Eric Marberg, *Bumping operators and insertion algorithms for queer supercrystals*, Selecta Math. (N.S.) **28** (2022), no. 2, article no. 36 (62 pages).
[21] Eric Marberg, *Shifted combinatorial Hopf algebras from K-theory*, Algebr. Comb. **7** (2024), no. 4, 1123–1156.

[22] Eric Marberg and Brendan Pawlowski, *K-theory formulas for orthogonal and symplectic orbit closures*, Adv. Math. **372** (2020), article no. 107299 (43 pages).
[23] Eric Marberg and Brendan Pawlowski, *On some properties of symplectic Grothendieck polynomials*, J. Pure Appl. Algebra **225** (2021), no. 1, article no. 106463 (22 pages).
[24] Eric Marberg and Travis Scrimshaw, *Key and Lascoux polynomials for symmetric orbit closures*, 2023, `https://arxiv.org/abs/2302.04226`.
[25] Eric Marberg and Kam Hung Tong, *Highest weight crystals for Schur Q-functions*, Comb. Theory **3** (2023), no. 2, article no. 6 (59 pages).
[26] Cara Monical, Oliver Pechenik, and Travis Scrimshaw, *Crystal structures for symmetric Grothendieck polynomials*, Transform. Groups **26** (2021), no. 3, 1025–1075.
[27] Masaki Nakagawa and Hiroshi Naruse, *Universal factorial Schur P, Q-functions and their duals*, 2018, `https://arxiv.org/abs/math/1812.03328`.
[28] Jianping Pan, Joseph Pappe, Wencin Poh, and Anne Schilling, *Uncrowding algorithm for hook-valued tableaux*, Ann. Comb. **26** (2022), no. 1, 261–301.
[29] Oliver Pechenik and Travis Scrimshaw, *K-theoretic crystals for set-valued tableaux of rectangular shapes*, Algebr. Comb. **5** (2022), no. 3, 515–536.
[30] Oliver Pechenik and Alexander Yong, *Genomic tableaux*, J. Algebraic Combin. **45** (2017), no. 3, 649–685.
[31] Luis Serrano, *The shifted plactic monoid*, Math. Z. **266** (2010), no. 2, 363–392.
[32] John R. Stembridge, *A local characterization of simply-laced crystals*, Trans. Amer. Math. Soc. **355** (2003), no. 12, 4807–4823.
[33] Tianyi Yu, *Set-valued tableaux rule for Lascoux polynomials*, Comb. Theory **3** (2023), no. 1, article no. 13 (31 pages).

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