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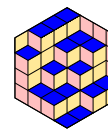


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ABSTRACT We point out to a connection of an old approach of Gondran and Minoux to ranks of matrices over the algebra $(\mathbb{R}, \max, +)$ and a recent work of Brandenburg, Loho, and Sinn on positive tropicalizations of algebraic varieties. This leads to examples of tropical varieties and their tropical bases which are not the corresponding positive tropical generating sets. In particular, we show that the $d \times d$ minors of a $d \times n$ matrix of variables form a positive tropical generating set if and only if either (1) $d = n$ or (2) $d \leq 4$.

1. POSITIVE TROPICALIZATIONS

A common object of tropical geometry is the field $\mathbb{C}\{\{t\}\}$ of Puiseux series with the corresponding valuation, which takes the formal sum

$$p = \sum_{\alpha \in \mathbb{Q}} p_{\alpha} t^{\alpha} \text{ with } p_{\alpha} \in \mathbb{C}$$

to the minimal a for which $p_a \neq 0$, that is, $a = \text{val } p$, and such an a exists because, by the definition, all exponents α with $p_{\alpha} \neq 0$ have a common denominator and minimal element. The *positive part* $\mathcal{C}^+$ is the set of all series $p \in \mathbb{C}\{\{t\}\}$ for which the coefficient p_a in the *leading term* $p_a t^a$ is real and positive [1, 3]. The mapping

$$(1.1) \quad \text{trop}_{\mathcal{C}^+}(X) = \overline{\{\text{val } x : x \in X \cap (\mathcal{C}^+)^n\}} \subseteq \mathbb{R}^n$$

sending any algebraic variety $X \subset (\mathbb{C}\{\{t\}\})^n$ to a subset of $\mathbb{R}^n$ is called the *positive tropicalization*, where the overline denotes the Euclidean closure [1, 3].

DEFINITION 1.1 ([3, Definition 2.3]). *A set of polynomials $\mathcal{F} \subset \mathbb{R}[x_1, \dots, x_n]$ is called a positive tropical generating set of an ideal $I \subset \mathbb{R}[x_1, \dots, x_n]$ if*

$$\text{trop}_{\mathcal{C}^+} V(I) = \bigcap_{f \in \mathcal{F}} \text{trop}_{\mathcal{C}^+} V(f),$$

where $V(I) \subseteq (\mathbb{C}\{\{t\}\})^n$ is the intersection of the zero loci of all polynomials in I .

As an alternative in [3, Definition 2.3], one replaces $\mathcal{C}^+$ by $(\mathcal{C}^+ \cap \mathbb{R}\{\{t\}\})$ in (1.1), but our results and arguments are valid in both settings. Instead, one can replace $\mathcal{C}^+$ with the set of all nonzero series in $\mathbb{C}\{\{t\}\}$, and the resulting version of Definition 1.1 gives the concept of *tropical bases*, see [6]. Indeed, there has been a known problem that asked one to describe the triples (d, n, r) for which the $(r+1) \times (r+1)$ minors of a $d \times n$ matrix of variables form a tropical basis of the variety that they generate, that

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is, the set of all $d \times n$ matrices of the rank at most r , see [5]. The study of this question begins in the work of Develin, Santos, Sturmfels [6], and its resolution culminates in the article of the present author, see [9] and references therein. The authors of [3] discuss a similar question in the case of positive generating sets.

QUESTION 1.2 ([3, Question 2.8]). *Is there a tropical variety $\mathcal{T}$ and a tropical basis B of $\mathcal{T}$ such that B is not a positive tropical generating set of $\mathcal{T}$?*

We answer this question in the positive, and we correct a statement in Table 1 in [3]. Indeed, with a reference to [3, Theorem 2.4], it is said that the $(r+1) \times (r+1)$ minors of a $d \times n$ matrix of variables form a positive tropical generating set, provided that $r+1 = d \leq n$, see [3, Table 1]. Of course, this reasoning is true if the additional condition $d = n$ is imposed. However, if one has $d < n$, the variety of rank-deficient matrices is not anymore cut by a single minor, and Theorem 2.4 in [3] may not be applicable. This issue was not addressed in the correction either [4], and, in fact, it turns out that, in the case $d < n$, the answer depends on (d, n) .

THEOREM 1.3. *Assume $d \leq n$. The $d \times d$ minors of a $d \times n$ matrix of variables form a positive tropical generating set if and only if either (1) $d = n$ or (2) $d \leq 4$.*

In particular, the set of all $d \times d$ minors is not a positive tropical generating set, for a $d \times n$ matrix of variables with $5 \leq d < n$. However, the $d \times d$ minors give tropical bases for $d \times n$ matrices, for all possible values with $d \leq n$, as explained in [6, 9], and hence we get a positive answer to Question 1.2.

2. GONDRAN–MINOUX RANK

As it turns out, the description of the set $\text{trop}_{\mathcal{C}^+} V(f)$ is quite well known in the case when f is the determinant of an $m \times m$ matrix, and the corresponding results, stated in different notation, arise to the works of Gondran and Minoux [7].

DEFINITION 2.1 ([2, Definition 3.5]). *The tropical bideterminant of a real $m \times m$ matrix A is the pair $(|A^+|, |A^-|)$ defined as follows, where $\mathcal{A}_m$ and $\mathcal{B}_m$ are the sets of all even and odd permutations of $\{1, \dots, m\}$, respectively:*

$$|A^+| = \min_{\sigma \in \mathcal{A}_m} \{a_{1\sigma_1} + \dots + a_{m\sigma_m}\}, \quad |A^-| = \min_{\tau \in \mathcal{B}_m} \{a_{1\tau_1} + \dots + a_{m\tau_m}\}.$$

DEFINITION 2.2 ([2, Definition 2.11]). *The rows of an $m \times n$ real matrix A are called linearly dependent in the sense of Gondran–Minoux (or just GM-dependent) if there is a partition of $\{1, \dots, m\}$ into two disjoint subsets I and J , and there are real numbers $(\lambda_1, \dots, \lambda_n)$ such that, for any $k \in \{1, \dots, n\}$, one has*

$$\min_{i \in I} \{\lambda_i + a_{ik}\} = \min_{j \in J} \{\lambda_j + a_{jk}\}.$$

The old result relating the bideterminant to GM-dependence is as follows.

THEOREM 2.3 (Gondran–Minoux theorem [2, 7]). *For an $m \times m$ real matrix A , one has $|A^+| = |A^-|$ if and only if the rows of A are GM-dependent.*

The Gondran–Minoux theorem is equivalent to the description of the positive tropicalizations of the varieties of singular matrices, which can appear in Section 8 of [6], Proposition 3.1 in [3], Theorem 4.1 in [1], and Lemma 2.6 in [10].

THEOREM 2.4. *If X is an $m \times m$ matrix of variables, then $\text{trop}_{\mathcal{C}^+} V(\det X)$ coincides with the set of all $m \times m$ matrices A such that $|A^+| = |A^-|$.*

In fact, Definition 2.2 gives a deeper description of the positive tropicalization of the variety of rank deficient matrices.

THEOREM 2.5. *Let d and n be integers with $1 \leq d \leq n$, and let $\mathcal{V}_{dn}$ be the variety of all $d \times n$ matrices which have the ranks less than d over $\mathbb{C}\{\{t\}\}$. For a $d \times n$ real matrix A , if the rows of A are GM-dependent, then $A \in \text{trop}_{\mathcal{C}^+}(\mathcal{V}_{dn})$.*

Proof. We apply Definition 2.2. Indeed, up to scaling and permutations of rows, we can assume that there is an index $s \in \{2, \dots, d-1\}$ such that the condition

$$(2.1) \quad \min\{a_{1k}, \dots, a_{sk}\} = \min\{a_{s+1k}, \dots, a_{dk}\}$$

holds for all $k \in \{1, \dots, n\}$, and we write $\hat{i}(k)$ to denote one of the indices that deliver the minimum to the left-hand side of (2.1). We proceed with the definition of the $d \times n$ matrix B with entries in $\mathcal{C}^+$ as follows. Indeed, for any $k \in \{1, \dots, n\}$, for any $i \in \{1, \dots, s\}$ with $i \neq \hat{i}(k)$, and, for any $j \in \{s+1, \dots, d\}$, we define

$$(b_{jk} = t^{a_{jk}}) \ \& \ (b_{ik} = \varepsilon \cdot t^{a_{ik}}) \ \& \ (b_{ik} = \sum_j b_{jk} - \sum_i b_{ik})$$

with $\varepsilon \in (0, 1/d)$. For all $q \in \{1, \dots, d\}$, we get $\text{val } b_{qk} = a_{qk}$ with positive leading terms, and the row sums over $\{1, \dots, s\}$ and $\{s+1, \dots, d\}$ coincide, so $\text{rk } B < d$. $\square$

The part (2) of Theorem 1.3 is clear from the following result. Here, we use

- the *bideterminantal Gondran–Minoux rank* $\delta(A)$, which is the largest order of a square submatrix M of a real matrix A with $|M^+| \neq |M^-|$, and
- the *Gondran–Minoux row rank* $\text{GMr}(A)$, which is the largest cardinality of a GM-independent family of the rows of A , see [2, 8].

THEOREM 2.6 ([8, Theorem 4.6]). *If $\delta(A) \leq 3$, then $\text{GMr}(A) = \delta(A)$.*

THEOREM 2.7. *Let d, n be integers with $d \in \{1, 2, 3, 4\}$ and $d \leq n$. Then the $d \times d$ minors of a $d \times n$ matrix of variables form a positive tropical generating set.*

Proof. Let A be a $d \times n$ real matrix. If $\delta(A) = d$, then some $d \times d$ submatrix M of A has $|M^+| \neq |M^-|$, and its determinant polynomial f satisfies $A \notin \text{trop}_{\mathcal{C}^+} V(f)$, due to Theorem 2.4. If $\delta(A) < d$, then we have $A \in \text{trop}_{\mathcal{C}^+} \mathcal{V}_{dn}$ by Theorems 2.5 and 2.6. Therefore, the family $\text{trop}_{\mathcal{C}^+} \mathcal{V}_{dn}$ is the intersection of all tropicalizations $\text{trop}_{\mathcal{C}^+} V(f)$ in which f runs over all determinants of the $d \times d$ minors, as desired. $\square$

We complete the proof of Theorem 1.3 with a study of the matrix in [8, Example 3.1]. Namely, we present a 5×6 matrix A each of whose 5×5 submatrices can be lifted to singular matrices over $\mathcal{C}^+$ with the degrees of the entries equal to the corresponding entries of A , but A cannot be lifted to a rank-four matrix itself.

EXAMPLE 2.8. The matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

satisfies $\delta(A) = 4$, but the positive tropicalization $\text{trop}_{\mathcal{C}^+} \mathcal{V}_{56}$ does not contain A .

Proof. The property $\delta(A) = 4$ can be checked straightforwardly, and, due to the symmetry of the construction, it is sufficient to check the property $|M^+| = |M^-|$ for the matrices M obtained from A by removing either the first column or last column. Also, the full details of the corresponding checking appear in [8, Example 3.1].

As to the second assertion, we argue by *absurdum* and take a 5×6 lifting B_ε with entries in $\mathbb{C}^+$ such that $\text{rk } B_\varepsilon \leq 4$ and $\text{val } b_{ij} \rightarrow a_{ij}$ as $\varepsilon \rightarrow 0$. Up to scaling, we get

$$B_\varepsilon = \begin{pmatrix} 1 & 0 & z_1 & z_3 & 0 & 0 \\ 1 & 0 & 0 & 0 & x_5 & x_7 \\ 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ 0 & 1 & x_2 & 0 & x_6 & 0 \\ 0 & 1 & 0 & x_4 & 0 & x_8 \end{pmatrix} + C$$

with real $x_k > 0$, $y_s > 0$, $z_t \geq 0$ and $\text{val } c_{ij} > 0$. If $\lambda_1 B_1 + \lambda_2 B_2 + \dots + \lambda_5 B_5 = 0$ is a linear combination of the rows of B_ε with $\lambda_i \in \mathbb{C}\{\{t\}\}$ that certifies the inequality $\text{rk } B_\varepsilon \leq 4$, then we use the scaling again and assume

$$(2.2) \quad \min\{\text{val } \lambda_1, \text{val } \lambda_2, \text{val } \lambda_3, \text{val } \lambda_4, \text{val } \lambda_5\} = 0,$$

and the constant terms of λ_i are denoted as $\mu_i \in \mathbb{C}$. We have $\mu_1 + \mu_2 = \mu_4 + \mu_5 = 0$ and $\mu_1 z_1 + \mu_4 x_2 = -y_1 \mu_3$, $\mu_1 z_3 + \mu_5 x_4 = -y_2 \mu_3$, $\mu_2 x_5 + \mu_4 x_6 = -y_3 \mu_3$, and also $\mu_2 x_7 + \mu_5 x_8 = -y_4 \mu_3$. Using the further notation $(\alpha, \beta, \gamma) = (\mu_1, \mu_4, -\mu_3)$, we get $z_1 \alpha + x_2 \beta = y_1 \gamma$, $z_3 \alpha - x_4 \beta = y_2 \gamma$, $-x_5 \alpha + x_6 \beta = y_3 \gamma$, $-x_7 \alpha - x_8 \beta = y_4 \gamma$ and

$$(2.3) \quad (x_2 x_5 + z_1 x_6) \beta = (x_5 y_1 + z_1 y_3) \gamma, \quad -(x_4 x_7 + z_3 x_8) \beta = (x_7 y_2 + z_3 y_4) \gamma$$

or $((x_4 x_7 + z_3 x_8) \cdot (x_5 y_1 + z_1 y_3) + (x_2 x_5 + z_1 x_6) \cdot (x_7 y_2 + z_3 y_4)) \cdot \gamma = 0$, and this gives $\gamma = 0$. Using (2.3), we get $\beta = 0$ and hence $\alpha = 0$, which implies that all coefficients $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ are zero, and this is a contradiction to (2.2). $\square$

Therefore, the $d \times d$ minors of a $d \times n$ matrix are not a positive tropical generating set if $n > d \geq 5$, which completes the proof of Theorem 1.3.

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