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Abstract Schubert polynomials form a basis of the polynomial ring $\mathbb{Q}[x_1, x_2, \dots]$. This basis and its structure constants have received extensive study. Recently, Pan and Yu initiated the study of top Lascoux polynomials. These polynomials form a basis of the vector space $\widehat{V}$, a sub-algebra of $\mathbb{Q}[x_1, x_2, \dots]$ where each graded piece has finite dimension. This paper connects Schubert polynomials and top Lascoux polynomials via a simple operator. We use this connection to show these two bases share the same structure constants. We also translate several results on Schubert polynomials to top Lascoux polynomials, including combinatorial formulas for their monomial expansions and supports.

1. Introduction

For a permutation w, Lascoux and Schützenberger [14] recursively defined the *Schubert polynomial* $\mathfrak{S}_w$ using *divided difference operators*. These polynomials represent Schubert cycles in flag varieties and have been extensively investigated from various perspectives. We summarize some significant results on Schubert polynomials relevant to this paper.

(1) The set of all Schubert polynomials forms a basis of the polynomial ring $\mathbb{Q}[x_1, x_2, \dots]$. Products of Schubert polynomials can be expanded positively into Schubert polynomials (i.e. the expansion only involves positive integer coefficients):
$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{u,v}^w \mathfrak{S}_w,$$
The coefficient $c_{u,v}^w$ is known as the *Schubert structure constant*. A major open problem in algebraic combinatorics is to compute $c_{u,v}^w$ combinatorially.

(2) Lam, Lee and Shimozono [13] introduced the *(reduced) bumpless pipedreams (BPD)* to compute the monomial expansion of Schubert polynomials.

(3) The Schubert polynomial can be expanded positively into *key polynomials* [25].

(4) The dual character of the flagged Weyl module of a diagram D is denoted as χ_D. The Schubert polynomial $\mathfrak{S}_w$ is $\chi_{RD(w)}$ where $RD(w)$ is the Rothe diagram of w [12].

(5) Adve, Robichaux, and Yong [1] introduced *perfect tableaux* to compute the support of Schubert polynomials.

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(6) The Schubert polynomials have the *saturated Newton polytope (SNP)* property [7].

The key polynomials mentioned above are denoted as κ_α, where α is a *weak composition*. They are the characters of Demazure modules [4]. Lascoux [15] introduced an inhomogeneous analogue of κ_α known as the *Lascoux polynomial* $\mathfrak{L}_\alpha$. The lowest-degree terms of $\mathfrak{L}_\alpha$ form κ_α. Recently, Pan and Yu [20] introduced the *top Lascoux polynomial* $\widehat{\mathfrak{L}}_\alpha$ which consists of the highest-degree terms of $\mathfrak{L}_\alpha$. Let $\widehat{V}$ be the $\mathbb{Q}$-span of all top Lascoux polynomials. Unlike the Schubert polynomials, the set of all top Lascoux polynomials is not linearly independent. To resolve this, Pan and Yu called a weak composition *snowy* if its positive entries are distinct. Then $\{\widehat{\mathfrak{L}}_\alpha : \alpha \text{ is snowy}\}$ forms a basis of $\widehat{V}$. By [20, Theorem 1.2], every top Lascoux polynomial is a top Lascoux indexed by snowy weak composition multiplied by an integer. In the rest of this paper, we only focus on $\widehat{\mathfrak{L}}_\alpha$ when α is snowy.

Pan and Yu showed that $\widehat{V}$ is closed under multiplication. Thus, $\widehat{V}$ can be viewed as a graded algebra where the grading is given by degrees of polynomials. Each graded piece of $\widehat{V}$ has finite dimension. Pan and Yu computed the Hilbert series of $\widehat{V}$. Intuitively, $\widehat{V}$ is much smaller than the polynomial ring $\mathbb{Q}[x_1, x_2, \dots]$ which has no Hilbert series.

Just like the Schubert polynomials, $\widehat{\mathfrak{L}}_\alpha$ can be defined recursively using divided difference operators (see (1)). This resemblance leads to a strong connection between Schubert polynomials and top Lascoux polynomials, which is the main focus of this paper.

Definition 1.1. *Define the following involution on polynomials in $\mathbb{Q}[x_1, \dots, x_n]$ where each variable has degree at most m:*

$$r_{m,n}(f) := (x_1 \cdots x_n)^m f(x_n^{-1}, \dots, x_1^{-1}).$$

In §3, we show that each top Lascoux polynomial can be realized as $r_{m,n}(\mathfrak{S}_w)$ for some m, n, w and vice versa. Following this connection, we translate the results on $\mathfrak{S}_w$ summarized above to $\widehat{\mathfrak{L}}_\alpha$.

(1) Products of top Lascoux polynomials can be expanded positively into top Lascoux polynomials:

$$\widehat{\mathfrak{L}}_\alpha \widehat{\mathfrak{L}}_\gamma = \sum_\sigma d_{\alpha,\gamma}^\sigma \widehat{\mathfrak{L}}_\sigma.$$

We call the coefficient *top Lascoux structure constants*. Every $d_{\alpha,\gamma}^\sigma$ is $c_{u,v}^w$ for some explicit permutations u, v, w and vice versa (see §4).

(2) We give a monomial expansion of top Lascoux polynomials using (modified) bumpless-pipedreams (see §5).

(3) The top Lascoux polynomials can be expanded positively into key polynomials (see §6).

(4) The top Lascoux polynomial $\widehat{\mathfrak{L}}_\alpha = \chi_{\mathsf{snow}(D(\alpha))}$ where $\mathsf{snow}(D(\alpha))$ is some diagram defined in [20](See §7).

(5) The support of top Lascoux polynomials can be computed using perfect tableaux (see §7).

(6) The top Lascoux polynomials have the SNP property (See §7).

These properties of the top Lascoux basis precisely mirror those of the Schubert basis. It may be interesting to do Schubert calculus in the graded ring $\widehat{V}$ which as a defined and understood Hilbert series. By (1), computing the Schubert structure constants is the same as computing the top Lascoux structure constants.

Another potential application of our results is to understand the Grothendieck polynomial $\mathfrak{G}_w$, the K-theoretic analogue of $\mathfrak{S}_w$. By Shimozono and Yu [27], $\mathfrak{G}_w$ positively into $\mathfrak{L}_\alpha$. Consequently, the top degree component of $\mathfrak{G}_w$, denoted as $\widehat{\mathfrak{G}}_w$, expands positively into $\widehat{\mathfrak{L}}_\alpha$. There has been a recent surge in the study of $\widehat{\mathfrak{G}}_w$ [2, 3, 5, 9, 10, 22, 19, 23, 24]. Together with the expansion of $\widehat{\mathfrak{G}}_w$ into $\widehat{\mathfrak{L}}_\alpha$, one would translate our results on $\widehat{\mathfrak{L}}_\alpha$ to $\widehat{\mathfrak{G}}_w$. In particular, when w is vexillary (2143-avoiding), Pechenik and Scrimshaw [21] showed that $\mathfrak{G}_w$ is just a Lascoux polynomial. We may then translate our results to $\widehat{\mathfrak{G}}_w$ for vexillary w. See Corollary 7.4 for one such application.

The rest of the paper is structured as follows. In §2, we provide an overview of the necessary background information. In §3, we use $r_{m,n}$ to relate the Schubert polynomials and top Lascoux polynomials. The subsequent sections explore various applications of this relationship. In §4, we examine the connection between the structure coefficients of top Lascoux polynomials and Schubert polynomials. In §5, we derive a combinatorial formula for top Lascoux polynomials from the BPD formula of Schubert polynomials. In §6, we translate the key expansion of Schubert polynomials to obtain a key expansion of top Lascoux polynomials. In §7, we show the top Lascoux polynomials are certain dual characters of the flagged Weyl modules and characterize the support of top Lascoux polynomials.

2. Background

2.1. Schubert polynomials. Let S_+ be the group of permutations of $\{1, 2, \dots\}$ where only finitely many elements are permuted. The simple transpositions $s_1, s_2, \dots$ where $s_i = (i, i+1)$ generate S_+. For any positive number n, S_n is a subgroup of S_+ consisting of w that only permutes $[n] = \{1, 2, \dots, n\}$. We represent $w \in S_+$ by its *one-line notation* $[w(1), \dots, w(n)]$ for some n large enough such that $w \in S_n$.

A *weak composition* $\alpha = (\alpha_1, \alpha_2, \dots)$ is an infinite sequence of non-negative numbers with finitely many positive entries. The *support* of α is $\mathsf{supp}(\alpha) := \{i : \alpha_i > 0\}$. We represent α as $(\alpha_1, \alpha_2, \dots, \alpha_n)$ where $\mathsf{supp}(\alpha) \subseteq [n]$. Let $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ and $|\alpha| := \sum_{i \geqslant 1} \alpha_i$.

We say (i, j) is an inversion of $w \in S_+$ if $i < j$ and $w(i) > w(j)$. The *inversion code* of w, denoted as $\mathsf{invcode}(w)$ is a weak composition defined as

$$\mathsf{invcode}(w)_i := |\{j : (i, j) \text{ is an inversion of } w\}|.$$

The *Schubert polynomials* $\mathfrak{S}_w$ are indexed by permutations from S_+. When a weak composition is weakly decreasing, we say it is a *partition*. When $\mathsf{invcode}(w)$ is a partition, we say w is a *dominant permutation*. Define the *Newton divided difference operator*:

$$\partial_i(f) := \frac{f - s_i f}{x_i - x_{i+1}},$$

where s_i is the operator that swaps x_i and x_{i+1}. Now we can define the *Schubert polynomial* of $w \in S_+$ recursively [14].

$$\mathfrak{S}_w = \begin{cases} x^{\mathsf{invcode}(w)} & \text{if } w \text{ is dominant} \\ \partial_i(\mathfrak{S}_{ws_i}) & \text{if } w(i) < w(i+1). \end{cases}$$

The set of Schubert polynomials forms a $\mathbb{Q}$-basis of the polynomial ring $\mathbb{Q}[x_1, x_2, \dots]$. For $u, v \in S_+$, the product $\mathfrak{S}_u \mathfrak{S}_v$ can be expanded into Schubert polynomials. Let $c_{u,v}^w$ be the coefficient of $\mathfrak{S}_w$ in this expansion. By geometric results, $c_{u,v}^w$ is a non-negative integer known as the *Schubert structure constants*.

2.2. KEY POLYNOMIALS AND TOP LASCOUX POLYNOMIALS. The *key polynomials* κ_α are indexed by weak compositions. Lascoux and Schützenberger [16] define the key polynomials recursively. using the operator $\pi_i(f) := \partial_i(x_i f)$:

$$\kappa_\alpha := \begin{cases} x^\alpha & \text{if } \alpha \text{ is a partition,} \\ \pi_i(\kappa_{s_i\alpha}) & \text{if } \alpha_i < \alpha_{i+1}, \end{cases}$$

where s_i swaps the i^{th} and $(i+1)^{\text{th}}$ entries of α.

The *top Lascoux polynomials* $\widehat{\mathfrak{L}}_\alpha$ are homogeneous polynomials indexed by *snowy* weak compositions: weak compositions whose positive entries are distinct. Following [20, Lemma 4.23], we may define these polynomials recursively. Define the operator $\widehat{\pi}_i$ as

$$\widehat{\pi}_i(f) := \pi_i(x_{i+1}f) = x_i x_{i+1}\partial_i(f).$$

Then define

$$\widehat{\mathfrak{L}}_\alpha := \begin{cases} x^\alpha & \text{if } \alpha \text{ is a partition,} \\ \widehat{\pi}_i(\widehat{\mathfrak{L}}_{s_i\alpha}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases} \tag{1}$$

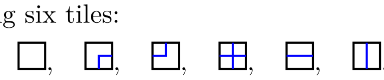
By [20], the vector space

$$\widehat{V} := \mathbb{Q}\text{-span}\{\widehat{\mathfrak{L}}_\alpha : \alpha \text{ is a snowy weak composition}\} \tag{2}$$

is a sub-algebra of $\mathbb{Q}[x_1, x_2, \dots]$. Its basis is given by the spanning set in (2) and its Hilbert series is $\prod_{m>0}\left(1 + \frac{q^m}{1-q}\right)$. In particular, each graded piece of $\widehat{V}$ has finite dimension.

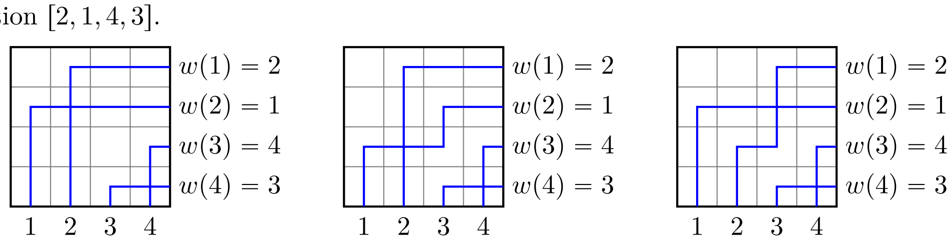
For weak compositions α, γ, and δ, let $d_{\alpha,\gamma}^{\delta}$ be the coefficient of $\widehat{\mathfrak{L}}_\delta$ in the expansion of $\widehat{\mathfrak{L}}_\alpha \times \widehat{\mathfrak{L}}_\gamma$. We call them the *top Lascoux structure constants*. Later in §4, we show each $d_{\alpha,\gamma}^{\delta}$ is the Schubert structure constant $c_{u,v}^{w}$ for some permutations u, v, w and vice versa.

2.3. BUMPLESS PIPEDREAMS. The *(reduced) bumpless pipedreams (BPD)*, introduced by Lam, Lee and Shimozono [13], are combinatorial objects that give a monomial expansion of a Schubert polynomial. For a permutation $w \in S_n$, a BPD is an $n \times n$ grid built by the following six tiles:



We adopt the convention that row 1 is the topmost row and column 1 is the leftmost column. For each $i \in [n]$, we require a pipe to enter from the bottom of column i and end at the rightmost edge of row $w(i)$. Moreover, two pipes cannot cross more than once.

EXAMPLE 2.1. There are three BPDs for the permutation $w \in S_4$ with one-line notation $[2, 1, 4, 3]$.



We let $\mathsf{BPD}(w)$ be the set of BPDs of a permutation w. We call $\square$ a *blank*. The *blank-weight* of a BPD D is a weak composition where the i^{th} entry counts the number of $\square$ in row i. We denote it as $\mathsf{wt}_\square(D)$ to emphasize that the weight comes from the blanks. Then BPD gives a combinatorial formula for Schubert polynomials.

THEOREM 2.2 ([13]). *For a permutation $w \in S_n$,*

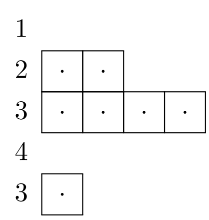
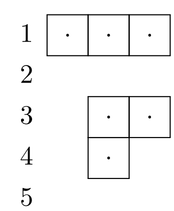
$$\mathfrak{S}_w = \sum_{D \in \mathsf{BPD}(w)} x^{\mathsf{wt}_\square(D)}.$$

Continuing Example 2.1, $\mathfrak{S}_{[2,1,4,3]} = x_1x_3 + x_1x_2 + x_1^2$.

2.4. DIAGRAMS. A *diagram* is a finite subset of $\mathbb{N} \times \mathbb{N}$. We may represent a diagram by putting a cell at row r and column c for each (r, c) in the diagram. The *weight* of a diagram D, denoted as $\mathsf{wt}(D)$, is a weak composition whose i^{th} entry is the number of boxes in its i^{th} row. Each weak composition α is associated with a diagram $D(\alpha)$, the unique left-justified diagram with weight α. Each permutation $w \in S_n$ or S_+ is associated with a diagram called the *Rothe diagram*

$$RD(w) := \{(r, c) : w(r) > c, w(i) \neq c \text{ for any } i \in [r]\}.$$

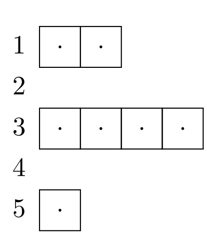
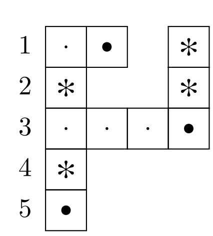
EXAMPLE 2.3. We provide examples of two diagrams. For clarity, we put an "i" on the left of the i^{th} row and put a small dot in each cell.

$D((0,2,4,0,1)) =$  , $RD([4,1,5,3,2]) =$  .

The Rothe diagram can characterize one special term in a Schubert polynomial. We consider the *tail-lexicographical order* on weak compositions: For two weak compositions α, γ, we say α is larger than γ if there exists i such that $\alpha_j = \gamma_j$ for all $j > i$ and $\alpha_i > \gamma_i$. For a polynomial f, the *support* of f, denoted as $\mathsf{supp}(f)$, is the set of weak compositions α such that x^α has non-zero coefficient in f. The *leading monomial* of f is x^α such that α is the largest in $\mathsf{supp}(\alpha)$. By Lascoux and Schützenberger [14], the leading monomial of $\mathfrak{S}_w$ is $x^{\mathsf{wt}(RD(w))}$ with coefficient 1.

To describe the leading monomial of a top Lascoux polynomial, Pan and Yu [20] introduce the *snow diagram*. For each diagram D, its snow diagram $\mathsf{snow}(D)$ is a diagram together with some labels in its cells. Each cell can be unlabeled, or labeled by $\bullet$ or $\circledast$. We only consider the snow diagram of $D(\alpha)$ where α is a snowy weak composition. In this case, $\mathsf{snow}(D(\alpha))$ can be defined as follows. In $D(\alpha)$, label the rightmost cell on each row with $\bullet$. Then put a cell labeled by $\circledast$ in empty spaces above each $\bullet$.

EXAMPLE 2.4. Let $\alpha = (2, 0, 4, 0, 1)$. Then $D(\alpha)$ and $\mathsf{snow}(D(\alpha))$ are depicted as follows.

$D(\alpha) =$  , $\mathsf{snow}(D(\alpha)) =$  .

For a snowy α, define

$$\mathsf{rajcode}(\alpha) := \mathsf{wt}(\mathsf{snow}(D(\alpha))). \text{ Equivalently, } \mathsf{rajcode}(\alpha)_i := \alpha_i + |\{j > i : \alpha_j > \alpha_i\}|.$$

By [20], $x^{\mathsf{rajcode}(\alpha)}$ is the leading monomial of $\widehat{\mathfrak{L}}_\alpha$. Moreover, any two distinct snowy weak compositions have different rajcode. We end this subsection with a simple property of rajcode that will be useful in §4.

Lemma 2.5. *Let α, γ be two snowy weak compositions. If α is larger than γ in tail-lexicographical order, then* $\mathsf{rajcode}(\alpha)$ *is also larger than* $\mathsf{rajcode}(\gamma)$.

Proof. Find i such that $\alpha_j = \gamma_j$ for all $j > i$ and $\alpha_i > \gamma_i$. Clearly, $\mathsf{rajcode}(\alpha)_j = \mathsf{rajcode}(\gamma)_j$ for all $j > i$ and $\mathsf{rajcode}(\alpha)_i > \mathsf{rajcode}(\gamma)_i$. □

2.5. Dual character of the flagged Weyl module. Let B be the group of $n \times n$ upper triangular matrices over $\mathbb{C}$. Each diagram D that lies in $[n] \times [n]$ is associated with a representation of B known as the *flagged Weyl module*. See [17, 26] for a detailed construction of this module.

Let D be a diagram. We use χ_D to denote the dual character of the flagged Weyl module associated with D. This is a family of polynomials in $\mathbb{Q}[x_1, x_2, \dots]$ containing the Schubert polynomials and key polynomials.

Theorem 2.6 ([4, 12, 18]). *For $w \in S_n$, $\chi_{RD(w)} = \mathfrak{S}_w$. For a weak composition α, $\chi_{D(\alpha)} = \kappa_\alpha$.*

The polynomials χ_D can be computed recursively for certain diagrams D. To characterize them, we define the following moves on diagrams.

Definition 2.7 ([18]). *Let D be a diagram. We say D' is obtained from D via an* orthodontic move *if one of the following holds.*

- *Suppose there is r such that $(1, r') \in D$ if and only if $r' \in [r]$. Then one may remove all cells in column 1 of D and shift all remaining cells leftward by 1 to obtain D'.*
- *Assume there exists r such that $(r, c) \in D$ implies $(r+1, c) \in D$ for any $c \in \mathbb{Z}_{>0}$. Then one may swap row r and row $r+1$ to obtain D'.*

The effect of each orthodontic move on χ_D can be characterized as follows.

Theorem 2.8 ([18]). *Let D be a diagram and D' is obtained from D via an orthodontic move.*

- *If D' is obtained from D by removing cells $(1,1), \dots, (r,c)$ and shifting all cells left, then $\chi_D = \chi_{D'} x_1 \cdots x_r$.*
- *If D' is obtained from D by swapping row r and $r+1$, then $\chi_D = \pi_i(\chi_{D'})$.*

Together with the base case $\chi_\varnothing = 1$, one may compute χ_D using Theorem 2.8 if one can obtain the empty diagram from D using the orthodontic moves. Such diagrams can be described as follows.

Definition 2.9 ([26]). *A diagram D is* percent-avoiding *if $(i_1, j_1), (i_2, j_2) \in D$ with $i_1 < i_2$ and $j_2 < j_1$ implies either $(i_1, j_2) \in D$ or $(i_2, j_1) \in D$.*

Theorem 2.10 ([26]). *Let D be a diagram. One may apply orthodontic moves on D repetitively and eventually obtain the empty diagram if and only if D is percent-avoiding. In particular, Rothe diagrams are such diagrams.*

We also need a well-known property of χ_D that follows immediately from its construction.

Theorem 2.11. *If D and D' differ by permuting the columns, then $\chi_D = \chi_{D'}$.*

3. Relations between top Lascoux polynomials and Schubert polynomials

This section describes the relationship between top Lascoux and Schubert polynomials.

3.1. The reverse complement involution on polynomials.

We introduce a linear operator on polynomials that later will be used to transform a top Lascoux polynomial into a Schubert polynomial. We begin with an involution on certain weak compositions.

Definition 3.1. *Let m, n be positive integers. Define the* reverse complement *operator $r_{m,n}$ on the set of weak compositions α such that $\mathsf{supp}(\alpha) \subseteq [n]$ and $\alpha_i \leqslant m$ for all i. We define*

$$r_{m,n}(\alpha) := (m - \alpha_n, \ldots, m - \alpha_1).$$

Next, we analogously define $r_{m,n}$ on certain polynomials.

Definition 3.2. *Let m, n be positive integers. We extend $r_{m,n}$ to the set of polynomials in $x_1, \ldots, x_n$ where the power of any x_i is at most m. We define it as the linear operator such that $r_{m,n}(x^\alpha) := x^{r_{m,n}(\alpha)}$. Equivalently, we can define $r_{m,n}$ as*

$$r_{m,n}(f) := x_1^m \cdots x_n^m f(x_n^{-1}, \ldots, x_1^{-1}).$$

Remark 3.3. The operator $r_{m,n}$ on polynomials is similar to the operator

$$f \mapsto x_1^n \cdots x_n^n f(x_1^{-1}, \ldots, x_n^{-1})$$

considered by Huh, Matherne, Mészáros and St. Dizier [11]. In [11, Theorem 6], the authors apply this operator on a Schubert polynomial $\mathfrak{S}_w$ with $w \in S_n$ and show the resulting polynomial is Lorentzian after normalization. Our $r_{m,n}$ is also similar to the operator

$$f \mapsto f(x_1^{-1}, \ldots, x_n^{-1})$$

which sends the character of a GL_n module to the character of its dual. A more generalized operator $f \mapsto x^\alpha f(x_1^{-1}, \ldots, x_n^{-1})$ and its action on Schubert polynomials was studied by Fan, Guo and Liu [6].

Next, we investigate how to swap $r_{m,n}$ with the operators: ∂_i, π_i, and $\widehat{\pi}_i$. Even though the proofs are straightforward, these identities are crucial in the rest of the paper.

Lemma 3.4. *Suppose $r_{m,n}$ is defined on a polynomial f. Take $i \in [n-1]$. Then $r_{m,n}$ is also defined on $\partial_i(f), \pi_i(f)$, and $\widehat{\pi}_i(f)$, with:*

$$\begin{aligned} r_{m,n}(\partial_i(f)) &= \widehat{\pi}_{n-i}(r_{m,n}(f)), \\ r_{m,n}(\pi_i(f)) &= \pi_{n-i}(r_{m,n}(f)), \\ r_{m,n}(\widehat{\pi}_i(f)) &= \partial_{n-i}(r_{m,n}(f)). \end{aligned}$$

Proof. It is enough to assume $f = x^\alpha$. Clearly, $\partial_i(f), \pi_i(f)$, and $\widehat{\pi}_i(f)$ will involve monomials in $x_1, \ldots, x_n$ where the power of any x_i is at most m. Thus, $r_{m,n}$ is defined on these polynomials. The three equations follow from a routine check. □

3.2. Relating Schubert polynomials to top Lascoux polynomials.

In this subsection, we establish that each Schubert polynomial is the reverse complement of a top Lascoux polynomial and vice versa. We start by describing a variation of the map introduced by Fulton [8, (3.4)].

Definition 3.5. *Let α be a snowy weak composition. Take any m, n such that $\mathsf{supp}(\alpha) \subseteq [n]$ and $m \geqslant \max(\alpha)$. The (m,n)-standardization of α, denoted as $\mathsf{std}_{m,n}(\alpha)$ is the unique permutation w satisfying $w(n+1) < w(n+2) < \cdots$ and*

$$w(i) := \begin{cases} r_{m+1,n}(\alpha)_i & \text{if } r_{m+1,n}(\alpha)_i \leqslant m \\ m + |\{j \in [i] : r_{m+1,n}(\alpha)_j = m+1\}| & \text{if } r_{m+1,n}(\alpha)_i = m+1, \end{cases}$$

for any $i \in [n]$.

Example 3.6. Let $\alpha = (2,4,0,6,0,0,1)$, $m = 6$ and $n = 7$. To compute $w = std_{m,n}(\alpha)$, we start with $r_{m+1,n}(\alpha) = (6,7,7,1,7,3,5)$. Based on the definition, the first n entries in the one-line notation of w can be obtained by changing those $m+1$ in $r_{m+1,n}(\alpha)$ into $m+1, m+2, \ldots,$. Thus, we get $(6,7,8,1,9,3,5)$. Finally, since $w(n+1) < w(n+2) < \cdots$, we know w has one-line notation $[6,7,8,1,9,3,5,2,4]$.

Then we can describe the relation between top Lascoux polynomials and Schubert polynomials.

Theorem 3.7. *Let α be a snowy weak composition. Take any m, n such that $\mathsf{supp}(\alpha) \subseteq [n]$ and $m \geqslant \max(\alpha)$. Let w be the permutation $\mathsf{std}_{m,n}(\alpha)$. Then*

$$r_{m,n}(\widehat{\mathfrak{L}}_\alpha) = \mathfrak{S}_w.$$

For instance, let $\alpha = (2,4,0,6,0,0,1)$, $m = 6$ and $n = 7$. The theorem concludes that $r_{6,7}(\widehat{\mathfrak{L}}_\alpha) = \mathfrak{S}_w$ where $w = \mathsf{std}_{m,n}(\alpha) = [6,7,8,1,9,3,5,2,4]$ from the computation in Example 3.6.

Proof. Prove by induction on α. For the base case, assume α is a partition with $\mathsf{supp}(\alpha) = [k]$. Then $r_{m+1,n}(\alpha) = (m+1, \ldots, m+1, m+1-\alpha_k, \ldots, m+1-\alpha_1)$. The first n numbers in the one-line notation of w are

$$m+1, m+2, \ldots, m+n-k, m+1-\alpha_k, \ldots, m+1-\alpha_1.$$

Thus, we have $\mathsf{invcode}(w) = (m, \ldots, m, m-\alpha_k, \ldots, m-\alpha_1) = r_{m,n}(\alpha)$, so w is a dominant permutation. By the definition of Schubert polynomials,

$$\mathfrak{S}_w = x^{r_{m,n}(\alpha)} = r_{m,n}(x^\alpha) = r_{m,n}(\widehat{\mathfrak{L}}_\alpha).$$

Now suppose $\alpha_i < \alpha_{i+1}$. It is routine to check $w(n-i) < w(n-i+1)$ and $\mathsf{std}_{m,n}(s_i\alpha) = ws_{n-i}$. Then the proof is finished by Lemma 3.4:

$$\begin{aligned} \mathfrak{S}_w &= \partial_{n-i}(\mathfrak{S}_{ws_{n-i}}) = \partial_{n-i}(\mathfrak{S}_{\mathsf{std}_{m,n}(s_i\alpha)}) \\ &= \partial_{n-i}(r_{m,n}(\widehat{\mathfrak{L}}_{s_i\alpha})) = r_{m,n}(\widehat{\pi}_i(\widehat{\mathfrak{L}}_{s_i\alpha})) = r_{m,n}(\widehat{\mathfrak{L}}_\alpha). \end{aligned}$$ □

Example 3.8. We can understand Theorem 3.7 via a commutative diagram. For instance, the equation $r_{4,5}(\widehat{\mathfrak{L}}_{(2,0,4,0,1)}) = \widehat{\mathfrak{L}}_{[4,5,1,6,3,2]}$ is implied by the following commutative diagram.

$$\begin{array}{ccccccccc} \mathfrak{S}_{[5,6,4,3,1,2]} & \xrightarrow{\partial_2} & \mathfrak{S}_{[5,4,6,3,1,2]} & \xrightarrow{\partial_1} & \mathfrak{S}_{[4,5,6,3,1,2]} & \xrightarrow{\partial_4} & \mathfrak{S}_{[4,5,6,1,3,2]} & \xrightarrow{\partial_3} & \mathfrak{S}_{[4,5,1,6,3,2]} \\ \updownarrow{\scriptstyle r_{4,5}} & & \updownarrow{\scriptstyle r_{4,5}} & & \updownarrow{\scriptstyle r_{4,5}} & & \updownarrow{\scriptstyle r_{4,5}} & & \updownarrow{\scriptstyle r_{4,5}} \\ \widehat{\mathfrak{L}}_{(4,2,1,0,0)} & \xrightarrow{\widehat{\pi}_3} & \widehat{\mathfrak{L}}_{(4,2,0,1,0)} & \xrightarrow{\widehat{\pi}_4} & \widehat{\mathfrak{L}}_{(4,2,0,0,1)} & \xrightarrow{\widehat{\pi}_1} & \widehat{\mathfrak{L}}_{(2,4,0,0,1)} & \xrightarrow{\widehat{\pi}_2} & \widehat{\mathfrak{L}}_{(2,0,4,0,1)} \end{array}$$

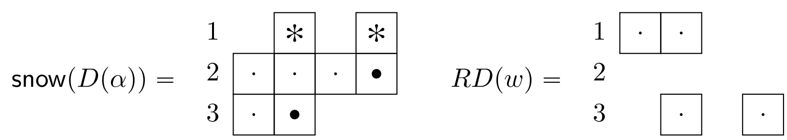
Consequently, every Schubert polynomial is the reverse complement of a top Lascoux polynomial.

Corollary 3.9. *Consider $w \in S_n$. Let $\alpha = (n+1-w(n), \ldots, n+1-w(2), n+1-w(1))$. Then $\mathfrak{S}_w = r_{n,n}(\widehat{\mathfrak{L}}_\alpha)$.*

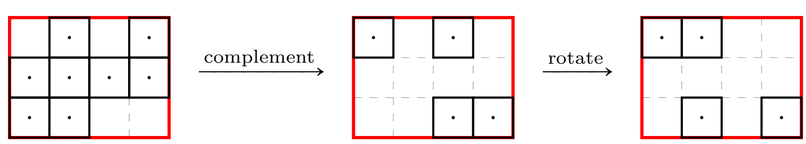
Proof. Notice that $\mathsf{std}_{n,n}(\alpha) = w$. Then the proof is finished by Theorem 3.7. □

3.3. A diagrammatic perspective of standardization. We can interpret the standardization map $\mathsf{std}_{m,n}$ in terms of diagrams. Recall that each snowy weak composition α is associated with a labeled diagram $\mathsf{snow}(D(\alpha))$. Each permutation w is associated with the Rothe diagram $RD(w)$. We describe the relationship between $\mathsf{snow}(D(\alpha))$ and $RD(\mathsf{std}_{m,n}(\alpha))$.

Example 3.10. Consider the snowy weak composition $\alpha = (0,4,2)$. Let $m = 4$ and $n = 3$. Let $w = \mathsf{std}_{m,n}(\alpha) = [3,1,5,2,4]$. We depict $\mathsf{snow}(D(\alpha))$ and $RD(w)$ as follows:



We observe that both diagrams live in the first 3 rows and 4 columns. Imagine that we ignore the labels of $\mathsf{snow}(D(\alpha))$ and put it in an 3×4 box. Then we take its complement within the box and rotate the box by $180°$. What we get is exactly $RD(w)$.



To characterize this relation in general, we define the operator $r_{m,n}$ on certain diagrams.

Definition 3.11. *Let D be a diagram that lives in the first n rows and first m columns. Let $r_{m,n}(D)$ be the diagram obtained from the following process: Place D in a $n \times m$ box, take its complement within the box, and rotate the box by $180°$.*

Lemma 3.12. *Let α be a snowy weak composition. Take n with $\mathsf{supp}(\alpha) \subseteq [n]$ and m with $\max(\alpha) \leqslant m$. Let $w = \mathsf{std}_{m,n}(\alpha)$. Then $RD(w) = r_{m,n}(\mathsf{snow}(D(\alpha)))$.*

Proof. If we ignore the labels, $\mathsf{snow}(D(\alpha))$ consists of cells (r,c) such that $\alpha_r \geqslant c$ or $\alpha_i = c$ for some $i > r$. Clearly, $\mathsf{snow}(D(\alpha))$ lies in the first n rows and m columns. If we complement $\mathsf{snow}(D(\alpha))$ within the $n \times m$ box, the resulting diagram consists of $(r,c) \in [m] \times [n]$ such that $\alpha_r < c$ and $\alpha_i \neq c$ for any $i > r$.

Let $D' = r_{m,n}(\mathsf{snow}(D(\alpha)))$. We check D' and $RD(w)$ agree row by row. They clearly agree under row n. Consider $r \in [n]$. If $\alpha_{n+1-r} = 0$, then D' has no cells in row r. Also, $w(r) \geqslant m+1$. By the definition of standardization map, there are no inversions of w of the form (r,r'). Thus, $RD(w)$ also has no cells in row r. If $\alpha_{n+1-r} > 0$, D' has cells in column c for c such that $c < m+1-\alpha_{n+1-r}$ and $c \neq m+1-\alpha_{n+1-i}$ for all $i \in [r]$. In other words, that is all c such that $c < w(r)$ and c is not in $w(1), \ldots, w(r)$. Therefore, D' and $RD(w)$ agree on row r. □

4. Relations between top Lascoux structure constants and the Schubert structure constants

Recall that $\{\widehat{\mathfrak{L}}_\alpha : \alpha \text{ is snowy}\}$ forms a basis of the algebra $\widehat{V}$. The top Lascoux structure constant $d^{\delta}_{\alpha,\gamma}$ is the coefficient of $\widehat{\mathfrak{L}}_\delta$ in the expansion of $\widehat{\mathfrak{L}}_\alpha \widehat{\mathfrak{L}}_\gamma$. At this point, we do not have any reason to believe that they are positive integers. Moreover, it is well-known that key polynomials do not have positive structure constants. Surprisingly, the connection between top Lascoux polynomials and Schubert polynomials establishes a bridge between $d^{\delta}_{\alpha,\gamma}$ and the Schubert structure constants $c^{w}_{u,v}$. First, we describe a necessary condition for $d^{\delta}_{\alpha,\gamma}$ to be non-zero.

Lemma 4.1. *Let α, γ and δ be snowy weak compositions. Find m_1, m_2 and n such that $m_1 \geqslant \max(\alpha)$, $m_2 \geqslant \max(\gamma)$, $\mathsf{supp}(\alpha) \subseteq [n]$ and $\mathsf{supp}(\gamma) \subseteq [n]$. If $d^{\delta}_{\alpha,\gamma} \neq 0$, we must have $\mathsf{supp}(\delta) \subseteq [n]$ and $\max(\delta) \leqslant m_1 + m_2$.*

The proof relies heavily on the statistic $\mathsf{rajcode}$.

Proof. First, expand

$$\widehat{\mathfrak{L}}_\alpha \times \widehat{\mathfrak{L}}_\gamma = \sum_\sigma d^\sigma_{\alpha,\gamma} \widehat{\mathfrak{L}}_\sigma. \tag{3}$$

We know the left hand side uses only variables $x_1, \dots, x_n$. Moreover, in any monomial on the left hand side, each variable has power at most $m_1 + m_2$. Let S be the set of all σ with $d^\sigma_{\alpha,\gamma} \neq 0$. Among S, find the largest σ in tail-lexicographical order. By Lemma 2.5, $\mathsf{rajcode}(\sigma)$ is also larger than $\mathsf{rajcode}(\sigma')$ for any $\sigma' \in S$. Thus, $x^{\mathsf{rajcode}(\sigma)}$ has non-zero coefficient on the right hand side of (3), so $\mathsf{supp}(\sigma) \subseteq [n]$. It follows that $\mathsf{supp}(\sigma') \subseteq [n]$ for any $\sigma' \in S$.

Now find $\sigma \in S$ with the largest $\max(\sigma)$, break ties by picking the largest in tail-lexicographical order. Say $\max(\sigma) = m$. In $\mathsf{rajcode}(\sigma)$, one entry is m. We can see $\mathsf{rajcode}(\sigma)$ cannot appear in $\widehat{\mathfrak{L}}_{\sigma'}$ for any other $\sigma' \in S$: If so, then σ' has an entry at least m and $\mathsf{rajcode}(\sigma')$ is larger than $\mathsf{rajcode}(\sigma)$, contradicting to the maximality of σ. Thus, $x^{\mathsf{rajcode}}(\sigma)$ has non-zero coefficient on the right hand side of (3), so $m \leqslant m_1 + m_2$. It follows that $\max(\sigma') \leqslant m_1 + m_2$ for any $\sigma' \in S$. □

Now we describe the main theorem of this section.

Theorem 4.2. *Let α, γ be snowy weak compositions. Find m_1, m_2 and n such that $m_1 \geqslant \max(\alpha)$, $m_2 \geqslant \max(\gamma)$, $\mathsf{supp}(\alpha) \subseteq [n]$ and $\mathsf{supp}(\gamma) \subseteq [n]$. Let $u = \mathsf{std}_{m_1,n}(\alpha)$ and $v = \mathsf{std}_{m_2,n}(\gamma)$. For any snowy weak composition δ with $\mathsf{supp}(\delta) \subseteq [n]$ and $\max(\delta) \leqslant m_1 + m_2$, we let $w = \mathsf{std}_{m_1+m_2,n}(\delta)$. Then $d^\delta_{\alpha,\gamma} = c^w_{u,v}$.*

Proof. First, we have $\widehat{\mathfrak{L}}_\alpha \times \widehat{\mathfrak{L}}_\gamma = \sum_\sigma d^\sigma_{\alpha,\gamma} \widehat{\mathfrak{L}}_\sigma$. By Lemma 4.1, the sum is over all σ with $\mathsf{supp}(\sigma) \subseteq [n]$ and $\max(\sigma) \leqslant m_1 + m_2$. Apply $r_{m_1+m_2,n}$ on both sides. Using Theorem 3.7, the left hand side becomes

$$r_{m_1+m_2,n}(\widehat{\mathfrak{L}}_\alpha \times \widehat{\mathfrak{L}}_\gamma) = r_{m_1,n}(\widehat{\mathfrak{L}}_\alpha) r_{m_2,n}(\widehat{\mathfrak{L}}_\gamma) = \mathfrak{S}_u \mathfrak{S}_v.$$

The right hand becomes

$$r_{m_1+m_2,n}\left(\sum_\sigma d^\sigma_{\alpha,\gamma} \widehat{\mathfrak{L}}_\sigma\right) = \sum_\sigma d^\sigma_{\alpha,\gamma} r_{m_1+m_2,n}(\widehat{\mathfrak{L}}_\sigma) = \sum_\sigma d^\sigma_{\alpha,\gamma} \mathfrak{S}_{\mathsf{std}_{m_1+m_2,n}(\sigma)}. \quad \square$$

Example 4.3. Let $\alpha = (2,3,1,4)$ and $\gamma = (2,1,4,3)$. We can let $m_1 = m_2 = n = 4$. Then $u = \mathsf{std}_{4,4}(\alpha) = [1,4,2,3]$ and $v = \mathsf{std}_{4,4}(\gamma) = [2,1,4,3]$. We compute

$$\widehat{\mathfrak{L}}_\alpha \times \widehat{\mathfrak{L}}_\gamma = \widehat{\mathfrak{L}}_{(8,6,5,7)} + \widehat{\mathfrak{L}}_{(6,8,4,7)} + \widehat{\mathfrak{L}}_{(7,8,5,6)} + \widehat{\mathfrak{L}}_{(7,6,8,5)} + \widehat{\mathfrak{L}}_{(6,7,8,4)}.$$
$$\mathfrak{S}_u \times \mathfrak{S}_v = \mathfrak{S}_{[2,4,3,1]} + \mathfrak{S}_{[2,5,1,3,4]} + \mathfrak{S}_{[3,4,1,2]} + \mathfrak{S}_{[4,1,3,2]} + \mathfrak{S}_{[5,1,2,3,4]}.$$

We check Theorem 4.2 when $\delta = (8,6,5,7)$. We have $d^\delta_{\alpha,\gamma} = 1$. Now we compute $w = \mathsf{std}_{4+4,4}(\delta) = [2,4,3,1]$. Indeed, $c^w_{u,v} = 1$.

Theorem 4.2 can express each Schubert structure constant as a top Lascoux structure constant.

Corollary 4.4. *Take $u, v \in S_n$ and $w \in S_{2n}$. If $w(n+1) < \cdots < w(2n)$. Then $c^w_{u,v} = d^{\delta}_{\alpha,\gamma}$, where*

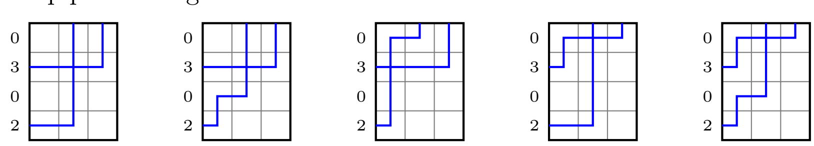
$$\alpha = (n+1-u(n), \ldots, n+1-u(1)),$$
$$\gamma = (n+1-v(n), \ldots, n+1-v(1)), \text{ and}$$
$$\delta = (n+1-w(n), \ldots, n+1-w(1)).$$

5. Bumpless pipedream formula

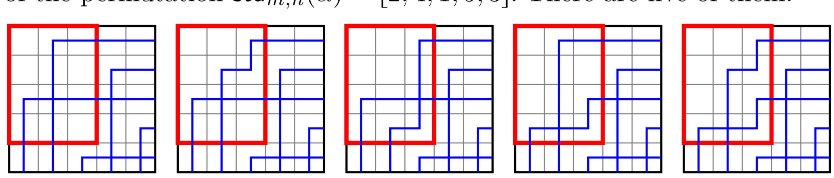
We discuss another application of the relationship between top Lascoux polynomials and Schubert polynomials. Bumpless pipedreams (BPD) give a formula to compute the monomial expansion of Schubert polynomials. After "reversing the BPDs", we get a formula for top Lascoux polynomials.

Definition 5.1. *Let α be a snowy weak composition. Find the smallest n such that $\mathsf{supp}(\alpha) \subseteq [n]$ and let $m = \max(\alpha)$. A left-to-top BPD of α is a grid with n rows and m columns built by tiles , , , , and . For each $i \in [n]$ with $\alpha_i > 0$, there is a pipe entering from the left in row i and goes to the top of column α_i. Moreover, no two pipes can cross more than once. Let $\mathsf{LTBPD}(\alpha)$ be the set of all left-to-top BPDs of α.*

Example 5.2. Consider $\alpha = (0, 3, 0, 2)$. Then $n = 4$ and $m = 3$. The set $\mathsf{LTBPD}(\alpha)$ has the following five elements. We write α vertically on the left. The 3 (resp. 2) indicates that the pipe from that row should end in column 3 (resp. 2). The 0 indicates that there is no pipe entering from that row.



Remark 5.3. Readers might notice that left-to-top BPDs look like the top left part of a BPD after rotation. Keep α, m and n from Example 5.2. Consider the classical BPDs of the permutation $\mathsf{std}_{m,n}(\alpha) = [2, 4, 1, 5, 3]$. There are five of them:



The top-to-left BPDs in Example 5.2 are obtained by rotating the red part of these BPDs.

This pattern holds in general.

Proposition 5.4. *Let α be a snowy weak composition. Find smallest n such that $\mathsf{supp}(\alpha) \in [n]$ and let $m = \max(\alpha)$. Let $w = \mathsf{std}_{m,n}(\alpha)$. Then $\mathsf{LTBPD}(\alpha)$ is formed by rotating the first n rows, m columns of BPDs in $\mathsf{BPD}(w)$.*

Proof. Immediate from Lemma 3.12. □

Now we are ready to introduce a combinatorial formula for the top Lascoux polynomials. We just need to specify the "weight" of a left-to-top BPD.

Definition 5.5. *Let D be a left-to-top BPD for some snowy weak composition. The non-blank weight of D, denoted as $\mathsf{wt}_{\boxdot}(D)$, is a weak composition where the i^{th} entry counts the number of* ***non-blank tiles*** *in row i of D.*

Theorem 5.6. *Let α be a snowy weak composition. Then*

$$\widehat{\mathfrak{L}}_\alpha = \sum_{D \in \mathsf{LTBPD}(\alpha)} \mathsf{wt}_{\boxdot}(D).$$

For instance, Example 5.2 yields

$$\widehat{\mathfrak{L}}_{(0,3,0,2)} = x_1^2x_2^3x_3x_4^2 + x_1^2x_2^3x_3^2x_4 + x_1^3x_2^3x_3x_4 + x_1^3x_2^2x_3x_4^2 + x_1^3x_2^2x_3^2x_4.$$

Proof. Follows from Theorem 3.7 and Proposition 5.4. ☐

6. Key expansion of top Lascoux polynomials

We expand top Lascoux polynomials positively into key polynomials. Our main tool is the Schubert-to-key expansion:

Theorem 6.1 ([25]). *For $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w$ can be expanded as $\sum_\alpha c_\alpha^w \kappa_\alpha$, where the coefficients c_α^w are non-negative integers counting certain tableaux.*

We will translate this expansion to top Lascoux polynomials. The first step is to understand how the $r_{m,n}$ operator affects a key polynomial.

Proposition 6.2. *Let α be any weak composition. Let m, n be positive integers. Then $r_{m,n}$ is defined on α if and only if $r_{m,n}$ is defined on κ_α. If this is the case, $r_{m,n}(\kappa_\alpha) = \kappa_{r_{m,n}(\alpha)}$.*

The proof is essentially the same as the proof of Theorem 3.7.

Proof. The first claim is immediate. We prove the equation by induction on α. For the base case, assume α is a partition. So is $r_{m,n}(\alpha)$. We have $r_{m,n}(\kappa_\alpha) = r_{m,n}(x^\alpha) = x^{r_{m,n}(\alpha)} = \kappa_{r_{m,n}(\alpha)}$.

Now assume $\alpha_i < \alpha_{i+1}$ for some i. For our inductive hypothesis, assume $\kappa_{s_i\alpha} = r_{m,n}(\kappa_{r_{m,n}(s_i\alpha)})$. By Lemma 3.4,

$$\begin{aligned}\kappa_\alpha &= \pi_i(\kappa_{s_i\alpha}) = \pi_i(r_{m,n}(\kappa_{r_{m,n}(s_i\alpha)}))\\ &= r_{m,n}(\pi_{n-i}(\kappa_{s_{n-i}(r_{m,n}(\alpha))})) = r_{m,n}(\kappa_{r_{m,n}(\alpha)}).\end{aligned}$$ ☐

Corollary 6.3. *Let α be a snowy weak composition. Find m, n such that $\alpha \subseteq [n]$ and $\max(\alpha) \leqslant m$. Let w be the permutation $\mathsf{std}_{m,n}(\alpha)$. The top Lascoux polynomial $\widehat{\mathfrak{L}}_\alpha$ can be expanded as $\sum_\gamma c_\gamma^w \kappa_{r_{m,n}(\gamma)}$ where the coefficient c_γ^w is the coefficient of κ_γ in the expansion of $\mathfrak{S}_w$.*

Proof. By Theorem 3.7, we know $\widehat{\mathfrak{L}}_\alpha = r_{m,n}(\mathfrak{S}_w)$. By the Schubert-to-key expansion and Proposition 6.2,

$$\widehat{\mathfrak{L}}_\alpha = r_{m,n}\left(\sum_\gamma c_\gamma^w \kappa_\gamma\right) = \sum_\gamma c_\gamma^w r_{m,n}(\kappa_\gamma) = \sum_\gamma c_\gamma^w \kappa_{r_{m,n}(\gamma)}.$$ ☐

7. Top Lascoux polynomials as dual-characters of flagged Weyl modules

We show $\widehat{\mathfrak{L}}_\alpha$ agrees with $\chi_{\mathsf{snow}(D(\alpha))}$. Our proof relies on the following relation between the $r_{m,n}$ on polynomials and the $r_{m,n}$ on diagrams.

Proposition 7.1. *Let D be a diagram in the first n rows and m columns. Assume one can obtain the empty diagram by applying orthodontic moves from D. Then*

$$\chi_{r_{m,n}(D)} = r_{m,n}(\chi_D). \tag{4}$$

Proof. We prove by induction on m. The case when $m = 0$ is trivial.

Suppose $m > 0$. If D is the empty diagram, then $r_{m,n}(D)$ is the $n \times m$ box. We have $\chi_{r_{m,n}(D)} = x_1^m \cdots x_n^m$, which agrees with $r_{m,n}(\chi_D) = r_{m,n}(1)$.

Now suppose D is not empty and let D' be a diagram obtained from D after one orthodontic move. It remains to check (4) while assuming $\chi_{r_{m,n}(D')} = r_{m,n}(\chi_{D'})$. We consider the two types of orthodontic moves.

- Suppose D' is obtained from D by removing cells $(1,1), \dots, (1,r)$ and shift all remaining cells leftward. Thus, D' lives in the $n \times (m-1)$ box. The diagram $r_{m,n}(D)$ is obtained by adding cells $(1,m), \dots, (n-r,m)$ to $r_{m-1,n}(D')$. Let D'' be the diagram obtained by moving column m of $r_{m,n}(D)$ to column 1 and shift all other column rightward by 1. By Theorem 2.11, $\chi_{r_{m,n}(D)} = \chi_{D''}$. Notice that $r_{m-1,n}(D')$ is obtained from D'' via an orthodontic move, so $\chi_{D''} = x_1 \cdots x_{n-r} \chi_{r_{m-1,n}(D')}$. We have
$$\begin{aligned} r_{m,n}(\chi_D) = r_{m,n}(x_1 \cdots x_r \chi_{D'}) &= x_1 \cdots x_{n-r} r_{m-1,n}(\chi_{D'}) \\ &= x_1 \cdots x_{n-r} \chi_{r_{m-1,n}(D')} = \chi_{r_{m,n}(D)}. \end{aligned}$$
- Suppose D' is obtained from D by swapping row r and row $r+1$. Then $r_{m,n}(D')$ is obtained from $r_{m,n}(D)$ via an orthodontic move that swaps row $n-r$ and row $n-r+1$. By Lemma 3.4, we have
$$r_{m,n}(\chi_D) = r_{m,n}(\pi_r(\chi_{D'})) = \pi_{n-r}(r_{m,n}(\chi_{D'})) = \chi_{r_{m,n}(D)}. \quad \square$$

Remark 7.2. Fix an $n \in \mathbb{Z}_{>0}$. The symmetrization operator π_{w_0} can be defined as
$$\pi_{w_0} := (\pi_1 \cdots \pi_{n-1})(\pi_1 \cdots \pi_{n-2}) \cdots (\pi_1 \pi_2)(\pi_1).$$
It turns a polynomial in $x_1, \dots, x_n$ into a polynomial symmetric in these variables. Magyar [18] showed that for any diagram D living in the first n rows and m columns,
$$\pi_{w_0}(\chi_{r_{m,n}(D)}) = r_{m,n}(\pi_{w_0}(\chi_D)).$$
Our statement in Proposition 7.1 can be viewed as the “non-symmetric refinement” of Magyar’s result.

Corollary 7.3. *Let α be a snowy weak composition. We have $\widehat{\mathfrak{L}}_\alpha = \chi_{\mathsf{snow}(D(\alpha))}$.*

Proof. Take n with $\mathsf{supp}(\alpha) \subseteq [n]$ and m with $\max(\alpha) \leqslant m$. Let $w = \mathsf{std}_{m,n}(\alpha)$. By Lemma 3.12, $RD(w) = r_{m,n}(\mathsf{snow}(D(\alpha))$. By Theorem 2.10 and Proposition 7.1, $r_{m,n}(\chi_{RD(w)}) = \chi_{\mathsf{snow}(D(\alpha))}$. On the other hand, by Theorem 2.6, $\chi_{RD(w)} = \mathfrak{S}_w$. Thus,
$$\widehat{\mathfrak{L}}_\alpha = r_{m,n}(\mathfrak{S}_w) = r_{m,n}(\chi_{RD(w)}) = \chi_{\mathsf{snow}(D(\alpha))}. \quad \square$$

In [10], the authors focused on Grothendieck polynomials indexed by vexillary (i.e. 2143-avoiding) permutations. For w vexillary, they constructed a diagram $D^{top}(w)$. Then they showed the polynomials $\widehat{\mathfrak{G}}_w$ and $\chi_{D^{top}(w)}$ have the same support in Theorem 1.2. Finally, they conjectured $\widehat{\mathfrak{G}}_w$ is a scalar multiple of $\chi_{D^{top}(w)}$ in Conjecture 1.5. We can resolve this conjecture using Corollary 7.3.

Corollary 7.4. *Conjecture 1.5 of* [10] *is correct.*

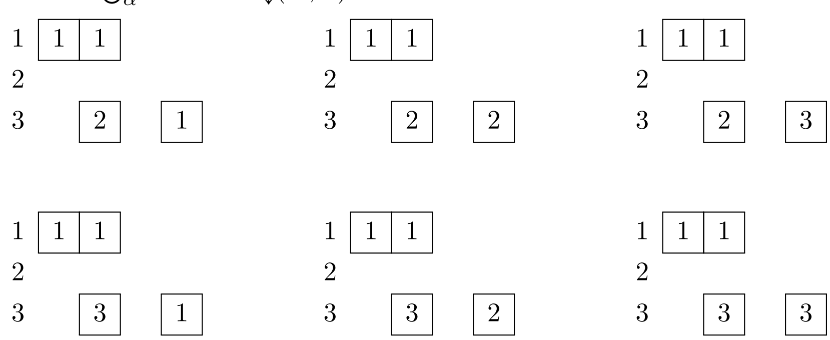
Proof. By Pechenik and Scrimshaw [21], in this case, $\mathfrak{G}_w$ is a Lascoux polynomial. Then by [20, Theorem 1.2], its top degree component $\widehat{\mathfrak{G}}_w$ is a scalar multiple of $\widehat{\mathfrak{L}}_\alpha$ for some snowy α. By Corollary 7.3, $\widehat{\mathfrak{G}}_w = \chi_{\mathsf{snow}(D(\alpha))}$. By [10, Theorem 1.2], $\mathsf{supp}(\chi_{D^{top}(w)}) = \mathsf{supp}(\widehat{\mathfrak{G}}_w) = \mathsf{supp}(\chi_{\mathsf{snow}(D(\alpha))})$. Finally, [10, Theorem 1.4] says that if the dual characters of the two diagrams have the same support, then the two dual characters must agree. We have $\chi_{D^{top}(w)} = \chi_{\mathsf{snow}(D(\alpha))}$, so $\widehat{\mathfrak{G}}_w$ is a multiple of $\chi_{D^{top}(w)}$. □

We conclude this section by deriving a few properties related to the support of $\widehat{\mathfrak{L}}_\alpha$, showing that its support has similar properties as the Schubert polynomials. Adve, Robichaux, and Yong [1] characterized the support of Schubert polynomials using certain fillings:

Definition 7.5. [1] *Let D be a diagram and α be a weak composition. Then* $\mathsf{PerfectTab}_\downarrow(D, \alpha)$ *is the set of fillings of D satisfying all of the following:*

- *For each k, the number of cells in D filled by k is α_k.*
- *In each column, numbers are increasing from top to bottom.*
- *Any entry in row i is at most i.*

Example 7.6. [1, Example 1.4] Consider $D = RD((3,1,5,2,4))$. We enumerate the six elements in $\bigcup_\alpha \mathsf{PerfectTab}_\downarrow(D, \alpha)$:



Theorem 7.7. [1, Theorem 1.3] *For a permutation w, the support of $\mathfrak{S}_w$ is the set of α such that* $\mathsf{PerfectTab}_\downarrow(RD(w), \alpha) \neq \emptyset$.

For instance, Example 7.6 says that $\mathsf{supp}(\mathfrak{S}_{(3,1,5,2,4)})$ consists of

$$(3,1), (2,2), (2,1,1), (3,0,1) \text{ and } (2,0,2).$$

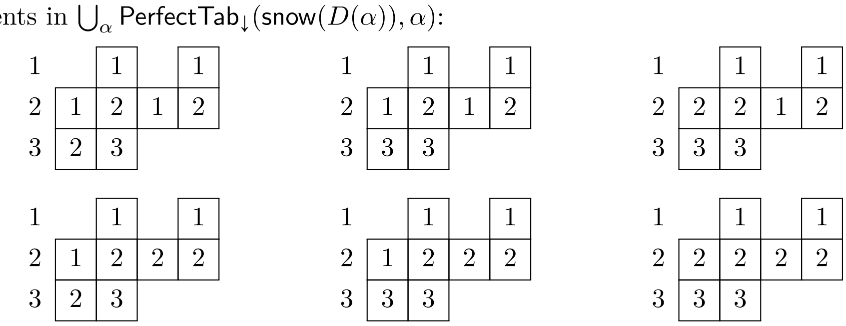
Fink, Mészáros, and St. Dizier [7] extended this characterization to χ_D.

Theorem 7.8. [7, Theorem 7] *For a diagram D, χ_D has SNP and its support χ_D is the set of α such that* $\mathsf{PerfectTab}_\downarrow(D, \alpha) \neq \emptyset$.

Consequently, the support of a top Lascoux polynomial can be characterized in the same manner.

Corollary 7.9. *For a snowy weak composition α, $\widehat{\mathfrak{L}}_\alpha$ has saturated Newton polytope. View* $\mathsf{snow}(D(\alpha))$ *as a diagram with labels erased. The support of $\widehat{\mathfrak{L}}_\alpha$ is the set of γ such that* $\mathsf{PerfectTab}_\downarrow(\mathsf{snow}(D(\alpha)), \gamma) \neq \emptyset$.

Example 7.10. Consider the snowy weak composition $(0,4,2)$. We enumerate the six elements in $\bigcup_\alpha \mathsf{PerfectTab}_\downarrow(\mathsf{snow}(D(\alpha)), \alpha)$:



By Proposition 7.9, the support of $\widehat{\mathfrak{L}}_{(0,4,2)}$ consists of $(4,3,1),(4,2,2),(3,3,2)$, $(3,4,1)$ and $(2,4,2)$.

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Tianyi Yu, Laboratoire d'Algèbre, de Combinatoire et d'Informatique Mathématique (LACIM), 201 avenue du Président Kennedy, Montréal, Québec, H2X 3Y7, Canada
E-mail : yu.tianyi@uqam.ca