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A Pieri formula and a factorization formula for sums of K-theoretic k-Schur functions

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Abstract We give a Pieri-type formula for the sum of K-k-Schur functions $\sum_{\mu \leqslant \lambda} g^{(k)}_\mu$ over a principal order ideal of the poset of k-bounded partitions under the strong Bruhat order, whose sum we denote by $\widetilde{g}^{(k)}_\lambda$. As an application of this, we also give a k-rectangle factorization formula $\widetilde{g}^{(k)}_{R_t \cup \lambda} = \widetilde{g}^{(k)}_{R_t} \widetilde{g}^{(k)}_\lambda$ where $R_t = (t^{k+1-t})$, analogous to that of k-Schur functions $s^{(k)}_{R_t \cup \lambda} = s^{(k)}_{R_t} s^{(k)}_\lambda$.

1. Introduction

Let k be a positive integer. *K-k-Schur functions* $g^{(k)}_\lambda$ are inhomogeneous symmetric functions parametrized by k-bounded partitions λ, namely by the weakly decreasing strictly positive integer sequences $\lambda = (\lambda_1, \dots, \lambda_l)$, $l \in \mathbb{Z}_{\geqslant 0}$, whose terms are all bounded by k. They are K-theoretic analogues of another family of symmetric functions called *k-Schur functions* $s^{(k)}_\lambda$, which are homogeneous and also parametrized by k-bounded partitions. The set of k-bounded partitions is denoted by $\mathcal{P}_k$.

In this paper we give a Pieri-type formula for a certain sum of K-k-Schur functions (Theorem 1.3 and 1.4) and a factorization formula (Theorem 1.5) involving the *k-rectangle partitions* R_t defined later, mainly using combinatorial properties of the strong (Bruhat) and weak orderings on the affine symmetric groups.

Historically, k-Schur functions were first introduced by Lapointe, Lascoux and Morse [18], and subsequent studies led to several (conjecturally equivalent) characterizations of $s^{(k)}_\lambda$: Lapointe and Morse [21] gave the Pieri-type formula, and Lam [12] proved that k-Schur functions correspond to the Schubert basis of homology of the affine Grassmannian. Moreover, Lam and Shimozono [17] showed that k-Schur functions play a central role in the explicit description of the Peterson isomorphism.

These developments have analogues in K-theory. Lam, Schilling and Shimozono [15] characterized the K-theoretic k-Schur functions as the Schubert basis of the K-homology of the affine Grassmannian, and Morse [23] investigated them from a combinatorial viewpoint, giving various properties including Pieri-type formulas using affine set-valued strips (the form using cyclically decreasing words are also given in [15]). In this paper we start from this combinatorial characterization (see Definition 2.19).

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Among the k-bounded partitions, those of the form

$$(t^{k+1-t}) = (\underbrace{t,\dots,t}_{k+1-t})$$

for $1 \leqslant t \leqslant k$, called *$k$-rectangle* and denoted by R_t, play a special role. A notable property is the *k-rectangle factorization* for k-Schur functions [21, Theorem 40]: if a k-bounded partition has the form $R_t \cup \lambda$, where the symbol $\cup$ denotes the operation of concatenating the two sequences and reordering the terms in the weakly decreasing order, then the corresponding k-Schur function factorizes as follows:

$$s^{(k)}_{R_t\cup\lambda} = s^{(k)}_{R_t} s^{(k)}_{\lambda}. \tag{1}$$

It is natural to consider K-theoretic version of this formula. For several reasons below, in this regard it seems to make more sense to consider the sum of K-k-Schur functions $\sum_{\mu\leqslant\lambda} g^{(k)}_\mu$ rather than K-k-Schur function $g^{(k)}_\lambda$ (here $\leqslant$ denotes the *strong order*, also known as the *Bruhat order*, which is transferred from that of the affine symmetric group $\widetilde{S}_{k+1}$ through the bijection $\mathcal{P}_k \simeq \widetilde{S}_{k+1}/S_{k+1}$. See Sections 2.1.1 and 2.2.3 for the details):

Connection to K-Peterson isomorphism. The (original) Peterson isomorphism, first presented by Peterson in his lectures at MIT and then published by Lam and Shimozono [16], states that the homology of the affine Grassmannian is isomorphic to the quantum cohomology of the flag variety after appropriate localization. As its K-theoretic version, an isomorphism between the K-homology of the affine Grassmannian and the quantum K-theory of the flag manifold, up to appropriate localization, is conjectured and called *K-Peterson isomorphism*:

- In their attempt in [14] to verify the coincidence of the Schubert structure constants in the K-homology of the affine Grassmannian and the quantum K-theory of the flag manifold on torus-equivariant settings, Lam, Li, Mihalcea and Shimozono proved a special case of Theorem 1.5 for SL_2 (i.e. the case $k = 1$) with explicit calculations, in the context of geometry:

$$\mathcal{O}_x\mathcal{O}_{t_{-\alpha^\vee}} = \mathcal{O}_{xt_{-\alpha^\vee}}, \tag{2}$$

  where x is any affine Grassmannian element in the affine Weyl group, $\mathcal{O}_x$ is the Schubert class of structure sheaves on the affine Grassmannian and $t_{-\alpha^\vee}$ is the translation by the negative of the simple coroot of SL_2. (See also Remark 2.14.)
- In [9], Ikeda, Iwao and Maeno gave an explicit ring isomorphism, after appropriate localization, between the K-homology of the affine Grassmannian and the presentation of the quantum K-theory of the flag manifold that is conjectured by Kirillov and Maeno and proved by Anderson, Chen, and Tseng [1], as well as a conjectural description of the image of the quantum Grothendieck polynomials, which is conjectured to be the quantum Schubert classes. These presentations notably involve the dual stable Grothendieck polynomials g_{R_t} and their sum $\sum_{\mu\subset R_t} g_\mu$ corresponding to the k-rectangles R_t. Note that $\mu \subset R_t \iff \mu \leqslant R_t$, and that it is conjectured that $g^{(k)}_\lambda = g_\lambda$ for $\lambda \subset R_t$.

Remark 1.1. After this article was submitted, there appeared a preprint [10] by Syu Kato in which he claims to have proved conjectures in [14] and in particular the factorization property for the structure sheaves in general type.

Natural appearances of $\sum_{\mu\leqslant\lambda} g^{(k)}_\mu$ in k-rectangle factorization formulas of $g^{(k)}_\lambda$. It is suggested in [15, Remark 7.4] that the K-k-Schur functions should also possess similar properties to (1), including the divisibility of $g^{(k)}_{R_t\cup\lambda}$ by $g^{(k)}_{R_t}$, for which the author's preceding work [26, 27] gives an affirmative answer.

Let us review the results of [26, 27]. It is proved that $g^{(k)}_{R_t}$ divides $g^{(k)}_{R_t\cup\lambda}$ in the ring $\Lambda_{(k)} = \mathbb{Z}[h_1,\dots,h_k]$, of which the K-k-Schur functions $\{g^{(k)}_\mu\}_{\mu\in\mathcal{P}_k}$ form a basis. However, unlike (1), the quotient $g^{(k)}_{R_t\cup\lambda}/g^{(k)}_{R_t}$ is not a single term $g^{(k)}_\lambda$ but in general a linear combination of K-k-Schur functions with leading term $g^{(k)}_\lambda$: for any $\lambda\in\mathcal{P}_k$,

$$g^{(k)}_{R_t\cup\lambda} = g^{(k)}_{R_t}\left(g^{(k)}_\lambda + \sum_\mu a_{\lambda\mu} g^{(k)}_\mu\right), \tag{3}$$

summed over $\mu\in\mathcal{P}_k$ such that $|\mu|<|\lambda|$, with some coefficients $a_{\lambda\mu}$ depending on R_t. A special yet important case is the factorization of multiple k-rectangles: for $1\leqslant t\leqslant k$ and $a>1$,

$$g^{(k)}_{R_t^a} = g^{(k)}_{R_t}\left(\sum_{\mu\subset R_t} g^{(k)}_\mu\right)^{a-1},$$

where $R_t^a = R_t\cup\cdots\cup R_t$ (a times). Note that $\mu\subset R_t \iff \mu\leqslant R_t$. Furthermore, it is conjectured that the set of μ appearing in (3) forms an interval under the strong order: namely, for any $\lambda\in\mathcal{P}_k$ and $1\leqslant t\leqslant k$, we expect there to exist $\nu\in\mathcal{P}_k$ such that

$$g^{(k)}_{R_t\cup\lambda} = g^{(k)}_{R_t}\sum_{\nu\leqslant\mu\leqslant\lambda} g^{(k)}_\mu.$$

These observations suggest the usefulness of Definition 1.2 below.

1.1. Main results. Let $\leqslant$, $\leqslant_L$, and $\leqslant_R$ be the strong, left weak, and right weak order on $\widetilde{S}_{k+1}$ (see Section 2.1.1 for the details).

From the observation above, we consider and denote by $\widetilde{g}^{(k)}_\lambda$ the sum of K-k-Schur functions over the order ideal generated by λ under the strong order $\leqslant$:

Definition 1.2. *For any $\lambda\in\mathcal{P}_k$, we write*

$$\widetilde{g}^{(k)}_\lambda = \sum_{\mu\leqslant\lambda} g^{(k)}_\mu.$$

Our first main theorem is a Pieri-type formula for $\widetilde{g}^{(k)}_\lambda$. We start with the Pieri rule for $g^{(k)}_\lambda$ [15, 23]: for $\lambda\in\mathcal{P}_k$ and $1\leqslant r\leqslant k$,

$$g^{(k)}_\lambda h_r = \sum_{(A,\mu)} (-1)^{|\lambda|+r-|\mu|} g^{(k)}_\mu,$$

summed over affine set-valued strips $(\mu/\lambda, A)$ of size r (See Definition 2.19 for more details). In terms of $\widetilde{g}^{(k)}_\lambda$, this rule becomes relatively simple:

Theorem 1.3. *Let $\lambda\in\mathcal{P}_k$ and $1\leqslant r\leqslant k$, and define $\widetilde{h}_r = h_0+h_1+\cdots+h_r$. Then*

$$\widetilde{g}^{(k)}_\lambda \widetilde{h}_r = \sum_\mu g^{(k)}_\mu,$$

summed over $\mu\in\mathcal{P}_k$ such that $\mu\leqslant\kappa$ for some $\kappa\in\mathcal{P}_k$ such that κ/λ is a weak strip of size r.

To express its right-hand side as a linear combination of $\{\widetilde{g}^{(k)}_\mu\}_\mu$, we recall that a weak strip over λ corresponds to a proper subset of $I=\{0,1,\dots,k\}$: for $\kappa\in\mathcal{P}_k$, κ/λ is a weak strip if and only if there exists $A\subsetneq I$ such that $\kappa = d_A\lambda \geqslant_L \lambda$, where d_A

is the cyclically decreasing permutation corresponding to A (see Sections 2.2.2, 2.2.3, and 2.2.4 for the details).

Theorem 1.4. *With the setting in Theorem* 1.3*, we let* $d_{A_1}\lambda, d_{A_2}\lambda, \dots$ *be the list of weak strips of size* r *over* λ*. Then*

$$\widetilde{g}^{(k)}_\lambda \widetilde{h}_r = \sum_{m\geqslant 1}(-1)^{m-1}\sum_{a_1<\cdots<a_m}\widetilde{g}^{(k)}_{d_{A_{a_1}\cap\cdots\cap A_{a_m}}\lambda}.$$

$$\left(= \sum_a \widetilde{g}^{(k)}_{d_{A_a}\lambda} - \sum_{a<b}\widetilde{g}^{(k)}_{d_{A_a\cap A_b}\lambda} + \sum_{a<b<c}\widetilde{g}^{(k)}_{d_{A_a\cap A_b\cap A_c}\lambda} - \cdots \right)$$

(Moreover $d_{A_a\cap A_b\cap\dots}\lambda = (d_{A_a}\lambda)\wedge(d_{A_b}\lambda)\wedge\dots$*, where* $\wedge$ *denotes the meet in the poset* $\mathcal{P}_k$ *with the strong order. See also Proposition* 1.6*.)*

Our second main theorem is the k-rectangle factorization formula for $\widetilde{g}^{(k)}_\lambda$, which holds in the same form as that for k-Schur functions (1):

Theorem 1.5. *For any* $\lambda\in\mathcal{P}_k$ *and* $1\leqslant t\leqslant k$*, we have*

$$\widetilde{g}^{(k)}_{R_t\cup\lambda} = \widetilde{g}^{(k)}_{R_t}\widetilde{g}^{(k)}_\lambda.$$

To deduce Theorem 1.5 from Theorem 1.4 is easy and discussed in Section 6. The proof of Theorem 1.3 and 1.4, on the other hand, is the technical heart of this paper and requires auxiliary work on the strong and weak orderings on the set of affine permutations as well as the structure of the set of weak strips, which are discussed in Section 3 and 4.

This paper is organized as follows.

In Section 2, we review notations and facts on combinatorial backgrounds. In Section 2.1 we treat arbitrary Coxeter groups and their strong and weak orderings. It also contains quick reviews on the generalized quotients [4] and the Demazure products. Section 2.2 contains notations specific to the affine symmetric groups and a review on their Young-diagrammatic treatment. In Section 2.3 we briefly review the Pieri-type formulas for k-Schur and K-k-Schur functions.

Section 3 contains technical lemmas on the strong and weak orders on arbitrary Coxeter groups. In Section 3.1 the lattice property of the weak order is reviewed. Although it is known that the quotient of an affine Weyl group by its corresponding finite Weyl group forms a lattice under the weak order [28], we include another proof for the type affine A using the k-Schur functions. Section 3.2 contains basic properties of the Demazure and anti-Demazure actions. In Section 3.3 we show the existence of $\min_\leqslant\{z\in W \mid x\leqslant z\geqslant_L y\}$ and $\max_\leqslant\{z\in W\mid x\geqslant_L z\leqslant y\}$, analogous to the join and meet. In Section 3.4 we consider an "interval-flipping" map $\Phi_z\colon [e,z]_L \longrightarrow [e,z]_R;\ x\mapsto zx^{-1}$ and show that Φ_z is anti-isomorphic under the strong order and sends strong-meets (if exist) to strong-joins. In Section 3.5 we show the Chain Property of lower weak intervals, analogous to the Chain Property of the generalized quotients.

In Section 4, we focus on the affine symmetric groups and give results on the structure of the set of weak strips, which includes:

Proposition 1.6 ($\subset$ Proposition 4.2). *For any* $\lambda\in\mathcal{P}_k$ *and* $A,B\subsetneq I$ *with* $d_A\lambda/\lambda$ *and* $d_B\lambda/\lambda$ *are weak strips,*

(1) $d_{A\cap B}\lambda/\lambda$ *and* $d_{A\cup B}\lambda/\lambda$ *are weak strips.*
(2) $d_{A\cap B}\lambda = d_A\lambda\wedge d_B\lambda$ *under the strong order.*

Proposition 1.7 ($\subset$ Proposition 4.12). *For any* $\lambda\in\mathcal{P}_k$*, there exists* $i_\lambda\in I$ $(=\{0,1,\dots,k\})$ *such that* $i_\lambda\notin A$ *for any weak strip* $d_A\lambda/\lambda$*.*

Section 5 and 6 are devoted to proving the Pieri-type formula for $\widetilde{g}_\lambda^{(k)}$ (Theorem 1.3 and 1.4) and the k-rectangle factorization formula for $\widetilde{g}_\lambda^{(k)}$ (Theorem 1.5), respectively.

2. Preliminaries

In this section we review some requisite combinatorial background.

2.1. Coxeter groups. For basic definitions for the Coxeter groups we refer the reader to [2] or [8].

2.1.1. *Strong and weak orderings.* Let (W, S) be a Coxeter group and $T = \{wsw^{-1} \mid w \in W\}$ its set of reflections. The *left weak order* (or simply *left order*) $\leqslant_L$, *right weak order* (or *right order*) $\leqslant_R$, and *strong order* (or *Bruhat order*) $\leqslant$ on W are generated by the covering relations:

$$\begin{aligned} u \lessdot_L v &\iff l(v) = l(u) + 1,\ v = su \text{ for some } s \in S, \\ u \lessdot_R v &\iff l(v) = l(u) + 1,\ v = us \text{ for some } s \in S, \\ u \lessdot v &\iff l(v) = l(u) + 1,\ v = tu \text{ for some } t \in T. \end{aligned}$$

Note that the definition of the strong order looks different from but coincides with the classical one.

It is a few immediate observations that, for $u, v \in W$,

$$u \leqslant_L v \iff l(vu^{-1}) + l(u) = l(v), \tag{4}$$

$$u \leqslant_R v \iff l(u) + l(u^{-1}v) = l(v), \tag{5}$$

$$u \leqslant_R uv \iff l(u) + l(v) = l(uv) \iff v \leqslant_L uv. \tag{6}$$

We often use these equivalences without any mention. Using this translation from the weak order to length conditions, we can easily prove the following lemma:

Lemma 2.1. *For $x, y, z \in W$, we have*

1. $z \leqslant_L yz \leqslant_L xyz \iff y \leqslant_L xy$ *and* $z \leqslant_L xyz$.
2. $z \geqslant_L yz \geqslant_L xyz \iff y \leqslant_L xy$ *and* $z \geqslant_L xyz$.

We often use the following notation taken from [4]: for $w \in W$ we let $\langle w \rangle$ denote any reduced expression for w, and $\langle u \rangle \langle v \rangle$ the concatenation of reduced expressions for u and v. Hence, saying that $\langle u \rangle \langle v \rangle$ is reduced means $l(u) + l(v) = l(uv)$.

For $u, v \in W$ with $u \leqslant_L v$ the set $\{w \in W \mid u \leqslant_L w \leqslant_L v\}$ is called a *left interval* and denoted by $[u, v]_L$. We define *right interval* $[u, v]_R$ and *strong (or Bruhat) interval* $[u, v]$ similarly. We shall use the notation $[u, \infty)_L$ to denote the set $\{w \in W \mid u \leqslant_L w\}$, and define $[u, \infty)_R$ and $[u, \infty)$ similarly.

In this paper we heavily use some well-known results on the strong and weak orderings on Coxeter groups described below. See, for example, [2] for details. Let $v, w \in W$.

Strong Exchange Property. Suppose $w = s_1 s_2 \dots s_k$ $(s_i \in S)$ and $t \in T$. If $l(tw) < l(w)$, then $tw = s_1 \dots \widehat{s_i} \dots s_k$ for some $i \in [k]$. Furthermore, if $s_1 s_2 \dots s_k$ is a reduced expression then i is uniquely determined.

Subword Property. Let $w = s_1 s_2 \dots s_k$ be a reduced expression. Then $v \leqslant w$ if and only if there exists a reduced expression $v = s_{i_1} s_{i_2} \dots s_{i_l}$ with $1 \leqslant i_1 < i_2 < \dots < i_l \leqslant k$.

Chain Property(1). If $v \leqslant w$, then there exists a chain $v = x_0 \lessdot x_1 \lessdot \dots \lessdot x_k = w$.

Lifting Property (also known as *Z-property*). Let $s \in S$. If $sv > v$ and $sw > w$, then the following are equivalent: (1) $v \leqslant w$, (2) $v \leqslant sw$, and (3) $sv \leqslant sw$.

(1)With the definition of $\leqslant$ we employed here, this is trivial.

2.1.2. *Generalized quotients.* For $V \subset W$, let

$$W/V = \{w \in W \mid l(wv) = l(w) + l(v) \text{ for all } v \in V\}.$$

The subsets of the form W/V are called *(right) generalized quotients* [4]. Similarly the sets of the form

$$V\backslash W = \{w \in W \mid l(vw) = l(v) + l(w) \text{ for all } v \in V\}$$

are called *left generalized quotients*. Note that, when $V = W_J$, the parabolic subgroup corresponding to $J \subset I$, the generalized quotient W/W_J is just the parabolic quotient W^J.

It is shown in [4, Lemma 2.2] that if $a, b, v \in W$ satisfy $l(av) = l(a) + l(v)$ and $l(bv) = l(b) + l(v)$, then $av < bv \iff a < b$. An immediate consequence is

$$W/\{v\} \simeq [v, \infty)_L; w \mapsto wv \tag{7}$$

under both the strong and left weak order.

Chain Property for generalized quotients ([4, Corollary 3.5]). If $v, w \in W/V$ and $v < w$, then there exists a chain $v = x_0 \lessdot x_1 \lessdot \ldots \lessdot x_k = w$ with $x_i \in W/V$ for all i.

2.1.3. 0*-Hecke algebra and Demazure product.* The 0*-Hecke algebra* H associated to (W, S) is the associative algebra generated by $\{v_s \mid s \in S\}$ subject to the quadratic relations $v_s^2 = -v_s$ and the braid relations of (W, S), that is,

$$\underbrace{v_s v_t v_s \ldots}_{m} = \underbrace{v_t v_s v_t \ldots}_{m} \quad \text{for } s, t \in S \text{ with } \underbrace{sts\ldots}_{m} = \underbrace{tst\ldots}_{m}.$$

For $w \in W$ we can define without ambiguity $v_w \in H$ to be $v_{s_1} \ldots v_{s_n}$ where $s_1 \ldots s_n$ is any reduced expression for w. Furthermore, the elements $\{v_w \mid w \in W\}$ form a basis of H. The *Demazure product* (or *Hecke product*) $*$ on W describes the multiplication of basis elements in H: $x * y$ is such that $v_x v_y = \pm v_{x*y}$. Some properties on the Demazure product can be found on [5, 11].

We explicitly prepare the notation to denote the left multiplication in the Demazure product: for $s \in S$, we define the Demazure action $\phi_s \colon W \longrightarrow W$ by

$$\phi_s(x) = s * x = \begin{cases} x & (\text{if } x > sx) \\ sx & (\text{if } x < sx). \end{cases}$$

Similarly we define the anti-Demazure action $\psi_s \colon W \longrightarrow W$ by

$$\psi_s(x) = \begin{cases} sx & (\text{if } x > sx) \\ x & (\text{if } x < sx). \end{cases}$$

These maps $\{\phi_s\}_s$ and $\{\psi_s\}_s$ satisfy the quadratic relations $\phi_s^2 = \phi_s$, $\psi_s^2 = \psi_s$ and the braid relations of (W, S); a direct proof (found on [25, Proposition 2.1]) of this (for ψ) is that both $\psi_s \psi_t \psi_s \ldots$ and $\psi_t \psi_s \psi_t \ldots$ (m terms for each), where $sts \ldots = tst \ldots$ (m terms for each), send $x \in W$ to the shortest (resp. longest, when we consider ϕ) element of the parabolic coset $W_{\{s,t\}}x$. Therefore we can define without ambiguity $\phi_x, \psi_x : W \longrightarrow W$ for $x \in W$ by $\phi_x = \phi_{s_1} \ldots \phi_{s_n}$ and $\psi_x = \psi_{s_1} \ldots \psi_{s_n}$ where $x = s_1 \ldots s_n$ is any reduced expression. Similarly we define right Demazure and anti-Demazure actions $\phi_s^R, \psi_s^R \colon W \longrightarrow W$ for $s \in S$ by $\phi_s^R(x) = \phi_s(x^{-1})^{-1}$ and $\psi_s^R(x) = \psi_s(x^{-1})^{-1}$, that is, $\phi_s^R(x) = xs$ if $x < xs$, etc. We also define ϕ_x^R and ψ_x^R to be $\phi_{s_n}^R \ldots \phi_{s_1}^R$ and $\psi_{s_n}^R \ldots \psi_{s_1}^R$ (be careful for the order of composition) where $x = s_1 \ldots s_n$ is any reduced expression. Note that $\phi_x(y) = x * y = \phi_y^R(x)$. When S is indexed with a set I, i.e. $S = \{s_i \mid i \in I\}$, we often write $\phi_i = \phi_{s_i}$ and $\psi_i = \psi_{s_i}$.

The following lemma is essentially given in [4, Theorem 4.2], and explicitly in [5, Proposition 3.1 (e)]:

Lemma 2.2. *Let $x, y, z \in W$ with $x * y = z$, that is, $\phi_x(y) = z = \phi_y^R(x)$. Let $x' = zy^{-1}$ and $y' = x^{-1}z$, that is, $z = xy' = x'y$. Then we have*

(1) $x, x' \leqslant_R z$.
(2) $y, y' \leqslant_L z$.
(3) $l(z) = l(x) + l(y') = l(x') + l(y)$.
(4) $x' \leqslant x$.
(5) $y' \leqslant y$.

Proof. It follows easily from the definition of $*$ and the Subword Property. □

The proof of the following lemma is easy and similar to that of Lemma 2.2.

Lemma 2.3. *Let $x, y, z \in W$ with $\psi_x(y) = z$. Let $x' = zy^{-1}$, that is, $z = x'y$. Then we have*

(1) $x' \leqslant x$.
(2) $z \leqslant_L y$.
(3) $x'^{-1} \leqslant_R y$.

We see more properties of ϕ_x, ψ_x in Section 3.2.

2.2. Affine symmetric groups. In this section we briefly review the connection between affine permutations, bounded partitions and core partitions. We refer the reader to [13, Chapter 2] and [6] for the details.

Hereafter we fix a positive integer k.

2.2.1. *Affine symmetric group.* Let $I = \mathbb{Z}_{k+1} = \{0, \dots, k\}$. Let $[p, q] = \{p, p+1, \dots, q-1, q\} \subsetneq I$ for $p \neq q-1$. For example, $[4, 2] = \{4, 5, 0, 1, 2\}$ where $k = 5$. A subset $A \subset I$ is called *connected* if $A = [p, q]$ for some p, q. A *connected component* of $A \subsetneq I$ is a maximal connected subset of A.

The *affine symmetric group* $\widetilde{S}_{k+1}$ is a group generated by the generators $\{s_i \mid i \in I\}$ subject to the relations $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_i$ for $i - j \not\equiv 0, \pm 1$, with all indices considered mod $(k+1)$. We often write $s_{ij\dots}$ instead of $s_i s_j \cdots$. The parabolic quotient $\widetilde{S}_{k+1}/S_{k+1}$, where S_{k+1} is the symmetric group $\langle s_1, \dots, s_k \rangle$ as a subgroup of $\widetilde{S}_{k+1}$, is denoted by $\widetilde{S}_{k+1}^{\circ}$ and its elements are called *affine Grassmannian elements.*

For $x \in \widetilde{S}_{k+1}$, the set of *right descents* $D_R(x)$ is $\{i \in I \mid x > xs_i\}$ ($\subsetneq I$). The set of *left descents* $D_L(x)$ is defined similarly. For $i \in I$, an element $w \in \widetilde{S}_{k+1}$ is called *i-dominant* if $D_R(w) \subset \{i\}$. Note that an affine permutation is 0-dominant if and only if it is affine Grassmannian.

2.2.2. *Cyclically decreasing elements.* A word $a = a_1 a_2 \dots a_m$ with letters from I is called *cyclically decreasing* (resp. *cyclically increasing*) if $a_1, a_2, \dots, a_m$ are distinct and any $j \in I$ does not precede $j+1$ (resp. $j-1$) in a. For $A \subsetneq I$, the *cyclically decreasing element* d_A is defined to be $s_{i_1} s_{i_2} \dots s_{i_m}$ where $A = \{i_1, i_2, \dots, i_m\}$ and the word $i_1 i_2 \dots i_m$ is cyclically decreasing. The *cyclically increasing element* $u_A = s_{i_m} s_{i_{m-1}} \dots s_{i_1}$ is defined similarly. Note that these definitions are independent of the choice of the word.

Example 2.4. Let $k = 5$ and $A = \{0, 1, 3, 5\} \subsetneq \mathbb{Z}_6$. The possible cyclically decreasing words for A are 1053, 1035, 1305 and 3105, and hence $d_A = s_1 s_0 s_5 s_3 = s_1 s_0 s_3 s_5 = s_1 s_3 s_0 s_5 = s_3 s_1 s_0 s_5$.

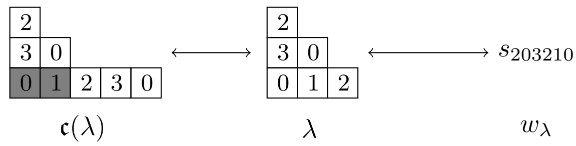


Figure 1. $k=3$, $\lambda=(3,2,1)\in\mathcal{P}_3$, $\mathfrak{c}(\lambda)=(5,2,1)\in\mathcal{C}_4$, and $w_\lambda=s_{203210}\in\widetilde{S}_4^\circ$.

2.2.3. *Connection to bounded partitions and core partitions.* In this section we review the bijection between the set of k-bounded partitions, $k+1$-core partitions and affine Grassmannian elements in $\widetilde{S}_{k+1}$. For the details see [13, Chapter 2] and references given there.

A partition λ is called *k-bounded* if $\lambda_1\leqslant k$. Let $\mathcal{P}_k$ be the set of all k-bounded partitions. An *r-core* (or simply a *core* if no confusion can arise) is a partition none of whose cells have a hook length equal to r. We denote by $\mathcal{C}_r$ the set of all r-core partitions.

Now we recall the bijection

$$\mathcal{C}_{k+1}\simeq\mathcal{P}_k\simeq\widetilde{S}_{k+1}^\circ. \tag{8}$$

The map $\mathfrak{p}\colon\mathcal{C}_{k+1}\longrightarrow\mathcal{P}_k;\kappa\mapsto\lambda$ is defined by

$$\lambda_i=\#\{j\mid(i,j)\in\kappa,\ \mathrm{hook}_{(i,j)}(\kappa)\leqslant k\}.$$

In fact $\mathfrak{p}$ is bijective and the inverse map $\mathfrak{c}=\mathfrak{p}^{-1}\colon\mathcal{P}_k\longrightarrow\mathcal{C}_{k+1}$ is algorithmically described as a "sliding cells" procedure.

The map $\mathfrak{s}\colon\widetilde{S}_{k+1}^\circ\longrightarrow\mathcal{C}_{k+1}$ is constructed via an action of $\widetilde{S}_{k+1}$ on $\mathcal{C}_{k+1}$: for $\kappa\in\mathcal{C}_{k+1}$ and $i\in I$, we define $s_i\cdot\kappa$ to be κ with all its addable (resp. removable) corners with residue i added (resp. removed), where the *residue* of a cell (i,j) is $j-i$ mod $k+1$. In fact this gives a well-defined $\widetilde{S}_{k+1}$-action on $\mathcal{C}_{k+1}$, which induces the bijection $\mathfrak{s}:\widetilde{S}_{k+1}^\circ\longrightarrow\mathcal{C}_{k+1};w\mapsto w\cdot\varnothing$.

The map $\mathcal{P}_k\longrightarrow\widetilde{S}_{k+1}^\circ;\lambda\mapsto w_\lambda$ is given by $w_\lambda=s_{i_1}s_{i_2}\dots s_{i_l}$, where $(i_1,i_2,\dots,i_l)$ is the sequence obtained by reading the residues of the cells in λ, from the shortest row to the largest, and within each row from right to left. See [20, Corollary 48] for the proof.

For $\lambda\in\mathcal{P}_k$, the *k-transpose* of λ is $\mathfrak{p}(\mathfrak{c}(\lambda)')$ and denoted by λ^{ω_k}. (Here μ' denotes the transpose of a partition μ.)

Example 2.5. Let $k=3$ and $\lambda=(3,2,1)\in\mathcal{P}_3$. The corresponding 4-core partition and affine permutation are $\mathfrak{c}(\lambda)=(5,2,1)\in\mathcal{C}_4$ and $w_\lambda=s_{203210}\in\widetilde{S}_4^\circ$. (See Figure 1.)

2.2.4. *Weak strips.*

Definition 2.6. *For $v,w\in\widetilde{S}_{k+1}^\circ$, we say v/w is a* weak strip *(or* affine strip*) of size r if $v=d_Aw\geqslant_L w$ for some $A\subsetneq I$ with $|A|=r$. We also say v is a weak strip of size r over w.*

Definition 2.7. *For $v,w\in\widetilde{S}_{k+1}^\circ$ and $A\subsetneq I$, we say $(v/w,A)$ is an* affine set-valued strip *of size r if $v=d_A*w\,(=\phi_{d_A}(w))$ and $|A|=r$. We also say (v,A) is an affine set-valued strip of size r over w.*

Note that if $(v/w,A)$ is an affine set-valued strip of size r then v/w is an affine strip of size $\leqslant r$.

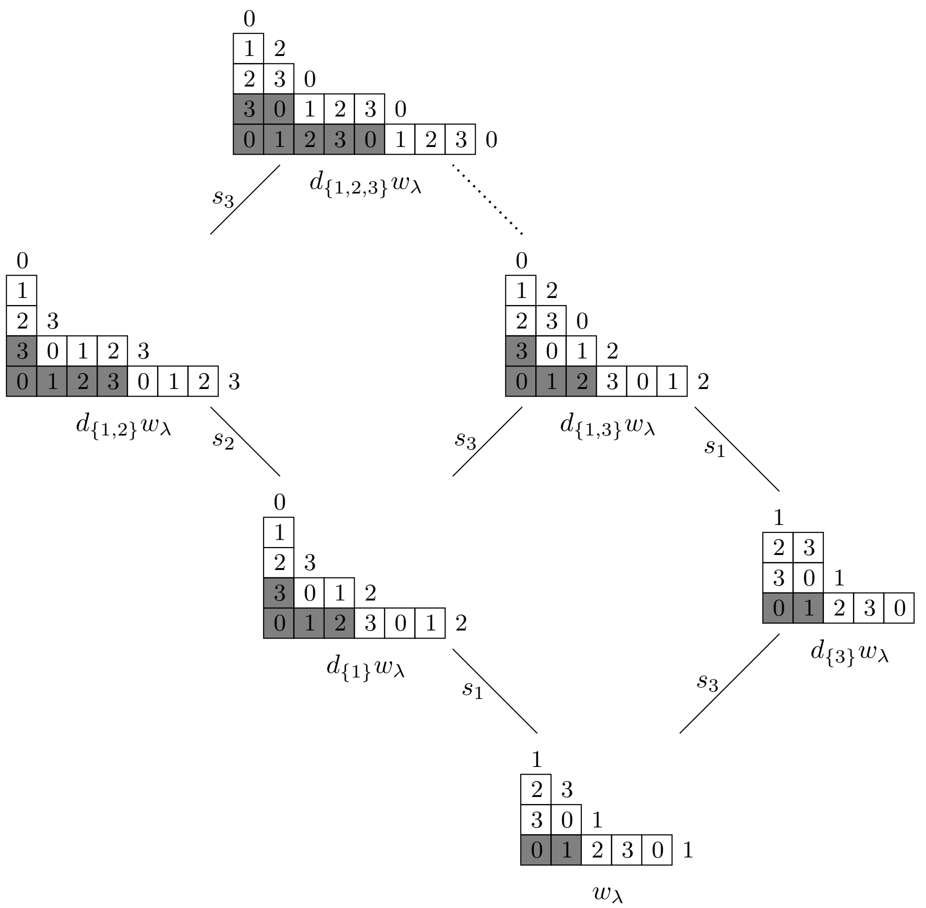


Figure 2. The weak strips over w_λ where $\lambda = (3, 2, 1)$. Left weak covers are represented as solid lines, and strong covers are solid or dotted lines. A solid edge between v and w is labelled with s_i if $v = s_i w$.

Remark 2.8. Identifying λ, $\mathfrak{c}(\lambda)$ and w_λ through the bijection $\mathcal{P}_k \simeq \mathcal{C}_{k+1} \simeq \widetilde{S}^\circ_{k+1}$, we often say μ/λ (resp. κ/γ) is a weak strip for $\lambda, \mu \in \mathcal{P}_k$ (resp. $\kappa, \gamma \in \mathcal{C}_{k+1}$), etc.

Remark 2.9. Regarding $v, w \in \widetilde{S}^\circ_{k+1}$ as bounded (or core) partitions as above, we see these notions are variants of the horizontal strip. For example, w_μ/w_λ is a weak strip if and only if the corresponding cores $\mathfrak{c}(\mu)/\mathfrak{c}(\lambda)$ form a horizontal strip and $w_\mu \geqslant_L w_\lambda$, and the term "affine set-valued" originates in affine set-valued tableaux. See, for example, [13, 23] for more details.

Example 2.10. Let $k = 3$ and $\lambda = (3, 2, 1) \in \mathcal{P}_3$, and thus $w_\lambda = s_{203210}$ and $\mathfrak{c}(\lambda) = (5, 2, 1)$. Figure 2 lists all v such that v/w_λ is a weak strip (the corresponding core partitions are displayed).

2.2.5. *k-code.* The content of this section is mostly cited from [6].

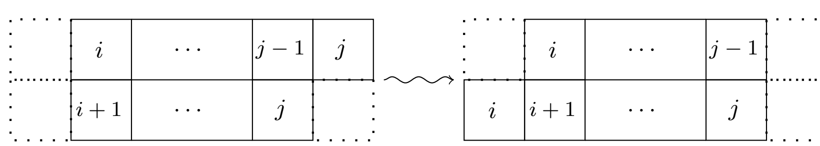
A *k-code* is a function $\alpha : I \longrightarrow \mathbb{Z}_{\geqslant 0}$ such that there exists at least one $i \in I$ with $\alpha(i) = 0$. We often write $\alpha_i = \alpha(i)$. The *diagram* of a k-code α is the Ferrers diagram on a cylinder with $k + 1$ columns indexed by I, where the i-th column contains α_i

boxes. A k-code α may be identified with its *filling*, which is the diagram of α with all its boxes marked with their residues, that is, $i-j$ ($\in I$) for one in the i-th column and j-th row.

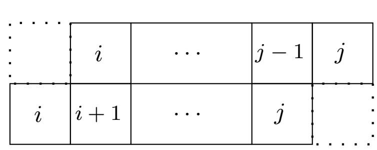
A *flattening* of the diagram of a k-code α is what is obtained by cutting out a column with no boxes (that is, column j with $\alpha_j = 0$). A *reading word* of α is obtained by reading the rows of a flattening of α from right to left, beginning with the last row. Note that, though a k-code may have multiple columns with no boxes, the affine permutation given by the reading word of α is independent of the choice of a flattening. Indeed, for a k-code α with m rows, letting A_i be the set of the residues of the boxes in the i-th row in the diagram of α, we have that $d_{A_m} \cdots d_{A_2} d_{A_1}$ is the affine permutation corresponding to α. In fact this correspondence is bijective (Theorem 2.11); an algorithm to obtain a k-code from an affine permutation is explained below.

Maximizing moves. For a cyclically decreasing decomposition $w = d_{A_m} \cdots d_{A_1}$, there corresponds a "skew k-code diagram", that is, a set of boxes in the cylinder with $k+1$ columns indexed by I for which A_i is the set of the residues of the boxes in the i-th row. To justify it to the bottom, we consider the following "two-row move": pick any consecutive two rows A_a and A_{a+1}, and let $i, j \in I$ with $j \neq i-1$. Then,

(1) if $i-1 \notin A_{a+1}$, $[i,j] \subset A_{a+1}$, $[i+1,j] \subset A_a$, and $i, j+1 \notin A_a$, then we replace A_a and A_{a+1} with $A_a \cup \{i\}$ and $A_{a+1} \setminus \{j\}$, reflecting the equation $(s_j s_{j-1} \dots s_i)(s_j \dots s_{i+1}) = (s_{j-1} \dots s_i)(s_j \dots s_{i+1} s_i)$.

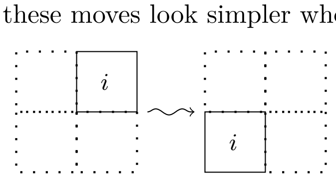


(2) if $i-1 \notin A_{a+1}$, $[i,j] \subset A_{a+1}$, $[i,j] \subset A_a$, and $j+1 \notin A_a$, then we conclude this decomposition does not give a reduced expression, reflecting the fact that $(s_j s_{j-1} \dots s_i)(s_j \dots s_{i+1} s_i)$ is not a reduced expression.



: not reduced

Note that these moves look simpler when $i = j$:

(1) 

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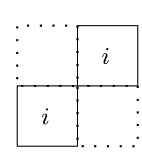
(2) 

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It is shown in [6, Section 3] that, for any decomposition $w = d_{A_m} \cdots d_{A_1}$ that gives a reduced expression, we can apply a finite series of moves of type (1) to justify its diagram to the bottom and obtain a k-code, which is in fact uniquely determined from w and denoted by $\mathrm{RD}(w)$, and gives the *maximal decreasing decomposition* $w = d_{B_n} \cdots d_{B_1}$, that is, the vector $(|B_1|, \dots, |B_n|)$ is maximal in the lexicographical order among such decompositions for w. Furthermore, this procedure bijectively maps affine permutations to k-codes:

Theorem 2.11 ([6, Theorem 38]). *The map $w \mapsto \mathrm{RD}(w)$ gives a bijection between $\widetilde{S}_{k+1}$ and the set of k-codes.*

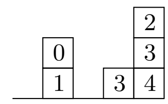


Figure 3. $\mathrm{RD}(w)$ where $k=3$ and $w=s_2s_{30}s_{431}$

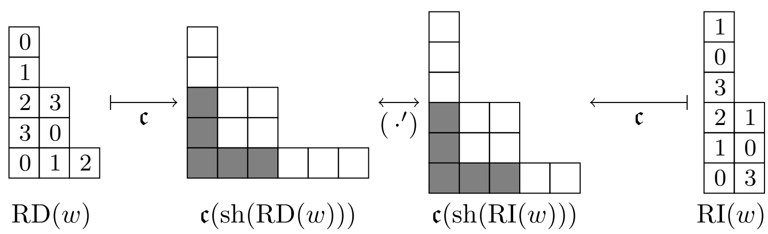


Figure 4.

Example 2.12. Let $k=3$ and $w=s_2s_{30}s_{431}$ (this expression gives the maximal decreasing decomposition). Then $\mathrm{RD}(w)=(0,2,0,1,3)$. (See Figure 3)

Note that this construction also works if maximal decreasing decomposition is replaced with *maximal increasing decompositions*, that is, the unique decomposition $w=u_{B_n}\cdots u_{B_1}$ into cyclically increasing elements with the vector $(|B_1|,\dots,|B_n|)$ being maximal in the lexicographical order, by modifying the notion of the filling of a k-code so that the box in the i-th column and j-th row is marked with $j-i$ instead of $i-j$. The resulting k-code is denoted by $\mathrm{RI}(w)$. The map $w\mapsto\mathrm{RI}(w)$ also gives a bijection between $\widetilde{S}_{k+1}$ and the set of k-codes.

It is proved [6, Corollary 39] that $w\in\widetilde{S}_{k+1}$ is i-dominant if and only if the flattening of the corresponding k-code $\mathrm{RD}(w)$ forms a k-bounded partition with residue i in its lower left box, that is, $\mathrm{RD}(w)_i\geqslant\mathrm{RD}(w)_{i+1}\geqslant\cdots\geqslant\mathrm{RD}(w)_{i-2}\geqslant\mathrm{RD}(w)_{i-1}=0$. When $i=0$, this mapping from 0-dominant permutations to k-bounded partitions coincides with the one described earlier in Section 2.2.3. Moreover, it is proved [6, Proposition 51] that, for $w\in W^\circ$ the two corresponding k-codes $\mathrm{RD}(w)$ and $\mathrm{RI}(w)$, regarded as k-bounded partitions, are transformed into each other by taking k-transpose: $\mathrm{sh}(\mathrm{RI}(w))=(\mathrm{sh}(\mathrm{RD}(w)))^{\omega_k}$ where $\mathrm{sh}(\alpha)\in\mathcal{P}_k$ is defined by $\mathrm{sh}(\alpha)_j=|\{i\mid\alpha_i\geqslant j\}|$.

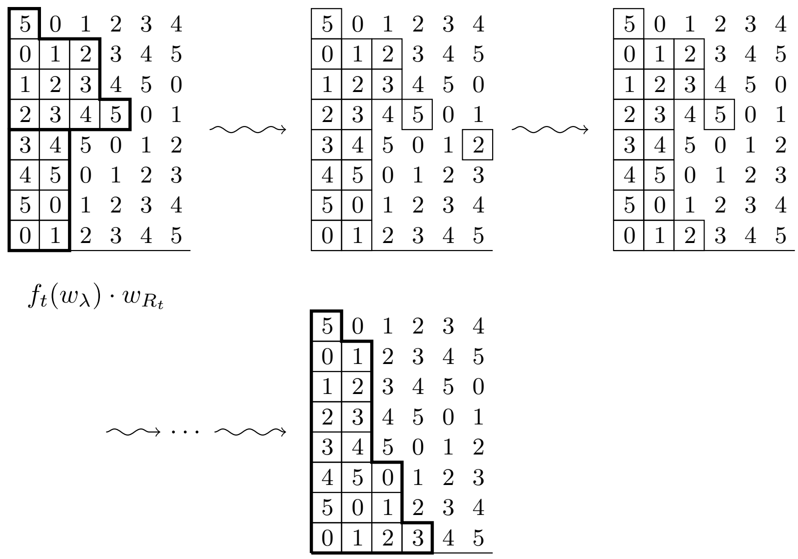
It is also proved in [6, Proposition 56] that if $x\leqslant_L y$ then $\mathrm{RD}(x)\subset\mathrm{RD}(y)$ and $\mathrm{RI}(x)\subset\mathrm{RI}(y)$.

Example 2.13. Let $k=3$ and $w=s_0s_1s_{32}s_{03}s_{210}=s_1s_0s_3s_{12}s_{01}s_{30}$ (these presentations give the maximal decreasing and increasing decompositions). Then $\mathrm{RD}(w)=(5,3,1,0)$ and $\mathrm{RI}(w)=(6,3,0,0)$, and thus $\mathrm{sh}(\mathrm{RD}(w))=(3,2,2,1,1)=(2,2,2,1,1,1)^{\omega_3}=\mathrm{sh}(\mathrm{RI}(w))^{\omega_3}$. (See Figure 4)

2.2.6. *k-rectangles.* The partition $(t^{k+1-t})=(t,t,\dots,t)\in\mathcal{P}_k$, for $1\leqslant t\leqslant k$, is denoted by R_t and called a *k-rectangle*.

Remark 2.14. Consider the affine permutation w_{R_i} corresponding to the k-rectangle R_i under the bijection (8). In fact w_{R_i} is congruent, in the extended affine Weyl group, to the translation $t_{-\varpi_i^\vee}$ by the negative of a fundamental coweight, modulo left multiplication by the length zero elements.

The next lemma describes the mapping $\lambda\mapsto R_t\cup\lambda$ in terms of affine permutations. For $A\subset I$ and $0\leqslant t\leqslant k$, we write $A+t=\{a+t\mid a\in A\}$ $(\subset I)$.



$f_t(w_\lambda) \cdot w_{R_t}$

$w_{R_t \cup \lambda}$

Figure 5. Justifying process with maximizing moves, where $k = 5$, $t = 2$, $R_2 = (2^4)$, and $\lambda = (4, 3, 3, 1)$.

Lemma 2.15. *Let $1 \leqslant t \leqslant k$. Define a group isomorphism*

$$f_t \colon \widetilde{S}_{k+1} \longrightarrow \widetilde{S}_{k+1} \ ; \ s_i \mapsto s_{i+t} \quad \text{for } i \in I.$$

For any $\lambda \in \mathcal{P}_k$, we have

$$w_{R_t \cup \lambda} = f_t(w_\lambda) w_{R_t}.$$

Proof. Let $d_{A_m} \cdots d_{A_1}$ and $d_{B_{k+1-t}} \cdots d_{B_1}$ be the maximal decreasing decompositions of w_λ and w_{R_t}. Then $d_{A_m+t} \cdots d_{A_1+t}$ is the maximal decomposition of $f_t(w_\lambda)$. Stacking the k-code diagram of $f_t(w_\lambda)$ on that of w_{R_t}, we obtain the diagram (not necessarily justified to the bottom) corresponding to the (not necessarily maximal) decreasing decomposition $f_t(w_\lambda) w_{R_t} = d_{A_m+t} \cdots d_{A_1+t} d_{B_{k+1-t}} \cdots d_{B_1}$ (See Figure 5). With maximizing moves, we can justify the diagram to obtain one with shape $R_t \cup \lambda$, which corresponds to the maximal decomposition of $w_{R_t \cup \lambda}$. $\square$

The next lemma explains the correspondence between weak strips over λ and weak strips over $R_t \cup \lambda$.

Lemma 2.16. *Let $\lambda \in \mathcal{P}_k$.*

(1) *For $A \subsetneq I$, if $d_A\lambda/\lambda$ is a weak strip then $R_t \cup (d_A\lambda) = d_{A+t}(R_t \cup \lambda)$.*

Moreover, let $d_{A_1}\lambda, d_{A_2}\lambda, \ldots$ be the list of all weak strips over λ (of size r).

(2) *$R_t \cup (d_{A_1}\lambda), R_t \cup (d_{A_2}\lambda), \ldots$ is the list of all weak strips over $R_t \cup \lambda$ (of size r).*

(3) *$d_{A_1+t}(R_t \cup \lambda), d_{A_2+t}(R_t \cup \lambda), \ldots$ is the list of all weak strips over $R_t \cup \lambda$ (of size r).*

Proof. (2) is [19, Theorem 20]. (3) follows from (1) and (2).

To prove (1), it suffices to show the case $|A| = 1$, that is, $R_t \cup (s_i\lambda) = s_{i+t}(R_t \cup \lambda)$. This is essentially shown in the process of proving [19, Theorem 20] by seeing correspondence between addable corners of $\mathfrak{c}(\lambda)$ with residue i and addable corners of $\mathfrak{c}(R_t \cup \lambda)$ with residue $i + t$, yet we here give another proof: by Lemma 2.15, it follows $w_{R_t \cup (s_i\lambda)} = f_t(w_{s_i\lambda}) w_{R_t} = f_t(s_i w_\lambda) w_{R_t} = s_{i+t} f_t(w_\lambda) w_{R_t} = s_{i+t} w_{R_t \cup \lambda}$. $\square$

2.3. Symmetric functions. For basic definitions for symmetric functions, see for instance [22, Chapter I].

2.3.1. *Symmetric functions.* Let $\Lambda = \mathbb{Z}[h_1, h_2, \dots]$ be the ring of symmetric functions, generated by the *complete symmetric functions* $h_r = \sum_{i_1 \leqslant i_2 \leqslant \dots \leqslant i_r} x_{i_1} \dots x_{i_r}$. For a partition λ we set $h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_{l(\lambda)}}$. The set $\{h_\lambda\}_{\lambda \in \mathcal{P}}$ forms a $\mathbb{Z}$-basis of Λ.

2.3.2. *Schur functions.* The *Schur functions* $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ are the family of symmetric functions satisfying the *Pieri rule*:

$$h_r s_\lambda = \sum_{\mu/\lambda:\text{horizontal strip of size } r} s_\mu.$$

2.3.3. *k-Schur functions.* We recall a characterization of k-Schur functions given in [21], since it is a model for and has a relationship with K-k-Schur functions.

Definition 2.17 (k-Schur function via k-Pieri rule). *k-Schur functions $\{s_w^{(k)}\}_{w \in \widetilde{S}_{k+1}^\circ}$ are the family of symmetric functions such that*

$$s_e^{(k)} = 1,$$
$$h_r s_w^{(k)} = \sum_v s_v^{(k)} \quad \text{for } 0 \leqslant r \leqslant k \text{ and } w \in \widetilde{S}_{k+1}^\circ,$$

summed over $v \in \widetilde{S}_{k+1}^\circ$ such that v/w is a weak strip of size r.

It is known that $\{s_w^{(k)}\}_{w \in \widetilde{S}_{k+1}^\circ}$ forms a basis of $\Lambda_{(k)} = \mathbb{Z}[h_1, \dots, h_k] \subset \Lambda$, and $s_w^{(k)}$ is homogeneous of degree $l(w)$. We regard $s_\lambda^{(k)}$ as $s_{w_\lambda}^{(k)}$ for $\lambda \in \mathcal{P}_k$. It is proved in [21, Theorem 40] that

Proposition 2.18 (k-rectangle property). *For $1 \leqslant t \leqslant k$ and $\lambda \in \mathcal{P}_k$, we have $s_{R_t \cup \lambda}^{(k)} = s_{R_t}^{(k)} s_\lambda^{(k)} (= s_{R_t} s_\lambda^{(k)})$.*

2.3.4. *K-k-Schur functions.* In this paper we employ the following characterization with the Pieri rule ([15, Corollary 7.6], [23, Corollary 50]) of the K-k-Schur function as its definition.

Definition 2.19 (K-k-Schur function via K-k-Pieri rule). *K-k-Schur functions $\{g_w^{(k)}\}_{w \in \widetilde{S}_{k+1}^\circ}$ are the family of symmetric functions such that $g_e^{(k)} = 1$ and*

$$h_r \cdot g_w^{(k)} = \sum_{(A,v)} (-1)^{r + l(w) - l(v)} g_v^{(k)},$$

for $w \in \widetilde{S}_{k+1}^\circ$ and $0 \leqslant r \leqslant k$, summed over $v \in \widetilde{S}_{k+1}^\circ$ and $A \subsetneq I$ such that $(v/w, A)$ is an affine set-valued strip of size r.

It is known that $\{g_w^{(k)}\}_{w \in \widetilde{S}_{k+1}^\circ}$ forms a basis of $\Lambda_{(k)}$. Besides, though $g_w^{(k)}$ is an inhomogeneous symmetric function in general, the degree of $g_w^{(k)}$ is $l(w)$ and its homogeneous part of highest degree is equal to $s_w^{(k)}$. In this paper, for $f = \sum_w c_w g_w^{(k)} \in \Lambda_{(k)}$ we write $[g_v^{(k)}](f) = c_v$.

3. Properties of the strong and weak orderings on Coxeter groups

In this section we let (W,S) be an arbitrary Coxeter group.

Recall that for a poset $(P,\leqslant)$ and a subset $A\subset P$, if the set $\{z\in P\mid z\leqslant y \text{ for any } y\in A\}$ has the maximum element z_0 then z_0 is called the *meet* of A and denoted by $\bigwedge A$, and if $\{z\in P\mid z\geqslant y \text{ for any } y\in A\}$ has the minimum element then it is called the *join* of A and denoted by $\bigvee A$. When $A=\{x,y\}$, its meet and join are simply called the meet and join of x and y, and denoted by $x\wedge y$ and $x\vee y$. A poset for which any nonempty subset has the meet is called a *complete meet-semilattice*. A poset for which any two elements have the meet and join is called a *lattice*. A subset of a complete meet-semilattice has the join if it has a common upper bound, since the join is the meet of all its common upper bounds then.

In this paper we denote the meet of $x,y\in W$ under the strong (resp. left, right) order by $x\wedge y$ (resp. $x\wedge_L y$, $x\wedge_R y$) and call it the *strong meet* (resp. *left meet*, *right meet*) of $\{x,y\}$. We define $x\vee y$, $x\vee_L y$ and $x\vee_R y$ similarly.

3.1. Lattice property of the weak order.

It is known that the weak order on any Coxeter group or its parabolic quotient forms complete meet-semilattices (see, for example, [2, Theorem 3.2.1]). The join of two elements in them, however, does not always exist, but it is known that the quotient of an affine Weyl group by its corresponding finite Weyl group forms a lattice under the weak order [28]. We here include another proof for the type affine A case for the sake of completeness.

Lemma 3.1. *For any $v,w\in\widetilde{S}^\circ_{k+1}$, their join $v\vee_L w$ under the left weak order exists.*

Proof. Since $\widetilde{S}^\circ_{k+1}$ is a complete meet-semilattice, it remains to show the existence of a common upper bound of v and w under the left order. Let $s^{(k)}_v$ and $s^{(k)}_w$ denote the k-Schur functions corresponding to v and w. In the expansion of their product in the k-Schur function basis $s^{(k)}_v s^{(k)}_w=\sum_u c^u_{vw}s^{(k)}_u$, every u appearing in the right-hand side satisfies $w\leqslant_L u$ because $s^{(k)}_v$ can be written as a polynomial in $h_1,\dots,h_k$ and by the Pieri rule $h_i s^{(k)}_x$ is in general a linear combination of $s^{(k)}_y$ with $y\geqslant_L x$. By the same reason we have $v\leqslant_L u$. □

We proved the following corollary in the proof of the lemma above:

Corollary 3.2. *For any $v,w\in\widetilde{S}^\circ_{k+1}$, every u appearing with a nonzero coefficient in the right-hand side of $s^{(k)}_v s^{(k)}_w=\sum_u c^u_{vw}s^{(k)}_u$ satisfies $u\geqslant_L v\vee_L w$.*

With the K-k-Pieri rule instead of the k-Pieri in hand, the same holds for the K-k-Schur functions:

Corollary 3.3. *For any $v,w\in\widetilde{S}^\circ_{k+1}$, every u appearing with a nonzero coefficient in the right-hand side of $g^{(k)}_v g^{(k)}_w=\sum_u d^u_{vw}g^{(k)}_u$ satisfies $u\geqslant_L v\vee_L w$.*

3.2. Properties of Demazure and anti-Demazure actions.

Lemma 3.4. *Let $x\in W$ and ϕ_x,ψ_x be the Demazure and anti-Demazure actions defined in Section* 2.1.3.

1. *$\phi_x(w)\geqslant_L w$ and $\psi_x(w)\leqslant_L w$ for any $w\in W$.*
2. *ϕ_x and ψ_x are order-preserving under $\leqslant$. Namely, if $v\leqslant w$ then $\phi_x(v)\leqslant\phi_x(w)$ and $\psi_x(v)\leqslant\psi_x(w)$.*
3. *For any $y\in W$, the map $(x\mapsto\phi_x(y))$ is order-preserving and the map $(x\mapsto\psi_x(y))$ is order-reversing under $\leqslant$.*
4. *$\phi_x\psi_{x^{-1}}(y)\geqslant y$ and $\psi_{x^{-1}}\phi_x(y)\leqslant y$ for any $y\in W$.*

(5) *ϕ_x preserves strong meets and ψ_x preserves strong joins. Namely, for $v, w \in W$,*
   (a) *if $v \wedge w$ exists then $\phi_x(v) \wedge \phi_x(w)$ exists and equals to $\phi_x(v \wedge w)$.*
   (b) *if $v \vee w$ exists then $\psi_x(v) \vee \psi_x(w)$ exists and equals to $\psi_x(v \vee w)$.*

Remark 3.5. This lemma also works for ϕ_x^R and ψ_x^R instead of ϕ_x and ψ_x.

Remark 3.6. For the statements on ϕ_x, (1) of this lemma is done in [5, Proposition 3.1 (d)]; (2) and (3) in [5, Proposition 3.1 (c)].

Proof. (1) is clear from the definition of ϕ_s and ψ_s. (2) is from the Lifting Property. (3) is clear from (1) and the Subword Property. For (4), the case $x = s \in S$ is clear from the definition of ϕ_s, ψ_s, and the general case follows from this and (2).

For (5a), it suffices to prove it when $x = s \in S$. Write simply $\phi = \phi_s$ and $\psi = \psi_s$. Assume $v \wedge w$ exists. We have $\phi(v \wedge w) \leqslant \phi(v), \phi(w)$ by (2). To show that $\phi(v \wedge w)$ is the meet of $\phi(v)$ and $\phi(w)$, take arbitrary u with $u \leqslant \phi(v), \phi(w)$. Then $\psi(u) \leqslant \psi(v), \psi(w)$ from the Lifting Property, and hence $\psi(u) \leqslant v, w$, which implies $\psi(u) \leqslant v \wedge w$. Applying ϕ, we have $\phi(u) = \phi(\psi(u)) \leqslant \phi(v \wedge w)$, and hence $u \leqslant \phi(v \wedge w)$. (5b) is essentially the same as (5a). □

Remark 3.7. The map ϕ_x (resp. ψ_x) does not preserve strong joins (resp. meets) in general. For example, letting $W = S_4$, we have $s_{212} \wedge s_{232} = s_2$ but $\psi_2(s_{212}) \wedge \psi_2(s_{232}) = s_{12} \wedge s_{32} = s_2 \neq \psi_2(s_2)$, where we write $s_{ij\dots}$ instead of $s_i s_j \cdots$. Mapping everything above via $x \mapsto x w_0$ where w_0 is the longest element of W, we obtain a counterexample for ϕ_x preserving joins.

Corollary 3.8. *Let $u, v, x, y \in W$ with $\langle u \rangle \langle x \rangle$ and $\langle v \rangle \langle y \rangle$ reduced and $ux = vy$ (or namely, $u \leqslant_L ux = vy \geqslant_L v$). Then $u \geqslant v \iff x \leqslant y$.*

Proof. By Lemma 3.4 (3) we have $u \geqslant v \iff u^{-1} \geqslant v^{-1} \implies (x =) \, \psi_{u^{-1}}(ux) \leqslant \psi_{v^{-1}}(vy) \, (= y)$. The other direction is similar. □

3.3. Half-strong, half-weak meets and joins. Analogous to the meets and joins under the weak order, we show the existence of the minimum element (under $\leqslant$) of the set

$$\{z \in W \mid x \leqslant z \geqslant_L y\},$$

and the maximum of

$$\{z \in W \mid x \geqslant_L z \leqslant y\}.$$

Remark 3.9. It seems that the existence of such elements has been known; for example, in his Sage implementation to compute the Deodhar lift [7], Shimozono explicitly used (1) of the following proposition. However we do not know about a reference, so we take the opportunity to give one here. The proof of (1) of the following proposition is by Shimozono [24].

Proposition 3.10. *Let $x, y \in W$.*

(1) *The set $\{u \in W \mid x \leqslant \phi_u(y)\}$ has the minimum element $\psi^R_{y^{-1}}(x)$ under the strong order.*
(2) *The set $\{u \in W \mid \psi_{u^{-1}}(x) \leqslant y\}$ has the minimum element $\psi^R_{y^{-1}}(x)$ under the strong order.*

Proof. (1). We prove it by induction on $l(y)$. The base case $l(y) = 0$ being clear, we assume $l(y) > 0$. Take $s \in S$ such that $y > ys$. Let $x' = \psi^R_s(x) \, (= \min(x, xs))$ and $y' = ys$. Since $y = y' * s$, for any u we see $u * y = u * y' * s$, whence by the Lifting Property $x \leqslant u * y \iff x' \leqslant u * y'$. Hence $D(x, y) = D(x', y')$ where

$$D(x, y) = \{u \in W \mid x \leqslant \phi_u(y) \, (= u * y)\}.$$

By the induction hypothesis it follows that $D(x,y) = D(x',y')$ has the minimum element $\psi^R_{y'^{-1}}(x')$, which equals to $\psi^R_{y^{-1}}(x)$.

(2). Let $E(x,y) = \{u \in W \mid \psi_{u^{-1}}(x) \leqslant y\}$. It suffices to show $D(x,y) = E(x,y)$. By Lemma 3.4 (2),(4) we have $x \leqslant \phi_u(y) \implies \psi_{u^{-1}}(x) \leqslant \psi_{u^{-1}}\phi_u(y) \leqslant y$ and $\psi_{u^{-1}}(x) \leqslant y \implies x \leqslant \phi_u\psi_{u^{-1}}(x) \leqslant \phi_u(y)$. ☐

Proposition 3.11. *Let $x, y \in W$.*

1. *The set $\{z \in W \mid x \leqslant z \geqslant_L y\}$ has the minimum element $\psi^R_{y^{-1}}(x)y$ under the strong order.*
2. *The set $\{z \in W \mid x \geqslant_L z \leqslant y\}$ has the maximum element $\left(\psi^R_{y^{-1}}(x)\right)^{-1}x$ under the strong order.*

Proof. (1). By (7), we have $D(x,y) \supset \{u \mid x \leqslant uy \geqslant_L y\} \simeq \{z \mid x \leqslant z \geqslant_L y\}$; $u \mapsto uy$, where the isomorphism is under $\leqslant$. The minimum element u of $D(x,y)$ satisfies $u * y = uy$ i.e. $uy \geqslant_L y$, since otherwise $(u*y)y^{-1}$ is a smaller element of $D(x,y)$. Hence by Proposition 3.10 (1) we have $\psi^R_{y^{-1}}(x)y = \min_{\leqslant}\{z \mid x \leqslant z \geqslant_L y\}$.

(2). By Corollary 3.8 we have $E(x,y) \supset \{u \mid x \geqslant_L u^{-1}x \leqslant y\} \underset{\text{anti}}{\simeq} \{z \mid x \geqslant_L z \leqslant y\}$; $u \mapsto u^{-1}x$, where the anti-isomorphism is under $\leqslant$. For a similar reason to (1) we have $\max_{\leqslant}\{z \mid x \geqslant_L z \leqslant y\} = (\min_{\leqslant} E(x,y))^{-1}x = (\psi^R_{y^{-1}}(x))^{-1}x$. ☐

From the proposition above, we define

$$x\,{}_S{\vee}_L\,y = y\,{}_L{\vee}_S\,x := \min_{\leqslant}\{z \in W \mid x \leqslant z \geqslant_L y\} = \psi^R_{y^{-1}}(x)y,$$

$$x\,{}_L{\wedge}_S\,y = y\,{}_S{\wedge}_L\,x := \max_{\leqslant}\{z \in W \mid x \geqslant_L z \leqslant y\} = \left(\psi^R_{y^{-1}}(x)\right)^{-1}x.$$

We define $x\,{}_S{\vee}_R\,y$ and $x\,{}_S{\wedge}_R\,y$ similarly.

3.4. Flipping lower weak intervals. For any $z \in W$, define the map

$$\Phi_z : [e,z]_L \longrightarrow [e,z]_R; x \mapsto zx^{-1}$$

and its inverse

$$\Psi_z : [e,z]_R \longrightarrow [e,z]_L; y \mapsto y^{-1}z.$$

Proposition 3.12 below demonstrates that these maps behave well along with the strong order on W and its meet/join operations.

Proposition 3.12. *Let $z \in W$.*

1. *Φ_z and Ψ_z are anti-isomorphisms under the strong order.*
2. *$l(\Phi_z(x)) = l(z) - l(x)$ for any $x \in [e,z]_L$ and $l(\Psi_z(y)) = l(z) - l(y)$ for any $y \in [e,z]_R$.*
3. *Φ_z and Ψ_z send strong meets to strong joins. Namely,*
   (a) *for $x, y \in [e,z]_L$ such that $x \wedge y$ exists and $x \wedge y \in [e,z]_L$, we have $\Phi_z(x \wedge y) = \Phi_z(x) \vee \Phi_z(y)$.*
   (b) *for $x, y \in [e,z]_R$ such that $x \wedge y$ exists and $x \wedge y \in [e,z]_R$, we have $\Psi_z(x \wedge y) = \Psi_z(x) \vee \Psi_z(y)$.*
   (Note that the meets and joins are not taken in $[e,z]_L$ or $[e,z]_R$ but in W.)

Proof. (1) is done in Corollary 3.8, and (2) is obvious.

For (3), we only prove (3a) since (3b) is shown similarly. Let $x, y, x \wedge y \in [e,z]_L$. From (1) it follows that $\Phi_z(x \wedge y) \geqslant \Phi_z(x), \Phi_z(y)$. To show the minimality of $\Phi_z(x \wedge y)$, let us take arbitrary $w \in W$ such that $w \geqslant \Phi_z(x), \Phi_z(y)$. From Proposition 3.11, we can let $w' = z\,{}_R{\wedge}_S\,w$. Since $\Phi_z(x), \Phi_z(y) \in [e,z]_R \cap [e,w]$, we have $\Phi_z(x), \Phi_z(y) \leqslant w'$. Since $w' \leqslant_R z$, applying Ψ_z $(= \Phi_z^{-1})$, we have $x, y \geqslant \Psi_z(w')$. Hence $x \wedge y \geqslant \Psi_z(w')$.

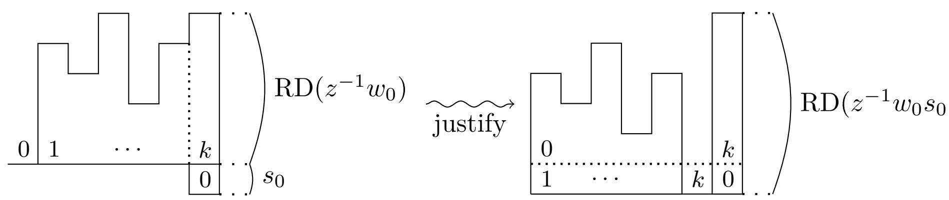


FIGURE 6. Inserting s_0 into $\mathrm{RD}(z^{-1}w_0)$ and justify it to obtain a k-code for $z^{-1}w_0s_0$.

Applying Φ_z, we have $\Phi_z(x \wedge y) \leqslant w'$, and hence $\Phi_z(x \wedge y) \leqslant w$. Therefore $\Phi_z(x \wedge y)$ is the join of $\{\Phi_z(x), \Phi_z(y)\}$. □

REMARK 3.13. It seems to be true that Φ_z and Ψ_z send strong joins to strong meets. Its proof would require that there be the strong-minimum element of $\{z \mid x \leqslant z \leqslant_L y\}$ and the strong-maximum of $\{z \mid x \leqslant_L z \leqslant y\}$ for any $x, y \in W$, analogous to Proposition 3.11.

3.5. CHAIN PROPERTY FOR LOWER WEAK INTERVALS. In this section we prove the Chain Property for the lower weak intervals $[e, u]_L$ and $[e, u]_R$ for arbitrary Coxeter group W and its element $u \in W$. This is similar to that for the generalized quotients, in that $[e, u]_L = \{x \mid x \leqslant_L u\}$ whereas $W/\{u\} \simeq \{x \mid x \geqslant_L u\}$. Besides it is shown in [4, Corollary 4.5] that the class of right generalized quotients and lower left intervals coincide for finite W. When W is infinite, however, these do not, as we give a counterexample below. Beforehand we recall [4, Theorem 4.10]: for any Coxeter group W, the left generalized quotients and the right generalized quotients are in bijection by $U \mapsto W/U$ and $V\backslash W \hookleftarrow V$, and a subset $U \subset W$ is a right generalized quotient if and only if $U = W/(U\backslash W)$.

EXAMPLE 3.14. Let $W = \widetilde{S}_{k+1} = \langle s_0, s_1, \dots, s_k\rangle$. Let w_0 be the longest element of $S_{k+1} = \langle s_1, \dots, s_k\rangle$. From the following claim we have $s_0w_0 \in \widetilde{S}_{k+1}/(S_{k+1}\backslash\widetilde{S}_{k+1})$, and thereby $S_{k+1} = [e, w_0]_L$ is not a right generalized quotient of $\widetilde{S}_{k+1}$.

CLAIM 3.15. *For any $z \in \widetilde{S}_{k+1}$, the product $\langle w_0\rangle\langle z\rangle$ is reduced if and only if $\langle s_0w_0\rangle\langle z\rangle$ is reduced.*

Proof of Claim 3.15. The "if" direction is clear. Toward the "only if" direction, assume $\langle w_0\rangle\langle z\rangle$ is reduced, that is, $\langle z^{-1}\rangle\langle w_0\rangle$ is reduced. Since $z^{-1}w_0 \geqslant_L w_0$, we have $\mathrm{RD}(z^{-1}w_0) \supset \mathrm{RD}(w_0)$. Hence, since the first row of $\mathrm{RD}(w_0)$ is $\{1, \dots, k\}$ and the rows of a k-code are proper subsets of $\{0, 1, \dots, k\}$, the first row of $\mathrm{RD}(z^{-1}w_0)$ is also $\{1, \dots, k\}$. Thus, inserting s_0 into $\mathrm{RD}(z^{-1}w_0)$ from the bottom and justifying it to the bottom with maximizing moves, we successfully obtain $\mathrm{RD}(z^{-1}w_0s_0)$, the i-th column of which is

- the k-th column of $\mathrm{RD}(z^{-1}w_0)$ with an s_0 added, when $i = 0$,
- the i-th column of $\mathrm{RD}(z^{-1}w_0)$ when $i = 1, \dots, k-1$,
- empty when $i = k$.

(See Figure 6.) In particular $\langle z^{-1}w_0\rangle\langle s_0\rangle$ is reduced. Combining this with that $\langle z^{-1}\rangle\langle w_0\rangle$ is reduced, we have $\langle z^{-1}\rangle\langle w_0\rangle\langle s_0\rangle$ is reduced, and hence so is $\langle s_0\rangle\langle w_0\rangle\langle z\rangle$, as desired. □

The proof of the following proposition is parallel to that of [4, Theorem 3.4]. Beforehand we recall that, for $x, y \in W$ with $x \geqslant y$ and any fixed reduced expression

$x = s_1 \dots s_m$, there exists $1 \leqslant j_1 < j_2 < \dots < j_l \leqslant m$ such that $x = y^{(0)} > y^{(1)} > \dots > y^{(l)} = y$ where

$$y^{(a)} = s_1 \dots \widehat{s_{j_1}} \dots \widehat{s_{j_a}} \dots s_m.$$

See, for example, [4, Section 3] or [3] for the details.

Proposition 3.16. *Let $u, x, y \in W$ with $xu, yu \leqslant_L u$ and $xu \leqslant yu$. Note that $xu \leqslant yu \iff x^{-1} \geqslant y^{-1} \iff x \geqslant y$ for $xu, yu \leqslant_L u$. Fix a reduced expression for $x = s_1 \dots s_m$ and take $y^{(0)}, \dots, y^{(l)}$ as right above. Then $y^{(a)}u \leqslant_L u$ for any a.*

Proof. Suppose to the contrary that there exists a such that $y^{(a)}u \not\leqslant_L u$. Since $y^{(l)}u = yu \leqslant_L u$, we can take such a that $y^{(a)}u \not\leqslant_L u$ and $y^{(a+1)}u \leqslant_L u$.

Since $xu \leqslant_L u$, we have $s_{j_a+1} \dots s_m u \leqslant_L u$. Hence there exists $p < j_a$ such that

$$s_p z u \not\leqslant_L u \quad \text{and} \quad zu \leqslant_L u, \tag{9}$$

where we put

$$z = s_{p+1} \dots \widehat{s_{j_a}} \dots s_{j_{a+1}} \dots s_m,$$

where there may be more indices omitted between s_{p+1} and $\widehat{s_{j_a}}$ (including s_{p+1}), according to the omissions in $y^{(a)} = s_1 \dots \widehat{s_{j_1}} \dots \widehat{s_{j_a}} \dots s_m$. Since $y^{(a+1)}u \leqslant_L u$, we have

$$s_p \widehat{z} u \leqslant_L u \quad \text{and} \quad \widehat{z} u \leqslant_L u, \tag{10}$$

where we put

$$\widehat{z} = s_{p+1} \dots \widehat{s_{j_a}} \dots \widehat{s_{j_{a+1}}} \dots s_m.$$

We have $zu < s_p zu$ by (9) and $\widehat{z}u > s_p \widehat{z} u$ by (10). Besides, since $y^{(a)} > y^{(a+1)}$ it follows $z > \widehat{z}$, and thereby $zu < \widehat{z}u$. Hence we have $s_p zu = \widehat{z}u$ by the Lifting Property and length arguments. Therefore $s_p z = \widehat{z} < z$, which contradicts the fact that $s_p z$ is a consecutive subword of a reduced expression for $y^{(a)}$. ☐

As a corollary, we have the Chain Property for weak lower intervals:

Theorem 3.17. *For any $u \in W$, the interval $[e, u]_L$ (resp. $[e, u]_R$) under the left (resp. right) weak order has the Chain Property.*

Proof. The statement for left lower intervals follows from Proposition 3.16 and that $\{x \in W \mid xu \leqslant_L u\} = [e, u^{-1}]_L$ for $u \in W$, which follows from $xu \leqslant_L u \iff x^{-1} \leqslant_R u \iff x \leqslant_L u^{-1}$. The statement for right intervals is proved parallelly. ☐

4. Properties of the weak strips

Hereafter we restrict our attention to $\widetilde{S}_{k+1}$ rather than general Coxeter groups and let $W = \widetilde{S}_{k+1}$ and $W^\circ = \widetilde{S}^\circ_{k+1}$. In Section 2.2 we put $I = \mathbb{Z}_{k+1} = \{0, 1, \dots, k\}$ and let d_A denote the cyclically decreasing element corresponding to $A \subsetneq I$.

In this section we prove some properties on weak strips. First we define for any $u \in W$,

$$\begin{aligned}
Z_{u,+} &= \{v \in W \mid v = d_A u \geqslant_L u \text{ for } \exists\, A \subsetneq I\}, \\
Z'_{u,+} &= \{A \subsetneq I \mid d_A u \geqslant_L u\} = \{A \subsetneq I \mid d_A u \in Z_{u,+}\}, \\
Z_{u,-} &= \{v \in W \mid v = d_A^{-1} u \leqslant_L u \text{ for } \exists\, A \subsetneq I\}, \\
Z'_{u,-} &= \{A \subsetneq I \mid d_A^{-1} u \leqslant_L u\} = \{A \subsetneq I \mid d_A^{-1} u \in Z_{u,-}\}.
\end{aligned}$$

It is an immediate observation from the Subword Property that

- The map $(Z'_{u,+}, \subset) \longrightarrow (Z_{u,+}, \leqslant)\,;\, A \mapsto d_A u$ is an isomorphism of posets.
- The map $(Z'_{u,-}, \subset) \longrightarrow (Z_{u,-}, \leqslant)\,;\, A \mapsto d_A^{-1} u$ is an anti-isomorphism of posets.

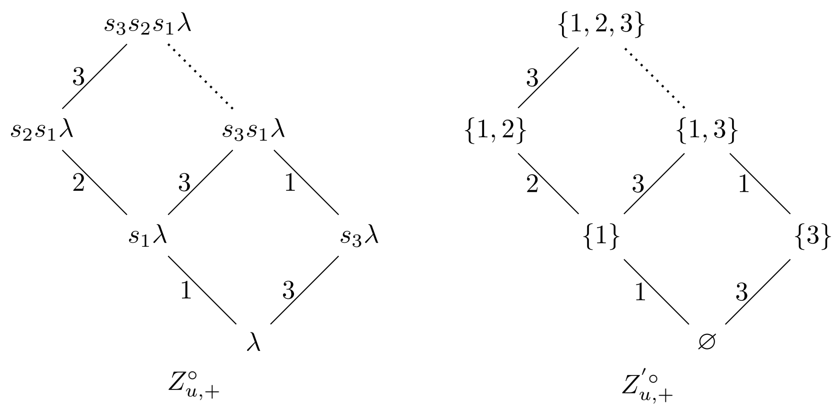


Figure 7. The posets $Z^\circ_{u,+}$ ($\simeq Z_{uw_0^J,+}$) and $Z'^\circ_{u,+}$ ($= Z'_{uw_0^J,+}$) for $u = w_\lambda$ where $k = 3$ and $\lambda = (3,2,1) \in \mathcal{P}_3$ (and w_0^J is the longest element of S_4). Left weak covers are represented as solid lines, and strong covers are solid or dotted lines. A solid edge between v and w is labelled with i if $v = s_i w$.

Since if $u \in W^\circ$ and $v \leqslant_L u$ then $v \in W^\circ$, for $u \in W^\circ$ we have

$$Z_{u,-} = \{v \mid u/v \text{ is a weak strip}\}.$$

On the other hand, the set $Z_{u,+}$ does not coincide with the set of v such that v/u is a weak strip. More precisely, for $u \in W^\circ$ we have by definition

$$v/u \text{ is a weak strip} \iff v \in Z_{u,+} \text{ and } v \in W^\circ.$$

Recalling that $v \in W^\circ \iff vw_0^J \geqslant_L w_0^J$ where $J = \{1,\dots,k\}$ and w_0^J is the longest element of $W_J = S_{k+1}$, by Lemma 2.1 we have

$$\begin{aligned} v/u \text{ is a weak strip} &\iff vw_0^J \in Z_{uw_0^J,+} \\ &\iff v = d_A u \text{ with } A \in Z'_{uw_0^J,+}. \end{aligned}$$

In other words, defining

$$\begin{aligned} Z^\circ_{u,+} &= \{v \mid v/u \text{ is a weak strip}\}, \\ Z'^\circ_{u,+} &= \{A \subsetneq I \mid d_A u/u \text{ is a weak strip}\} = \{A \subsetneq I \mid d_A u \in Z^\circ_{u,+}\}, \end{aligned}$$

we have

$$\begin{aligned} Z^\circ_{u,+} &\simeq Z_{uw_0^J,+} \ ; \ v \mapsto vw_0^J, \\ Z'^\circ_{u,+} &= Z'_{uw_0^J,+}. \end{aligned}$$

Example 4.1. Figure 7 illustrates the same example as Example 2.10.

From the example above, we would expect these properties:

(i) $Z'_{u,\pm}$ is closed under intersection and union.
(ii) $Z'_{u,\pm}$ has the maximum element.
(iii) $Z_{u,\pm}$ and $Z'_{u,\pm}$ have the Chain Property. (See Section 4.3 for the details.)

(i), (ii), (iii) are proved in Sections 4.1, 4.2, 4.3, respectively.

4.1. Intersection and union. In this section we prove the following proposition as the compilation of Lemmas 4.5, 4.9 and 4.10.

Proposition 4.2. *For $u \in W$, we have*

(1) $A, B \in Z'_{u,\pm}$ *and* $A \cup B \neq I \implies A \cup B \in Z'_{u,\pm}$.
(2) $A, B \in Z'_{u,\pm} \implies A \cap B \in Z'_{u,\pm}$.
(3) $A, B \in Z'_{u,+} \implies d_{A\cap B} u = (d_A u) \wedge (d_B u)$.
(4) $A, B \in Z'_{u,-} \implies d^{-1}_{A\cap B} u = (d^{-1}_A u) \vee (d^{-1}_B u)$.

In this section we say $A, B \subset I$ are *strongly disjoint* if for any $i \in A$ and $j \in B$ it holds that $i - j \not\equiv 0, \pm 1$, and $x, y \in W$ are *strongly commutative* if any Coxeter generator s_i appearing in a reduced expression of x and any s_j appearing in that of y satisfy $i - j \not\equiv 0, \pm 1$. The next lemma is straightforward.

Lemma 4.3. *Let $A, B \subsetneq I$ and $x, y \in W$.*

(1) *If A, B are strongly disjoint, then d_A, d_B are strongly commutative.*
(2) *For the decomposition $A = A_1 \sqcup \cdots \sqcup A_m$ into connected components, $A_1, \dots, A_m$ are pairwise strongly disjoint and $d_{A_1}, \dots, d_{A_m}$ are pairwise strongly commutative.*
(3) *For $x' \leqslant x$ and $y' \leqslant y$, if x, y are strongly commutative then so are x', y'.*
(4) *If x, y are strongly commutative, then x, y are commutative and $l(xy) = l(x) + l(y)$.*

Lemma 4.4. *Let $x, y, z \in W$ with x, y strongly commutative. Then*

(1) $z \leqslant_L xyz \iff z \leqslant_L xz, yz$.
(2) $z \geqslant_L xyz \iff z \geqslant_L xz, yz$.

Proof. We only prove (1) since (2) is shown similarly.

The "only if" direction immediately follows by the definition of the weak order and commutativity of x, y. We prove the "if" direction by induction on $l(x) + l(y)$. It is clear when $l(x) = 0$ or $l(y) = 0$. In particular the case $l(x) + l(y) \leqslant 1$ is done and we may assume $l(x) + l(y) \geqslant 2$ and $l(x), l(y) > 0$.

Step A: the case $l(x) + l(y) = 2$, i.e. $l(x) = l(y) = 1$. We can write $x = s_i$ and $y = s_j$ with $s_i \neq s_j$, $s_i s_j = s_j s_i$ from the strong commutativity. We have $s_i z, s_j z \geqslant_L z$ by the assumption. Hence $z \in W/W_{\{i,j\}}$, where $W_{\{i,j\}} = \langle s_i, s_j \rangle = \{e, s_i, s_j, s_i s_j\}$. Therefore $s_i s_j z \geqslant_L z$.

Step B: the case $l(x) + l(y) > 2$. From the commutativity of x, y we may assume $l(y) \geqslant l(x)$; in particular $l(y) > 1$. Take a reduced expression of $y = s_{i_1} \dots s_{i_l}$ and put $y' = s_{i_1} \dots s_{i_{l-1}}$, $z' = s_{i_l} z$. Since $z \leqslant_L yz$ and $s_{i_l} \leqslant_L y$, we have $z \leqslant_L z'$. Now we can obtain $z' \leqslant_L xy'z'$, which implies $z \leqslant_L z' \leqslant_L xy'z' = xyz$ as desired, by applying the induction hypothesis for $(x, y, z) := (x, y', z')$, having its assumption satisfied as follows:

- x, y' are strongly commutative.
  Proof. From Lemma 4.3 (3).
- $z' \leqslant_L y'z'$.
  Proof. Since $z \leqslant_L yz$ and $s_{i_l} \leqslant_L y$, by Lemma 2.1 (1) we have $z' = s_{i_l} z \leqslant_L yz = y'z'$.
- $z' \leqslant_L xz'$.
  Proof. Since $l(x) + l(y) > l(x) + l(s_{i_l})$, we can obtain $z \leqslant_L xz'$ by applying the induction hypothesis for $(x, y, z) := (x, s_{i_l}, z)$, having that its assumption described below is clearly satisfied:
  - x and s_{i_l} are strongly commutative.

– $z \leqslant_L xz$.
– $z \leqslant_L s_{i_l} z$.

Besides $s_{i_l} \leqslant_L x s_{i_l}$, hence we have $z' \leqslant_L xz'$ by Lemma 2.1 (1). □

Lemma 4.5. *Let $w \in W$ and $A, B \subsetneq I$ with $w \leqslant_L d_A w, d_B w$.*

(1) *$w \leqslant_L d_{A \cap B} w$.*
(2) *The element $d_{A \cap B} w$ is the strong meet of $d_A w$ and $d_B w$.*

Remark 4.6. The same statement with all d_X replaced with u_X is proved similarly.

Remark 4.7. It does not generally hold that if $w \leqslant_L xw, yw$ and $x \wedge y$ exists then $w \leqslant_L (x \wedge y) w$; a counterexample is $W = S_4$, $x = s_{21}$, $y = s_{23}$, $w = s_2$.

Proof. (1): Within this proof we say $x \in W$ satisfies $(*)$ if $w \leqslant_L xw$.

Decomposing A, B into connected components $A = A_1 \sqcup \cdots \sqcup A_m$ and $B = B_1 \sqcup \cdots \sqcup B_n$, we have $A \cap B = \bigsqcup_{i,j} (A_i \cap B_j)$. Each nonempty $A_i \cap B_j$ has at most two connected components, each component C of which satisfies $d_{A_i} = x d_C$ for some $x \in W$ or $d_{B_j} = y d_C$ for some $y \in W$ as easily seen. Having that both d_A ($\geqslant_L d_{A_i}$) and d_B ($\geqslant_L d_{B_j}$) satisfy $(*)$ and that lower bounds in $\leqslant_L$ inherit $(*)$, we see each d_C satisfies $(*)$. Besides $(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) = (A_i \cap A_{i'}) \cap (B_j \cap B_{j'})$ is empty unless $(i, j) = (i', j')$, we thus have $A \cap B$ decomposes as $A \cap B = C_1 \sqcup \cdots \sqcup C_l$ into connected components, where each d_{C_i} satisfies $(*)$. Now it follows from Lemma 4.4 (1) that $d_{A \cap B} = d_{C_1} \ldots d_{C_l}$ satisfies $(*)$, as desired.

(2): By the Subword Property we have $d_{A \cap B} = d_A \wedge d_B$. From the assumption and (1), we have $\phi^R_w(d_X) = d_X w$ for $X = A, B, A \cap B$. Hence by Lemma 3.4 (5) we have $d_{A \cap B} w = d_A w \wedge d_B w$. □

Corollary 4.8. *Let $\lambda \in \mathcal{P}_k$, and $\kappa^{(1)}, \kappa^{(2)}$ be weak strips over λ. Write $\kappa^{(i)} = d_{A_i} \lambda$ for each i with $A_i \subsetneq I$. Then $d_{A_1 \cap A_2} \lambda$ is a weak strip over λ and is the meet of $\kappa^{(1)}, \kappa^{(2)}$ in the poset $\mathcal{P}_k$ with the strong order: $\kappa^{(1)} \wedge \kappa^{(2)} = d_{A_1 \cap A_2} \lambda$.*

Proof. Let $w_\lambda \in W^\circ$ be the affine Grassmannian permutation corresponding to λ, and w_0 the longest element of S_{k+1}. By Lemma 2.1, the condition $d_A \lambda / \lambda$ is a weak strip is equivalent to $d_A w_\lambda w_0 \geqslant_L w_\lambda w_0$. From this and Lemma 4.5 (1) we see $d_{A_1 \cap A_2} \lambda / \lambda$ is a weak strip. From Lemma 4.5 (2) we have $d_{A_1 \cap A_2} w_\lambda = (d_{A_1} w_\lambda) \wedge (d_{A_2} w_\lambda)$ in W. Since $W^\circ \subset W$ is a subposet, this is also the meet in $W^\circ \simeq \mathcal{P}_k$. □

Lemma 4.9. *Let $w \in W$ and $A, B \subsetneq I$ with $d_A^{-1} w, d_B^{-1} w \leqslant_L w$.*

(1) *$d_{A \cap B}^{-1} w \leqslant_L w$.*
(2) *The element $d_{A \cap B}^{-1} w$ is the strong join of $d_A^{-1} w$ and $d_B^{-1} w$.*

Proof. (1) is proved parallelly to Lemma 4.5 (1), making use of Lemma 4.4 (2) instead of Lemma 4.4 (1).

Next we show (2). We have $d_A^{-1} w, d_B^{-1} w, d_{A \cap B}^{-1} w \in [e, w]_L$ by (1). The map Φ_w in Lemma 3.12 sends $d_A^{-1} w, d_B^{-1} w, d_{A \cap B}^{-1} w$ to $d_A, d_B, d_{A \cap B}$ respectively. Since $d_{A \cap B} = d_A \wedge d_B$, sending them back via Ψ_w, we have $d_{A \cap B}^{-1} w = (d_A^{-1} w) \vee (d_B^{-1} w)$ by Lemma 3.12 (3). □

Lemma 4.10. *Let $u \in W$ and $A, B \subsetneq I$ with $A \cup B \neq I$.*

(1) *If $d_A u, d_B u \geqslant_L u$, then $d_{A \cup B} u \geqslant_L u$.*
(2) *If $d_A^{-1} u, d_B^{-1} u \leqslant_L u$, then $d_{A \cup B}^{-1} u \leqslant_L u$.*

Proof. We only give a proof of (1) since that of (2) is quite similar.

Assume $d_A u, d_B u \geqslant_L u$. Take the decomposition $A = A_1 \sqcup \cdots \sqcup A_m$ and $B = B_1 \sqcup \cdots \sqcup B_n$ into connected components. Since $d_{A_i} \leqslant_L d_A$, we have $d_{A_i} u \geqslant_L u$ for any i, and similarly $d_{B_j}u \geqslant_L u$ for any j. Since $A \cup B = (\dots(A \cup B_1) \cup \dots) \cup B_n$, we only need to prove it when B is connected. Assume B is connected. It is also easy to see, from Lemma 4.3 and Lemma 4.4 (1), that it suffices to prove it when A, B and $A \cup B$ are connected. We therefore assume A, B and $A \cup B$ are connected. The case $A \subset B$ or $B \subset A$ being clear, we assume $A \not\subset B$ and $B \not\subset A$; namely we let $A = [i, j]$ and $B = [p, q]$ with $p \leqslant i \leqslant q + 1 \leqslant j + 1$ without loss of generality, where we employ an ordering $r + 1 < \dots < k < 0 < \dots < r - 1$ of $I \smallsetminus \{r\}$ with an arbitrarily fixed element $r \in I \smallsetminus (A \cup B)$. Since $d_B = s_q \dots s_p \geqslant_L s_{i-1} \dots s_p = d_{B \smallsetminus A}$ and $d_B u \geqslant_L u$, we have $d_{B \smallsetminus A} u \geqslant_L u$. Hence we may replace B by $B \smallsetminus A$ $(= [p, i-1])$.

Let $B' = B \smallsetminus \{i-1\} = [p, i-2]$ and $u' = d_{B'}u$. Since $d_{B'} \leqslant_L d_B$ and $d_B u \geqslant_L u$, it follows that $u' \geqslant_L u$. Since $s_{i-1}u' = d_B u \geqslant_L u$ and $d_A u' = d_A d_{B'} u \geqslant_L u$, the latter of which is from Lemma 4.4 (1), it easily follows that $s_{i-1}u' \geqslant_L u'$ and $d_A u' \geqslant_L u'$ from Lemma 2.1.

Toward a contradiction, suppose $d_{A \cup B} u \not\geqslant_L u$. Then we have $d_A s_{i-1} u' \not\geqslant_L u'$ since $d_{A \cup B} u = d_A s_{i-1} u'$ and $u \leqslant_L u'$. Since $s_{i-1}u' \geqslant_L u'$, there exists $a \in [i, j]$ such that $x s_{i-1} u' \geqslant_L u'$ and $s_a x s_{i-1} u' \not\geqslant_L u'$, which implies $s_a x s_{i-1} u' \lessdot x s_{i-1} u'$, where we write $x = s_{a-1} s_{a-2} \dots s_{i+1} s_i$. On the other hand, since $d_A u' \geqslant_L u'$ we have $s_a x u' \geqslant_L u'$ and $x u' \geqslant_L u'$. Besides we have $x s_{i-1} u' \gtrdot x u'$ from the Subword Property. Hence the Lifting Property implies that $x u' \leqslant s_a x s_{i-1} u'$, which is actually an equality since both sides have the same length. Therefore we have $(s_{a-1} s_{a-2} \dots s_{i+1} s_i =)$ $x = s_a x s_{i-1}$ $(= s_a s_{a-1} \dots s_i s_{i-1})$, which is absurd. $\square$

Remark 4.11. Unlike the "cap" case, it does not always hold that $d_{A \cup B} u = (d_A u) \vee (d_B u)$ in (1), or $d_{A \cup B}^{-1} u = (d_A^{-1} u) \wedge (d_B^{-1} u)$ in (2).

A counterexample for (1) is given by $W = S_3$, $u = e$, $A = \{1\}$ and $B = \{2\}$.

4.2. Non-appearing indices.

Proposition 4.12.

(1) *For any $w \in W$, there exists $i_w^- \in I$ such that $i_w^- \notin A$ for any $A \subsetneq I$ with $d_A^{-1} w \leqslant_L w$.*

(2) *For any $w \in W^\circ$, there exists $i_w^+ \in I$ such that $i_w^+ \notin A$ for any $A \subsetneq I$ with $d_A w \geqslant_L w$ and $d_A w \in W^\circ$.*

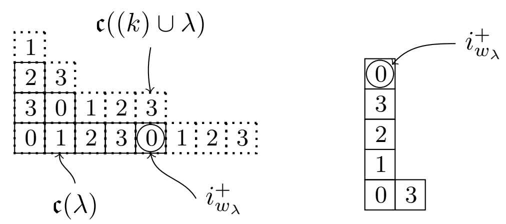
Proof. (1). For any $A \subsetneq I$, we have

$$\begin{aligned} d_A^{-1} w \leqslant_L w &\iff d_A \leqslant_R w \\ &\iff u_A \leqslant_L w^{-1} \\ &\implies \mathrm{RI}(u_A) \subset \mathrm{RI}(w^{-1}), \end{aligned}$$

and the last condition is equivalent to A being included by the first row of $\mathrm{RI}(w^{-1})$. Hence we can take i_w^- from the complement of the first row of $\mathrm{RI}(w^{-1})$.

(2). By Lemma 3.1 we may take $z := \bigvee_L \{d_A w \mid A \subsetneq I \text{ s.t. } d_A w \geqslant_L w,\, d_A w \in W^\circ\}$, the left join of all weak strips over w. Take any $A \subsetneq I$ such that $d_A w \geqslant_L w$ and $d_A w \in W^\circ$. Since $w, d_A w \leqslant_L z$, we have $z w^{-1} \geqslant_R z(d_A w)^{-1} = z w^{-1} u_A$, which is equivalent to $w z^{-1} \geqslant_L d_A w z^{-1}$. Hence, similarly to the proof of (1) we have that A is a subset of the first row of $\mathrm{RD}((w z^{-1})^{-1}) = \mathrm{RD}(z w^{-1})$, which is a proper subset of I and independent of A, and therefore we can take i_w^+ from its complement. $\square$

Remark 4.13. The index i_w^+ in (2) above is in fact uniquely determined as follows: a bounded partition $\lambda \in \mathcal{P}_k$, corresponding to a 0-dominant affine permutation $w_\lambda \in W^\circ$, has the unique weak strip of size k, namely $(k) \cup \lambda$. Since the corresponding core $\mathfrak{c}((k) \cup \lambda)$ has k more boxes in the first row than $\mathfrak{c}(\lambda)$ does, the only possibility for $i_{w_\lambda}^+$ is what is determined by the following equivalent descriptions:



$\mathfrak{c}(\lambda)$ and $\mathfrak{c}((k) \cup \lambda)$ $\qquad$ $\mathrm{RI}(w_\lambda) = \lambda^{\omega_3}$

Figure 8. An example where $k = 3$, $\lambda = (3, 2, 1)$ and $\mathfrak{c}(\lambda) = (5, 2, 1)$. The dotted shape on the left figure represents $\mathfrak{c}((k) \cup \lambda)$, and the solid one does $\mathfrak{c}(\lambda)$. In this case $w_{(k)\cup\lambda} = s_3 s_2 s_1 w_\lambda = d_{\{1,2,3\}} w_\lambda$ and therefore $i^+_{w_\lambda} = 0$.

- The residue of the rightmost box in the first row of $\mathfrak{c}(\lambda)$.
- The negative of the residue written in the leftmost box in the last row of $\mathrm{RI}(w_\lambda) = \lambda^{\omega_k}$.
- $m - 1$, where $w_\lambda = u_{A_m} \dots u_{A_1}$ is the maximal increasing decomposition for w_λ. (Note that $A_m = \{i, i+1, \dots, m-2, m-1\}$ for some i.)

Remark 4.14. We cannot drop the assumption on 0-dominantness of $d_A w$ in (2) of the proposition. For example, let $k = 3$ and $w = s_3 s_0$. Then $w = u_{\{3,0\}}$ is the maximal increasing decomposition and hence i^+_w should be 0, but $d_{\{0\}} w = s_0 s_3 s_0 \geqslant_L w$.

Corollary 4.15. *Let $u \in W$.*

(1) *$Z'_{u,+}$ has the maximum element under $\subset$. Hence, $Z_{u,+}$ has the maximum element under $\leqslant$.*
(2) *$Z'_{u,-}$ has the maximum element under $\subset$. Hence, $Z_{u,-}$ has the minimum element under $\leqslant$.*

Proof. By Proposition 4.2 (1) and Proposition 4.12. □

4.3. Chain Property. Recall that an *order ideal* of a poset P is a subset $X \subset P$ such that if $x \in X$ and $y \leqslant x$ then $y \in X$, and an *order filter* of P is a subset $X \subset P$ such that if $x \in X$ and $y \geqslant x$ then $y \in X$.

Proposition 4.16. *The sets $Z_{u,+}$ and $Z_{u,-}$ have the Chain Property. Namely, for any $x, y \in Z_{u,\pm}$ such that $x \leqslant y$, there exists a sequence $x = z^{(0)} \lessdot z^{(1)} \lessdot \dots \lessdot z^{(l)} = y$ such that $z^{(i)} \in Z_{u,\pm}$ for any i.*

Proof. First we note a few immediate observations:

- For a poset P and a subposet $Q \subset P$, if $A \subset P$ is an order ideal then $A \cap Q$ is an order ideal of Q.
- If a subset X of a Coxeter group W has the Chain Property and $Y \subset X$ is an order ideal, then Y also has the Chain Property.

Let $D = \{d_A \mid A \subsetneq I\}$. Since $D \subset W$ is an order ideal, the set $\{d_A \mid d_A \leqslant_R u\} = D \cap [e, u]_R$ is an order ideal of $[e, u]_R$ and hence has the Chain Property since so does $[e, u]_R$ as proved in Theorem 3.17. Hence $Z_{u,-}$ also has the Chain Property since it is the image of $\{d_A \mid d_A \leqslant_R u\}$ under the anti-isomorphism $\Psi_u \colon [e, u]_R \longrightarrow [e, u]_L; x \mapsto x^{-1}u$.

Similarly, the set $\{d_A \mid d_A u \geqslant_L u\} = D \cap (W/\{u\})$ has the Chain Property since it is an order ideal of $W/\{u\}$, which has the Chain Property [4, Corollary 3.5]. Hence, since $Z_{u,+}$ is the image of $\{d_A \mid d_A u \geqslant_L u\}$ under the isomorphism $(\cdot u)\colon W/\{u\} \longrightarrow [u,\infty)_L$, we conclude that $Z_{u,+}$ has the Chain Property. $\square$

From the isomorphism $(Z_{u,+},\leqslant)\simeq(Z'_{u,+},\subset)$ and the anti-isomorphism $(Z_{u,-},\leqslant)\underset{\text{anti}}{\simeq}(Z'_{u,-},\subset)$, we have the Chain Property for $Z'_{u,\pm}$:

Corollary 4.17. *The sets $Z'_{u,+}$ and $Z'_{u,-}$ have the Chain Property. Namely, for any $A,B\in Z'_{u,\pm}$ with $A\subset B$, there exists a sequence $A=C^{(0)}\subset\!\!\cdot\; C^{(1)}\subset\!\!\cdot\;\dots\subset\!\!\cdot\; C^{(l)}=B$ such that $C^{(i)}\in Z'_{u,\pm}$ for any i.*

5. Proof of the Pieri rule for $\widetilde{g}^{(k)}_\lambda$

This section is devoted for the proof of Theorem 1.3 and 1.4.

5.1. Outline. Recall the K-k-Pieri rule (Definition 2.19): for $v\in W^\circ$ and $0\leqslant i\leqslant k$,

$$g^{(k)}_v h_i = \sum_{\substack{A\subset I,|A|=i\\ d_A*v\in W^\circ}} (-1)^{i-(l(d_A*v)-l(v))} g^{(k)}_{d_A*v}. \tag{11}$$

Let $w=w_\lambda\in W^\circ$ be the affine Grassmannian element corresponding to λ. Summing (11) up over $v\in W^\circ\cap[e,w]$ and $i\in\{0,1,\dots,r\}$, we have

$$\widetilde{g}^{(k)}_w\widetilde{h}_r = \sum_{\substack{v\leqslant w\\ v\in W^\circ}}\sum_{\substack{A\subset I,|A|\leqslant r\\ d_A*v\in W^\circ}} (-1)^{|A|-(l(d_A*v)-l(v))} g^{(k)}_{d_A*v},$$

and its coefficient of $g^{(k)}_u$ (for $u\in W^\circ$) is

$$[g^{(k)}_u](\widetilde{g}^{(k)}_w\widetilde{h}_r) = \sum_{\substack{v\leqslant w\\ v\in W^\circ}}\sum_{\substack{A\subset I,|A|\leqslant r\\ u=d_A*v}} (-1)^{|A|-(l(u)-l(v))}. \tag{12}$$

We shall illustrate, in the example below, that if the summation above is not empty then there are exactly one larger number of pairs (v,A) with $(-1)^{|A|-(l(u)-l(v))}=+1$ than those with $(-1)^{|A|-(l(u)-l(v))}=-1$, and consequently the value of the summation in (12) is equal to 1.

Example 5.1. Let $k=3$ and $u=s_{310}=w_\lambda\in\widetilde{S}^\circ_4$ where $\lambda=(2,1)\in\mathcal{P}_3$. Table 1 lists the pairs (v,A) such that $d_A*v=u$, organized according to the size of A. Apparently there are the same number of pairs (v,A) with $|A|=r'$ and $(-1)^{|A|-(l(u)-l(v))}=+1$, and those with $|A|=r'$ and $(-1)^{|A|-(l(u)-l(v))}=-1$, for each fixed $r'>0$. Furthermore, introducing the condition $v\leqslant w$ for $w=s_{210}$, say, we see that the set of the pairs (v,A) with $d_A*v=u$ and $v\leqslant w$ is $\{(s_{10},\{3\}),(s_0,\{1,3\}),(s_{10},\{1,3\})\}$, and that the number of such pairs (v,A) with $|A|=r'$ and $(-1)^{|A|-(l(u)-l(v))}=+1$ and those with $|A|=r'$ and $(-1)^{|A|-(l(u)-l(v))}=-1$ coincide whenever $r'\neq1$, and differ by 1 when $r'=1$.

According to the observation above, for $u\in W^\circ$ and $A\subsetneq I$ we let

$$\begin{aligned} X_{A,u} &= \{v\in W \mid d_A*v=u\}=\{v\in W\mid \phi_{d_A}(v)=u\},\\ Y_{A,u} &= X_{A,u}\cap[e,w], \end{aligned}$$

and

$$\begin{aligned} X'_{A,u} &= \{B\subsetneq I \mid d_B^{-1}u\in X_{A,u}\},\\ Y'_{A,u} &= \{B\subsetneq I \mid d_B^{-1}u\in Y_{A,u}\}. \end{aligned}$$

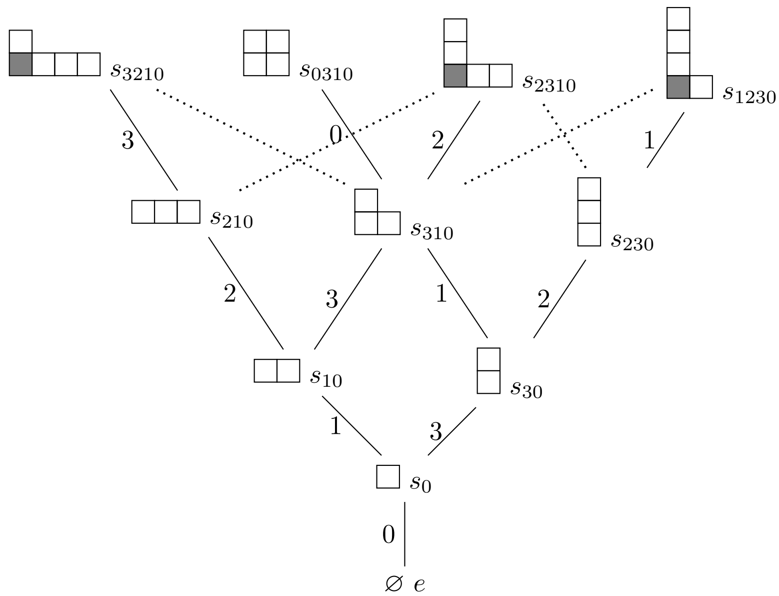


FIGURE 9. The poset of 4-cores (and corresponding elements in $\widetilde{S_4^\circ}$) up to those of size 4. The left weak covers are represented by solid lines, and the strong covers are dotted or solid lines. A solid edge labelled with i corresponds to the left multiplication by s_i.

	(v, A)	$(-1)^{\lvert A\rvert-(l(u)-l(v))}$
$\lvert A\rvert = 0$	$(s_{310}, \varnothing)$	$+1$
$\lvert A\rvert = 1$	$(s_{30}, \{1\})$	$+1$
	$(s_{310}, \{1\})$	-1
	$(s_{10}, \{3\})$	$+1$
	$(s_{310}, \{3\})$	-1
$\lvert A\rvert = 2$	$(s_0, \{1,3\})$	$+1$
	$(s_{10}, \{1,3\})$	-1
	$(s_{30}, \{1,3\})$	-1
	$(s_{310}, \{1,3\})$	$+1$

TABLE 1. The list of (v, A) such that $d_A * v = u$, where $u = s_{310}$.

Note that, for any $v \in X_{A,u}$, Lemma 3.4 (1) implies $v \leqslant_L u$, and hence it follows from $u \in W^\circ$ that $v \in W^\circ$. Hence

$$[g_u^{(k)}](\widetilde{g}_w^{(k)}\widetilde{h}_r) = \sum_{|A|\leqslant r} \sum_{v \in Y_{A,u}} (-1)^{|A|-(l(u)-l(v))}. \tag{13}$$

The flow of the proof is as follows:

Step 1. Every element of $X_{A,u}$ has the form $d_B^{-1}u$ with $B \subset A$, and thereby $X_{A,u}$ is anti-isomorphic to the subposet $X'_{A,u}$ of $[\varnothing, A]$ by $d_B^{-1}u \mapsto B$.

Step 2. The poset $X'_{A,u} \subset [\varnothing, A]$ has the minimum element B and is a boolean poset; $X'_{A,u} = [B, A]$.

$$
\begin{array}{cccccc}
 & B & \longmapsto & d_B^{-1}u & & \\
 & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\
[\varnothing, I] \supset & Z'_{u,-} & \underset{\text{anti}}{\simeq} & Z_{u,-} & \subset & [e,u]_L \\
\cup & \cup & & \cup & & \cup \\
[\varnothing, A] \supset & X'_{A,u} & \underset{\text{anti}}{\simeq} & X_{A,u} & \subset & [e,u]_L \cap [d_A^{-1}u, u] \\
 & \cup & & \cup & & \\
 & Y'_{A,u} & \underset{\text{anti}}{\simeq} & Y_{A,u} & = & X_{A,u} \cap [e,w]
\end{array}
$$

Figure 10. Relation between $Z_{u,-}, Z'_{u,-}, X_{A,u}, X'_{A,u}, Y_{A,u}, Y'_{A,u}$.

Step 3. The subset $Y_{A,u}$ of $X_{A,u}$ being an order ideal, its image $Y'_{A,u}$ under $X_{A,u} \simeq X'_{A,u}$ is an order filter of $X'_{A,u}$. Moreover $Y'_{A,u}$ is closed under intersection, reflecting join-closedness of $Y_{A,u}$. Hence $Y'_{A,u}$ is also a boolean poset. Therefore, the value of the summation over $v \in Y_{A,u}$ in (13) is 0 unless $|Y_{A,u}| = 1$ since its summands cancel out, and 1 if $|Y_{A,u}| = 1$.

Step 4. If $u \leqslant d_B w$ for some $B \subsetneq I$ with $|B| = r$ and $d_B w \geqslant_L w$, then there uniquely exists A such that $|Y_{A,u}| = 1$, and hence the value of the right-hand side in (13) is 1. If there does not exist such B, then neither does such A, and hence (13) is 0.

Remark 5.2. The set $X_{A,u}$ is a fiber of the Demazure action ϕ_{d_A}. In Step 2 (Corollary 5.14) this fiber is shown to be a boolean poset. Meanwhile, for the longest element w_J of a finite parabolic subgroup W_J, any fiber of its Demazure action ϕ_{w_J} is a parabolic coset $W_J x$, whence isomorphic to W_J. More generally it might be interesting to find fibers of the Demazure action ϕ_w of an arbitrary element w.

5.2. Proof of Theorem 1.3 and 1.4. We fix $u \in W^\circ$.

5.2.1. *Step 1.* We fix $A \subsetneq I$. Since $Y_{A,u} \subset X_{A,u}$, the summation over v in (13) is 0 when $X_{A,u} = \varnothing$. We hence assume $X_{A,u} \neq \varnothing$, since otherwise such A does not contribute to the value of the right-hand side of (13). Take arbitrary $v \in X_{A,u}$. From Lemma 2.2 and the definition of $X_{A,u}$ we have

(1) $v, d_A^{-1}u \leqslant_L u$,
(2) $d_A^{-1}u \leqslant v$.

From Proposition 3.12 (1) and (1) above, (2) is equivalent to

(3) $uv^{-1} \leqslant d_A$.

The Subword Property and (3) imply $uv^{-1} = d_B$, or equivalently $v = d_B^{-1}u$, for some $B \subset A$. We have $A, B \in Z'_{u,-}$ from (1).

The argument above is restated as follows (see also Figure 10):

Lemma 5.3.

(1) $X_{A,u} \neq \varnothing \implies A \in Z'_{u,-}$.
(2) $X_{A,u} \subset [d_A^{-1}u, u]$.
(3) $(X'_{A,u}, \subset)$ *and* $(X_{A,u}, \leqslant)$ *are anti-isomorphic by* $B \mapsto d_B^{-1}u$.
(4) $X'_{A,u} \subset [\varnothing, A]$.
(5) $X'_{A,u} \subset Z'_{u,-}$.

Proof. It remains to show that the mapping $B \mapsto d_B^{-1}u$ in (3) is order-reversing, which follows from Proposition 3.12 (1) and the Subword Property. □

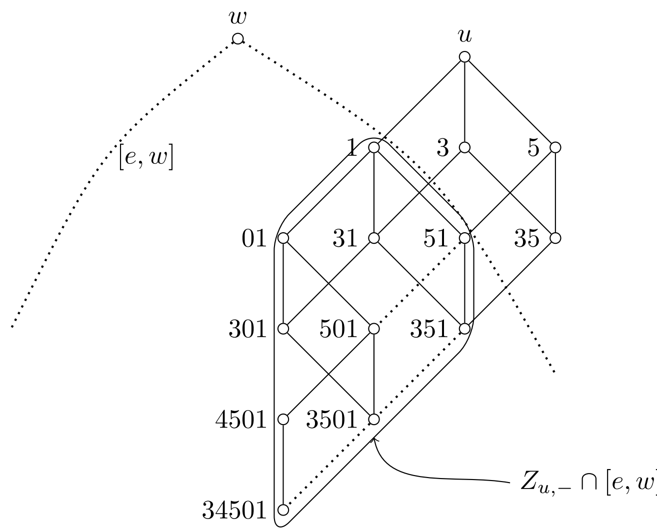


Figure 11. Each vertex labelled with $i_1\dots i_m$ represents $s_{i_1}\dots s_{i_m}u \in Z_{u,-}$. Left covers are represented by solid edges, and strong covers are dotted or solid edges.

5.2.2. *Step 2 and 3.* Let us start with an example to describe the situation.

Example 5.4. Let $k=5$, $\lambda=(5,3,2,1)$, $\mu=(5,2,2,2)$, $u=w_\lambda$ and $w=w_\mu$ (see Figure 11). When $A=\{5,0,1\}^{(2)}$, for example, $X_{A,u}=Y_{A,u}=\{s_1u, s_{01}u, s_{51}u, s_{501}u\}$ and $X'_{A,u}=Y'_{A,u}=[\{1\},\{5,0,1\}]$. Similarly, when $A=\{3,5,1\}$ we see $X'_{A,u}=[\varnothing,\{3,5,1\}]$ and $Y'_{A,u}=[\{1\},\{3,5,1\}]$.

Lemma 5.5. *X_A and Y_A are convex under the strong order. Namely, if $v\leqslant v'\leqslant v''$ and $v,v''\in X_A$ (resp. Y_A) then $v'\in X_A$ (resp. Y_A).*

Proof. It follows from Lemma 3.4(2). □

Remark 5.6. It is not a very immediate consequence of Lemma 5.5 that $X'_{A,u}$ and $Y'_{A,u}$ are convex in the boolean poset $[\varnothing, I]$, yet it is shown to be true in Corollary 5.14.

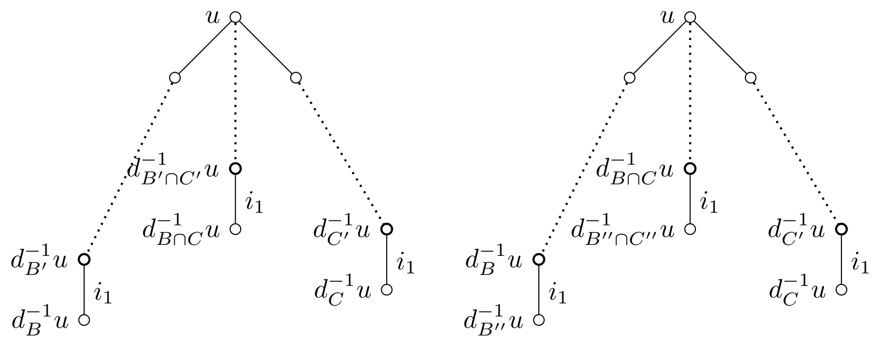
In this section we write $\{i_1,\dots,i_m\}_<$ to denote the set $\{i_1,\dots,i_m\}$ for which the condition that $(i_1,\dots,i_m)$ is cyclically increasing is imposed.

Lemma 5.7.

(1) $B,C\in X'_{A,u} \implies B\cap C\in X'_{A,u}$.
(2) $B,C\in Y'_{A,u} \implies B\cap C\in Y'_{A,u}$.

Proof. We prove (1) by induction on $|A|$. The base case $A=\varnothing$ is clear. Assume $|A|=m>0$. Write $A=\{i_1,\dots,i_m\}_<$. We need to show $\phi_{d_A}(d_{B\cap C}^{-1}u)=u$ if $\phi_{d_A}(d_B^{-1}u)=u$ and $\phi_{d_A}(d_B^{-1}u)=u$ for $B,C\subset A$. Note that $B\cap C\in Z'_{u,-}$ by Lemma 5.3(5). Let $A'=A\smallsetminus\{i_1\}$, $B'=B\smallsetminus\{i_1\}$, $C'=C\smallsetminus\{i_1\}$, $B''=B\cup\{i_1\}$ and $C''=C\cup\{i_1\}$. Note that $\phi_{d_A}=\phi_{i_m}\dots\phi_{i_1}=\phi_{d_{A'}}\phi_{i_1}$.

(2)In this example we follow the cyclic ordering $3<4<5<0<1$ on $I\smallsetminus\{2\}$, as we see $i_u^-=2$, i.e. every element of $Z'_{u,-}$ is a subset of $I\smallsetminus\{2\}$.



(When $i_1 \in B \cap C$) (When $i_1 \notin B$ and $i_1 \in C$)

Figure 12. For Lemma 5.7

Claim 5.8.

(1) $\phi_{i_1}(d_B^{-1}u) = d_{B'}^{-1}u$ *and* $\phi_{i_1}(d_C^{-1}u) = d_{C'}^{-1}u$.
(2) $B'', C'' \in Z'_{u,-}$.

Proof of Claim 5.8. We only give a proof of the statement for B since that for C is the same.

Case 1. When $i_1 \in B$, we see $d_{B''}^{-1}u = d_B^{-1}u = s_{i_1}d_{B'}^{-1}u \lessdot d_{B'}^{-1}u$, and hence both (1) and (2) are clear.

Case 2. When $i_1 \notin B$, we claim that $s_{i_1}d_B^{-1}u < d_B^{-1}u$; suppose, on the contrary, $s_{i_1}d_B^{-1}u > d_B^{-1}u$. Then we have $s_{i_1}d_B^{-1}u \not\leqslant_L u$ since $l(s_{i_1}d_B^{-1}u) > l(u) - l(s_{i_1}d_B^{-1})$. On the other hand, $u = \phi_{d_A}(d_B^{-1}u) = \phi_{d_{A'}}(s_{i_1}d_B^{-1}u)$ since $s_{i_1}d_B^{-1}u > d_B^{-1}u$, and therefore $s_{i_1}d_B^{-1}u \leqslant_L u$ by Lemma 2.2, which is in contradiction.

Therefore $s_{i_1}d_B^{-1}u < d_B^{-1}u$. Now (1) is clear since $d_B^{-1}u = d_{B'}^{-1}u$, and (2) follows from $d_{B''}^{-1}u = s_{i_1}d_B^{-1}u$. □

Claim 5.9. $\phi_{i_1}(d_{B\cap C}^{-1}u) = d_{B'\cap C'}^{-1}u$.

Proof of Claim 5.9. By Claim 5.8 (2) and Proposition 4.2 (2), we have $B''\cap C'' \in Z'_{u,-}$, that is, $u \geqslant_L d_{B''\cap C''}^{-1}u$. Since $B'' \cap C'' = (B' \cap C') \cup \{i_1\}$, we have $d_{B''\cap C''}^{-1} = s_{i_1}d_{B'\cap C'}^{-1} > d_{B'\cap C'}^{-1}$, and hence $d_{B''\cap C''}^{-1}u = s_{i_1}d_{B'\cap C'}^{-1}u \lessdot d_{B'\cap C'}^{-1}u$ by Lemma 2.1. Noting that $B \cap C = B' \cap C'$ or $B'' \cap C''$, in either case $\phi_{i_1}(d_{B\cap C}^{-1}u) = d_{B'\cap C'}^{-1}u$. □

Claim 5.10. $B' \cap C' \in X'_{A',u}$.

Proof of Claim 5.10. By Claim 5.8 (1) and that $B \in X'_{A,u}$, we have $u = \phi_{d_A}(d_B^{-1}u) = \phi_{d_{A'}}\phi_{i_1}(d_B^{-1}u) = \phi_{d_{A'}}(d_{B'}^{-1}u)$, and hence $B' \in X'_{A',u}$. Similarly $C' \in X'_{A',u}$. Hence $B' \cap C' \in X'_{A',u}$ by the induction hypothesis. □

Now we have

$$\begin{aligned} \phi_{d_A}(d_{B\cap C}^{-1}u) &= \phi_{d_{A'}}\phi_{i_1}(d_{B\cap C}^{-1}u) \\ &= \phi_{d_{A'}}(d_{B'\cap C'}^{-1}u) && \text{(by Claim 5.9)} \\ &= u. && \text{(by Claim 5.10)} \end{aligned}$$

Hence (1) is proved. (2) follows from (1) and the definition of join and $Y_{A,u}$. □

Lemma 5.11. *Let* $A, A' \in Z'_{u,-}$ *with* $A' \subset A$ *and* $|A \setminus A'| = 1$. *Then* $A' \in X'_{A,u}$.

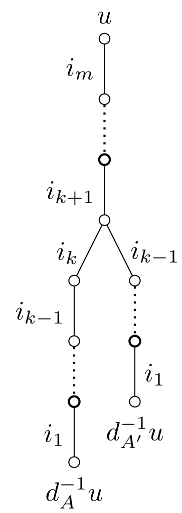


Figure 13. For Lemma 5.11

Proof. Let $A = \{i_1, \dots, i_m\}_<$ and $A' = \{i_1, \dots, \widehat{i_k}, \dots, i_m\}_<$.

Since $u \geqslant_L d_{A'}^{-1}u = s_{i_1} \dots \widehat{s_{i_k}} \dots s_{i_m} u$,

- $\phi_{i_j}(s_{i_j} \dots s_{i_{k-1}} s_{i_{k+1}} \dots s_{i_m} u) = s_{i_{j+1}} \dots s_{i_{k-1}} s_{i_{k+1}} \dots s_{i_m} u$ for $1 \leqslant j < k$,
- $\phi_{i_j}(s_{i_j} \dots s_{i_m} u) = s_{i_{j+1}} \dots s_{i_m} u$ for $k < j \leqslant m$.

Since $u \geqslant_L d_A^{-1}u = s_{i_1} \dots s_{i_m} u$,

- $\phi_{i_k}(s_{i_{k+1}} \dots s_{i_m} u) = s_{i_{k+1}} \dots s_{i_m} u$.

Hence

$$\begin{aligned}
\phi_{d_A}(d_{A'}^{-1}u) &= \phi_{i_m} \dots \phi_{i_{k+1}} \phi_{i_k} \phi_{i_{k-1}} \dots \phi_{i_1}(s_{i_1} \dots s_{i_{k-1}} s_{i_{k+1}} \dots s_{i_m} u) \\
&= \phi_{i_m} \dots \phi_{i_{k+1}} \phi_{i_k}(s_{i_{k+1}} \dots s_{i_m} u) \\
&= \phi_{i_m} \dots \phi_{i_{k+1}}(s_{i_{k+1}} \dots s_{i_m} u) \\
&= u.
\end{aligned}$$

□

Lemma 5.12. *Let $A = \{i_1, \dots, i_m\}_< \in Z'_{u,-}$ and $B \in X'_{A,u}$. By Lemma 5.3(4) we can write $B = \{i_1, \dots, \widehat{i_{j_1}}, \dots, \widehat{i_{j_l}}, \dots, i_m\}$ for some $1 \leqslant j_1 < \dots < j_l \leqslant m$. Let $A^{(a)} = \{i_{j_a+1}, i_{j_a+2}, \dots, i_{m-1}, i_m\}$ and $B^{(a)} = B \cap A^{(a)} = \{i_{j_a+1}, \dots, \widehat{i_{j_{a+1}}}, \dots, \widehat{i_{j_l}}, \dots, i_m\}$ for each $a \in \{1, \dots, l\}$. Then, for each $1 \leqslant a \leqslant l$,*

$$s_{i_{j_a}} d_{B^{(a)}}^{-1} u < d_{B^{(a)}}^{-1} u.$$

Proof. We carry out induction on $l = |A \smallsetminus B|$, with trivial base case $l = 0$. Assume $l > 0$. From Lemma 5.3(5), we have $u \geqslant_L d_B^{-1}u = s_{i_1} \dots s_{i_{j_1-1}} d_{B^{(1)}}^{-1} u$, and hence $d_{B^{(1)}}^{-1} u \geqslant_L s_{i_1} \dots s_{i_{j_1-1}} d_{B^{(1)}}^{-1} u$ by Lemma 2.1. Hence

$$\begin{aligned}
u &= \phi_{d_A}(d_B^{-1}u) \\
&= \phi_{d_{A^{(1)}}} \phi_{i_{j_1}} \phi_{i_{j_1-1}} \dots \phi_{i_1}(s_{i_1} \dots s_{i_{j_1-1}} d_{B^{(1)}}^{-1} u) \\
&= \phi_{d_{A^{(1)}}} \phi_{i_{j_1}}(d_{B^{(1)}}^{-1} u). \qquad (14)
\end{aligned}$$

We now claim $s_{i_{j_1}} d_{B^{(1)}}^{-1} u < d_{B^{(1)}}^{-1} u$; suppose to the contrary that $s_{i_{j_1}} d_{B^{(1)}}^{-1} u > d_{B^{(1)}}^{-1} u$. Then we have $s_{i_{j_1}} d_{B^{(1)}}^{-1} u \not\leqslant_L u$ since $l(s_{i_{j_1}} d_{B^{(1)}}^{-1} u) > l(u) - l(s_{i_{j_1}} d_{B^{(1)}}^{-1})$. On the other hand, $s_{i_{j_1}} d_{B^{(1)}}^{-1} u > d_{B^{(1)}}^{-1} u$ implies $\phi_{i_{j_1}}(d_{B^{(1)}}^{-1} u) = s_{i_{j_1}} d_{B^{(1)}}^{-1} u$, which implies

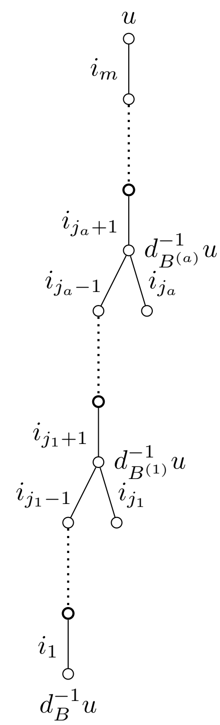


Figure 14. For Lemma 5.12

$\phi_{d_{A^{(1)}}}(s_{i_{j_1}} d_{B^{(1)}}^{-1} u) = u$ by (14), which implies $s_{i_{j_1}} d_{B^{(1)}}^{-1} u \leqslant_L u$ by Lemma 2.2, which is in contradiction.

Therefore $s_{i_{j_1}} d_{B^{(1)}}^{-1} u < d_{B^{(1)}}^{-1} u$, that is, $\phi_{i_{j_1}}(d_{B^{(1)}}^{-1} u) = d_{B^{(1)}}^{-1} u$, and hence $\phi_{d_{A^{(1)}}}(d_{B^{(1)}}^{-1} u) = u$ by (14). Hence, since $|A^{(1)} \smallsetminus B^{(1)}| = |A \smallsetminus B| - 1$, we obtain $s_{i_{j_a}} d_{B^{(a)}}^{-1} u < d_{B^{(a)}}^{-1} u$ for $a = 2, \dots, l$ by the induction hypothesis applied for $(A, B) := (A^{(1)}, B^{(1)})$. □

Lemma 5.13. *Let $A, B \in Z'_{u,-}$ with $B \subset A$. The following are equivalent*:

(1) $B \in X'_{A,u}$.
(2) $B \cup \{i\} \in Z'_{u,-}$ *for any* $i \in A \smallsetminus B$.
(3) $B \cup \{i\} \in X'_{A,u}$ *for any* $i \in A \smallsetminus B$.
(4) $A \smallsetminus \{i\} \in Z'_{u,-}$ *for any* $i \in A \smallsetminus B$.
(5) $A \smallsetminus \{i\} \in X'_{A,u}$ *for any* $i \in A \smallsetminus B$.
(6) $[B, A] \subset Z'_{u,-}$.
(7) $[B, A] \subset X'_{A,u}$.

Proof. (2)⇔(4)⇔(6). (6)⇒(4) and (6)⇒(2) are obvious. (2)⇒(4)⇒(6) is from Lemma 4.2 (1).

(1)⇒(2). We use the notations $A^{(a)}$ and $B^{(a)}$ in Lemma 5.12. From Lemma 5.12 we have $\{i_{j_a}\} \cup B^{(a)} \in Z'_{u,-}$ for any a, and hence $B \cup \{i_{j_a}\} = (\{i_{j_a}\} \cup B^{(a)}) \cup B \in Z'_{u,-}$ by Proposition 4.2 (1).

(1)⇒(7). We already proved (1)⇒(2)⇔(6). Hence, since $A, B \in X'_{A,u}$ and $[B, A] \subset Z'_{u,-}$, by Lemma 5.5 we have $[B, A] \subset X'_{A,u}$.

(1)⇔(3)⇔(5)⇔(7). It is obvious that (7)⇒(3), (5). From Lemma 5.7 (1) we have (3)⇒(1) and (5)⇒(1). Besides we already proved (1)⇒(7).

(4)⇒(5). By Lemma 5.11. □

We write $\bigcap X = \bigcap_{x \in X} x$ for a set X of sets.

Corollary 5.14. *We have $X'_{A,u} = [\bigcap X'_{A,u}, A]$ if $X'_{A,u} \neq \varnothing$, and $Y'_{A,u} = [\bigcap Y'_{A,u}, A]$ if $Y'_{A,u} \neq \varnothing$. In particular, $X'_{A,u}$ and $Y'_{A,u}$ are isomorphic to boolean posets, and therefore so are $X_{A,u}$ and $Y_{A,u}$.*

Proof. Assume $Y'_{A,u}$ is nonempty. Then $Y'_{A,u}$ has the minimum element $C = \bigcap Y'_{A,u}$ by Lemma 5.7 (2). By Lemma 5.13 (1)⇒(7) we have $[C, A] \subset X'_{A,u}$. Moreover, since $Y_{A,u}$ is an order ideal of $X_{A,u}$ we have $Y'_{A,u}$ is an order filter of $X'_{A,u}$, and therefore $[C, A] \subset Y'_{A,u}$. The opposite inclusion $Y'_{A,u} \subset [C, A]$ is implied by minimality of C. Therefore $Y'_{A,u} = [C, A]$.

It is proved similarly that $X'_{A,u} = [\bigcap X'_{A,u}, A]$ whenever $X'_{A,u} \neq \varnothing$. □

Therefore we have

$$\begin{aligned}
\sum_{v \in Y_{A,u}} (-1)^{|A|-(l(u)-l(v))} &= \sum_{B \in Y'_{A,u}} (-1)^{|A|-(l(u)-l(d_B^{-1}u))} \\
&= \sum_{B \in Y'_{A,u}} (-1)^{|A|-|B|} \\
&= \begin{cases} 1 & \text{if } |Y'_{A,u}| = 1, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} 1 & \text{if } |Y_{A,u}| = 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{15}$$

5.2.3. *Step 4.* Next we discuss which A satisfies the condition $|Y_{A,u}| = 1$.

Since $Z_{u,-} \subset [e, u]_L$ is an order filter, so is $Z_{u,-} \cap [e, w] \subset [e, u]_L \cap [e, w]$. Hence, if $(w_S \wedge_L u =) \max([e, u]_L \cap [e, w]) \notin Z_{u,-}$, then $Z_{u,-} \cap [e, w] = \varnothing$, and hence $Y_{A,u} = \varnothing$ for any A since $Y_{A,u} = X_{A,u} \cap [e, w] \subset Z_{u,-} \cap [e, w]$. We hence assume $w_S \wedge_L u \in Z_{u,-}$ and write $w_S \wedge_L u = d_{A_0}^{-1} u$ with $A_0 \in Z'_{u,-}$. Write $Z_{u,-}^{\leqslant w} = Z_{u,-} \cap [e, w]$. Note that $w_S \wedge_L u = \max Z_{u,-}^{\leqslant w}$.

Example 5.15. Recall Example 5.4. In that case $\max(Z_{u,-} \cap [e, w]) = s_1 u$ and hence $A_0 = \{1\}$. It is easily checked that $X_{\{1\},u} = \{u, s_1 u\}$ and $Y_{\{1\},u} = \{s_1 u\}$.

Lemma 5.16. $|Y_{A,u}| = 1 \iff A = A_0$.

Proof. ($\Longrightarrow$). Clearly $d_{A_0}^{-1} u \in Y_{A_0,u}$. On the contrary, take any $v \in Y_{A_0,u}$. Then $v = d_B^{-1} u$ for some $B \in Y'_{A_0,u}$. Since $Y'_{A_0,u} \subset X'_{A_0,u} \subset [\varnothing, A_0]$, we have $B \subset A_0$. On the other hand, since $v \in Y_{A_0,u} = X_{A_0,u} \cap [e, w] \subset Z_{u,-}^{\leqslant w}$, we have $v \leqslant \max Z_{u,-}^{\leqslant w} = d_{A_0}^{-1} u$, and hence $B \supset A_0$. Therefore $B = A_0$.

($\Longleftarrow$). If $A \notin Z'_{u,-}$, then $|Y_{A,u}| \leqslant |X_{A,u}| = 0$ from Lemma 5.3 (1). We hence assume $A \in Z'_{u,-}$. Then $d_A^{-1} u \in Z_{u,-}$.

If $d_A^{-1} u \not\leqslant w$, then $Y_{A,u} = \varnothing$ since $d_A^{-1} u$ is the minimum element of $X_{A,u}$ and $Y_{A,u} = X_{A,u} \cap [e, w]$ is an order ideal of $X_{A,u}$.

Hence we assume $d_A^{-1} u \leqslant w$. Since $d_{A_0}^{-1} u = \max Z_{u,-}^{\leqslant w}$, we have $d_A^{-1} u \leqslant d_{A_0}^{-1} u$, and hence $A_0 \subset A$. Suppose $A_0 \subsetneq A$. By Corollary 4.17 there exists an $A' \in Z'_{u,-}$ such that $A_0 \subset A' \subset A$ and $|A \setminus A'| = 1$. By Lemma 5.11 and that $d_{A'}^{-1}u \leqslant d_{A_0}^{-1}u \leqslant w$ we have $d_{A'}^{-1}u \in Y_{A,u}$. Hence $Y_{A,u} \supset \{d_A^{-1}u, d_{A'}^{-1}u\}$. □

Therefore, substituting (15) and the result of Lemma 5.16 into the right-hand side of (13) and noting that $|A_0| = l(u) - l(w_S \wedge_L u)$, we have

$$[g_u^{(k)}](\widetilde{g}_w^{(k)}\widetilde{h}_r) = \begin{cases} 1 & \text{if } w_S \wedge_L u \in Z_{u,-} \text{ and } l(u) - l(w_S \wedge_L u) \leqslant r, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we show the following:

Lemma 5.17. *The following are equivalent*:

(1) $w_S \wedge_L u \in Z_{u,-}$ *and* $l(u) - l(w_S \wedge_L u) \leqslant r$.
(2) *There exists* A *such that* $|A| \leqslant r$ *and* $u \geqslant_L d_A^{-1}u \leqslant w$.
(3) *There exists* A *such that* $|A| \leqslant r$ *and* $u \leqslant d_A w \geqslant_L w$.
(4) *There exists* A *such that* $|A| = r$ *and* $u \leqslant d_A w \geqslant_L w$.

Proof. (1)⇔(2). Clear.

(3)⇔(4). (4)⇒(3) is obvious. (3)⇒(4) follows from the fact that $Z'_{u,+}$ has the Chain Property and the maximum element of size k, which corresponds to the maximum element of $Z_{u,+}$.

(2)⇒(3). Assume $u \geqslant_L d_A^{-1}u \leqslant w$. Then $u = \phi_{d_A}(d_A^{-1}u) \leqslant \phi_{d_A}(w)$ by Lemma 3.4 (2). Besides, we have $\phi_{d_A}(w) = d_B w \geqslant_L w$ for some $B \subset A$ by Lemma 2.2, and $|B| \leqslant |A| \leqslant r$.

(3)⇒(2). Proved similarly to (2)⇒(3), with Lemma 2.3 instead of Lemma 2.2. □

Now we finished, from Lemma 5.17 (1)⇔(4), the proof of Theorem 1.3:

$$\widetilde{g}_w^{(k)}\widetilde{h}_r = \sum_u g_u^{(k)},$$

summed over $u \in W^\circ$ such that $u \leqslant d_A w$ for some $A \subsetneq I$ with $|A| = r$ and $d_A w \geqslant_L w$.

Theorem 1.4 follows from Theorem 1.3, Corollary 4.8, and the Inclusion-Exclusion Principle.

6. Proof of the k-rectangle factorization formula

This section is devoted for the proof of Theorem 1.5.

The idea of the proof is similar to that of Proposition 2.18; we consider a linear map $\Theta : \Lambda_{(k)} \longrightarrow \Lambda_{(k)}$ extending $\widetilde{g}_\lambda^{(k)} \mapsto \widetilde{g}_{R_t \cup \lambda}^{(k)}$, having that $\{\widetilde{g}_\lambda^{(k)}\}_{\lambda \in \mathcal{P}_k}$ forms a basis of $\Lambda_{(k)}$. It suffices to show Θ is a $\Lambda_{(k)}$-homomorphism, since it implies $\widetilde{g}_{R_t \cup \lambda}^{(k)} = \Theta(\widetilde{g}_\lambda^{(k)}) = \widetilde{g}_\lambda^{(k)}\Theta(1) = \widetilde{g}_\lambda^{(k)}\Theta(\widetilde{g}_\varnothing^{(k)}) = \widetilde{g}_\lambda^{(k)}\widetilde{g}_{R_t}^{(k)}$. Since $\{\widetilde{h}_i\}_{1 \leqslant i \leqslant k}$ generate $\Lambda_{(k)}$, we only need to show

$$\Theta(\widetilde{h}_r \widetilde{g}_\lambda^{(k)}) = \widetilde{h}_r \Theta(\widetilde{g}_\lambda^{(k)}). \tag{16}$$

Let $d_{A_1}\lambda, d_{A_2}\lambda, \dots$ be the list of all weak strips over λ of size r. Applying Theorem 1.4 to both sides of (16), we have

$$\begin{aligned} \text{(LHS)} &= \Theta\left(\sum_a \widetilde{g}_{d_{A_a}\lambda}^{(k)} - \sum_{a<b} \widetilde{g}_{d_{A_a \cap A_b}\lambda}^{(k)} + \cdots\right) \\ &= \sum_a \widetilde{g}_{R_t \cup (d_{A_a}\lambda)}^{(k)} - \sum_{a<b} \widetilde{g}_{R_t \cup (d_{A_a \cap A_b}\lambda)}^{(k)} + \cdots, \end{aligned} \tag{17}$$

and by Lemma 2.16 (3) we have

$$\begin{aligned}(\mathrm{RHS}) &= \widetilde{h}_r \widetilde{g}^{(k)}_{R_t \cup \lambda} \\ &= \sum_a \widetilde{g}^{(k)}_{d_{A_a+t}(R_t\cup\lambda)} - \sum_{a<b} \widetilde{g}^{(k)}_{d_{(A_a+t)\cap(A_b+t)}(R_t\cup\lambda)} + \cdots . \end{aligned} \tag{18}$$

Since $(A_a + t) \cap (A_b + t) \cap \cdots = (A_a \cap A_b \cap \cdots) + t$, by Lemma 2.16 (1) we have (17) = (18).

Now Theorem 1.5 is proved.

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References

[1] David Anderson, Linda Chen, and Hsian-Hua Tseng, *On the quantum K-ring of the flag manifold*, https://arxiv.org/abs/1711.08414, 2017.

[2] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, 2005.

[3] Anders Björner and Michelle Wachs, *Bruhat order of Coxeter groups and shellability*, Adv. Math. **43** (1982), no. 1, 87–100.

[4] ———, *Generalized quotients in Coxeter groups*, Trans. Am. Math. Soc. **308** (1988), no. 1, 1–37.

[5] Anders S. Buch and Leonardo C. Mihalcea, *Curve neighborhoods of Schubert varieties*, J. Differ. Geom. **99** (2015), no. 2, 255–283.

[6] Tom Denton, *Canonical decompositions of affine permutations, affine codes, and split k-Schur functions*, Electron. J. Comb. **19** (2012), no. 4, 19 (41 pages).

[7] Vinay V. Deodhar, *A splitting criterion for the Bruhat orderings on Coxeter groups*, Commun. Algebra **15** (1987), no. 9, 1889–1894.

[8] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, 1990.

[9] Takeshi Ikeda, Shinsuke Iwao, and Toshiaki Maeno, *Peterson isomorphism in K-theory and relativistic Toda lattice*, Int. Math. Res. Not. (2018), rny051.

[10] Syu Kato, *Loop structure on equivariant K-theory of semi-infinite flag manifolds*, https://arxiv.org/abs/1805.01718, 2018.

[11] Allen Knutson and Ezra Miller, *Subword complexes in Coxeter groups*, Adv. Math. **184** (2004), no. 1, 161–176.

[12] Thomas Lam, *Schubert polynomials for the affine Grassmannian*, J. Am. Math. Soc. **21** (2008), no. 1, 259–281.

[13] Thomas Lam, Luc Lapointe, Jennifer Morse, Anne Schilling, Mark Shimozono, and Mike Zabrocki, *k-Schur functions and affine Schubert calculus*, Fields Institute Monographs, vol. 33, The Fields Institute for Research in the Mathematical Sciences, 2014.

[14] Thomas Lam, Changzheng Li, Leonardo C. Mihalcea, and Mark Shimozono, *A conjectural Peterson isomorphism in K-theory*, J. Algebra **513** (2018), 326–343.

[15] Thomas Lam, Anne Schilling, and Mark Shimozono, *K-theory Schubert calculus of the affine Grassmannian*, Compos. Math. **146** (2010), no. 4, 811–852.

[16] Thomas Lam and Mark Shimozono, *Quantum cohomology of G/P and homology of affine Grassmannian*, Acta Math. **204** (2010), no. 1, 49–90.

[17] ———, *From quantum Schubert polynomials to k-Schur functions via the Toda lattice*, Math. Res. Lett. **19** (2012), no. 1, 81–93.

[18] Luc Lapointe, Alain Lascoux, and Jennifer Morse, *Tableau atoms and a new Macdonald positivity conjecture*, Duke Math. J. **116** (2003), no. 1, 103–146.

[19] Luc Lapointe and Jennifer Morse, *Order ideals in weak subposets of Young's lattice and associated unimodality conjectures*, Ann. Comb. **8** (2004), no. 2, 197–219.

[20] ———, *Tableaux on $k+1$-cores, reduced words for affine permutations, and k-Schur expansions*, J. Comb. Theory, Ser. A **112** (2005), no. 1, 44–81.
[21] ———, *A k-tableau characterization of k-Schur functions*, Adv. Math. **213** (2007), no. 1, 183–204.
[22] Ian G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Science Publications, Clarendon Press, 1995, With contributions by A. Zelevinsky.
[23] Jennifer Morse, *Combinatorics of the K-theory of affine Grassmannians*, Adv. Math. **229** (2012), no. 5, 2950–2984.
[24] Mark Shimozono, private communication, 2018.
[25] John R. Stembridge, *A short derivation of the Möbius function for the Bruhat order*, J. Algebr. Comb. **25** (2007), no. 2, 141–148.
[26] Motoki Takigiku, *Factorization formulas of K-k-Schur functions I*, https://arxiv.org/abs/1704.08643, 2017.
[27] ———, *Factorization formulas of K-k-Schur functions II*, https://arxiv.org/abs/1704.08660, 2017.
[28] Debra J. Waugh, *Upper bounds in affine Weyl groups under the weak order*, Order **16** (1999), no. 1, 77–87.

Motoki Takigiku, Graduate School of Mathematical Sciences, the University of Tokyo, Japan
E-mail : takigiku@ms.u-tokyo.ac.jp