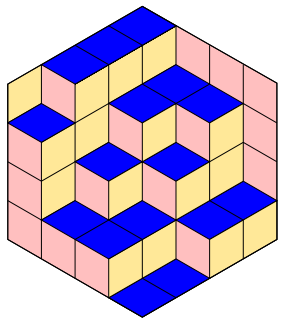

ALGEBRAIC COMBINATORICS

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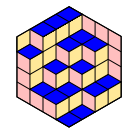

Decomposition numbers for weight 3 blocks of Iwahori–Hecke algebras of type B

Matthew Fayers & Lorenzo Putignano

Abstract Let B be a weight-3 block of an Iwahori–Hecke algebra of type B over any field. We develop the combinatorics of B to prove that the decomposition numbers for B are all 0 or 1.

1. Introduction

In the representation theory of the symmetric groups, a fundamental problem is to determine the *decomposition numbers* in characteristic p: the composition multiplicities of p-modular simple representations in p-modular reductions of ordinary simple representations. A complete solution to this problem seems far out of reach, but solutions are known in various special cases. One of these results (due to the first author [10]) says that the decomposition numbers in a p-block of weight 3 in characteristic $p \geqslant 5$ are all either 0 or 1; this in particular allows these decomposition numbers to be computed quickly using the Jantzen–Schaper formula.

The decomposition number problem naturally extends to the Iwahori–Hecke algebras of type A, where we seek the composition multiplicities of the simple modules in the Specht modules. Many of the known results for symmetric groups have been proved for the Iwahori–Hecke algebras as well, including the result for blocks of weight 3. More recently, this problem has been extended to the Iwahori–Hecke algebras $\mathcal{H}_n$ of type B. These were first systematically studied by Dipper, James and Murphy [6], who introduced appropriate definitions of Specht modules labelled by bipartitions of n. This naturally leads to a combinatorial approach to the corresponding decomposition number problem, underpinned by the combinatorics of bipartitions. The first author's combinatorial definition of the weight of a bipartition, and hence the weight of a block [8], allows us to approach this problem systematically by addressing blocks of a fixed weight. As in type A, the blocks of weight 0 are precisely the simple blocks, and blocks of weight 1 admit a very simple description (analogous to symmetric group blocks with cyclic defect groups). The first author [7] gave a combinatorial formula for decomposition number for blocks of weight 2, analogous to Richards's formula [19] for type A.

In this paper we address blocks of weight 3. Our main result is as follows.

Theorem 1.1. *Suppose $\mathbb{F}$ is a field, $q \in \mathbb{F}$ is non-zero, and $k_1, k_2 \in \mathbb{Z}$. Let $\mathcal{H}_n$ denote the Iwahori–Hecke algebra of type B over $\mathbb{F}$ with parameters q, q^{k_1}, q^{k_2}, and let B be a block of $\mathcal{H}_n$ of weight 3. Then the decomposition numbers for B are all either 0 or 1.*

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We note that (unlike in type A) Theorem 1.1 applies even when $\mathbb{F}$ has characteristic 2 or 3. In particular, following the proof of Theorem 1.1 we deduce the following interesting result.

Corollary 1.2. *The decomposition numbers for a weight 3 block of $\mathcal{H}_n$ are independent of the characteristic of the field $\mathbb{F}$.*

We conjecture that this result extends to all weight 3 blocks of *Ariki–Koike algebras*; these are higher-level Hecke algebras generalising the Iwahori–Hecke algebras of types A and B. A particular case of this conjecture (the case of 'tree blocks') has already been proved by Lyle and Ruff in [15, Corollary A.5]. (Note: this result does not appear in the published version of the paper, but in the arXiv version that we cite here.)

The proof of Theorem 1.1 is by induction on n, using the Brundan–Kleshchev branching rules which govern induction and restriction between $\mathcal{H}_n$ and $\mathcal{H}_m$, for $m < n$. An important part of our work is a combinatorial description of all the bipartitions labelling Specht modules in a given block, and an analysis of which Specht modules S_λ cannot be shown to be multiplicity-free via the inductive hypothesis. For the most difficult cases, we need to use the cyclotomic Jantzen–Schaper formula due to James and Mathas [11].

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2. Background

In this section we summarise some of the background results we shall need. For notation we mostly follow the first author's paper [7], where more details and motivation may be found.

2.1. Basic notation. In this paper $\mathbb{N}$ denotes the set of positive integers. Throughout, we fix an integer $e \geqslant 2$. For any integer a, we write $\overline{a} = a + e\mathbb{Z} = \{a + em \mid m \in \mathbb{Z}\}$, and $\mathbb{Z}/e\mathbb{Z} = \{\overline{a} \mid a \in \mathbb{Z}\}$. The set $\mathbb{Z}/e\mathbb{Z}$ admits an additive action of $\mathbb{Z}$ via $a + \overline{b} = \overline{a+b}$.

In general, given any tuple $\{x_i \mid i \in \mathbb{Z}/e\mathbb{Z}\}$ of objects indexed by $\mathbb{Z}/e\mathbb{Z}$, and given an integer a, we may write x_a to mean $x_{\overline{a}}$. Similarly, given any terminology involving an element of $\mathbb{Z}/e\mathbb{Z}$ (such as 'i-node'), we may substitute an integer a in place of $\overline{a}$ (so we may define an a-node to mean an $\overline{a}$-node).

Throughout the paper, we fix a pair $\kappa = (\kappa_1, \kappa_2) \in (\mathbb{Z}/e\mathbb{Z})^2$. A pair of integers k_1, k_2 such that $\kappa_1 = \overline{k_1}$ and $\kappa_2 = \overline{k_2}$ is called a *bicharge* for κ.

If M and N are modules for an algebra, then we write $M \sim N$ to mean that M and N have the same composition factors with multiplicity.

2.2. Partitions and bipartitions. A *partition* is an infinite weakly-decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of integers which are eventually zero. We write $|\lambda|$ for the sum $\lambda_1 + \lambda_2 + \dots$, and say that λ is a partition of $|\lambda|$. When writing partitions, we omit trailing zeroes and group together equal parts with a superscript. We write $\varnothing$ for the unique partition of 0. For any partition λ, we write $h(\lambda)$ for the number of non-zero parts of λ, which we call the *length* of λ.

A *bipartition* is an ordered pair $\lambda = \left(\lambda^{(1)} \mid \lambda^{(2)}\right)$ of partitions, which we call the *components* of λ. We write $|\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}|$, and say that λ is a bipartition of $|\lambda|$.

The Young diagram of a bipartition λ is the set

$$[\lambda] = \left\{ (r, c, a) \in \mathbb{N}^2 \times \{1, 2\} \;\middle|\; c \leqslant \lambda_r^{(a)} \right\},$$

whose elements we call the *nodes* of λ. In general, a node means an element of $\mathbb{N}^2 \times \{1, 2\}$. A node (r, c, a) of λ is *removable* if it can be removed from $[\lambda]$ to leave the Young diagram of a smaller bipartition (that is, if $c = \lambda_r^{(a)} > \lambda_{r+1}^{(a)}$), while a node not in λ is an addable node of λ if it can be added to $[\lambda]$ to give a Young diagram.

We depict $[\lambda]$ by drawing the Young diagram of $\lambda^{(1)}$ above the Young diagram of $\lambda^{(2)}$ (using the English convention). Accordingly, we will say that a node (r, c, a) is *above* or *higher than* a node (s, d, b) if either $a < b$ or $a = b$ and $r < s$.

If λ and μ are two bipartitions of n, then we say that λ *dominates* μ (and write $\lambda \trianglerighteqslant \mu$) if

$$\lambda_1^{(1)} + \cdots + \lambda_r^{(1)} \geqslant \mu_1^{(1)} + \cdots + \mu_r^{(1)} \quad \text{and} \quad |\lambda^{(1)}| + \lambda_1^{(2)} + \cdots + \lambda_r^{(2)} \geqslant |\mu^{(1)}| + \mu_1^{(2)} + \cdots + \mu_r^{(2)}$$

for every $r \geqslant 1$. Another way of saying this is that $\lambda = \mu$ or $[\mu]$ can be obtained from $[\lambda]$ by moving one or more nodes to lower positions.

If λ is a partition, the *conjugate* partition λ' is defined by

$$(\lambda')_r = |\{ c \in \mathbb{N} \mid \lambda_c \geqslant r \}|.$$

If λ is a bipartition, the conjugate bipartition is $\lambda' = \left(\lambda^{(2)'} \;\middle|\; \lambda^{(1)'}\right)$. In other words, λ' is obtained by interchanging the two components, and then replacing each component with the conjugate partition. Note then that if λ, μ are two bipartitions, then $\lambda \trianglerighteqslant \mu$ if and only if $\mu' \trianglerighteqslant \lambda'$.

Now suppose we have fixed a pair $\kappa = (\kappa_1, \kappa_2) \in (\mathbb{Z}/e\mathbb{Z})^2$. The *residue* of a node (r, c, a), denoted $\operatorname{res}(r, c, a)$, is $c - r + \kappa_a$. If $i \in \mathbb{Z}/e\mathbb{Z}$, then an *$i$-node* means a node of residue i. If λ is a bipartition, then the *content* of λ is the multiset of the residues of the nodes of λ. For any $i \in \mathbb{Z}/e\mathbb{Z}$, we write $\operatorname{rem}_i(\lambda)$ for the number of removable i-nodes of λ.

We will also need to consider rim hooks. The *rim* of a bipartition λ is the set of nodes $(r, c, a) \in [\lambda]$ such that $(r + 1, c + 1, a) \notin [\lambda]$. A *rim hook* of λ is a connected subset of the rim which can be removed to leave a smaller bipartition. If H is a rim hook, the *hand node* $\operatorname{hand}(H)$ is the top-rightmost node of H, and the *leg length* $\operatorname{ll}(H)$ is the difference in height between the highest and lowest nodes in H.

2.3. Beta-sets and the abacus. We often use James's abacus for partition combinatorics. With e fixed as above, we take an abacus with e vertical runners labelled $0, \ldots, e - 1$ from left to right, and with positions marked $0, 1, \ldots$ from left to right along successive rows of the abacus, working down the page. Now given a partition λ and an integer $k \geqslant h(\lambda)$, we define the *beta-set for* λ with charge k to be the set $\{ \lambda_r + k - r \mid 1 \leqslant r \leqslant k \}$. The *abacus display* for λ with charge k is constructed by placing a bead on the abacus at position $\lambda_r + k - r$ for each $1 \leqslant r \leqslant k$.

Given a bipartition $\lambda = \left(\lambda^{(1)} \;\middle|\; \lambda^{(2)}\right)$ and a bicharge $(k_1, k_2) \in \mathbb{N}^2$ with $k_a \geqslant h(\lambda^{(a)})$ for $a = 1, 2$, the abacus display for λ with bicharge (k_1, k_2) is obtained by placing the abacus display for $\lambda^{(1)}$ with charge k_1 above the abacus display for $\lambda^{(2)}$ with charge k_2.

The abacus is good for visualising rim hooks (and in particular, removable nodes). Given an abacus display for a bipartition λ, removable nodes in component a correspond to positions $x > 0$ on the abacus such that in component a there is a bead at position x but no bead at position $x - 1$. Given such a x, the residue of the corresponding removable node is $x + e\mathbb{Z}$. Similarly, addable nodes correspond to positions x such that there is a bead at position $x - 1$ but not at position x.

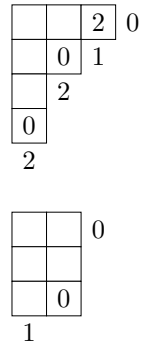
More generally, rim hooks in component a of λ correspond to pairs of positions x, y where $x > y$ and there is a bead at position x but not at position y in component a. The residue of the hand node of the corresponding rim hook is $x + e\mathbb{Z}$, the leg length is the number of beads between positions x and y (not including position x), and removing the rim hook corresponds to moving the bead at position x to position y.

2.4. Restricted bipartitions. In [3], Ariki and Mathas introduced a set of multipartitions to provide a combinatorial labelling for simple modules of Ariki–Koike algebras. They called these *Kleshchev multipartitions*, but here we prefer the term *restricted multipartitions* introduced by Brundan–Kleshchev [4]. We now recall the definition here, specialising to the case of bipartitions. We will use the original recursive definition of Ariki–Mathas, though we note that a more efficient definition was found by Ariki, Kreiman and Tsuchioka [2].

The definition of restricted bipartitions depends on our fixed $\kappa \in (\mathbb{Z}/e\mathbb{Z})^2$. Take a bipartition λ and a residue $i \in \mathbb{Z}/e\mathbb{Z}$. Define the *$i$-signature* of λ to be the sequence of signs obtained by reading the Young diagram of λ from top to bottom, writing $+$ for each addable i-node and $-$ for each removable i-node. The *reduced i-signature* is the subsequence obtained by repeatedly deleting adjacent pairs $-+$. The removable nodes corresponding to the $-$ signs are called the *normal* i-nodes of λ, and the highest of these (if there are any) is the *good* i-node. The addable nodes corresponding to the $+$ signs in the reduced i-signature are the *conormal* i-nodes of λ, and the lowest of these (if there are any) is the *cogood* i-node.

For any bipartition μ and any $i \in \mathbb{Z}/e\mathbb{Z}$, we write $\mathrm{nor}_i(\mu)$ for the number of normal i-nodes of μ.

Example 2.1. Suppose $e = 3$, $\kappa = (\bar{0}, \bar{1})$ and $\lambda = \left(3, 2, 1^2 \mid 2^3\right)$. The Young diagram of λ is shown below, with the residues of the addable and removable nodes indicated.



If we let $i = \bar{0}$, then we see that the i-signature of λ is $+ - - + -$. So the reduced i-signature is $+ - -$. So there are two normal i-nodes $(2, 2, 1)$ and $(3, 2, 2)$, with $(2, 2, 1)$ being the good i-node. There is a unique conormal (and therefore cogood) i-node $(1, 4, 1)$.

Now given two bipartitions λ and μ, write $\mu \xrightarrow{i} \lambda$ if μ is obtained from λ by removing the good i-node for $i \in \mathbb{Z}/e\mathbb{Z}$, or equivalently if λ is obtained from μ by adding the cogood i-node. We write $\mu \longrightarrow \lambda$ to mean that $\mu \xrightarrow{i} \lambda$ for some i, and let $\longleftrightarrow$ be the equivalence relation generated by $\longrightarrow$. Now define the set of restricted bipartitions (with respect to κ) to be the bipartitions in the $\longleftrightarrow$-class containing $\left(\varnothing \mid \varnothing\right)$.

The definition of restricted bipartitions derives from the theory of crystals for highest-weight representations of quantum groups. It follows from this theory (in particular, the uniqueness of highest-weight vectors) that the only restricted bipartition

with no normal nodes of any residue is $\big(\varnothing \;\big|\; \varnothing\big)$. So to test whether a bipartition λ is restricted, one can repeatedly remove arbitrarily-chosen good nodes until there are no good nodes remaining; then λ is restricted if and only if the resulting bipartition is $\big(\varnothing \;\big|\; \varnothing\big)$.

We also make an observation which will be helpful later on. Say that a partition λ is *e-restricted* if $\lambda_r - \lambda_{r+1} < e$ for all r. Now the following is a well-known result; it reflects that restricted bipartitions label the vertices of a level-2 highest-weight crystal obtained as a component of the tensor product of two level-1 crystals, which themselves have vertices labelled by e-restricted partitions.

LEMMA 2.2. *Suppose λ is a restricted bipartition. Then both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e-restricted partitions.*

Proof. It is an easy exercise to check that if λ, μ are bipartitions with $\mu \longrightarrow \lambda$, and $a \in \{1, 2\}$, then $\lambda^{(a)}$ is e-restricted if and only if $\mu^{(a)}$ is. Since the components of the empty bipartition $\big(\varnothing \;\big|\; \varnothing\big)$ are both e-restricted, the result follows. □

We will also need to consider *regular* bipartitions. These are described in [7, 1.3.3], where they are called *conjugate Kleshchev* bipartitions. We say that a bipartition λ is *regular* if λ' is restricted. In fact we can modify the above definition of restricted bipartitions to give an alternative description of regular bipartitions, which will be useful later. To do this, define the i-signature of a bipartition as above, and define the *antireduced* i-signature by successively deleting pairs $+-$. The removable nodes corresponding to $-$ signs in the antireduced i-signature are called the *antinormal* i-nodes of λ, and the lowest of these (if there are any) is the *antigood* i-node. Similarly, the addable nodes corresponding to $+$ signs are the *anticonormal* i-nodes of λ, and the highest of these (if there are any) is the *anticogood* i-node. Given two bipartitions λ and μ, we write $\mu \overset{i}{\Longrightarrow} \lambda$ if μ is obtained from λ by removing the antigood i-node for some i. Equivalently, if we write $i' = \kappa_1 + \kappa_2 - i$, then $\mu \overset{i}{\Longrightarrow} \lambda$ if and only if $\mu' \xrightarrow{i'} \lambda'$. Now let $\Longrightarrow$ be the union of the relations $\overset{i}{\Longrightarrow}$, and let $\Longleftrightarrow$ be the equivalence relation generated by $\Longrightarrow$. The bipartitions in the $\Longleftrightarrow$-class containing $\big(\varnothing \;\big|\; \varnothing\big)$ are the regular bipartitions.

2.5. IWAHORI–HECKE ALGEBRAS. We fix a field $\mathbb{F}$ with q, Q_1, Q_2 non-zero elements of $\mathbb{F}$, and let $\mathcal{H}_n$ denote the Iwahori–Hecke algebra of type B with parameters q, Q_1, Q_2. This is the unital associative $\mathbb{F}$-algebra with generators $T_0, \dots, T_{n-1}$, subject to relations

$$\begin{aligned}
(T_0 - Q_1)(T_0 - Q_2) &= 0, \\
(T_i - q)(T_i + 1) &= 0 && \text{for } 1 \leqslant i < n, \\
T_i T_j &= T_j T_i && \text{if } j - i > 1, \\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i < n - 1.
\end{aligned}$$

The definition of $\mathcal{H}_n$ is unaffected if we interchange Q_1 and Q_2, but it will be important for us to consider (Q_1, Q_2) as an ordered pair; in particular, the construction of Specht modules depends on this ordered pair. (In Section 2.7 below, we consider in more detail the effect of interchanging Q_1 and Q_2.)

As explained in [7, 1.3.2], using a Morita equivalence result of Dipper–James [5], we can restrict attention to the case where Q_1 and Q_2 are powers of q. We also assume that q is a primitive eth root of unity, where $e \geqslant 2$; the case where q is not a root of unity works in the same way as the case where $e > n$ (and in fact most of

the particular cases of blocks that we consider only arise if q is an eth root of unity for $e \leqslant n$).

So with our fixed pair $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}/e\mathbb{Z}^2$, we will assume henceforth that $Q_1 = q^{k_1}$ and $Q_2 = q^{k_2}$, where (k_1, k_2) is any bicharge for κ.

The representation theory of $\mathcal{H}_n$ is based on the theory of *Specht modules*. For each bipartition λ of n, we write S_λ for the corresponding Specht module, as defined in Mathas's survey article [17]. The Specht modules are precisely the cell modules with respect to a particular cellular basis of $\mathcal{H}_n$, and a central problem in the representation theory of $\mathcal{H}_n$ is to determine their composition factors. To do this, we need a classification of the simple $\mathcal{H}_n$-modules. If λ is a restricted bipartition, then S_λ has a unique simple quotient D_λ, and the simple modules D_λ arising in this way give a complete irredundant set of simple $\mathcal{H}_n$-modules. Thus the problem of determining the composition factors of the Specht modules amounts to determining the *decomposition numbers* $[\mathrm{S}_\lambda : \mathrm{D}_\mu]$, where λ, μ are bipartitions of n with μ restricted.

A fundamental result on decomposition numbers is the following.

Proposition 2.3 ([6, Theorem 6.5]). *If λ and μ are bipartitions of n with μ restricted, then $[\mathrm{S}_\mu : \mathrm{D}_\mu] = 1$, while if $[\mathrm{S}_\lambda : \mathrm{D}_\mu] > 0$ then $\lambda \trianglerighteq \mu$.*

2.6. Blocks and weight. An important tool for studying the decomposition number problem is the classification of blocks, since $[\mathrm{S}_\lambda : \mathrm{D}_\mu] = 0$ unless S_λ and D_μ lie in the same block. It is a consequence of the general theory of cellular algebras that a Specht module always belongs to a single block, and we will abuse notation by saying that λ lies in a block B to mean that S_λ lies in B. Of course, if λ is restricted this means that D_λ lies in this block too, so to classify the blocks of $\mathcal{H}_n$ it is sufficient to determine when two Specht modules lie in the same block. The block classification for $\mathcal{H}_n$ is a special case of the result for Ariki–Koike algebras proved by Lyle and Mathas.

Proposition 2.4 ([14, Theorem 2.11]). *Suppose λ and μ are bipartitions of n. Then S_λ and S_μ lie in the same block of $\mathcal{H}_n$ if and only if λ and μ have the same content.*

As explained in [8, 3.2], this result can also be formulated in terms of the abacus: if we choose a bicharge (k_1, k_2) for κ with k_1, k_2 sufficiently large, and construct the abacus displays for λ and μ with this bicharge, then λ and μ have the same content if and only if for each i the number of beads on runner i (across both components) is the same in each abacus display. This leads to another useful way to determine whether two bipartitions belong to the same block. If λ is a bipartition, then for each $i \in \mathbb{Z}/e\mathbb{Z}$ we define $\delta_i(\lambda)$ to be the number of removable i-nodes of λ minus the number of addable i-nodes of λ. Now we have the following result.

Proposition 2.5 ([8, Proposition 3.2]). *Suppose λ and μ are bipartitions of n. Then λ and μ lie in the same block if and only if $\delta_i(\lambda) = \delta_i(\mu)$ for all $i \in \mathbb{Z}/e\mathbb{Z}$.*

As a consequence we can define $\delta_i(B)$ for a block B, meaning $\delta_i(\lambda)$ for any bipartition λ in B. If $a \in \mathbb{Z}$, then we may write $\delta_a(B)$ to mean $\delta_{a+e\mathbb{Z}}(B)$.

In [8], the first author introduced the notion of the *weight* $\mathrm{wt}(\lambda)$ of a bipartition λ. Since the weight of λ depends only on the content of λ, it is the same for all λ in a given block, and therefore one may define the weight of a block B to be the weight of any bipartition in B. The weight of B is a non-negative integer which measures the complexity of B, and a fruitful approach to the representation theory of $\mathcal{H}_n$ is to consider blocks of a given small weight.

We will not give the definition of weight in terms of content here, but we will use the results from [7, 1.3.5] showing how to calculate the weight of a bipartition using the abacus. We will introduce the necessary notation in Section 3.

We will also use the following lemma several times.

LEMMA 2.6 ([8, Lemma 3.6]). *Suppose λ is a bipartition, and that μ is a bipartition obtained by removing u removable i-nodes from λ. Then*

$$\mathrm{wt}(\mu) = \mathrm{wt}(\lambda) + u(\delta_i(\lambda) - u).$$

This paper is motivated by the following result.

THEOREM 2.7. *Suppose B is a block of $\mathcal{H}_n$ of weight 0, 1 or 2, and λ, μ are bipartitions in B with μ restricted. Then $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$.*

Proof. The result for blocks of weight 0 follows from [8, Theorem 4.1], and the result for weight 1 is [8, Theorem 4.2]. For blocks of weight 2, the result is given in [7, Theorems 3.18 & 4.13]. □

Our aim in this paper is to prove the same result for blocks of weight 3. In fact for blocks of weight at most 2 explicit formulæ for the decomposition numbers are known; for weight 3 we just show that the decomposition numbers are at most 1 (which is enough to enable the decomposition numbers to be computed quickly using the Jantzen–Schaper formula). This is analogous to the situation for Hecke algebras of type A [10].

2.7. CONJUGATION AND COMPONENT-SWITCHING. The set of bipartitions of n has very natural symmetries, given by conjugation or by simply interchanging components. These symmetries provide useful tools for studying decomposition numbers, which we describe here.

We begin with conjugation, which relates to automorphisms. Let θ be the automorphism defined defined by mapping $T_0 \mapsto q^{k_1+k_2}T_0^{-1}$ and $T_i \mapsto -qT_i^{-1}$ for $i \geqslant 1$. For any $\mathcal{H}_n$-module M, define the module M^θ by twisting the $\mathcal{H}_n$-action by θ. Obviously if M is simple then so is M^θ, and so we can define a bijection $\mu \mapsto \mu^\diamond$ from the set of restricted bipartitions to the set of regular bipartitions by $(\mathrm{D}_\mu)^\theta \cong \mathrm{D}_{(\mu^\diamond)'}$. Understanding what the automorphism θ does to Specht modules gives us more information on decomposition numbers, as follows.

PROPOSITION 2.8. *Suppose λ and μ are bipartitions of n with μ restricted. Then $[\mathrm{S}_\lambda : \mathrm{D}_\mu] = [\mathrm{S}_{\lambda'} : \mathrm{D}_{(\mu^\diamond)'}]$.*

Proof. Our argument is based on [16, Sections 4 and 5], where Mathas recalls the *dual Specht module* $\mathrm{S}'(\lambda)$ for each λ. The dual Specht modules are related to the usual Specht modules in two ways. On the one hand, one can check that $\mathrm{S}'(\lambda)$ is isomorphic to $(\mathrm{S}_\lambda)^\theta$. To see this, we recall how S_λ is constructed: a certain element $m_\lambda \in \mathcal{H}_n$ is defined, and S_λ is defined to be the right ideal generated by m_λ, modulo its intersection with the two-sided ideal generated by the elements m_ν for $\nu \rhd \lambda$. The dual Specht module $\mathrm{S}'(\lambda)$ is defined similarly using a different element n_λ. But one can check that $\theta(m_\lambda)$ equals n_λ times an invertible element of $\mathcal{H}_n$, so that the (right or two-sided) ideal generated by $\theta(m_\lambda)$ coincides with the (right or two-sided) ideal generated by n_λ. As a consequence, we obtain $[\mathrm{S}_\lambda : \mathrm{D}_\mu] = [\mathrm{S}'(\lambda) : (\mathrm{D}_\mu)^\theta]$ for all λ, μ.

On the other hand, Mathas considers contragredient duality $M \mapsto M^\circledast$ induced by the antiautomorphism of $\mathcal{H}_n$ that fixes each T_i. In [16, Corollary 5.7] Mathas shows that $\mathrm{S}'(\lambda) \cong (\mathrm{S}_{\lambda'})^\circledast$. Since contragredient duality preserves composition multiplicities, and the simple modules are self-dual (which follows from the cellularity of $\mathcal{H}_n$), we find that $[\mathrm{S}'(\lambda) : \mathrm{D}_\mu] = [\mathrm{S}_{\lambda'} : \mathrm{D}_\mu]$ for every λ, μ. The result follows. □

COROLLARY 2.9. *Suppose λ and μ are bipartitions of n, with μ restricted. Then $[\mathrm{S}_{\mu^\diamond} : \mathrm{D}_\mu] = 1$, while if $[\mathrm{S}_\lambda : \mathrm{D}_\mu] > 0$ then $\mu^\diamond \trianglerighteq \lambda$.*

Proof. This follows from Proposition 2.3 and Proposition 2.8 using the fact that conjugation reverses the dominance order on bipartitions. □

Later in the paper we will need to be able to calculate the bijection $^\diamond$ combinatorially. Recall the relations $\xrightarrow{i}$ and $\overset{i}{\Longrightarrow}$ from Section 2.4.

Proposition 2.10. *Suppose λ and μ are restricted bipartitions. Then $\mu \xrightarrow{i} \lambda$ if and only if $\mu^\diamond \overset{i}{\Longrightarrow} \lambda^\diamond$.*

Proof. The Brundan–Kleshchev modular branching rules show that the relation $\xrightarrow{i}$ can be described algebraically via

$$\mu \xrightarrow{i} \lambda \qquad \text{if and only if} \qquad \mathrm{Hom}_{\mathcal{H}_{n-1}}(\mathrm{D}_\mu, \mathrm{e}_i \, \mathrm{D}_\lambda) \neq 0,$$

where e_i is the ith restriction functor (see Section 2.8 below). The definition of e_i and the automorphism θ shows that $(\mathrm{e}_i M)^\theta = \mathrm{e}_{i'}(M^\theta)$ for any module M, where as above $i' = \kappa_1 + \kappa_2 - i$. Hence

$$\mathrm{Hom}_{\mathcal{H}_{n-1}}(\mathrm{D}_\mu, \mathrm{e}_i \, \mathrm{D}_\lambda) \neq 0 \qquad \text{if and only if} \qquad \mathrm{Hom}_{\mathcal{H}_{n-1}}(\mathrm{D}_{(\mu^\diamond)'}, \mathrm{e}_{i'} \, \mathrm{D}_{(\lambda^\diamond)'}) \neq 0,$$

and therefore

$$\mu \xrightarrow{i} \lambda \qquad \text{if and only if} \qquad (\mu^\diamond)' \xrightarrow{i'} (\lambda^\diamond)',$$

and from Section 2.4 this is equivalent to $\mu^\diamond \overset{i}{\Longrightarrow} \lambda^\diamond$. □

Now we come to component-switching, where we don't need to say as much. If λ is a bipartition, let $\lambda^\leftrightarrow$ denote the bipartition $\left(\lambda^{(2)} \mid \lambda^{(1)}\right)$ obtained by switching the two components of λ. Note that although the isomorphism type of $\mathcal{H}_n$ only requires the unordered pair (κ_1, κ_2), the definition of the Specht module (and the associated combinatorics of residues, blocks, restricted bipartitions etc) depends on the ordered pair (κ_1, κ_2). We let $\mathrm{S}^\leftrightarrow_\lambda$ denote the Specht module constructed using the bipartition $\lambda^\leftrightarrow$ and the ordered pair (κ_2, κ_1).

Lemma 2.11. *Suppose λ is a bipartition of n. Then S_λ and $\mathrm{S}^\leftrightarrow_\lambda$ have the same composition factors (with multiplicity).*

Proof. This uses a specialisation argument. Let $\hat{q}, \hat{Q}_1, \hat{Q}_2$ be algebraically independent indeterminates over $\mathbb{F}$, and consider the Iwahori–Hecke algebra $\hat{\mathcal{H}}_n$ over $\mathbb{F}(\hat{q}, \hat{Q}_1, \hat{Q}_2)$ with parameters $\hat{q}, \hat{Q}_1, \hat{Q}_2$. Then $\hat{\mathcal{H}}_n$ is semisimple by [1, Main Theorem], with the Specht modules being the simple modules. If we define the Specht module $\hat{\mathrm{S}}_\lambda$ for $\hat{\mathcal{H}}_n$ using the bipartition λ and the parameters $\hat{q}, \hat{Q}_1, \hat{Q}_2$, and we define the Specht module $\hat{\mathrm{S}}^\leftrightarrow_\lambda$ using the bipartition $\lambda^\leftrightarrow$ and the parameters $\hat{q}, \hat{Q}_2, \hat{Q}_1$, then $\hat{\mathrm{S}}_\lambda$ and $\hat{\mathrm{S}}^\leftrightarrow_\lambda$ have the same character (a recursive formula for characters of semisimple Iwahori–Hecke algebras of type B is given by Pfeiffer [18]), and are therefore isomorphic. The $\mathcal{H}_n$-modules S_λ and $\mathrm{S}^\leftrightarrow_\lambda$ are obtained by specialising $\hat{q}$ to q, $\hat{Q}_1$ to q^{k_1} and $\hat{Q}_2$ to q^{k_2}, where $\kappa_1 = \overline{k_1}$, $\kappa_2 = \overline{k_2}$. It is a standard result on specialisation that specialisations of isomorphic modules have the same composition factors. □

We remark that since the simple modules are labelled by a different set of partitions in each case, S_λ and $\mathrm{S}^\leftrightarrow_\lambda$ have the same composition factors but the labels of these may not agree.

Lemma 2.11 allows us to reduce the work we do in considering different types of blocks: given a block B of $\mathcal{H}_n$, we can study the same block by replacing κ with (κ_2, κ_1) and replacing each bipartition λ in B with $\lambda^\leftrightarrow$. The truth of Theorem 1.1 for B is then independent of which way we view B.

2.8. Branching rules. Now we summarise some essential background on the Brundan–Kleshchev branching rules; the most comprehensive reference for these is [4]. Recall that the content of a bipartition is the multiset of the residues of its nodes, and that two bipartitions lie in the same block if and only if they have the same content. So we may define the content $\mathcal{C}_B$ of a block B to be the content of any bipartition in B.

Now suppose M is an $\mathcal{H}_n$-module lying in a block B, and $i \in \mathbb{Z}/e\mathbb{Z}$. If there is a block A of $\mathcal{H}_{n-1}$ such that $\mathcal{C}_A$ is obtained by removing a copy of i from $\mathcal{C}_B$, then we define $\mathrm{e}_i M$ to be the block component of $M\downarrow_{\mathcal{H}_{n-1}}$ lying in A; otherwise, we set $\mathrm{e}_i M = 0$.

The functors e_i are defined for all n, so we can define powers e_i^r. In fact, it is possible to define *divided powers*: for any $\mathcal{H}_n$-module M and for any r there is an $\mathcal{H}_{n-r}$-module $\mathrm{e}_i^{(r)} M$ such that $\mathrm{e}_i^r M \cong (\mathrm{e}_i^{(r)} M)^{\oplus r!}$. For any non-zero module M, we set $\epsilon_i M = \max\{r \mid \mathrm{e}_i^r M \neq 0\}$.

The following results are essential tools in the study of decomposition numbers. Recall that we write $\mathrm{nor}_i(\mu)$ for the number of normal i-nodes of a bipartition μ and $\mathrm{rem}_i(\lambda)$ for the number of removable i-nodes of λ.

Proposition 2.12.

(1) *Suppose λ is a bipartition of n. Then $\mathrm{e}_i^{(r)} \mathrm{S}_\lambda$ has a filtration in which the factors are the Specht modules S_ν, for all bipartitions ν that can be obtained from λ by removing r removable i-nodes. In particular, $\epsilon_i \mathrm{S}_\lambda = \mathrm{rem}_i(\lambda)$, and $\mathrm{e}_i^{(\epsilon_i \mathrm{S}_\lambda)} \mathrm{S}_\lambda$ is isomorphic to the Specht module S_{λ^-}, where λ is the bipartition obtained by removing all the removable i-nodes from λ.*

(2) *Suppose μ is a restricted bipartition of n. Then $\epsilon_i \mathrm{D}_\mu = \mathrm{nor}_i(\mu)$ and $\mathrm{e}_i^{(\epsilon_i \mathrm{D}_\mu)} \mathrm{D}_\mu \cong \mathrm{D}_{\mu^-}$, where μ^- is the bipartition obtained by removing all the normal i-nodes from μ.*

2.9. The cyclotomic Jantzen–Schaper formula. The cyclotomic Jantzen–Schaper formula introduced by James and Mathas [11] will be a very useful tool. We describe a special case here, specialising to the case of Hecke algebras of type B.

First we note that because every field is a splitting field for $\mathcal{H}_n$, we can extend the field $\mathbb{F}$ without affecting the representation theory. So we assume that the subfield of $\mathbb{F}$ generated by q is a proper subfield, and we fix an element x of $\mathbb{F}$ lying outside this subfield.

Let $\hat{q}$ be an indeterminate over $\mathbb{F}$, and let $R = \mathbb{F}[\hat{q}^{\pm 1}]$. Choose a bicharge (k_1, k_2) for κ for which $|k_1 - k_2| > n$. Let $\mathfrak{p}$ be the prime ideal in R generated by $\hat{q} - q$. We want to consider the Iwahori–Hecke algebra of type B over R with parameters $\hat{q}, \hat{Q}_1, \hat{Q}_2$, where

$$\hat{Q}_1 = \hat{q}^{k_1}, \qquad \hat{Q}_2 = \hat{q}^{k_2} + x(\hat{q} - q).$$

Now given a node $(r, c, a) \in \mathbb{N}^2 \times \{1, 2\}$ define

$$\widehat{\mathrm{res}}(r, c, a) = \hat{q}^{c-r} \hat{Q}_a \in R.$$

Now suppose λ and ν are bipartitions of n with $\lambda \rhd \nu$. Let $G(\lambda, \nu)$ be the set of all pairs (L, N) such that

- L is a rim hook of λ and N a rim hook of ν,
- $[\lambda] \setminus L = [\nu] \setminus N$, and
- $\mathrm{res}(\mathrm{hand}(L)) = \mathrm{res}(\mathrm{hand}(N))$.

Given a pair $(L, N) \in G(\lambda, \nu)$, define $\epsilon_{LN} = (-1)^{\mathrm{ll}(L) - \mathrm{ll}(N)}$, and let

$$j_{\lambda\nu} = \prod_{(L,N) \in G(\lambda,\nu)} \big(\widehat{\mathrm{res}}(\mathrm{hand}(L)) - \widehat{\mathrm{res}}(\mathrm{hand}(N))\big)^{\epsilon_{LN}}.$$

Now for any pair of bipartitions (λ, μ) with μ restricted, we define

$$J_{\lambda\mu} = \sum_{\nu \lhd \lambda} \nu_{\mathfrak{p}}(j_{\lambda\nu})[\mathrm{S}_\nu : \mathrm{D}_\mu].$$

The following statement is a special case of the Jantzen–Schaper formula [11, Theorem 4.6].

Theorem 2.13 ([11, Theorem 4.6]). *Suppose λ and μ are bipartitions of n with μ restricted. Then the decomposition number $[\mathrm{S}_\lambda : \mathrm{D}_\mu]$ is at most $J_{\lambda\mu}$, and is non-zero if and only if $J_{\lambda\mu}$ is non-zero.*

Our particular choice of $\hat{q}, \hat{Q}_1, \hat{Q}_2$ means that the values $\nu_{\mathfrak{p}}(j_{\lambda\nu})$ will be easy to compute, and will not depend on the characteristic of $\mathbb{F}$ for the cases we will need to consider. In particular, suppose (r, c, a) and (s, d, b) are nodes. Then it is easy to check that:

- if $a \neq b$, then
$$\nu_{\mathfrak{p}}(\widehat{\mathrm{res}}(r, c, a) - \widehat{\mathrm{res}}(s, d, b)) = \begin{cases} 1 & \text{if } \mathrm{res}(r, c, a) = \mathrm{res}(s, d, b), \\ 0 & \text{otherwise;} \end{cases}$$
- if $a = b$, then
$$\nu_{\mathfrak{p}}(\widehat{\mathrm{res}}(r, c, a) - \widehat{\mathrm{res}}(s, d, b)) = \begin{cases} 1 & \text{if } c - r = d - s \pm e, \\ 0 & \text{if } c - r \not\equiv d - s \pmod{e}. \end{cases}$$

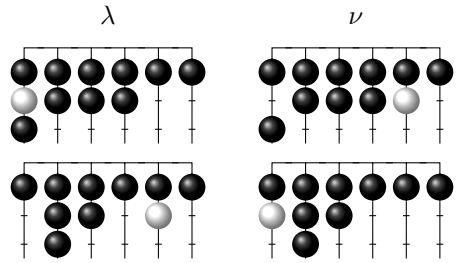
We can use the Jantzen–Schaper formula to refine Proposition 2.3 and corollary 2.9, by using a coarser order than the dominance order. We write $\lambda \blacktriangleright \nu$ if $\lambda \rhd \nu$ and $\nu_{\mathfrak{p}}(j_{\lambda\nu}) \neq 0$, and we extend $\blacktriangleright$ reflexively and transitively to give a partial order $\trianglerighteq$, which we call the *Jantzen–Schaper dominance* order; note that this order depends on our fixed parameters e and κ. It is easy to see that the usual dominance order $\trianglerighteq$ is a refinement of $\trianglerighteq$, and that conjugation of bipartitions reverses the Jantzen–Schaper dominance order. Moreover, Theorem 2.13 allows us to strengthen Proposition 2.3 and corollary 2.9.

Proposition 2.14. *Suppose λ and μ are bipartitions with μ restricted and $[\mathrm{S}_\lambda : \mathrm{D}_\mu] > 0$, then $\mu^\diamond \trianglerighteq \lambda \trianglerighteq \mu$.*

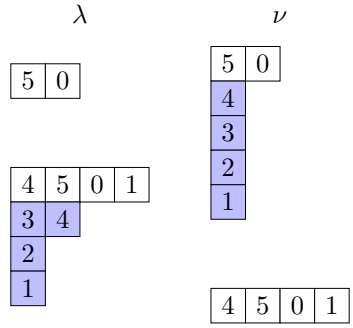
Proof. The fact that $\lambda \blacktriangleright \mu$ is immediate from Theorem 2.13. This then implies that $\mu^\diamond \blacktriangleright \lambda$, using Proposition 2.8 and the fact that the Jantzen–Schaper dominance order is reversed by conjugation. □

With this result in mind, we will use the Jantzen–Schaper order almost exclusively from now on. Although this order is harder to work with, it will help us to narrow down the cases we need to consider. Note that when trying to decide whether $\lambda \blacktriangleright \mu$, the usual dominance order can be used as a "first pass": if $\lambda \not\trianglerighteq \mu$, then certainly $\lambda \not\trianglerighteq \mu$. To examine the Jantzen–Schaper order more closely, it will be helpful to be able to visualise it on the abacus. Take two bipartitions λ, ν, and take the abacus displays for these bipartitions. In order to obtain $\nu_{\mathfrak{p}}(j_{\lambda\nu}) \neq 0$, we need to be able to get from the abacus display for λ to the abacus display for ν by moving a bead from position b_1 to position c_1 on component i_1, and a bead from position c_2 to position b_2 on component i_2, where $i_1, i_2 \in \{1, 2\}$, $b_1 - c_1 = b_2 - c_2 > 0$, and $b_1 \equiv b_2 \pmod{e}$. We then have $\lambda \blacktriangleright \nu$ if and only if either $b_1 > b_2$ or $i_1 < i_2$. The leg length of the hook L removed from λ is the number of beads between positions b_1 and c_1, and the leg length of the hook removed from ν is the number of beads between positions b_2 and c_2.

For example, take $e = 6$, $\kappa = (\overline{5}, \overline{4})$, $\lambda = \left(2 \mid 4, 2, 1^2\right)$ and $\nu = \left(2, 1^4 \mid 4\right)$. Taking the bicharge $(11, 10)$, we obtain the following abacus displays. (We draw the beads that have moved in white.)



Taking $b_1 = b_2 = 10$, $c_1 = c_2 = 6$, $i_1 = 1$ and $i_2 = 2$, we see that $j_{\lambda\nu} \neq 0$ and $\lambda \blacktriangleleft \nu$. Alternatively, we can see this by looking at the Young diagrams (in which we label nodes with their residues, and shade the involved rim hooks).



3. Blocks of weight 3

3.1. Weight and the abacus. In this section we summarise results from [8] showing how to calculate the weight of a bipartition from the abacus, and then use these to describe bipartitions of weight 3.

Suppose λ is a bipartition, and construct the abacus display for λ using the bicharge (k_1, k_2). We compute the weight of λ recursively, in three stages.

First suppose there is a bead in the abacus display with an unoccupied position immediately above. Define a new abacus display (and hence a new bipartition λ^-) by moving the bead up into this empty space (this corresponds to removing a rim e-hook from λ). Then [8, Corollary 3.4] gives $\mathrm{wt}(\lambda^-) = \mathrm{wt}(\lambda) - 2$.

Applying this repeatedly, we can assume that λ is a *bicore*; that is, it has no rim e-hooks. Now for each $x \in \mathbb{Z}/e\mathbb{Z}$ define $\gamma_x(\lambda)$ to equal the number of beads on runner x in component 1 minus the number of beads on runner x in component 2 of the abacus display. (Note that $\gamma_x(\lambda)$ depends on the choice of bicharge (k_1, k_2), but the difference $\gamma_x(\lambda) - \gamma_y(\lambda)$ for $x, y \in \mathbb{Z}/e\mathbb{Z}$ does not.) Now for $x, y \in \mathbb{Z}/e\mathbb{Z}$ with $x \neq y$ define the bicore $s_{xy}(\lambda)$ by moving a bead from component 1 to component 2 on runner x, and a bead from component 2 to component 1 on runner y of the abacus display. Then [8, Lemma 3.7] shows that $\mathrm{wt}(s_{xy}(\lambda)) = \mathrm{wt}(\lambda) - 2(\gamma_x(\lambda) - \gamma_y(\lambda) - 2)$.

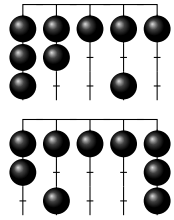
This shows that whenever there are x, y with $\gamma_x(\lambda) - \gamma_y(\lambda) \geqslant 3$, we can replace λ with a bipartition $s_{xy}(\lambda)$ of smaller weight. By doing this repeatedly (and using the fact that weight is always non-negative) we can reduce to the case where λ is a bicore with $\gamma_x(\lambda) - \gamma_y(\lambda) \leqslant 2$ for all $x, y \in \mathbb{Z}/e\mathbb{Z}$. In this case we use [8, Proposition 3.8],

which gives $\mathrm{wt}(\lambda) = \min\{|X|, |Y|\}$, where

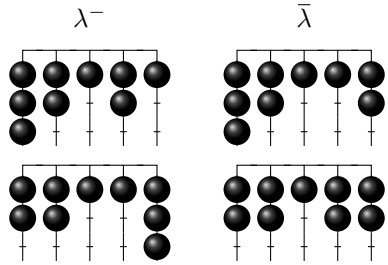
$$X = \{\, x \in \mathbb{Z}/e\mathbb{Z} \mid \gamma_x(\lambda) - \gamma_y(\lambda) = 2 \text{ for some } y \in \mathbb{Z}/e\mathbb{Z}\},$$
$$Y = \{\, y \in \mathbb{Z}/e\mathbb{Z} \mid \gamma_x(\lambda) - \gamma_y(\lambda) = 2 \text{ for some } x \in \mathbb{Z}/e\mathbb{Z}\}.$$

In particular, we note that $\mathrm{wt}(\lambda) = 0$ if and only if λ is a bicore with $\gamma_x(\lambda) - \gamma_y(\lambda) \leqslant 1$ for all $x, y \in \mathbb{Z}/e\mathbb{Z}$.

Example 3.1. Suppose $e = 5$, $(\kappa_1, \kappa_2) = (\overline{4}, \overline{4})$ and $(k_1, k_2) = (9, 9)$, and let λ be the bipartition $\big(5, 3 \;\big|\; 6, 4, 3\big)$, with the following abacus display.



Define a new bipartition $\lambda^- = \big(2, 1 \;\big|\; 6, 2\big)$ by moving all beads as up as possible in their runners, and then define $\overline{\lambda} = s_{34}(\lambda^-) = \big(2^2 \;\big|\; 1^2\big)$.



Then $\mathrm{wt}(\lambda) = \mathrm{wt}(\lambda^-) + 4$ and $\mathrm{wt}(\lambda^-) = \mathrm{wt}(\overline{\lambda}) + 2$, since $\gamma_{\overline{3}}(\lambda^-) - \gamma_{\overline{4}}(\lambda^-) = 3$. Now $\overline{\lambda}$ satisfies $\gamma_{\overline{0}}(\overline{\lambda}) - \gamma_{\overline{3}}(\overline{\lambda}) = 2$ and $\gamma_x(\overline{\lambda}) - \gamma_y(\overline{\lambda}) \leqslant 1$ for all other $x, y \in \mathbb{Z}/5\mathbb{Z}$. In the above notation we have that $|X| = |Y| = 1$ which implies $\mathrm{wt}(\overline{\lambda}) = 1$. We conclude that $\mathrm{wt}(\lambda) = 7$.

We can use these results to describe blocks of weight 3; this is very similar to the description of blocks of weight 2 in [7, Section 2.1]. The above results show that if λ is a bipartition in a block B of weight 3, then one of the following occurs.

1a. λ has a rim e-hook, and removing this leaves a bipartition of weight 1.
1b. λ is a bicore, and there are $x, y \in \mathbb{Z}/e\mathbb{Z}$ such that $\gamma_x(\lambda) - \gamma_y(\lambda) = 3$ and $s_{xy}(\lambda)$ has weight 1.
2. λ is a bicore with $\gamma_i(\lambda) - \gamma_j(\lambda) \leqslant 2$ for all i, j, and $\min\{|X|, |Y|\} = 3$, where X and Y are as defined above.

We observe that if λ^- is obtained by removing a rim e-hook from λ, then $\delta_i(\lambda^-) = \delta_i(\lambda)$ for all i, because $\delta_i(\lambda)$ is determined by the total number of beads on each runner (across both components) in the abacus display for λ, which does not change when we move a bead up its runner. Similarly, if λ is a bicore then $\delta_i(s_{xy}(\lambda)) = \delta_i(\lambda)$ for all i. In view of [8, Lemma 3.3], we then obtain $|s_{xy}(\lambda)| = n - e$. So in cases (1a) and (1b) there is a block A of some $\mathcal{H}_{n-e}$ with weight 1 and with $\delta_i(A) = \delta_i(B)$ for all i.

If B contains a bipartition satisfying (2) above, then (following [9]) we say that B is a *core block*. By [9, Theorem 3.1], this is equivalent to the statement that there is no block A of $\mathcal{H}_m$ for any m with weight less than 3, and with $\delta_i(A) = \delta_i(B)$ for all i.

What this means is that if B is a core block of weight 3 then the bipartitions in B all satisfy (2), while if B is a non-core block then the bipartitions in B all satisfy (1a) or (1b).

In Section 3.3 we will see that our main theorem is easy to prove for core blocks, but we need to consider non-core blocks in detail.

3.2. The bipartitions in a non-core weight 3 block. Suppose B is a non-core block of weight 3. In this section we show how to construct all the bipartitions in B. Suppose λ is a bipartition in B, and take an abacus display for λ with bicharge (k_1, k_2). Then we can construct a bipartition ν lying in a block A of weight 1 with $\delta_i(\nu) = \delta_i(\lambda)$ for all i (either by removing a rim e-hook or by applying s_{xy} for some x, y). We call A the *weight* 1 *block underlying* B. Now $\gamma_x(\nu) - \gamma_y(\nu) \leqslant 2$ for all x and y, and if we define the sets X and Y as in Section 3.1 (with ν in place of λ), then $\min\{|X|, |Y|\} = 1$. We will assume for the rest of the paper that $|Y| = 1$; the case $|X| = 1$ follows similarly using the component-switching arguments of Section 2.7. We set $Z = X \cup Y$ and $\mathcal{C}(Z) = \mathbb{Z}/e\mathbb{Z} \smallsetminus Z$; note in particular that $|Z| \geqslant 2$. Now define a new abacus display (with bicharge $(k_1 + 1, k_2 - 1)$) by moving a bead from component 2 to component 1 on runner y (where $Y = \{y\}$). We denote the resulting bipartition ξ_A, and call it the *nucleus* of A. Then ξ_A is independent of the choice of ν (and hence independent of the choice of λ), and the $(\kappa_1 + 1, \kappa_2 - 1)$-weight of ξ_A is 0. Moreover, if we define the integers $\gamma_x(\xi_A)$ using the abacus display with bicharge $(k_1 + 1, k_2 - 1)$, then

$$\gamma_x(\xi_A) - \gamma_y(\xi_A) = \begin{cases} 1 & \text{if } x \in Z,\ y \in \mathcal{C}(Z) \\ -1 & \text{if } x \in \mathcal{C}(Z),\ y \in Z \\ 0 & \text{otherwise.} \end{cases}$$

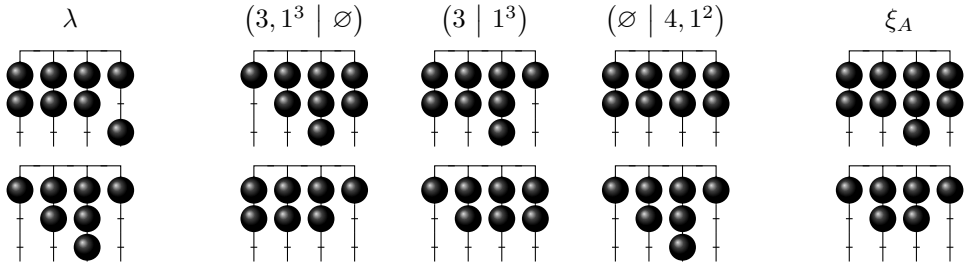
Now for each $z \in Z$ define a bipartition ν_z in A by taking the abacus display for ξ_A and moving a bead from component 1 to component 2 on runner z. Then the description of the bipartitions in a weight 1 block in [8, Section 4.2.1] shows that the bipartitions ν_z for $z \in X \cup Y$ are precisely the bipartitions in A. Now the results in Section 3.1 show that each bipartition in B is obtained from a bipartition ν_z in A either by adding a rim e-hook or by applying the operator s_{xy} for some x, y for which $\gamma_x(\nu_z) - \gamma_y(\nu_z) = 1$. But the above formula for $\gamma_x(\xi_A) - \gamma_y(\xi_A)$ shows that $\gamma_x(\nu_z) - \gamma_y(\nu_z) = 1$ if and only if either $x \in Z \smallsetminus \{z\}$ and $y \in \mathcal{C}(Z)$, or $x \in \mathcal{C}(Z)$ and $y = z$.

This enables us to describe (and label) all the bipartitions in B. These come in three families, constructed from ξ_A as follows.

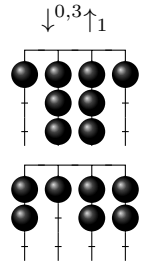
- For each $z \in Z$, $x \in \mathbb{Z}/e\mathbb{Z}$ and $a \in \{1, 2\}$, we obtain a bipartition $\mathrm{a}_x^{(a)}{\downarrow^z}$ from ξ_A by moving a bead from component 1 to component 2 on runner z, then moving the lowest bead on runner x on component a down one position.
- For each $x \in \mathcal{C}(Z)$, we obtain a bipartition $\downarrow^x$ from ξ_A by moving a bead from component 1 to component 2 on runner x.
- For each $w, z \in Z$ with $w \neq z$ and for each $y \in \mathcal{C}(Z)$, we obtain a bipartition ${\downarrow^{w,z}}{\uparrow_y}$ from ξ_A by moving beads from component 1 to component 2 on runners w and z, and then moving a bead from component 2 to component 1 on runner y.

Example 3.2. Suppose $e = 4$, $(\kappa_1, \kappa_2) = (\overline{0}, \overline{3})$ and $(k_1, k_2) = (8, 7)$. Let B be the weight-3 block of $\mathcal{H}_{10}$ containing the bipartition $\lambda = \left(4 \;\middle|\; 4, 1^2\right)$. Then the underlying weight-1 block A comprises the bipartitions $\left(3, 1^3 \;\middle|\; \varnothing\right)$, $\left(3 \;\middle|\; 1^3\right)$ and $\left(\varnothing \;\middle|\; 4, 1^2\right)$, and the nucleus of A is $\xi_A = \left(2 \;\middle|\; 1^2\right)$. These bipartitions have the following abacus

displays.



We see that $Z = \{\bar{0}, \bar{2}, \bar{3}\}$. So there are 28 bipartitions in B: the bipartitions $\mathrm{a}_x^{(a)}\downarrow^z$ for $z \in \{\bar{0}, \bar{2}, \bar{3}\}$, $x \in \mathbb{Z}/4\mathbb{Z}$ and $a \in \{1, 2\}$, together with the four bipartitions $\downarrow^1$, $\downarrow^{0,2}\uparrow_1$, $\downarrow^{0,3}\uparrow_1$ and $\downarrow^{2,3}\uparrow_1$. For example, the bipartition λ above is $\mathrm{a}_3^{(1)}\downarrow^2$, while $\downarrow^{0,3}\uparrow_1$ is the bipartition $\left(3^2, 1^2 \mid 1^2\right)$ shown below.



3.3. Inductive approach to the main theorem. Now we give some results to facilitate a proof of Theorem 1.1 by induction. We begin with an analogue of a result for partitions in [12], which greatly restricts the set of bipartitions we need to consider. We give the result for blocks of weight 3 only, but a more general result along the lines of [12, Proposition 2.1] can easily be proved.

Lemma 3.3. *Suppose B is a weight 3 block of $\mathcal{H}_n$, and that Theorem 1.1 holds with n replaced by any smaller integer. Suppose λ, μ are bipartitions of n with μ restricted, and $i \in \mathbb{Z}/e\mathbb{Z}$.*

(1) *If* $\mathrm{nor}_i(\mu) \geqslant \max\{1, \mathrm{rem}_i(\lambda)\}$, *then* $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$.
(2) *If* $\mathrm{nor}_i(\mu) \geqslant \max\{1, \delta_i(B) + 1\}$, *then* $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$.

Proof. Recall the notation $\epsilon_i M$ from Section 2.8, and that (from Proposition 2.12) $\epsilon_i \mathrm{S}_\lambda = \mathrm{rem}_i(\lambda)$ and $\epsilon_i \mathrm{D}_\mu = \mathrm{nor}_i(\mu)$.

(1) Since e_i is an exact functor, if M is a module and N is a subquotient of M, then $\epsilon_i N \leqslant \epsilon_i M$. In particular, if $\mathrm{nor}_i(\mu) > \mathrm{rem}_i(\lambda)$, then $\epsilon_i \mathrm{D}_\mu > \epsilon_i \mathrm{S}_\lambda$, and therefore D_μ cannot be a composition factor of S_λ.

Suppose instead that $\mathrm{nor}_i(\mu) = \mathrm{rem}_i(\lambda) = r$, say. Then $r = \epsilon_i \mathrm{S}_\lambda = \epsilon_i \mathrm{D}_\mu$. Now $\mathrm{e}_i^{(r)}(\mathrm{S}_\lambda) \cong \mathrm{S}_{\lambda^-}$, and $\mathrm{e}_i^{(r)}(\mathrm{D}_\mu) \cong \mathrm{D}_{\mu^-}$, where λ^- is obtained by removing all the removable i-nodes from λ, and μ^- is obtained by removing all the normal i-nodes from μ. Therefore

$$[\mathrm{S}_{\lambda^-} : \mathrm{D}_{\mu^-}] = [\mathrm{e}_i^{(r)} \mathrm{S}_\lambda : \mathrm{D}_{\mu^-}] = \sum_\nu [\mathrm{S}_\lambda : \mathrm{D}_\nu][\mathrm{e}_i^{(r)} \mathrm{D}_\nu : \mathrm{D}_{\mu^-}] \geqslant [\mathrm{S}_\lambda : \mathrm{D}_\mu],$$

summing over restricted bipartitions ν of n. Because $r = \mathrm{rem}_i(\lambda)$, we obtain $r \geqslant \delta_i(\lambda)$. So by Lemma 2.6 the weight of λ^- is at most 3. If λ^- has weight 3, then by the inductive assumption $[\mathrm{S}_{\lambda^-} : \mathrm{D}_{\mu^-}] \leqslant 1$. If λ^- has weight less than 3, then $[\mathrm{S}_{\lambda^-} : \mathrm{D}_{\mu^-}] \leqslant 1$ from Theorem 2.7. Either way, we obtain $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$.

(2) We show that the number $r = \mathrm{rem}_i(\lambda)$ is at most $\max\{1, \delta_i(B)+1\}$; then in particular $r \leqslant \mathrm{nor}_i(\mu)$, and we can use part (1) of the lemma. Suppose for a contradiction that $r \geqslant 2$ and $r \geqslant \delta_i(B)+2$, and let λ^- be obtained by removing all the removable i-nodes from λ. Then by Lemma 2.6 the weight of λ^- equals $3 - r(r - \delta_i(B))$. Since by assumption $r \geqslant 2$ and $r - \delta_i(B) \geqslant 2$, this makes the weight of λ^- negative, a contradiction. □

We deduce two very useful corollaries.

Corollary 3.4. *Suppose B is a weight* 3 *block of $\mathcal{H}_n$ with $\delta_i(B) \leqslant 0$ for all $i \in \mathbb{Z}/e\mathbb{Z}$, and that Theorem 1.1 holds with n replaced by any smaller integer. Then Theorem 1.1 holds for B.*

Proof. Suppose λ, μ are bipartitions in B with μ restricted. Then $\mathrm{D}_\mu \downarrow_{\mathcal{H}_{n-1}} \neq 0$, so $\mathrm{e}_i \mathrm{D}_\mu \neq 0$ for some $i \in \mathbb{Z}/e\mathbb{Z}$. This means that $\mathrm{nor}_i(\mu) \geqslant 1$, so $\mathrm{nor}_i(\mu) \geqslant \max\{1, \delta_i(B)+1\}$, and therefore $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$ by Lemma 3.3(2). □

Corollary 3.5. *Suppose B is a weight* 3 *block of $\mathcal{H}_n$, and that Theorem 1.1 holds with n replaced by any smaller integer. If λ is a bipartition in B and $i \in \mathbb{Z}/e\mathbb{Z}$ such that $\delta_i(B) \geqslant 1$ and λ has no addable i-nodes, then $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$ for all μ.*

Proof. The assumption that λ has no addable i-nodes means that $\mathrm{rem}_i(\lambda) = \delta_i(B)$, while $\mathrm{nor}_i(\mu) \geqslant \delta_i(B)$. So $\mathrm{nor}_i(\mu) \geqslant \max\{1, \mathrm{rem}_i(\lambda)\}$, and the result follows from Lemma 3.3(1). □

So to prove Theorem 1.1 by induction using Corollary 3.5, it suffices to consider only those λ that have at least one addable i-node for *every* i for which $\delta_i(B) \geqslant 1$. Call such λ *exceptional*. The next few lemmas will show that in most cases B has no exceptional bipartitions.

Lemma 3.6. *Suppose B is a core block, and that $\delta_i(B) \geqslant 1$ for some $i \in \mathbb{Z}/e\mathbb{Z}$. Then B contains no exceptional bipartitions.*

Proof. Suppose λ is a bipartition in B. Then we claim that λ has no addable i-nodes. Assume for a contradiction that λ has at least one addable i-node. Then because $\delta_i(B) \geqslant 1$, we know that λ has at least two removable i-nodes. The two components of λ are both e-cores, and an e-core cannot have addable and removable nodes of the same residue, so the addable i-nodes of λ are in one component and the removable i-nodes in the other. We assume that the removable i-nodes are in component 1 (the other case is similar). This means that in an abacus display for λ there are at least two positions b on runner i such that there is a bead in position b but not in position $b-1$. So there are at least two more beads on runner i than on runner $i-1$ in component 1 (or at least three more, in the case $i = \bar{0}$). Similarly, in component 2 of the abacus display, there is at least one position b on runner $i-1$ such that there is a bead at position b but no bead at position $b+1$. So there are more beads on runner $i-1$ than on runner i in component 2 (or at least as many beads on runner $i-1$ as on runner i, in the case $i = \bar{0}$). We deduce that $\gamma_i(\lambda) - \gamma_{i-1}(\lambda) \geqslant 3$. But the assumption that B is a core block gives $\gamma_i(\lambda) - \gamma_{i-1}(\lambda) \leqslant 2$, a contradiction. □

In view of Lemma 3.6, Theorem 1.1 holds for core blocks. These blocks have been extensively studied; a closed formula for the decomposition numbers is given in [13].

We consider non-core blocks from now on.

Lemma 3.7. *Suppose $\delta_i(B) \geqslant 3$ for some $i \in \mathbb{Z}/e\mathbb{Z}$. Then B contains no exceptional bipartitions.*

Proof. If λ is a bipartition in B, then we claim that λ cannot have an addable i-node: if it did, then by Lemma 2.6 the weight of a bipartition obtained by adding an addable i-node to λ would be $3-(\delta_i(\lambda)+1)$, which is negative, a contradiction. □

Lemma 3.8. *Suppose there are $i,j \in \mathbb{Z}/e\mathbb{Z}$ such that $\delta_i(B), \delta_j(B) \geqslant 1$, and $j \neq i, i\pm 1$. Then B contains no exceptional bipartitions.*

Proof. Suppose λ is an exceptional bipartition in B. Then λ has both an addable i-node $\mathfrak{m}$ and an addable j-node $\mathfrak{n}$. Define bipartitions μ, ν by $[\mu] = [\lambda] \cup \{\mathfrak{m}\}$ and $[\nu] = [\mu] \cup \{\mathfrak{n}\}$. Lemma 2.6 then gives $\mathrm{wt}(\mu) = 3 - (\delta_i(B)+1) \leqslant 1$. Because $j \neq i, i\pm 1$, adding the i-node to λ does not affect the addable or removable j-nodes, so $\delta_j(\mu) = \delta_j(\lambda)$. Hence $\mathrm{wt}(\nu) = \mathrm{wt}(\mu) - (\delta_j(B)+1) \leqslant -1$, a contradiction. □

Lemma 3.9. *Suppose there are $i,j \in \mathbb{Z}/e\mathbb{Z}$ such that $i \neq j$, $\delta_i(B) \geqslant 1$ and $\delta_j(B) \geqslant 2$. Then B contains no exceptional bipartitions.*

Proof. If $j \neq i \pm 1$ then the result follows from Lemma 3.8, so assume that $j = i+1$ (the case $j = i-1$ is similar). Suppose λ is an exceptional bipartition in B. Then λ has both an addable i-node $\mathfrak{m}$ and an addable $(i+1)$-node $\mathfrak{n}$. Define bipartitions μ, ν by $[\mu] = [\lambda] \cup \{\mathfrak{m}\}$ and $[\nu] = [\mu] \cup \{\mathfrak{n}\}$. Lemma 2.6 now gives $\mathrm{wt}(\mu) = 3 - (\delta_i(B)+1) \leqslant 1$. Now from [8, Lemma 3.1] we obtain $\delta_{i+1}(\mu) = \delta_{i+1}(\lambda) - 1$, so that $\mathrm{wt}(\nu) = \mathrm{wt}(\mu) - \delta_{i+1}(B) \leqslant -1$, and again we have a contradiction. □

These lemmas mean that there are only four types of non-core block B in which there can be exceptional bipartitions. These are exactly analogous to the four types of weight 3 blocks of symmetric groups considered in [12, Section 4]. These types can be defined according to the values $\delta_i(B)$, as follows.

I. $\delta_i(B) \leqslant 0$ for all $i \in \mathbb{Z}/e\mathbb{Z}$.
II. There is $i \in \mathbb{Z}/e\mathbb{Z}$ such that $\delta_i(B) = 1$ while $\delta_j(B) \leqslant 0$ for all $j \neq i$.
III. There is $i \in \mathbb{Z}/e\mathbb{Z}$ such that $\delta_i(B) = \delta_{i+1}(B) = 1$ while $\delta_j(B) \leqslant 0$ for all $j \neq i, i+1$.
IV. There is $i \in \mathbb{Z}/e\mathbb{Z}$ such that $\delta_i(B) = 2$ while $\delta_j(B) \leqslant 0$ for all $j \neq i$.

We know from Corollary 3.4 that Theorem 1.1 holds for blocks of type I. To prove Theorem 1.1, we need to show that if B is a block of $\mathcal{H}_n$ of type II, III or IV and if Theorem 1.1 holds with n replaced by any smaller integer, then it holds for B. We address the three types of block in Sections 4 to 6.

4. Blocks of type II

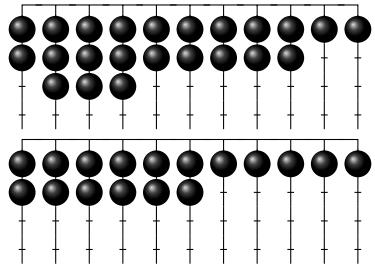
Throughout this section, we assume B is a block of $\mathcal{H}_n$ and $i \in \mathbb{Z}$ is such that $\delta_i(B) = 1$ while $\delta_j(B) \leqslant 0$ for all $j \not\equiv i \pmod{e}$. Our aim is to show that if Theorem 1.1 holds with n replaced by any smaller integer, then it holds for B.

4.1. The abacus for a block of type II. Let ξ be the nucleus of B. Whenever we refer to the abacus display for ξ, we will mean the (k_1+1, k_2-1)-abacus display, and whenever we refer to the residues of nodes for ξ, we mean with respect to (κ_1+1, κ_2-1). With this convention in mind, we have $\delta_i(\xi) = 1$ while $\delta_j(\xi) \leqslant 0$ for $j \not\equiv i \pmod{e}$. This means that ξ has exactly one removable node, which has residue i (because by Lemma 2.6 a bipartition of weight 0 cannot have an addable and a removable node of the same residue). We will assume that this removable node lies in component 1 (in the opposite case we can replace κ with (κ_2, κ_1) and appeal to the results in Section 2.7).

By considering the abacus display for ξ and using the fact that ξ has (κ_1+1, κ_2-1)-weight 0, we find that there are integers $i \leqslant j \leqslant k \leqslant l \leqslant e+i-2$ such that in component 1 of ξ there is a removable i-node, and addable nodes of residues $j+1$ and

$l+1$, while in component 2 there is just an addable node of residue $k+1$ (so in fact $\kappa_1 = \overline{j+l+1-i}$, and $\kappa_2 = \overline{k+2}$). We therefore have $Z = \{\bar{i}, \dots, \bar{j}\} \cup \{\overline{k+1}, \dots, \bar{l}\}$. The fact that $|Z| \geqslant 2$ implies that either $i < j$ or $k < l$.

EXAMPLE 4.1. Suppose $e = 11$, $\kappa = (\bar{0}, \bar{7})$ and $\xi = \left(3^3 \mid \varnothing\right)$, with the following abacus display.



We can take $(i, j, k, l) = (1, 3, 5, 8)$, and $Z = \{\bar{1}, \bar{2}, \bar{3}, \bar{6}, \bar{7}, \bar{8}\}$.

4.2. EXCEPTIONAL BIPARTITIONS IN B. Suppose λ is a bipartition in B. From the results in Section 3.3, we need only consider Specht modules labelled by exceptional bipartitions; that is, those with addable i-nodes. By examining the abacus displays of the bipartitions in B, we can easily classify these. There are $3|Z|$ exceptional bipartitions, which we label $\alpha_x, \beta_x, \gamma_x$ for $x \in Z$ as follows:

$$\alpha_x = \begin{cases} \mathrm{a}_i^{(2)}{\downarrow^x} & \text{if } x \neq i \\ {\downarrow^{i-1}} & \text{if } x = i, \end{cases} \quad \beta_x = \begin{cases} \mathrm{a}_{i-1}^{(2)}{\downarrow^x} & \text{if } x \neq i \\ \mathrm{a}_i^{(1)}{\downarrow^i} & \text{if } x = i, \end{cases} \quad \gamma_x = \begin{cases} {\downarrow^{i,x}\uparrow_{i-1}} & \text{if } x \neq i \\ \mathrm{a}_{i-1}^{(1)}{\downarrow^i} & \text{if } x = i. \end{cases}$$

It will be helpful to write these bipartitions explicitly:

$$\alpha_x = \begin{cases} \left((e+i-l)^{j-i+1}, 1^{l-i+1} \mid e+i-k-2\right) & \text{if } x = i \\ \left((e+i-l)^{j-x}, (e+i-l-1)^{x-i} \mid e+x-k-1, e+i-k, 1^{k-i}\right) & \text{if } i < x \leqslant j \\ \left((e+i-l)^{j-i+1}, 1^{l-x} \mid e+i-k-1, x-k, 1^{k-i}\right) & \text{if } k < x \leqslant l, \end{cases}$$

$$\beta_x = \begin{cases} \left((e+i-l)^{j-i+1}, 1^{l-i} \mid e+i-k-1\right) & \text{if } x = i \\ \left((e+i-l)^{j-x}, (e+i-l-1)^{x-i} \mid e+x-k-1, e+i-k-1, 1^{k-i+1}\right) & \text{if } i < x \leqslant j \\ \left((e+i-l)^{j-i+1}, 1^{l-x} \mid e+i-k-2, x-k, 1^{k-i+1}\right) & \text{if } k < x \leqslant l, \end{cases}$$

$$\gamma_x = \begin{cases} \left((e+i-l)^{j-i}, e+i-l-1, 1^{l-i+1} \mid e+i-k-1\right) & \text{if } x = i \\ \left((e+i-l)^{j-x}, (e+i-l-1)^{x-i-1}, e+i-l-2 \middle| e+x-k-1, e+i-k, 1^{k-i+1}\right) & \text{if } i < x \leqslant j \\ \left((e+i-l)^{j-i}, e+i-l-1, 1^{l-x} \mid e+i-k-1, x-k, 1^{k-i+1}\right) & \text{if } k < x \leqslant l. \end{cases}$$

Each exceptional bipartition has exactly one addable i-node, and the labelling is chosen so that $\alpha_x, \beta_x, \gamma_x$ all yield the same bipartition when the addable i-node is added, with $\alpha_x \blacktriangleright \beta_x \blacktriangleright \gamma_x$.

4.3. PROVING THEOREM 1.1 IN B. Our aim is to show that $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$ for all bipartitions λ, μ in B, given the inductive assumption. We will do this by looking at one restricted bipartition μ at a time. The next lemma will help in narrowing down the cases we need to consider.

LEMMA 4.2. *Suppose μ is a restricted bipartition in B, and that $[\mathrm{S}_\lambda : \mathrm{D}_\mu] > 1$ for some λ. Then $\mathrm{nor}_i(\mu) = 1$, and there is some $x \in Z$ for which $[\mathrm{S}_{\alpha_x} : \mathrm{D}_\mu]$, $[\mathrm{S}_{\beta_x} : \mathrm{D}_\mu]$, $[\mathrm{S}_{\gamma_x} : \mathrm{D}_\mu]$ are all non-zero; in particular, $\mu^\diamond \blacktriangleright \alpha_x, \beta_x, \gamma_x \blacktriangleright \mu$.*

Proof. If λ is such that $[\mathrm{S}_\lambda : \mathrm{D}_\mu] > 1$, then λ is exceptional, so $\lambda \in \{\alpha_x, \beta_x, \gamma_x\}$ for some x. The conclusion that μ has only one normal node comes from Lemma 3.3(2).

Now we claim that $[\mathrm{S}_{\alpha_x} : \mathrm{D}_\mu], [\mathrm{S}_{\beta_x} : \mathrm{D}_\mu], [\mathrm{S}_{\gamma_x} : \mathrm{D}_\mu] \geqslant 1$, which gives the result using Proposition 2.14. Each of $\alpha_x, \beta_x, \gamma_x$ has two removable i-nodes, and the branching rules show that there are bipartitions $\hat{\alpha}_x, \hat{\beta}_x, \hat{\gamma}_x$ of $n-1$ such that

$$\begin{aligned} \mathrm{e}_i \, \mathrm{S}_{\alpha_x} &\sim \mathrm{S}_{\hat{\alpha}_x} + \mathrm{S}_{\hat{\beta}_x}, \\ \mathrm{e}_i \, \mathrm{S}_{\beta_x} &\sim \mathrm{S}_{\hat{\alpha}_x} + \mathrm{S}_{\hat{\gamma}_x}, \\ \mathrm{e}_i \, \mathrm{S}_{\gamma_x} &\sim \mathrm{S}_{\hat{\beta}_x} + \mathrm{S}_{\hat{\gamma}_x}. \end{aligned}$$

Because $\mathrm{nor}_i(\mu) = 1$, there is a restricted bipartition $\tilde{\mu}$ such that $\mathrm{e}_i \, \mathrm{D}_\mu \cong \mathrm{D}_{\tilde{\mu}}$. Now (analogously to the proof of [10, Proposition 2.8(2)]) we obtain

$$[\mathrm{S}_{\alpha_x} : \mathrm{D}_\mu] + [\mathrm{S}_{\hat{\gamma}_x} : \mathrm{D}_{\tilde{\mu}}] = [\mathrm{S}_{\beta_x} : \mathrm{D}_\mu] + [\mathrm{S}_{\hat{\beta}_x} : \mathrm{D}_{\tilde{\mu}}] = [\mathrm{S}_{\gamma_x} : \mathrm{D}_\mu] + [\mathrm{S}_{\hat{\alpha}_x} : \mathrm{D}_{\tilde{\mu}}].$$

Since by assumption the decomposition numbers for weight 3 blocks of $\mathcal{H}_{n-1}$ are all 0 or 1, this means that the decomposition numbers $[\mathrm{S}_{\alpha_x} : \mathrm{D}_\mu]$, $[\mathrm{S}_{\beta_x} : \mathrm{D}_\mu]$, $[\mathrm{S}_{\gamma_x} : \mathrm{D}_\mu]$ differ by at most 1. So if one of them is greater than 1, then they are all positive. □

Lemma 4.2 greatly restricts the set of restricted bipartitions μ that we need to consider. To apply Lemma 4.2 we look at the Jantzen–Schaper dominance order among the exceptional bipartitions, which is easy to check given the explicit description above: if ϵ is any of the symbols α, β, γ, then

$$\epsilon_i \blacktriangleright \epsilon_{k+1} \blacktriangleright \epsilon_{k+2} \blacktriangleright \cdots \blacktriangleright \epsilon_l \blacktriangleright \epsilon_{i+1} \blacktriangleright \epsilon_{i+2} \blacktriangleright \cdots \blacktriangleright \epsilon_j.$$

In particular, the most dominant γ_x is γ_i. So to show that $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$ for all μ, we need only consider bipartitions μ for which:

(1) μ is restricted;
(2) $\gamma_i \not\blacktriangleright \mu$; and
(3) $\mathrm{nor}_i(\mu) = 1$, and $\mathrm{nor}_j(\mu) = 0$ for all $j \neq i$.

Analysis of the possible abacus configurations enables a classification of the bipartitions μ satisfying (2) and (3). By working through the different bipartitions in B, we can check that the only bipartitions μ satisfying these conditions are $\mathrm{a}_{i-1}^{(2)}{\downarrow^i}$ and $\mathrm{a}_i^{(2)}{\downarrow^i}$.

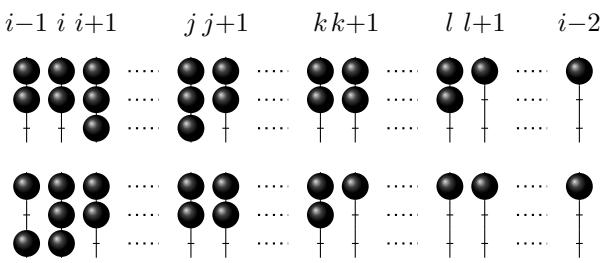
(Note that if $i < k$ then the bipartition $\mu = \mathrm{a}_k^{(2)}{\downarrow^i}$ does not satisfy (2): in this case

$$\begin{aligned} \mu &= \big((e+i-l)^{j-i} \;\big|\; e-1, e+i-k\big), \\ \gamma_i &= \big((e+i-l)^{j-i}, e+i-l-1, 1^{l-i+1} \;\big|\; e+i-k-1\big), \end{aligned}$$

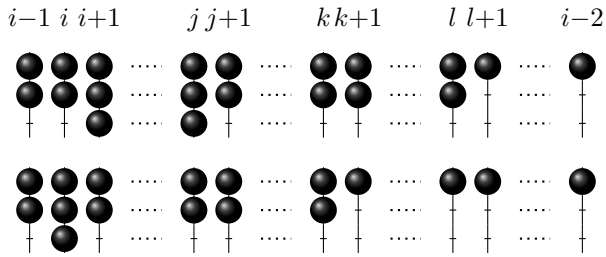
so $\gamma_i \rhd \mu$. But by examining residues we can check that $\gamma_i \not\blacktriangleright \mu$.)

Now we introduce condition (1). First, from Lemma 2.2 we know that $\mathrm{a}_i^{(2)}{\downarrow^i}$ is not restricted, since its second component is not e-restricted. So it remains to check when $\mu = \mathrm{a}_{i-1}^{(2)}{\downarrow^i}$ is restricted. We do this by removing good nodes, using the results of Section 2.4.

We begin by drawing an abacus display for μ (where we adjust the numbers of beads in order to make runner $i-1$ the leftmost runner).



Now we remove good nodes of residues $i, i+1, \dots, k,\ i-1, i-2, \dots, k+1$ in turn, to obtain the bipartition $\xi = \big((e+i-l)^{j-i} \;\big|\; e+i-k-1\big)$.

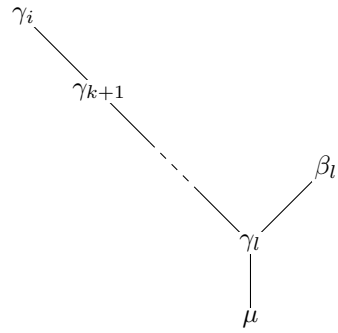


This bipartition lies in the weight 1 block underlying B, and (from the results in Section 2.4) is restricted if and only if μ is. Hence we need to understand whether ξ is restricted or not. [8, Lemma 4.9] shows that a bipartition of weight 1 is restricted if and only if it is not the most dominant bipartition in its block. Using the explicit description of weight 1 blocks [8, Theorem 4.4], we find all the bipartitions in the same block as ξ and we see that ξ is the most dominant bipartition in its block if and only if $k = l$. It follows that μ is restricted if and only if $k < l$.

We deal with this case using the Jantzen–Schaper formula. From Lemma 4.2 we need only consider exceptional bipartitions $\alpha_x, \beta_x, \gamma_x$ for which $\gamma_x \trianglerighteq \mu$, and by looking at the explicit expressions for the exceptional bipartitions, we find that this happens if and only if $x = i$ or $k < x \leqslant l$. So to show that $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$ for all λ it suffices to show that

$$[\mathrm{S}_{\gamma_i} : \mathrm{D}_\mu] = [\mathrm{S}_{\gamma_{k+1}} : \mathrm{D}_\mu] = \cdots = [\mathrm{S}_{\gamma_{l-1}} : \mathrm{D}_\mu] = [\mathrm{S}_{\beta_l} : \mathrm{D}_\mu] = 0.$$

To do this using the Jantzen–Schaper order, we first find all the bipartitions λ such that $\beta_l \trianglerighteq \lambda \trianglerighteq \mu$ or $\gamma_x \trianglerighteq \lambda \trianglerighteq \mu$ for some $x \in \{i\} \cup \{k+1, \dots, l\}$. In fact by examining the explicit forms of these bipartitions, we easily find that the only such bipartitions λ are μ, β_l, γ_i and $\gamma_{k+1}, \dots, \gamma_l$, with the Jantzen–Schaper order on these bipartitions given by the following diagram.



The Jantzen–Schaper coefficients we need are given by

$$j_{\mu\gamma_x} = \begin{cases} (-1)^{l-x} & (k < x \leqslant l) \\ (-1)^{l-k} & (x = i), \end{cases} \qquad j_{\mu\beta_l} = -1,$$

$$j_{\gamma_l\gamma_x} = \begin{cases} (-1)^{l-x+1} & (k < x < l) \\ (-1)^{l-k+1} & (x = i), \end{cases} \qquad j_{\gamma_l\beta_l} = 1.$$

Now we deduce that $[\mathrm{S}_{\gamma_l} : \mathrm{D}_\mu] = 1$, and then (successively) $[\mathrm{S}_{\gamma_{l-1}} : \mathrm{D}_\mu] = \cdots = [\mathrm{S}_{\gamma_{k+1}} : \mathrm{D}_\mu] = [\mathrm{S}_{\gamma_i} : \mathrm{D}_\mu] = 0$ as well as $[\mathrm{S}_{\beta_l} : \mathrm{D}_\mu] = 0$.

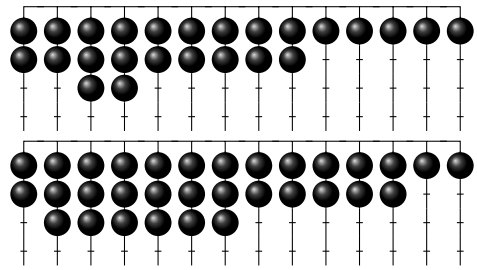
5. Blocks of type III

Throughout this section, we assume B is a block of $\mathcal{H}_n$ and $i \in \mathbb{Z}$ is such that $\delta_i(B) = \delta_{i+1}(B) = 1$ while $\delta_j(B) \leqslant 0$ for all $j \not\equiv i, i+1 \pmod e$. Our aim is to show that if Theorem 1.1 holds with n replaced by any smaller integer, then it holds for B.

5.1. The abacus for a block of type III. Take ξ the nucleus of B and consider its (k_1+1, k_2-1)-abacus display following the same convention of Section 4. We have $\delta_i(\xi) = \delta_{i+1}(\xi) = 1$ while $\delta_j(\xi) \leqslant 0$ for $j \not\equiv i, i+1 \pmod e$. This means that ξ has exactly two removable nodes, of residues i and $i+1$. We will assume that the removable $(i+1)$-node lies in component 1 (in the opposite case we can replace κ with (κ_2, κ_1) and appeal to the results in Section 2.7). This means in particular that $\gamma_{i+1}(\xi) = \gamma_i(\xi)+1$. Now we claim that the removable i-node of ξ must lie in component 2; if not, then we obtain $\gamma_i(\xi) = \gamma_{i-1}(\xi) + 1$ and hence $\gamma_{i+1}(\xi) - \gamma_{i-1}(\xi) = 2$, and now the results in Section 3.1 show that ξ has positive weight, a contradiction.

Now we can describe the abacus display for ξ more precisely: there are integers $i+1 \leqslant j \leqslant k \leqslant l \leqslant m \leqslant e+i-2$ such that in component 1 of ξ there is a removable node of residue $i+1$ and addable nodes of residues $j+1$ and $l+1$, while in component 2 there is a removable node of residue i and addable nodes of residues $k+1$ and $m+1$. (So in fact $\kappa_1 = \overline{j+l-i}$, and $\kappa_2 = \overline{k+m+3-i}$). We therefore have $Z = \{\overline{i+1}, \dots, \overline{j}\} \cup \{\overline{k+1}, \dots, \overline{l}\} \cup \{\overline{m+1}, \dots, \overline{i-1}\}$.

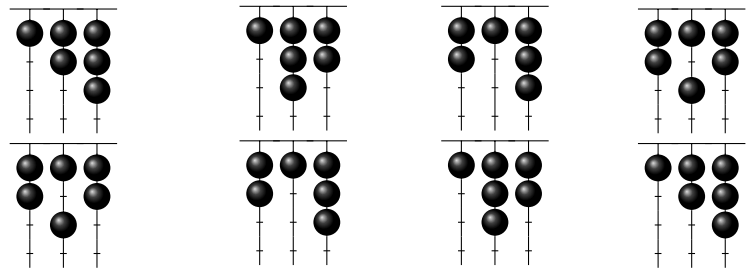
Example 5.1. Suppose $e = 14$, $\kappa = (\overline{10}, \overline{5})$ and $\xi = \left(7^2 \;\middle|\; 3^6\right)$, with the following abacus display.



We can take $(i, j, k, l, m) = (1, 3, 6, 8, 11)$, and $Z = \{\overline{0}, \overline{2}, \overline{3}, \overline{7}, \overline{8}, \overline{12}, \overline{13}\}$.

5.2. Exceptional bipartitions in B. The exceptional bipartitions in B are those with both addable i-nodes and addable $(i+1)$-nodes. By examining the abacus displays of all the bipartitions in B, we find that there are four exceptional bipartitions, which we label as follows, showing the configuration of runners $i-1, i, i+1$ of their abacus displays:

$$\alpha\beta = \mathrm{a}_i^{(2)}{\downarrow}^{i-1} \qquad \alpha\gamma = {\downarrow}^{i-1,i+1}{\uparrow}_i \qquad \gamma\alpha = {\downarrow}^{i} \qquad \beta\gamma = \mathrm{a}_i^{(1)}{\downarrow}^{i+1}$$



Observe that $\alpha\beta \blacktriangleright \alpha\gamma, \gamma\alpha \blacktriangleright \beta\gamma$ (and in fact $\alpha\gamma$ and $\gamma\alpha$ are incomparable even in the usual dominance order).

5.3. Proving Theorem 1.1 in B. Now we consider what conditions we need on a restricted bipartition μ in order to have $[S_\lambda : D_\mu] > 1$ for some λ.

Lemma 5.2. *Suppose μ is a restricted bipartition in B, and $[S_\lambda : D_\mu] > 1$ for some λ. Then $\mathrm{nor}_i(\mu) = \mathrm{nor}_{i+1}(\mu) = 1$, and at least three of the decomposition numbers $[S_{\alpha\beta} : D_\mu]$, $[S_{\alpha\gamma} : D_\mu]$, $[S_{\gamma\alpha} : D_\mu]$ and $[S_{\beta\gamma} : D_\mu]$ are positive. Hence $\mu^\diamond \blacktriangleright \alpha\gamma, \gamma\alpha \blacktriangleright \mu$.*

Proof. In the same way as in the proof of Lemma 4.2, we can apply the restriction functor e_i to show that μ has only one normal i-node and that if either of the decomposition numbers $[S_{\alpha\beta} : D_\mu]$ and $[S_{\gamma\alpha} : D_\mu]$ is greater than 1 then they are both positive, while if either of $[S_{\alpha\gamma} : D_\mu]$ and $[S_{\beta\gamma} : D_\mu]$ is greater than 1 then they are both positive. But in the same way we can use e_{i+1} to show that $\mathrm{nor}_{i+1}(\mu) = 1$ and either $[S_{\alpha\beta} : D_\mu]$ and $[S_{\alpha\gamma} : D_\mu]$ are both positive or $[S_{\gamma\alpha} : D_\mu]$ and $[S_{\beta\gamma} : D_\mu]$ are both positive. Combining these statements shows that at least three of the given decomposition numbers are positive. Now Proposition 2.14, together with the Jantzen–Schaper dominance order on the bipartitions $\alpha\beta, \alpha\gamma, \gamma\alpha, \beta\gamma$ gives the second statement. □

So to prove our main result for B we can focus on bipartitions μ in B for which:

(1) μ is restricted;
(2) $\alpha\gamma, \gamma\alpha \blacktriangleright \mu$;
(3) $\mu^\diamond \blacktriangleright \alpha\gamma, \gamma\alpha$; and
(4) $\mathrm{nor}_i(\mu) = \mathrm{nor}_{i+1}(\mu) = 1$, and $\mathrm{nor}_j(\mu) = 0$ for $j \neq i, i+1$.

We begin by finding the bipartitions μ for which conditions (2) and (4) are satisfied. A careful analysis of the possible abacus configurations shows that this happens in the cases given in the following table.

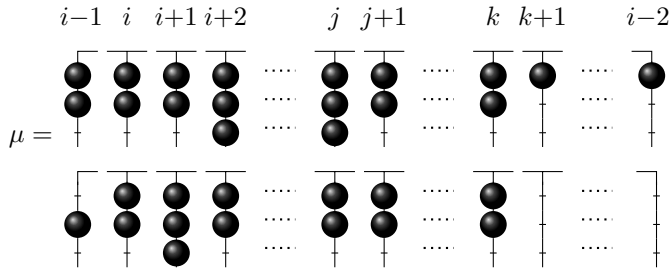
μ	conditions
$a_m^{(1)}\downarrow^{i+1}$	$l < m$
$a_{i-1}^{(2)}\downarrow^{i+1}$	–
$a_i^{(2)}\downarrow^{i+1}$	–
$a_l^{(2)}\downarrow^{i+1}$	$k < l = m$
$a_m^{(2)}\downarrow^{i+1}$	$l < m$
$\downarrow^{i+1,i+2}\uparrow_m$	$i+1 < j,\ l < m$
$\downarrow^{i+1,k+1}\uparrow_m$	$k < l < m$

(Note in particular that the bipartitions $\mu = a_{i+1}^{(2)}\downarrow^{i+1}$ and $a_k^{(2)}\downarrow^{i+1}$ (with $i+1 < k$) do not appear; these bipartitions satisfy (2) and (4) if $\blacktriangleright$ is replaced with the usual dominance order, but $\alpha\gamma \not\blacktriangleright \mu$.)

We next want to check which of these bipartitions are restricted. To do this we use (and extend) the technique that we used for the bipartition $a_{i-1}^{(2)}\downarrow^i$ in Section 4.3. Given a bipartition μ in the above table, we repeatedly remove good nodes until we reach a bipartition ξ which has weight 0 or 1, or has no normal nodes. If ξ has no normal nodes (but is not the empty bipartition) then ξ is not restricted, and so neither is μ. If ξ has weight 0 then it is restricted; if ξ has weight 1, then it is restricted if and only if it is not the most dominant bipartition in its block.

If we determine that μ is restricted via this technique, we can also use it to construct $\mu^\diamond$. Suppose we have reached the restricted bipartition ξ of weight 0 or 1 by removing good nodes of residues $r_1, \ldots, r_t$ in turn. The bipartition $\xi^\diamond$ can easily be identified: if ξ has weight 0 then $\xi^\diamond = \xi$, while if ξ has weight 1 then (from [8, Lemma 4.9]) $\xi^\diamond$ is the minimal bipartition in the same block as ξ for which $\xi^\diamond \blacktriangleright \xi$. Having identified $\xi^\diamond$, we add anticogood nodes of residues $r_t, \ldots, r_1$ in turn, and by Proposition 2.10 we obtain $\mu^\diamond$.

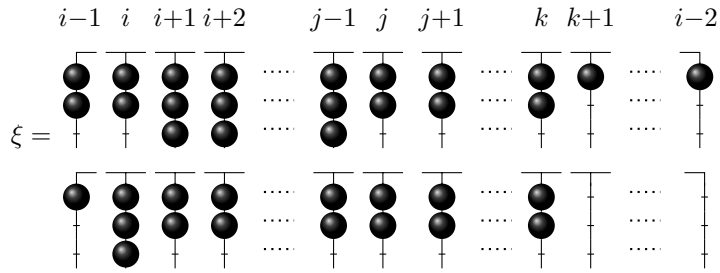
Example 5.3. Suppose $k = m$ and let $\mu = \mathrm{a}_{i-1}^{(2)}{\downarrow}^{i+1}$, with the following abacus display. (We modify the abacus display by adjusting the numbers of beads in order to make runner $i-1$ the leftmost runner).



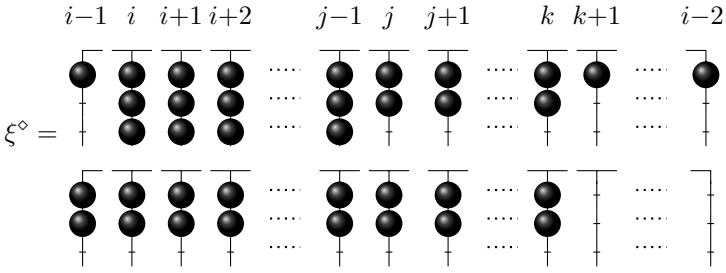
Now we remove good nodes of residues

$$i,\ i+1, i+1, i+2, i+2, \ldots, j, j,\ j+1, \ldots, k,\ i-1, i-2, \ldots, k+1$$

in turn, to obtain the bipartition ξ with the following abacus display.



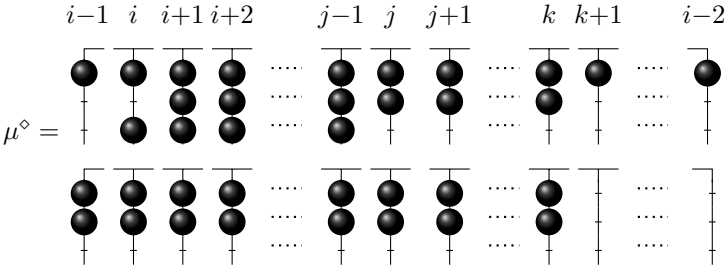
By calculating the values $\gamma_i(\xi)$ we find that ξ has weight 1. We now look at the other bipartitions in the same block as ξ, and we find that ξ is restricted, and that $\xi^\diamond$ is obtained by moving a bead from component 2 to component 1 on runner i, and from component 1 to component 2 on runner $i-1$. This implies in particular that μ is restricted.



Now adding anticogood nodes of residues

$$k+1, k+2, \ldots, i-1,\ k, k-1, \ldots, j+1,\ j, j, j-1, j-1, \ldots, i+1, i+1,\ i$$

in turn yields the bipartition $\mu^\diamond = \mathrm{a}_i^{(1)}{\downarrow}^{i-1}$.



Following this procedure for all the cases in the above table yields the following list of restricted bipartitions μ.

	μ	conditions	$\mu^\diamond$
1.	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i+1}$	$k<l<m$	$\mathrm{a}_{l+1}^{(2)}{\downarrow}^{i-1}$
2.	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i+1}$	$k<l=m$	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i-1}$
3.	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i+1}$	$k=l<m<e+i-2$	${\downarrow}^{i-1,i-2}{\uparrow}_{k+1}$
4.	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i+1}$	$k=l<m=e+i-2$	$\mathrm{a}_{k+1}^{(1)}{\downarrow}^{i-1}$
5.	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i+1}$	$k=l=m$	$\mathrm{a}_{i}^{(1)}{\downarrow}^{i-1}$
6.	$\mathrm{a}_{i}^{(2)}{\downarrow}^{i+1}$	$j<k<l-1$	${\downarrow}^{l-1,l}{\uparrow}_{j+1}$
7.	$\mathrm{a}_{i}^{(2)}{\downarrow}^{i+1}$	$j<k=l-1$	$\mathrm{a}_{j+1}^{(1)}{\downarrow}^{k+1}$
8.	$\mathrm{a}_{i}^{(2)}{\downarrow}^{i+1}$	$j<k=l$	$\mathrm{a}_{j+1}^{(1)}{\downarrow}^{i-1}$
9.	$\mathrm{a}_{i}^{(2)}{\downarrow}^{i+1}$	$j=k<l$	$\mathrm{a}_{i+1}^{(1)}{\downarrow}^{l}$
10.	$\mathrm{a}_{i}^{(2)}{\downarrow}^{i+1}$	$j=k=l$	$\mathrm{a}_{i+1}^{(1)}{\downarrow}^{i-1}$
11.	$\mathrm{a}_{l}^{(2)}{\downarrow}^{i+1}$	$k<l-1,\ l=m$	${\downarrow}^{l-1,l}{\uparrow}_{i}$
12.	$\mathrm{a}_{l}^{(2)}{\downarrow}^{i+1}$	$k=l-1,\ l=m$	$\mathrm{a}_{i}^{(1)}{\downarrow}^{k+1}$
13.	$\mathrm{a}_{m}^{(2)}{\downarrow}^{i+1}$	$k<l-1,\ l<m$	${\downarrow}^{l-1,l}{\uparrow}_{i}$
14.	$\mathrm{a}_{m}^{(2)}{\downarrow}^{i+1}$	$k=l-1,\ l<m$	$\mathrm{a}_{i}^{(1)}{\downarrow}^{k+1}$
15.	$\mathrm{a}_{m}^{(2)}{\downarrow}^{i+1}$	$k=l<m$	$\mathrm{a}_{i}^{(1)}{\downarrow}^{i-1}$
16.	${\downarrow}^{i+1,i+2}{\uparrow}_{m}$	$i+1<j,\ k<l<m$	$\mathrm{a}_{i}^{(2)}{\downarrow}^{l}$
17.	${\downarrow}^{i+1,i+2}{\uparrow}_{m}$	$i+1<j,\ k=l<m$	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i-1}$
18.	${\downarrow}^{i+1,k+1}{\uparrow}_{m}$	$k<l<m$	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i-1}$

Our next task is to check which of these cases satisfy $\mu^\diamond \blacktriangleright \alpha\gamma, \gamma\alpha$. We find that this happens in cases 5, 8, 10 and 15. (Note that in case 4 it is critical that we use the Jantzen–Schaper order: in this case $\mu^\diamond \trianglerighteq \alpha\gamma$, but $\mu^\diamond \not\blacktriangleright \alpha\gamma$.) Now we consider the remaining four cases.

5.4. Cases 8 and 10. Assume $k=l$ and let $\mu = \mathrm{a}_i^{(2)}{\downarrow}^{i+1}$. In this case we will show that $[\mathrm{S}_{\alpha\gamma} : \mathrm{D}_\mu] = [\mathrm{S}_{\gamma\alpha} : \mathrm{D}_\mu] = 0$, so that $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$ for all λ by Lemma 5.2. To do this, we use the Jantzen–Schaper formula. First we find all bipartitions in the set

$$D = \{\lambda \mid \lambda \blacktriangleright \mu \text{ and either } \alpha\gamma \blacktriangleright \lambda \text{ or } \gamma\alpha \blacktriangleright \lambda\}.$$

By examining abacus displays, we find that

$$D = \left\{ \mathrm{a}_x^{(2)}{\downarrow}^{i+1} \;\middle|\; i \leqslant x \leqslant k \right\} \cup \{{\downarrow}^x \mid j+1 \leqslant x \leqslant k\}$$
$$\cup \left\{ \mathrm{a}_x^{(1)}{\downarrow}^{x} \;\middle|\; i+1 \leqslant x \leqslant j \right\} \cup \{\beta\gamma, \alpha\gamma, \gamma\alpha\}.$$

The Jantzen–Schaper order on these bipartitions is given in Figure 1. Now we compute the decomposition numbers $[\mathrm{S}_\lambda : \mathrm{D}_\mu]$ for $\lambda \in D$ using the Jantzen–Schaper formula. In Table 3 we collate the needed Jantzen–Schaper coefficients $j_{\lambda\nu}$, and compute the bounds $J_{\lambda\mu}$, which give the decomposition numbers. (Note that for $\lambda = \mathrm{a}_{i+1}^{(1)}{\downarrow}^{i+1}$, the bound $J_{\lambda\mu}$ equals 2, but we use the fact that (by the inductive assumption, and because λ is not exceptional) $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$, and therefore $[\mathrm{S}_\lambda : \mathrm{D}_\mu] = 1$.)

We conclude that if $\lambda \in D$, then

$$[\mathrm{S}_\lambda : \mathrm{D}_\mu] = \begin{cases} 1 & \text{if } \lambda \in \{\mu, \beta\gamma, \mathrm{a}_{i+2}^{(2)}{\downarrow}^{i+1}, \mathrm{a}_{i+2}^{(1)}{\downarrow}^{i+2}, \mathrm{a}_{i+1}^{(1)}{\downarrow}^{i+1}\} \\ 0 & \text{otherwise.} \end{cases}$$

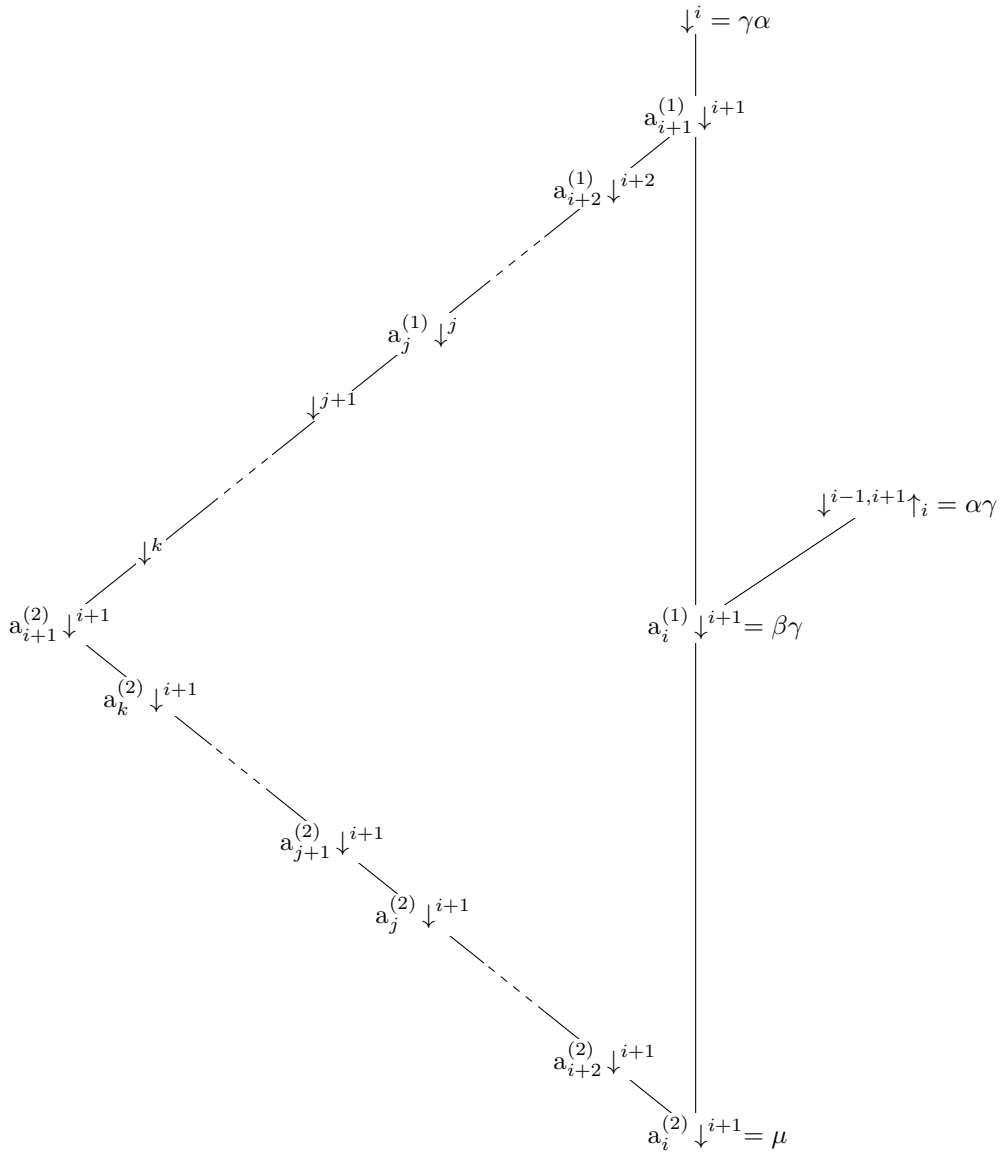


FIGURE 1. Jantzen–Schaper interval for Cases 8 and 10

Now the fact that $[\mathrm{S}_{\alpha\gamma} : \mathrm{D}_\mu] = [\mathrm{S}_{\gamma\alpha} : \mathrm{D}_\mu] = 0$, combined with Lemma 5.2, shows that $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$ for all λ.

5.5. CASE 5. Suppose $k = l = m$, and let $\mu = \mathrm{a}^{(2)}_{i-1}\downarrow^{i+1}$, so that $\mu^\diamond = \mathrm{a}^{(1)}_{i}\downarrow^{i-1}$. In this case we use a conjugation argument. If we let B' be the block containing the bipartition $(\mu^\diamond)'$, then B' is also a block of type III, with nucleus ξ', and can be treated in the same way as the block B, with the integers i, j, k, l, m replaced by

λ	ν μ	$\beta\gamma$	$\mathrm{a}^{(2)}_{i+2}{\downarrow}^{i+1}$	$\mathrm{a}^{(1)}_{i+2}{\downarrow}^{i+2}$	$\mathrm{a}^{(1)}_{i+1}{\downarrow}^{i+1}$	$J_{\lambda\mu}$	$[\mathrm{S}_\lambda : \mathrm{D}_\mu]$
$\beta\gamma$	1					1	1
$\alpha\gamma$	-1	1				0	0
$\mathrm{a}^{(2)}_{i+2}{\downarrow}^{i+1}$	1					1	1
$\mathrm{a}^{(2)}_{x}{\downarrow}^{i+1}$ $(i+3 \leqslant x \leqslant k)$	$(-1)^{x-i}$		$(-1)^{x-i+1}$			0	0
$\mathrm{a}^{(2)}_{i+1}{\downarrow}^{i+1}$	$(-1)^{k-i+1}$		$(-1)^{k-i}$			0	0
${\downarrow}^{x}$ $(k \geqslant x \geqslant j+1)$	0		0			0	0
$\mathrm{a}^{(1)}_{x}{\downarrow}^{x}$ $(j \geqslant x \geqslant i+3)$	0		0			0	0
$\mathrm{a}^{(1)}_{i+2}{\downarrow}^{i+2}$	0		1			1	1
$\mathrm{a}^{(1)}_{i+1}{\downarrow}^{i+1}$	0	1	0	1		2	1
$\gamma\alpha$	-1	1	0	-1	1	0	0

TABLE 3. Jantzen–Schaper calculations for cases 8 and 10

i', j', k', l', m', where

$$\begin{aligned} i' &= k_1 + k_2 - i - 1, \\ j' &= k_1 + k_2 + e - m, \\ k' &= k_1 + k_2 + e - l, \\ l' &= k_1 + k_2 + e - k, \\ m' &= k_1 + k_2 + e - j. \end{aligned}$$

In the block B', $(\mu^\diamond)'$ is the bipartition $\mathrm{a}^{(2)}_{i'}{\downarrow}^{i'+1}$, with $j' = k' = l'$. So using case 10 dealt with above, we can deduce that $[\mathrm{S}_\lambda : \mathrm{D}_{(\mu^\diamond)'}] \leqslant 1$ for all λ, and therefore (by Proposition 2.8) $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$ for all λ.

5.6. CASE 15. In this case we use a conjugation argument, as we did for case 5 above. In this case $(\mu^\diamond)'$ satisfies the conditions of case 8 above, therefore we can deduce that $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$ for all λ in the same way.

This deals with all cases, and so our inductive step for type III blocks is complete.

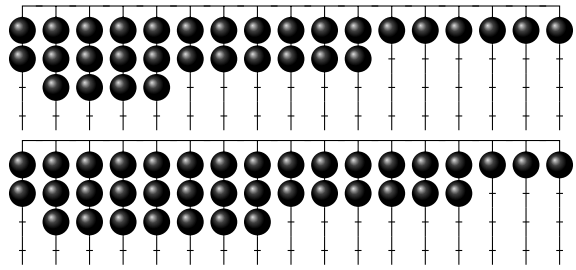
6. BLOCKS OF TYPE IV

The last situation we have to investigate is when B is a block of $\mathcal{H}_n$ and $i \in \mathbb{Z}$ is such that $\delta_i(B) = 2$ and $\delta_j(B) \leqslant 0$ for every $j \not\equiv i \pmod e$. Again we will follow an inductive approach to show that Theorem 1.1 holds for B.

6.1. THE ABACUS FOR A BLOCK OF TYPE IV. Take ξ the nucleus of B and consider its (k_1+1, k_2-1)-abacus display following the same convention of Sections 4 and 5. Then $\gamma_i(\xi) = 2$ and $\gamma_j(\xi) \leqslant 0$ for every $j \not\equiv i \pmod e$ and we note that ξ has exactly two removable i-nodes (again because ξ has weight 0 and by Lemma 2.6 it cannot have both removable and addable nodes of the same residue). We now observe that these two removable nodes occur in different components of ξ. Assuming the contrary we would have that $\gamma_i(\xi) - \gamma_{i-1}(\xi) = 2$ but ξ has weight 0 and $\gamma_x(\xi) - \gamma_y(\xi) \leqslant 1$ for every $x, y \in \mathbb{Z}/e\mathbb{Z}$.

Similarly to the previous sections we draw the (k_1+1, k_2-1)-abacus display of ξ finding integers $i \leqslant j \leqslant k \leqslant l \leqslant m \leqslant e+i-2$ such that ξ has a removable i-node in both components and addable nodes of residue $j+1$ and $l+1$ in component 1 and of residue $k+1$ and $m+1$ in component 2 (in particular $\kappa_1 = \overline{j+l-i+1}$, and $\kappa_2 = \overline{k+m+3-i}$). Now $Z = \{\bar{i}, \dots, \bar{j}\} \cup \{\overline{k+1}, \dots, \bar{l}\} \cup \{\overline{m+1}, \dots, \overline{i-1}\}$.

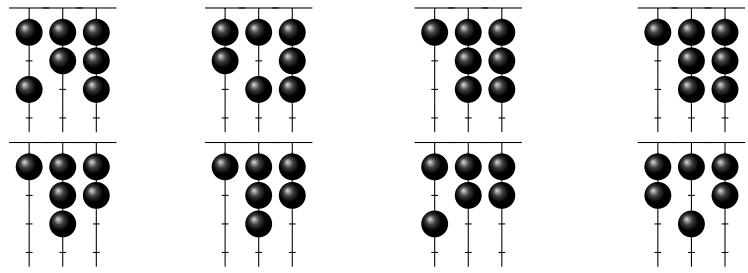
Example 6.1. Suppose $e = 17$, $\kappa = (\overline{14}, \bar{5})$ and $\xi = \left(7^4 \;\middle|\; 4^7\right)$, with the following abacus display.



We can take $(i, j, k, l, m) = (1, 4, 7, 10, 13)$, so that $Z = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{8}, \bar{9}, \overline{10}, \overline{14}, \overline{15}, \overline{16}\}$.

6.2. Exceptional bipartitions in B. There are four exceptional bipartitions in B, which we label $\beta_1, \beta_2, \beta_3, \beta_4$, with abacus displays (for runners $i-1, i, i+1$) as follows.

$\beta_1 = \mathrm{a}_{i-1}^{(1)}\downarrow^i$ $\quad$ $\beta_2 = \mathrm{a}_{i}^{(1)}\downarrow^i$ $\quad$ $\beta_3 = \mathrm{a}_{i-1}^{(2)}\downarrow^{i-1}$ $\quad$ $\beta_4 = \mathrm{a}_{i}^{(2)}\downarrow^{i-1}$



In contrast to Case III, these bipartitions are totally ordered by the Jantzen–Schaper dominance order; in particular $\beta_4 \blacktriangleright \beta_3 \blacktriangleright \beta_2 \blacktriangleright \beta_1$.

6.3. Proving Theorem 1.1 in B. Also in this situation it is useful to establish a lemma that brings out a useful property of those restricted bipartitions μ such that $[\mathrm{S}_\lambda : \mathrm{D}_\mu] > 1$ for some λ. By Lemma 3.3(2) we note that such bipartitions have exactly two normal i-nodes.

Lemma 6.2. *Suppose μ is a restricted bipartition in B, and $[\mathrm{S}_\lambda : \mathrm{D}_\mu] > 1$ for some λ. Then the decomposition numbers $[\mathrm{S}_{\beta_1} : \mathrm{D}_\mu], [\mathrm{S}_{\beta_2} : \mathrm{D}_\mu], [\mathrm{S}_{\beta_3} : \mathrm{D}_\mu], [\mathrm{S}_{\beta_4} : \mathrm{D}_\mu]$ are all positive; in particular $\mu^\diamond \trianglerighteq \beta_4 \blacktriangleright \beta_3 \blacktriangleright \beta_2 \blacktriangleright \beta_1 \trianglerighteq \mu$.*

Proof. Let λ be a bipartition such that $[\mathrm{S}_\lambda : \mathrm{D}_\mu] > 1$. Then $\lambda = \beta_x$ for some $x \in \{1, 2, 3, 4\}$. We follow the same idea as in Lemma 4.2 and Lemma 5.2 with the only difference that here we need to apply the restriction functor e_i to $\mathrm{S}_{\beta_1}, \dots, \mathrm{S}_{\beta_4}$ 'twice', namely we apply the functor e_i^2 described in Section 2.8. By the branching rules we

find that there are bipartitions $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4, \tilde{\mu}$ of $n-2$ such that

$$\begin{aligned}
\mathrm{e}_i^2\,\mathrm{S}_{\beta_1} &\sim (\mathrm{S}_{\hat{\beta}_1})^2 + (\mathrm{S}_{\hat{\beta}_2})^2 + (\mathrm{S}_{\hat{\beta}_3})^2,\\
\mathrm{e}_i^2\,\mathrm{S}_{\beta_2} &\sim (\mathrm{S}_{\hat{\beta}_1})^2 + (\mathrm{S}_{\hat{\beta}_2})^2 + (\mathrm{S}_{\hat{\beta}_4})^2,\\
\mathrm{e}_i^2\,\mathrm{S}_{\beta_3} &\sim (\mathrm{S}_{\hat{\beta}_1})^2 + (\mathrm{S}_{\hat{\beta}_3})^2 + (\mathrm{S}_{\hat{\beta}_4})^2,\\
\mathrm{e}_i^2\,\mathrm{S}_{\beta_4} &\sim (\mathrm{S}_{\hat{\beta}_2})^2 + (\mathrm{S}_{\hat{\beta}_3})^2 + (\mathrm{S}_{\hat{\beta}_4})^2,\\
\mathrm{e}_i^2\,\mathrm{D}_{\mu} &\sim \mathrm{D}_{\tilde{\mu}}\,.
\end{aligned}$$

Analogously as in [10, Proposition 2.10(4)], we find

$$\begin{aligned}
[\mathrm{S}_{\beta_1} : \mathrm{D}_\mu] + [\mathrm{S}_{\hat{\beta}_4} : \mathrm{D}_{\tilde{\mu}}] &= [\mathrm{S}_{\beta_2} : \mathrm{D}_\mu] + [\mathrm{S}_{\hat{\beta}_3} : \mathrm{D}_{\tilde{\mu}}]\\
&= [\mathrm{S}_{\beta_3} : \mathrm{D}_\mu] + [\mathrm{S}_{\hat{\beta}_2} : \mathrm{D}_{\tilde{\mu}}]\\
&= [\mathrm{S}_{\hat{\beta}_4} : \mathrm{D}_\mu] + [\mathrm{S}_{\hat{\beta}_1} : \mathrm{D}_{\tilde{\mu}}],
\end{aligned}$$

and we conclude by induction that if one of the decomposition numbers $[\mathrm{S}_{\beta_x} : \mathrm{D}_\mu]$ is greater than 1, then they are all positive. The last sentence follows from Proposition 2.14. □

Summing up, it remains to consider those bipartitions μ in B such that

(1) μ is restricted;
(2) $\beta_1 \trianglerighteq \mu$;
(3) $\mu^\diamond \trianglerighteq \beta_4$; and
(4) $\mathrm{nor}_i(\mu) = 2$, and $\mathrm{nor}_j(\mu) = 0$ for $j \neq i$.

Firstly we list all possible bipartitions μ which satisfy (2) and (4) and in which both components are e-restricted (which is a necessary condition for μ to be restricted, by Lemma 2.2).

μ	conditions
$\mathrm{a}_m^{(1)}\downarrow^i$	$l < m$
$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	–
$\mathrm{a}_m^{(2)}\downarrow^i$	$l < m$
$\downarrow^{i,i+1}\uparrow_m$	$i < j,\ l < m$
$\downarrow^{i,k+1}\uparrow_m$	$k < l < m$

Now we find which bipartitions in this list satisfy condition (1), i.e. we find the restricted bipartitions. To do this we use the same technique as in the previous sections. In each case where μ is restricted we find $\mu^\diamond$ using the procedure explained in Section 5.3.

	μ	conditions	$\mu^\diamond$
1.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$k < l < m$	$\mathrm{a}_{l+1}^{(2)}\downarrow^{i-1}$
2.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$k < l = m$	$\mathrm{a}_{i-1}^{(2)}\downarrow^{i-1}$
3.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$k = l < m < e+i-2$	$\downarrow^{i-1,i-2}\uparrow_{k+1}$
4.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$k = l < m = e+i-2$	$\mathrm{a}_{k+1}^{(1)}\downarrow^{i-1}$
5.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$j < k = l = m$	$\mathrm{a}_{j+1}^{(1)}\downarrow^{i-1}$
6.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$j = k = l = m$	$\mathrm{a}_{i}^{(1)}\downarrow^{i-1}$

7.	$\mathrm{a}_m^{(2)}\downarrow^i$	$j<k<l-1,\ l<m$	$\downarrow^{l-1,l}\uparrow_{j+1}$
8.	$\mathrm{a}_m^{(2)}\downarrow^i$	$j<k=l-1,\ l<m$	$\mathrm{a}_{j+1}^{(1)}\downarrow^{k+1}$
9.	$\mathrm{a}_m^{(2)}\downarrow^i$	$j<k=l<m$	$\mathrm{a}_{j+1}^{(1)}\downarrow^{i-1}$
10.	$\mathrm{a}_m^{(2)}\downarrow^i$	$j=k<l<m$	$\mathrm{a}_i^{(1)}\downarrow^l$
11.	$\mathrm{a}_m^{(2)}\downarrow^i$	$j=k=l<m$	$\mathrm{a}_i^{(1)}\downarrow^{i-1}$
12.	$\downarrow^{i,i+1}\uparrow_m$	$i<j,\ k<l<m$	$\mathrm{a}_i^{(2)}\downarrow^l$
13.	$\downarrow^{i,i+1}\uparrow_m$	$i<j,\ k=l<m$	$\mathrm{a}_{i-1}^{(2)}\downarrow^{i-1}$
14.	$\downarrow^{i,k+1}\uparrow_m$	$k<l<m$	$\mathrm{a}_{i-1}^{(2)}\downarrow^{i-1}$

Finally we cut our list with condition (3). It turns out that there are four cases left: 5, 6, 9, and 11. (Note that case 4 is the only one where we need to use the Jantzen–Schaper dominance order in condition (3) instead of the usual dominance order).

6.4. Cases 5 and 11. To address cases 5 and 11, we use a conjugation argument similar to that used for cases 5 and 15 in Section 5. If μ satisfies the conditions of case 5, then $\mu^\diamond = \mathrm{a}_{j+1}^{(1)}\downarrow^{i-1}$. So $(\mu^\diamond)'$ satisfies the conditions of case 11 for the block with nucleus ξ', and conversely. So, by Proposition 2.8, showing that $[\mathrm{S}_\lambda : \mathrm{D}_\mu] \leqslant 1$ for all λ in case 5 is equivalent to doing the same in case 11.

In fact, we will need to consider both cases simultaneously: we will obtain a partial result in case 11, and use this in our analysis of case 5 to obtain a complete result. So we assume first that μ satisfies the conditions of case 11. Our objective here will be to prove the following.

Lemma 6.3. *Suppose μ satisfies the conditions of case* 11. *Then* $[\mathrm{S}_{\beta_1} : \mathrm{D}_\mu] = 1$, *while* $[\mathrm{S}_{\beta_2} : \mathrm{D}_\mu] \geqslant [\mathrm{S}_{\beta_3} : \mathrm{D}_\mu] \geqslant 1$.

To prove Lemma 6.3, we compute decomposition numbers using the Jantzen–Schaper formula. For this, we need to consider the set of bipartitions

$$D = \{\lambda \mid \beta_3 \trianglerighteq \lambda \trianglerighteq \mu\}.$$

By considering possible abacus displays, we find that

$$D = \left\{\mathrm{a}_x^{(2)}\downarrow^i \;\middle|\; i \leqslant x \leqslant j\right\} \cup \left\{\mathrm{a}_x^{(1)}\downarrow^x \;\middle|\; i+1 \leqslant x \leqslant j\right\} \cup \left\{\downarrow^{i,x}\uparrow_m \;\middle|\; m+1 \leqslant x \leqslant e+i-2\right\}$$
$$\cup \left\{\mathrm{a}_x^{(1)}\downarrow^i \;\middle|\; m \leqslant x \leqslant e+i-2\right\} \cup \left\{\mu, \mathrm{a}_m^{(2)}\downarrow^{i-1}, \downarrow^{i,i-1}\uparrow_m, \beta_1, \beta_2, \beta_3\right\},$$

and that the Jantzen–Schaper order on these bipartitions is given in Figure 2. Now, as in the cases in Section 5 we collect in a table the integers $j_{\lambda\nu}$ and $J_{\lambda\mu}$ needed to calculate decomposition numbers $[\mathrm{S}_\lambda : \mathrm{D}_\mu]$. We obtain the results in Table 6, from which Lemma 6.3 follows.

Having proved Lemma 6.3, we now analyse case 5 fully. Assume $j < k = l = m$ and $\mu = \mathrm{a}_{i-1}^{(2)}\downarrow^i$. First we use Proposition 2.8 to derive some consequences of Lemma 6.3: since conjugation reverses the (Jantzen–Schaper) dominance order on bipartitions, the conjugates of the bipartitions $\beta_1, \beta_2, \beta_3, \beta_4$ from case 11 are the bipartitions $\beta_4, \beta_3, \beta_2, \beta_1$ in case 5. So Proposition 2.8 and lemma 6.3 imply that in case 5

$$[\mathrm{S}_{\beta_4} : \mathrm{D}_\mu] = 1 \qquad \text{and} \qquad 1 \leqslant [\mathrm{S}_{\beta_2} : \mathrm{D}_\mu] \leqslant [\mathrm{S}_{\beta_3} : \mathrm{D}_\mu].$$

We define τ be the bipartition $\mathrm{a}_{i+1}^{(2)}\downarrow^{i-1}$, and we want to consider the decomposition numbers $[\mathrm{S}_\lambda : \mathrm{D}_\mu]$ for bipartitions λ in the set

$$D = \{\lambda \mid \tau \trianglerighteq \lambda \trianglerighteq \mu\}.$$

We can compute

$$D = \left\{ \mathrm{a}_x^{(2)}\downarrow^i \;\middle|\; i-1 \leqslant x \leqslant k \right\} \cup \{\downarrow^x \mid j+1 \leqslant x \leqslant k\} \cup \left\{ \mathrm{a}_x^{(1)}\downarrow^x \;\middle|\; i+1 \leqslant x \leqslant j \right\} \cup \{\beta_1, \beta_2, \beta_3, \beta_4, \tau\}$$

and the Jantzen–Schaper order on the bipartitions in D is given in Figure 3. Now we apply the Jantzen–Schaper formula to compute decomposition numbers $[\mathrm{S}_\lambda : \mathrm{D}_\mu]$ for $\beta_2 \blacktriangleright \lambda \not\trianglerighteq \mu$. These are given in Table 7.

We also use the Jantzen–Schaper formula to estimate $[\mathrm{S}_\tau : \mathrm{D}_\mu]$: we obtain

$$J_{\tau\mu} = -[\mathrm{S}_{\mathrm{a}_{i+1}^{(2)}\downarrow^i} : \mathrm{D}_\mu] + [\mathrm{S}_{\mathrm{a}_{i+1}^{(1)}\downarrow^{i+1}} : \mathrm{D}_\mu] - [\mathrm{S}_{\beta_3} : \mathrm{D}_\mu] + [\mathrm{S}_{\beta_4} : \mathrm{D}_\mu] = 1 - [\mathrm{S}_{\beta_3} : \mathrm{D}_\mu],$$

forcing $[\mathrm{S}_{\beta_3} : \mathrm{D}_\mu] = 1$. Therefore we have $[\mathrm{S}_{\beta_x} : \mathrm{D}_\mu] = 1$ for $x = 1, 2, 3, 4$.

6.5. Case 9. The analysis of case 9 is very similar to the cases in Section 6.4. We begin by proving an exact analogue of Lemma 6.3 for case 9. The calculation is very

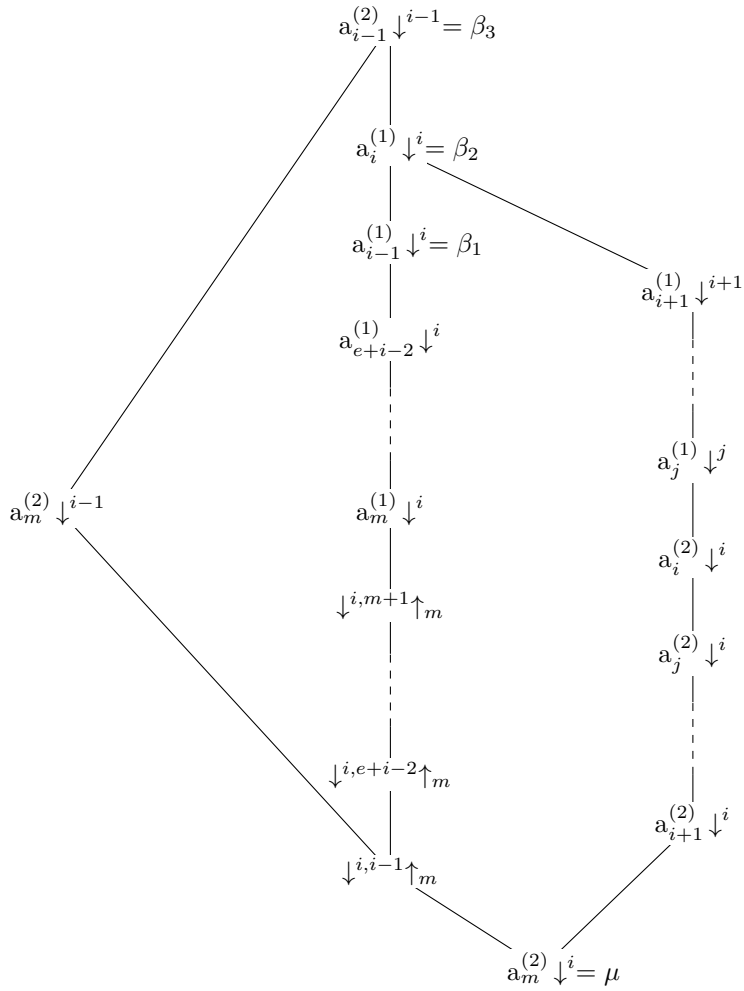


Figure 2. Jantzen–Schaper interval for case 11

similar to the calculation used to prove Lemma 6.3: if $\beta_3 \blacktriangleright \lambda \blacktriangleright \mu$, then we find that

$$J_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda \in \{\mathrm{a}^{(2)}_{i+1}{\downarrow}^i,\ \mathrm{a}^{(1)}_{i+1}{\downarrow}^{i+1},\ {\downarrow}^{i,i-1}{\uparrow}_m,\ \beta_1\}, \\ 2 & \text{if } \lambda = \beta_2, \\ [\mathrm{S}_{\beta_2} : \mathrm{D}_\mu] & \text{if } \lambda = \beta_3, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that if $i = j$ then a slight modification is necessary: the bipartition ${\downarrow}^{i+1}$ replaces $\mathrm{a}^{(1)}_{i+1}{\downarrow}^{i+1}$ in the above formula.)

Having established the analogue of Lemma 6.3, we apply duality: case 9 is self-dual, so we deduce that

$$[\mathrm{S}_{\beta_1} : \mathrm{D}_\mu] = 1, \qquad [\mathrm{S}_{\beta_2} : \mathrm{D}_\mu] = [\mathrm{S}_{\beta_3} : \mathrm{D}_\mu] \geqslant 1, \qquad [\mathrm{S}_{\beta_4} : \mathrm{D}_\mu] = 1$$

whenever μ satisfies the conditions of case 9. We complete case 9 in the same way as case 5 above, letting τ be the bipartition $\mathrm{a}^{(2)}_{i+1}{\downarrow}^{i-1}$. As in case 5, we find that $J_{\tau\mu} = 1 - [\mathrm{S}_{\beta_3} : \mathrm{D}_\mu]$, forcing $[\mathrm{S}_{\beta_3} : \mathrm{D}_\mu] = 1$.

6.6. Case 6. This is the last case we have to deal with. Assume that $j = k = l = m$ and $\mu = \mathrm{a}^{(2)}_{i-1}{\downarrow}^i$. In this case we also begin by proving a direct analogue of Lemma 6.3. This is proved as in the previous cases: the bipartitions λ for which $\beta_3 \blacktriangleright \lambda \blacktriangleright \mu$ are shown in Figure 4 (ignoring the red portion of the diagram), and calculating Jantzen–Schaper coefficients as in previous cases we find that if $\beta_3 \blacktriangleright \lambda \blacktriangleright \mu$, then

$$J_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda \in \{\mathrm{a}^{(2)}_{i+1}{\downarrow}^i,\ \mathrm{a}^{(1)}_{i+1}{\downarrow}^{i+1}, \beta_1\}, \\ 2 & \text{if } \lambda = \beta_2, \\ [\mathrm{S}_{\beta_2} : \mathrm{D}_\mu] & \text{if } \lambda = \beta_3, \\ 0 & \text{otherwise.} \end{cases}$$

λ	μ	ν $\mathrm{a}^{(2)}_{i+1}{\downarrow}^i$	$\mathrm{a}^{(1)}_{i+1}{\downarrow}^{i+1}$	${\downarrow}^{i,i-1}{\uparrow}_m$	β_1	β_2	$J_{\lambda\mu}$	$[\mathrm{S}_\lambda : \mathrm{D}_\mu]$
$\mathrm{a}^{(2)}_{i+1}{\downarrow}^i$	1						1	1
$\mathrm{a}^{(2)}_x{\downarrow}^i$ $(i+2 \leqslant x \leqslant j)$	$(-1)^{x-i+1}$	$(-1)^{x-i}$					0	0
$\mathrm{a}^{(2)}_i{\downarrow}^i$	$(-1)^{j-i}$	$(-1)^{j-i+1}$					0	0
$\mathrm{a}^{(1)}_x{\downarrow}^x$ $(j \geqslant x \geqslant i+2)$	0	0					0	0
$\mathrm{a}^{(1)}_{i+1}{\downarrow}^{i+1}$	0	1					1	1
${\downarrow}^{i,i-1}{\uparrow}_m$	1						1	1
$\mathrm{a}^{(2)}_m{\downarrow}^{i-1}$	-1		1				0	0
${\downarrow}^{i,x}{\uparrow}_m$ $(e+i-2 \geqslant x \geqslant m+1)$	$(-1)^{e+x-i+1}$		$(-1)^{e+x-i}$				0	0
$\mathrm{a}^{(1)}_m{\downarrow}^i$	$(-1)^{e+m-i+1}$		$(-1)^{e+m-i}$				0	0
$\mathrm{a}^{(1)}_x{\downarrow}^i$ $(m+1 \leqslant x \leqslant e+i-2)$	0		0				0	0
β_1	0			1			1	1
β_2	0	0	1	0	1		2	
β_3	0	0	-1	0	1	1	$[\mathrm{S}_{\beta_2} : \mathrm{D}_\mu]$	

Table 6. Jantzen–Schaper calculations for case 11

So an analogue of Lemma 6.3 follows. As with case 9, this case is self-dual, so we deduce that

$$[\mathrm{S}_{\beta_1} : \mathrm{D}_\mu] = 1, \qquad [\mathrm{S}_{\beta_2} : \mathrm{D}_\mu] = [\mathrm{S}_{\beta_3} : \mathrm{D}_\mu] \geqslant 1, \qquad [\mathrm{S}_{\beta_4} : \mathrm{D}_\mu] = 1$$

whenever μ satisfies the conditions of case 6.

Now assume that $j < e + i - 2$. With this assumption we complete case 6 in the same way as in the preceding cases. Let τ be the bipartition $\mathrm{a}_i^{(2)}{\downarrow^{e+i-2}}$. Figure 4 (now including the red part) shows the Jantzen–Schaper order on bipartitions λ for which $\tau \blacktriangleright \lambda \blacktriangleright \mu$.

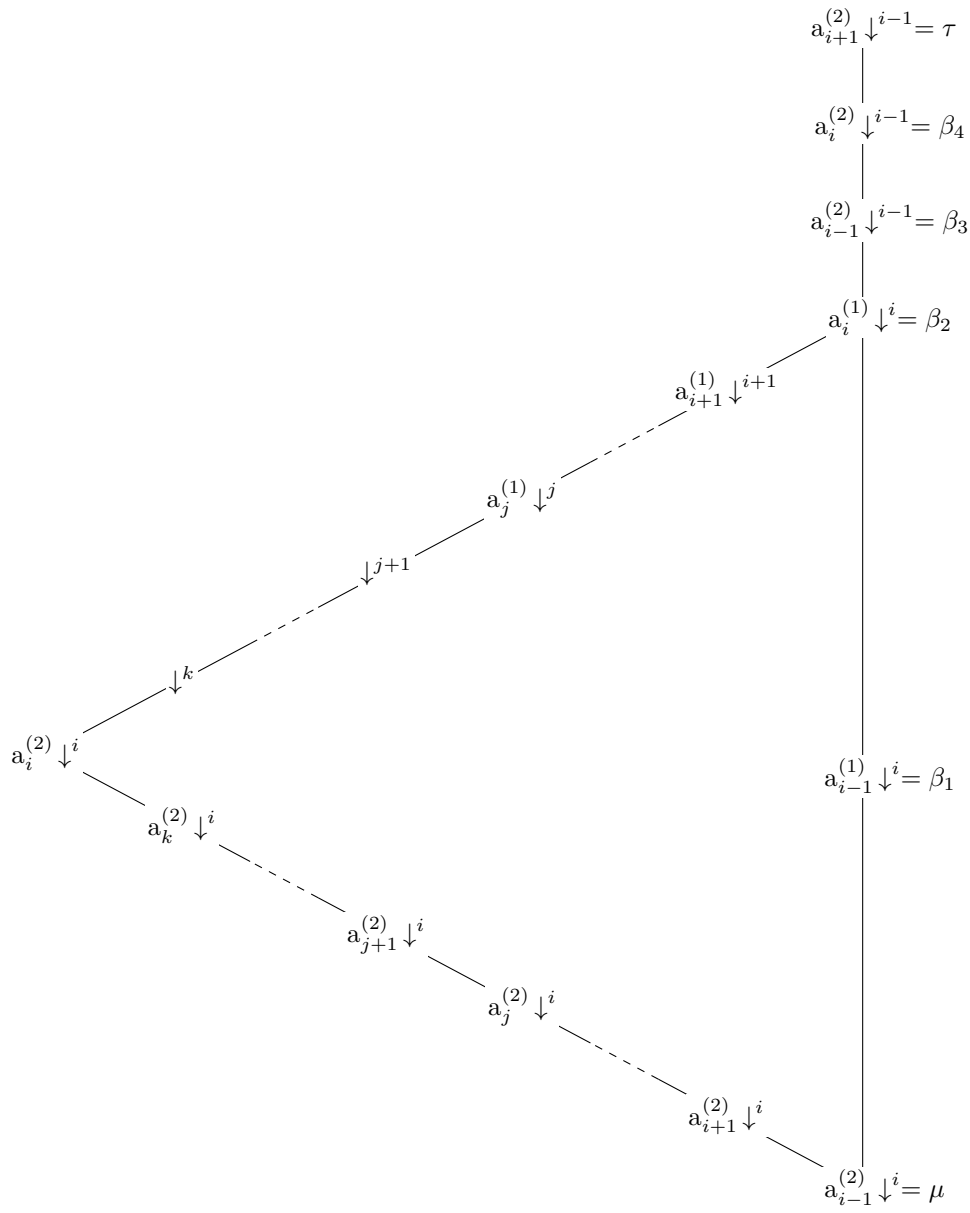


FIGURE 3. Jantzen–Schaper interval for case 5

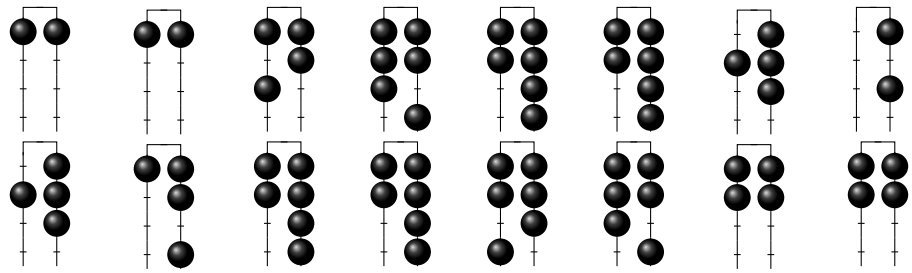
λ	ν / μ	$\mathrm{a}_{i+1}^{(2)}{\downarrow}^{i}$	$J_{\lambda\mu}$	$[\mathrm{S}_\lambda : \mathrm{D}_\mu]$
$\mathrm{a}_{i+1}^{(2)}{\downarrow}^{i}$	1		1	1
$\mathrm{a}_{x}^{(2)}{\downarrow}^{i}$ $(i+2 \leqslant x \leqslant k)$	$(-1)^{x-i+1}$	$(-1)^{x-i}$	0	0
$\mathrm{a}_{i}^{(2)}{\downarrow}^{i}$	$(-1)^{k-i}$	$(-1)^{k-i+1}$	0	0
${\downarrow}^{x}$ $(k \geqslant x \geqslant j+1)$	0	0	0	0
$\mathrm{a}_{x}^{(1)}{\downarrow}^{x}$ $(j \geqslant x \geqslant i+2)$	0	0	0	0
$\mathrm{a}_{i+1}^{(1)}{\downarrow}^{i+1}$	0	1	1	1
β_1	1		1	1

TABLE 7. Jantzen–Schaper calculations for case 5

As in the preceding cases, we find that $J_{\tau\mu} = 1 - [\mathrm{S}_{\beta_3} : \mathrm{D}_\mu]$, forcing $[\mathrm{S}_{\beta_3} : \mathrm{D}_\mu] = [\mathrm{S}_{\beta_2} : \mathrm{D}_\mu] = 1$, so that we are done when $j < e + i - 2$. By duality, case 6 is also complete when $i < j$.

Then the only situation left to be checked is when $i = j = k = l = m = e+i-2$. In this case $e = 2$ and B is the block of $\mathcal{H}_5$ which contains eight Specht modules labelled by the following bipartitions:

μ	$\mathrm{a}_{i}^{(2)}{\downarrow}^{i}$	β_1	β_2	β_3	β_4	$\mathrm{a}_{i-1}^{(1)}{\downarrow}^{i-1}$	$\mu^\diamond$
$(\varnothing \mid 2, 1^3)$	$(\varnothing \mid 4, 1)$	$(1^2 \mid 2, 1)$	$(2 \mid 2, 1)$	$(2, 1 \mid 1^2)$	$(2, 1 \mid 2)$	$(2, 1^3 \mid \varnothing)$	$(4, 1 \mid \varnothing)$



and two simple modules labelled by μ and β_1. It is easy to solve this case by hand. We use a dimension argument here. First we note that the two Specht modules S_{β_1} and S_{β_2} have the same dimension. From our calculations above we know that

$$[\mathrm{S}_{\beta_1} : \mathrm{D}_\mu] = 1,$$

while Proposition 2.14 and a very simple calculation with the Jantzen–Schaper formula show that

$$[\mathrm{S}_{\beta_1} : \mathrm{D}_{\beta_1}] = [\mathrm{S}_{\beta_2} : \mathrm{D}_{\beta_1}] = 1,$$

forcing $[\mathrm{S}_{\beta_2} : \mathrm{D}_\mu] = 1$.

This completes the proof of Case IV, and of our main theorem.

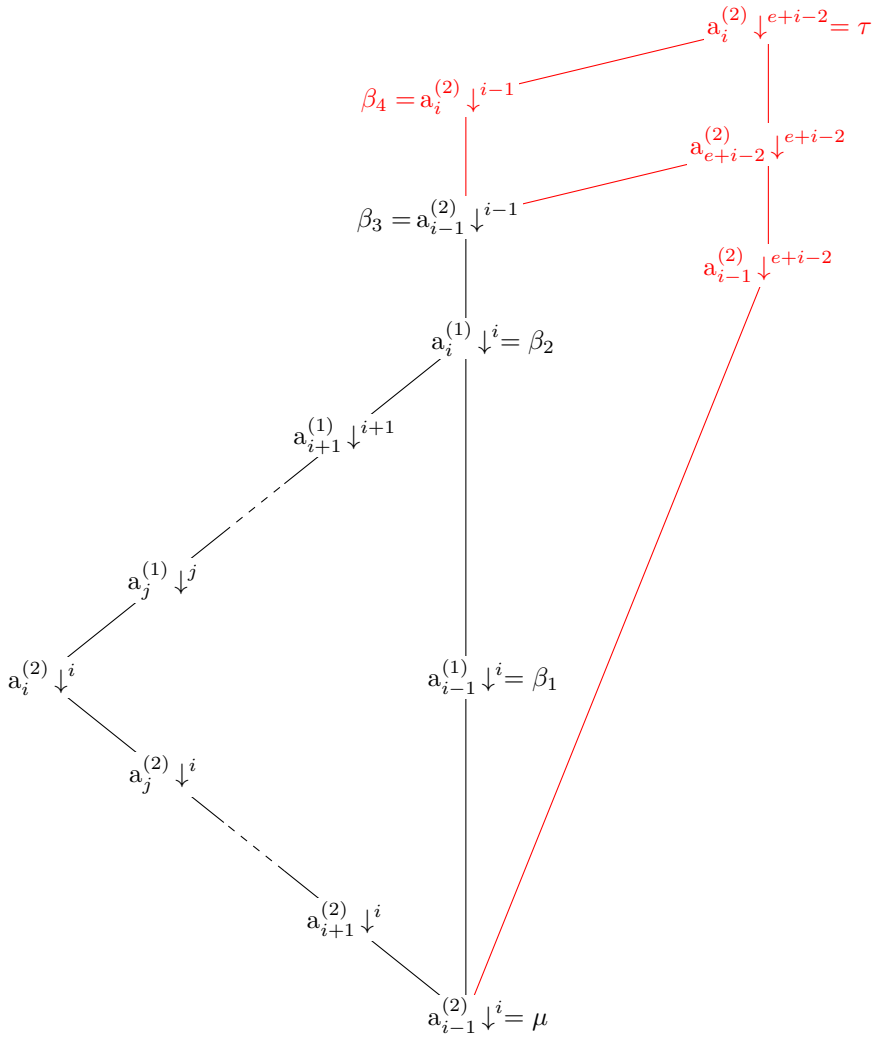


FIGURE 4. Jantzen–Schaper interval for case 6

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