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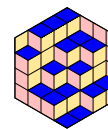


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ABSTRACT We investigate the “stratified Ehrhart ring theory” for periodic graphs, which gives an algorithm for determining the growth sequences of periodic graphs. The growth sequence $(s_{\Gamma, x_0, i})_{i \geq 0}$ is defined for a graph Γ and its fixed vertex x_0 , where $s_{\Gamma, x_0, i}$ is defined as the number of vertices of Γ at distance i from x_0 . Although the sequences $(s_{\Gamma, x_0, i})_{i \geq 0}$ for periodic graphs are known to be of quasi-polynomial type, their determination had not been established, even in dimension two. Our theory and algorithm can be applied to arbitrary periodic graphs of any dimension. As an application of the algorithm, we determine the growth sequences in several new examples.

1. INTRODUCTION

An n -dimensional periodic graph (Γ, L) is a pair of a directed graph Γ (that may have loops and multiple edges) and a free abelian group L of rank n such that L freely acts on Γ and its quotient graph Γ/L is finite (see Definition 3.2). For a vertex x_0 of Γ , the *growth sequence* $(s_{\Gamma, x_0, i})_{i \geq 0}$ (resp. *cumulative growth sequence* $(b_{\Gamma, x_0, i})_{i \geq 0}$) is defined as the number of vertices of Γ whose distance from x_0 is i (resp. at most i). Periodic graphs naturally appear in crystallography, also appear as periodic tilings in combinatorics, and as Cayley graphs of virtually abelian groups in geometric group theory (see [10, Section 13]). Furthermore, it is shown in [15] that the theory of growth sequence of periodic graphs potentially includes the Ehrhart theory of rational polytopes.

In [11], Grosse-Kunstleve, Brunner and Sloane conjectured that the growth sequences of periodic graphs are of quasi-polynomial type, i.e., there exist an integer M and a quasi-polynomial $f_s : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $s_{\Gamma, x_0, i} = f_s(i)$ holds for any $i \geq M$ (see Definition 3.9). In [20], the second author, Sakamoto, Mase, and Nakagawa prove that this conjecture is true for any periodic graphs (Theorem 3.8). Although it was proved to be of quasi-polynomial type, determining the explicit formulae of growth sequences is still difficult. Thus, the following natural question arises.

QUESTION 1.1 (cf. [15, Question 1.1]). *Find an effective algorithm to determine the explicit formulae of the growth sequences.*

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So far, various computational methods have been established for several special classes of periodic graphs. In [6], Conway and Sloane study the growth sequences of the contact graphs of some lattices from the viewpoint of the Ehrhart theory, and they give explicit formulae for the root lattices A_d . In [3], Bacher, de la Harpe, and Venkov give the proof for the conjectural formulae for the root lattices B_d , C_d and D_d (cf. [2]). We note that the periodic graphs (Γ, L) obtained from the lattices satisfy $\#(V_\Gamma/L) = 1$, where V_Γ denotes the set of vertices of Γ . When $\#(V_\Gamma/L) = 1$, V_Γ has a monoid structure, and the growth sequence can be directly studied by the Hilbert series of the corresponding graded monoid. However, when $\#(V_\Gamma/L) > 1$, it is more difficult to study the growth sequence. In [10], Goodman-Strauss and Sloane proposed “the coloring book approach” and obtained the growth sequence for some periodic tilings. In [21, 22], Shutov and Maleev obtained the growth sequences for tilings satisfying certain conditions that contain the 20 2-uniform tilings. In [15], the authors introduce a class of periodic graphs called “well-arranged”, and they give an algorithm to calculate their growth sequences. However, as far as we know, no algorithm has been proposed that can be applied to any periodic graph, even in dimension two.

The purpose of this paper is to give an algorithm for determining the growth sequences that can be applied to all periodic graphs in all dimensions. In Subsection 4.1, we define invariants $\beta \in \mathbb{R}_{\geq 0}$ and $\text{cpx}_\Gamma \in \mathbb{Z}_{> 0}$ from combinatorial information of Γ . In Corollary 4.23, we prove that the growth sequence $(s_{\Gamma, x_0, i})_{i \geq 0}$ is a quasi-polynomial on $i > \beta$ and cpx_Γ is its quasi-period. More precisely, we prove that its generating function $G_{\Gamma, x_0}(t)$ is given by

$$G_{\Gamma, x_0}(t) := \sum_{i \geq 0} s_{\Gamma, x_0, i} t^i = \frac{Q(t)}{(1 - t^{\text{cpx}_\Gamma})^n}$$

with some polynomial $Q(t)$ of $\deg Q \leq \beta + n \cdot \text{cpx}_\Gamma$. On the other hand, with the help of a computer program (breadth-first search algorithm), we can compute the first few terms of $(s_{\Gamma, x_0, i})_{i \geq 0}$. After we compute the first $\lfloor \beta \rfloor + n \cdot \text{cpx}_\Gamma + 1$ terms, we can determine the generating function as follows: For $\gamma := \lfloor \beta \rfloor + n \cdot \text{cpx}_\Gamma$, we have

$$G_{\Gamma, x_0}(t) = \frac{(\text{The terms of } (1 - t^{\text{cpx}_\Gamma})^n \sum_{i=0}^{\gamma} s_{\Gamma, x_0, i} t^i \text{ of degree } \gamma \text{ or less})}{(1 - t^{\text{cpx}_\Gamma})^n}.$$

Corollary 4.23 follows from Theorem 4.20, which we call the “stratified Ehrhart ring theory”. Theorem 4.20 gives an algebraic meaning to the growth sequence, and by using it, Corollary 4.23 can be proved by a standard technique in algebraic combinatorics. In what follows, we will outline the statement of Theorem 4.20.

First, we briefly review the Ehrhart ring of a rational polytope and its structure. Let $Q \subset \mathbb{R}^n$ be a rational polytope. Then, the Ehrhart ring of a rational polytope $Q \subset \mathbb{R}^n$ is defined as the group ring $k[A]$ corresponding to the (commutative) monoid

$$A := \{(d, y) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n \mid y \in dQ\}.$$

This A has a graded monoid structure with respect to the degree function $\deg : \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n \rightarrow \mathbb{Z}_{\geq 0} : (d, y) \mapsto d$. When $0 \in \text{int}(Q)$, the monoid A has a nice algebraic structure as follows:

FACT 1.2 (cf. [4, Section 3.2]). *Suppose $0 \in \text{int}(Q)$. For each $\sigma \in \text{Facet}(Q)$, we take a triangulation T_σ of σ satisfying $V(\Delta) \subset V(\sigma)$ for any $\Delta \in T_\sigma$ (see Section 2 for the notation of triangulations). For $v \in V(Q)$, let a_v be the minimum positive integer satisfying $a_v v \in \mathbb{Z}^n$. For $\sigma \in \text{Facet}(Q)$, $\Delta \in T_\sigma$ and $\Delta' \in \text{Face}(\Delta)$, we define*

$$L_{\Delta'} := \mathbb{R}_{\geq 0} \Delta' \subset \mathbb{R}^n, \quad A(L_{\Delta'}) := A \cap (\mathbb{Z}_{\geq 0} \times L_{\Delta'}).$$

We also define a free submonoid $M_{\sigma, \Delta'} \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n$ by

$$M_{\sigma, \Delta'} := \mathbb{Z}_{\geq 0}(1, 0) + \sum_{v \in V(\Delta')} \mathbb{Z}_{\geq 0}(a_v, a_v v).$$

Then, for $\sigma \in \text{Facet}(Q)$, $\Delta \in T_\sigma$ and $\Delta' \in \text{Face}(\Delta)$, it follows that

- each $A(L_{\Delta'})$ is a free $M_{\sigma, \Delta'}$ -module. Furthermore, $A(L_{\Delta'})$ is freely generated by finitely many elements whose degrees are less than $\sum_{v \in V(\Delta')} a_v$.

By using this structure (and the inclusion-exclusion principle), we can prove the counting function

$$\mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}; d \mapsto \# \{y \in \mathbb{Z}^n \mid y \in dQ\} = \# \{y \in \mathbb{Z}^n \mid (d, y) \in A\}$$

is a quasi-polynomial.

Next, we shall explain the “stratified Ehrhart ring theory” for periodic graphs, which is an analogy of Fact 1.2. Let (Γ, L) be an n -dimensional strongly connected periodic graph, and let x_0 be a vertex of Γ . Let V_Γ denote the set of vertices of Γ . We define

$$B := \{(d, y) \in \mathbb{Z}_{\geq 0} \times V_\Gamma \mid d_\Gamma(x_0, y) \leq d\}.$$

Unlike the case of the Ehrhart ring, B itself does not have a monoid structure when $\#(V_\Gamma/L) > 1$, and the situation is more complicated. In the statement of Theorem 4.20, the *growth polytope* P_Γ plays an important role. The growth polytope $P_\Gamma \subset L_\mathbb{R} := L \otimes_\mathbb{Z} \mathbb{R}$ is a rational polytope and is canonically defined from the periodic graph Γ (Definition 3.10). Furthermore, we fix a *periodic realization* $\Phi : V_\Gamma \rightarrow L_\mathbb{R}$, that is a map satisfying $\Phi(u + y) = u + \Phi(y)$ for any $y \in V_\Gamma$ and $u \in L$ (see Definition 3.4).

THEOREM 1.3 (cf. Theorem 4.20). *There exist*

- $\beta \in \mathbb{R}_{\geq 0}$,
- a triangulation T_σ for each $\sigma \in \text{Facet}(P_\Gamma)$, and
- $\text{cpx}_\sigma(v) \in \mathbb{Z}_{> 0}$ and $\beta_{\Delta, v} \in \mathbb{R}_{\geq 0}$ for any $\sigma \in \text{Facet}(P_\Gamma)$, $\Delta \in T_\sigma$ and $v \in V(\Delta)$

with the condition ($\spadesuit$) below: For $\sigma \in \text{Facet}(P_\Gamma)$, $\Delta \in T_\sigma$, $\Delta' \in \text{Face}(\Delta)$ and $\Delta'' \in \text{Face}(\Delta')$, we define $L_{\Delta'}, L_{\Delta, \Delta', \Delta''} \subset L_\mathbb{R}$ by

$$L_{\Delta'} := \mathbb{R}_{\geq 0}\Delta', \quad L_{\Delta, \Delta', \Delta''} := \sum_{v \in V(\Delta'')} (\beta_{\Delta, v}, \infty) \cdot v + \sum_{v \in V(\Delta') \setminus V(\Delta'')} [0, \beta_{\Delta, v}] \cdot v.$$

(Note that we have $L_{\Delta'} = \bigsqcup_{\Delta'' \in \text{Face}(\Delta')} L_{\Delta, \Delta', \Delta''}$.) For a subset $F \subset L_\mathbb{R}$, we define

$B(F) := B \cap (\mathbb{Z}_{\geq 0} \times \Phi^{-1}(F))$. We also define a free submonoid $M_{\sigma, \Delta'} \subset \mathbb{Z}_{\geq 0} \times L$ by

$$M_{\sigma, \Delta'} := \mathbb{Z}_{\geq 0}(1, 0) + \sum_{v \in V(\Delta')} \mathbb{Z}_{\geq 0}(\text{cpx}_\sigma(v), \text{cpx}_\sigma(v)v).$$

Then, for $\sigma \in \text{Facet}(P_\Gamma)$, $\Delta \in T_\sigma$, $\Delta' \in \text{Face}(\Delta)$ and $\Delta'' \in \text{Face}(\Delta')$, it follows that

- ($\spadesuit$) each $B(L_{\Delta, \Delta', \Delta''})$ is a free $M_{\sigma, \Delta''}$ -module. Furthermore, $B(L_{\Delta, \Delta', \Delta''})$ is freely generated by finitely many elements whose degrees are at most $\beta + \sum_{v \in V(\Delta'')} \text{cpx}_\sigma(v)$.

We define

$$\text{cpx}_\Gamma := \text{LCM}\{\text{cpx}_\sigma(v) \mid \sigma \in \text{Facet}(P_\Gamma), \Delta \in T_\sigma, v \in V(\Delta)\}.$$

In Corollary 4.23, by using the algebraic structure in Theorem 1.3 (and the inclusion-exclusion principle), we prove that the generating function $G_{\Gamma, x_0}(t)$ is given by the form

$$G_{\Gamma, x_0}(t) = \frac{Q(t)}{(1 - t^{\text{cpx}_\Gamma})^n}$$

with some polynomial $Q(t)$ of $\deg Q \leq \beta + n \cdot \text{cpx}_\Gamma$.

In [22] and [10], similar ideas of calculating the growth sequence by partitioning V_Γ into finite regions, as in Theorem 1.3, also appear (although in both papers, the construction is explained only for a specific class of planar graphs and not for general graphs in arbitrary dimension). Theorem 1.3 can be said to be a rigorous formulation and a generalization of their ideas.

In [15], the authors introduce a class of periodic graphs called “well-arranged”. For well-arranged periodic graphs, they prove that $\beta = 0$ and $\beta_{\Delta,v} = 0$ satisfy the statement of Theorem 1.3 (see [15, Claim 4.10]). As a consequence, they prove that their growth sequences are quasi-polynomial on $i > 0$.

In the two-dimensional case, it is relatively simple to implement our algorithm into a computer program since there is no need to consider the complicated triangulations of the facets of P_Γ (see Subsection 4.2 and Remark 5.1). In fact, using the computer program, the growth sequences of the examples treated in [10] can be computed automatically without separate consideration (see Subsection 5.1). In Subsection 5.1, we examine a 6-uniform tiling using our algorithm.

In higher dimensions, although it is possible to implement the algorithm in a computer program, it is practical to give the triangulation by hand. In Subsections 5.2 and 5.3, we illustrate the algorithm for the periodic graphs corresponding to two carbon allotropes called K6 and CFS. There, the triangulation and other calculations are partially given by hand, and the rest is done by a computer program. As far as we know, this is the first time the growth series of these two examples have been determined with proofs.

The paper is organized as follows: in Section 3, we introduce notations related to periodic graphs following [15]. In Subsection 4.1, we define the invariants β and cpx_Γ . In Subsection 4.2, we explain the invariants for 2-dimensional periodic graphs. In Subsection 4.3, we prove the main theorem (Theorem 4.20 and Corollary 4.23). In Section 5, we apply our algorithm to some specific periodic graphs. In Appendix A, we briefly describe an input form of periodic graphs to implement the algorithm in a computer program.

2. NOTATION

- For a set X , $\#X$ denotes the cardinality of X , and 2^X denotes the power set of X .
- For a finite subset $S \subset \mathbb{Z}_{>0}$, $\text{LCM}(S)$ denotes the least common multiple of the elements of S .
- For a polytope $P \subset \mathbb{R}^N$, $\text{Facet}(P)$ denotes the set of facets of P , $\text{Face}(P)$ denotes the set of faces of P , and $V(P)$ denotes the set of vertices of P . Note that both P itself and the empty set $\emptyset$ are considered as faces of P .
- For a set $C \subset \mathbb{R}^N$, $\text{int}(C)$ denotes the interior of C , and $\text{relint}(C)$ denotes the relative interior of C .
- For a polytope $\sigma \subset \mathbb{R}^N$ of dimension d , a *triangulation* T_σ means a finite collection of d -simplices with the following two conditions:
 - $\sigma = \bigcup_{\Delta \in T_\sigma} \Delta$.
 - For any $\Delta_1, \Delta_2 \in T_\sigma$, $\Delta_1 \cap \Delta_2$ is a face of Δ_1 .
- Let M be a set equipped with a binary operation $*$. For $u \in M$ and subsets $X, Y \subset M$, we define subsets $u * X, X * Y \subset M$ by

$$u * X := \{u * x \mid x \in X\}, \quad X * Y := \{x * y \mid (x, y) \in X \times Y\}.$$
- In this paper, monoids always mean commutative monoids. We refer the reader to [5] and [20] for the terminology of monoid and its module theory.

3. PRELIMINARIES

Following [15], we introduce notations related to periodic graphs.

3.1. GRAPHS AND WALKS. In this paper, a *graph* means a directed weighted graph which may have loops and multiple edges. A graph $\Gamma = (V_\Gamma, E_\Gamma, s_\Gamma, t_\Gamma, w_\Gamma)$ consists of the set V_Γ of vertices, the set E_Γ of edges, the source function $s_\Gamma : E_\Gamma \rightarrow V_\Gamma$, the target function $t_\Gamma : E_\Gamma \rightarrow V_\Gamma$, and the weight function $w_\Gamma : E_\Gamma \rightarrow \mathbb{Z}_{>0}$. We often abbreviate s_Γ , t_Γ and w_Γ to s , t and w when no confusion can arise.

DEFINITION 3.1. Let $\Gamma = (V_\Gamma, E_\Gamma, s, t, w)$ be a graph.

- (1) Γ is said to be *unweighted* if $w(e) = 1$ holds for any $e \in E_\Gamma$. Γ is said to be *undirected* when there exists an involution $E_\Gamma \rightarrow E_\Gamma$; $e \mapsto e'$ such that $s(e) = t(e')$, $t(e) = s(e')$ and $w(e) = w(e')$.
- (2) A *walk* p in Γ is a sequence $e_1 e_2 \cdots e_\ell$ of edges e_i of Γ satisfying $t(e_i) = s(e_{i+1})$ for each $i = 1, \dots, \ell - 1$. We define

$$s(p) := s(e_1), \quad t(p) := t(e_\ell), \quad w(p) := \sum_{i=1}^{\ell} w(e_i), \quad \text{length}(p) := \ell.$$

Note that we have $w(p) = \text{length}(p)$ if Γ is unweighted.

We say that “ p is a walk from x to y ” when $x = s(p)$ and $y = t(p)$. We also define the *support* $\text{supp}(p) \subset V_\Gamma$ of p by

$$\text{supp}(p) := \{s(e_1), t(e_1), t(e_2), \dots, t(e_\ell)\} \subset V_\Gamma.$$

By convention, each vertex $v \in V_\Gamma$ is also considered as a walk of length 0. This is called the *trivial walk* at v and denoted by $\emptyset_v$: i.e., we define

$$s(\emptyset_v) := v, \quad t(\emptyset_v) := v, \quad w(\emptyset_v) := 0, \quad \text{length}(\emptyset_v) := 0, \quad \text{supp}(\emptyset_v) := \{v\}.$$

- (3) A *path* in Γ is a walk $e_1 \cdots e_\ell$ such that $s(e_1), t(e_1), t(e_2), \dots, t(e_\ell)$ are distinct. A walk of length 0 is considered as a path.
- (4) A *cycle* in Γ is a walk $e_1 \cdots e_\ell$ with $s(e_1) = t(e_\ell)$ such that $t(e_1), t(e_2), \dots, t(e_\ell)$ are distinct. A walk of length 0 is **NOT** considered as a cycle. Cyc_Γ denotes the set of cycles in Γ .
- (5) For $x, y \in V_\Gamma$, $d_\Gamma(x, y) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ denotes the smallest weight $w(p)$ of any walk p from x to y . By convention, we define $d_\Gamma(x, y) = \infty$ when there is no walk from x to y . A graph Γ is said to be *strongly connected* when we have $d_\Gamma(x, y) < \infty$ for any $x, y \in V_\Gamma$. When Γ is undirected, we have $d_\Gamma(x, y) = d_\Gamma(y, x)$ for any $x, y \in V_\Gamma$.
- (6) $C_1(\Gamma, \mathbb{Z})$ denotes the group of 1-chains on Γ with coefficients in $\mathbb{Z}$, i.e., $C_1(\Gamma, \mathbb{Z})$ is a free abelian group generated by E_Γ . For a walk $p = e_1 \cdots e_\ell$ in Γ , let $\langle p \rangle$ denote the 1-chain $\sum_{i=1}^{\ell} e_i \in C_1(\Gamma, \mathbb{Z})$. $H_1(\Gamma, \mathbb{Z}) \subset C_1(\Gamma, \mathbb{Z})$ denotes the 1-st homology group, i.e., $H_1(\Gamma, \mathbb{Z})$ is a subgroup generated by $\langle p \rangle$ for $p \in \text{Cyc}_\Gamma$. We refer the reader to [24] for more detail.

3.2. PERIODIC GRAPHS.

DEFINITION 3.2. Let n be a positive integer. An *n -dimensional periodic graph* (Γ, L) is a graph Γ and a free abelian group $L \simeq \mathbb{Z}^n$ of rank n with the following two conditions:

- L freely acts on both V_Γ and E_Γ , and their quotients V_Γ/L and E_Γ/L are finite sets.
- This action preserves the edge relations and the weight function, i.e., for any $u \in L$ and $e \in E_\Gamma$, we have $s_\Gamma(u(e)) = u(s_\Gamma(e))$, $t_\Gamma(u(e)) = u(t_\Gamma(e))$ and $w_\Gamma(u(e)) = w_\Gamma(e)$.

If (Γ, L) is an n -dimensional periodic graph, then the *quotient graph* $\Gamma/L = (V_{\Gamma/L}, E_{\Gamma/L}, s_{\Gamma/L}, t_{\Gamma/L}, w_{\Gamma/L})$ is defined by $V_{\Gamma/L} := V_{\Gamma}/L$, $E_{\Gamma/L} := E_{\Gamma}/L$, and the functions $s_{\Gamma/L} : E_{\Gamma/L} \rightarrow V_{\Gamma/L}$, $t_{\Gamma/L} : E_{\Gamma/L} \rightarrow V_{\Gamma/L}$, and $w_{\Gamma/L} : E_{\Gamma/L} \rightarrow \mathbb{Z}_{>0}$ induced from s_{Γ} , t_{Γ} , and w_{Γ} . Note that the functions $s_{\Gamma/L}$, $t_{\Gamma/L}$ and $w_{\Gamma/L}$ are well-defined due to the second condition in Definition 3.2.

DEFINITION 3.3. Let (Γ, L) be an n -dimensional periodic graph.

- (1) Since L is an abelian group, we use the additive notation: for $u \in L$, $x \in V_{\Gamma}$, $e \in E_{\Gamma}$ and a walk $p = e_1 \cdots e_{\ell}$,

$$u + x := u(x), \quad u + e := u(e), \quad u + p := u(e_1) \cdots u(e_{\ell})$$

denote their translations by u .

- (2) For any $x \in V_{\Gamma}$ and $e \in E_{\Gamma}$, let $\bar{x} \in V_{\Gamma/L}$ and $\bar{e} \in E_{\Gamma/L}$ denote their images in $V_{\Gamma/L} = V_{\Gamma}/L$ and $E_{\Gamma/L} = E_{\Gamma}/L$. For a walk $p = e_1 \cdots e_{\ell}$ in Γ , we define its image in Γ/L by $\bar{p} := \bar{e}_1 \cdots \bar{e}_{\ell}$.
- (3) When $x, y \in V_{\Gamma}$ satisfy $\bar{x} = \bar{y}$, there exists an element $u \in L$ such that $u + x = y$. Since the action is free, such $u \in L$ uniquely exists and is denoted by $y - x$.
- (4) For a walk p in Γ with $\overline{s(p)} = \overline{t(p)}$, we define

$$\text{vec}(p) := t(p) - s(p) \in L.$$

DEFINITION 3.4. Let (Γ, L) be an n -dimensional periodic graph. We define $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$.

- (1) A *periodic realization* $\Phi : V_{\Gamma} \rightarrow L_{\mathbb{R}}$ is a map satisfying $\Phi(u + x) = u + \Phi(x)$ for any $u \in L$ and $x \in V_{\Gamma}$.
- (2) Let Φ be a periodic realization of (Γ, L) . For an edge e and a walk p in Γ , we define

$$\text{vec}_{\Phi}(e) := \Phi(t(e)) - \Phi(s(e)) \in L_{\mathbb{R}},$$

$$\text{vec}_{\Phi}(p) := \Phi(t(p)) - \Phi(s(p)) \in L_{\mathbb{R}}.$$

It is easy to see that the value $\text{vec}_{\Phi}(e) \in L_{\mathbb{R}}$ depends only on the class $\bar{e} \in E_{\Gamma/L}$, and therefore, the map

$$\mu_{\Phi} : E_{\Gamma/L} \rightarrow L_{\mathbb{R}}; \quad \bar{e} \mapsto \text{vec}_{\Phi}(e)$$

is well-defined. It can be extended to a homomorphism

$$\mu_{\Phi} : C_1(\Gamma/L, \mathbb{Z}) \rightarrow L_{\mathbb{R}}; \quad \sum a_i \bar{e}_i \mapsto \sum a_i \mu_{\Phi}(\bar{e}_i).$$

By construction, we have $\mu_{\Phi}(\langle \bar{p} \rangle) = \text{vec}_{\Phi}(p)$ for any walk p in Γ .

- (3) It is known that the restriction map $\mu_{\Phi}|_{H_1(\Gamma/L, \mathbb{Z})} : H_1(\Gamma/L, \mathbb{Z}) \rightarrow L_{\mathbb{R}}$ is independent of the choice of Φ and that its image is contained in L (see [15, Lemma 2.7]). This restriction map is denoted by $\mu : H_1(\Gamma/L, \mathbb{Z}) \rightarrow L$.

REMARK 3.5. In Definition 3.4(2), we have $\text{vec}_{\Phi}(p) = \text{vec}(p)$ for any p satisfying $\overline{s(p)} = \overline{t(p)}$.

We finish this subsection with an observation from [15] on the decomposition and the composition of walks. The notation differs slightly from that in [15], but is essentially the same.

DEFINITION 3.6 (cf. [15, Definition 2.11]). Let (Γ, L) be an n -dimensional periodic graph. Let q_0 be a path in Γ/L , and let $a : \text{Cyc}_{\Gamma/L} \rightarrow \mathbb{Z}_{\geq 0}$ be a function. The pair (q_0, a) is said to be *walkable* if there exists a walk q' in Γ/L such that $\langle q' \rangle = \langle q_0 \rangle + \sum_{q \in \text{Cyc}_{\Gamma/L}} a(q) \langle q \rangle$.

LEMMA 3.7 ([15, Lemma 2.12, Remark 2.13]). Let (Γ, L) be an n -dimensional periodic graph.

- (1) For a walk q' in Γ/L , there exists a walkable pair (q_0, a) such that $\langle q' \rangle = \langle q_0 \rangle + \sum_{q \in \text{Cyc}_{\Gamma/L}} a(q) \langle q \rangle$. Furthermore, if q' satisfies $s(q') = t(q')$, then q_0 should be a trivial path.
- (2) Let q_0 be a path in Γ/L , and let $a : \text{Cyc}_{\Gamma/L} \rightarrow \mathbb{Z}_{\geq 0}$ be a function. Then, (q_0, a) is walkable if and only if there exists a sequence $q_1, \dots, q_\ell \in a^{-1}(\mathbb{Z}_{>0})$ satisfying $a^{-1}(\mathbb{Z}_{>0}) = \{q_1, \dots, q_\ell\}$ such that

$$\left(\text{supp}(q_0) \cup \bigcup_{1 \leq i \leq k} \text{supp}(q_i) \right) \cap \text{supp}(q_{k+1}) \neq \emptyset$$

holds for any $0 \leq k \leq \ell - 1$.

- (3) For a walk q in Γ/L and a vertex $x \in V_\Gamma$ satisfying $s(q) = \bar{x}$, there exists a unique walk p in Γ such that $q = \bar{p}$ and $s(p) = x$.

3.3. GROWTH SEQUENCES OF PERIODIC GRAPHS. Let Γ be a locally finite graph, and let $x_0 \in V_\Gamma$. For $i \in \mathbb{Z}_{\geq 0}$, we define subsets $B_{\Gamma, x_0, i}, S_{\Gamma, x_0, i} \subset V_\Gamma$ by

$$B_{\Gamma, x_0, i} := \{y \in V_\Gamma \mid d_\Gamma(x_0, y) \leq i\}, \quad S_{\Gamma, x_0, i} := \{y \in V_\Gamma \mid d_\Gamma(x_0, y) = i\}.$$

Let $b_{\Gamma, x_0, i} := \#B_{\Gamma, x_0, i}$ and $s_{\Gamma, x_0, i} := \#S_{\Gamma, x_0, i}$ denote their cardinalities. The sequences $(s_{\Gamma, x_0, i})_i$ and $(b_{\Gamma, x_0, i})_i$ are called the *growth sequence* and the *cumulative growth sequence* of Γ with the start point x_0 , respectively.

The *growth series* $G_{\Gamma, x_0}(t)$ of Γ with the start point x_0 is the generating function

$$G_{\Gamma, x_0}(t) := \sum_{i \geq 0} s_{\Gamma, x_0, i} t^i$$

of the growth sequence $(s_{\Gamma, x_0, i})_i$.

The growth sequences of periodic graphs are known to be of quasi-polynomial type.

THEOREM 3.8 ([20, Theorem 2.2]). Let (Γ, L) be a periodic graph, and let $x_0 \in V_\Gamma$. Then, the functions $b : i \mapsto b_{\Gamma, x_0, i}$ and $s : i \mapsto s_{\Gamma, x_0, i}$ are of quasi-polynomial type (see Definition 3.9 below). In particular, its growth series is rational.

DEFINITION 3.9 (cf. [23, Chapter 0]).

- (1) A function $f : \mathbb{Z} \rightarrow \mathbb{C}$ is called a *quasi-polynomial* if there exist a positive integer N and polynomials $Q_0, \dots, Q_{N-1} \in \mathbb{C}[x]$ such that

$$f(n) = \begin{cases} Q_0(n) & \text{if } n \equiv 0 \pmod{N}, \\ Q_1(n) & \text{if } n \equiv 1 \pmod{N}, \\ \vdots & \\ Q_{N-1}(n) & \text{if } n \equiv N-1 \pmod{N}. \end{cases}$$

- (2) A function $g : \mathbb{Z} \rightarrow \mathbb{C}$ is said to be of *quasi-polynomial type* if there exists a non-negative integer $M \in \mathbb{Z}_{\geq 0}$ and a quasi-polynomial f such that $g(n) = f(n)$ holds for any $n > M$. The positive integer N is called a *quasi-period* of g when f is of the form in (1). Note that the notion of quasi-period is not unique. The minimum quasi-period is called the *period* of g . We say that the function g is a *quasi-polynomial on* $n \geq m$ if $g(n) = f(n)$ holds for $n \geq m$.

3.4. GROWTH POLYTOPE. In this subsection, according to [15], we define the *growth polytope* $P_\Gamma \subset L_\mathbb{R}$ for a periodic graph (Γ, L) . The concept of a growth polytope has been defined and studied in various papers [16, 17, 25, 7, 18, 8, 1].

DEFINITION 3.10. Let (Γ, L) be an n -dimensional periodic graph.

- (1) We define the *normalization map* $\nu : \text{Cyc}_{\Gamma/L} \rightarrow L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ by

$$\nu : \text{Cyc}_{\Gamma/L} \rightarrow L_{\mathbb{R}}; \quad p \mapsto \frac{\mu(\langle p \rangle)}{w(p)}.$$

We define the *growth polytope*

$$P_{\Gamma} := \text{conv}(\text{Im}(\nu) \cup \{0\}) \subset L_{\mathbb{R}}$$

as the convex hull of the set $\text{Im}(\nu) \cup \{0\} \subset L_{\mathbb{R}}$.

- (2) For a polytope $Q \subset L_{\mathbb{R}}$ and $y \in L_{\mathbb{R}}$, we define

$$d_Q(y) := \min\{t \in \mathbb{R}_{\geq 0} \mid y \in tQ\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

When $0 \in \text{int}(Q)$, we have $d_Q(y) < \infty$ for any $y \in L_{\mathbb{R}}$.

- (3) For a periodic realization $\Phi : V_{\Gamma} \rightarrow L_{\mathbb{R}}$, we define

$$d_{P_{\Gamma}, \Phi}(x, y) := d_{P_{\Gamma}}(\Phi(y) - \Phi(x))$$

for $x, y \in V_{\Gamma}$.

REMARK 3.11. Note that P_{Γ} is a rational polytope (i.e., P_{Γ} is a polytope whose vertices are on $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$). This is because $\text{Cyc}_{\Gamma/L}$ is a finite set, and we have $\text{Im}(\nu) \subset L_{\mathbb{Q}}$. Furthermore, when Γ is strongly connected, we have $0 \in \text{int}(P_{\Gamma})$ by [8, Proposition 21] (cf. [15, Lemma A.1]).

We define $C_1(\Gamma, \Phi, x_0)$ and $C_2(\Gamma, \Phi, x_0)$ as invariants that measure the difference between d_{Γ} and $d_{P_{\Gamma}, \Phi}$.

DEFINITION 3.12. Let (Γ, L) be a strongly connected periodic graph. Let $\Phi : V_{\Gamma} \rightarrow L_{\mathbb{R}}$ be a periodic realization, and let $x_0 \in V_{\Gamma}$. Then, we define

$$C_1(\Gamma, \Phi, x_0) := \sup_{y \in V_{\Gamma}} (d_{P_{\Gamma}, \Phi}(x_0, y) - d_{\Gamma}(x_0, y)),$$

$$C_2(\Gamma, \Phi, x_0) := \sup_{y \in V_{\Gamma}} (d_{\Gamma}(x_0, y) - d_{P_{\Gamma}, \Phi}(x_0, y)).$$

By [15, Theorem A.2], we have $C_1(\Gamma, \Phi, x_0) < \infty$ and $C_2(\Gamma, \Phi, x_0) < \infty$.

REMARK 3.13. As in Proposition 3.14(1), it is easy to determine $C_1(\Gamma, \Phi, x_0)$. However, it is not easy to determine $C_2(\Gamma, \Phi, x_0)$ in general, and we just give an upper bound of it in Proposition 3.14(2). In Theorem 4.26, we give an algorithm to determine $C_2(\Gamma, \Phi, x_0)$ using the invariants defined in Section 4.

PROPOSITION 3.14 (cf. [15, Theorem A.2]). *Let (Γ, L) be a strongly connected periodic graph. Let $\Phi : V_{\Gamma} \rightarrow L_{\mathbb{R}}$ be a periodic realization, and let $x_0 \in V_{\Gamma}$.*

- (1) *We have*

$$C_1(\Gamma, \Phi, x_0) = \max_{y \in B'_{c-1}} (d_{P_{\Gamma}, \Phi}(x_0, y) - d_{\Gamma}(x_0, y)),$$

where $c := \#(V_{\Gamma}/L)$ and

$$B'_{c-1} := \{y \in V_{\Gamma} \mid \text{there exists a walk } p \text{ from } x_0 \text{ to } y \text{ with } \text{length}(p) \leq c-1\}.$$

- (2) *We have $C_2(\Gamma, \Phi, x_0) \leq C'_2$ when we define C'_2 as follows:*

- *First, we define $d_v := \min_{q \in \nu^{-1}(v)} w(q)$ for each $v \in V(P_{\Gamma})$.*
- *For each $\sigma \in \text{Facet}(P_{\Gamma})$, we fix a triangulation T_{σ} of σ such that $V(\Delta) \subset V(\sigma)$ holds for any $\Delta \in T_{\sigma}$.*
- *We define a bounded set $Q \subset L_{\mathbb{R}}$ as follows:*

$$Q := \bigcup_{\substack{\sigma \in \text{Facet}(P_{\Gamma}), \\ \Delta \in T_{\sigma}}} \left(\sum_{v \in V(\Delta)} [0, 1) d_v v \right) \subset L_{\mathbb{R}}.$$

- For $y \in V_\Gamma$, we define $d'(x_0, y)$ as the smallest weight $w(p)$ of a walk p from x_0 to y satisfying $\text{supp}(\bar{p}) = V_\Gamma/L$.
- Then, we set

$$C'_2 := \max\{d'(x_0, y) - d_{P_\Gamma, \Phi}(x_0, y) \mid y \in V_\Gamma, \Phi(y) - \Phi(x_0) \in Q\}.$$

Proof. In the proof of [15, Theorem A.2], (1) and (2) are proved. $\square$

4. STRATIFIED EHRHART RING THEORY

Throughout this section, we fix a strongly connected n -dimensional periodic graph (Γ, L) . We also fix $x_0 \in V_\Gamma$. Let $(b(d))_d$ be the cumulative growth sequence of Γ with the start point x_0 . By Theorem 3.8, it is known that the function $b : d \mapsto b(d)$ is of quasi-polynomial type. The goal of this section is to give an algorithm for finding a quasi-period and an integer m such that the function b is a quasi-polynomial on $d \geq m$.

In Subsection 4.1, we will define invariants $\text{cpx}_\Gamma \in \mathbb{Z}_{>0}$ and $\beta \in \mathbb{R}_{\geq 0}$. In Subsection 4.3, we will prove that the function b is a quasi-polynomial on $d > \beta - 1$, and cpx_Γ is its quasi-period (Corollary 4.23).

4.1. INVARIANTS. Let $P := P_\Gamma$ be the growth polytope of Γ . We define $\nu : \text{Cyc}_{\Gamma/L} \rightarrow L_\mathbb{R}$ as in Definition 3.10. We also fix a periodic realization $\Phi : V_\Gamma \rightarrow L_\mathbb{R}$ such that $\Phi(x_0) = 0$. Let $C_1 := C_1(\Gamma, \Phi, x_0)$ and $C_2 := C_2(\Gamma, \Phi, x_0)$ (see Definition 3.12).

We will define β and cpx_Γ in Subsection 4.1.5:

- In Subsection 4.1.1, for each $\sigma \in \text{Facet}(P)$, we take a triangulation T_σ with conditions $(\diamond)_1$ and $(\diamond)_2$.
- In Subsection 4.1.2, for each $\sigma \in \text{Facet}(P)$ and $v \in \sigma$, we define a subset $\mathcal{H}''_{\sigma, v} \subset 2^{\text{Im}(\nu) \cap \sigma}$.
- In Subsection 4.1.3, for $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$, $v \in V(\Delta)$ and $F \in \mathcal{H}''_{\sigma, v}$, we take invariants $\text{cpx}_\sigma(v) \in \mathbb{Z}_{>0}$, $r_{\sigma, v}^F \in \mathbb{Z}_{\geq 0}$ and $s_{\sigma, v}^F \in \mathbb{Z}_{\geq 0}$.
- In Subsection 4.1.4, for $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$, $v \in V(\Delta)$ and $F \in \mathcal{H}''_{\sigma, v}$, we take invariants $a_{\Delta, v}^F(v) \in (0, 1)$ and $h_{\Delta, v}^F \in (0, 1)$.
- In Subsection 4.1.5, using the invariants above, we define invariants $\beta \in \mathbb{R}_{\geq 0}$ and $\text{cpx}_\Gamma \in \mathbb{Z}_{>0}$.

4.1.1. Triangulation T_σ . For each facet σ of P , we take a triangulation T_σ of σ (see Section 2) with the following two conditions:

- $(\diamond)_1$ $V(\Delta) \subset L_\mathbb{Q}$ holds for any $\Delta \in T_\sigma$.
- $(\diamond)_2$ For any $\Delta \in T_\sigma$ and any subset $F \subset \text{Im}(\nu) \cap \sigma$, $\Delta \cap \text{conv}(F)$ is a face of Δ .

See Lemma 4.1 for the existence of such a triangulation. The rationality condition $(\diamond)_1$ will be used in the proof of Lemma 4.7. Condition $(\diamond)_2$ will be used in the proof of Lemma 4.10 (see Lemma 4.9).

LEMMA 4.1. *For each $\sigma \in \text{Facet}(P)$, there exists a triangulation T_σ of σ with the conditions $(\diamond)_1$ and $(\diamond)_2$.*

Proof. We fix $\sigma \in \text{Facet}(P)$. First, we define a set $\mathcal{S}$ of polytopes by

$$\mathcal{S} := \left\{ \text{conv}(F) \mid F \in 2^{\text{Im}(\nu) \cap \sigma} \right\}.$$

Then, since $\mathcal{S}$ is a finite set of polytopes, by taking a subdivision, we can construct a triangulation T_σ of σ with the following condition:

- (i) For any $Q \in \mathcal{S}$, there exist $\Delta'_1, \dots, \Delta'_\ell \in \{\Delta' \mid \Delta \in T_\sigma, \Delta' \in \text{Face}(\Delta)\}$ such that $Q = \bigcup_{i=1}^\ell \Delta'_i$.

Since $\text{Im}(\nu) \subset L_{\mathbb{Q}}$, $\mathcal{S}$ consists of rational polytopes. Therefore, we may take such a triangulation T_{σ} to satisfy $(\diamond)_1$.

We prove that the T_{σ} satisfies $(\diamond)_2$. Let $F \subset \text{Im}(\nu) \cap \sigma$ and $\Delta \in T_{\sigma}$. By (i), $\Delta \cap \text{conv}(F)$ is a union of some faces of Δ . Since $\Delta \cap \text{conv}(F)$ is a convex set, $\Delta \cap \text{conv}(F)$ should be a face of Δ . $\square$

EXAMPLE 4.2. Figure 1 shows the example of T_{σ} . In this example, we assume that σ is a quadrilateral with vertices $u_1, \dots, u_4$ and $\text{Im}(\nu) \cap \sigma = V(\sigma) = \{u_1, \dots, u_4\}$. Then, $T_{\sigma} = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ in the figure satisfies the condition $(\diamond)_2$.

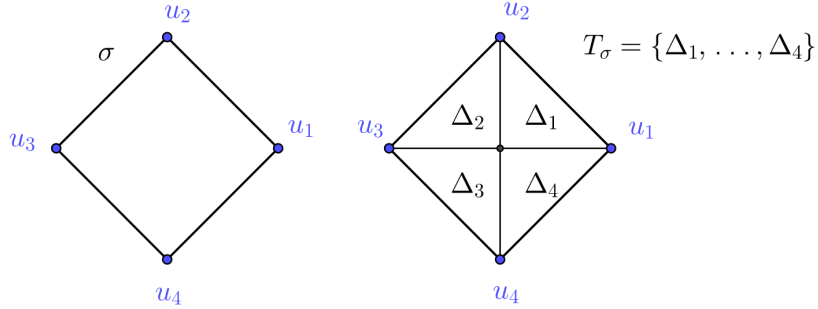


FIGURE 1. T_{σ} for a quadrilateral σ .

4.1.2. $\mathcal{H}''_{\sigma,v}$.

DEFINITION 4.3. Let $\sigma \in \text{Facet}(P)$. Let $\bar{\sigma}$ denote the hyperplane of $L_{\mathbb{R}}$ that contains σ . For $v \in \bar{\sigma}$, let $\mathcal{H}_{\sigma,v}$ denote the set of all closed half-spaces H of $\bar{\sigma}$ satisfying $v \in \partial H := H \setminus \text{relint}(H)$. Let

$$\mathcal{H}'_{\sigma,v} := \{\text{Im}(\nu) \cap H \mid H \in \mathcal{H}_{\sigma,v}\} \subset 2^{\text{Im}(\nu) \cap \sigma}.$$

Note that $\mathcal{H}'_{\sigma,v}$ is a finite set since $\text{Im}(\nu)$ is a finite set. Let $\mathcal{H}''_{\sigma,v} \subset \mathcal{H}'_{\sigma,v}$ be the set of minimal elements of $\mathcal{H}'_{\sigma,v}$ with respect to the inclusion.

See Lemma 4.4 for a property of $\mathcal{H}''_{\sigma,v}$.

LEMMA 4.4. Let $\sigma \in \text{Facet}(P)$ and $v \in \sigma$. For each $F \in \mathcal{H}''_{\sigma,v}$, we pick arbitrary $u_F \in F$. For any choice of u_F , we have $v \in \text{conv}(\{u_F \mid F \in \mathcal{H}''_{\sigma,v}\})$.

Proof. Suppose the contrary that $v \notin \text{conv}(\{u_F \mid F \in \mathcal{H}''_{\sigma,v}\})$. Then, $\text{conv}(\{u_F \mid F \in \mathcal{H}''_{\sigma,v}\})$ should be contained in an open half-space of $\bar{\sigma}$ whose boundary passes through v , and hence, we have

$$\{u_F \mid F \in \mathcal{H}''_{\sigma,v}\} \cap H = \emptyset$$

for some $H \in \mathcal{H}_{\sigma,v}$. Let $F_H := \text{Im}(\nu) \cap H \in \mathcal{H}'_{\sigma,v}$. Then, by the definition of $\mathcal{H}''_{\sigma,v}$, there exists $F' \in \mathcal{H}''_{\sigma,v}$ such that $F' \subset F_H$. Since we have

$$u_{F'} \in \{u_F \mid F \in \mathcal{H}''_{\sigma,v}\}, \quad u_{F'} \in F' \subset F_H \subset H,$$

we get a contradiction. $\square$

EXAMPLE 4.5. For example, in Figure 2, we have

$$\begin{aligned}\mathcal{H}'_{\sigma,v} &= \{\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_6\}, \{x_2, x_6\}, \{x_2, x_6, x_3\}, \\ &\quad \{x_2, x_6, x_3, x_4\}, \{x_2, x_6, x_3, x_4, x_5\}, \{x_6, x_3, x_4, x_5\}, \\ &\quad \{x_3, x_4, x_5\}, \{x_3, x_4, x_5, x_1\}, \{x_4, x_5, x_1\}, \{x_5, x_1\}\}, \\ \mathcal{H}''_{\sigma,v} &= \{\{x_1\}, \{x_2, x_6\}, \{x_3, x_4, x_5\}\}.\end{aligned}$$

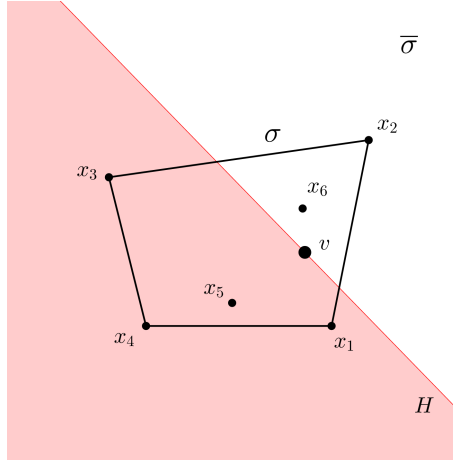


FIGURE 2. $\text{Im}(\nu) \cap \sigma = \{x_1, \dots, x_6\}$.

4.1.3. $\text{cpx}_\sigma(v)$, $r_{\sigma,v}^F$ and $s_{\sigma,v}^F$.

DEFINITION 4.6. For a subset $F \subset L_{\mathbb{R}}$, we define $\text{Cyc}_{\Gamma/L}(F) := \nu^{-1}(F) \subset \text{Cyc}_{\Gamma/L}$.

We fix $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$ and $v \in V(\Delta)$. We pick $\text{cpx}_\sigma(v) \in \mathbb{Z}_{\geq 0}$ and $r_{\sigma,v}^F, s_{\sigma,v}^F \in \mathbb{Z}_{\geq 0}$ for each $F \in \mathcal{H}''_{\sigma,v}$ satisfying the following two conditions (R) and (S):

(R) Suppose that a function $a : \text{Cyc}_{\Gamma/L}(\sigma) \rightarrow \mathbb{Z}_{\geq 0}$ satisfies

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q) > r_{\sigma,v}^F$$

for any $F \in \mathcal{H}''_{\sigma,v}$. Then, there exists a function $b : \text{Cyc}_{\Gamma/L}(\sigma) \rightarrow \mathbb{Z}_{\geq 0}$ satisfying

- $b^{-1}(\mathbb{Z}_{>0}) \subset a^{-1}(\mathbb{Z}_{>0})$, and
- $\sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot \mu(\langle q \rangle) = \text{cpx}_\sigma(v)v$.

(S) Suppose that a function $a : \text{Cyc}_{\Gamma/L}(\sigma) \rightarrow \mathbb{Z}_{\geq 0}$ satisfies

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q) > s_{\sigma,v}^F$$

for any $F \in \mathcal{H}''_{\sigma,v}$. Then, there exists a function $b : \text{Cyc}_{\Gamma/L}(\sigma) \rightarrow \mathbb{Z}_{\geq 0}$ satisfying

- $b(q) \leq a(q)$ holds for any $q \in \text{Cyc}_{\Gamma/L}(\sigma)$,
- $\sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot \mu(\langle q \rangle) = \text{cpx}_\sigma(v)v$, and
- for any $q \in b^{-1}(\mathbb{Z}_{>0})$, there exists $q' \in \text{Cyc}_{\Gamma/L}(\sigma)$ such that $b(q') < a(q')$ and $\text{supp}(q) \subset \text{supp}(q')$.

See Lemma 4.7 for the existence and a construction of $\text{cpx}_\sigma(v)$, $r_{\sigma,v}^F$ and $s_{\sigma,v}^F$. These technical conditions will be used in the proof of Theorems 4.16 and 4.17.

LEMMA 4.7. *Fix $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$ and $v \in V(\Delta)$. Then, there exist $\text{cpx}_\sigma(v) \in \mathbb{Z}_{>0}$ and $r_{\sigma,v}^F, s_{\sigma,v}^F \in \mathbb{Z}_{\geq 0}$ for each $F \in \mathcal{H}_{\sigma,v}''$ satisfying the conditions (R) and (S).*

Proof. We set $S_{\sigma,v} := \bigcup_{F \in \mathcal{H}_{\sigma,v}''} F \subset \text{Im}(\nu) \cap \sigma$.

We define $\text{cpx}_\sigma(v)$ as the minimum positive integer with the following condition.

- If a subset $G \subset \text{Cyc}_{\Gamma/L}(S_{\sigma,v})$ satisfies $v \in \text{conv}(\nu(G))$, then we have

$$\text{cpx}_\sigma(v)v \in \sum_{q \in G} \mathbb{Z}_{\geq 0} \cdot \mu(\langle q \rangle).$$

Such $\text{cpx}_\sigma(v) \in \mathbb{Z}_{>0}$ exists since we have $v \in L_\mathbb{Q}$ by the assumed condition $(\diamond)_1$ on T_σ , and the choice of G has finitely many possibilities.

Next, for $u \in S_{\sigma,v}$, we define $m_{\sigma,v}(u)$ as the minimum positive integer ℓ with the following condition.

- Let $G \subset S_{\sigma,v}$ be a set of $\mathbb{R}$ -linear independent elements satisfying $u \in G$ and $v \in \text{conv}(G)$. For $u' \in G$, we define $\text{proj}_{G,u'}(v) \in \mathbb{R}_{\geq 0}$ by the unique expression $v = \sum_{u' \in G} \text{proj}_{G,u'}(v) \cdot u'$. Then, we require ℓ to satisfy $\text{cpx}_\sigma(v) \cdot \text{proj}_{G,u}(v) \leq \ell$.

For $u \in \text{Im}(\nu)$ and $d \in \mathbb{Z}_{>0}$, we define

$$\begin{aligned} \text{Len}(\{u\}) &:= \{w(q) \mid q \in \text{Cyc}_{\Gamma/L}(\{u\})\}, \\ \text{Cyc}_{\Gamma/L}^d(\{u\}) &:= \{q \in \text{Cyc}_{\Gamma/L}(\{u\}) \mid w(q) = d\}, \\ \text{num}(u, d) &:= \#\{\text{supp}(q) \mid q \in \text{Cyc}_{\Gamma/L}^d(\{u\})\}. \end{aligned}$$

Then, we define $s_{\sigma,v}^F$ by

$$s_{\sigma,v}^F := \sum_{u \in \text{Im}(\nu) \cap F} \sum_{d \in \text{Len}(\{u\})} (m_{\sigma,v}(u) + d(\text{num}(u, d) - 1)).$$

In what follows, we shall show that $\text{cpx}_\sigma(v)$ and $s_{\sigma,v}^F$ satisfy condition (S).

Suppose that a function $a : \text{Cyc}_{\Gamma/L}(\sigma) \rightarrow \mathbb{Z}_{\geq 0}$ satisfies

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q) > s_{\sigma,v}^F$$

for each $F \in \mathcal{H}_{\sigma,v}''$. Note that

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q) = \sum_{u \in \text{Im}(\nu) \cap F} \sum_{d \in \text{Len}(\{u\})} \left(d \cdot \sum_{q \in \text{Cyc}_{\Gamma/L}^d(\{u\})} a(q) \right).$$

Therefore, for each F , we can pick $u_F \in \text{Im}(\nu) \cap F$ and $d_F \in \text{Len}(\{u_F\})$ such that

$$d_F \cdot \sum_{q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\})} a(q) > m_{\sigma,v}(u_F) + d_F(\text{num}(u_F, d_F) - 1).$$

By Lemma 4.4, we have $v \in \text{conv}(\{u_F \mid F \in \mathcal{H}_{\sigma,v}''\})$. We take a minimal subset $I \subset \mathcal{H}_{\sigma,v}''$ such that

$$v \in \text{conv}(\{u_F \mid F \in I\}).$$

Then by the minimality of I , the elements of $\{u_F \mid F \in I\}$ are $\mathbb{R}$ -linear independent. Therefore, we may uniquely write

$$\text{cpx}_\sigma(v)v = \sum_{F \in I} c_F d_F u_F$$

with $c_F \in \mathbb{R}_{>0}$. By the choice of $\text{cpx}_\sigma(v)$, we have $c_F \in \mathbb{Z}_{>0}$. Moreover, by the choice of $m_{\sigma,v}(u_F)$, we have $c_F d_F \leq m_{\sigma,v}(u_F)$. In particular, we have

$$c_F \leq \frac{m_{\sigma,v}(u_F)}{d_F} < 1 - \text{num}(u_F, d_F) + \sum_{q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\})} a(q).$$

For each F , by the definition of $\text{num}(u_F, d_F)$, we can take $q_1, \dots, q_\ell \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\})$ such that

- $\ell \leq \text{num}(u_F, d_F)$,
- $a(q_1), \dots, a(q_\ell) > 0$, and
- $\{\text{supp}(q) \mid q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\}), a(q) > 0\} = \{\text{supp}(q_1), \dots, \text{supp}(q_\ell)\}$.

Then we define a function $f'_F : \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\}) \rightarrow \mathbb{Z}_{\geq 0}$ by

$$f'_F(q) := \begin{cases} a(q) - 1 & \text{if } q = q_i \text{ for some } i = 1, \dots, \ell, \\ a(q) & \text{otherwise.} \end{cases}$$

Then, we have

$$\begin{aligned} \text{(i)} \quad & \{\text{supp}(q) \mid q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\}), f'_F(q) > 0\} \\ & \subset \{\text{supp}(q_1), \dots, \text{supp}(q_\ell)\} \\ & = \{\text{supp}(q) \mid q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\}), a(q) > f'_F(q)\}. \end{aligned}$$

Since we have

$$\begin{aligned} c_F & \leq -\text{num}(u_F, d_F) + \sum_{q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\})} a(q) \\ & \leq -\ell + \sum_{q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\})} a(q) \\ & = \sum_{q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\})} f'_F(q), \end{aligned}$$

we can take a function $f_F : \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\}) \rightarrow \mathbb{Z}_{\geq 0}$ satisfying the following two conditions:

- (ii) $\sum_{q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\})} f_F(q) = c_F$.
- (iii) $f_F(q) \leq f'_F(q)$ holds for any $q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\})$.

By (i) and (iii), we have

$$\begin{aligned} \text{(iv)} \quad & \{\text{supp}(q) \mid q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\}), f_F(q) > 0\} \\ & \subset \{\text{supp}(q) \mid q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\}), a(q) > f_F(q)\}. \end{aligned}$$

We define a function $b : \text{Cyc}_{\Gamma/L}(\sigma) \rightarrow \mathbb{Z}_{\geq 0}$ by

$$b(q) := \begin{cases} f_F(q) & \text{if } q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\}) \text{ for some } F \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by (iii) and (iv), the function b satisfies the following two conditions:

- $b(q) \leq a(q)$ holds for any $q \in \text{Cyc}_{\Gamma/L}(\sigma)$.
- For any $q \in b^{-1}(\mathbb{Z}_{>0})$, there exists $q' \in \text{Cyc}_{\Gamma/L}(\sigma)$ such that $b(q') < a(q')$ and $\text{supp}(q) = \text{supp}(q')$ (in particular, $\text{supp}(q) \subset \text{supp}(q')$).

By (ii), we have

$$\begin{aligned}
 \sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot \mu(\langle q \rangle) &= \sum_{F \in I} \sum_{q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\})} f_F(q) \cdot \mu(\langle q \rangle) \\
 &= \sum_{F \in I} \sum_{q \in \text{Cyc}_{\Gamma/L}^{d_F}(\{u_F\})} f_F(q) d_F u_F \\
 &= \sum_{F \in I} c_F d_F u_F \\
 &= \text{cpx}_{\sigma}(v)v.
 \end{aligned}$$

Therefore, this b satisfies the condition (S).

We define $r_{\sigma,v}^F := 0$ for any $F \in \mathcal{H}_{\sigma,v}''$. Then by the same argument as for $s_{\sigma,v}^F$, we can show that $\text{cpx}_{\sigma}(v)$ and $r_{\sigma,v}^F$'s satisfy the condition (R). $\square$

4.1.4. $a_{\Delta,v}^F(v')$ and $h_{\Delta,v}^F$.

DEFINITION 4.8. For a subset $F \subset L_{\mathbb{R}}$ that consists of $\mathbb{R}$ -linear independent n elements, we define

$$H(F) := \left\{ \sum_{x \in F} a_x x \mid a_x \in \mathbb{R} \text{ with } \sum_{x \in F} a_x \leq 1 \right\}.$$

This is the closed half-space H of $L_{\mathbb{R}}$ satisfying $0 \in H$ and $F \subset \partial H := H \setminus \text{relint}(H)$.

We fix $\sigma \in \text{Facet}(P)$, $\Delta \in T_{\sigma}$, $v \in V(\Delta)$ and $F \in \mathcal{H}_{\sigma,v}''$. For each $v' \in V(\Delta)$, we pick $a_{\Delta,v}^F(v') \in (0, 1]$ with the following conditions:

- (A1) $a_{\Delta,v}^F(v) \in (0, 1)$.
- (A2) $\text{Im}(\nu) \setminus F \subset H(\{a_{\Delta,v}^F(v')v' \mid v' \in V(\Delta)\})$.

See Lemma 4.10 for the existence of such $a_{\Delta,v}^F(v')$'s.

We set

$$(H) \quad h_{\Delta,v}^F := \max\{\alpha \in \mathbb{R}_{\geq 0} \mid \alpha P \subset H(\{a_{\Delta,v}^F(v')v' \mid v' \in V(\Delta)\})\}.$$

We have $0 < h_{\Delta,v}^F \leq a_{\Delta,v}^F(v) < 1$ since $a_{\Delta,v}^F(v') > 0$ for each $v' \in V(\Delta)$.

Lemma 4.9 below will be used in the proof of Lemma 4.10. We prove Lemma 4.9 following the proof of the hyperplane separation theorem (cf. [13, Theorem 1.17]). In the proof below, the finiteness of F' and the assumption $(\diamond)_2$ on the triangulation T_{σ} will be important.

LEMMA 4.9. Fix $\sigma \in \text{Facet}(P)$, $\Delta \in T_{\sigma}$ and $v \in V(\Delta)$. Let $F' \subset \text{Im}(\nu) \cap \sigma$ be a subset satisfying $v \notin \text{conv}(F')$. Then, there exists a closed halfspace H' of $\bar{\sigma}$ such that

$$v \notin H', \quad \text{relint}(\Delta) \cap H' = \emptyset, \quad F' \subset H'.$$

Proof. Note that we have $\text{relint}(\Delta) \cap \text{conv}(F') = \emptyset$. Otherwise, we have $\Delta \subset \text{conv}(F')$ by the assumption $(\diamond)_2$, and it contradicts $v \notin \text{conv}(F')$. If $\Delta \cap \text{conv}(F') = \emptyset$, then the assertion follows from the hyperplane separation theorem (cf. [13, Theorem 1.17]). In what follows, we assume that $\Delta \cap \text{conv}(F') \neq \emptyset$.

We identify $\bar{\sigma} = \mathbb{R}^{n-1}$, and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^{n-1}$. Let $v_1 = v, v_2, \dots, v_n$ be the vertices of Δ , and let $x_1, \dots, x_{\ell}$ be the elements of F' . For $1 \leq i \leq n$ and $1 \leq j \leq \ell$, we define $y_{ij} := x_j - v_i$. We set

$$Q := \text{conv}(\{y_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq \ell\}).$$

Since $\Delta \cap \text{conv}(F') \neq \emptyset$, we have $0 \in Q$. Furthermore, since $\text{relint}(\Delta) \cap \text{conv}(F') = \emptyset$, we have $0 \notin \text{relint}(Q)$. Let G be the minimal face of Q satisfying $0 \in G$. Then, there exists $u \in \mathbb{R}^{n-1} \setminus \{0\}$ such that

$$Q \subset \{z \in \mathbb{R}^{n-1} \mid \langle z, u \rangle \geq 0\}, \quad \{z \in \mathbb{R}^{n-1} \mid \langle z, u \rangle = 0\} \cap Q = G.$$

We set $\alpha := \min_{1 \leq j \leq \ell} \langle x_j, u \rangle$. Then we have

$$(1) \quad F' \subset \{z \in \mathbb{R}^{n-1} \mid \langle z, u \rangle \geq \alpha\}.$$

Since $Q \subset \{z \in \mathbb{R}^{n-1} \mid \langle z, u \rangle \geq 0\}$, we have $\langle x_j, u \rangle \geq \langle v_i, u \rangle$ for any $1 \leq i \leq n$ and $1 \leq j \leq \ell$. Therefore, we have $\max_{1 \leq i \leq n} \langle v_i, u \rangle \leq \alpha$, and hence, we have

$$(2) \quad \Delta \subset \{z \in \mathbb{R}^{n-1} \mid \langle z, u \rangle \leq \alpha\}.$$

In what follows, we prove $\langle v, u \rangle < \alpha$. We define

$$I := \{(i, j) \in \{1, \dots, n\} \times \{1, \dots, \ell\} \mid y_{ij} \in G\}.$$

By the minimality of G , we have $0 \in \text{relint}(G)$. Therefore, there exist $c_{ij} \in \mathbb{R}_{>0}$ for $(i, j) \in I$ such that

$$\sum_{(i,j) \in I} c_{ij} y_{ij} = 0, \quad \sum_{(i,j) \in I} c_{ij} = 1.$$

Then, we have

$$(3) \quad \sum_{(i,j) \in I} c_{ij} v_i = \sum_{(i,j) \in I} c_{ij} x_j.$$

Let Δ' be the face of Δ such that $V(\Delta') = \{v_i \mid (i, j) \in I\}$. Then, by (3), we have $\text{relint}(\Delta') \cap \text{conv}(F') \neq \emptyset$. Therefore, by $(\diamond)_2$, we have $\Delta' \subset \text{conv}(F')$. Since $v \notin \text{conv}(F')$, we can conclude that $v_1 = v \notin V(\Delta')$. Therefore, by the definition of I , we have $y_{11}, \dots, y_{1\ell} \notin G$, and hence, $\langle y_{1j}, u \rangle > 0$ for any $1 \leq j \leq \ell$. Thus, for j' satisfying $\langle x_{j'}, u \rangle = \alpha$, we have

$$(4) \quad \langle v_1, u \rangle = \langle x_{j'}, u \rangle - \langle y_{1j'}, u \rangle < \alpha.$$

By (1), (2) and (4), we conclude that

$$H' := \{z \in \mathbb{R}^{n-1} \mid \langle z, u \rangle \geq \alpha\}$$

satisfies the desired conditions. $\square$

LEMMA 4.10. Fix $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$, $v \in V(\Delta)$ and $F \in \mathcal{H}_{\sigma,v}''$. Then, for each $v' \in V(\Delta)$, there exists $a_{\Delta,v}^F(v') \in (0, 1]$ such that

- (A1) $a_{\Delta,v}^F(v) \in (0, 1)$, and
- (A2) $\text{Im}(\nu) \setminus F \subset H(\{a_{\Delta,v}^F(v')v' \mid v' \in V(\Delta)\})$.

Proof. We set $F'_\sigma := (\text{Im}(\nu) \cap \sigma) \setminus F$. By the definition of $\mathcal{H}_{\sigma,v}''$, we have $v \notin \text{conv}(F'_\sigma)$. Therefore, by Lemma 4.9, there exists a closed halfspace H' of $\bar{\sigma}$ such that

$$v \notin H', \quad \text{relint}(\Delta) \subset \bar{\sigma} \setminus H', \quad F'_\sigma \subset H'.$$

Let $\ell := \partial(H') = H' \setminus \text{relint}(H')$.

For $a \in \mathbb{R}_{>0}$, let $H(\ell, av)$ denote the closed halfspace of $L_\mathbb{R}$ uniquely determined by the following conditions:

$$0 \in H(\ell, av), \quad \ell \subset \partial(H(\ell, av)), \quad av \in \partial(H(\ell, av)),$$

where $\partial(H(\ell, av)) := H(\ell, av) \setminus \text{relint}(H(\ell, av))$. Note that $\partial(H(\ell, v)) = \bar{\sigma}$. Therefore, we have $\text{Im}(\nu) \setminus \sigma \subset \text{relint}(H(\ell, v))$. Since $\text{Im}(\nu)$ is a finite set, there exists $\epsilon \in (0, 1)$ such that

$$\text{Im}(\nu) \setminus \sigma \subset H(\ell, (1 - \epsilon)v).$$

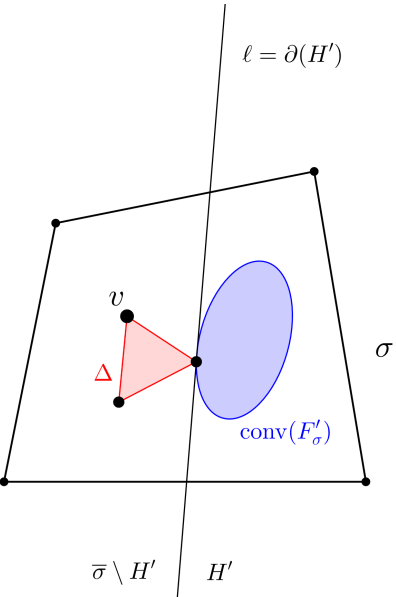


FIGURE 3. H' and ℓ .

For each $v' \in V(\Delta)$, we take $a(v') \in \mathbb{R}_{>0}$ so that

$$H(\ell, (1 - \epsilon)v) = H(\{a(v')v' \mid v' \in V(\Delta)\}).$$

This $a(v')$ can also be determined by $a(v')v' \in \partial(H(\ell, (1 - \epsilon)v))$. We have $a(v) = 1 - \epsilon \in (0, 1)$. Note that on $\bar{\sigma}$, Δ is on the same side as v with respect to the hyperplane ℓ . Therefore, we have $a(v') \leq 1$ for any $v' \in V(\Delta)$.

Since $v \notin H'$ and $1 - \epsilon < 1$, we have

$$H' = H(\ell, (1 - \epsilon)v) \cap \bar{\sigma}.$$

Therefore, we have

$$\begin{aligned} \text{Im}(\nu) \setminus F &= (\text{Im}(\nu) \setminus \sigma) \cup F'_\sigma \\ &\subset (\text{Im}(\nu) \setminus \sigma) \cup H' \\ &\subset H(\ell, (1 - \epsilon)v) = H(\{a(v')v' \mid v' \in V(\Delta)\}). \end{aligned}$$

Therefore, $a_{\Delta,v}^F(v') := a(v') \in (0, 1]$ satisfies the desired conditions. □

4.1.5. C'_2 , β and cpx_Γ . We define

$$W := \max\{w(e) \mid e \in E_\Gamma\}.$$

We have $W < \infty$ by the definition of periodic graphs. We pick $C'_2 \in \mathbb{R}_{\geq 0}$ satisfying $C_2 \leq C'_2$ (cf. Proposition 3.14(2)).

Let $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$ and $v \in V(\Delta)$. For $F \in \mathcal{H}''_{\sigma,v}$, we set $\alpha_{\Delta,v}^F, \alpha'_{\Delta,v} \in \mathbb{R}_{\geq 0}$ by

$$\begin{aligned} \alpha_{\Delta,v}^F &:= \frac{a_{\Delta,v}^F(v)}{1 - a_{\Delta,v}^F(v)} \left(\frac{C_1}{h_{\Delta,v}^F} + C'_2 + \frac{1 - h_{\Delta,v}^F}{h_{\Delta,v}^F} (r_{\sigma,v}^F + W(\#(V_\Gamma/L) - 1)) \right), \\ \alpha'_{\Delta,v} &:= \frac{a_{\Delta,v}^F(v)}{1 - a_{\Delta,v}^F(v)} \left(\frac{C_1}{h_{\Delta,v}^F} + C'_2 + \frac{1 - h_{\Delta,v}^F}{h_{\Delta,v}^F} (s_{\sigma,v}^F + W(\#(V_\Gamma/L) - 1)) \right). \end{aligned}$$

Then, we define $\alpha_{\Delta,v}, \alpha'_{\Delta,v} \in \mathbb{R}_{\geq 0}$ as follows.

$$\alpha_{\Delta,v} := \max_{F \in \mathcal{H}''_{\sigma,v}} \alpha_{\Delta,v}^F, \quad \alpha'_{\Delta,v} := \max_{F \in \mathcal{H}''_{\sigma,v}} \alpha_{\Delta,v}^{'F}.$$

We define $\beta_{\Delta,v}, \beta'_{\Delta,v} \in \mathbb{R}_{\geq 0}$ by

$$\beta_{\Delta,v} := \max\{\alpha_{\Delta,v}, \alpha'_{\Delta,v} - \text{cpx}_{\sigma}(v)\}, \quad \beta'_{\Delta,v} := \beta_{\Delta,v} + \text{cpx}_{\sigma}(v).$$

Furthermore, for $\Delta' \in \text{Face}(\Delta)$, we define

$$\beta_{\Delta,\Delta'} := \sum_{v \in V(\Delta')} \beta_{\Delta,v}.$$

We also define $\beta \in \mathbb{R}_{\geq 0}$ by

$$\beta := C'_2 + \max\{\beta_{\Delta,\Delta} \mid \sigma \in \text{Facet}(P), \Delta \in T_{\sigma}\}.$$

Let $\sigma \in \text{Facet}(P)$, $\Delta \in T_{\sigma}$ and $\Delta' \in \text{Face}(\Delta)$. Then we define $\text{cpx}_{\sigma}(\Delta') \in \mathbb{Z}_{>0}$ by

$$\text{cpx}_{\sigma}(\Delta') := \text{LCM}\{\text{cpx}_{\sigma}(v) \mid v \in V(\Delta')\}.$$

We also define $\text{cpx}_{\Gamma} \in \mathbb{Z}_{>0}$ by

$$\text{cpx}_{\Gamma} := \text{LCM}\{\text{cpx}_{\sigma}(\Delta) \mid \sigma \in \text{Facet}(P), \Delta \in T_{\sigma}\}.$$

REMARK 4.11. (1) The notations $\beta_{\Delta,\Delta'}$ and $\text{cpx}_{\sigma}(\Delta')$ for $\Delta' \neq \Delta$ were not used to define β and cpx_{Γ} , but they will be used in Theorem 4.22. The notation $\beta'_{\Delta,v}$ will be used in Definition 4.19.

(2) When we use the invariants $\text{cpx}_{\sigma}(v)$, $r_{\sigma,v}^F$ and $s_{\sigma,v}^F$ constructed in the proof of Lemma 4.7, we have

$$\beta_{\Delta,v} = \alpha'_{\Delta,v} - \text{cpx}_{\sigma}(v)$$

by Lemma 4.12 below.

LEMMA 4.12. Let $\sigma \in \text{Facet}(P)$, $\Delta \in T_{\sigma}$, $v \in V(\Delta)$ and $F \in \mathcal{H}''_{\sigma,v}$. Suppose that $\text{cpx}_{\sigma}(v)$, $r_{\sigma,v}^F$ and $s_{\sigma,v}^F$ are the invariants constructed in the proof of Lemma 4.7. Then, we have $\alpha'_{\Delta,v} - \alpha_{\Delta,v}^F \geq \text{cpx}_{\sigma}(v)$, and hence, $\beta_{\Delta,v} = \alpha'_{\Delta,v} - \text{cpx}_{\sigma}(v)$.

Proof. Note that we set $r_{\sigma,v}^F = 0$ in the construction. Therefore, it is sufficient to show the inequality

$$(i) \quad \frac{a_{\Delta,v}^F(v)}{1 - a_{\Delta,v}^F(v)} \cdot \frac{1 - h_{\Delta,v}^F}{h_{\Delta,v}^F} \cdot s_{\sigma,v}^F \geq \text{cpx}_{\sigma}(v).$$

When $v \in F$, we have $s_{\sigma,v}^F \geq m_{\sigma,v}(v) = \text{cpx}_{\sigma}(v)$ by the construction. Since $h_{\Delta,v}^F \leq a_{\Delta,v}^F(v) < 1$, we obtain the desired inequality (i).

In what follows, we assume $v \notin F$. Take any $v_F \in F$. For each $F' \in \mathcal{H}''_{\sigma,v} \setminus \{F\}$, we take any $v_{F'} \in F' \setminus F$. Then by Lemma 4.4, we have

$$v \in \text{conv}(\{v_{F'} \mid F' \in \mathcal{H}''_{\sigma,v}\}).$$

Take a minimal subset $I \subset \mathcal{H}''_{\sigma,v}$ satisfying $v \in \text{conv}(\{v_{F'} \mid F' \in I\})$. Then, by the minimality of I , the elements of $\{v_{F'} \mid F' \in I\}$ are $\mathbb{R}$ -linear independent. Furthermore, by the definition of $\mathcal{H}''_{\sigma,v}$, we conclude $F \in I$ since $v_{F'} \notin F$ holds for each $F' \in I \setminus \{F\}$.

Since $v \in \text{conv}(\{v_{F'} \mid F' \in I\})$, there exist $w \in \text{conv}(\{v_{F'} \mid F' \in I \setminus \{F\}\})$ and $t \in [0, 1]$ such that

$$v = tv_F + (1 - t)w.$$

We have $t \neq 1$ by the assumption $v \notin F$. We also have $t \neq 0$ by the minimality of I . Therefore, we conclude that $t \in (0, 1)$. Furthermore, by the construction of $s_{\sigma,v}^F$ in the proof of Lemma 4.7, we have

$$(ii) \quad s_{\sigma,v}^F \geq m_{\sigma,v}(v_F) \geq t \operatorname{cpx}_{\sigma}(v).$$

Let $H := H(\{a_{\Delta,v}^F(v')v' \mid v' \in V(\Delta)\})$ be the half-space defined in Definition 4.8. By the conditions (A1) and (A2) on $a_{\Delta,v}^F(v')$'s in Subsection 4.1.4, we have

$$v \notin H, \quad w \in \operatorname{conv}((\operatorname{Im}(\nu) \cap \sigma) \setminus F) \subset H.$$

Therefore, ∂H intersects with the line through v and w . Let x be the intersection point. Then the points v_F , v , x , and w are on the line in this order. Let $b \in \mathbb{R}$ be the unique real number such that $bv_F \in \partial H$. Then, we have $h_{\Delta,v}^F \leq b < 1$ since $v_F \notin H$ and $h_{\Delta,v}^F v_F \in H$. Therefore, we have

$$(iii) \quad \frac{a_{\Delta,v}^F(v)}{1 - a_{\Delta,v}^F(v)} \cdot \frac{1 - h_{\Delta,v}^F}{h_{\Delta,v}^F} \geq \frac{|a_{\Delta,v}^F(v)v - 0|}{|v - a_{\Delta,v}^F(v)v|} \cdot \frac{|v_F - bv_F|}{|bv_F - 0|} \\ = \frac{|v_F - x|}{|x - v|} \\ \geq \frac{|v_F - w|}{|w - v|} = \frac{1}{t}.$$

Here, we used Menelaus's theorem for the first equality. From (ii) and (iii), we get the desired inequality (i). $\square$

4.2. INVARIANTS FOR 2-DIMENSIONAL PERIODIC GRAPHS. In general, it is not easy to find the invariants in Subsection 4.1. However, it is easy for 2-dimensional periodic graphs. In what follows, we assume $n = 2$.

Let $\sigma \in \operatorname{Facet}(P)$, and let $\{v_1, \dots, v_{\ell}\} = \operatorname{Im}(\nu) \cap \sigma$. Since $\dim \sigma = 1$, the points $v_1, \dots, v_{\ell}$ determine the triangulation T_{σ} of σ satisfying $\bigcup_{\Delta \in T_{\sigma}} V(\Delta) = \{v_1, \dots, v_{\ell}\}$. Furthermore, since $\dim \sigma = 1$, this T_{σ} satisfies the required conditions $(\diamond)_1$ and $(\diamond)_2$.

Since the triangulation T_{σ} is concretely given, $\operatorname{cpx}_{\sigma}(v)$ and $s_{\sigma,v}^F$ can be easily computed according to the construction in the proof of Lemma 4.7. We shall explain the construction of the invariant $a_{\sigma,v}^F$ below.

Suppose that the points $v_1, \dots, v_{\ell}$ are on σ in this order as in Figure 4. For $1 \leq i \leq \ell - 1$, let $\Delta_i \in T_{\sigma}$ denote the 1-simplex determined by $V(\Delta_i) = \{v_i, v_{i+1}\}$. For $1 \leq i \leq \ell$, we set

$$F_i^- := \{v_1, \dots, v_i\}, \quad F_i^+ := \{v_i, \dots, v_{\ell}\}.$$

Then we have

$$\mathcal{H}_{\sigma,v_i}'' = \begin{cases} \{F_i^-\} = \{\{v_1\}\} & (\text{if } i = 1) \\ \{F_i^-, F_i^+\} & (\text{if } 2 \leq i \leq \ell - 1) \\ \{F_i^+\} = \{\{v_{\ell}\}\} & (\text{if } i = \ell) \end{cases}.$$

By symmetry, it is sufficient to see the construction of $a_{\Delta_{i-1},v_i}^{F_i^+}(v_{i-1})$ and $a_{\Delta_{i-1},v_i}^{F_i^+}(v_i)$ for $2 \leq i \leq \ell$, and $a_{\Delta_i,v_i}^{F_i^+}(v_i)$ and $a_{\Delta_i,v_i}^{F_i^+}(v_{i+1})$ for $2 \leq i \leq \ell - 1$. We fix $2 \leq i \leq \ell$. First, we define $t_1 \in (0, 1)$ by

$$t_1 := \max \left\{ \epsilon, \min \{ \alpha \in \mathbb{R}_{\geq 0} \mid \operatorname{Im}(\nu) \setminus F_i^+ \subset H(v_{i-1}, \alpha v_i) \} \right\},$$

where ϵ is any real number satisfying $\epsilon \in (0, 1)$. Then, $a_{\Delta_{i-1}, v_i}^{F_i^+}(v_i) := t_1$ and $a_{\Delta_{i-1}, v_i}^{F_i^+}(v_{i-1}) := 1$ satisfy the conditions (A1) and (A2) in Subsection 4.1.4 for Δ_{i-1} and v_i . When $i \leq \ell - 1$, we define $t_2 \in (0, 1)$ to satisfy

$$H(v_{i-1}, t_1 v_i) = H(t_1 v_i, t_2 v_{i+1}).$$

Then, $a_{\Delta_i, v_i}^{F_i^+}(v_i) := t_1$ and $a_{\Delta_i, v_i}^{F_i^+}(v_{i+1}) := t_2$ satisfy the conditions (A1) and (A2) in Subsection 4.1.4 for Δ_i and v_i .

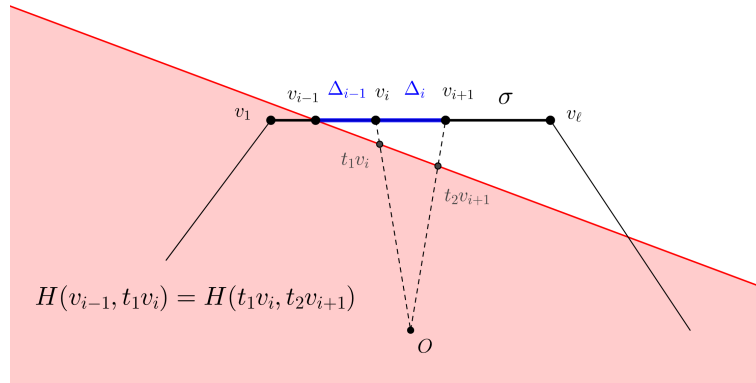


FIGURE 4. 2-dimensional case.

EXAMPLE 4.13. We shall calculate the invariants in Subsection 4.1 for the Wakatsuki graph and the start point $x_0 = v'_2$ in Figure 5. See [15, Example 2.6] for the detailed definition of the Wakatsuki graph.

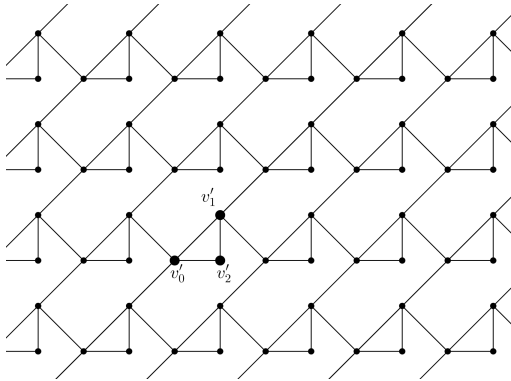


FIGURE 5. The Wakatsuki graph Γ .

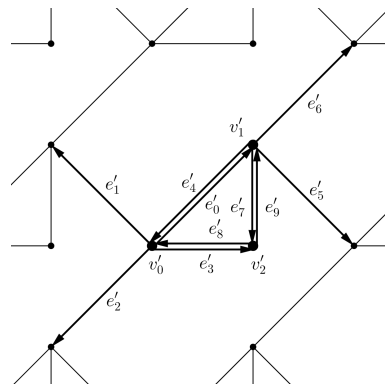


FIGURE 6. $e'_0, \dots, e'_9$.

It is known that $C_1 = 1$ and we can take $C'_2 = 3$ ([15, Example A.3]). $\text{Im}(\nu)$ consists of 11 points as in Figure 7 ([15, Example 2.22]).

Let $\sigma_1, \sigma_2, \sigma_3 \in \text{Facet}(P)$ and $u_1, u_2, u_3, u_4 \in V(P)$ as in Figure 7. Then, we have $\text{cpx}_\sigma(v) = 2$ for any $v \in V(P)$ and $\sigma \in \text{Facet}(P)$, and the values $s_{\sigma, v}^{\{v\}}$, $a_{\sigma, v}^{\{v\}}(v)$, $\beta_{\sigma, v}$ and $\beta_{\sigma, \sigma}$ are given by Table 1. By symmetry, we have $\text{cpx}_\Gamma = 2$ and $\beta = C'_2 + \max_{i=1,2,3} \beta_{\sigma_i, \sigma_i} = 25$.

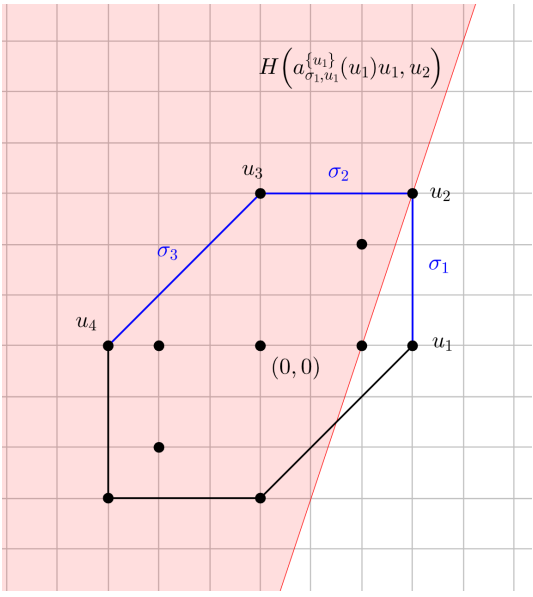


FIGURE 7. $\sigma_1, \sigma_2, \sigma_3$, and u_1, u_2, u_3, u_4 .

(σ, v)	(σ_1, u_1)	(σ_1, u_2)	(σ_2, u_2)	(σ_2, u_3)	(σ_3, u_3)	(σ_3, u_4)
$s_{\sigma, v}^{\{v\}}$	2	2	2	2	2	2
$a_{\sigma, v}^{\{v\}}(v)$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$
$\beta_{\sigma, v}$	11	11	11	7	7	11
$\beta_{\sigma, \sigma}$	22		18		18	

TABLE 1. Invariants for the Wakatsuki graph.

We shall briefly explain the values $\text{cpx}_{\sigma_1}(u_1)$, $s_{\sigma_1, u_1}^{\{u_1\}}$ and $a_{\sigma_1, u_1}^{\{u_1\}}(u_1)$. First, we can see that

$$\text{Cyc}_{\Gamma/L}(\{u_1\}) = \left\{ \overline{e'_0} \, \overline{e'_5}, \, \overline{e'_5} \, \overline{e'_0} \right\}.$$

Therefore, we have $\text{cpx}_{\sigma_1}(u_1) = \text{LCM}\{2, 2\} = 2$. Since $\text{supp} \left(\overline{e'_0} \, \overline{e'_5} \right) = \left\{ \overline{v'_0}, \overline{v'_1} \right\} = \text{supp} \left(\overline{e'_5} \, \overline{e'_0} \right)$, we have $\text{num}(u_1, 2) = 1$, and therefore, we have $s_{\sigma_1, u_1}^{\{u_1\}} = \text{cpx}_{\sigma_1}(u_1) = 2$. Next, $a_{\sigma_1, u_1}^{\{u_1\}}(u_1)$ is defined as the minimum real number α satisfying

$$\text{Im}(\nu) \setminus \{u_1\} \subset H(\alpha u_1, u_2).$$

$H(\alpha u_1, u_2)$ is illustrated in Figure 7.

4.3. MAIN THEOREM. We keep the notations in Subsection 4.1.

DEFINITION 4.14. Let $\sigma \in \text{Facet}(P)$ and $\Delta \in T_\sigma$. For $x \in L_\mathbb{R}$ and $v \in V(\Delta)$, we define $\text{proj}_{\Delta, v}(x) \in \mathbb{R}$ by the unique expression $x = \sum_{v \in V(\Delta)} \text{proj}_{\Delta, v}(x)v$.

The following lemma explains why we defined $\alpha_{\Delta, v}$ and $\alpha'_{\Delta, v}$ as in Subsection 4.1.5.

LEMMA 4.15. *Let $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$ and $v \in V(\Delta)$. Let $y \in V_\Gamma \cap \Phi^{-1}(\mathbb{R}_{\geq 0}\Delta)$, and let p be a shortest walk in Γ from x_0 to y . By applying Lemma 3.7(1) to $\bar{p}$, we take a path q_0 in Γ/L and a function $a : \text{Cyc}_{\Gamma/L} \rightarrow \mathbb{Z}_{\geq 0}$ such that $\langle \bar{p} \rangle = \langle q_0 \rangle + \sum_{q \in \text{Cyc}_{\Gamma/L}} a(q)\langle q \rangle$. Then the following assertions hold.*

(1) We fix any $F \in \mathcal{H}_{\sigma,v}''$ and $t \in \mathbb{R}_{\geq 0}$. If $\text{proj}_{\Delta,v}(\Phi(y)) > f^F(t)$ holds for

$$f^F(t) := \frac{a_{\Delta,v}^F(v)}{1 - a_{\Delta,v}^F(v)} \left(\frac{C_1}{h_{\Delta,v}^F} + C'_2 + \frac{1 - h_{\Delta,v}^F}{h_{\Delta,v}^F} (t + W(\#(V_{\Gamma}/L) - 1)) \right),$$

then we have

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q) > t.$$

(2) In particular, if $\text{proj}_{\Delta,v}(\Phi(y)) > \alpha_{\Delta,v}$ (resp. $\text{proj}_{\Delta,v}(\Phi(y)) > \alpha'_{\Delta,v}$) holds, then we have

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q) > r_{\sigma,v}^F \quad (\text{resp.} \quad \sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q) > s_{\sigma,v}^F)$$

for each $F \in \mathcal{H}_{\sigma,v}''$.

Proof. We prove (1). Let $F \in \mathcal{H}_{\sigma,v}''$ and $t \in \mathbb{R}_{\geq 0}$. We assume

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q) \leq t.$$

First, we have

- $\mu(\langle q \rangle) \in w(q) \cdot H(\{a_{\Delta,v}^F(v')v' \mid v' \in V(\Delta)\})$ for $q \in \text{Cyc}_{\Gamma/L} \setminus \text{Cyc}_{\Gamma/L}(F)$,
- $\mu(\langle q \rangle) \in w(q) \cdot P$ for $q \in \text{Cyc}_{\Gamma/L}$, and
- $\mu(\langle q_0 \rangle) \in (C_1 + w(q_0))P$.

For simplicity, we put

$$\begin{aligned} \ell_0 &:= w(q_0), \\ \ell_1 &:= \sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q), \\ \ell_2 &:= \sum_{q \in \text{Cyc}_{\Gamma/L} \setminus \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q), \\ H' &:= H(\{a_{\Delta,v}^F(v')v' \mid v' \in V(\Delta)\}). \end{aligned}$$

Then, we have

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot \mu(\langle q \rangle) \in \ell_1 P, \quad \sum_{q \in \text{Cyc}_{\Gamma/L} \setminus \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot \mu(\langle q \rangle) \in \ell_2 H'.$$

Furthermore, we have

- $\ell_0 + \ell_1 + \ell_2 = w(p) = d_{\Gamma}(x_0, y)$,
- $\ell_0 \leq W \cdot \text{length}(q_0) \leq W(\#(V_{\Gamma}/L) - 1)$, and
- $\ell_1 = \sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q) \leq t$.

Therefore, we have

$$\begin{aligned} \Phi(y) &= \mu(\langle \bar{p} \rangle) \\ &= \mu(\langle q_0 \rangle) + \sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot \mu(\langle q \rangle) + \sum_{q \in \text{Cyc}_{\Gamma/L} \setminus \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot \mu(\langle q \rangle) \\ &\in (C_1 + \ell_0)P + \ell_1 P + \ell_2 H'. \end{aligned}$$

Since we have $P \subset (h_{\Delta,v}^F)^{-1}H'$ by the choice of $h_{\Delta,v}^F$, we have

$$\Phi(y) \in ((h_{\Delta,v}^F)^{-1}(C_1 + \ell_0 + \ell_1) + \ell_2)H'.$$

Since we have

$$\begin{aligned} & (h_{\Delta,v}^F)^{-1}(C_1 + \ell_0 + \ell_1) + \ell_2 \\ &= (h_{\Delta,v}^F)^{-1}C_1 + ((h_{\Delta,v}^F)^{-1} - 1)(\ell_0 + \ell_1) + d_\Gamma(x_0, y) \\ &\leq (h_{\Delta,v}^F)^{-1}C_1 + ((h_{\Delta,v}^F)^{-1} - 1)(t + W(\#(V_\Gamma/L) - 1)) + d_\Gamma(x_0, y), \end{aligned}$$

we have

$$\Phi(y) \in \left((h_{\Delta,v}^F)^{-1}C_1 + ((h_{\Delta,v}^F)^{-1} - 1)(t + W(\#(V_\Gamma/L) - 1)) + d_\Gamma(x_0, y) \right) H'.$$

Therefore, we have

$$\sum_{v' \in V(\Delta)} \frac{\text{proj}_{\Delta,v'}(\Phi(y))}{a_{\Delta,v'}^F(v')} \leq \frac{C_1}{h_{\Delta,v}^F} + \frac{1 - h_{\Delta,v}^F}{h_{\Delta,v}^F} (t + W(\#(V_\Gamma/L) - 1)) + d_\Gamma(x_0, y).$$

On the other hand, by the definition of C_2 , we have

$$d_\Gamma(x_0, y) \leq C'_2 + d_{P,\Phi}(x_0, y) = C'_2 + \sum_{v' \in V(\Delta)} \text{proj}_{\Delta,v'}(\Phi(y)).$$

Since $a_{\Delta,v}^F(v') \leq 1$ holds for any $v' \in V(\Delta)$, we have

$$\begin{aligned} & \frac{1 - a_{\Delta,v}^F(v)}{a_{\Delta,v}^F(v)} \text{proj}_{\Delta,v}(\Phi(y)) \\ &\leq \left(\sum_{v' \in V(\Delta)} \frac{\text{proj}_{\Delta,v'}(\Phi(y))}{a_{\Delta,v'}^F(v')} \right) - \left(\sum_{v' \in V(\Delta)} \text{proj}_{\Delta,v'}(\Phi(y)) \right) \\ &\leq \frac{C_1}{h_{\Delta,v}^F} + C'_2 + \frac{1 - h_{\Delta,v}^F}{h_{\Delta,v}^F} (t + W(\#(V_\Gamma/L) - 1)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \text{proj}_{\Delta,v}(\Phi(y)) &\leq \frac{a_{\Delta,v}^F(v)}{1 - a_{\Delta,v}^F(v)} \left(\frac{C_1}{h_{\Delta,v}^F} + C'_2 + \frac{1 - h_{\Delta,v}^F}{h_{\Delta,v}^F} (t + W(\#(V_\Gamma/L) - 1)) \right) \\ &= f^F(t), \end{aligned}$$

which completes the proof of (1).

(2) follows from (1) and the definitions of $\alpha_{\Delta,v}$ and $\alpha'_{\Delta,v}$. $\square$

THEOREM 4.16. *Let $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$ and $v \in V(\Delta)$. Then for any $y \in V_\Gamma \cap \Phi^{-1}(\mathbb{R}_{\geq 0}\Delta)$ with $\text{proj}_{\Delta,v}(\Phi(y)) > \alpha_{\Delta,v}$, we have*

$$d_\Gamma(x_0, y + \text{cpx}_\sigma(v)v) \leq d_\Gamma(x_0, y) + \text{cpx}_\sigma(v).$$

Proof. Suppose that $y \in V_\Gamma \cap \Phi^{-1}(\mathbb{R}_{\geq 0}\Delta)$ satisfies $\text{proj}_{\Delta,v}(\Phi(y)) > \alpha_{\Delta,v}$. Let p be a shortest walk in Γ from x_0 to y . By applying Lemma 3.7(1) to $\bar{p}$, we can take a path q_0 in Γ/L and a function $a : \text{Cyc}_{\Gamma/L} \rightarrow \mathbb{Z}_{\geq 0}$ such that $\langle \bar{p} \rangle = \langle q_0 \rangle + \sum_{q \in \text{Cyc}_{\Gamma/L}} a(q) \langle q \rangle$. Then, by Lemma 4.15, we have

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q) > r_{\sigma,v}^F$$

for each $F \in \mathcal{H}''_{\sigma,v}$. Then, by the condition (R) in Subsection 4.1.3, there exists a function $b : \text{Cyc}_{\Gamma/L}(\sigma) \rightarrow \mathbb{Z}_{\geq 0}$ such that $b^{-1}(\mathbb{Z}_{>0}) \subset a^{-1}(\mathbb{Z}_{>0})$ and that

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot \mu(\langle q \rangle) = \text{cpx}_\sigma(v)v.$$

Since any $q \in \text{Cyc}_{\Gamma/L}(\sigma)$ satisfies $\frac{\mu(\langle q \rangle)}{w(q)} \in \sigma$, we also have

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot w(q) = \text{cpx}_{\sigma}(v).$$

We define a function $a' : \text{Cyc}_{\Gamma/L} \rightarrow \mathbb{Z}_{\geq 0}$ by

$$a'(q) := \begin{cases} a(q) + b(q) & \text{if } q \in \text{Cyc}_{\Gamma/L}(\sigma), \\ a(q) & \text{if } q \in \text{Cyc}_{\Gamma/L} \setminus \text{Cyc}_{\Gamma/L}(\sigma). \end{cases}$$

Then, we have $a'^{-1}(\mathbb{Z}_{>0}) = a^{-1}(\mathbb{Z}_{>0})$ since $b^{-1}(\mathbb{Z}_{>0}) \subset a^{-1}(\mathbb{Z}_{>0})$. Therefore, (q_0, a') is also a walkable pair by Lemma 3.7(2). Hence, there exists a walk p' in Γ with $s(p') = x_0$ such that

$$\langle \bar{p}' \rangle = \langle q_0 \rangle + \sum_{q \in \text{Cyc}_{\Gamma/L}} a'(q) \cdot \langle q \rangle = \langle \bar{p} \rangle + \sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot \langle q \rangle.$$

Hence, we have

$$\begin{aligned} w(p') &= w(p) + \sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot w(q) = d_{\Gamma}(x_0, y) + \text{cpx}_{\sigma}(v), \\ t(p') &= t(p) + \sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot \mu(\langle q \rangle) = y + \text{cpx}_{\sigma}(v)v. \end{aligned}$$

Therefore, p' is a walk from x_0 to $y + \text{cpx}_{\sigma}(v)v$ of weight equal to $d_{\Gamma}(x_0, y) + \text{cpx}_{\sigma}(v)$, which shows the desired inequality. $\square$

THEOREM 4.17. *Let $\sigma \in \text{Facet}(P)$, $\Delta \in T_{\sigma}$ and $v \in V(\Delta)$. Then for any $y \in V_{\Gamma} \cap \Phi^{-1}(\mathbb{R}_{\geq 0}\Delta)$ with $\text{proj}_{\Delta, v}(\Phi(y)) > \alpha'_{\Delta, v}$, we have*

$$d_{\Gamma}(x_0, y - \text{cpx}_{\sigma}(v)v) \leq d_{\Gamma}(x_0, y) - \text{cpx}_{\sigma}(v).$$

Proof. Suppose that $y \in V_{\Gamma} \cap \Phi^{-1}(\mathbb{R}_{\geq 0}\Delta)$ satisfies $\text{proj}_{\Delta, v}(\Phi(y)) > \alpha'_{\Delta, v}$. Let p be a shortest walk in Γ from x_0 to y . By applying Lemma 3.7(1) to $\bar{p}$, we can take a path q_0 in Γ/L and a function $a : \text{Cyc}_{\Gamma/L} \rightarrow \mathbb{Z}_{\geq 0}$ such that $\langle \bar{p} \rangle = \langle q_0 \rangle + \sum_{q \in \text{Cyc}_{\Gamma/L}} a(q) \langle q \rangle$. Then, by Lemma 4.15, we have

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(F)} a(q) \cdot w(q) > s_{\sigma, v}^F$$

for each $F \in \mathcal{H}_{\sigma, v}''$. Then, by the condition (S) in Subsection 4.1.3, there exists a function $b : \text{Cyc}_{\Gamma/L}(\sigma) \rightarrow \mathbb{Z}_{\geq 0}$ such that

- $b(q) \leq a(q)$ holds for any $q \in \text{Cyc}_{\Gamma/L}(\sigma)$,
- $\sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot \mu(\langle q \rangle) = \text{cpx}_{\sigma}(v)v$, and
- for any $q \in b^{-1}(\mathbb{Z}_{>0})$, there exists $q' \in \text{Cyc}_{\Gamma/L}(\sigma)$ such that $b(q') < a(q')$ and $\text{supp}(q) \subset \text{supp}(q')$.

Since any $q \in \text{Cyc}_{\Gamma/L}(\sigma)$ satisfies $\frac{\mu(\langle q \rangle)}{w(q)} \in \sigma$, we also have

$$\sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot w(q) = \text{cpx}_{\sigma}(v).$$

We define a function $a' : \text{Cyc}_{\Gamma/L} \rightarrow \mathbb{Z}_{\geq 0}$ by

$$a'(q) := \begin{cases} a(q) - b(q) & \text{(if } q \in \text{Cyc}_{\Gamma/L}(\sigma)) \\ a(q) & \text{(if } q \in \text{Cyc}_{\Gamma/L} \setminus \text{Cyc}_{\Gamma/L}(\sigma)). \end{cases}$$

Then, by the third condition on b above, (q_0, a') is also a walkable pair by Lemma 3.7(2). Hence, there exists a walk p' in Γ with $s(p') = x_0$ such that

$$\langle \bar{p}' \rangle = \langle q_0 \rangle + \sum_{q \in \text{Cyc}_{\Gamma/L}} a'(q) \cdot \langle q \rangle = \langle \bar{p} \rangle - \sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot \langle q \rangle.$$

Therefore, we have

$$\begin{aligned} w(p') &= w(p) - \sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot w(q) = d_{\Gamma}(x_0, y) - \text{cpx}_{\sigma}(v), \\ t(p') &= t(p) - \sum_{q \in \text{Cyc}_{\Gamma/L}(\sigma)} b(q) \cdot \mu(\langle q \rangle) = y - \text{cpx}_{\sigma}(v)v. \end{aligned}$$

Therefore, p' is a walk from x_0 to $y - \text{cpx}_{\sigma}(v)v$ of weight equal to $d_{\Gamma}(x_0, y) - \text{cpx}_{\sigma}(v)$, which shows the desired inequality. $\square$

THEOREM 4.18. *Let $\sigma \in \text{Facet}(P)$, $\Delta \in T_{\sigma}$ and $v \in V(\Delta)$. Then for any $y \in V_{\Gamma} \cap \Phi^{-1}(\mathbb{R}_{\geq 0}\Delta)$ with $\text{proj}_{\Delta,v}(\Phi(y)) > \beta_{\Delta,v}$, we have*

$$d_{\Gamma}(x_0, y + \text{cpx}_{\sigma}(v)v) = d_{\Gamma}(x_0, y) + \text{cpx}_{\sigma}(v).$$

Proof. Since $\beta_{\Delta,v} = \max\{\alpha_{\Delta,v}, \alpha'_{\Delta,v} - \text{cpx}_{\sigma}(v)\}$, we have

$$\begin{aligned} \text{proj}_{\Delta,v}(\Phi(y)) &> \beta_{\Delta,v} \geq \alpha_{\Delta,v}, \\ \text{proj}_{\Delta,v}(\Phi(y + \text{cpx}_{\sigma}(v)v)) &> \beta_{\Delta,v} + \text{cpx}_{\sigma}(v) \geq \alpha'_{\Delta,v}. \end{aligned}$$

Therefore, the assertion follows from Theorems 4.16 and 4.17. $\square$

DEFINITION 4.19.

(1) We define

$$B := \{(d, y) \in \mathbb{Z}_{\geq 0} \times V_{\Gamma} \mid d_{\Gamma}(x_0, y) \leq d\} \subset \mathbb{Z}_{\geq 0} \times V_{\Gamma}.$$

(2) For any subset $F \subset L_{\mathbb{R}}$, we define

$$B(F) := B \cap (\mathbb{Z}_{\geq 0} \times \Phi^{-1}(F)).$$

We also define $S(F)$ by

$$S(F) := \{(d_{\Gamma}(x_0, y), y) \mid y \in V_{\Gamma} \cap \Phi^{-1}(F)\}.$$

Note that we have $B(F) = \mathbb{Z}_{\geq 0}(1, 0) + S(F)$.

(3) Let $\sigma \in \text{Facet}(P)$, $\Delta \in T_{\sigma}$ and $\Delta' \in \text{Face}(\Delta)$. We define

$$L_{\Delta'} := \mathbb{R}_{\geq 0}\Delta' \subset L_{\mathbb{R}}.$$

For $v \in V(\Delta)$, we also define

$$\begin{aligned} L_{\Delta,v}^{>\beta} &:= \{z \in L_{\Delta} \mid \text{proj}_{\Delta,v}(z) > \beta_{\Delta,v}\}, \\ L_{\Delta,v}^{\leq\beta} &:= \{z \in L_{\Delta} \mid \text{proj}_{\Delta,v}(z) \leq \beta_{\Delta,v}\}, \\ L_{\Delta,v}^{(\beta, \beta']} &:= \{z \in L_{\Delta} \mid \beta'_{\Delta,v} \geq \text{proj}_{\Delta,v}(z) > \beta_{\Delta,v}\} \subset L_{\Delta,v}^{>\beta}. \end{aligned}$$

For $\Delta'' \in \text{Face}(\Delta')$, we define

$$\begin{aligned} L_{\Delta, \Delta', \Delta''} &:= L_{\Delta'} \cap \left(\bigcap_{v \in V(\Delta') \setminus V(\Delta'')} L_{\Delta,v}^{\leq\beta} \right) \cap \left(\bigcap_{v \in V(\Delta'')} L_{\Delta,v}^{>\beta} \right) \subset L_{\Delta'}, \\ \bar{L}_{\Delta, \Delta', \Delta''} &:= L_{\Delta'} \cap \left(\bigcap_{v \in V(\Delta') \setminus V(\Delta'')} L_{\Delta,v}^{\leq\beta} \right) \cap \left(\bigcap_{v \in V(\Delta'')} L_{\Delta,v}^{(\beta, \beta']} \right) \subset L_{\Delta, \Delta', \Delta''}. \end{aligned}$$

Then we have

$$L_{\Delta'} = \bigsqcup_{\Delta'' \in \text{Face}(\Delta')} L_{\Delta, \Delta', \Delta''}.$$

(4) For $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$ and $\Delta' \in \text{Face}(\Delta)$, we define a monoid $M_{\sigma, \Delta'}$ by

$$M_{\sigma, \Delta'} := \mathbb{Z}_{\geq 0}(1, 0) + \sum_{v \in V(\Delta')} \mathbb{Z}_{\geq 0}(\text{cpx}_\sigma(v), \text{cpx}_\sigma(v)v) \subset \mathbb{Z}_{\geq 0} \times L.$$

This is a free submonoid of $\mathbb{Z}_{\geq 0} \times L$ generated by $(1, 0)$ and $(\text{cpx}_\sigma(v), \text{cpx}_\sigma(v)v)$ for $v \in V(\Delta')$.

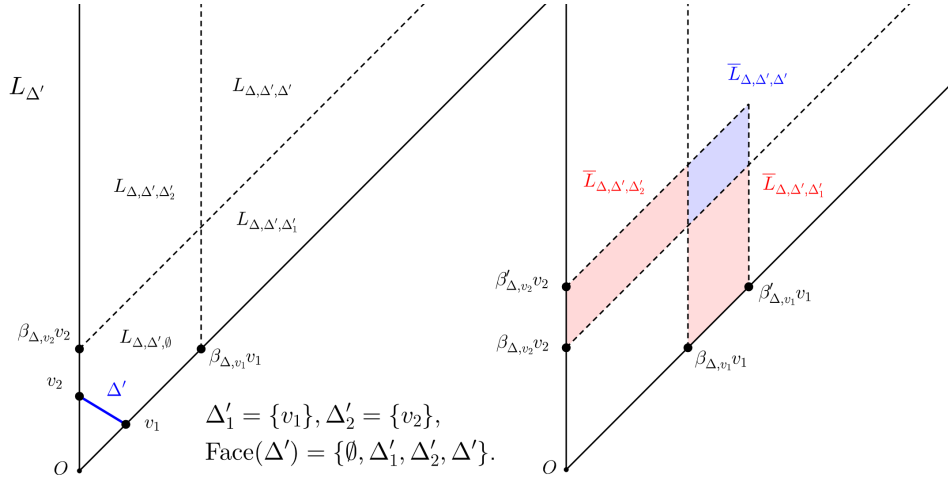


FIGURE 8. $L_{\Delta'}$, $L_{\Delta, \Delta', \Delta''}$, and $\bar{L}_{\Delta, \Delta', \Delta''}$.

THEOREM 4.20. *Let $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$, $\Delta' \in \text{Face}(\Delta)$ and $\Delta'' \in \text{Face}(\Delta')$. Then, the following assertions hold.*

- (1) $B(L_{\Delta, \Delta', \Delta''})$ is a free $M_{\sigma, \Delta''}$ -module with basis $S(\bar{L}_{\Delta, \Delta', \Delta''})$.
- (2) For any $(d, y) \in S(\bar{L}_{\Delta, \Delta', \Delta''})$, we have $d \leq C'_2 + \beta_{\Delta, \Delta'} + \sum_{v \in V(\Delta'')} \text{cpx}_\sigma(v)$.

Proof. We prove (1). First, we prove that $B(L_{\Delta, \Delta', \Delta''})$ is an $M_{\sigma, \Delta''}$ -module. By definition, it is easy to see that $B(L_{\Delta, \Delta', \Delta''}) + (1, 0) \subset B(L_{\Delta, \Delta', \Delta''})$. We fix $(d, y) \in B(L_{\Delta, \Delta', \Delta''})$ and $v \in V(\Delta'')$. By the definition of $L_{\Delta, \Delta', \Delta''}$, we have $\text{proj}_{\Delta, v}(\Phi(y)) > \beta_{\Delta, v}$. Therefore, By Theorem 4.18, we have

$$d_\Gamma(x_0, y + \text{cpx}_\sigma(v)v) = d_\Gamma(x_0, y) + \text{cpx}_\sigma(v) \leq d + \text{cpx}_\sigma(v).$$

Furthermore, we have $\Phi(y + \text{cpx}_\sigma(v)v) \in L_{\Delta, \Delta', \Delta''}$. Hence, we conclude that

$$(d, y) + (\text{cpx}_\sigma(v), \text{cpx}_\sigma(v)v) \in B(L_{\Delta, \Delta', \Delta''}).$$

We fix $(d, y) \in B(L_{\Delta, \Delta', \Delta''})$. By the definitions of $L_{\Delta, \Delta', \Delta''}$ and $\bar{L}_{\Delta, \Delta', \Delta''}$, there exist $y' \in V_\Gamma \cap \Phi^{-1}(\bar{L}_{\Delta, \Delta', \Delta''})$ and $b_v \in \mathbb{Z}_{\geq 0}$ for $v \in V(\Delta'')$ such that

$$y = y' + \sum_{v \in V(\Delta'')} b_v \cdot \text{cpx}_\sigma(v) \cdot v.$$

By Theorem 4.18, we have

$$d_\Gamma(x_0, y) = d_\Gamma(x_0, y') + \sum_{v \in V(\Delta'')} b_v \cdot \text{cpx}_\sigma(v).$$

Therefore, for $d' := d_\Gamma(x_0, y')$, we have $(d', y') \in S(\bar{L}_{\Delta, \Delta', \Delta''})$ and

$$\begin{aligned} (d, y) &= (d', y') + (d - d_\Gamma(x_0, y)) \cdot (1, 0) + \sum_{v \in V(\Delta'')} b_v \cdot (\text{cpx}_\sigma(v), \text{cpx}_\sigma(v)) \\ &\in (d', y') + M_{\sigma, \Delta''}. \end{aligned}$$

We have proved that $B(L_{\Delta, \Delta', \Delta''})$ is an $M_{\sigma, \Delta''}$ -module generated by $S(\bar{L}_{\Delta, \Delta', \Delta''})$.

Suppose that $(d, y), (d', y') \in S(\bar{L}_{\Delta, \Delta', \Delta''})$ satisfies

$$\begin{aligned} (d, y) + a \cdot (1, 0) + \sum_{v \in V(\Delta'')} b_v \cdot \text{cpx}_\sigma(v) \\ = (d', y') + a' \cdot (1, 0) + \sum_{v \in V(\Delta'')} b'_v \cdot \text{cpx}_\sigma(v) \end{aligned}$$

with some $a, a' \in \mathbb{Z}_{\geq 0}$ and $b_v, b'_v \in \mathbb{Z}_{\geq 0}$ for $v \in V(\Delta'')$. Then, for each $v \in V(\Delta'')$, we have

$$\text{proj}_{\Delta, v}(\Phi(y)) + b_v \cdot \text{cpx}_\sigma(v) = \text{proj}_{\Delta, v}(\Phi(y')) + b'_v \cdot \text{cpx}_\sigma(v).$$

Here, by the definition of $\bar{L}_{\Delta, \Delta', \Delta''}$, we have

$$\text{proj}_{\Delta, v}(\Phi(y)), \text{proj}_{\Delta, v}(\Phi(y')) \in (\beta_{\Delta, v}, \beta'_{\Delta, v}].$$

Since $\beta'_{\Delta, v} - \beta_{\Delta, v} = \text{cpx}_\sigma(v)$, we have $y = y'$ and $b_v = b'_v$ for any $v \in V(\Delta'')$. Since $y = y'$, we have $d = d_\Gamma(x_0, y) = d_\Gamma(x_0, y') = d'$ by the definition of $S(\bar{L}_{\Delta, \Delta', \Delta''})$. Thus, we also have $a = a'$. Therefore, we conclude that $B(L_{\Delta, \Delta', \Delta''})$ is freely generated by $S(\bar{L}_{\Delta, \Delta', \Delta''})$. We complete the proof of (1).

Next, we prove (2). For $y \in V_\Gamma \cap \Phi^{-1}(\bar{L}_{\Delta, \Delta', \Delta''})$, we have

$$\begin{aligned} d_\Gamma(x_0, y) &\leq C'_2 + d_{P, \Phi}(x_0, y) \\ &= C'_2 + \sum_{v \in V(\Delta')} \text{proj}_{\Delta, v}(\Phi(y)) \\ &\leq C'_2 + \sum_{v \in V(\Delta'')} \beta'_{\Delta, v} + \sum_{v \in V(\Delta') \setminus V(\Delta'')} \beta_{\Delta, v} \\ &\leq C'_2 + \beta_{\Delta, \Delta'} + \sum_{v \in V(\Delta'')} \text{cpx}_\sigma(v), \end{aligned}$$

which completes the proof. $\square$

DEFINITION 4.21. For $d \in \mathbb{Z}_{\geq 0}$ and a subset $S \subset \mathbb{Z}_{\geq 0} \times V_\Gamma$, we define

$$S_d := \{x \in V_\Gamma \mid (d, x) \in S\}.$$

Furthermore, we define a function $f_S : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ by $f_S(d) := \#S_d$.

THEOREM 4.22. Let $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$, $\Delta' \in \text{Face}(\Delta)$ and $\Delta'' \in \text{Face}(\Delta')$. Then the following assertions hold.

(1) The generating function of $f_{B(L_{\Delta, \Delta', \Delta''})}$ is given by

$$\sum_{i \geq 0} f_{B(L_{\Delta, \Delta', \Delta''})}(i) t^i = \frac{Q(t)}{(1-t) \prod_{v \in V(\Delta'')} (1 - t^{\text{cpx}_\sigma(v)})},$$

where $Q(t)$ is a polynomial with $\deg Q \leq C'_2 + \beta_{\Delta, \Delta'} + \sum_{v \in V(\Delta'')} \text{cpx}_\sigma(v)$.

(2) The generating function of $f_{B(L_{\Delta'})}$ is given by

$$\sum_{i \geq 0} f_{B(L_{\Delta'})}(i) t^i = \frac{Q(t)}{(1-t) \prod_{v \in V(\Delta')} (1 - t^{\text{cpx}_\sigma(v)})},$$

where $Q(t)$ is a polynomial with $\deg Q \leq C'_2 + \beta_{\Delta, \Delta'} + \sum_{v \in V(\Delta')} \text{cpx}_\sigma(v)$.

(3) The generating function of f_B is given by

$$\sum_{i \geq 0} f_B(i) t^i = \frac{Q(t)}{(1-t)R(t)},$$

where

$$R(t) := \text{LCM} \left\{ \prod_{v \in V(\Delta)} (1 - t^{\text{cpx}_\sigma(v)}) \mid \sigma \in \text{Facet}(P), \Delta \in T_\sigma \right\},$$

and $Q(t)$ is a polynomial with $\deg Q \leq \beta + \deg R$.

Proof. We prove (1). By Theorem 4.20(1), $B(L_{\Delta, \Delta', \Delta''})$ is freely generated by the finite set $S(\bar{L}_{\Delta, \Delta', \Delta''})$ as an $M_{\sigma, \Delta''}$ -module. Note that $M_{\sigma, \Delta''}$ is a free monoid with basis $(1, 0)$ and $(\text{cpx}_\sigma(v), \text{cpx}_\sigma(v)v)$ for $v \in V(\Delta'')$. Therefore, we have

$$\sum_{i \geq 0} f_{B(L_{\Delta, \Delta', \Delta''})}(i) t^i = \frac{Q(t)}{(1-t) \prod_{v \in V(\Delta'')} (1 - t^{\text{cpx}_\sigma(v)})},$$

where

$$Q(t) = \sum_{i \geq 0} f_{S(\bar{L}_{\Delta, \Delta', \Delta''})}(i) t^i.$$

Here, by Theorem 4.20(2), we have

$$\begin{aligned} \deg Q(t) &= \max \{d \mid (d, y) \in S(\bar{L}_{\Delta, \Delta', \Delta''})\} \\ &\leq C'_2 + \beta_{\Delta, \Delta'} + \sum_{v \in V(\Delta'')} \text{cpx}_\sigma(v), \end{aligned}$$

which completes the proof of (1).

Since we have $L_{\Delta'} = \bigsqcup_{\Delta'' \in \text{Face}(\Delta')} L_{\Delta, \Delta', \Delta''}$, (2) follows from (1).

(3) follows from (2) by the inclusion-exclusion principle as detailed below. We fix a face τ of σ . We set

$$\Xi_\tau := \{\Delta' \mid \Delta \in T_\sigma, \Delta' \in \text{Face}(\Delta), \Delta' \subset \tau\}.$$

Then we have $\tau = \bigcup_{\Delta' \in \Xi_\tau} \Delta'$, and furthermore, the set Ξ_τ is closed under taking intersection. Therefore, by (2), the generating function of $f_{B(\mathbb{R}_{\geq 0}\tau)}$ is given by the form

$$\frac{Q(t)}{(1-t)R(t)},$$

where $Q(t)$ is a polynomial with $\deg Q \leq \beta + \deg R$. Since the set $\{\tau \mid \tau \in \text{Face}(P)\}$ is closed under taking intersection, we can conclude that the generating function of f_B is also given by the same form. We complete the proof of (3). $\square$

Here, we will summarize our main theorem.

COROLLARY 4.23. *Let (Γ, L) be a strongly connected n -dimensional periodic graph, and let $x_0 \in V_\Gamma$. Let $(b(d))_d$ and $(s(d))_d$ be the cumulative growth sequence and the growth sequence of Γ with the start point x_0 (see Subsection 3.3). Let $\beta \in \mathbb{R}_{\geq 0}$ and $\text{cpx}_\Gamma \in \mathbb{Z}_{>0}$ be as in Subsection 4.1. Then, the following assertions hold.*

(1) The generating function of $b : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}; d \mapsto b(d)$ is given by

$$\sum_{i \geq 0} b(i) t^i = \frac{Q(t)}{(1-t)(1 - t^{\text{cpx}_\Gamma})^n},$$

where $Q(t)$ is a polynomial with $\deg Q \leq \beta + n \cdot \text{cpx}_\Gamma$. In particular, the function $b : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}; d \mapsto b(d)$ is a quasi-polynomial on $d > \beta - 1$, and cpx_Γ is its quasi-period.

(2) The generating function of $s : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}; d \mapsto s(d)$ is given by

$$\sum_{i \geq 0} s(i)t^i = \frac{Q(t)}{(1 - t^{\text{cpx}_\Gamma})^n},$$

where $Q(t)$ is a polynomial with $\deg Q \leq \beta + n \cdot \text{cpx}_\Gamma$. In particular, the function $s : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}; d \mapsto s(d)$ is a quasi-polynomial on $d > \beta$, and cpx_Γ is its quasi-period.

Proof. Note that the polynomial $R(t)$ in Theorem 4.22(3) divides $(1 - t^{\text{cpx}_\Gamma})^n$. Therefore, (1) follows from Theorem 4.22(3) since $b = f_B$. (2) follows from (1). $\square$

REMARK 4.24. The proof of Corollary 4.23 does not rely on the result in [20]. Therefore, the proof of Corollary 4.23 also gives a different proof of Theorem 3.8.

EXAMPLE 4.25. We saw in Example 4.13 that $C'_2 = 3$, $\beta = 25$ and $\text{cpx}_\Gamma = 2$ for the Wakatsuki graph Γ with the start point $x_0 = v'_2$. Therefore, Corollary 4.23 shows that the growth sequence $(s_{\Gamma, x_0, i})_i$ is a quasi-polynomial on $i > \beta = 25$, and $\text{cpx}_\Gamma = 2$ is its quasi-period. Actually, it is known that $(s_{\Gamma, x_0, i})_i$ is a quasi-polynomial on $i \geq 3$, and 2 is its period (see [15, Example 2.18]).

As a corollary of Theorem 4.18, we give an algorithm to compute the precise value of $C_2(\Gamma, \Phi, x_0)$.

THEOREM 4.26. For $\sigma \in \text{Facet}(P)$ and $\Delta \in T_\sigma$, we define

$$L_\Delta^{\leq \beta'} := \{z \in L_\Delta \mid \text{proj}_{\Delta, v}(z) \leq \beta'_{\Delta, v} \text{ holds for any } v \in V(\Delta)\}.$$

We also define

$$L^{\leq \beta'} := \bigcup_{\substack{\sigma \in \text{Facet}(P), \\ \Delta \in T_\sigma}} L_\Delta^{\leq \beta'}.$$

Then, we have

$$C_2(\Gamma, \Phi, x_0) = \max \left\{ d_\Gamma(x_0, y) - d_{P, \Phi}(x_0, y) \mid y \in V_\Gamma \cap \Phi^{-1}(L^{\leq \beta'}) \right\}.$$

In particular, we have

$$C_2(\Gamma, \Phi, x_0) = \max \{ d_\Gamma(x_0, y) - d_{P, \Phi}(x_0, y) \mid y \in B_{\lfloor \beta' \rfloor} \},$$

where we define

$$\beta' := C'_2 + \max \left\{ \sum_{v \in V(\Delta)} \beta'_{\Delta, v} \mid \sigma \in \text{Facet}(P), \Delta \in T_\sigma \right\}.$$

Proof. Let $y \in V_\Gamma$. Take $\sigma \in \text{Facet}(P)$ and $\Delta \in T_\sigma$ such that $\Phi(y) \in L_\Delta$. For each $v \in V(\Delta)$, we define

$$b_v := \max \left\{ 0, \left\lceil \frac{\text{proj}_{\Delta, v}(\Phi(y)) - \beta'_{\Delta, v}}{\text{cpx}_\sigma(v)} \right\rceil \right\}.$$

We define

$$y' := y - \sum_{v \in V(\Delta)} b_v \text{cpx}_\sigma(v)v.$$

Then, we have $\Phi(y') \in L_\Delta^{\leq \beta'}$. By Theorem 4.18, we have

$$d_\Gamma(x_0, y') = d_\Gamma(x_0, y) - \sum_{v \in V(\Delta)} b_v \text{cpx}_\sigma(v).$$

On the other hand, we have

$$d_{P,\Phi}(x_0, y') = d_{P,\Phi}(x_0, y) - \sum_{v \in V(\Delta)} b_v \text{cpx}_\sigma(v).$$

Therefore, we have

$$d_\Gamma(x_0, y') - d_{P,\Phi}(x_0, y') = d_\Gamma(x_0, y) - d_{P,\Phi}(x_0, y).$$

Hence, we have

$$\begin{aligned} C_2(\Gamma, \Phi, x_0) &= \sup\{d_\Gamma(x_0, y) - d_{P,\Phi}(x_0, y) \mid y \in V_\Gamma\} \\ &= \max\left\{d_\Gamma(x_0, y) - d_{P,\Phi}(x_0, y) \mid y \in V_\Gamma \cap \Phi^{-1}(L^{\leq \beta'})\right\}, \end{aligned}$$

which proves the first assertion.

For $y \in V_\Gamma$ with $\Phi(y) \in L_\Delta^{\leq \beta'}$, we have

$$d_\Gamma(x_0, y) \leq C'_2 + d_{P,\Phi}(x_0, y) \leq C'_2 + \sum_{v \in V(\Delta)} \beta'_{\Delta, v}.$$

Therefore, we have $V_\Gamma \cap \Phi^{-1}(L^{\leq \beta'}) \subset B_{\lfloor \beta' \rfloor}$, which proves the second assertion. $\square$

5. EXAMPLES

In this section, using Corollary 4.23, we calculate the growth series for some specific periodic graphs. In Subsection 5.1, we examine a 6-uniform tiling. In Subsections 5.2 and 5.3, we treat two 3-dimensional periodic graphs obtained by carbon allotropes. As far as we know, this is the first time the growth series in these three examples have been determined with proofs.

5.1. 2-DIMENSIONAL PERIODIC GRAPHS. We begin this subsection with the following remark on the computability of invariants of 2-dimensional periodic graphs.

REMARK 5.1. When $n = 2$, it is not difficult to implement our algorithm to compute the invariants β and cpx_Γ as follows (cf. Subsection 4.2):

- (1) When $n = 2$, each facet σ of $P := P_\Gamma$ is one dimensional. Therefore, the points in $\text{Im}(\nu) \cap \sigma$ give a triangulation T_σ of σ with the required conditions $(\diamond)_1$ and $(\diamond)_2$, and no further subdivision is necessary.
- (2) For $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$ and $v \in V(\Delta)$, the set $\mathcal{H}''_{\sigma, v}$ can be concretely given (see Subsection 4.2).
- (3) For $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$, $v \in V(\Delta)$ and $F \in \mathcal{H}''_{\sigma, v}$, the invariants $\text{cpx}_\sigma(v)$, $r_{\Delta, v}^F$ and $s_{\Delta, v}^F$ can be computed by the construction in the proof of Lemma 4.15.
- (4) For $\sigma \in \text{Facet}(P)$, $\Delta \in T_\sigma$, $v \in V(\Delta)$ and $F \in \mathcal{H}''_{\sigma, v}$, the invariants $a_{\Delta, v}^F(v)$ and $h_{\Delta, v}^F$ can be computed by the construction in Subsection 4.2.

Furthermore, the first few terms of the growth sequence can be computed by the breadth-first search algorithm. Therefore, it is not difficult to implement an algorithm to compute the growth series of two dimensional periodic graphs.

REMARK 5.2. In [10], Goodman-Strauss and Sloane determine the growth sequences for seven specific periodic tilings. With the help of a computer program (Remark 5.1 and Appendix A), we can automatically compute the invariants and their growth series.

We examine a 6-uniform tiling illustrated in Figure 9, which is the tiling #673 in the Galebach list of 673 6-uniform tilings [9] (see also [A313961](#) in the OEIS [14]). The invariants and the growth series are computed as in Table 2. From the form of these growth series, it can be determined that the actual (minimum) period is 36 for all cases. To the best of our knowledge, this is the first time the growth series for this example has been determined with a proof.

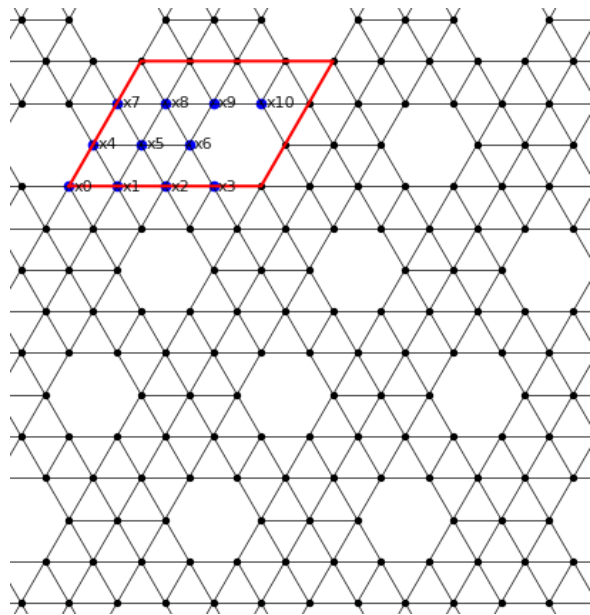


FIGURE 9. The 6-uniform tiling (#673).

5.2. THE CARBON ALLOTROPE K6. In this subsection, we consider the 3-dimensional periodic graph Γ shown in Figure 10. This graph corresponds to a carbon allotrope called the K6 carbon (#12 in SACADA database). In SACADA database [12], the fundamental region of the K6 carbon is taken as shown in Figure 10. In what follows, we will proceed with this fundamental region, although it is possible to replace it with a smaller fundamental region. Then, we have $\#(V_\Gamma/L) = 12$. Note that all vertices of Γ are symmetric, and hence, the growth sequence does not depend on the choice of its start point x_0 .

With the help of a computer program, $\text{Im}(\nu)$ can be computed as in Figure 11. The growth polytope $P := P_\Gamma$ has 14 vertices (marked in red) and 24 facets, and each facet is a triangle as in Figure 12. Note that $\text{Im}(\nu)$ has a symmetry such that the 24 facets of P are symmetric.

With the help of a computer program, C_1 and C'_2 can be computed according to Proposition 3.14 as follows:

$$C_1 = 0.5, \quad C'_2 = 13.$$

We define $v_0, v_1, v_2, v_3 \in \text{Im}(\nu)$ as in Figure 11. Let $\sigma \in \text{Facet}(P)$ denote the face of P satisfying $V(\sigma) = \{v_0, v_1, v_2\}$. Then, we have $\text{Im}(\nu) \cap \sigma = \{v_0, v_1, v_2, v_3\}$. We take a triangulation $T_\sigma = \{\Delta_1, \Delta_2\}$ such that $V(\Delta_1) = \{v_1, v_2, v_3\}$ and $V(\Delta_2) = \{v_0, v_1, v_3\}$. Then, T_σ satisfies the required conditions $(\diamond)_1$ and $(\diamond)_2$. Note that v_3 is the midpoint of v_2 and v_0 . Furthermore, $\text{Im}(\nu)$ has a symmetry such that the three

Start	OEIS#	C_1	C'_2	β	cpx_Γ	$\deg R$ (see Theorem 4.22(3))
				Growth series		
x_0, x_9	<u>A314154</u>	5/6	11	2648.4...	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	132
					$\frac{1+5t+10t^2+10t^3+5t^4-4t^5-9t^6-12t^7-6t^8-6t^{10}-12t^{11}-9t^{12}-4t^{13}+5t^{14}+10t^{15}+10t^{16}+5t^{17}+t^{18}}{(1-t^2)(1-t^3)(1-t^4)(1-t^9)}$	
x_1, x_8	<u>A315346</u>	3/4	11	2644.1...	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	132
					$\frac{1+6t+10t^2+15t^3+14t^4+7t^5-5t^6-15t^7-15t^8-11t^9-14t^{10}-11t^{11}-15t^{12}-15t^{13}-5t^{14}+7t^{15}+14t^{16}+15t^{17}+10t^{18}+6t^{19}+t^{20}}{(1-t^3)(1-t^4)^2(1-t^9)}$	
x_2, x_7	<u>A315356</u>	1/2	11	2631.3...	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	132
					$\frac{1+6t+9t^2+10t^3+6t^4-7t^5-9t^6-10t^7-6t^8-6t^{10}-10t^{11}-9t^{12}-7t^{13}+6t^{14}+10t^{15}+9t^{16}+6t^{17}+t^{18}}{(1-t^2)(1-t^3)(1-t^4)(1-t^9)}$	
x_3, x_{10}	<u>A313961</u>	7/12	11	2635.6...	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	132
					$\frac{1+5t+10t^2+16t^3+14t^4+6t^5-3t^6-12t^7-18t^8-13t^9-13t^{10}-11t^{11}-14t^{12}-14t^{13}-6t^{14}+3t^{15}+11t^{16}+19t^{17}+13t^{18}+6t^{19}+2t^{20}-t^{21}-t^{22}}{(1-t^3)(1-t^4)^2(1-t^9)}$	
x_4, x_6	<u>A314064</u>	2/3	11	2639.8...	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	132
					$\frac{1+5t+10t^2+9t^3+4t^4-t^5-10t^6-9t^7-5t^8-8t^9-4t^{10}-8t^{11}-10t^{12}-3t^{13}+2t^{14}+8t^{15}+11t^{16}+6t^{17}+3t^{18}-t^{21}}{(1-t^2)(1-t^3)(1-t^4)(1-t^9)}$	
x_5	<u>A315238</u>	5/12	11	2627.0...	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	132
					$\frac{1+6t+10t^2+13t^3+16t^4+6t^5-4t^6-12t^7-17t^8-13t^9-12t^{10}-11t^{11}-15t^{12}-12t^{13}-8t^{14}+2t^{15}+14t^{16}+15t^{17}+12t^{18}+10t^{19}+t^{20}-2t^{22}}{(1-t^3)(1-t^4)^2(1-t^9)}$	

TABLE 2. Growth series of the 6-uniform tiling #673 (Fig. 9).

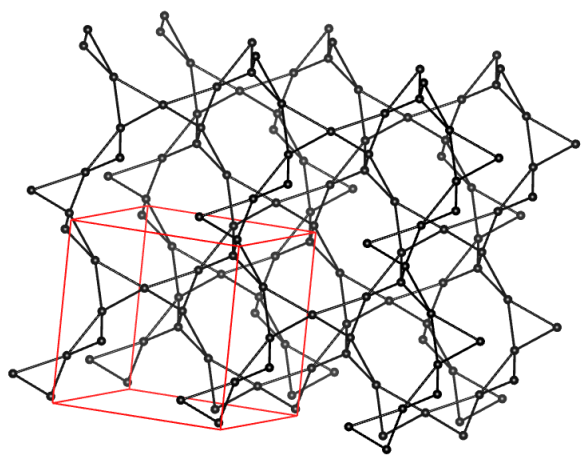


FIGURE 10. The K6 carbon.

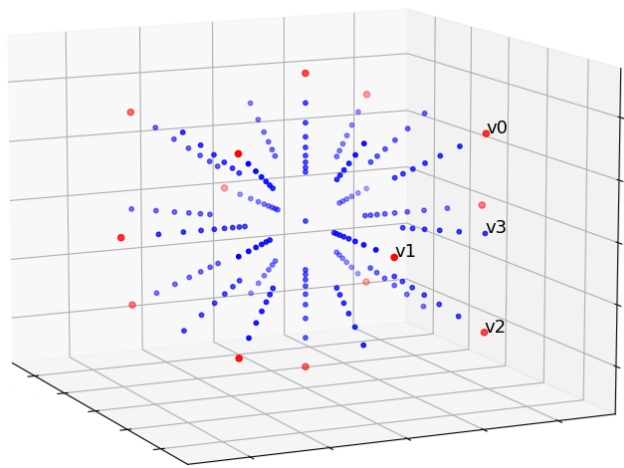


FIGURE 11. $\text{Im}(\nu)$.

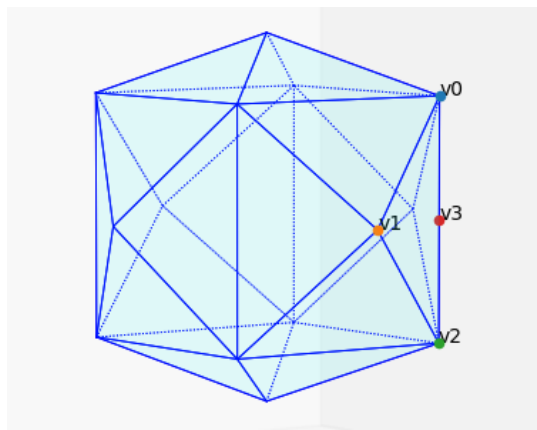


FIGURE 12. P .

points v_1 , v_2 and v_3 are translated to v_1 , v_0 and v_3 . In what follows, we shall compute the invariants for Δ_1 .

First, we shall calculate the invariants s_{σ, v_i}^F 's according to the construction in the proof of Lemma 4.7. We have

$$\mathcal{H}_{\sigma, v_1}'' = \{\{v_1\}\}, \quad \mathcal{H}_{\sigma, v_2}'' = \{\{v_2\}\}, \quad \mathcal{H}_{\sigma, v_3}'' = \{\{v_0, v_3\}, \{v_2, v_3\}\}.$$

With the help of a computer program, we have

$$\begin{aligned} \text{Len}(\{v_1\}) &= \{4\}, & \text{Len}(\{v_0\}) &= \text{Len}(\{v_2\}) = \text{Len}(\{v_3\}) = \{6\}, \\ \text{num}(v_0, 6) &= \text{num}(v_2, 6) = 1, & \text{num}(v_1, 4) &= \text{num}(v_3, 6) = 2. \end{aligned}$$

Since v_3 is the midpoint of v_2 and v_0 , we have

$$\begin{aligned} \text{cpx}_{\sigma}(v_1) &= m_{\sigma, v_1}(v_1) = 4, & \text{cpx}_{\sigma}(v_2) &= m_{\sigma, v_2}(v_2) = 6, \\ \text{cpx}_{\sigma}(v_3) &= m_{\sigma, v_3}(v_3) = 12, & m_{\sigma, v_3}(v_0) &= m_{\sigma, v_3}(v_2) = 6. \end{aligned}$$

Therefore, we have

$$s_{\sigma, v_1}^{\{v_1\}} = 8, \quad s_{\sigma, v_2}^{\{v_2\}} = 6, \quad s_{\sigma, v_3}^{\{v_0, v_3\}} = s_{\sigma, v_3}^{\{v_2, v_3\}} = 6 + 18 = 24.$$

Next, we have

$$\begin{aligned} \text{Im}(\nu) \setminus \{v_1\} &\subset H\left(\frac{4}{5}v_1, v_2, v_3\right), \\ \text{Im}(\nu) \setminus \{v_2\} &\subset H\left(v_1, \frac{6}{7}v_2, v_3\right), \\ \text{Im}(\nu) \setminus \{v_0, v_3\} &\subset H\left(v_1, v_2, \frac{6}{7}v_0\right) = H\left(v_1, v_2, \frac{12}{13}v_3\right), \\ \text{Im}(\nu) \setminus \{v_2, v_3\} &\subset H\left(v_0, v_1, \frac{6}{7}v_2\right) = H\left(v_1, \frac{6}{7}v_2, \frac{12}{13}v_3\right). \end{aligned}$$

Therefore, we can take the invariants $a_{\Delta_1, v}^F(v)$'s and $h_{\Delta_1, v}^F$'s as in Table 3. According to these $a_{\Delta_1, v}^F(v)$'s and $h_{\Delta_1, v}^F$'s, the invariants $\alpha_{\Delta_1, v}'^F$'s and $\alpha_{\Delta_1, v}'$'s are computed as in Table 3.

	v	v_1	v_2	v_3		
	u	v_1	v_2	v_0	v_2	v_3
	$\text{cpx}_{\sigma}(v)$	4	6	12		
	$m_{\sigma,v}(u)$	4	6	6	6	12

v	v_1	v_2	v_3	
F	$\{v_1\}$	$\{v_2\}$	$\{v_0, v_3\}$	$\{v_2, v_3\}$
$s_{\sigma,v}^F(v)$	8	6	24	24
$a_{\Delta_1,v}^F(v)$	$\frac{4}{5}$	$\frac{6}{7}$	$\frac{12}{13}$	$\frac{12}{13}$
$h_{\Delta_1,v}^F$	$\frac{4}{5}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{6}{7}$
$\alpha_{\Delta_1,v}'^F$	$5C_1 + 4C_2' + 19$	$7C_1 + 6C_2' + 17$	$14C_1 + 12C_2' + 70$	$14C_1 + 12C_2' + 70$
$\alpha_{\Delta_1,v}'$	$5C_1 + 4C_2' + 19$	$7C_1 + 6C_2' + 17$	$14C_1 + 12C_2' + 70$	
$\beta_{\Delta_1,\Delta_1}$	$26C_1 + 22C_2' + 84$			
β	$26C_1 + 23C_2' + 84 = 396$			

TABLE 3. Invariants for Γ .

Hence, we have

$$\beta_{\Delta_1, \Delta_1} = \sum_{i=1,2,3} (\alpha'_{\Delta_1, v_i} - \text{cpx}_\sigma(v_i)) = 26C_1 + 22C'_2 + 84.$$

By symmetry, we have

$$\beta = C'_2 + \beta_{\Delta_1, \Delta_1} = 26C_1 + 23C'_2 + 84 = 396.$$

Therefore, by Corollary 4.23 (and Theorem 4.22(3)), it follows that the growth series G_{Γ, x_0} is of the form

$$G_{\Gamma, x_0}(t) = \frac{Q(t)}{(1 - t^4)(1 - t^6)(1 - t^{12})},$$

where $Q(t)$ is a polynomial of degree $\deg Q \leq \beta + 22$.

With the help of a computer program (breadth-first search algorithm), the first $\beta + 22 + 1$ ($= 419$) terms of the growth sequence $(s_{\Gamma, x_0, i})_{i \geq 0}$ can be computed. Using them, we can calculate $G_{\Gamma, x_0}(t)$ as follows:

$$\begin{aligned} G_{\Gamma, x_0}(t) &= \frac{(\text{The terms of } (1 - t^4)(1 - t^6)(1 - t^{12}) \sum_{i=0}^{418} s_{\Gamma, x_0, i} t^i \text{ of degree 418 or less.})}{(1 - t^4)(1 - t^6)(1 - t^{12})} \\ &= \frac{1 + 4t + 8t^2 + 14t^3 + 23t^4 + 34t^5 + 31t^6 + 28t^7 + 4t^8 - 4t^9 + t^{10} - 8t^{11} + 8t^{12}}{(1 - t^3)^2(1 - t^4)}. \end{aligned}$$

By the form of $G_{\Gamma, x_0}(t)$, we can conclude that the growth sequence $(s_{\Gamma, x_0, i})_{i \geq 0}$ is a quasi-polynomial on $i \geq 3$, and 12 is its period.

REMARK 5.3. By Theorem 4.26, we can obtain $C_2 = 1.25$.

5.3. THE CARBON ALLOTROPE CFS. In this subsection, we consider the 3-dimensional periodic graph Γ shown in Figure 13. This graph corresponds to a carbon allotrope called CFS (#29 in SACADA database). In SACADA database, the fundamental region of CFS is taken as shown in Figure 13. In what follows, we will proceed with this fundamental region. Then, we have $\#(V_\Gamma/L) = 6$. Note that all vertices of Γ are symmetric, and hence, the growth sequence does not depend on the choice of its start point x_0 .

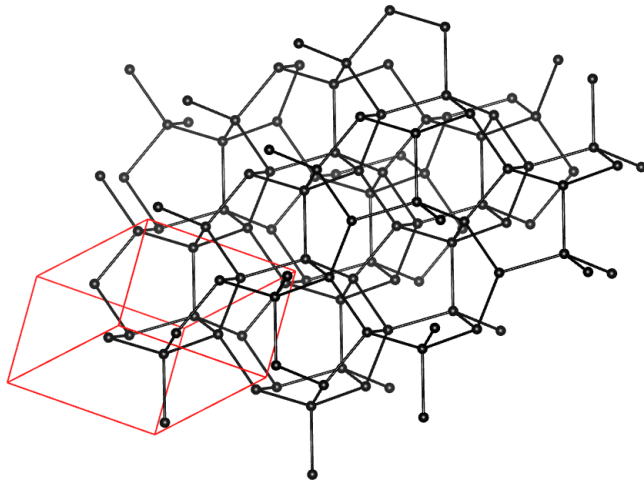


FIGURE 13. The carbon allotrope CFS.

With the help of a computer program, $\text{Im}(\nu)$ can be computed as in Figure 14. The growth polytope $P := P_\Gamma$ has 32 vertices (marked in red) and 54 facets (48 triangles and 6 quadrilaterals) as in Figure 15.

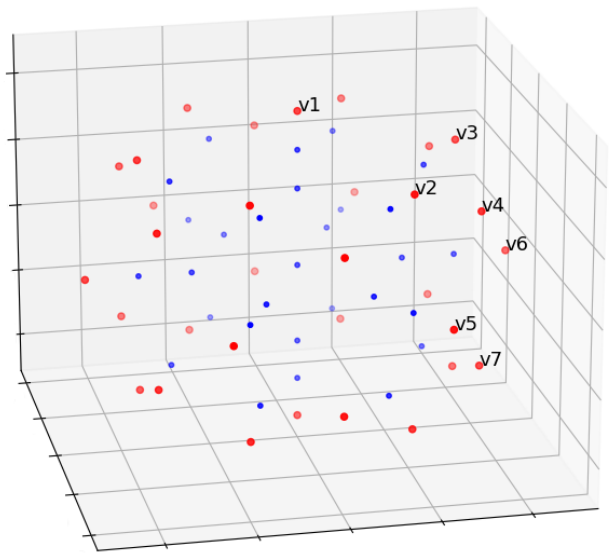


FIGURE 14. $\text{Im}(\nu)$.

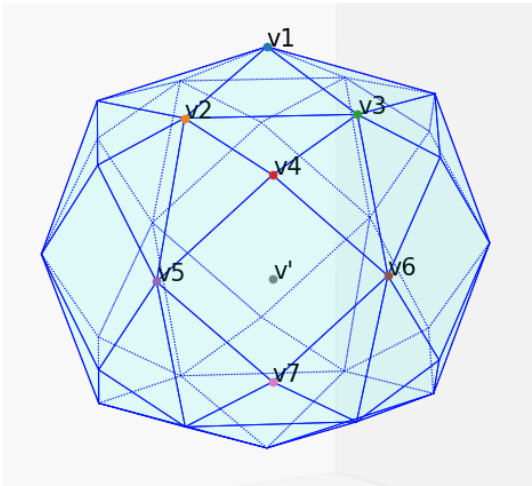


FIGURE 15. P .

With the help of a computer program, C_1 and C'_2 can be computed according to Proposition 3.14 as follows:

$$C_1 = 0.6, \quad C'_2 = 7.$$

We define $v_1, v_2, \dots, v_7 \in \text{Im}(\nu)$ as in Figure 14. Let $\sigma_1, \dots, \sigma_4 \in \text{Facet}(P)$ denote the face of P satisfying

$$\begin{aligned} V(\sigma_1) &= \{v_1, v_2, v_3\}, & V(\sigma_2) &= \{v_2, v_3, v_4\}, \\ V(\sigma_3) &= \{v_2, v_4, v_5\}, & V(\sigma_4) &= \{v_4, v_5, v_6, v_7\}. \end{aligned}$$

Then, for each i , we have $\text{Im}(\nu) \cap \sigma_i = V(\sigma_i)$. For each $i = 1, 2, 3$, σ_i is a simplex, and hence σ_i has the trivial triangulation $T_{\sigma_i} = \{\sigma_i\}$. The facet σ_4 is a rhombus. Let v' be the intersection point of the diagonals v_4v_7 and v_5v_6 . We take a triangulation $T_{\sigma_4} = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ such that

$$\begin{aligned} V(\Delta_1) &= \{v', v_4, v_5\}, & V(\Delta_2) &= \{v', v_7, v_5\}, \\ V(\Delta_3) &= \{v', v_4, v_6\}, & V(\Delta_4) &= \{v', v_7, v_6\}. \end{aligned}$$

Note that $\text{Im}(\nu)$ has a symmetry such that any facet of P can be translated to either σ_1 , σ_2 , σ_3 or σ_4 . Furthermore, $\text{Im}(\nu)$ also has a symmetry such that Δ_1 can be translated to Δ_2 , Δ_3 or Δ_4 . In what follows, we shall calculate the invariants for the simplices σ_1 , σ_2 , σ_3 and Δ_1 .

First, we shall compute the invariants $s_{\sigma, v}^F$'s according to the construction in the proof of Lemma 4.7. For each $i = 1, \dots, 4$, if $v \in V(\sigma_i)$, we have $\mathcal{H}_{\sigma_i, v}'' = \{\{v\}\}$. We also have

$$\mathcal{H}_{\sigma_4, v'}'' = \{\{v_4, v_5\}, \{v_5, v_7\}, \{v_7, v_6\}, \{v_6, v_4\}\}.$$

With the help of a computer program, we have

$$\begin{aligned} \text{Len}(\{v_1\}) &= \text{Len}(\{v_5\}) = \text{Len}(\{v_6\}) = \{3\}, \\ \text{Len}(\{v_2\}) &= \text{Len}(\{v_3\}) = \{4\}, & \text{Len}(\{v_4\}) &= \text{Len}(\{v_7\}) = \{6\}, \\ \text{num}(v_1, 3) &= \text{num}(v_5, 3) = \text{num}(v_6, 3) = 2 \\ \text{num}(v_2, 4) &= \text{num}(v_3, 4) = \text{num}(v_4, 6) = \text{num}(v_7, 6) = 1. \end{aligned}$$

For each $i = 1, \dots, 4$, if $v \in V(\sigma_i)$, we have $m_{\sigma_i, v}(v) = \text{cpx}_{\sigma_i}(v)$. Here, $\text{cpx}_{\sigma_i}(v)$ is determined as follows:

$$\begin{aligned} \text{cpx}_{\sigma_1}(v_1) &= \text{cpx}_{\sigma_3}(v_5) = \text{cpx}_{\sigma_4}(v_5) = \text{cpx}_{\sigma_4}(v_6) = 3, \\ \text{cpx}_{\sigma_1}(v_2) &= \text{cpx}_{\sigma_2}(v_2) = \text{cpx}_{\sigma_3}(v_2) = \text{cpx}_{\sigma_1}(v_3) = \text{cpx}_{\sigma_2}(v_3) = 4, \\ \text{cpx}_{\sigma_2}(v_4) &= \text{cpx}_{\sigma_3}(v_4) = \text{cpx}_{\sigma_4}(v_4) = \text{cpx}_{\sigma_4}(v_7) = 6. \end{aligned}$$

Since v' is the center of the rhombus σ_4 , we have

$$\text{cpx}_{\sigma_4}(v') = 12, \quad m_{\sigma_4, v'}(v_4) = m_{\sigma_4, v'}(v_5) = m_{\sigma_4, v'}(v_6) = m_{\sigma_4, v'}(v_7) = 6.$$

Therefore, the invariants $s_{\sigma, v}^F$'s are computed as in Table 4.

Next, we have

$$\begin{aligned} \text{Im}(\nu) \setminus \{v_1\} &\subset H\left(\frac{3}{4}v_1, v_2, v_3\right), \\ \text{Im}(\nu) \setminus \{v_2\} &\subset H\left(v_1, \frac{4}{5}v_2, v_3\right) = H\left(\frac{4}{5}v_2, v_3, v_4\right) \subset H\left(\frac{8}{9}v_2, v_4, v_5\right), \\ \text{Im}(\nu) \setminus \{v_3\} &\subset H\left(v_1, v_2, \frac{4}{5}v_3\right) = H\left(v_2, \frac{4}{5}v_3, v_4\right), \\ \text{Im}(\nu) \setminus \{v_4\} &\subset H\left(v_2, v_3, \frac{6}{7}v_4\right) = H\left(v_2, \frac{6}{7}v_4, v_5\right) = H\left(\frac{6}{7}v_4, v_5, v'\right), \\ \text{Im}(\nu) \setminus \{v_5\} &\subset H\left(v_2, v_4, \frac{3}{4}v_5\right) = H\left(v_4, \frac{3}{4}v_5, v'\right). \end{aligned}$$

v	v_1	v_2	v_3
$\text{cpx}_{\sigma_1}(v) = m_{\sigma_1,v}(v)$	3	4	4
$s_{\sigma_1}^{\{v\}}(v)$	6	4	4
$a_{\sigma_1,v}^{\{v\}}(v) = h_{\sigma_1,v}^{\{v\}}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{4}{5}$
$\alpha_{\sigma_1,v}'^{\{v\}} = \alpha'_{\sigma_1,v}$	$4C_1 + 3C_2' + 11$	$5C_1 + 4C_2' + 9$	$5C_1 + 4C_2' + 9$
$\beta_{\sigma_1,\sigma_1}$	$14C_1 + 11C_2' + 18$		

v	v_2	v_3	v_4
$\text{cpx}_{\sigma_2}(v) = m_{\sigma_2,v}(v)$	4	4	6
$s_{\sigma_2}^{\{v\}}(v)$	4	4	6
$a_{\sigma_2,v}^{\{v\}}(v) = h_{\sigma_2,v}^{\{v\}}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{6}{7}$
$\alpha_{\sigma_2,v}'^{\{v\}} = \alpha'_{\sigma_2,v}$	$5C_1 + 4C_2' + 9$	$5C_1 + 4C_2' + 9$	$7C_1 + 6C_2' + 11$
$\beta_{\sigma_2,\sigma_2}$	$17C_1 + 14C_2' + 15$		

v	v_2	v_4	v_5
$\text{cpx}_{\sigma_3}(v) = m_{\sigma_3,v}(v)$	4	6	3
$s_{\sigma_3}^{\{v\}}(v)$	4	6	6
$a_{\sigma_3,v}^{\{v\}}(v) = h_{\sigma_3,v}^{\{v\}}$	$\frac{8}{9}$	$\frac{6}{7}$	$\frac{3}{4}$
$\alpha_{\sigma_3,v}'^{\{v\}} = \alpha'_{\sigma_3,v}$	$9C_1 + 8C_2' + 9$	$7C_1 + 6C_2' + 11$	$4C_1 + 3C_2' + 11$
$\beta_{\sigma_3,\sigma_3}$	$20C_1 + 17C_2' + 18$		

v	v_4	v_5	v_6	v_7	v'			
u	v_4	v_5	v_6	v_7	v_4	v_5	v_6	v_7
$\text{cpx}_{\sigma_4}(v)$	6	3	3	6	12			
$m_{\sigma_4,v}(u)$	6	3	3	6	6	6	6	6

v	v_4	v_5	v'			
F	$\{v_4\}$	$\{v_5\}$	$\{v_4, v_5\}$	$\{v_5, v_7\}$	$\{v_7, v_6\}$	$\{v_6, v_4\}$
$s_{\sigma_4,v}^F(v)$	6	6	15	15	15	15
$a_{\Delta_1,v}^F(v)$	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{12}{13}$	$\frac{12}{13}$	$\frac{12}{13}$	$\frac{12}{13}$
$h_{\Delta_1,v}^F$	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{6}{7}$
$\alpha'_{\Delta_1,v}$	$7C_1 + 6C_2' + 11$	$4C_1 + 3C_2' + 11$	$14C_1 + 12C_2' + 40$			
$\beta_{\Delta_1,\Delta_1}$	$25C_1 + 21C_2' + 41$					

TABLE 4. Invariants for Γ .

Therefore, the invariants $a_{\sigma_i,v}^{\{v\}}(v)$'s and $h_{\sigma_i,v}^{\{v\}}$'s for $i = 1, 2, 3$ and $a_{\Delta_1,v}^{\{v\}}(v)$'s and $h_{\Delta_1,v}^{\{v\}}$'s for $v = v_4, v_5$ can be given as in Table 4. Furthermore, we have

$$\text{Im}(\nu) \setminus \{v_4, v_5\} \subset H(v_2, v_6, v_7).$$

Note that the six points

$$v_2, \quad v_6, \quad v_7, \quad \frac{6}{7}v_4, \quad \frac{6}{7}v_5, \quad \frac{12}{13}v'$$

are on a same hyperplane. Therefore, we have

$$\begin{aligned} \text{Im}(\nu) \setminus \{v_4, v_5\} &\subset H\left(\frac{6}{7}v_4, \frac{6}{7}v_5, \frac{12}{13}v'\right) = H\left(v_7, \frac{6}{7}v_5, \frac{12}{13}v'\right) \\ &= H\left(\frac{6}{7}v_4, v_6, \frac{12}{13}v'\right) = H\left(v_6, v_7, \frac{12}{13}v'\right). \end{aligned}$$

Therefore, we can take

$$\begin{aligned} a_{\Delta_1, v'}^{\{v_4, v_5\}}(v') &= a_{\Delta_2, v'}^{\{v_4, v_5\}}(v') = a_{\Delta_3, v'}^{\{v_4, v_5\}}(v') = a_{\Delta_4, v'}^{\{v_4, v_5\}}(v') = \frac{12}{13}, \\ h_{\Delta_1, v'}^{\{v_4, v_5\}} &= h_{\Delta_2, v'}^{\{v_4, v_5\}} = h_{\Delta_3, v'}^{\{v_4, v_5\}} = h_{\Delta_4, v'}^{\{v_4, v_5\}} = \frac{6}{7}. \end{aligned}$$

By symmetry, we can also take

$$a_{\Delta_1, v'}^{\{v_5, v_7\}}(v') = a_{\Delta_1, v'}^{\{v_7, v_6\}}(v') = a_{\Delta_1, v'}^{\{v_6, v_4\}}(v') = \frac{12}{13}, \quad h_{\Delta_1, v'}^{\{v_5, v_7\}} = h_{\Delta_1, v'}^{\{v_7, v_6\}} = h_{\Delta_1, v'}^{\{v_6, v_4\}} = \frac{6}{7}.$$

Hence, the invariants $\alpha'_{\Delta, v}$'s and $\beta_{\Delta, \Delta}$'s are computed as in Table 4. By symmetry, we have

$$\beta = C'_2 + \beta_{\Delta_1, \Delta_1} = 25C_1 + 22C'_2 + 41 = 210.$$

Therefore, by Corollary 4.23 (and Theorem 4.22(3)), it follows that the growth series G_{Γ, x_0} is of the form

$$G_{\Gamma, x_0}(t) = \frac{Q(t)}{(1-t^6)(1-t^{12})^2},$$

where $Q(t)$ is a polynomial of degree $\deg Q \leq \beta + 30$.

With the help of a computer program (breadth-first search algorithm), the first $\beta + 30 + 1 (= 241)$ terms of the growth sequence $(s_{\Gamma, x_0, i})_{i \geq 0}$ can be computed. Using them, we can calculate $G_{\Gamma, x_0}(t)$ as follows:

$$\begin{aligned} G_{\Gamma, x_0}(t) &= \frac{(\text{The terms of } (1-t^6)(1-t^{12})^2 \sum_{i=0}^{240} s_{\Gamma, x_0, i} t^i \text{ of degree 240 or less.})}{(1-t^6)(1-t^{12})^2} \\ &= \frac{1 + 4t + 12t^2 + 25t^3 + 38t^4 + 52t^5 + 54t^6 + 44t^7 + 27t^8 + 8t^9 - t^{11} + 2t^{12} + 2t^{13}}{(1-t^3)(1-t^4)^2}. \end{aligned}$$

By the form of $G_{\Gamma, x_0}(t)$, we can conclude that the growth sequence $(s_{\Gamma, x_0, i})_{i \geq 0}$ is a quasi-polynomial on $i \geq 3$, and 12 is its period.

REMARK 5.4. By Theorem 4.26, we can obtain $C_2 = 1.2414\dots$.

APPENDIX A. IMPLEMENTATION OF THE ALGORITHM

We prepare an implementation of the algorithm in Python to compute the growth series of unweighted 2-dimensional periodic graphs according to Remark 5.1. For details, see the source code:

https://github.com/yokozuna57/Ehrhart_on_PG

Here, we will only explain the input format. Let (Γ, L) be an n -dimensional periodic graph. Let $c := \#V_\Gamma/L$. When we choose representatives of V_Γ/L and label them as $0, 1, \dots, c-1$, we can identify $V_\Gamma = \{0, 1, \dots, c-1\} \times \mathbb{Z}^{\dim}$, where we set $\dim := n$.

Thanks to the definition of the periodic graph, we can recover all the combinatorial data E_Γ , s_Γ and t_Γ from only the neighborhoods of vertices $(0, \mathbf{0})$, $(1, \mathbf{0})$, $\dots$,

$(c-1, \mathbf{0})$. In the case of the Wakatsuki graph (introduced in Example 4.13 and [15, Example 2.6]), the input is as follows:

```

dim=2
c=3
edges=[
  [(1,(0,0)),(1,(-1,0)),(1,(-1,-1)),(2,(0,0))],
  [(0,(0,0)),(0,(1,0)),(0,(1,1)),(2,(0,0))],
  [(0,(0,0)),(1,(0,0))]
]
pos=[(0,0),(0.5,0.5),(0.5,0)]

```

Here, we choose v'_0 , v'_1 and v'_2 in Figure 5 as representatives. The variable `edge` represents the edges of the quotient graph Γ/L . More precisely, `edges[i]` is the list of vertices that have an edge from $v'_i = (i, (0, 0))$ to them. For example,

$$\text{edges}[0] = [(1, (0, 0)), (1, (-1, 0)), (1, (-1, -1)), (2, (0, 0))]$$

represents the list of edges e with $s(e) = v'_0$, namely, e'_0 , e'_1 , e'_2 and e'_3 in Figure 6. The right hand side is the list of the target vertices of e'_0 , e'_1 , e'_2 and e'_3 .

In addition, the variable `pos` is used to give information on the periodic realization $\Phi : V_\Gamma \rightarrow L_\mathbb{R} := L \otimes_\mathbb{Z} \mathbb{R}$. More precisely, the variable `pos` presents the coordinates of $\Phi((0, \mathbf{0}))$, $\Phi((1, \mathbf{0}))$, $\dots$, $\Phi((c-1, \mathbf{0})) \in L_\mathbb{R}$. In this example,

$$\text{edges}[0] = [(0, 0), (0.5, 0.5), (0.5, 0)]$$

represents

$$\Phi(v'_0) = (0, 0), \quad \Phi(v'_1) = (1/2, 1/2), \quad \Phi(v'_2) = (1/2, 0)$$

as in Figure 5. Note that the choice of `pos` does not affect the graph structure of Γ , and hence, it does not affect the growth series of the periodic graph, but it is used in the program when calculating the values of C_1 and C'_2 . For the program to work correctly, each element of `pos` should be chosen in $[0, 1)$. If no specific realization is in mind, you can always choose `pos=[(0,0,...,0)]*dim`.

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