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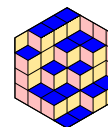


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Permutations with restricted cycle lengths

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ABSTRACT We find exact and asymptotic formulas for the number of pairs (s, z) of N -cycles s and z such that all cycles of the product $s \cdot z$ have lengths from a given set of integers. We then apply one of these results to prove a surprisingly high lower bound for the number of permutations whose block transposition distance from the identity is at least $(N + 1)/2$.

1. INTRODUCTION

1.1. PERMUTATIONS WITH RESTRICTED CYCLE LENGTHS. We are interested in counting *some* of the permutations of a set $[N]$ whose cycle lengths belong to a given subset $A \subseteq \{1, 2, \dots\}$. Permutations of this kind were studied by Sachkov [12], under the name of A -permutations. We stress “some,” since our focus is on permutations of $[N]$ that are products of pairs of N -long cycles. Some special cases have direct consequences in the theory of biologically motivated sorting algorithms. We will explore a very surprising one of these, related to sorting a permutation by block transpositions, in Section 4. One of our permutation-counting results will imply the existence of a high number of permutations of length N that take at least $(N + 1)/2$ block transpositions to sort. This is remarkable, because for most values of N , there are no known examples of permutations of length N that would take more than that many block transpositions to sort.

Let $\mathbb{P}_N(A)$ denote the total number of *all* A -permutations. It is well known (cf. Bóna [3]) that

$$(1) \quad \mathbb{P}_N(A) = N! \cdot [x^N] \exp(F_A(x)), \quad F_A(x) := \sum_{r \in A} \frac{x^r}{r};$$

here, and elsewhere, $[x^m]f(x)$ stands for the coefficient by x^m in the Taylor expansion of a function $f(x)$, analytic at an open disc centered at $x = 0$. Equivalently, $p_N(A)$, the probability that the uniformly random permutation of $[N]$ has all its cycle lengths from A , is given by

$$(2) \quad p_N(A) = [x^N] \exp(F_A(x)).$$

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We will show that this leads to the explicit formulas

$$(3) \quad \begin{aligned} p_k(A) &= \sum_{\substack{\mu_r \geq 0, r \in A^c \\ \sum_r r \mu_r \leq k}} \prod_{r \in A^c} \frac{(-1)^{\mu_r}}{r^{\mu_r} \mu_r!}, \\ p_\ell(A^c) &= \sum_{\substack{r \in A^c \\ \sum_r r \mu_r = \ell}} \prod_{r \in A^c} \frac{1}{r^{\mu_r} \mu_r!}, \end{aligned}$$

where $A^c := \{1, 2, \dots\} \setminus A$ is the complement of A in the set of all positive integers. In the context of sorting algorithms, particularly relevant are the cases $A = \mathcal{E}$, the set of even numbers, and $A = \mathcal{O}$, the set of odd numbers.

The formulas for $\mathbb{P}_N(\mathcal{E})$ and $\mathbb{P}_N(\mathcal{O})$ are well known, [3, 16]:

$$(4) \quad F_{\mathcal{E}}(x) = -\frac{1}{2} \log(1 - x^2), \quad F_{\mathcal{O}}(x) = \frac{1}{2} \log \frac{1+x}{1-x}.$$

It follows that

$$(5) \quad \begin{aligned} p_N(\mathcal{E}) &= 2^{-N} \binom{N}{N/2}, \quad N \text{ even}, \\ p_N(\mathcal{O}) &= \begin{cases} 2^{-N} \binom{N}{N/2}, & N \text{ is even}, \\ 2^{-N+1} \binom{N-1}{(N-1)/2}, & N \text{ is odd}. \end{cases} \end{aligned}$$

We point out that the enumeration formulas that follow from (4) and (5) have known combinatorial proofs. The interested reader should consult the references given in the solution of Exercise 5.10 in [16].

It is significantly more challenging to obtain a tractable counterpart of (2) for $q_N(A)$, the probability that the product of two independent, uniformly random N -long cycles has all its cycle lengths from a given set A . Our main result consists of two complementary parts. First, for an arbitrary $A \subset \{1, 2, \dots\}$, we will get a sum-type formula for the numbers $q_N(A)$ in terms of the numbers $p_k(A)$ and $p_\ell(A^c)$, including a wide range extension of well known results for $A = \{2, 3, \dots\}$. While explicit, the expression does not lend itself easily to asymptotic analysis, since the summands have alternating signs. For two important cases, A being either the set of even numbers or the set of odd numbers, we find a way to obtain the sum-type formulas with positive summands only, much more conducive to asymptotic computations. We will present these formulas and their proofs next.

1.2. PRODUCTS OF MAXIMAL CYCLES. The following theorems summarize the formulas we will obtain for $q_N(A)$.

THEOREM 1.1. *For all positive integers N , the following hold.*

(a) *For a given $A \subset \{1, 2, \dots\}$,*

$$(6) \quad q_N(A) = \frac{N}{N+1} \left(\sum_{\substack{k+\ell=N \\ 0 \leq k, \ell \leq N}} (-1)^\ell \binom{N}{\ell}^{-1} p_k(A) (p_\ell(A^c) - p_{\ell-1}(A^c)) \right),$$

where $p_0(A) = p_0(A^c) = 1$, $p_{-1}(A^c) := 0$; for $k, \ell \geq 1$, $p_k(A)$, $p_\ell(A^c)$ are given by (3).

(b) For $A = \mathcal{E}$ (even positive integers) and $A = \mathcal{O}$ (odd positive integers), we have

$$\begin{aligned}
 q_N(\mathcal{E}) &= \frac{2N}{(N+1)2^N} \sum_{\substack{2k+2\ell=N, \\ k \geq 1, 0 \leq k, \ell \leq N}} \frac{1}{k} \binom{2(k-1)}{k-1} \binom{2\ell}{\ell} \left(1 - \binom{N}{2\ell}^{-1}\right), \quad N \text{ even}, \\
 q_N(\mathcal{O}) &= \frac{N}{2^N(N+1)} \sum_{\substack{2k+2\ell=N, \\ 0 \leq k, \ell \leq N}} \binom{2k}{k} \binom{2\ell}{\ell} \binom{N}{2\ell}^{-1} \\
 &\quad + \frac{N}{2^{N-2}(N+1)} \sum_{\substack{2k+2\ell=N-2, \\ 0 \leq k, \ell \leq N}} \binom{2k}{k} \binom{2\ell}{\ell} \binom{N}{2\ell+1}^{-1}, \quad N \text{ even}, \\
 q_N(\mathcal{O}) &= \frac{N}{(N+1)2^{N-1}} \sum_{\substack{2\ell+2k=N-1, \\ 0 \leq k, \ell \leq N}} \binom{2\ell}{\ell} \binom{2k}{k} \left(\binom{N}{2\ell}^{-1} + \binom{N}{2\ell+1}^{-1} \right), \quad N \text{ odd}.
 \end{aligned}
 \tag{7}$$

In each of the above sums the dominant terms are those with the smallest admissible k . Using this observation, and the Stirling formula for factorials, we obtain that $q_N(\mathcal{E}), q_N(\mathcal{O}) \sim (\pi N/2)^{-1/2}$, as $N \rightarrow \infty$.

The set $\mathcal{E}$ is a special case of $\mathcal{D}$, the full set of positive integers divisible by a fixed $d > 1$. So, we count the permutations whose cycle lengths are all divisible by d . Here

$$F_{\mathcal{D}} = -\frac{1}{d} \log(1 - x^d),$$

and consequently, for $N \equiv 0 \pmod{d}$,

$$p_N(\mathcal{D}) = \frac{1}{(N/d)!} \prod_{r=0}^{N/d-1} (r + 1/d).$$

THEOREM 1.2. For all positive integers N and $d > 1$, the equality

$$\begin{aligned}
 q_N(\mathcal{D}) &= \frac{N}{d(N+1)} \sum_{\substack{dk+d\ell=N, \\ 0 \leq k, \ell \leq N}} \frac{1}{k!\ell!} \left(1 - (-1)^{dk} \binom{N}{d\ell}^{-1}\right) \\
 &\quad \times \prod_{r=1}^{k-1} (r - 1/d) \prod_{r'=0}^{\ell-1} (r' + 1/d)
 \end{aligned}
 \tag{9}$$

holds.

For $d = 2$ the equation (9) yields the top formula in (7).

We note the following. For $A = \{2, 3, \dots\}$, a permutation s of $[N]$ is type-A if and only if s has no cycles of length 1, that is, when s is a derangement. In this case, the formula (3) becomes

$$p_k(A) = \sum_{\mu=0}^k \frac{(-1)^\mu}{\mu!}, \quad p_\ell(A^c) = \frac{1}{\ell!}.$$

With some work (see Appendix B), the equations (6) and (10) imply

$$(N-1)! q_N(A) = (-1)^{N-1} \sum_{j=0}^{N-2} \frac{(-1)^j}{j!} \sum_{j \leq k < N} (-1)^k k!.$$

The reason we are interested in $(N-1)! q_N(A)$ is that it equals the number of N -long cycles p such that the product $(12 \dots N)p$ is a derangement. (Comfortingly, the right-hand side of (11) is indeed an integer for all N , since $k!/j!$ is an integer for $k \geq j$.) On

the other hand, it is well-known that a N -long cycle satisfies the above property if and only there is no $i \in [N]$ such that $p(i) = i - 1 \pmod{N}$, see for instance Stanley [14], or Charalambides [5]. It follows from the inclusion-exclusion principle that C_N , the total number of such maximal cycles p is given by

$$(12) \quad C_N = N! \sum_{\mu=0}^{N-1} \frac{(-1)^\mu}{(N-\mu)\mu!} + (-1)^N.$$

Consequently, $(N-1)!q_N(A) = C_N$, which is not obvious at all. Using Maple, we checked that this identity holds for all $N \leq 10$. At this moment it is unclear to us how to prove this identity formally, that is, without using the fact that both sides count the same objects.

2. PROOF OF THEOREM 1.1.

We start with the following general formula of Pittel [11] and Zagier [17]. Let S_N be the set of all permutations on $[N]$, and let $C_1, \dots, C_k$ be some conjugacy classes from S_N . Let $\sigma_1, \dots, \sigma_k$ be the random *independent* elements of S_N , where σ_j is distributed *uniformly* on C_j . For $\sigma := \sigma_1 \cdots \sigma_k$, we have; for an arbitrary permutation s ,

$$(13) \quad \mathbb{P}(\sigma = s) = P_\sigma(s) := \frac{1}{N!} \sum_{\lambda} (f^\lambda)^{-k+1} \chi^\lambda(s) \prod_{j=1}^k \chi^\lambda(C_j).$$

Here the sum is taken over all integer partitions (Young diagrams) of N , and f^λ is the dimension of the irreducible representation of S_N associated with λ , given by the hook formula

$$(14) \quad f^\lambda = \frac{N!}{\prod_{(i,j) \in \lambda} h_{i,j}}.$$

Here $h_{i,j}$ is the cardinality of the hook for a cell (i, j) , that is, 1 plus the number of cells in λ to the east and to the south from (i, j) . Furthermore, $\chi^\lambda(s)$ and $\chi^\lambda(C_j)$ are the values of the character χ^λ associated with λ , at s and at the elements of the conjugacy classes C_j .

Let $\mathcal{C}_{\mathbf{p}}$ be a generic conjugacy class consisting of all permutations of $[N]$, with cycle lengths $p_1, p_2, \dots$, with $\sum_j p_j = N$. By the Murnaghan-Nakayama rule, (see Macdonald [10], Sagan [13], Stanley [15]), the common value of $\chi^\lambda(s)$ for $s \in \mathcal{C}_{\mathbf{p}}$ is given by

$$(15) \quad \chi^\lambda(\mathcal{C}_{\mathbf{p}}) = \sum_T (-1)^{\text{ht}(T)}.$$

Here the sum is over all *rim hook* diagrams T of shape λ and type $\mathbf{p}$, that is, over all ways to empty the diagram λ by successive deletions of *border strips*, (strips consisting of only current border cells), one strip at a time, of *weakly increasing* lengths p_j . Further, $\text{ht}(T)$ is the sum of heights of the individual border strips (number of their rows minus 1) in T . We will see that, contrary to its forbidding appearance, the formula (15) *alone* leads to a surprisingly simple expression for the values $\chi^\lambda(\mathcal{C})$, if $\mathcal{C}$ is the conjugacy class of one-cycle permutations.

We focus on $k = 2$, and $C_1 = C_2 = \mathcal{C}_N$, the set of all maximal, N -long, cycles. Here the composition $\mathbf{p}$ consists of a single component N , and the range of T is empty unless the diagram λ is a single hook λ^* , with one horizontal row of length λ_1 and one column of height λ^1 , $\lambda_1 + \lambda^1 = N + 1$, in which case $\chi^{\lambda^*}(\mathcal{C}_N) = (-1)^{\lambda^1-1}$. For that diagram, we obtain

$$(16) \quad f^{\lambda^*} = \binom{N-1}{\lambda_1-1}$$

either by the hook formula (14), or by remembering the well-known fact [13] that f^{λ^*} is the number of Standard Young Tableaux of shape λ^* and noting that a Standard Young Tableau of this shape is completely determined by the set of its entries located in the first row.

Furthermore, by (15), given one-hook diagram λ^* , the value of $\chi^{\lambda^*}(s)$ depends on a generic permutation s only through $\nu = \nu(s)$, where $\nu_r = \nu_r(s)$ is the total number of r -long cycles in s . We need the following Lemma, [11].

LEMMA 2.1. *Let $\mathcal{C}$ be a generic conjugacy class consisting of all permutations s of $[N]$, with ν_j cycles of length j , $\sum_j j\nu_j = N$. Then, for every $s \in \mathcal{C}$,*

$$(17) \quad \chi^{\lambda^*}(s) = (-1)^{\lambda^1} [x^{\lambda_1}]_{\frac{x}{1-x}} \prod_{r \geq 1} (x^r - 1)^{\nu_r},$$

where λ_1 and λ^1 are the arm length and the leg length of the hook λ^* .

For the reader's convenience, we put an edited proof of (17) into Appendix A. Substituting formulas (16) and (17) into (13), we obtain the key formula

$$(18) \quad \mathbb{P}(\sigma = s) = \frac{1}{N!} \sum_{\lambda_1 \in [N]} \binom{N-1}{\lambda_1-1}^{-1} (-1)^{\lambda^1} [x^{\lambda_1}]_{\frac{x}{1-x}} \prod_r (x^r - 1)^{\nu_r};$$

here σ is the product of two independent, uniformly random cycles of length N , and ν_j is the total number of cycles of length j in a generic permutation s of $[N]$.

Furthermore, using the terminology in Sachkov [12], for a given $A \subseteq \{1, 2, \dots\}$, we call s an A -permutation if $\nu_r = 0$ for all $r \notin A$. Let us use (18) to obtain a hopefully tractable formula for $q_N(A)$, the probability that the product s of two N -cycles is an A -permutation. To this end, we need to sum the right-hand side of (18) over all ν with $\nu_r = 0$ for $r \notin A$. To begin, notice that the right-hand side of (18) depends on the cycle-type $\{\nu_r(s)\}$ of the permutation s only via the rightmost product $\prod_r (x^r - 1)^{\nu_r}$. Observing that the total number of permutations with given cycle counts $\nu = \{\nu_r\}$ is

$$\frac{N!}{\prod_r (r!)^{\nu_r} \nu_r!} \cdot \prod_{r'} ((r' - 1)!)^{\nu_{r'}} = \frac{N!}{\prod_r r^{\nu_r} \nu_r!},$$

we have

$$\begin{aligned} \sum_{\substack{\nu: \nu_r=0, \forall r \notin A \\ \sum_r r\nu_r=N}} \frac{N!}{\prod_r r^{\nu_r} \nu_r!} \prod_r (x^r - 1)^{\nu_r} &= N! \sum_{\substack{\nu: \nu_r=0, \forall r \notin A \\ \sum_r r\nu_r=N}} \prod_r \left(\frac{x^r-1}{r}\right)^{\nu_r} \frac{1}{\nu_r!} \\ &= N! [y^N] \prod_{r \in A} \sum_{m \geq 0} \left(\frac{x^r-1}{r}\right)^m \cdot \frac{y^{mr}}{m!} = N! [y^N] \prod_{r \in A} \exp\left(\frac{y^r(x^r-1)}{r}\right) \\ &= N! [y^N] \exp(F(xy) - F(y)), \quad F(\eta) := \sum_{r \in A} \frac{\eta^r}{r}. \end{aligned}$$

Consequently,

$$\begin{aligned}
 (19) \quad q_N(A) &= \sum_{s \text{ of type } A} \mathbb{P}(\sigma = s) \\
 &= \frac{1}{N!} \sum_{\lambda_1 \in [N]} \binom{N-1}{\lambda_1-1}^{-1} (-1)^{\lambda_1} [x^{\lambda_1}]_{\frac{x}{1-x}} \sum_{\substack{\nu: \nu_r=0, \forall r \notin A \\ \sum_r r\nu_r=N}} \prod_r \frac{N!}{r^{\nu_r} \nu_r!} \prod_r (x^r - 1)^{\nu_r} \\
 &= [y^N] e^{-F(y)} \sum_{\lambda_1 \in [N]} \binom{N-1}{\lambda_1-1}^{-1} (-1)^{\lambda_1} [x^{\lambda_1}]_{\frac{x}{1-x}} e^{F(xy)}.
 \end{aligned}$$

Here

$$\begin{aligned}
 \binom{N-1}{\lambda_1-1}^{-1} &= \frac{(\lambda_1-1)!(N-\lambda_1)!}{(N-1)!} = N \int_0^1 z^{\lambda_1-1} (1-z)^{N-\lambda_1} dz, \\
 (-1)^{\lambda_1} [x^{\lambda_1}]_{\frac{x}{1-x}} e^{F(xy)} &= (-1)^{N+1-\lambda_1} [x^{\lambda_1-1}] \frac{e^{F(xy)}}{1-x} \\
 &= (-1)^N (-1)^{\lambda_1-1} [x^{\lambda_1-1}] \frac{e^{F(xy)}}{1-x} = (-1)^N [x^{\lambda_1-1}] \frac{\exp(F(-xy))}{1+x}.
 \end{aligned}$$

Therefore the sum in the last row of (19) equals

$$\begin{aligned}
 &(-1)^N N \sum_{\lambda_1 \in [N]} \left(\int_0^1 z^{\lambda_1-1} (1-z)^{N-\lambda_1} dz \right) [x^{\lambda_1-1}] \frac{\exp(F(-xy))}{1+x} \\
 &= (-1)^N N \int_0^1 \sum_{\lambda_1 \in [N]} z^{\lambda_1-1} [x^{\lambda_1-1}] \frac{\exp(F(-xy))}{1+x} (1-z)^{(N-1)-(\lambda_1-1)} dz \\
 &= (-1)^N N \int_0^1 \sum_{\lambda_1 \in [N]} [x^{\lambda_1-1}] \frac{\exp(F(-xyz))}{1+xz} \times [x^{(N-1)-(\lambda_1-1)}] \frac{1}{1-x(1-z)} dz \\
 &= (-1)^N N \int_0^1 [x^{N-1}] \frac{\exp(F(-xyz))}{(1+xz)(1-x(1-z))} dz \\
 &= -N \int_0^1 [x^{N-1}] \frac{\exp(F(xyz))}{(1-xz)(1+x(1-z))} dz.
 \end{aligned}$$

In combination with (19) we have proved

$$(20) \quad q_N(A) = -N[y^N] e^{-F(y)} \int_0^1 [x^{N-1}] \frac{e^{F(xyz)}}{(1-xz)(1+x(1-z))} dz.$$

(a) Let us compute the integral on the right-hand side of (20) for a general A . Since

$$\begin{aligned}
 (21) \quad \frac{1}{(1-xz)(1+x(1-z))} &= \frac{1}{x} \frac{1}{1-xz} - \frac{1}{x} \cdot \frac{1}{1+x(1-z)} \\
 &= \frac{1}{x} \sum_{j \geq 0} (xz)^j - \frac{1}{x} \sum_{j \geq 0} (-1)^j x^j (1-z)^j,
 \end{aligned}$$

we have

$$\begin{aligned} [x^{N-1}] \frac{\exp(F(xyz))}{(1-xz)(1+x(1-z))} &= [x^N] e^{F(xyz)} \left(\sum_{j \geq 0} (xz)^j - \sum_{j \geq 0} (-1)^j x^j (1-z)^j \right) \\ &= \sum_{k+j=N} (yz)^k [u^k] e^{F(u)} (z^j - (-1)^j (1-z)^j). \end{aligned}$$

So, the integral on the right-hand side of (20) equals

$$\begin{aligned} \sum_{k+j=N} y^k [u^k] e^{F(u)} \int_0^1 z^N dz - \sum_{k+j=N} (-1)^j y^k [u^k] e^{F(u)} \int_0^1 z^k (1-z)^j dz \\ = \frac{1}{N+1} \sum_{k+j=N} y^k [u^k] e^{F(u)} - \frac{1}{N+1} \sum_{k+j=N} (-1)^j \binom{N}{j}^{-1} y^k [u^k] e^{F(u)}. \end{aligned}$$

Then, since $e^{-F(y)} = \sum_{\ell} y^{\ell} [v^{\ell}] e^{-F(v)}$, the factor $-N$ aside the rest of the right-hand side in (20) equals

$$\begin{aligned} [y^N] e^{-F(y)} \int_0^1 [x^{N-1}] \frac{e^{F(xyz)}}{(1-xz)(1+x(1-z))} dz &= \frac{1}{N+1} \sum_{k+\ell=N} [v^{\ell}] e^{-F(v)} [u^k] e^{F(u)} \\ &\quad - \frac{1}{N+1} \sum_{\substack{k+j=N \\ k+\ell=N}} (-1)^j \binom{N}{j}^{-1} [v^{\ell}] e^{-F(v)} [u^k] e^{F(u)} \\ &= \frac{1}{N+1} \sum_{k+\ell=N} [v^{\ell}] e^{-F(v)} [u^k] e^{F(u)} \\ &\quad - \frac{1}{N+1} \sum_{k+\ell=N} (-1)^{\ell} \binom{N}{\ell}^{-1} [v^{\ell}] e^{-F(v)} [u^k] e^{F(u)}. \end{aligned}$$

The first sum equals $[x^N] e^{-F(x)+F(x)} = 0$. In the second sum, $[u^k] e^{F(u)} = p_k(A)$ for $k \geq 1$, and $[u^0] e^{F(u)} = 1 =: p_0(A)$. Furthermore, recalling that A^c denotes the complement of A in $\{1, 2, \dots\}$, we get that

$$[v^{\ell}] e^{-F(v)} = [v^{\ell}] \exp\left(-\log \frac{1}{1-v} + F_{A^c}(v)\right), \quad F_{A^c}(v) = \sum_{r \in A^c} \frac{v^r}{r}.$$

So recalling that $p_0(A^c) := 1$ and $p_{-1}(A^c) := 0$, we have

$$[v^{\ell}] e^{-F(v)} = [v^{\ell}] (1-v) e^{F_{A^c}(v)} = p_{\ell}(A^c) - p_{\ell-1}(A^c), \quad \ell \geq 0.$$

Collecting the pieces, we obtain

$$(22) \quad q_N(A) = \frac{N}{N+1} \sum_{k+\ell=N} (-1)^{\ell} \binom{N}{\ell}^{-1} p_k(A) (p_{\ell}(A^c) - p_{\ell-1}(A^c)).$$

Thus to evaluate the probability that the product of two cycles is an A -permutation, it suffices to know the probabilities that uniformly distributed permutations of various length are A -permutations or A^c -permutations. Here those probabilities are, tailored for the case when A^c is “small”:

$$\begin{aligned}
 p_k(A) &:= [x^k] \exp(F_A(x)) = [x^k] \exp\left(\sum_{r>1} \frac{x^r}{r} - \sum_{r \in A^c} \frac{x^r}{r}\right) \\
 &= [x^k] \frac{1}{1-x} \exp\left(-\sum_{r \in A^c} \frac{x^r}{r}\right) = \sum_{j \geq 0} [x^{k-j}] \exp\left(-\sum_{r \in A^c} \frac{x^r}{r}\right) \\
 &= \sum_{\mu \geq 0} \frac{(-1)^\mu}{\mu!} \sum_{j \geq 0} [x^{k-j}] \left(\sum_{r \in A^c} \frac{x^r}{r}\right)^\mu \\
 &= \sum_{\mu \geq 0} (-1)^\mu \sum_{j \leq k} \sum_{\substack{\mu_r \geq 0, r \in A^c \\ \sum_r \mu_r = \mu, \sum_r r \mu_r = k-j}} \prod_{r \in A^c} \frac{1}{r^{\mu_r} \mu_r!} \\
 &= \sum_{\substack{\mu_r \geq 0, r \in A^c \\ \sum_r r \mu_r \leq k}} \prod_{r \in A^c} \frac{(-1)^{\mu_r}}{r^{\mu_r} \mu_r!},
 \end{aligned}$$

and

$$\begin{aligned}
 p_\ell(A^c) &= [x^\ell] \exp\left(\sum_{r \in A^c} \frac{x^r}{r}\right) = \sum_j [x^\ell] \frac{\left(\sum_{r \in A^c} \frac{x^r}{r}\right)^j}{j!} \\
 &= \sum_{\sum_r r \mu_r = \ell} \prod_{r \in A^c} \frac{1}{r^{\mu_r} \mu_r!}.
 \end{aligned}$$

(b) For $A = \mathcal{E}$ and $A = \mathcal{O}$, we get explicit formulas for $q_N(A)$ by combining (22) with (5) for those A 's. However, it turns out that it is possible to obtain considerably simpler, sum-type, formulas for $q_N(\mathcal{E})$ and $q_N(\mathcal{O})$ with only positive summands. This can be done by properly changing the order of steps in the derivation above.

Start with $q_N(\mathcal{E})$; so $q_N(\mathcal{E}) = 0$ if N is odd. Let N be even. In this case, we carry the operation $[y^N]$ inside the integration in (20) and using (4) we obtain:

$$\begin{aligned}
 (23) \quad q_N(\mathcal{E}) &= -N \int_0^1 [x^{N-1} y^N] \left(\frac{(1-y^2)^{1/2}}{(1-(xyz)^2)^{1/2}} \cdot \frac{1}{(1-xz)(1+x(1-z))} \right) dz \\
 &= -N \int_0^1 [x^{N-1}] \left(H_N(xz) \cdot \frac{1}{(1-xz)(1+x(1-z))} \right) dz,
 \end{aligned}$$

where

$$\begin{aligned}
 (24) \quad H_N(xz) &:= [y^N] \frac{(1-y^2)^{1/2}}{(1-(xyz)^2)^{1/2}} \\
 &= [y^N] \left(1 + \sum_{k \geq 1} \binom{1/2}{k} (-y^2)^k \right) \cdot \left(\sum_{\ell \geq 0} \binom{-1/2}{\ell} (-(xyz)^2)^\ell \right) \\
 &= [y^N] \left(1 - \frac{1}{2} \sum_{k \geq 1} \frac{y^{2k}}{k!} \prod_{r=1}^{k-1} (r - 1/2) \right) \cdot \left(\sum_{\ell \geq 0} \frac{(xyz)^{2\ell}}{\ell!} \prod_{r'=0}^{\ell-1} (r' + 1/2) \right) \\
 &= \frac{(xz)^N}{(N/2)!} \prod_{r'=0}^{N/2-1} (r' + 1/2) - \frac{1}{2} \sum_{\substack{k \geq 1 \\ 2k+2\ell=N}} \frac{(xz)^{2\ell}}{k! \ell!} \prod_{r=1}^{k-1} (r - 1/2) \prod_{r'=0}^{\ell-1} (r' + 1/2).
 \end{aligned}$$

Thus, $H_N(xz)$ is a polynomial function of xz . So, combining (24) and (21), we evaluate

$$\begin{aligned} & [x^N]H_N(xz) \sum_{j \geq 0} x^j [z^j - (-1)^j (1-z)^j] \\ &= -\frac{1}{2} \sum_{\substack{k \geq 1, 2k+2\ell=N, \\ j+2\ell=N}} \frac{z^{2\ell}}{k!\ell!} [z^j - (-1)^j (1-z)^j] \prod_{r=1}^{k-1} (r-1/d) \prod_{r'=0}^{\ell-1} (r'+1/d) \\ &= -\frac{1}{2} \sum_{k \geq 1, 2k+2\ell=N} \frac{1}{k!\ell!} [z^N - (-1)^{2k} (1-z)^{2k} z^{2\ell}] \prod_{r=1}^{k-1} (r-1/2) \prod_{r'=0}^{\ell-1} (r'+1/2). \end{aligned}$$

Integrating over $z \in [0, 1]$, using

$$\int_0^1 (1-z)^{2k} z^{2\ell} dz = \frac{1}{N+1} \binom{N}{2\ell}^{-1}, \quad (2k+2\ell=N),$$

and multiplying by $-N$, we obtain

$$\begin{aligned} q_N(\mathcal{E}) &= \frac{N}{2(N+1)} \sum_{k \geq 1, 2k+2\ell=N} \frac{1}{k!\ell!} \left(1 - \binom{N}{2\ell}^{-1} \right) \\ &\quad \times \prod_{r=1}^{k-1} (r-1/2) \prod_{r'=0}^{\ell-1} (r'+1/2). \end{aligned}$$

Since

$$\frac{1}{k!} \prod_{r=1}^{k-1} (r-1/2) = \frac{1}{2^{2(k-1)k}} \binom{2(k-1)}{k-1}, \quad \frac{1}{\ell!} \prod_{r'=0}^{\ell-1} (r'+1/2) = \frac{1}{2^{2\ell}} \binom{2\ell}{\ell},$$

we conclude that

$$q_N(\mathcal{E}) = \frac{2N}{(N+1)2^N} \sum_{k \geq 1, 2k+2\ell=N} \frac{1}{k} \binom{2(k-1)}{k-1} \binom{2\ell}{\ell} \left(1 - \binom{N}{2\ell}^{-1} \right), \quad N \text{ even.}$$

Consider $A = \mathcal{O}$. This time, again by (4),

$$\begin{aligned} \exp(F_{\mathcal{O}}(xyz) - F_{\mathcal{O}}(y)) &= \sqrt{\frac{1+xyz}{1-xyz}} \cdot \sqrt{\frac{1-y}{1+y}} \\ &= \frac{1+xyz}{\sqrt{1-(xyz)^2}} \cdot \frac{1-y}{\sqrt{1-y^2}}, \end{aligned}$$

and (20) yields

$$\begin{aligned} q_N(\mathcal{O}) &= -N[x^{N-1}y^N] \int_0^1 \frac{1+xyz}{\sqrt{1-(xyz)^2}} \cdot \frac{1-y}{\sqrt{1-y^2}} \cdot \frac{1}{(1-xz)(1+x(1-z))} dz \\ &= -N[x^{N-1}] \int_0^1 H_N(xz) \cdot \frac{1}{(1-xz)(1+x(1-z))} dz, \end{aligned}$$

where

$$\begin{aligned} (25) \quad H_N(xz) &:= [y^N] \frac{1+xyz}{\sqrt{1-(xyz)^2}} \cdot \frac{1-y}{\sqrt{1-y^2}} \\ &= [y^N] (1+xyz)(1-y) \sum_{\ell \geq 0} \binom{2\ell}{\ell} \left(\frac{(xyz)^2}{4} \right)^\ell \cdot \sum_{k \geq 0} \binom{2k}{k} \left(\frac{y^2}{4} \right)^k, \end{aligned}$$

i.e., again $H_N(xz)$ is a polynomial function of xz .

(b1) If N is even, then (25) yields

$$\begin{aligned} H_N(xz) &= [y^N](1 - xy^2z) \sum_{\ell \geq 0} \binom{2\ell}{\ell} \left(\frac{(xyz)^2}{4}\right)^\ell \cdot \sum_{k \geq 0} \binom{2k}{k} \left(\frac{y^2}{4}\right)^k \\ &= 2^{-N} \sum_{2k+2\ell=N} \binom{2k}{k} \binom{2\ell}{\ell} (xz)^{2\ell} - 2^{-N+2} \sum_{2k+2\ell=N-2} \binom{2k}{k} \binom{2\ell}{\ell} (xz)^{2\ell+1}. \end{aligned}$$

Therefore, recalling (21),

$$\begin{aligned} [x^N]H_N(xz) \sum_{j \geq 0} (xz)^j &= 2^{-N} z^N \sum_{2k+2\ell=N} \binom{2k}{k} \binom{2\ell}{\ell} \\ &\quad - 2^{-N+2} z^N \sum_{2k+2\ell=N-2} \binom{2k}{k} \binom{2\ell}{\ell}, \end{aligned}$$

and

$$\begin{aligned} [x^N]H_N(xz) \sum_{j \geq 0} (-1)^j x^j (1-z)^j &= 2^{-N} \sum_{2k+2\ell=N} \binom{2k}{k} \binom{2\ell}{\ell} (1-z)^{2k} z^{2\ell} \\ &\quad + 2^{-N+2} \sum_{2k+2\ell=N-2} \binom{2k}{k} \binom{2\ell}{\ell} (1-z)^{2k+1} z^{2\ell+1}. \end{aligned}$$

Therefore

$$\begin{aligned} q_N(\mathcal{O}) &= -N \int_0^1 [x^{N-1}] \left(H_N(xz) \cdot \frac{1}{(1-xz)(1+x(1-z))} \right) dz \\ &= -N \int_0^1 \left[2^{-N} z^N \sum_{2k+2\ell=N} \binom{2k}{k} \binom{2\ell}{\ell} - 2^{-N+2} z^N \sum_{2k+2\ell=N-2} \binom{2k}{k} \binom{2\ell}{\ell} \right. \\ &\quad \left. - 2^{-N} \sum_{2k+2\ell=N} \binom{2k}{k} \binom{2\ell}{\ell} (1-z)^{2k} z^{2\ell} \right. \\ &\quad \left. - 2^{-N+2} \sum_{2k+2\ell=N-2} \binom{2k}{k} \binom{2\ell}{\ell} (1-z)^{2k+1} z^{2\ell+1} \right] dz. \end{aligned}$$

Integrating over $z \in [0, 1]$, we obtain

$$\begin{aligned} q_N(\mathcal{O}) &= -\frac{N}{2^N(N+1)} \sum_{2k+2\ell=N} \binom{2k}{k} \binom{2\ell}{\ell} + \frac{N}{2^{N-2}(N+1)} \sum_{2k+2\ell=N-2} \binom{2k}{k} \binom{2\ell}{\ell} \\ &\quad + \frac{N}{2^N(N+1)} \sum_{2k+2\ell=N} \binom{2k}{k} \binom{2\ell}{\ell} \binom{N}{2\ell}^{-1} \\ &\quad + \frac{N}{2^{N-2}(N+1)} \sum_{2k+2\ell=N-2} \binom{2k}{k} \binom{2\ell}{\ell} \binom{N}{2\ell+1}^{-1}. \end{aligned}$$

Here

$$\begin{aligned} \sum_{2k+2\ell=N} \binom{2k}{k} \binom{2\ell}{\ell} &= [\eta^{N/2}] \left(\sum_{k \geq 0} \binom{2k}{k} \eta^k \right)^2 \\ &= [\eta^{N/2}] \frac{1}{1-4\eta} = 4^{N/2} = 2^N, \end{aligned}$$

and likewise

$$\sum_{2k+2\ell=N-2} \binom{2k}{k} \binom{2\ell}{\ell} = 2^{N-2}.$$

Therefore the formula for $q_N(\mathcal{O})$ simplifies to

$$(26) \quad q_N(\mathcal{O}) = \frac{N}{2^N(N+1)} \sum_{2k+2\ell=N} \binom{2k}{k} \binom{2\ell}{\ell} \binom{N}{2\ell}^{-1} \\ + \frac{N}{2^{N-2}(N+1)} \sum_{2k+2\ell=N-2} \binom{2k}{k} \binom{2\ell}{\ell} \binom{N}{2\ell+1}^{-1}.$$

(b2) If N is odd, then (25) yields

$$H_N(xz) = [y^N](y(xz-1)) \sum_{\ell \geq 0} \binom{2\ell}{\ell} \left(\frac{(xyz)^2}{4}\right)^\ell \cdot \sum_{k \geq 0} \binom{2k}{k} \left(\frac{y^2}{4}\right)^k \\ = \frac{xz-1}{2^{N-1}} \sum_{2\ell+2k=N-1} \binom{2\ell}{\ell} \binom{2k}{k} (xz)^{2\ell}.$$

Consequently

$$[x^N]H_N(xz) \sum_{j \geq 0} (xz)^j = \frac{1}{2^{N-1}} \left[z[x^{N-1}] \sum_{2\ell+2k=N-1} \binom{2\ell}{\ell} \binom{2k}{k} (xz)^{2\ell} \sum_{j \geq 0} (xz)^j \right. \\ \left. - [x^N] \sum_{2\ell+2k=N-1} \binom{2\ell}{\ell} \binom{2k}{k} (xz)^{2\ell} \sum_{j \geq 0} (xz)^j \right] \\ = \frac{1}{2^{N-1}} \left[z \sum_{2\ell+2k=N-1} \binom{2\ell}{\ell} \binom{2k}{k} z^{2\ell+2k} - z^N \sum_{2\ell+2k=N-1} \binom{2\ell}{\ell} \binom{2k}{k} \right] = 0,$$

and similarly

$$[x^N]H_N(xz) \sum_{j \geq 0} (-1)^j x^j (1-z)^j \\ = \frac{1}{2^{N-1}} \left[\sum_{2\ell+2k=N-1} \binom{2\ell}{\ell} \binom{2k}{k} z^{2\ell+1} (1-z)^{2k} \right. \\ \left. + \sum_{2\ell+2k=N-1} \binom{2\ell}{\ell} \binom{2k}{k} z^{2\ell} (1-z)^{2k+1} \right].$$

So, integrating $-[x^N]H_N(xz) \sum_{j \geq 0} (-1)^j x^j (1-z)^j$ over $z \in [0, 1]$ and multiplying the result by $-N$, we obtain

$$(27) \quad q_N(\mathcal{O}) = \frac{N}{(N+1)2^{N-1}} \sum_{2\ell+2k=N-1} \binom{2\ell}{\ell} \binom{2k}{k} \left(\binom{N}{2\ell}^{-1} + \binom{N}{2\ell+1}^{-1} \right).$$

The proof of the part (b) is complete, and so is the proof of Theorem 1.1.

Remarks. (1) The probabilities $q_N(A)$ do not change if instead of considering all pairs (p, q) of cycles of length N , selected from $\mathcal{C}_N \times \mathcal{C}_N$ uniformly at random, we set $p = (12 \cdots N)$ and we select q from $\mathcal{C}_N$ uniformly at random.

3. PROOF OF THEOREM 1.2

The argument mimics the proof the formula for $q_N(\mathcal{E})$, i.e. the case $d = 2$. This time N is divisible by d . The equation (23) holds for $q_N(\mathcal{D})$, where

$$\begin{aligned}
 (28) \quad H_N(xz) &:= [y^N] \frac{(1-y^d)^{1/d}}{(1-(xyz)^d)^{1/d}} \\
 &= [y^N] \left(1 + \sum_{k \geq 1} \binom{1/d}{k} (-y^d)^k \right) \cdot \left(\sum_{\ell \geq 0} \binom{-1/d}{\ell} (-(xyz)^d)^\ell \right) \\
 &= [y^N] \left(1 - \frac{1}{d} \sum_{k \geq 1} \frac{y^{dk}}{k!} \prod_{r=1}^{k-1} (r-1/d) \right) \cdot \left(\sum_{\ell \geq 0} \frac{(xyz)^{d\ell}}{\ell!} \prod_{r'=0}^{\ell-1} (r'+1/d) \right) \\
 &= \frac{(xz)^N}{(N/d)!} \prod_{r'=0}^{N/d-1} (r'+1/d) - \frac{1}{d} \sum_{k \geq 1, dk+d\ell=N} \frac{(xz)^{d\ell}}{k!\ell!} \prod_{r=1}^{k-1} (r-1/d) \prod_{r'=0}^{\ell-1} (r'+1/d).
 \end{aligned}$$

So, according to the equation (23) for $q_N(\mathcal{D})$, and (21), we evaluate

$$\begin{aligned}
 [x^N] H_N(xz) &\sum_{j \geq 0} x^j [z^j - (-1)^j (1-z)^j] \\
 &= -\frac{1}{d} \sum_{\substack{k \geq 1, dk+d\ell=N, \\ j+d\ell=N}} \frac{z^{d\ell}}{k!\ell!} [z^j - (-1)^j (1-z)^j] \prod_{r=1}^{k-1} (r-1/d) \prod_{r'=0}^{\ell-1} (r'+1/d) \\
 &= -\frac{1}{d} \sum_{k \geq 1, dk+d\ell=N} \frac{1}{k!\ell!} [z^N - (-1)^{dk} (1-z)^{dk} z^{d\ell}] \prod_{r=1}^{k-1} (r-1/d) \prod_{r'=0}^{\ell-1} (r'+1/d).
 \end{aligned}$$

Integrating over $z \in [0, 1]$ and multiplying by $-N$, we obtain

$$\begin{aligned}
 q_N(\mathcal{D}) &= \frac{N}{d(N+1)} \sum_{k \geq 1, dk+d\ell=N} \frac{1}{k!\ell!} \left(1 - (-1)^{dk} \binom{N}{d\ell} \right)^{-1} \\
 &\quad \times \prod_{r=1}^{k-1} (r-1/d) \prod_{r'=0}^{\ell-1} (r'+1/d).
 \end{aligned}$$

This completes the proof of Theorem 1.2

4. BLOCK TRANSPOSITION SORTING

Our motivation for obtaining exact and asymptotic results regarding the product of two long cycles comes from a biologically motivated sorting algorithms called *block transposition sorting*, Bafna and Pevzner [1]. A step in this sorting algorithm consists of swapping *adjacent* blocks of entries. The lengths of the two blocks do not have to be the same. That is,

$$u = u_1 u_2 \cdots u_i (u_{i+1} \cdots u_j) (u_{j+1} \cdots u_k) u_{k+1} \cdots u_N$$

is turned into the permutation

$$v = u_1 u_2 \cdots u_i (u_{j+1} \cdots u_k) (u_{i+1} \cdots u_j) u_{k+1} \cdots u_N.$$

If $u, v \in S_N$, and t is the smallest integer so that there exists a sequence of t block transpositions that turn u into v , then we say that $t = \text{btd}(u, v)$ is the *block transposition distance* of u and v .

The block transposition distance of two permutations is *left-invariant*, that is, if u , v and r are permutations of length N , then $\text{btd}(u, v) = \text{btd}(ru, rv)$. Selecting $r = v^{-1}$, this means that $\text{btd}(u, v) = \text{btd}(v^{-1}u, \text{id})$, where id denotes the identity permutation. This reduces our distance-measuring problem to a *sorting problem*, that is, we can focus, without loss of generality, on the block transposition distance of a given permutation from the *identity permutation*. Therefore, we will write $\text{btd}(u)$ for $\text{btd}(u, \text{id})$.

Several natural and difficult questions arise. Which permutation u is the furthest away from id ? What is $\text{td}(N) = \max_{u \in S_N} \text{btd}(u)$? It is known, Eriksson et al., [9], that $\text{td}(N) \leq \lfloor 2N/3 \rfloor$. On the lower end [9], $\text{btd}(N \cdots 21) = \lceil (N+1)/2 \rceil$, which led to the conjecture $(\text{td})(N) = \lceil (N+1)/2 \rceil$. However, this has been disproved, Elias and Hartman [8], for $N = 4k+1 \geq 17$. For such N , there is a permutation $u \in S_N$ so that $\text{btd}(u) = 2k+2 = \lceil (N+1)/2 \rceil + 1$. So there is a considerable gap between the best known lower and upper bound for $\text{td}(N)$.

As there are almost no known permutations u satisfying the inequality $\text{btd}(u) > (N+1)/2$, there is intrinsic interest in finding out how many permutations v satisfy $\text{btd}(v) = \lceil (N+1)/2 \rceil$. In this paper, we will find a lower bound for the number of such permutations. This lower bound will be significantly higher than what could be deduced before, based on the results in Cranston et al. [7] and Christie [6].

Crucially, there is a bijection C from the set of all permutations of length N to the set of all permutations of length $N+1$ that are of the form $s \cdot z$, where $s = (12 \cdots (N+1))$ and $\nu(z) = 1$, that is, z is a cycle of length $N+1$. This bijection is defined [6] with the help of a graph called the *cycle graph*, but we will not need that definition in this paper. Let $\nu_{\text{odd}}(\pi)$ be the number of odd cycles of the permutation π . The following lemma shows the importance of the bijection C for us.

LEMMA 4.1. [1] *For all permutations u of length N , the inequality $\text{btd}(u) \geq \frac{N+1-\nu_{\text{odd}}(C(u))}{2}$ holds.*

Clearly, the upper bound of Lemma 4.1 is the strongest when $\nu_{\text{odd}}(C(u)) = 0$, that is, when all cycles of $C(u)$ are even. We are now in a position to prove a lower bound for the number of such permutations u .

THEOREM 4.2. *The number of permutations of length N that take at least $\lceil (N+1)/2 \rceil$ block transpositions to sort is at least $N! q_{N+1}(\mathcal{E})$, where $q_{N+1}(\mathcal{E})$ is defined by the first equation in (7).*

Consequently, the fraction of such permutations among all permutations of length N is asymptotically at least $1/\sqrt{\pi N/2}$.

Proof. Lemma 4.1 shows that $\text{btd}(u) \geq \lceil (N+1)/2 \rceil$ when all cycles of $C(u)$ are even. The existence of bijection C shows that the number of such permutations u equals the number of permutations q that consist of a single cycle of length $N+1$ so that $(12 \cdots (N+1)) \cdot q$ has even cycles only. And, by the definition of $q_{N+1}(\mathcal{E})$, the number of such pairs is $N! q_{N+1}(\mathcal{E})$, and their portion among all $N!$ permutations is asymptotically equal to $(\pi N/2)^{-1/2}$, see Theorem 1.1. $\square$

Note that Theorem 4.2 is significant for two reasons.

First, the result of Theorem 4.2 is significantly stronger than what was known before. In particular, it follows from Lemma 4.1 that $\text{btd}(u) \geq \lfloor N/2 \rfloor$ holds when $C(u)$ has exactly one odd cycle. The number of such permutations u is known, Boccara [2], Bóna and Pittel [4], to be $2N!/(N+2)$ if N is even, and 0 if N is odd, so the ratio of such permutations among all permutations of length N is $2/(N+2)$ if n is even. This lower bound is weaker than what we have just proved, by a factor of $c/\sqrt{N}$. Another lower bound, also significantly weaker than the one we just proved, can be deduced from the results in [7], where the authors considered *cut-and-paste* sorting,

an operation that is stronger than block transposition sorting. However, that lower bound just shows that the relevant ratio is at least $\sqrt{N}/2^N$, which is again weaker than the result above.

Second, and probably more importantly, we have just proved the existence of a lot of permutations that are very close to the known worst case. The decreasing permutation $N \cdots 21$ takes $\lceil (N+1)/2 \rceil$ block transpositions to sort, and as we mentioned before, no permutation of length N is known to be harder to sort unless $N = 4k+1 \geq 17$, and even in that case, only one permutation is known to be harder to sort, and that permutation needs only one additional block transposition to be sorted [8]. So, Theorem 4.2 shows that at least $1/\sqrt{\pi N}/2$ of all permutations are either equal to the strongest known construction for worst case, or are just one step away from it.

5. FURTHER DIRECTIONS

It is natural to ask if other special cases of Theorem 1.1 have applications in sorting algorithms. The answer is affirmative. A *block interchange* is like a block transposition except that the blocks that are swapped do not have to be consecutive. Then a theorem in [6] shows that $(N+1-\nu(C(u)))/2$ is *equal* to the block interchange distance of the permutation u of length n to the identity. It follows that if $C(u)$ has at least one even cycle, then the block interchange distance and the block transposition distance of u to the identity must be different. We hope to explore this fact in a subsequent paper.

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APPENDIX A.

Proof of Lemma 2.1. In a generic rim hook diagram T , the last rim hook is a hook diagram with a row and a column of sizes μ_1 and μ^1 , with $\mu_1 \leq \lambda_1$ and $\mu^1 \leq \lambda^1$. Define ℓ by $\ell+1 = \mu_1 + \mu^1$. All the preceding rim hooks in T are horizontal and vertical segments of the arm and of the leg of λ^* . Let us evaluate the total number of the rim hooks diagrams with h_r , the number of the horizontal hooks of size r ; so $h_r \in [0, \nu_r]$ for $r \neq \ell$ and $h_\ell \in [0, \nu_\ell - 1]$. The admissible $\mathbf{h} := \{h_r\}$ must meet the additional constraint

$$(29) \quad \sum_r h_r r + \mu_1 = \lambda_1.$$

If v_r is similarly defined as the total number of vertical hooks of size r , then the preceding constraint implies that $\sum_r v_r r + \mu^1 = \lambda^1$. The total number of the rim hook diagrams T , with parameters μ_1 , μ^1 , and $\mathbf{h}$, is

$$\binom{\nu_\ell - 1}{h_\ell} \prod_{r \neq \ell} \binom{\nu_r}{h_r} = \prod_r \binom{\nu_r - \delta_{\ell,r}}{h_r}, \quad \delta_{\ell,r} = \delta_{\ell,r} := \mathbb{I}(r = \ell).$$

We do not have to multiply the above product by the multinomial coefficient $\binom{\nu-1}{\nu_1, \dots, \nu_{\ell-1}, \nu_\ell-1, \nu_{\ell+1}, \dots}$ since the rim hooks are deleted in the fixed, increasing order of their lengths. Further, the μ^1 -long leg of the last (longest) hook contributes $\mu^1 - 1$ to the height T , while the total contribution of the vertical rim hooks is $(\lambda^1 - \mu^1)$, the sum of their sizes, minus $\sum_r (\nu_r - \delta_{\ell,r} - h_r)$, their total number. So

$$\begin{aligned} ht(T) &= (\mu^1 - 1) + (\lambda^1 - \mu^1) - \sum_r (\nu_r - \delta_{\ell,r} - h_r) \\ &\equiv \lambda^1 - 1 + \sum_r (\nu_r - \delta_{\ell,r} - h_r) \pmod{2}. \end{aligned}$$

Therefore the total contribution to $\chi^{\lambda^*}(s)$ from the rim hook diagrams with the last rim hook μ^* is

$$\begin{aligned} & (-1)^{\lambda^1-1} \sum_{\mathbf{h} \text{ meets (29)}} \prod_r (-1)^{\nu_r - \delta_{\ell,r} - h_r} \binom{\nu_r - \delta_{\ell,r}}{h_r} \\ &= (-1)^{\lambda^1-1} [\xi^{\lambda_1 - \mu_1}] \prod_r \sum_{h_r} (-1)^{\nu_r - \delta_{\ell,r} - h_r} (\xi^r)^{h_r} \binom{\nu_r - \delta_{\ell,r}}{h_r} \\ &= (-1)^{\lambda^1-1} [\xi^{\lambda_1 - \mu_1}] \prod_r (\xi^r - 1)^{\nu_r - \delta_{\ell,r}}. \end{aligned}$$

Since $\lambda_1 - \mu_1 \in [\lambda_1 - \min(\ell, \lambda_1), \lambda_1 - 1]$, we have

$$\begin{aligned} \chi^{\lambda^*}(s) &= (-1)^{\lambda^1-1} \sum_{t=\lambda_1 - \min(\ell, \lambda_1)}^{\lambda_1-1} [\xi^t] \prod_r (\xi^r - 1)^{\nu_r - \delta_{\ell,r}} \\ &= (-1)^{\lambda^1-1} [\xi^{\lambda_1-1}] \sum_{\ell} \left(\sum_{u=0}^{\min(\ell, \lambda_1)-1} \xi^u \right) \prod_r (\xi^r - 1)^{\nu_r - \delta_{\ell,r}} \\ &= (-1)^{\lambda^1-1} [\xi^{\lambda_1-1}] \sum_{\ell} \frac{\xi^{\min(\ell, \lambda_1)-1}}{(\xi-1)(\xi^{\ell}-1)} \prod_r (\xi^r - 1)^{\nu_r}. \end{aligned}$$

Here

$$\frac{\xi^{\min(\ell, \lambda_1)-1}}{(\xi-1)(\xi^{\ell}-1)} = \begin{cases} \frac{1}{\xi-1}, & \ell \leq \lambda_1, \\ \frac{\xi^{\lambda_1-1}-1}{(\xi-1)(\xi^{\ell}-1)}, & \ell > \lambda_1. \end{cases}$$

Since

$$\frac{\xi^{\lambda_1-1}-1}{(\xi-1)(\xi^{\ell}-1)} = \frac{1}{\xi-1} + \sum_{j \geq \lambda_1} c_j \xi^j,$$

we see that

$$\chi^{\lambda^*}(s) = (-1)^{\lambda^1-1} [\xi^{\lambda_1-1}] \frac{1}{\xi-1} \prod_r (\xi^r - 1)^{\nu_r}.$$

The proof is complete. $\square$

APPENDIX B.

Proof of (11). Using (6) and (10), we obtain

$$\begin{aligned} (N-1)! q_N(A) &= \frac{1}{N+1} \sum_{k=0}^N (-1)^{N-k-1} (N-k-1) k! \sum_{j \leq k} \frac{(-1)^j}{j!} \\ &= \frac{1}{N+1} \sum_{j=0}^N \frac{(-1)^j}{j!} \left[\sum_{j \leq k \leq N} (-1)^{N-k-1} (N-k-1) k! \right]. \end{aligned}$$

Here

$$\begin{aligned} & \sum_{j \leq k \leq N} (-1)^{N-k-1} (N-k-1) k! \\ &= N(-1)^{N-1} \sum_{j \leq k \leq N} (-1)^k k! - (-1)^N \sum_{j+1 \leq k \leq N+1} (-1)^k k! \\ &= (-1)^{N-1} \left[(N+1) \sum_{j \leq k \leq N} (-1)^k k! - (-1)^j j! + (-1)^{N+1} (N+1)! \right]. \end{aligned}$$

Therefore

$$\begin{aligned}
(N-1)!q_N(A) &= \frac{(-1)^{N-1}}{N+1} \sum_{j=0}^N \frac{(-1)^j}{j!} \\
&\quad \times \left[(N+1) \sum_{j \leq k \leq N} (-1)^k k! - (-1)^j j! + (-1)^{N+1} (N+1)! \right] \\
&= (-1)^{N-1} \left[\sum_{j=0}^N \frac{(-1)^j}{j!} \sum_{j \leq k \leq N} (-1)^k k! - 1 \right] + N! \sum_{j=0}^N \frac{(-1)^j}{j!} \\
&= (-1)^{N-1} \left[\sum_{j=0}^{N-1} \frac{(-1)^j}{j!} \sum_{j \leq k \leq N} (-1)^k k! \right] + N! \sum_{j=0}^N \frac{(-1)^j}{j!} \\
&= (-1)^{N-1} \left[\sum_{j=0}^{N-1} \frac{(-1)^j}{j!} \sum_{j \leq k \leq N-1} (-1)^k k! \right] - N! \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} + N! \sum_{j=0}^N \frac{(-1)^j}{j!} \\
&= (-1)^{N-1} \left[\sum_{j=0}^{N-1} \frac{(-1)^j}{j!} \sum_{j \leq k \leq N-1} (-1)^k k! - 1 \right] \\
&= (-1)^{N-1} \sum_{j=0}^{N-2} \frac{(-1)^j}{j!} \sum_{j \leq k \leq N-1} (-1)^k k!.
\end{aligned}$$

□

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