ALGEBRAIC COMBINATORICS

Kenji Tanino, Tomoki Tamaru, Masatake Hirao & Masanori Sawa

More on the corner-vector construction for spherical designs

Volume 8, issue 5 (2025), p. 1387-1414.
https://doi.org/10.5802/alco.441

Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 8, issue 5 (2025), p. 1387–1414
https://doi.org/10.5802/alco.441

More on the corner-vector construction for spherical designs

Kenji Tanino, Tomoki Tamaru, Masatake Hirao & Masanori Sawa

Abstract This paper explores a full generalization of the classical corner-vector method for constructing weighted spherical designs, which we call the *generalized corner-vector method*. First we establish a uniform upper bound for the degree of designs obtained from the proposed method. Our proof is a hybrid argument that employs techniques in analysis and combinatorics, especially a famous result by Xu (1998) on the interrelation between spherical designs and simplicial designs, and the cross-ratio comparison method for Hilbert identities introduced by Nozaki and Sawa (2013). We extensively study conditions for the existence of designs obtained from our method, and present many curious examples of degree 7 through 13, some of which are, to our surprise, characterized in terms of integral lattices.

1. Introduction

Delsarte et al. [7] introduces *spherical design* as a spherical analogue of balanced incomplete block (BIB) design in applied statistics and combinatorial t-design in discrete mathematics. A finite subset X of a unit sphere $\mathbb{S}^{n-1}$ is defined to be a *(weighted) spherical t-design*, if for every polynomial of degree at most t, the (weighted) average of the function values on X is equal to the integration with respect to the surface measure ρ. Spherical design is closely related to various objects including, but not limited to, spherical cubature in numerical analysis, isometric embeddings of the classical finite-dimensional Banach spaces in functional analysis and optimum experimental designs in applied statistics (see, for example, [14, 27] and [28]).

One of the most fundamental problems in the spherical design theory is the construction as well as existence of designs. Bondarenko et al. [4] shows a general existence theorem for designs of small sizes, and gives an affirmative answer to the Korevaar-Meyers conjecture [13]. Although significant progress has been made in the establishment of existence theorems, recent developments of the construction theory have been comparatively modest in scale, even for designs of small degrees.

There are two classical approaches to the construction of spherical designs: *product rule* and *invariant rule*. Roughly speaking, the former is a recursive approach that generates a spherical design from designs on simpler spaces, and the latter is an

Manuscript received 21st January 2025, revised 22nd May 2025, accepted 25th May 2025.

Keywords. Cubature formula, explicit construction, Hilbert identity, integral lattice, simplicial design, spherical design.

Acknowledgements. This work was supported by JSPS KAKENHI Grant Number 24K06871 (with the third author as principal investigator and the fourth author as co-investigator), and by the Early Support Program for Grant-in-Aid for Scientific Research of Kobe University (for the fourth author).

algebraic approach, based on the invariant theory for polynomial spaces, that directly generates a highly symmetric design from group orbits. A great advantage of the product rule is its simplicity, however, most of the known product rules strongly depend on the assumption of the existence of interval designs for Gegenbauer measure $(1-t^2)^{\lambda-1/2}dt, \lambda > -1/2$ (see, for example, [1, 18, 26]). As pointed out by Xiang [26], the construction of such interval designs is still an open problem. On the other hand, the invariant rule is a more sophisticated approach, which makes it possible to provide an explicit construction of spherical designs. We refer the reader to Xiang [26] for discussion on the explicitness of designs. A major difficulty of the invariant rule is the selection of finitely many points whose group orbits form a design of a given degree. A traditional and popular criterion is the utilization of *corner vectors* for the hyperoctahedral group B_n.

The aim of this paper is to make further progress on the *corner-vector method* by considering designs of more general type

$$(1) \quad \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f(y)d\rho = \sum_{i=1}^{k} W_i \sum_{x \in v_{a_i,s_i}^{B_n}} f(x) \text{ for every polynomial } f \text{ of degree} \leqslant t.$$

We here denote by $v_{a_i,s_i}^{B_n}$ the B_n-orbit of *generalized corner vector*

$$v_{a_i,s_i} = \frac{1}{\sqrt{a_i^2+s_i}}(a_i,1,\ldots,1,0,\ldots,0) \in \mathbb{S}^{n-1}, \quad a_i > 0, \quad s_i \in \{0,1,\ldots,n-1\}.$$

A point $v_{a,s}$ with $a = 1$ is just an $(s+1)$-dimensional *corner vector*, namely the barycentre of an $(s+1)$-dimensional face of an n-dimensional cross-polytope (see Definition 2.11). The corner-vector method is a traditional way of constructing designs with only corner vectors. While this method can respond to the preference for simplicity of construction, it has the drawback that the generated designs have degree at most 7 (see Bajnok [2]).

A classical result on spherical designs is that the vertex set of a regular $(t+1)$-gon is a t-design on $\mathbb{S}^1$ (see, for example, [11]). This means that generalized corner vectors can generate designs of any degree in dimension 2, since if $t \equiv 0 \pmod 4$, the vertex set consists of the B_2-orbits of $t/4$ generalized corner vectors. Heo and Xu [10, Theorem 2.1] establishes a systematic treatment of the three-dimensional case, and in particular shows that the degree of the generated design is uniformly upper bounded by 17. Schur [8, p.721] makes the first advance for $n \geqslant 4$, who discovers a weighted 11-design on $\mathbb{S}^3$ with a single *proper orbit*, namely an orbit $v_{a,s}^{B_4}$ with $a \neq 1$. The original version of Schur's formula appeared in the context of Waring problem in number theory, as briefly explained in Section 2.2. Sawa and Xu [21] discusses a higher dimensional extension of Schur's formula, and proves that the degree of any design with a single proper orbit is uniformly upper bounded by 11. This bound is sharp, since many examples of such 11-designs have been found in dimensions 3 through 23. Thus a natural question is to ask what happens to designs with two or more proper orbits.

This paper is organized as follows. Section 2 gives preliminaries where we review basic facts on weighted spherical designs, with particular emphasis on the connections to Hilbert identities and simplicial designs. Sections 3 through 6 are the main body of this paper. Section 3 demonstrates that for $n \geqslant 4$, the degree of any design of type (1) is uniformly upper bounded by 15 (Theorem 3.1). Our proof is a hybrid argument employing techniques in analysis and combinatorics, especially a famous theorem by Xu [27] on the interrelation between spherical and simplicial designs, and the *cross-ratio comparison* for Hilbert identity (see for example [17, Theorem 6.6]). Section 4 establishes a characterization theorem for 7-designs of type (1) with a single proper orbit (Proposition 4.1). As an important corollary of this result, we first show that there exist finitely many such designs including two sporadic examples $v_{2,8}^{B_{16}}$ and $v_{2,11}^{B_{23}}$ (Remark 4.5), and then prove that the degree of all such designs can never be pushed up to 9 (Proposition 4.6). Section 5 gives a two-orbit version of Proposition 4.1 (see Propositions 5.2 and 5.5), and thereby obtain some theoretical results, e.g. a classification of such 9-designs for $n = 3$ (see Table 3). Section 6 explores the exhibition of interesting examples of 11- and 13-designs with more than 2 proper orbits. Finally, Section 7 is the Conclusion, where some connections between our designs and integral lattices will be also made.

2. Preliminary

In this section we review basic facts concerning spherical designs, with particular emphasis on the connection with Hilbert identity and cubature on simplex, and then quickly overview the background of the *corner vector method*. Some of the materials appearing in this section are available in [19, 20].

2.1. Polynomial space. Let n be a positive integer with $n \geqslant 2$. Let $\mathbb{S}^{n-1}$ be the $(n-1)$-dimensional unit sphere centered at the origin, namely,

$$\mathbb{S}^{n-1} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1 \right\}.$$

We denote by $|\mathbb{S}^{n-1}|$ the surface area of $\mathbb{S}^{n-1}$, and by ρ the surface measure on $\mathbb{S}^{n-1}$. For convention, we write y^λ for $y_1^{\lambda_1} \cdots y_n^{\lambda_n}$, where $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_{\geqslant 0}^n$. Then we have

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} y^\lambda d\rho = \begin{cases} \frac{\Gamma(\frac{n}{2})}{2^{|\lambda|_1} \Gamma(\frac{|\lambda|_1 + n}{2})} \cdot \frac{\prod_{i=1}^n (\lambda_i)!}{\prod_{i=1}^n (\frac{\lambda_i}{2})!} & \lambda_1 \equiv \cdots \equiv \lambda_n \equiv 0 \pmod 2, \\ 0 & \text{otherwise}, \end{cases}$$

where $|\lambda|_1 = \lambda_1 + \cdots + \lambda_n$ for $\lambda \in \mathbb{Z}_{\geqslant 0}^n$.

We denote by $\mathcal{P}(\mathbb{R}^n)$ the space of all polynomials with real coefficients on $\mathbb{R}^n$, and by $\mathcal{P}_t(\mathbb{R}^n)$ the subspace of all polynomials of degree at most t. We also denote by $\mathrm{Hom}_t(\mathbb{R}^n)$ or $\mathrm{Harm}_t(\mathbb{R}^n)$, the subspace of $\mathcal{P}_t(\mathbb{R}^n)$ consisting of homogeneous polynomials or harmonic homogeneous polynomials (sometimes called h-harmonics) respectively. In summary,

$$\mathcal{P}_t(\mathbb{R}^n) = \Big\{ \sum_{\substack{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_{\geqslant 0}^n \\ \lambda_1 + \cdots + \lambda_n \leqslant t}} c_{\lambda_1, \ldots, \lambda_n} x_1^{\lambda_1} \cdots x_n^{\lambda_n} \mid c_{\lambda_1, \ldots, \lambda_n} \in \mathbb{R} \Big\},$$

$$\mathrm{Hom}_t(\mathbb{R}^n) = \Big\{ \sum_{\substack{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_{\geqslant 0}^n \\ \lambda_1 + \cdots + \lambda_n = t}} c_{\lambda_1, \ldots, \lambda_n} x_1^{\lambda_1} \cdots x_n^{\lambda_n} \mid c_{\lambda_1, \ldots, \lambda_n} \in \mathbb{R} \Big\},$$

$$\mathrm{Harm}_t(\mathbb{R}^n) = \Big\{ f \in \mathrm{Hom}_t(\mathbb{R}^n) \mid \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f = 0 \Big\}.$$

It is then obvious that It is not entirely obvious but shown $\mathcal{P}_t(\mathbb{R}^n) = \oplus_{\ell=0}^t \mathrm{Hom}_\ell(\mathbb{R}^n)$. We use the notation $\mathcal{P}(A)$ for the restriction of $\mathcal{P}(\mathbb{R}^n)$ to $A \subseteq \mathbb{R}^n$, and similarly for $\mathcal{P}(A)$, $\mathrm{Hom}_t(A)$ and $\mathrm{Harm}_t(A)$.

The following proposition is classical in spherical harmonics (see, e.g., [9, 15]), which is a basic tool in the study of spherical designs (see, e.g., [3, 7]), and is related to the dimension formula for a special subspace of $\mathcal{P}(\mathbb{S}^{n-1})$ (see Proposition 2.14).

Proposition 2.1 (cf. [7, 15]). *It holds that*

$$\mathcal{P}(\mathbb{S}^{n-1}) = \bigoplus_{i \geqslant 0} \mathrm{Harm}_i(\mathbb{S}^{n-1}). \tag{2}$$

Moreover (2) is the orthogonal direct sum with respect to the inner product $(\phi, \psi) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \phi\psi d\rho$.

2.2. Spherical design, Hilbert identity, simplicial design. Delsarte et al. [7] introduces the notion of spherical design.

Definition 2.2 (Spherical cubature and design). *Let X be a finite subset of $\mathbb{S}^{n-1}$, with a weight function $w : X \to \mathbb{R}_{>0}$. An integration formula of type*

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} f(y) d\rho = \sum_{x \in X} w(x) f(x) \quad \text{for every } f \in \mathcal{P}_t(\mathbb{S}^{n-1}) \tag{3}$$

is called a cubature of degree t on $\mathbb{S}^{n-1}$. The pair (X, w) is called an (equi-weighted) spherical t-design if $w(x)$ takes the constant $1/|X|$, and a weighted spherical t-design otherwise.

We shall look at some examples.

Example 2.3 (cf. [3, 7]). For any positive integer k, let X be the vertex set of a regular $(4k)$-gon inscribed in $\mathbb{S}^1$. An equi-weighted formula of type

$$\frac{1}{|\mathbb{S}^1|} \int_{\mathbb{S}^1} f(y_1, y_2) d\rho = \frac{1}{4k} \sum_{i=0}^{4k-1} f\Big(\cos \frac{(2i+1)\pi}{4k}, \sin \frac{(2i+1)\pi}{4k} \Big)$$

is a cubature of degree $4k - 1$ on $\mathbb{S}^1$.

Example 2.4. An equi-weighted formula of type

$$\frac{1}{|\mathbb{S}^2|} \int_{\mathbb{S}^2} f(y_1, y_2, y_3) d\rho = \frac{1}{6} \sum_{x \in (1,0,0)^{B_3}} f(x) \tag{4}$$

is a cubature of degree 3 on $\mathbb{S}^2$. Here x^{B_3} denotes the orbit of $x \in \mathbb{R}^3$ under the action of the octahedral group B_3. In the present example, we have

$$(1, 0, 0)^{B_3} = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}. \tag{5}$$

This is the vertex set of a regular octahedron in $\mathbb{R}^3$, which is a class of the *corner vectors* of the group B_3 (see Section 2.3 for the detail). An equi-weighted formula of type

$$\frac{1}{|\mathbb{S}^2|} \int_{\mathbb{S}^2} f(y_1, y_2, y_3) d\rho = \frac{1}{6} \sum_{x \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)^{B_3}} f(x)$$

is also a cubature of degree 3 on $\mathbb{S}^2$. In this example, we have

$$\Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\Big)^{B_3} = \Big\{ \Big(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, 0\Big), \Big(0, \pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\Big), \Big(\pm \frac{1}{\sqrt{2}}, 0, \pm \frac{1}{\sqrt{2}}\Big) \Big\}, \tag{6}$$

which is just the set of the midpoints of the edges of a regular octahedron, again a class of the corner vectors of B_3.

Example 2.5 (Schur's formula). The following is a cubature of degree 11 on $\mathbb{S}^3$:

$$\begin{aligned} \frac{1}{|\mathbb{S}^3|} \int_{\mathbb{S}^3} f(y_1, y_2, y_3, y_4) d\rho &= \frac{9}{640} \sum_{x \in (\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0)^{B_4}} f(x) \\ &+ \frac{1}{60} \sum_{x \in (1,0,0,0)^{B_4}} f(x) + \frac{1}{96} \sum_{x \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)^{B_4}} f(x) + \frac{1}{60} \sum_{x \in (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^{B_4}} f(x). \end{aligned} \tag{7}$$

This point configuration is not equi-weighted.

Formula (7) originally stems from Schur (cf. [8, p.721]), who finds a certain polynomial identity called *Hilbert identity* and thereby makes a significant contribution to Waring problem for 10th powers of integers.

Definition 2.6. *A polynomial identity of type*

$$(X_1^2 + \cdots + X_n^2)^t = \sum_{i=1}^{N} c_i (a_{i1} X_1 + \cdots + a_{in} X_n)^{2t}$$

where $c_i \in \mathbb{R}_{>0}$ and $a_{ij} \in \mathbb{R}$, is called a Hilbert identity. In particular this is called a rational (Hilbert) identity if $c_i \in \mathbb{Q}_{>0}$ and $a_{ij} \in \mathbb{Q}$.

The following result by Lyubich and Vaserstein [14] directly relates spherical cubature to Hilbert identity. We also refer the reader to [22] for a brief explanation of the Lyubich-Vaserstein theorem.

Theorem 2.7 (Lyubich-Vaserstein theorem). *Let n, t be positive integers and*

$$c_{n,t} = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} y_1^{2t} d\rho.$$

Let $x_i = (x_{i,1}, \ldots, x_{i,n}) \in \mathbb{S}^{n-1}$ and $c_i \in \mathbb{R}_{>0}$ for $i = 1, \ldots, N$. The following are equivalent:

(i) *The points x_i and weights c_i give a (weighted) spherical design of index $2t$ on $\mathbb{S}^{n-1}$, namely*

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} f(y) d\rho = \sum_{i=1}^{N} c_i f(x_i) \quad \text{for every } f \in \mathrm{Hom}_{2t}(\mathbb{S}^{n-1});$$

(ii)

$$c_{n,t}(X_1^2 + \cdots + X_n^2)^t = \sum_{i=1}^{N} c_i (x_{i,1} X_1 + \cdots + x_{i,n} X_n)^{2t}.$$

Example 2.8 (Schur's identity). By Theorem 2.7, the spherical cubature (7) is equivalent to the rational Hilbert identity

$$\begin{aligned} 22680(X_1^2 + X_2^2 + X_3^2 + X_4^2)^5 &= \sum_{48} (2X_i \pm X_j \pm X_k)^{10} \\ +9 \sum_{4} (2X_i)^{10} + 180 \sum_{12} &(X_i \pm X_j)^{10} + 9 \sum_{8} (X_1 \pm X_2 \pm X_3 \pm X_4)^{10}, \end{aligned} \tag{8}$$

where each summation is taken among all combinations of sign changes and permutations of X_1, X_2, X_3, X_4.

A spherical cubature is closely connected to a simplicial cubature. We consider the standard orthogonal simplex in $\mathbb{R}^n$, namely

$$T^n = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_1 \geqslant 0, \ldots, y_n \geqslant 0,\ |y|_1 \leqslant 1\},$$

where $|y|_1 = y_1 + \cdots + y_n$. For $y \in T^n$, we define

$$W(y_1, \ldots, y_n) = \frac{1}{\sqrt{(1 - |y|_1)\left(\prod_{i=1}^{n} y_i\right)}}.$$

The following result can be found in Xu [27].

Theorem 2.9 ([27]). *If for every $f \in \mathcal{P}_t(T^{n-1})$,*

$$\frac{\displaystyle\int_{T^{n-1}} f(y_1,\dots,y_{n-1})W(y_1,\dots,y_{n-1})\,dy_1\cdots dy_{n-1}}{\displaystyle\int_{T^{n-1}} W(y_1,\dots,y_{n-1})\,dy_1\cdots dy_{n-1}} = \sum_{i=1}^{N} c_i f(x_{i,1},\dots,x_{i,n-1}),$$

then for every $f \in \mathcal{P}_{2t+1}(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} f(y_1,\dots,y_{n-1},y_n)\,d\rho = \sum_{i=1}^{N} \frac{c_i}{2^{\mathrm{wt}(x_i)}} \sum_{\pm} f(\pm\sqrt{x_{i,1}},\dots,\pm\sqrt{x_{i,n-1}},\pm\sqrt{1-|x_i|_1}),$$

where $\mathrm{wt}(x_i) = |\{j \mid x_{i,j} \neq 0\}|$ *for* $i = 1,\dots,N$. *Moreover, the converse direction also holds.*

Example 2.10 (Example 2.4, revisited). By Theorem 2.9, the formula (4) of degree 3 on $\mathbb{S}^2$ is equivalent to a simplicial cubature of type

$$\begin{aligned}\frac{1}{\int_{T^2} W(y_1,y_2)dy_1dy_2} &\int_{T^2} f(y_1,y_2)W(y_1,y_2)dy_1dy_2 \\ &= \frac{1}{3}\left(f(1,0)+f(0,1)+f(0,0)\right) \quad \textit{for every } f \in \mathcal{P}_1(T^2).\end{aligned}$$

We note that the 6 vertices of a regular octahedron (see (5)) is transformed to the vertex set of the standard orthogonal simplex T^2. Meanwhile, the formula of type

$$\frac{1}{|\mathbb{S}^2|}\int_{\mathbb{S}^2} f(y_1,y_2,y_3)d\rho = \frac{1}{6}\sum_{x\in(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)^{B_3}} f(x)$$

is also a cubature of degree 3 on $\mathbb{S}^2$, which is equivalent to a simplicial cubature of type

$$\begin{aligned}\frac{1}{\int_{T^2} W(y_1,y_2)dy_1dy_2} &\int_{T^2} f(y_1,y_2)W(y_1,y_2)dy_1dy_2 \\ &= \frac{1}{3}\Big(f\big(\frac{1}{2},0\big)+f\big(0,\frac{1}{2}\big)+f\big(\frac{1}{2},\frac{1}{2}\big)\Big) \quad \textit{for every } f \in \mathcal{P}_1(T^2).\end{aligned}$$

Again, we note that the midpoints of the edges of a regular octahedron (see (6)) is transformed to those of the standard orthogonal simplex T^2. Theorem 2.9 preserves geometric information about corner vectors; more details will be available in Remark 2.21.

2.3. The corner-vector method.

A most classical method of constructing spherical designs is the *corner-vector method* for the symmetry group B_n of a regular hyperoctahedron in $\mathbb{R}^n$. Hereafter we write x^{B_n} for the orbit of $x \in \mathbb{R}^n$ under the action of the hyperoctahedral group B_n.

Definition 2.11. *Let $e_1,\dots,e_n$ be the standard basis vectors in $\mathbb{R}^n$. For $k = 1,\dots,n$, let v_k be a vector of type*

$$v_k = \frac{1}{\sqrt{k}}\sum_{i=1}^{k} e_i \in \mathbb{S}^{n-1}.$$

The corner vectors (for B_n) are the elements of $v_1^{B_n},\dots,v_n^{B_n}$.

Example 2.12 (Example 2.4, revisited). For $n = 2$, the corner vectors are given by

$$v_1^{B_2} = \{(\pm1,0),(0,\pm1)\}, \quad v_2^{B_2} = \Big\{\Big(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\Big)\Big\},$$

which are the vertices and midpoints of edges of a square in $\mathbb{R}^2$, respectively. For $n=3$, the corner vectors are given by

$$v_1^{B_3} = \{(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)\},$$
$$v_2^{B_3} = \Big(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\Big)^{B_3} = \Big\{\Big(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}},0\Big),\Big(0,\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\Big),\Big(\pm\frac{1}{\sqrt{2}},0,\pm\frac{1}{\sqrt{2}}\Big)\Big\},$$
$$v_3^{B_3} = \Big\{\Big(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\Big)\Big\}.$$

The first and second sets are the vertices and midpoints of edges of a regular octahedron in $\mathbb{R}^3$ respectively, and the third is the set of barycenters of faces.

The utility of the corner vector construction is verified from the following more sophisticated result. Below we denote by $\mathcal{P}_t(A)^{B_n}$ the space consisting of all B_n-invariant polynomials in $\mathcal{P}_t(A)$, and similarly for $\mathrm{Harm}_i(A)^{B_n}$.

Theorem 2.13 (Sobolev's theorem). *Let $x_1^{B_n},\dots,x_m^{B_n} \subset \mathbb{S}^{n-1}$. Let $w : \bigcup_{i=1}^m x_i^{B_n} \to \mathbb{R}_{>0}$ be a weight function such that $w(x)=w(x')$ for every $x,x' \in x_i^{B_n}$ and $i=1,\dots,m$. Then the following are equivalent:*

(i) *(X,w) is a (weighted) spherical t-design on $\mathbb{S}^{n-1}$;*
(ii)
$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f(y)d\rho = \sum_{x\in X} w(x)f(x) \quad \textit{for every } f\in\mathcal{P}_t(\mathbb{S}^{n-1})^{B_n}.$$

A great advantage of Theorem 2.13 is the reduction of the computational cost in order to check the defining property (3) of weighted spherical designs. With Proposition 2.1 in mind, we have to compute the dimension of $\mathrm{Harm}_i(\mathbb{S}^{n-1})^{B_n}$.

Proposition 2.14 (cf. Appendix of [19]). *The harmonic Molien series $\sum_{i=0}^\infty p_i\lambda^i$ where $p_i = \dim \mathrm{Harm}_i(\mathbb{S}^{n-1})^{B_n}$, is given by*

$$\sum_{i=0}^\infty p_i\lambda^i = \frac{1}{(1-\lambda^4)(1-\lambda^6)\cdots(1-\lambda^{2n})}.$$

In particular,

$$\begin{aligned}\sum_{i=0}^\infty p_i\lambda^i &= \frac{1}{(1-\lambda^4)(1-\lambda^6)\cdots(1-\lambda^{2n})}\\ &= \begin{cases} 1+\lambda^4+\lambda^6+\lambda^8+\lambda^{10}+\cdots & \textit{if } n=3;\\ 1+\lambda^4+\lambda^6+2\lambda^8+\lambda^{10}+\cdots & \textit{if } n=4;\\ 1+\lambda^4+\lambda^6+2\lambda^8+2\lambda^{10}+\cdots & \textit{if } n\geqslant 5.\end{cases}\end{aligned}$$

Corollary 2.15. *It holds that*

$$\dim \mathrm{Harm}_4(\mathbb{S}^{n-1})^{B_n} = \dim \mathrm{Harm}_6(\mathbb{S}^{n-1})^{B_n} = 1 \quad \textit{for } n\geqslant 3,$$
$$\dim \mathrm{Harm}_8(\mathbb{S}^{n-1})^{B_n} = \begin{cases} 1 \textit{ for } n=3,\\ 2 \textit{ for } n\geqslant 4,\end{cases}$$
$$\dim \mathrm{Harm}_{10}(\mathbb{S}^{n-1})^{B_n} = \begin{cases} 1 \textit{ for } n=3,4,\\ 2 \textit{ for } n\geqslant 5,\end{cases}$$
$$\dim \mathrm{Harm}_{2k+1}(\mathbb{S}^{n-1})^{B_n} = 0 \quad \textit{for } k=0,1,\dots$$

Let $\mathcal{S}_n$ be the symmetric group of order n. It is convenient to use the notation $\mathrm{sym}(f)$ for a symmetric polynomial as defined by

$$\mathrm{sym}(f) = \frac{1}{|(\mathcal{S}_n)_f|}\sum_{\gamma\in\mathcal{S}_n} f(x_{\gamma(1)},\dots,x_{\gamma(n)}) \tag{9}$$

where

$$(\mathcal{S}_n)_f = \{\gamma \in \mathcal{S}_n \mid f(x_{\gamma(1)}, \dots, x_{\gamma(n)}) = f(x_1, \dots, x_n) \text{ for all } (x_1, \dots, x_n) \in \mathbb{R}^n\}.$$

The following lemmas will be utilized for further arguments in the main body of this paper (see, for example, Lemma 3.5).

Lemma 2.16 ([19]). *We define homogeneous polynomials f_4, f_6, $f_{8,1}$, $f_{8,2}$, $f_{10,1}$, $f_{10,2}$ as*

$$\begin{aligned}
f_4(x) &= \mathrm{sym}(x_1^4) - \frac{6}{n-1}\mathrm{sym}(x_1^2x_2^2) \quad \text{for } n \geqslant 3,\\
f_6(x) &= \mathrm{sym}(x_1^6) - \frac{15}{n-1}\mathrm{sym}(x_1^2x_2^4) + \frac{180}{(n-1)(n-2)}\mathrm{sym}(x_1^2x_2^2x_3^2) \quad \text{for } n \geqslant 3,\\
f_{8,1}(x) &= \mathrm{sym}(x_1^8) - \frac{28}{n-1}\mathrm{sym}(x_1^2x_2^6) + \frac{70}{n-1}\mathrm{sym}(x_1^4x_2^4) \quad \text{for } n \geqslant 3,\\
f_{8,2}(x) &= \mathrm{sym}(x_1^4x_2^4) - \frac{6}{n-2}\mathrm{sym}(x_1^2x_2^2x_3^4)\\
&\quad + \frac{108}{(n-2)(n-3)}\mathrm{sym}(x_1^2x_2^2x_3^2x_4^2) \quad \text{for } n \geqslant 4,\\
f_{10,1}(x) &= \mathrm{sym}(x_1^{10}) - \frac{45}{n-1}\mathrm{sym}(x_1^2x_2^8) + \frac{42}{n-1}\mathrm{sym}(x_1^4x_2^6)\\
&\quad + \frac{1008}{(n-1)(n-2)}\mathrm{sym}(x_1^2x_2^2x_3^6) - \frac{1260}{(n-1)(n-2)}\mathrm{sym}(x_1^2x_2^4x_3^4) \quad \text{for } n \geqslant 3,\\
f_{10,2}(x) &= \mathrm{sym}(x_1^4x_2^6) - \frac{6}{n-2}\mathrm{sym}(x_1^2x_2^2x_3^6) - \frac{30}{n-2}\mathrm{sym}(x_1^2x_2^4x_3^4)\\
&\quad + \frac{450}{(n-2)(n-3)}\mathrm{sym}(x_1^2x_2^2x_3^2x_4^4)\\
&\quad - \frac{10800}{(n-2)(n-3)(n-4)}\mathrm{sym}(x_1^2x_2^2x_3^2x_4^2x_5^2) \quad \text{for } n \geqslant 5.
\end{aligned}$$

Then f_4, f_6, $f_{8,1}$, $f_{8,2}$, $f_{10,1}$, $f_{10,2}$ are B_n-invariant h-harmonics.

Lemma 2.17. *Let*

$$\begin{aligned}
\tilde{f}_4(v_{a,s}) &:= (a^2+s)^2 f_4(v_{a,s}), \quad \tilde{f}_6(v_{a,s}) := (a^2+s)^3 f_6(v_{a,s})\\
\tilde{f}_{8,1}(v_{a,s}) &:= (a^2+s)^4 f_{8,1}(v_{a,s}), \quad \tilde{f}_{8,2}(v_{a,s}) := (a^2+s)^4 f_{8,2}(v_{a,s})\\
\tilde{f}_{10,1}(v_{a,s}) &:= (a^2+s)^5 f_{10,1}(v_{a,s}), \quad \tilde{f}_{10,2}(v_{a,s}) := (a^2+s)^5 f_{10,2}(v_{a,s}).
\end{aligned}$$

It holds that

$$\begin{aligned}
\tilde{f}_4(v_{a,s}) &= a^4 + s - 6a^2\frac{s}{n-1} - 3\frac{s(s-1)}{n-1} \quad \textit{for } n \geqslant 3,\\
\tilde{f}_6(v_{a,s}) &= a^6 + s - 15(a^4 + a^2 + s - 1)\frac{s}{n-1} + 90a^2\frac{s(s-1)}{(n-1)(n-2)}\\
&\qquad + 30\frac{s(s-1)(s-2)}{(n-1)(n-2)} \quad \textit{for } n \geqslant 3,\\
\tilde{f}_{8,1}(v_{a,s}) &= a^8 + s - 28(a^6 + a^2 + s - 1)\frac{s}{n-1} + 70(a^4 + \frac{s-1}{2})\frac{s}{n-1} \quad \textit{for } n \geqslant 3,\\
\tilde{f}_{8,2}(v_{a,s}) &= a^4 + \frac{s-1}{2} - 3(a^4 + 2a^2 + s - 2)\frac{s-1}{n-2}\\
&\qquad + \frac{9}{2}(4a^2 + s - 3)\frac{(s-1)(s-2)}{(n-2)(n-3)} \quad \textit{for } n \geqslant 4,\\
\tilde{f}_{10,1}(v_{a,s}) &= a^{10} + s - 3(15a^8 - 14a^6 - 14a^4 + 15a^2 + s - 1)\frac{s}{n-1}\\
&\qquad + 126(4a^6 - 10a^4 + 3a^2 - s + 2)\frac{s(s-1)}{(n-1)(n-2)} \quad \textit{for } n \geqslant 3,\\
\tilde{f}_{10,2}(v_{a,s}) &= a^4 s + a^6 s + s(s-1) - 3(a^6 + 10a^4 + 7a^2 + 6s - 12)\frac{s(s-1)}{(n-2)}\\
&\qquad + 75(a^4 + 3a^2 + s - 3)\frac{s(s-1)(s-2)}{(n-2)(n-3)}\\
&\qquad\quad - 90(5a^2 + s - 4)\frac{s(s-1)(s-2)(s-3)}{(n-2)(n-3)(n-4)} \quad \textit{for } n \geqslant 5.
\end{aligned}$$

Now, a crucial demerit of the corner-vector construction is that it cannot generate spherical cubature of degree larger than 8, as shown in the following result.

Theorem 2.18 (Bajnok's theorem, Proposition 15 in [2]). *Let $n \geqslant 3$ be a positive integer. Assume that $\bigcup_{i=1}^m v_{k_i}^{B_n}$ is a B_n-invariant weighted spherical t-design. Then it holds that $t \leqslant 7$.*

To push up the maximum degree of design, we give a generalization of the notion of corner vectors.

Definition 2.19. *Let $e_1, \dots, e_n$ be the standard basis vectors in $\mathbb{R}^n$. Let a be a positive real number and $s = 0, \dots, n-1$. Let*

$$v_{a,s} = \frac{1}{\sqrt{a^2+s}}(ae_1 + \sum_{i=1}^{s} e_{i+1}) \in \mathbb{S}^{n-1}.$$

A generalized corner vector (for B_n) is an element of $v_{a,s}^{B_n}$. In particular when $a = 1$, $v_{1,s}$ exactly coincides with the corner vector v_{s+1}. Moreover, when $a \neq 1$, we refer to $v_{a,s}^{B_n}$ as a proper orbit.

Example 2.20. For $n = 2$, take generalized corner vectors of type

$$v_1^{B_2} = \{(\pm 1, 0), (0, \pm 1)\}, \quad v_{\sqrt{3},1}^{B_2} = \Big\{\Big(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\Big), \Big(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\Big)\Big\}.$$

As already seen in Example 2.3, these are the vertices of a regular dodecagon inscribed in $\mathbb{S}^1$, respectively.

The following explains a geometric interpretation of generalized corner vectors.

Remark 2.21 (See, for example, Heo-Xu [10]). For $n = 3$, we consider the map

$$\psi : \mathbb{S}^2 \to T^2, \quad \psi(x_1, x_2, x_3) = (x_1^2, x_2^2).$$

For a finite subset X of $\mathbb{S}^2$, we write $\psi(X)$ for $\{\psi(x) \mid x \in X\}$. Then $\psi(v_{a,0}^{B_3})$ is the vertex set of T^2, namely, $\psi(v_{a,0}^{B_3}) = \{(0,0),(1,0),(0,1)\}$. Moreover, $\psi(v_{a,1}^{B_3})$ is included in the boundary of T^2, namely

$$\begin{aligned}\psi(v_{a,1}^{B_3}) \subset \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 = -x_1 + 1,\ 0 \leqslant x_1 \leqslant 1\}\\ \cup\{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 = 0,\ 0 \leqslant x_1 \leqslant 1\}\\ \cup\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = 0,\ 0 \leqslant x_2 \leqslant 1\}.\end{aligned}$$

Also, $\psi(v_{a,2}^{B_3})$ is included in the medians of T^2, and more precisely

$$\begin{aligned}\psi(v_{a,2}^{B_3}) \subset \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 = x_1,\ 0 \leqslant x_1 \leqslant 1/2\}\\ \cup\{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 = -2x_1 + 1,\ 0 \leqslant x_1 \leqslant 1/2\}\\ \cup\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = -2x_2 + 1,\ 0 \leqslant x_2 \leqslant 1/2\}.\end{aligned}$$

$\psi(v_{1,s}^{B_3})$ coincides with the set of the barycenters of s-dimensional faces of T^2 (see Figure 1 below). These observations can also be found in Heo-Xu [10], where the situation is explained for regular triangles embedded in $\mathbb{R}^3$. A more general treatment of Remark 2.21 will be established in Section 3.

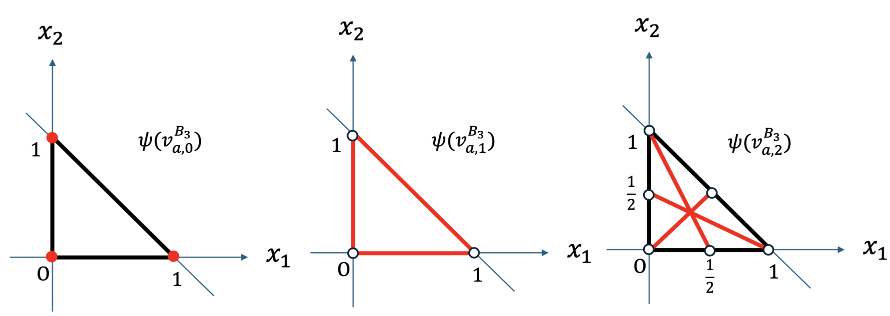

Figure 1. $\psi(v_{a,s}^{B_3})$ $(s = 0, 1, 2)$

Hereafter we restrict our attention to the case where $n \geqslant 3$. Let $s_1, \ldots, s_k$ be integers with $0 \leqslant s_i \leqslant n-1$ for all i, and let $a_1, \ldots, a_k$ be positive real numbers. Then, as a generalization of the corner-vector method, we consider a design of type

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f(y)d\rho = \sum_{i=1}^{k} W_i \sum_{x \in v_{a_i,s_i}^{B_n}} f(x) \quad \textit{for every } f \in \mathcal{P}_t(\mathbb{S}^{n-1}) \tag{10}$$

Example 2.22. (Example 2.5, revisited). For $n = 4$, Schur's formula (8) can be rewritten in terms of generalized corner vectors as follows:

$$\begin{aligned}&\frac{1}{|\mathbb{S}^3|}\int_{\mathbb{S}^3} f(y_1,y_2,y_3,y_4)d\rho\\ &= \frac{9}{640}\sum_{x \in v_{2,2}^{B_4}} f(x) + \frac{1}{60}\sum_{x \in v_1^{B_4}} f(x) + \frac{1}{96}\sum_{x \in v_2^{B_4}} f(x) + \frac{1}{60}\sum_{x \in v_4^{B_4}} f(x).\end{aligned}$$

This can be checked by applying Theorem 2.13 to $f_4, f_6, f_{8,1}, f_{8,2}$ and $f_{10,1}$.

What is remarkable here is that Schur's formula has degree 11, contrary to the situation of Bajnok's theorem. More generally, Sawa and Xu [21] establishes that the maximum degree of a B_n-invariant design with only one proper orbit $v_{a,s}^{B_n}$ can be pushed up to 11. Then what about designs with two or more proper orbits $v_{a_1,s_1}^{B_n}, v_{a_2,s_2}^{B_n}, \ldots$? This is the main subject of this paper, although in Section 4, some new results are established for designs with a single proper orbit $v_{a,s}^{B_n}$.

3. A uniform upper bound for the degree of our designs

Let n, s_i be integers with $n \geqslant 3$ and $0 \leqslant s_i \leqslant n-1$, and let $a_i > 0$. In this section we prove an upper bound for the degree of a weighted spherical design of type (10). The presentation in this section is conscious of algebraic and geometric interpretations of generalized corner vectors.

3.1. Uniform bound.

Theorem 3.1. *Let $n \geqslant 4$. Suppose that*

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f(y)d\rho = \sum_{i=1}^{k} W_i \sum_{x \in v_{a_i,s_i}^{B_n}} f(x) \quad \text{for every } f \in \mathcal{P}_t(\mathbb{S}^{n-1}). \tag{11}$$

Then it holds that $t \leqslant 15$.

Remark 3.2. As shown by Heo and Xu [10, Theorem 2.1], the maximum degree of weighted spherical designs on $\mathbb{S}^2$ of type (11) is upper bounded by 17.

Since designs of type (11) are centrally symmetric, it suffices to take care of even degrees $2, 4, \ldots, 2\lfloor t/2 \rfloor$. As a corollary of Theorems 3.1 and 2.7, we obtain the following result.

Corollary 3.3. *Suppose that*

$$c_{n,r}(X_1^2 + \cdots + X_n^2)^r = \sum_{i=1}^{k} \lambda_i \sum_{\sigma,\pm} (a_i X_{\sigma(1)} \pm X_{\sigma(2)} \pm \cdots \pm X_{\sigma(s_i+1)})^{2r} \tag{12}$$

where the second summation is taken over all permutations $\sigma \in \mathcal{S}_n$ and all sign changes $\pm$. Then it holds that $r \leqslant 7$.

The proof of Theorem 3.1 is substantially divided into three parts. The first part is an application of the *cross-ratio comparison* that was first introduced by Nozaki and Sawa [17, Theorem 6.6]. The remaining two parts involve elementary calculus on B_n-invariant h-harmonics of degree 8 and Xu's characterization theorem (see Theorem 2.9), respectively.

3.2. Proof of Theorem 3.1.

As briefly mentioned in Section 3.1, we need three preliminary lemmas, each including a key idea that can also be applied in the study of design of a different type than (11).

Lemma 3.4. *Let $n \geqslant 4$. Suppose that there exists a weighted t-design of type (11) with $s_i \geqslant 3$ for some i. Then it holds that $t \leqslant 15$.*

Proof. The proof is based on the cross-ratio comparison for monomials $X_1^2X_2^2X_3^6X_4^6$, $X_1^2X_2^4X_3^4X_4^6$ and $X_1^4X_2^4X_3^4X_4^4$.

First we compare the ratio of the coefficients of $X_1^2X_2^2X_3^6X_4^6$ and $X_1^2X_2^4X_3^4X_4^6$ on the both sides of (12). On the left side, we have

$$\binom{8}{1}\binom{7}{3}\binom{4}{1} : \binom{8}{1}\binom{7}{3}\binom{4}{2} = \binom{4}{1} : \binom{4}{2} = 2 : 3.$$

Whereas, on the right side, we have

$$\binom{16}{2}\binom{14}{6}\binom{8}{2}\sum_i A_i(2a_i^2+2a_i^6+s_i-3) : \binom{16}{2}\binom{14}{6}\binom{8}{4}\sum_i A_i(a_i^2+2a_i^4+a_i^6+s_i-3)$$
$$= 2\sum_i A_i(2a_i^2+2a_i^6+s_i-3) : 5\sum_i A_i(a_i^2+2a_i^4+a_i^6+s_i-3)$$

where $A_i = \lambda_i 2^{s_i}\binom{n-4}{s_i-3}$ for $i = 1, \ldots, k$. Then

$$3\sum_i A_i(2a_i^2+2a_i^6+s_i-3) = 5\sum_i A_i(a_i^2+2a_i^4+a_i^6+s_i-3)$$

and therefore

$$\sum_i A_i(a_i^6-10a_i^4+a_i^2) = 2\sum_i A_i(s_i-3). \tag{13}$$

Next we compare the ratio of the coefficients of $X_1^2X_2^4X_3^4X_4^6$ and $X_1^4X_2^4X_3^4X_4^4$ on the both sides of (12). On the left side, we have

$$\binom{8}{2}\binom{6}{2}\binom{4}{3} : \binom{8}{2}\binom{6}{2}\binom{4}{2} = \binom{4}{3} : \binom{4}{2} = 2:3.$$

Whereas on the right side, we have

$$\binom{16}{4}\binom{12}{4}\binom{8}{6}\sum_i A_i(a_i^2+2a_i^4+a_i^6+s_i-3) : \binom{16}{4}\binom{12}{4}\binom{8}{4}\sum_i A_i(4a_i^4+s_i-3)$$
$$= 2\sum_i A_i(a_i^2+2a_i^4+a_i^6+s_i-3) : 5\sum_i A_i(4a_i^4+s_i-3).$$

Then

$$5\sum_i A_i(4a_i^4+s_i-3) = 3\sum_i A_i(a_i^2+2a_i^4+a_i^6+s_i-3)$$

and therefore

$$\sum_i A_i(3a_i^6-14a_i^4+3a_i^2) = 2\sum_i A_i(s_i-3). \tag{14}$$

In summary, by subtracting (14) from (13), we have

$$0 = \sum_i A_i(a_i^6-2a_i^4+a_i^2) = \sum_i A_i a_i^2(a_i^2-1)^2,$$

which implies $a_i \in \{0,1\}$. This is a contradiction to Bajnok's theorem (see Theorem 2.18). ☐

Lemma 3.5. *Let $n \geqslant 8$. Suppose that there exists a weighted t-design of type (11) with $s_i \in \{0,1,2\}$ for all i. Then it holds that $t \leqslant 7$.*

Proof. By combining (11) with Lemma 2.17 for $\tilde{f}_{8,1}$, we have

$$0 = \sum_i \tilde{W}_i\Big(a_i^8+s_i+7(-4a_i^6+10a_i^4-4a_i^2+s_i-1)\frac{s_i}{n-1}\Big),$$

where $\tilde{W}_i = W_i|v_{a_i,s_i}^{B_n}| > 0$. For $i = 1, \ldots, k$, let

$$A_{a_i,s_i} = a_i^8+s_i+7(-4a_i^6+10a_i^4-4a_i^2+s_i-1)\frac{s_i}{n-1}.$$

Since $\tilde{W}_i > 0$ for every i, there exists j such that $A_{a_j,s_j} < 0$. Clearly $s_j \neq 0$. It thus follows from elementary calculus that

$$n-1 < \begin{cases} \dfrac{28a_j^6 - 70a_j^4 + 28a_j^2}{a_j^8+1} < 4 & \text{if } s_j = 1; \\ \dfrac{56a_j^6 - 140a_j^4 + 56a_j^2 - 14}{a_j^8+2} < 7 & \text{if } s_j = 2. \end{cases}$$

□

Lemma 3.6. *Let $n \geqslant 3$. Suppose that there exists a weighted t-design of type (11) with $s_i \leqslant n-2$ for all i. Then it holds that $t \leqslant 2n-1$.*

Proof. Let $n \geqslant 3$, and suppose that there exists a weighted $(2n+1)$-design of type (11) with $s_i \leqslant n-2$ for all i. As in Remark 2.21, we consider the map

$$\psi : \mathbb{S}^{n-1} \to T^{n-1}, \quad \psi(x_1, \ldots, x_{n-1}, x_n) = (x_1^2, \ldots, x_{n-1}^2).$$

Then by Theorem 2.9, the set

$$S = \Big\{ \psi(x) \in T^{n-1} \mid x \in \bigcup_{i=1}^{k} v_{a_i,s_i}^{B_n} \Big\}$$

is a weighted simplicial design of type

$$\sum_{s\in S} c_s f(z_s) = \frac{1}{\int_{T^{n-1}} W(y) dy_1 \cdots dy_{n-1}} \int_{T^{n-1}} f(y) W(y) dy_1 \cdots dy_{n-1}$$
$$\text{for every } f \in \mathcal{P}_n(T^{n-1}).$$

Let $x = (x_1, \ldots, x_n) \in \bigcup_{i=1}^{k} v_{a_i,s_i}^{B_n}$. Since $s_i \leqslant n-2$ for all i, there exists at least one nonzero coordinate of x. If $x_n = 0$, then $\psi(x)$ is included in the affine hyperplane $\pi_0 : x_1 + \cdots + x_n = 1$. If $x_i = 0$ where $i = 1, \ldots, n-1$, then $\psi(x)$ is included in the hyperplane $\pi_i : x_i = 0$. Then the polynomial

$$f(x_1, \ldots, x_n) = \Big(\prod_{i=1}^{n-1} x_i \Big) \Big(1 - \sum_{j=1}^{n} x_j \Big)$$

has degree n, which vanishes at the boundary of T^{n-1} but takes positive values at the interior of T^{n-1}. Hence it follows that

$$0 < \frac{1}{\int_{T^{n-1}} W(y) dy_1 \cdots dy_{n-1}} \int_{T^{n-1}} f(y) W(y) dy_1 \cdots dy_{n-1} = \sum_{s\in S} c_s f(z_s) = 0,$$

which is a contradiction. □

We are now in a position to complete the proof of Theorem 3.1

Proof of Theorem 3.1. The result is proved by Lemma 3.4 for $s_i \geqslant 3$ for some i, and by Lemmas 3.5, 3.6 for $s_i < 3$ for all i. (see Table 1) □

4. Designs with a single proper orbit

Throughout this section we assume $n \geqslant 3$ and consider a design of type

$$(15) \qquad \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} f(y) d\rho = \frac{1}{|v_{a,s}^{B_n}|} \sum_{x \in v_{a,s}^{B_n}} f(x) \quad \textit{for every } f \in \mathcal{P}_t(\mathbb{S}^{n-1}).$$

Table 1. Upper bound for the maximum degree t_n of weighted designs on $\mathbb{S}^{n-1}$ with generalized corner vectors

n	4	5	6	7	8	9
$\exists i$ $(s_i \geqslant 3)$ Lemma 3.4	$\boldsymbol{t_n \leqslant 15}$	$\boldsymbol{t_n \leqslant 15}$	$\boldsymbol{t_n \leqslant 15}$	$\boldsymbol{t_n \leqslant 15}$	$\boldsymbol{t_n \leqslant 15}$	$\boldsymbol{t_n \leqslant 15}$
$\forall i$ $(s_i = 0, 1, 2)$ Lemma 3.5	-	-	-	-	$\boldsymbol{t_n \leqslant 7}$	$\boldsymbol{t_n \leqslant 7}$
$\forall i$ $(s_i = 0, 1, 2)$ Lemma 3.6	$\boldsymbol{t_n \leqslant 7}$	$\boldsymbol{t_n \leqslant 9}$	$\boldsymbol{t_n \leqslant 11}$	$\boldsymbol{t_n \leqslant 13}$	$t_n \leqslant 15$	$t_n \leqslant 17$

Proposition 4.1. *Let $n \geqslant 3$. A 7-design of type (15) exists if and only if,*

$$a^2 = \frac{3s \pm \sqrt{(2+n)s(1-n+3s)}}{n-1}, \tag{16}$$

$$\begin{aligned} h_{\pm}(n,s) = n^3 + (2-9s)n^2 + (-7-9s+12s^2)n + 6s^2 + 18s + 4 \\ \pm(n^2 - 3(-2+s)n - 3s - 7)\sqrt{(2+n)s(1-n+3s)} = 0. \end{aligned} \tag{17}$$

Proof. By Theorem 2.13, Corollary 2.15 and Lemma 2.17, a 7-design of type (15) exists if and only if $\tilde{f}_4(v_{a,s}) = \tilde{f}_6(v_{a,s}) = 0$.

Solving $\tilde{f}_4(v_{a,s}) = 0$ for a^2, we obtain (16). Substituting (16) into $\tilde{f}_6(v_{a,s}) = 0$, we also obtain (17). □

Remark 4.2. When $s = 0$ in (17), we have

$$0 = n^3 + 2n^2 - 7n + 4 = (n-1)^2(n+4),$$

whose roots are 1 and -4, contradicting the assumption that $n \geqslant 3$.

Remark 4.3. Let $n \geqslant 3$. If a weighted 7-design of type (15) exists, then $n^2 - 3(-2+s)n - 3s - 7 = 0$ or $(2+n)s(1-n+3s)$ is a square number. But $n^2 - 3(-2+s)n - 3s - 7 = 0$ does not hold. Suppose contrary. Since n and s are integers, the discriminant of $n^2 - 3(-2+s)n - 3s - 7 = 0$ is a square, that is, there exists an integer m such that

$$9s^2 - 24s + 64 = m^2. \tag{18}$$

Then, the solutions of (18) are $(s,m) = (0, \pm 8), (1, \pm 7), (5, \pm 13)$, for which $n^2 - 3(-2+s)n - 3s - 7 = 0$ has no integer solutions.

Theorem 4.4. *There exist only finitely many pairs (n,s) that satisfy the condition (17).*

Proof. Substituting $n = x$, $s = y$ into (17), we have

$$\begin{aligned} h_+(x,y)h_-(x,y) &= \{n^3 + (2-9s)n^2 + (-7-9s+12s^2)n + 6s^2 + 18s + 4\}^2 \\ &\quad - \{n^2 - 3(-2+s)n - 3s - 7)\sqrt{(n+2)s(1-n+3s)}\}^2 \\ &= (-1+x)^2(1+y)\big\{x^4 + (6-9y)x^3 + (27y^2 - 30y + 1)x^2 \\ &\quad - (27y^3 - 54y^2 - 9y + 24)x - 18y^3 - 36y^2 + 30y + 16\big\} = 0. \end{aligned}$$

By noting $x \geqslant 3$ and $y \geqslant 0$, we obtain

$$\begin{aligned} f(x,y) := x^4 + (6-9y)x^3 + (27y^2 - 30y + 1)x^2 - (27y^3 - 54y^2 - 9y + 24)x \\ -18y^3 - 36y^2 + 30y + 16 = 0. \end{aligned} \tag{19}$$

Suppose $f_y(x,y)=0$. Since

$$\begin{aligned} f_y(x,y) &= -9x^3+6x^2(9y-5)+x(-81y^2+108y+9)-54y^2-72y+30 \\ &= -3(x-3y+1)(3x^2-9xy+7x-6y-10), \end{aligned}$$

we have

$$y=\frac{x+1}{3} \quad \text{or} \quad y=\frac{-10+7x+3x^2}{3(2+3x)} \text{ and } x\neq -\frac{2}{3}. \tag{20}$$

Substituting (20) into

$$f_x(x,y)=4x^3-9x^2(3y-2)+x(54y^2-60y+2)-27y^3+54y^2+9y-24=0,$$

we obtain $(x,y)=(2,1)$ or $(1,0)$. Since $f(2,1)=0$ and $f(1,0)=0$, these are the only singular points of the curve C defined by (19). It is not entirely obvious but shown that $(2,1)$ or $(1,0)$ are ordinary multiple points of multiplicity 2, and therefore C has genus one. By Siegel's theorem [23], there exist finitely many pairs (n,s) for which (17) holds. ☐

Remark 4.5. By using Mathematica, we find that a weighted 7-design of type (15) exists if $(n,a,s)=(16,2,8),(23,2,11)$.

In Section 7, we characterize $v_{2,8}^{B_{16}}$ and $v_{2,11}^{B_{23}}$ in terms of shells of integral lattices.

Contrary to the situation of 7-designs, there do not exist 9-designs, as shown in the following result.

Proposition 4.6. *Let $n\geqslant 3$. A weighted 9-design of type (15) does not exist.*

Proof. By Remark 4.2, we may assume $s\geqslant 1$. If a weighted 9-design of type (15) exists, then by Theorem 2.13, Corollary 2.15 and Lemma 2.17, we have $\tilde{f}_4(v_{a,s})=\tilde{f}_6(v_{a,s})=\tilde{f}_{8,1}(v_{a,s})=0$. By computing a Groebner basis for the ideal $\langle \tilde{f}_4,\tilde{f}_6,\tilde{f}_{8,1}\rangle$, we have

$$-277504-32752n-1412n^2-200n^3+3n^4=0.$$

However this equation has no integer solutions. ☐

5. Designs with two proper orbits

Throughout this section we assume that $n\geqslant 3$. Let $a_1,a_2>0$, and let s_1,s_2 be integers with $0\leqslant s_i\leqslant n-1$. We consider a weighted design of type

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f(y)d\rho = W_1\sum_{x\in v_{a_1,s_1}^{B_n}} f(x)+W_2\sum_{x\in v_{a_2,s_2}^{B_n}} f(x) \quad \textit{for every } f\in\mathcal{P}_t(\mathbb{S}^{n-1}). \tag{21}$$

Let

$$\begin{cases} G_{4,6}(a_1,a_2,s_1,s_2) &= f_4(v_{a_1,s_1})f_6(v_{a_2,s_2})-f_6(v_{a_1,s_1})f_4(v_{a_2,s_2}),\\ G_{4,8,i}(a_1,a_2,s_1,s_2) &= f_4(v_{a_1,s_1})f_{8,i}(v_{a_2,s_2})-f_{8,i}(v_{a_1,s_1})f_4(v_{a_2,s_2}), \quad i=1,2,\\ G_{6,8,i}(a_1,a_2,s_1,s_2) &= f_6(v_{a_1,s_1})f_{8,i}(v_{a_2,s_2})-f_{8,i}(v_{a_1,s_1})f_6(v_{a_2,s_2}), \quad i=1,2,\\ G_{8,1,8,2}(a_1,a_2,s_1,s_2) &= f_{8,1}(v_{a_1,s_1})f_{8,2}(v_{a_2,s_2})-f_{8,2}(v_{a_1,s_1})f_{8,1}(v_{a_2,s_2}). \end{cases}$$

Remark 5.1. By substituting $a_1=a_2=1$, $s_1=k_1-1$ and $s_2=k_2-1$ into $G_{4,6}(a_1,a_2,s_1,s_2)$, it can be confirmed that this function coincides, up to a constant multiple, with the function $G(k_1,k_2)$ defined in Bajnok [2, p.390].

5.1. Characterization of 7-designs. In this subsection, we show a characterization theorem for 7-designs with two orbits.

Proposition 5.2. *Let $n \geqslant 3$. A weighted 7-design of type (21) exists if and only if one of the following cases (i)-(iii) holds:*

(i)

$$f_4(v_{a_1,s_1})f_4(v_{a_2,s_2}) < 0\ ,\ G_{4,6}(a_1,a_2,s_1,s_2) = 0;$$

(ii)

$$f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = 0\ ,\ f_6(v_{a_1,s_1})f_6(v_{a_2,s_2}) < 0;$$

(iii)

$$f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = f_6(v_{a_1,s_1}) = f_6(v_{a_2,s_2}) = 0. \tag{22}$$

Proof. We solve the following system of liner equations

$$\begin{bmatrix} 1 & 1 \\ f_4(v_{a_1,s_1}) & f_4(v_{a_2,s_2}) \\ f_6(v_{a_1,s_1}) & f_6(v_{a_2,s_2}) \end{bmatrix} \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{23}$$

where $\tilde{W}_i = W_i|v_{a_i,s_i}^{B_n}| > 0,\ (i = 1,2)$. We divide the situation into three cases: (a) $f_4(v_{a_1,s_1}) \neq f_4(v_{a_2,s_2})$, (b) $f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}),\ f_6(v_{a_1,s_1}) \neq f_6(v_{a_2,s_2})$ and (c) $f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}),\ f_6(v_{a_1,s_1}) = f_6(v_{a_2,s_2})$.

We first consider Case (a). Applying the row reduction in the standard linear algebra to the augmented coefficient matrix in (23), we have

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 \\ f_4(v_{a_1,s_1}) & f_4(v_{a_2,s_2}) & 0 \\ f_6(v_{a_1,s_1}) & f_6(v_{a_2,s_2}) & 0 \end{bmatrix} &\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & f_4(v_{a_2,s_2}) - f_4(v_{a_1,s_1}) & -f_4(v_{a_1,s_1}) \\ 0 & f_6(v_{a_2,s_2}) - f_6(v_{a_1,s_1}) & -f_6(v_{a_1,s_1}) \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 0 & \frac{f_4(v_{a_2,s_2})}{f_4(v_{a_2,s_2}) - f_4(v_{a_1,s_1})} \\ 0 & 1 & -\frac{f_4(v_{a_1,s_1})}{f_4(v_{a_2,s_2}) - f_4(v_{a_1,s_1})} \\ 0 & 0 & \frac{G_{4,6}(a_1,a_2,s_1,s_2)}{f_4(v_{a_2,s_2}) - f_4(v_{a_1,s_1})} \end{bmatrix}. \end{aligned}$$

Thus in this case, a weighted 7-design of type (21) exists if and only if

$$\begin{cases} f_4(v_{a_1,s_1}) \neq f_4(v_{a_2,s_2}), \\ \frac{f_4(v_{a_2,s_2})}{f_4(v_{a_2,s_2}) - f_4(v_{a_1,s_1})} > 0, \\ -\frac{f_4(v_{a_1,s_1})}{f_4(v_{a_2,s_2}) - f_4(v_{a_1,s_1})} > 0, \\ G_{4,6}(a_1,a_2,s_1,s_2) = 0. \end{cases}$$

This is equivalent to

$$\begin{cases} f_4(v_{a_1,s_1})f_4(v_{a_2,s_2}) < 0, \\ G_{4,6}(a_1,a_2,s_1,s_2) = 0. \end{cases} \tag{24}$$

Next we consider Case (b). Applying the row reduction arguments again, the augmented coefficient matrix can be reduced to

$$\begin{bmatrix} 1 & 0 & \frac{f_6(v_{a_2,s_2})}{f_6(v_{a_2,s_2}) - f_6(v_{a_1,s_1})} \\ 0 & 1 & -\frac{f_6(v_{a_1,s_1})}{f_6(v_{a_2,s_2}) - f_6(v_{a_1,s_1})} \\ 0 & 0 & -f_4(v_{a_1,s_1}) \end{bmatrix}.$$

Thus in this case, a weighted 7-design of type (21) exists if and only if

$$\begin{cases} f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = 0, \\ f_6(v_{a_2,s_2}) \neq f_6(v_{a_1,s_1}), \\ \frac{f_6(v_{a_2,s_2})}{f_6(v_{a_2,s_2})-f_6(v_{a_1,s_1})} > 0, \\ -\frac{f_6(v_{a_1,s_1})}{f_6(v_{a_2,s_2})-f_6(v_{a_1,s_1})} > 0. \end{cases}$$

This is equivalent to

$$\begin{cases} f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = 0, \\ f_6(v_{a_1,s_1}) f_6(v_{a_2,s_2}) < 0. \end{cases} \tag{25}$$

Finally, we consider Case (c). Applying the same arguments again, the augmented coefficient matrix can be reduced to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -f_4(v_{a_1,s_1}) \\ 0 & 0 & -f_6(v_{a_1,s_1}) \end{bmatrix}.$$

Thus, in Case (c), a weighted 7-design of type (21) exists if and only if

$$f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = f_6(v_{a_1,s_1}) = f_6(v_{a_2,s_2}) = 0. \tag{26}$$

By summarizing (24)-(26), we obtain all the desired conditions. □

Theorem 5.3. *Let $n \geqslant 3$. Case (iii) of Proposition 5.2 does not occur if $s_1 \neq s_2$.*

We prove Theorem 5.3 in Section 5.3.

Remark 5.4. We list $\tilde{W}_1, \tilde{W}_2$ of Cases (i), (ii) listed in Table 2

Table 2. Weights of 7-design

	$\tilde{W}_1$	$\tilde{W}_2$
Case (i)	$\frac{f_4(v_{a_2,s_2})}{f_4(v_{a_2,s_2})-f_4(v_{a_1,s_1})}$	$-\frac{f_4(v_{a_1,s_1})}{f_4(v_{a_2,s_2})-f_4(v_{a_1,s_1})}$
Case (ii)	$\frac{f_6(v_{a_2,s_2})}{f_6(v_{a_2,s_2})-f_6(v_{a_1,s_1})}$	$-\frac{f_6(v_{a_1,s_1})}{f_6(v_{a_2,s_2})-f_6(v_{a_1,s_1})}$

5.2. Characterization of 9-designs.

We start with a characterization theorem for 9-designs with two orbits.

Proposition 5.5. *Let $n \geqslant 3$. A weighted 9-design of type (21) exists if and only if one of the following cases (i)-(v) holds:*

(i)

$$f_4(v_{a_1,s_1}) f_4(v_{a_2,s_2}) < 0,$$
$$G_{4,6}(a_1,a_2,s_1,s_2) = G_{4,8,1}(a_1,a_2,s_1,s_2) = G_{4,8,2}(a_1,a_2,s_1,s_2) = 0.$$

(ii)

$$f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = 0, \ f_6(v_{a_1,s_1}) f_6(v_{a_2,s_2}) < 0,$$
$$G_{6,8,1}(a_1,a_2,s_1,s_2) = G_{6,8,2}(a_1,a_2,s_1,s_2) = 0.$$

(iii)

$$f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = f_6(v_{a_1,s_1}) = f_6(v_{a_2,s_2}) = 0,$$
$$f_{8,1}(v_{a_1,s_1}) f_{8,1}(v_{a_2,s_2}) < 0 \ , \ G_{8,1,8,2}(a_1,a_2,s_1,s_2) = 0.$$

(iv)

$$f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = f_6(v_{a_1,s_1}) = f_6(v_{a_2,s_2}) = f_{8,1}(v_{a_1,s_1}) = f_{8,1}(v_{a_2,s_2}) = 0,$$
$$f_{8,2}(v_{a_1,s_1})f_{8,2}(v_{a_2,s_2}) < 0.$$

(v)

$$f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = f_6(v_{a_1,s_1}) = f_6(v_{a_2,s_2}) = 0,$$
$$f_{8,1}(v_{a_1,s_1}) = f_{8,1}(v_{a_2,s_2}) = f_{8,2}(v_{a_1,s_1}) = f_{8,2}(v_{a_2,s_2}) = 0.$$

Remark 5.6. When $n = 3$, we may ignore all conditions involving $f_{8,2}$. For example, both Case (iv) and Case (v) can be reduced to

$$f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = f_6(v_{a_1,s_1}) = f_6(v_{a_2,s_2}) = 0, \quad f_{8,1}(v_{a_1,s_1}) = f_{8,1}(v_{a_2,s_2}) = 0.$$

Theorem 5.7. *Cases (iii), (iv) and (v) of Proposition* 5.5 *do not occur.*

We prove Theorem 5.7 in Section 5.3, where a proof of Proposition 5.5 is given in the Appendix.

We close this subsection by discussing weighted 9-designs of type $(v^{B_3}_{a_1,s_1} \cup v^{B_3}_{a_2,s_2}, \{\tilde{W}_1, \tilde{W}_2\})$. Such 9-designs can be classified by two examples listed in Table 3, where numerical parameters are considered with six significant digits. (Throughout this paper, numerical results are treated with the same level of precision.) What is remarkable here is that all these examples belong to Case (ii), which implies that Case (i) may not occur even for $n \geqslant 4$.

Table 3. Parameters of weighted 9-design

s_1	s_2	a_1	a_2	W_1	W_2
1	2	0.396751	0.470350	0.0220088	0.0196579
2	2	0.389041	3.89103	0.0193973	0.0222694

The orbit $v^{B_3}_{a,1}$ can be understood as the vertices of a convex polyhedron obtained by uniformly removing the 6 square pyramid parts from a regular octahedron, or by uniformly *truncating* the 6 vertices of a regular octahedron in such a way that each edge of the octahedron is divided into the ratio $a : 1 - a : a$ if $0 < a < 1$, and $1 : a - 1 : 1$ if $1 < a$. The values of a_1, a_2, and W_1, appearing in Case 1 (see Figure 2), satisfy the equations

$$\begin{cases} 17 - 92A_2 + 78A_2^2 - 44A_2^3 + 5A_2^4 = 0, \\ 24 - 167A_1 + 24A_1^2 + 57A_1A_2 - 39A_1A_2^2 + 5A_1A_2^3 = 0, \\ -845 - 522A_2 + 369A_2^2 - 40A_2^3 + 1785W_1 = 0, \end{cases}$$

where $A_1 = a_1^2$ and $A_2 = a_2^2$.

The orbit $v^{B_3}_{a,2}$ can be characterized in terms of vertex-truncation and edge-truncation. When $0 < a < 1$, the orbit $v^{B_3}_{a,2}$ can be obtained by uniformly removing the 8 rectangular triangular pyramids from a cube in such a way that each edge of the cube is divided into the ratio $1 - a : 2a : 1 - a$. If $1 < a$, the orbit $v^{B_3}_{a,2}$ can be obtained by uniformly cutting off the 8 right-angle isosceles prisms including the edges of a cube in such a way that each edge of the cube is divided into the ratio $a - 1 : 2 : a - 1$. The values of a_1, a_2, and W_1, appearing in Case 2 (see Figure 3),

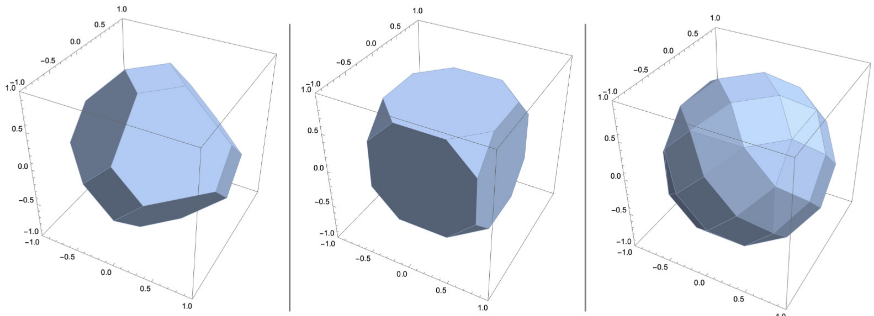

FIGURE 2. $v^{B_3}_{a_1,1}$(left), $v^{B_3}_{a_2,2}$(middle), $v^{B_3}_{a_1,1} \cup v^{B_3}_{a_2,2}$(right)

satisfy the equations

$$\begin{cases} -19 + 116A_2 + 66A_2^2 - 20A_2^3 + A_2^4 = 0, \\ -175 + 12A_1 - 47A_2 + 19A_2^2 - A_2^3 = 0, \\ -3956 + 2291A_2 - 349A_2^2 + 13A_2^3 + 7770W_1 = 0, \end{cases}$$

where $A_1 = a_1^2$ and $A_2 = a_2^2$.

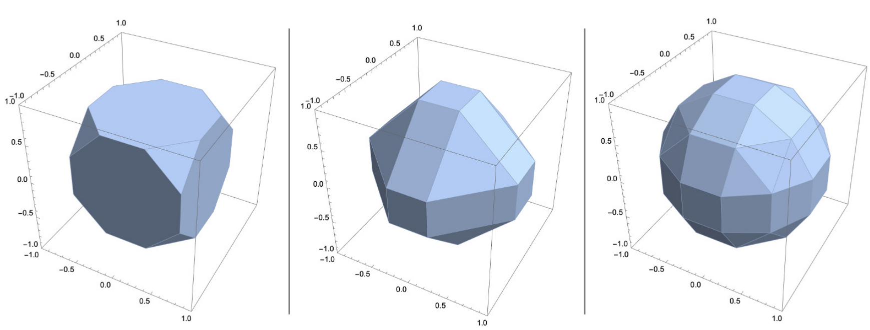

FIGURE 3. $v^{B_3}_{a_1,2}$(left), $v^{B_3}_{a_2,2}$(middle), $v^{B_3}_{a_1,2} \cup v^{B_3}_{a_2,2}$(right)

Note that all 24-point configurations consisting of each single proper orbit are 3-designs.

5.3. PROOFS OF THEOREMS 5.3 AND 5.7.

Proof of Theorem 5.3. Suppose that $s_1 \neq s_2$. By the similar argument that used in Proposition 4.1, it holds that

$$\begin{aligned} h(n, s_i) &= n^3 + (2 - 9s_i)n^2 + (12s_i^2 - 9s_i - 7)n + 6s_i^2 + 18s_i + 4 \\ &\quad \pm (n^3 - 3(s_i - 2) - 3s_i - 7)\sqrt{(n+2)s_i(1 - n + 3s_i)} = 0, \quad i = 1, 2. \end{aligned}$$

Since $(n^3 + (2 - 9s_i)n^2 + (12s_i^2 - 9s_i - 7)n + 6s_i^2 + 18s_i + 4)^2 = ((n^3 - 3(s_i - 2) - 3s_i - 7)\sqrt{(n+2)s_i(1 - n + 3s_i)})^2$ for each i, by noting $n \geqslant 3$ and $s_i \geqslant 0$, we obtain

$$\begin{aligned} 0 = n^4 &+ (6 - 9s_i)n^3 + (1 - 30s_i + 27s_i^2)n^2 - (24 - 9s_i - 54s_i^2 + 27s_i^3)n \\ &- 18s_i^3 - 36s_i^2 + 30s_i + 16, \quad i = 1, 2. \end{aligned}$$

By subtracting them and then dividing it by $3(s_2 - s_1)$, we obtain the Diophantine equation

$$0 = 3n^3 + (10 - 9s_1 - 9s_2)n^2 - 3(1 + 6s_1 + 6s_2 - 3s_1^2 - 3s_1s_2 - 3s_2^2)n \\ -10 + 12s_1 + 12s_2 + 6s_1^2 + 6s_1s_2 + 6s_2^2. \tag{27}$$

By using Mathematica, the positive integer solutions of (27) are $(n, s_1, s_2) = (8, 3, 3)$, $(5, 2, 2)$, $(2, 1, 1)$, each of which does not satisfy the restriction $s_1 \neq s_2$. □

Next we prove Theorem 5.7 by focusing on Cases (iii), (iv), and (v) of Proposition 5.5.

We first consider the case where $s_1 \neq s_2$. In this case, Theorem 5.3 implies the nonexistence of a weighted 9-design. Thus, our interest goes to the case where $s_1 = s_2$.

To prove Theorem 5.7, we prepare the following two lemmas.

Lemma 5.8. *Let $n \geqslant 3$, $a_1 \neq a_2$ and $s_1 = s_2$ (say s). Suppose $f_4(v_{a_1,s}) = f_4(v_{a_2,s}) = f_6(v_{a_1,s}) = f_6(v_{a_2,s}) = 0$. Then*

$$\frac{5n^2 + 15n - 20}{12n + 6} < s \leqslant \frac{5n^2 + 15n - 20}{9n + 12}.$$

Proof. Suppose (22) with $a_1 \neq a_2$ and $s = s_1 = s_2$. Since $\tilde{f}_4(v_{a_1,s}) - \tilde{f}_4(v_{a_2,s}) = 0$, we have

$$a_1^2 + a_2^2 = \frac{6s}{n-1}. \tag{28}$$

Since $\tilde{f}_6(v_{a_1,s}) - \tilde{f}_6(v_{a_2,s}) = 0$, we have

$$(a_1^4 + a_1^2a_2^2 + a_2^4)(-2+n)(-1+n) - 60s - 15(a_1^2 + a_2^2)(-2+n)s - 15ns + 90s^2 = 0. \tag{29}$$

Substituting (28) into (29), we have

$$a_1^2a_2^2 = \frac{3s\{20 - 5n^2 + 6s + 3n(-5 + 4s)\}}{(-2+n)(-1+n)^2}. \tag{30}$$

Since $a_1^2a_2^2 > 0$, we obtain

$$\frac{5n^2 + 15n - 20}{12n + 6} < s \leqslant n - 1. \tag{31}$$

Meanwhile, solving for a_1^2 and a_2^2 from (28) and (30), we know that a_1^2 (or a_2^2) is equal to

$$\frac{3s}{n-1} \pm \frac{\sqrt{3(-2+n)s\{5(-1+n)(4+n) - 3(4+3n)s\}}}{(-2+n)(-1+n)}.$$

Since a_1^2 and a_2^2 are positive reals, by noting $0 \leqslant s \leqslant n-1$, we have

$$s\big(5n^2 + n(15 - 9s) - 4(5 + 3s)\big) \geqslant 0 \quad \text{and} \quad 0 \leqslant s \leqslant n - 1.$$

Then we obtain

$$0 \leqslant s \leqslant \frac{5n^2 + 15n - 20}{9n + 12}. \tag{32}$$

Thus, by combining (31) and (32), we obtain the desired result. □

Lemma 5.9. *Assume $a_1 \neq a_2$. Then the following hold:*

(i) *If $n \geqslant 4$ and $f_4(v_{a_1,s}) = f_4(v_{a_2,s}) = f_{8,1}(v_{a_1,s}) = f_{8,1}(v_{a_2,s}) = f_{8,2}(v_{a_1,s}) = f_{8,2}(v_{a_2,s}) = 0$, then*

$$n = 6s - 3.$$

(ii) *If* $n = 3$ *and* $f_4(v_{a_1,s}) = f_4(v_{a_2,s}) = f_{8,1}(v_{a_1,s}) = f_{8,1}(v_{a_2,s}) = 0$, *then*

$$a_1 = \sqrt{\frac{3 \mp \sqrt{5}}{2}},\ a_2 = \frac{\pm 1 + \sqrt{5}}{2},\ s = 1. \tag{33}$$

Proof. Suppose $n \geqslant 4$. By recalling (28), $\tilde{f}_4(v_{a_1,s}) - \tilde{f}_4(v_{a_2,s}) = 0$ and $\tilde{f}_{8,i}(v_{a_1,s}) - \tilde{f}_{8,i}(v_{a_2,s}) = 0$ $(i = 1, 2)$, we have

$$\begin{aligned}
g_1(a_1, a_2, s) &= -(a_1^2 + a_2^2) + (a_1^2 + a_2^2)n - 6s = 0,\\
g_2(a_1, a_2, s) &= -a_1^6 - a_1^4 a_2^2 - a_1^2 a_2^4 - a_2^6 + a_1^6 n + a_1^4 a_2^2 n\\
&\quad + a_1^2 a_2^4 n + a_2^6 n - 28s + 70a_1^2 s - 28a_1^4 s + 70a_2^2 s - 28a_1^2 a_2^2 s - 28a_2^4 s = 0,\\
g_3(a_1, a_2, s) &= -18 + 3a_1^2 + 3a_2^2 - 6n + 2a_1^2 n + 2a_2^2 n - a_1^2 n^2\\
&\quad - a_2^2 n^2 + 36s - 9a_1^2 s - 9a_2^2 s + 6ns + 3a_1^2 ns + 3a_2^2 ns - 18s^2 = 0.
\end{aligned}$$

By computing a Groebner basis $\mathcal{G}$ for the ideal $\langle g_1, g_2, g_3 \rangle$, we can check that

$$\{(3 + n - 6s)(-1 + n - s),\ (5 - 4a_2 + a_2^2)(5 + 4a_2 + a_2^2)(-1 + n)(3 + n - 6s)\} \subset \mathcal{G} \tag{34}$$

Thus it holds that

$$n \in \{6s - 3, s + 1\}.$$

By substituting $n = s + 1$ into (34), we have

$$(5 - 4a_2 + a_2^2)(5 + 4a_2 + a_2^2)(4 - 5s)s = 0.$$

This equation holds true only when $s = 0$, but this is a contradiction. Thus we obtain $n = 6s - 3$.

Next, suppose $n = 3$. By computing a Groebner basis $\mathcal{H}$ for the ideal $\langle g_1, g_2 \rangle$, we have

$$\begin{aligned}
h_1(a_1, a_2, s) &= s(14 + 8a_2^4 - 105s - 24a_2^2 s + 99s^2) = 0,\\
h_2(a_1, a_2, s) &= a_1^2 + a_2^2 - 3s = 0.
\end{aligned}$$

We solve $14 + 8a_2^4 - 105s - 24a_2^2 s + 99s^2 = 0$. This equation holds only when

$$a_2^2 = \frac{3s}{2} \pm \frac{\sqrt{-14 + 105s - 81s^2}}{2\sqrt{2}}, \quad \frac{35 - \sqrt{721}}{54} < s < \frac{35 + \sqrt{721}}{54}.$$

Therefore, we obtain $s = 1$ and

$$a_1 = \sqrt{\frac{3 \mp \sqrt{5}}{2}},\ a_2 = \frac{\pm 1 + \sqrt{5}}{2}.$$

□

We are ready to prove Theorem 5.7.

Proof of Theorem 5.7. Let $n \geqslant 3$, and suppose that there exists a weighted 9-design of type (21). By Theorem 5.3 we may consider the case where $s_1 = s_2$ (say s).

First, we consider Case (iii). Then

$$\begin{cases} f_4(v_{a_1,s}) = f_4(v_{a_2,s}) = f_6(v_{a_1,s}) = f_6(v_{a_2,s}) = 0,\\ f_{8,1}(v_{a_1,s}) f_{8,1}(v_{a_2,s}) < 0. \end{cases} \tag{35}$$

Then by Lemma 5.8, we have

$$\frac{5n^2 + 15n - 20}{12n + 6} < s \leqslant \frac{5n^2 + 15n - 20}{9n + 12}. \tag{36}$$

By using Mathematica, we find that n satisfying (35) and (36) is upper bounded by 691. However, for $n \leqslant 691$, we can also confirm that there exists no integer s such that

$$f_4(v_{a_1,s}) = f_6(v_{a_2,s}) = G_{8,1,8,2}(a_1, a_2, s, s) = 0$$

Next, we consider Cases (iv) and (v). Let $n \geqslant 4$. By Theorem 2.18 we may assume that $s > 0$. By combining (28), (29) and Lemma 5.9, we obtain

$$a_1^2 a_2^2 = -\frac{3s(-5 - 15s + 27s^2)}{(-2 + 3s)^2(-5 + 6s)} < 0,$$

which is a contradiction. Let $n = 3$. Substituting a_1, a_2, s as in (33) into (29), we have

$$(a_1^4 + a_1^2 a_2^2 + a_2^4)(-2+n)(-1+n) - 60s - 15(a_1^2 + a_2^2)(-2+n)s - 15ns + 90s^2 = -44 \neq 0,$$

which is again a contradiction. $\square$

6. Designs with more than two proper orbits on $\mathbb{S}^3$

In Theorem 3.1, we have obtained a uniform bound for the existence of designs of type (1), namely we have shown that if $n \geqslant 4$, and if there exists a t-design on $\mathbb{S}^{n-1}$ of type (1), then $t \leqslant 15$.

Schur's formula (7) on $\mathbb{S}^3$ has degree 11. Sawa and Xu [21] discusses a higher-dimensional extension of Schur's design, and discovers many examples of weighted 11-designs in dimensions 3 through 23. Note that all these designs contain only one proper orbit. Sawa and Xu [21] moreover establishes that designs of type (1) with only one proper orbit have degree at most 11.

Then what about designs with at least two proper orbits? The following is a weighted 11-design on $\mathbb{S}^3$ with 3 proper orbits:

- $(v_{a_1,2}^{B_4} \cup v_{a_2,3}^{B_4} \cup v_{a_3,3}^{B_4}, \{W_1, W_2, W_3\})$,

$$\begin{cases} a_1 = 0.470499, \quad a_2 = 1.17310, \quad a_3 = 3.78381, \\ W_1 = 0.00522948, \quad W_2 = 0.00192016, \quad W_3 = 0.00586062. \end{cases}$$

We could not have found 11-designs, and even 9-designs with 2 proper orbits.

The use of more than 3 proper orbits enables us to obtain 13-designs. Indeed, the following two are weighted 13-designs on $\mathbb{S}^3$ with 5 orbits:

- $(v_{1,3}^{B_4} \cup v_{a_2,1}^{B_4} \cup v_{a_3,2}^{B_4} \cup v_{a_4,3}^{B_4} \cup v_{a_5,3}^{B_4}, \{W_1, W_2, W_3, W_4, W_5\})$,

$$\begin{cases} a_2 = 0.444883, \quad a_3 = 0.509692, \quad a_4 = 9.68607, \quad a_5 = 2.53788, \\ W_1 = 0.00475002, \quad W_2 = 0.00407904, \quad W_3 = 0.00435065, \\ W_4 = 0.000536920, \quad W_5 = 0.00431532. \end{cases}$$

- $(v_{1,0}^{B_4} \cup v_{a_2,1}^{B_4} \cup v_{a_3,2}^{B_4} \cup v_{a_4,2}^{B_4} \cup v_{a_5,3}^{B_4}, \{W_1, W_2, W_3, W_4, W_5\})$,

$$\begin{cases} a_2 = 0.597599, \quad a_3 = 0.521840, \quad a_4 = 3.07756, \quad a_5 = 1.85216, \\ W_1 = 0.00262253, \quad W_2 = 0.00244207, \quad W_3 = 0.00370960, \\ W_4 = 0.00256322, \quad W_5 = 0.00405640. \end{cases}$$

At this point we could not have succeeded in finding a 15-design on $\mathbb{S}^3$ with more than 5 proper orbits, though there actually exist 15-designs with negative weights W_i (see for example Keast [12]). The following is a challenging open question, which is left for future work.

Problem 6.1. *Does there exist a weighted* 15*-design on* $\mathbb{S}^3$ *of type* (1) *with more than* 5 *proper orbits?*

The situation is quite different in dimensions 3 and 4. Indeed, we obtain weighted 9-designs with two proper orbits, as already seen in Section 5.2 (Table 3). As in the four-dimensional case, one can obtain weighted 11-designs on $\mathbb{S}^2$ with 3 proper orbits, e.g.

- $(v_{a_1,1}^{B_3} \cup v_{a_2,1}^{B_3} \cup v_{a_3,2}^{B_3}, \{W_1, W_2, W_3\})$,

$$\begin{cases} a_1 = 1.51610, \quad a_2 = 4.44671, \quad a_3 = 1.79726, \\ W_1 = 0.0147251, \quad W_2 = 0.00936568, \quad W_3 = 0.0175759. \end{cases}$$

A remarkable gap between the three and four dimensional cases is that there exists a weighted 17-design on $\mathbb{S}^2$ of type (1). Indeed, the following example with 6 proper orbits can be found in Table 2.1 of Heo-Xu [10, p.275]:

- $(v_{1,0}^{B_3} \cup v_{1,2}^{B_3} \cup v_{a_3,1}^{B_3} \cup v_{a_4,2}^{B_3} \cup v_{a_5,2}^{B_3} \cup v_{a_6,2}^{B_3}, \{W_1, W_2, W_3, W_4, W_5, W_6\})$,

$$\begin{cases} a_3 = 0.544741, \quad a_4 = 5.21363, \quad a_5 = 2.0945, \quad a_6 = 0.312791, \\ W_1 = 0.00382827, \quad W_2 = 0.00979374, \quad W_3 = 0.009695, \\ W_4 = 0.00821174, \quad W_5 = 0.00959547, \quad W_6 = 0.00994281. \end{cases}$$

7. Conclusion and further remarks

In this paper we have explored a full generalization of the classical corner-vector method for constructing weighted spherical designs, and have extensively studied the existence of designs of type (1). We have first established a uniform upper bound for the degree of such designs (Theorem 3.1). Our proof is a hybrid argument combining the cross-ratio comparison technique for Hilbert identity and Xu's theorem on the interrelation between spherical designs and simplicial designs, which appears to be useful in the study of designs of a different type than (1). Moreover, we have made a detailed observation about the existence of 7-designs with one proper orbit, 9-designs with two proper orbits, and 11- and 13-designs with more than two proper orbits.

In the rest of this paper we explore the connections between integral lattices and some of our designs. Since the pioneering paper by Venkov [25], there have been numerous publications on the construction that derives spherical designs from shells of integral lattices. Such *lattice-based constructions* are also significant for coding theorists as a spherical analogue of the Assmus-Mattson theorem that directly relates blocks designs to linear codes. For a brief introduction to the connections among designs, codes and lattices, we refer the reader to Bannai and Bannai [3].

We shall make a brief explanation of the lattice-based construction, together with examples for D_4-root lattice. The *D_4-root lattice*, its dual lattice D_4^* and $\sqrt{2}$-normalization D_4' are defined by

$$D_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1 + x_2 + x_3 + x_4 \equiv 0 \pmod 2\},$$

$$D_4^* = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 \mid \sum_{i=1}^4 x_i y_i \in \mathbb{Z},\ (x_1, x_2, x_3, x_4) \in D_4\},$$

$$D_4' = \sqrt{2} D_4^*.$$

Given a set E of $\mathbb{R}^4$, the *shell of (square) norm m*, say E_m, is defined to be the intersection of E and the concentric sphere $\mathbb{S}_m^3$ of radius m. We also use this terminology for the n-dimensional case.

Example 7.1. Let $E = D_4 \cup D_4'$. Then the first two shells for D_4 are given by

$$(D_4)_2 = (1,1,0,0)^{B_4}, \quad (D_4)_6 = (2,1,1,0)^{B_4},$$

which can be represented in terms of generalized corner vectors as $(\sqrt{2}v_{1,1})^{B_4}$ and $(\sqrt{6}v_{2,2})^{B_4}$, respectively. Similarly, the first two shells of D_4' are given as

$$(D_4')_2 = (\sqrt{2},0,0,0)^{B_4} \cup (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})^{B_4} = (\sqrt{2}v_{1,0})^{B_4} \cup (\sqrt{2}v_{1,3})^{B_4},$$
$$(D_4')_6 = (\sqrt{2},\sqrt{2},\sqrt{2},0)^{B_4} \cup (\frac{3}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})^{B_4} = (\sqrt{6}v_{1,2})^{B_4} \cup (\sqrt{6}v_{3,3})^{B_4}.$$

The following theorem is proved in de la Harpe et al. [6].

Theorem 7.2 (Theorem D_4, [6]). *With the notation given above, the following holds:*
 (i) *Any shell E_{2m} is an (equi-weighted) spherical 7-design.*
 (ii) *The union of $\frac{1}{\sqrt{2}}E_2$ and $\frac{1}{\sqrt{6}}E_6$ is a weighted spherical 11-design.*

Remark 7.3. Theorem 7.2 (i) is a corollary of Bajnok's theorem (Theorem 2.18), and Theorem 7.2 (ii) is just Schur's formula described in Example 2.5.

We now clarify the connection between our 7-design $v_{2,8}^{B_{16}}$ (see Remark 4.5) and a certain integral lattice called the *Barnes-Wall lattice* in $\mathbb{R}^{16}$. The Barnes-Wall lattice in $\mathbb{R}^2$ is the standard $\mathbb{Z}^2$-lattice, and the Barnes-Wall lattice in $\mathbb{R}^4$ is just the D_4-root lattice. In general the Barnes-Wall lattice in $\mathbb{R}^{2^k}$ can be realized as the rational part of the lattice $M_1^{\otimes_k}$, where M_1 is the set of all $\mathbb{Z}[\sqrt{2}]$-integer combinations of $(\sqrt{2},0)$ and $(1,1)$ (see [16, Theorem 2.1] for the details). In particular the lattice in $\mathbb{R}^{16}$ is generated by the rows of the matrix

$$X = \begin{bmatrix}
4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
4&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
4&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0\\
2&2&2&2&0&0&0&0&0&0&0&0&0&0&0&0\\
4&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0\\
2&2&0&0&2&2&0&0&0&0&0&0&0&0&0&0\\
2&0&2&0&2&0&2&0&0&0&0&0&0&0&0&0\\
2&2&2&2&2&2&2&2&0&0&0&0&0&0&0&0\\
4&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0\\
2&2&0&0&0&0&0&0&2&2&0&0&0&0&0&0\\
2&0&2&0&0&0&0&0&2&0&2&0&0&0&0&0\\
2&2&2&2&0&0&0&0&2&2&2&2&0&0&0&0\\
2&0&0&0&2&0&0&0&2&0&0&0&2&0&0&0\\
2&2&0&0&2&2&0&0&2&2&0&0&2&2&0&0\\
2&0&2&0&2&0&2&0&2&0&2&0&2&0&2&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{bmatrix}.$$

Theorem 7.4. *The 7-design $(8\sqrt{3}v_{2,8})^{B_{16}}$ is a subset of some shell of the Barnes-Wall lattice in $\mathbb{R}^{16}$.*

Proof. We note that $4e_1,\dots,4e_{16}$, where $e_1,\dots,e_{16}$ are the standard basis vectors, are all integer combinations of the rows of the generator matrix X. The orbit $(8\sqrt{3}v_{2,8})^{B_{16}} = (8,\underbrace{4,\dots,4}_{8\text{ times}},0,\dots,0)^{B_{16}}$ is thus included in some shell of the lattice. □

Next we come to the connection between our 7-design $v_{2,11}^{B_{23}}$ (see Remark 4.5) and a certain integral lattice in $\mathbb{R}^{23}$. The *Leech lattice* Λ_{24} is an even unimodular lattice, which often appears to be the densest sphere packing in $\mathbb{R}^{24}$. As briefly explained in [5, Chapter 24], the lattice Λ_{24} has an explicit expression as

$$\Lambda_{24} = \Big\{ \frac{1}{\sqrt{8}}(2\boldsymbol{c} + 4\boldsymbol{x}) \mid \boldsymbol{c} \in G_{24} \pmod 2,\ \boldsymbol{x} \in \mathbb{Z}^{24} \text{ with } \sum_{i=1}^{24} x_i \equiv 0 \pmod 2 \Big\}$$

$$\cup \Big\{ \frac{1}{\sqrt{8}}(\mathbf{1} + 2\boldsymbol{c} + 4\boldsymbol{x}) \mid \boldsymbol{c} \in G_{24} \pmod 2,\ \boldsymbol{x} \in \mathbb{Z}^{24} \text{ with } \sum_{i=1}^{24} x_i \equiv 0 \pmod 2 \Big\}$$

where $\mathbf{1}$ is the 24-dimensional all-one vector and G_{24} is the set of all 24-dimensional vectors, considered as vectors in $\mathbb{R}^{24}$, of the *extended Golay codes* over $\mathbb{F}_2$; see, for example, van Lint and Wilson [24, § 20] for the definition of the (extended) Golay code over $\mathbb{F}_2$. The minimal shell consists of 1104 points of type $(\pm 4^2, 0^{22})$, 97152 points of type $(\pm 2^8, 0^{16})$, 98304 points of type $(\mp 3^1, \pm 1^{23})$, and totally 196560 points.

The *shorter Leech lattice*, O_{23}, is the unique odd (up to isometry) unimodular lattice with minimal norm 3 in $\mathbb{R}^{23}$ [5, Chapter 19]. Given a minimal vector $v \in \Lambda_{24}$ of norm 4, the lattice O_{23} can also be identified as the orthogonal projection of the set of points in Λ_{24} that have an even inner product with v onto

$$v^{\perp} = \{u \in \mathbb{R}^{24} \mid \sum_{i=1}^{24} u_i v_i = 0\};$$

see for example [5, p.179].

Consider a hyperplane H given by

$$H := \{(x_1, \ldots, x_{23}, 0) \in \mathbb{R}^{24} \mid x_1, \ldots, x_{23} \in \mathbb{R}\}.$$

Then, a linear transformation $\Upsilon : H \to \mathbb{R}^{24}$ is defined as follows:
(37)

$$\Upsilon((x_1, \ldots, x_{23}, 0)) := \left(\frac{x_1}{\sqrt{2}}, \frac{x_1}{\sqrt{2}}, \frac{x_2 - x_3}{\sqrt{2}}, \frac{x_2 + x_3}{\sqrt{2}}, \ldots, \frac{x_{22} - x_{23}}{\sqrt{2}}, \frac{x_{22} + x_{23}}{\sqrt{2}} \right) \in \mathbb{R}^{24}.$$

Since it can be easily verified that Υ preserves inner products, it follows that Υ is an orthogonal transformation.

Theorem 7.5. *With the orthogonal transformation Υ, the point set $\{\Upsilon((\boldsymbol{y}, 0)) \mid \boldsymbol{y} \in (4\sqrt{15}v_{2,11})^{B_{23}}\}$ is a subset of some shell of the shorter Leech lattice O_{23}.*

Proof. For $v = \frac{1}{\sqrt{8}}(4, -4, 0, \ldots, 0) \in \Lambda_{24}$ with $\|v\|^2 = 4$, we set

$$v^{\perp} = \{\boldsymbol{x} = (x_1, \ldots, x_{24}) \in \Lambda_{24} \mid x_1 - x_2 = 0\}.$$

The orthogonal projection onto $v^{\perp}$ is defined by

$$\pi_{v^{\perp}}(\boldsymbol{t}) := \boldsymbol{t} - \frac{\langle \boldsymbol{t}, v\rangle}{\|v\|^2} v = \boldsymbol{t} - \frac{t_1 - t_2}{\sqrt{8}} v$$

for $\boldsymbol{t} = (t_1, \ldots, t_{24}) \in \mathbb{R}^{24}$. By the definition of the shorter Leech lattice O_{23}, we have

$$\begin{aligned} O_{23} &= \{\pi_{v^{\perp}}(\boldsymbol{x}) \mid \boldsymbol{x} \in \Lambda_{24},\ \langle \boldsymbol{x}, v\rangle \in 2\mathbb{Z}\} \\ &\supset \{\pi_{v^{\perp}}(\boldsymbol{x}) \mid \boldsymbol{x} = (x_1, \ldots, x_{24}) \in \Lambda_{24},\ x_1 = x_2\} \\ &= \{(x_1, x_1, x_3, \ldots, x_{24}) \in \Lambda_{24}\}. \end{aligned}$$

Our goal is to show that the point set $\{(\boldsymbol{y}, 0) \mid \boldsymbol{y} \in (4\sqrt{15}v_{2,11})^{B_{23}}\} \subset H$ is transformed under the orthogonal transformation Υ into $\{(x_1, x_1, x_3, \ldots, x_{24}) \in \Lambda_{24}\}$ ($\subset$ O_{23}).

By noting

$$(4\sqrt{15}v_{2,11})^{B_{23}} = \frac{1}{\sqrt{8}}\big(16\sqrt{2}, \underbrace{8\sqrt{2}, \ldots, 8\sqrt{2}}_{11 \text{ times}}, 0, \ldots, 0\big)^{B_{23}}$$

and letting $(y_1,\dots,y_{24}) \in \{(\boldsymbol{y},0) \mid \boldsymbol{y} \in (4\sqrt{15}v_{2,11})^{B_{23}}\}$, we observe from equation (37) that each of $y_i/\sqrt{2}$ and $(y_i \pm y_j)/\sqrt{2}$ belongs to the set $\frac{1}{\sqrt{8}}\{0,\pm 8,\pm 16,\pm 24\}$. Hence, each resulting vector

$$\boldsymbol{z} \in \{\Upsilon((\boldsymbol{y},0)) \mid \boldsymbol{y} \in (4\sqrt{15}v_{2,11})^{B_{23}}\}$$

lies in $\frac{1}{\sqrt{8}}\mathbb{Z}^{24}$, and in fact we can write $\boldsymbol{z} = 4\boldsymbol{z}'/\sqrt{8}$ for some $\boldsymbol{z}' = (z_1', z_1', z_3', \dots, z_{24}') \in \mathbb{Z}^{24}$, with all $z_i' \in \{0,\pm 2,\pm 4,\pm 6\}$.

Since all entries of $\boldsymbol{z}'$ are even integers, their sum is also even, and thus $\boldsymbol{z}$ satisfies the parity condition required for membership in Λ_{24}. Therefore, we conclude that $\boldsymbol{z} \in \Lambda_{24}$, which completes the proof. $\square$

Appendix A. Proof of Proposition 5.5

In this section, we give a proof of Proposition 5.5. Suppose $n \geqslant 4$. We solve the following system of linear equations

$$\begin{bmatrix} 1 & 1 \\ f_4(v_{a_1,s_1}) & f_4(v_{a_2,s_2}) \\ f_6(v_{a_1,s_1}) & f_6(v_{a_2,s_2}) \\ f_{8,1}(v_{a_1,s_1}) & f_{8,1}(v_{a_2,s_2}) \\ f_{8,2}(v_{a_1,s_1}) & f_{8,2}(v_{a_2,s_2}) \end{bmatrix} \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $\tilde{W}_i = W_i |v_{a_i,s_i}^{B_n}| > 0$ $(i=1,2)$. As in the proof of Proposition 5.2, we divide the situation into five cases:

$$\begin{cases} (a)\ f_4(v_{a_1,s_1}) \neq f_4(v_{a_2,s_2}), \\ (b)\ f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2})\ ,\ f_6(v_{a_1,s_1}) \neq f_6(v_{a_2,s_2}), \\ (c)\ f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2})\ ,\ f_6(v_{a_1,s_1}) = f_6(v_{a_2,s_2})\ ,\ f_{8,1}(v_{a_1,s_1}) \neq f_{8,1}(v_{a_2,s_2}), \\ (d)\ f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2})\ ,\ f_6(v_{a_1,s_1}) = f_6(v_{a_2,s_2}), \\ \qquad f_{8,1}(v_{a_1,s_1}) = f_{8,1}(v_{a_2,s_2})\ ,\ f_{8,2}(v_{a_1,s_1}) \neq f_{8,2}(v_{a_2,s_2}), \\ (e)\ f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2})\ ,\ f_6(v_{a_1,s_1}) = f_6(v_{a_2,s_2}), \\ \qquad f_{8,1}(v_{a_1,s_1}) = f_{8,1}(v_{a_2,s_2})\ ,\ f_{8,2}(v_{a_1,s_1}) = f_{8,2}(v_{a_2,s_2}). \end{cases}$$

In Case (a), the augmented coefficient matrix has rank 2 and can be reduced to

$$\begin{bmatrix} 1 & 0 & \frac{f_4(v_{a_2,s_2})}{f_4(v_{a_2,s_2})-f_4(v_{a_1,s_1})} \\ 0 & 1 & -\frac{f_4(v_{a_1,s_1})}{f_4(v_{a_2,s_2})-f_4(v_{a_1,s_1})} \\ 0 & 0 & \frac{G_{4,6}(a_1,a_2,s_1,s_2)}{f_4(v_{a_2,s_2})-f_4(v_{a,s})} \\ 0 & 0 & \frac{G_{4,8,1}(a_1,a_2,s_1,s_2)}{f_4(v_{a_2,s_2})-f_4(v_{a,s})} \\ 0 & 0 & \frac{G_{4,8,2}(a_1,a_2,s_1,s_2)}{f_4(v_{a_2,s_2})-f_4(v_{a,s})} \end{bmatrix}.$$

Thus, in Case (a), a weighted 9-design of type (21) exists if and only if

$$\begin{cases} \frac{f_4(v_{a_2,s_2})}{f_4(v_{a_2,s_2})-f_4(v_{a_1,s_1})} > 0, \\ -\frac{f_4(v_{a_1,s_1})}{f_4(v_{a_2,s_2})-f_4(v_{a_1,s_1})} > 0, \\ G_{4,6}(a_1,a_2,s_1,s_2) = G_{4,8,1}(a_1,a_2,s_1,s_2) = G_{4,8,2}(a_1,a_2,s_1,s_2) = 0. \end{cases}$$

Since $f_4(v_{a_1,s_1}) \neq f_4(v_{a_2,s_2})$, this is equivalent to

$$\begin{cases} f_4(v_{a_1,s_1}) f_4(v_{a_2,s_2}) < 0, \\ G_{4,6}(a_1,a_2,s_1,s_2) = G_{4,8,1}(a_1,a_2,s_1,s_2) = G_{4,8,2}(a_1,a_2,s_1,s_2) = 0. \end{cases}$$

Similarly, in each of the remaining three cases (b), (c), (d), the augmented coefficient matrix has rank 2 and obtain the desired equivalence case.

In the last case, Case (e), the augmented coefficient matrix can be reduced to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -f_4(v_{a_1,s_1}) \\ 0 & 0 & -f_6(v_{a_1,s_1}) \\ 0 & 0 & -f_{8,1}(v_{a_1,s_1}) \\ 0 & 0 & -f_{8,2}(v_{a_1,s_1}) \end{bmatrix}.$$

Thus, a weighted 9-design of type (21) exists if and only if

$$\begin{cases} f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = f_6(v_{a_1,s_1}) = f_6(v_{a_2,s_2}) = 0, \\ f_{8,1}(v_{a_1,s_1}) = f_{8,1}(v_{a_2,s_2}) = f_{8,2}(v_{a_1,s_1}) = f_{8,2}(v_{a_2,s_2}) = 0. \end{cases}$$

It remains to consider the case of $n = 3$. In this case we may ignore $f_{8,2}$ and so reduce the size of the augmented coefficient matrix to 4×3 from 5×3.

Acknowledgements. This is a subsequent work of Bajnok [2] and Sawa and Xu [21], which was first started by the first and fourth authors. They would like to thank Yuan Xu and Tsuyoshi Miezaki for many valuable comments and suggestions to the former version of this paper. We are grateful to the anonymous referees and the editor for their careful reading and constructive feedback, which have helped to improve the quality of the manuscript.

References

[1] Bela Bajnok, *Construction of spherical t-designs*, Geom. Dedicata **43** (1992), no. 2, 167–179.

[2] Béla Bajnok, *Orbits of the hyperoctahedral group as Euclidean designs*, J. Algebraic Combin. **25** (2007), no. 4, 375–397.

[3] Eiichi Bannai and Etsuko Bannai, *A survey on spherical designs and algebraic combinatorics on spheres*, European J. Combin. **30** (2009), no. 6, 1392–1425.

[4] Andriy Bondarenko, Danylo Radchenko, and Maryna Viazovska, *Optimal asymptotic bounds for spherical designs*, Ann. of Math. (2) **178** (2013), no. 2, 443–452.

[5] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, third ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 290, Springer-Verlag, New York, 1999.

[6] P. de la Harpe, C. Pache, and B. Venkov, *Construction of spherical cubature formulas using lattices*, St. Petersbg. Math. J. **18** (2007), no. 1, 119–139.

[7] P. Delsarte, J. M. Goethals, and J. J. Seidel, *Spherical codes and designs*, Geometriae Dedicata **6** (1977), no. 3, 363–388.

[8] Leonard Eugene Dickson, *History of the theory of numbers. Vol. II: Diophantine analysis*, Chelsea Publishing Co., New York, 1966.

[9] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi, *Higher transcendental functions. Vol. II*, Robert E. Krieger Publishing Co., Inc., Melbourne, FL, 1981.

[10] Sangwoo Heo and Yuan Xu, *Constructing fully symmetric cubature formulae for the sphere*, Math. Comp. **70** (2001), no. 233, 269–279.

[11] Yiming Hong, *On spherical t-designs in* $\mathbf{R}^2$, European J. Combin. **3** (1982), no. 3, 255–258.

[12] Patrick Keast, *Cubature formulas for the surface of the sphere*, J. Comput. Appl. Math. **17** (1987), no. 1-2, 151–172.

[13] J. Korevaar and J. L. H. Meyers, *Spherical Faraday cage for the case of equal point charges and Chebyshev-type quadrature on the sphere*, Integral Transform. Spec. Funct. **1** (1993), no. 2, 105–117.

[14] Yuri I. Lyubich and Leonid N. Vaserstein, *Isometric embeddings between classical Banach spaces, cubature formulas, and spherical designs*, Geom. Dedicata **47** (1993), no. 3, 327–362.

[15] Claus Müller, *Spherical harmonics*, Lecture Notes in Mathematics, vol. 17, Springer-Verlag, Berlin-New York, 1966.

[16] G. Nebe, E. M. Rains, and N. J. A. Sloane, *A simple construction for the Barnes-Wall lattices*, in Codes, graphs, and systems, Boston, MA: Kluwer Academic Publishers, 2002, pp. 333–342.

[17] H. Nozaki and M. Sawa, *Remarks on Hilbert identities, isometric embeddings, and invariant cubature*, Algebra i Analiz **25** (2013), no. 4, 139–181.

[18] Patrick Rabau and Bela Bajnok, *Bounds for the number of nodes in Chebyshev type quadrature formulas*, J. Approx. Theory **67** (1991), no. 2, 199–214.

[19] Masanori Sawa, Masatake Hirao, and Kanami Ito, *Geometric designs and rotatable designs I*, Graphs Combin. **37** (2021), no. 5, 1605–1651.

[20] Masanori Sawa, Masatake Hirao, and Sanpei Kageyama, *Euclidean design theory*, SpringerBriefs in Statistics, Springer, Singapore, 2019, JSS Research Series in Statistics.

[21] Masanori Sawa and Yuan Xu, *On positive cubature rules on the simplex and isometric embeddings*, Math. Comp. **83** (2014), no. 287, 1251–1277.

[22] J. J. Seidel, *Isometric embeddings and geometric designs*, Discrete Math. **136** (1994), no. 1, 281–293.

[23] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen.*, Abh. Preuß. Akad. Wiss., Phys.-Math. Kl. **1929** (1929), no. 1, 70 s.

[24] J. H. van Lint and R. M. Wilson, *A course in combinatorics*, second ed., Cambridge University Press, Cambridge, 2001.

[25] B. B. Venkov, *On even unimodular extremal lattices*, Proc. Steklov Inst. Math. **165** (1985), 47–52.

[26] Ziqing Xiang, *Explicit spherical designs*, Algebr. Comb. **5** (2022), no. 2, 347–369.

[27] Yuan Xu, *Orthogonal polynomials and cubature formulae on spheres and on simplices*, Methods Appl. Anal. **5** (1998), no. 2, 169–184.

[28] Hirotaka Yamamoto, Masatake Hirao, and Masanori Sawa, *A construction of the fourth order rotatable designs invariant under the hyperoctahedral group*, J. Statist. Plann. Inference **200** (2019), 63–73.

Kenji Tanino, Graduate School of System Informatics, Kobe University, 1-1 Riokkodai, Nada, Kobe, Hyogo, 657-8501, Japan
E-mail : `kenji303t@icloud.com`

Tomoki Tamaru, Graduate School of System Informatics, Kobe University, 1-1 Riokkodai, Nada, Kobe, Hyogo, 657-8501, Japan
E-mail : `249x053x@stu.kobe-u.ac.jp`

Masatake Hirao, School of Information and Science Technology, Aichi Prefectural University, 1522-3 Ibaragabasama, Nagakute, Aichi, 480-1198, Japan
E-mail : `hirao@ist.aichi-pu.ac.jp`

Masanori Sawa, Graduate School of System Informatics, Kobe University, 1-1 Riokkodai, Nada, Kobe, Hyogo, 657-8501, Japan
E-mail : `sawa@people.kobe-u.ac.jp`