



ALGEBRAIC COMBINATORICS

Serena An, Katherine Tung & Yuchong Zhang

Newton polytopes of dual Schubert polynomials

Volume 8, issue 6 (2025), p. 1459-1472.

<https://doi.org/10.5802/alco.450>

© The author(s), 2025.



This article is licensed under the
CREATIVE COMMONS ATTRIBUTION (CC-BY) 4.0 LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486





Newton polytopes of dual Schubert polynomials

Serena An, Katherine Tung & Yuchong Zhang

ABSTRACT The M-convexity of dual Schubert polynomials was first proven by Huh, Matherne, Mészáros, and St. Dizier in 2022. We give a full characterization of the support of dual Schubert polynomials, which yields an elementary alternative proof of the M-convexity result, and furthermore strengthens it by explicitly characterizing the vertices of their Newton polytopes combinatorially. Using this characterization, we give a polynomial-time algorithm to determine if a coefficient of a dual Schubert polynomial is zero, analogous to a result of Adve, Robichaux, and Yong for Schubert polynomials.

1. INTRODUCTION

The dual Schubert polynomials $\{D^w\}_{w \in S_n}$, introduced by Bernstein, Gelfand, and Gelfand [5], are given by the following combinatorial formula [17]. Let the edge $u \triangleleft ut_{ab}$ in the (strong) Bruhat order have weight

$$m(u \triangleleft ut_{ab}) := x_a + x_{a+1} + \cdots + x_{b-1},$$

and let the saturated chain $C = (u_0 \triangleleft u_1 \triangleleft \cdots \triangleleft u_\ell)$ have weight

$$m_C := m(u_0 \triangleleft u_1)m(u_1 \triangleleft u_2) \cdots m(u_{\ell-1} \triangleleft u_\ell).$$

DEFINITION 1.1. For $w \in S_n$, the dual Schubert polynomial D^w is defined by

$$D^w(x_1, \dots, x_{n-1}) := \frac{1}{\ell(w)!} \sum_C m_C(x_1, \dots, x_{n-1}),$$

where $\ell(w)$ denotes the Coxeter length of w , and the sum is over all saturated chains C from id to w .

Dual Schubert polynomials possess deep geometric and algebraic properties. For example, the λ -degree of a Schubert variety X_w can be expressed as $\ell(w)! \cdot D^w(\lambda)$ [17]. Additionally, dual Schubert polynomials form a dual basis to Schubert polynomials under a certain natural pairing $\langle \cdot, \cdot \rangle$ on polynomials [17].

In recent years, dual Schubert polynomials have attracted increasing interest [9, 11, 12]. We continue the study of dual Schubert polynomials by fully characterizing their supports (set of exponent vectors) and *Newton polytopes* (convex hull of the support).

Manuscript received 31st January 2025, revised and accepted 28th July 2025.

KEYWORDS. Newton polytope, dual Schubert polynomial, M-convex.

THEOREM 1.2. *The support of the dual Schubert polynomial D^w is*

$$\text{supp}(D^w) = \sum_{(a,b) \in \text{Inv}(w)} \{e_a, e_{a+1}, \dots, e_{b-1}\},$$

where the right-hand side is a Minkowski sum of sets of elementary basis vectors. The sum is over pairs of indices (a, b) for which there is an inversion in w .

A polynomial f has a *saturated Newton polytope (SNP)* if its support is all integer points in its Newton polytope. The SNP property was first defined by Monical, Tokcan and Yong [14]. Many polynomials with algebraic combinatorial significance are known to have SNP, such as Schur polynomials [18] and resultants [10]. Monical, Tokcan, and Yong proved SNP for additional families of polynomials, including cycle index polynomials, Reutenauer’s symmetric polynomials linked to the free Lie algebra and to Witt vectors, Stembridge’s symmetric polynomials associated to totally nonnegative matrices, and symmetric Macdonald polynomials. Subsequent work of Fink, Mészáros, and St. Dizier proved SNP for key polynomials and Schubert polynomials [8], work of Castillo, Cid Ruiz, Mohammadi, and Montaña proved SNP for double Schubert polynomials [7], and work of Huh, Matherne, Mészáros, and St. Dizier [12] proved Lorentzian-ness, which implies SNP, for dual Schubert polynomials.

Proving SNP can be difficult given that many polynomial operations, such as multiplication, do not preserve SNP. However, a wide range of techniques have been harnessed to prove SNP. For instance, Rado uses elementary combinatorial techniques [18], Fink, Mészáros, and St. Dizier rely on representation theory [8], and Castillo et al. as well as Huh et al. use results from algebraic geometry [7, 12].

As a corollary of Theorem 1.2, we obtain an elementary proof of the following result.

COROLLARY 1.3. (cf. [12, Proposition 18]) *Dual Schubert polynomials are M-convex; equivalently, they have SNP and their Newton polytopes are generalized permutahedra.*

Strengthening this result, we fully characterize the vertices of $\text{Newton}(D^w)$.

COROLLARY 1.4. *For $w \in S_n$, the vertices of $\text{Newton}(D^w)$ are*

$$\{\alpha \in \mathbb{Z}_{\geq 0}^{n-1} \mid x^\alpha \text{ has coefficient 1 in } \prod_{(a,b) \in \text{Inv}(w)} (x_a + x_{a+1} + \dots + x_{b-1})\}.$$

Furthermore, in light of [16, Corollary 8.2], we give a combinatorial characterization of the vertices of $\text{Newton}(D^w)$ using rectangle tilings of staircase Young diagrams.

This paper is organized as follows. In Section 2, we give the necessary relevant background and definitions. In Section 3, we prove that dual Schubert polynomials are M-convex. In Section 4, we characterize the vertices of Newton polytopes of dual Schubert polynomials in two different ways. In Section 5, we solve the vanishing problem for dual Schubert polynomials in polynomial time, extending a result in [1].

2. PRELIMINARIES

2.1. BRUHAT ORDER. In this paper, we use standard terminology for the (strong) Bruhat order. The readers are referred to [6, Chapter 2] for a more detailed exposition.

Let S_n be the symmetric group on the set $[n] := \{1, \dots, n\}$. We write each permutation $w \in S_n$ in one-line notation as $w = w(1)w(2) \dots w(n)$.

Each permutation $w \in S_n$ can be expressed as a product of *simple transpositions* $s_i = \{(i, i + 1) : 1 \leq i \leq n - 1\}$. If $w = s_{i_1} \dots s_{i_\ell}$ is expressed as a product of simple transpositions with ℓ minimal among all such expressions, then the string $s_{i_1} \dots s_{i_\ell}$ is called a *reduced decomposition* of w . We call $\ell := \ell(w)$ the (Coxeter) length of w .

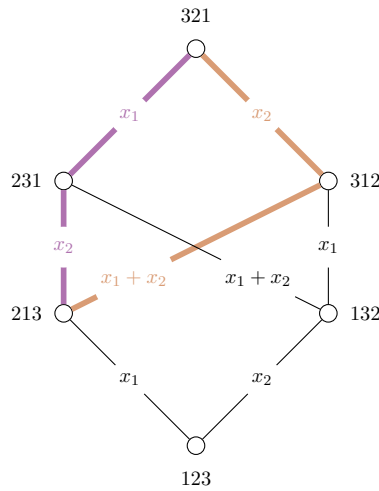


FIGURE 1. Calculating the Postnikov–Stanley polynomial D_{213}^{321} .

An *inversion* of w is an ordered pair $(a, b) \in [n]^2$ such that $a < b$ and $w(a) > w(b)$. We denote the set of all inversions of w by $\text{Inv}(w)$. It is well-known that $\ell(w) = |\text{Inv}(w)|$. Given $1 \leq a < b \leq n$, let t_{ab} act on $w \in S_n$ such that wt_{ab} is the permutation obtained by transposing the two numbers in positions a and b in w .

DEFINITION 2.1. ([6]) We define a partial order \leq on S_n called the strong Bruhat order as follows. Let $u, v \in S_n$ and $\ell = \ell(v)$. We have $u \leq v$ if and only if for any reduced decomposition $s_{i_1} \cdots s_{i_\ell}$ of v , there exists a reduced decomposition $s_{j_1} \cdots s_{j_k}$ of u such that $s_{j_1} \cdots s_{j_k}$ is a substring of $s_{i_1} \cdots s_{i_\ell}$. If additionally $\ell(v) = \ell(u) + 1$, we write $u < v$. Alternatively, we may characterize the covering relation as follows. For $u, v \in S_n$, we have $u < v$ if and only if $v = ut_{ab}$ and $\ell(v) = \ell(u) + 1$.

For $u \leq w$ in the Bruhat order of S_n , the (Bruhat) interval $[u, w]$ is the subposet containing all $v \in S_n$ such that $u \leq v \leq w$.

2.2. POSTNIKOV–STANLEY POLYNOMIALS. For $w \in S_n$, we define Postnikov–Stanley polynomials, which generalize dual Schubert polynomials.

DEFINITION 2.2. ([17]) Given $u \leq w$ in S_n , the Postnikov–Stanley polynomial D_u^w is defined by

$$D_u^w(x_1, \dots, x_{n-1}) := \frac{1}{(\ell(w) - \ell(u))!} \sum_C m_C(x_1, \dots, x_{n-1}),$$

where the sum is over all saturated chains C from u to w .

As defined in the introduction, the dual Schubert polynomial D^w is given by $D^w := D_{\text{id}}^w$.

EXAMPLE 2.3. Let $u = 213$ and $w = 321$. In the Bruhat order of S_3 , there are two saturated chains from u to w , namely $213 < 231 < 321$ and $213 < 312 < 321$, as shown in Figure 1. The first chain has weight x_1x_2 and the second chain has weight $(x_1 + x_2) \cdot x_2$. Thus, $D_{213}^{321} = \frac{1}{2!}(x_1x_2 + (x_1 + x_2) \cdot x_2)$.

REMARK 2.4. Dual Schubert polynomials D^w and Postnikov–Stanley polynomials D_u^w are defined for u, w in any Weyl group W [17]. However, we only consider the case

$W = S_n$ in this paper. For a treatment of dual Schubert polynomials and Postnikov–Stanley polynomials in their full generality, please see our follow-up paper [3].

2.3. NEWTON POLYTOPES. We formalize the parenthetical definitions of the Newton polytope given in the introduction.

DEFINITION 2.5. For a tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, let x^α denote the monomial

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{R}[x_1, \dots, x_n].$$

We call α the exponent vector of x^α .

DEFINITION 2.6. Let $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_\alpha x^\alpha \in \mathbb{R}[x_1, \dots, x_n]$ be a polynomial. The support of f , denoted $\text{supp}(f)$, is the set of exponent vectors α of the nonzero terms of f .

DEFINITION 2.7. The Newton polytope of a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$, denoted $\text{Newton}(f)$, is the convex hull of $\text{supp}(f)$ in \mathbb{R}^n .

The support and Newton polytope of a polynomial behave nicely with respect to polynomial addition and multiplication as follows.

PROPOSITION 2.8. For two polynomials $f, g \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$, we have

$$\begin{aligned} \text{supp}(fg) &= \text{supp}(f) + \text{supp}(g) \\ \text{supp}(f + g) &= \text{supp}(f) \cup \text{supp}(g), \end{aligned}$$

where $+$ in the first line denotes the Minkowski sum.

PROPOSITION 2.9 ([14]). For two polynomials $f, g \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$, we have

$$\begin{aligned} \text{Newton}(fg) &= \text{Newton}(f) + \text{Newton}(g) \\ \text{Newton}(f + g) &= \text{conv}(\text{Newton}(f) \cup \text{Newton}(g)), \end{aligned}$$

where $+$ in the first line denotes the Minkowski sum, and conv denotes the convex hull.

DEFINITION 2.10. ([14]) A polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ has saturated Newton polytope (SNP) if $\text{supp}(f)$ consists of all integer points in $\text{Newton}(f)$.

EXAMPLE 2.11. As computed in Example 2.3, $D_{213}^{321} = x_1x_2 + \frac{1}{2}x_2^2$, so its Newton polytope $\text{Newton}(D_{213}^{321})$ is the line segment from $(1, 1)$ to $(0, 2)$ in \mathbb{R}^2 . There are no integer points on this line segment besides the endpoints, so D_{213}^{321} has SNP.

REMARK 2.12. A nonexample for SNP is the polynomial

$$f = x^{(0,1,3)} + x^{(0,3,1)} + x^{(1,0,3)} + x^{(1,3,0)} + x^{(3,0,1)} + x^{(3,1,0)},$$

because $\text{Newton}(f)$ contains the integer point $(0, 2, 2)$, but there is no $cx^{(0,2,2)}$ monomial in f for c nonzero.

2.4. MATROID POLYTOPES. Matroid polytopes, a special type of Newton polytope, were heavily used in [7] to prove that double Schubert polynomials have SNP and characterize their Newton polytopes. We also leverage the following results about matroid polytopes to discuss products of linear polynomials in Section 3.1 and characterize the Newton polytope of dual Schubert polynomials in Section 3.3.

DEFINITION 2.13. A matroid $M = (E, \mathcal{B})$ consists of a finite set E and a nonempty collection of subsets \mathcal{B} of E , called the bases of M , which satisfy the basis exchange axiom: if $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 \setminus B_2$, then there exists $b_2 \in B_2 \setminus B_1$ such that $B_1 \setminus \{b_1\} \cup \{b_2\} \in \mathcal{B}$.

In the remainder of this section, we let $E = [n]$.

DEFINITION 2.14. The matroid polytope $P(M)$ of a matroid $M = ([n], \mathcal{B})$ is

$$P(M) = \text{conv}(\{\zeta^B : B \in \mathcal{B}\}),$$

where $\zeta^B = (\mathbb{1}_{i \in B})_{i=1}^n$ denotes the indicator vector of B .

Generalized permutahedra were first studied in [16], and many nice connections to matroid polytopes have been found since.

A standard permutahedron (or permutohedron) is the convex hull in \mathbb{R}^n of the vector $(0, 1, \dots, n - 1)$ and all permutations of its entries. Generalized permutahedra, which are deformations of standard permutahedra, are defined as follows.

DEFINITION 2.15. The generalized permutahedron $P_n^z(\{z_I\})$ associated to the collection of real numbers $\{z_I\}$ for $I \subseteq [n]$ is given by

$$P_n^z(\{z_I\}) = \left\{ t \in \mathbb{R}^n : \sum_{i \in I} t_i \geq z_I \text{ for } I \neq [n], \sum_{i=1}^n t_i = z_{[n]} \right\}.$$

PROPOSITION 2.16 ([4]). Generalized permutahedra are closed under Minkowski sums:

$$P_n^z(\{z_I\}) + P_n^z(\{z'_I\}) = P_n^z(\{z_I + z'_I\}).$$

DEFINITION 2.17. For a matroid $M = ([n], \mathcal{B})$, the rank function $r_M : 2^{[n]} \rightarrow \mathbb{Z}_{\geq 0}$ on subsets of $[n]$ is given by

$$r_M(S) = \max\{\#(S \cap B) : B \in \mathcal{B}\}.$$

PROPOSITION 2.18 ([4]). Matroid polytopes are generalized permutahedra with

$$\begin{aligned} P(M) &= P_n^z(\{r_M([n]) - r_M([n] \setminus I)\}) \\ &= \left\{ t \in \mathbb{R}^n : \sum_{i \in I} t_i \leq r_M(I) \text{ for } I \neq [n], \sum_{i=1}^n t_i = r_M([n]) \right\}. \end{aligned}$$

DEFINITION 2.19. ([15]) A subset $\mathcal{J} \subset \mathbb{Z}^n$ is M -convex if for any index $i \in [n]$ and any $\alpha, \beta \in \mathcal{J}$ whose i th coordinates satisfy $\alpha_i > \beta_i$, there is an index $j \in [n]$ satisfying

$$\alpha_j < \beta_j \quad \text{and} \quad \alpha - e_i + e_j \in \mathcal{J} \quad \text{and} \quad \beta - e_j + e_i \in \mathcal{J}.$$

A polynomial is M -convex if $\text{supp}(f)$ is an M -convex set.

PROPOSITION 2.20. ([15, Theorem 4.15], [12]) A homogeneous polynomial f is M -convex if and only if f has SNP and $\text{Newton}(f)$ is a generalized permutahedron.

3. M-CONVEXITY OF DUAL SCHUBERT POLYNOMIALS

In this section, we prove Theorem 1.2 and Corollary 1.3. Our key insight is that the support of any dual Schubert polynomial D^w always matches the support of the weight of a specific chain in $[\text{id}, w]$. After showing that the weight of any chain has SNP and has a Newton polytope that is a generalized permutahedron, we obtain the desired result.

3.1. THE SINGLE CHAIN NEWTON POLYTOPE (SCNP) PROPERTY. We introduce the single chain Newton polytope property and show that it implies, but is not equivalent to, the SNP property.

DEFINITION 3.1. The Postnikov–Stanley polynomial D_u^w has single chain Newton polytope (SCNP) if there exists a saturated chain C in the interval $[u, w]$ such that

$$\text{supp}(m_C) = \text{supp}(D_u^w).$$

Such a saturated chain C is called a dominant chain of the interval $[u, w]$.

EXAMPLE 3.2. Given the Postnikov–Stanley polynomial $D_{213}^{321} = \frac{1}{2!}(x_1x_2 + (x_1 + x_2) \cdot x_2)$, the saturated chain $C = (213 \triangleleft 312 \triangleleft 321)$ has weight $m_C = (x_1 + x_2) \cdot x_2$, which satisfies

$$\text{supp}(m_C) = \text{supp}(D_{213}^{321}).$$

Thus, C is a dominant chain of $[213, 321]$, and D_{213}^{321} has SCNP.

REMARK 3.3. The SCNP property is stronger than the SNP property: for example, D_{1324}^{4231} has SNP but not SCNP, according to calculations with SageMath [19]. As Postnikov–Stanley polynomials do not necessarily have SCNP, the method of using SCNP to prove SNP for dual Schubert polynomials does not generalize to Postnikov–Stanley polynomials.

The following propositions motivate the definition of SCNP.

PROPOSITION 3.4. *If a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ can be written as a product of nonnegative linear combinations of x_1, \dots, x_n , then f has SNP.*

Proof. This follows from [2, Theorem 3.4]. □

PROPOSITION 3.5. *If D_u^w has SCNP, then D_u^w has SNP.*

Proof. By Proposition 3.4, the chain weight m_C has SNP for any saturated chain C in the Bruhat order. If, by the definition of SCNP, there exists a saturated chain C in the interval $[u, w]$ such that $\text{supp}(m_C) = \text{supp}(D_u^w)$, then D_u^w also has SNP. □

3.2. USING SCNP TO PROVE SNP. In this section, we prove that dual Schubert polynomials have SCNP. As a corollary, we obtain Theorem 1.2 and the result that dual Schubert polynomials have SNP.

DEFINITION 3.6. *For a Bruhat interval $[u, w]$, a saturated chain*

$$u = w_0 \triangleleft w_1 \triangleleft w_2 \triangleleft \dots \triangleleft w_\ell = w$$

from u to w is a greedy chain if it satisfies the following condition for all $i \in [\ell]$: writing $w_i = w_{i-1}t_{ab}$ for $a < b$, there does not exist $w'_{i-1} \triangleleft w_i$ with $w'_{i-1} \in [u, w]$ such that

- (i) $w_i = w'_{i-1}t_{ab'}$ for $b' > b$, or
- (ii) $w_i = w'_{i-1}t_{a'b}$ for $a' < a$.

EXAMPLE 3.7. In the Bruhat interval $[123, 321]$, the saturated chain $123 \triangleleft 132 \triangleleft 231 \triangleleft 321$ is greedy, while $123 \triangleleft 213 \triangleleft 231 \triangleleft 321$ is not. This chain fails to be greedy because

$$w_2 = 231 = 213t_{23} = w_1t_{23}$$

but $132 \triangleleft w_2$ with $231 = 132t_{13}$, violating condition (ii). In general, greedy chains are not unique. For example, $123 \triangleleft 213 \triangleleft 312 \triangleleft 321$ is another greedy chain in $[123, 321]$.

LEMMA 3.8. *There exists a greedy chain in every Bruhat interval $[u, w]$.*

Proof. We construct a greedy chain $u = u_\ell \triangleleft u_{\ell-1} \triangleleft \dots \triangleleft u_0 = w$ inductively, from the top down. Given a greedy chain $u_{i-1} \triangleleft \dots \triangleleft u_0 = w$ for $i \in [\ell]$, set $u_i := u_{i-1}t_{ab}$ such that (a, b) cannot be replaced by some longer (a', b) or (a, b') . □

DEFINITION 3.9. *For a permutation $w \in S_n$, the global weight $\text{GW}(w)$ of w is the polynomial*

$$\text{GW}(w) = \prod_{(a,b) \in \text{Inv}(w)} (x_a + x_{a+1} + \dots + x_{b-1}).$$

LEMMA 3.10. *Given $w \in S_n$, the weight of every greedy chain in $[\text{id}, w]$ is $\text{GW}(w)$.*

Proof. We induct on $\ell(w)$, with the base case of $\ell(w) = 1$ clear. Suppose we have proven the statement for all $w' \in S_n$ with $\ell(w') < \ell$, and let $\ell(w) = \ell$. Let

$$C = (\text{id} = w_0 \triangleleft w_1 \triangleleft w_2 \triangleleft \cdots \triangleleft w_\ell = w)$$

be a greedy chain in $[\text{id}, w]$, and suppose $w = w_{\ell-1}t_{ab}$ for $a < b$. Since

$$C' = (\text{id} = w_0 \triangleleft w_1 \triangleleft w_2 \triangleleft \cdots \triangleleft w_{\ell-1})$$

is a greedy chain in $[\text{id}, w_{\ell-1}]$, it suffices to show by the inductive hypothesis that

$$\text{GW}(w) = \text{GW}(w_{\ell-1}) \cdot (x_a + x_{a+1} + \cdots + x_{b-1}).$$

We compare $\text{Inv}(w)$ and $\text{Inv}(w_{\ell-1})$. After swapping $w(a)$ and $w(b)$ in w , the pair (a, b) is no longer in $\text{Inv}(w_{\ell-1})$. It suffices to show that $\text{Inv}(w_{\ell-1}) = \text{Inv}(w) \setminus \{(a, b)\}$. Note that every pair $(c, d) \in \text{Inv}(w)$ satisfying $c \neq a, b$ and $d \neq a, b$ is in $\text{Inv}(w_{\ell-1})$. The remaining pairs in $\text{Inv}(w)$ are of one of the following forms:

- (i) (a, j) for $a < j < b$
- (ii) (a, j) for $a < b < j$
- (iii) (j, b) for $a < j < b$
- (iv) (j, b) for $j < a < b$.

Since $\ell(wt_{ab}) = \ell(w) - 1$, we have $w(j) \notin [w(b), w(a)]$ for every $j \in [a + 1, b - 1]$. Thus, all pairs under cases (i) and (iii) are in $\text{Inv}(w_{\ell-1})$.

For case (ii), suppose $(a, j) \in \text{Inv}(w) \setminus \text{Inv}(w_{\ell-1})$, so $w(j) \in [w(b), w(a)]$. Let $r > b$ be the smallest integer such that $w(r) \in [w(b), w(a)]$; note that such an r exists because $j > b$ has this property. Then $\ell(wt_{ar}) = \ell(w) - 1$, which contradicts the fact that C is a greedy chain. Case (iv) is similar to case (ii).

In conclusion, we have $\text{Inv}(w_{\ell-1}) = \text{Inv}(w) \setminus \{(a, b)\}$, which implies

$$\text{GW}(w) = \text{GW}(w_{\ell-1}) \cdot (x_a + x_{a+1} + \cdots + x_{b-1}),$$

as desired. □

DEFINITION 3.11. We define a partial order \preceq on $\{(a, b) \in \mathbb{N}^2 \mid a < b\}$ such that $(a, b) \preceq (c, d)$ if and only if $[a, b] \subseteq [c, d]$.

Given an integer ℓ , we define a partial order \preceq_ℓ on multisets of $\{(a, b) \in \mathbb{N}^2 \mid a < b\}$ with ℓ elements as follows: for two multisets G and H , we say $G \preceq_\ell H$ if and only if there exists a pairing of elements in G and H such that for each pair $((a_i, b_i), (c_i, d_i)) \in G \times H$ in this pairing, we have $(a_i, b_i) \preceq (c_i, d_i)$. Such a pairing is called a dominant pairing.

DEFINITION 3.12. For a saturated chain $C = (u_0 \triangleleft u_1 \triangleleft \cdots \triangleleft u_\ell)$ in the Bruhat interval $[u_0, u_\ell]$ in S_n , we define its generating set G_C to be the multiset containing the pairs of the positions (a_i, b_i) swapped along edges in C :

$$G_C = \{(a_i, b_i) \in [n] \times [n] \mid u_i = u_{i-1}t_{a_i b_i}, a_i < b_i, i \in [\ell]\}.$$

EXAMPLE 3.13. For the saturated chain $123 \triangleleft 132 \triangleleft 231 \triangleleft 321$, the generating set is $\{(2, 3), (1, 3), (1, 2)\}$.

LEMMA 3.14. Given $w \in S_n$ with $\ell(w) = \ell$, for every saturated chain

$$C = (\text{id} = w_0 \triangleleft w_1 \triangleleft w_2 \triangleleft \cdots \triangleleft w_\ell = w)$$

in $[\text{id}, w]$, we have $G_C \preceq_\ell \text{Inv}(w)$.

Proof. We induct on ℓ , with the base case of $\ell(w) = 1$ clear. Suppose we have proven the statement for all $w' \in S_n$ with $\ell(w') < \ell$, and let $\ell(w) = \ell$. Let $a < b$ satisfy $w = w_{\ell-1}t_{ab}$, and let

$$C' = (\text{id} = w_0 \triangleleft w_1 \triangleleft w_2 \triangleleft \cdots \triangleleft w_{\ell-1}).$$

Since $G_C = G_{C'} \cup \{(a, b)\}$, and we know $G_{C'} \preceq_{\ell-1} \text{Inv}(w_{\ell-1})$ from the inductive hypothesis, we have

$$G_C = G_{C'} \cup \{(a, b)\} \preceq_{\ell} \text{Inv}(w_{\ell-1}) \cup \{(a, b)\}.$$

It now suffices to show that $\text{Inv}(w_{\ell-1}) \cup \{(a, b)\} \preceq_{\ell} \text{Inv}(w)$, which we do by constructing a dominant pairing \mathcal{P} .

The multisets $\text{Inv}(w_{\ell-1}) \cup \{(a, b)\}$ and $\text{Inv}(w)$ are in fact both sets, since $(a, b) \notin \text{Inv}(w_{\ell-1})$. Note that every pair $(c, d) \in \text{Inv}(w)$ satisfying $c \neq a, b$ and $d \neq a, b$ is in $\text{Inv}(w_{\ell-1})$, and (a, b) is in $\text{Inv}(w_{\ell-1}) \cup \{(a, b)\}$. The remaining pairs in $\text{Inv}(w)$ are of one of the following forms:

- (i) (a, j) for $a < j < b$
- (ii) (a, j) for $a < b < j$
- (iii) (j, b) for $a < j < b$
- (iv) (j, b) for $j < a < b$.

Since $\ell(w_{ab}) = \ell(w) - 1$, we have $w(j) \notin [w(b), w(a)]$ for every $j \in [a + 1, b - 1]$. Then the pairs from cases (i) and (iii) are also in $\text{Inv}(w_{\ell-1})$. We put these pairs appearing in both $\text{Inv}(w_{\ell-1}) \cup \{(a, b)\}$ and $\text{Inv}(w)$ together in \mathcal{P} .

Now suppose there is a pair $(a, j) \in \text{Inv}(w)$ falling under case (ii). If (b, j) is also in $\text{Inv}(w)$, then $w(j) < w(b) < w(a)$. Hence (a, j) is in $\text{Inv}(w_{\ell-1})$, and we let $((a, j), (a, j)) \in \mathcal{P}$. Otherwise if $(b, j) \notin \text{Inv}(w)$, then $w(b) < w(j) < w(a)$. So $(b, j) \in \text{Inv}(w_{\ell-1}) \setminus \text{Inv}(w)$, and we can let $((b, j), (a, j)) \in \mathcal{P}$ since $[b, j] \subseteq [a, j]$. Case (iv) is similar to case (ii). \square

LEMMA 3.15. *Given $w \in S_n$, for every saturated chain C in $[\text{id}, w]$, we have*

$$\text{supp}(m_C) \subseteq \text{supp}(\text{GW}(w)).$$

Proof. Let $\ell = \ell(w)$. By Lemma 3.14, we have $G_C \preceq_{\ell} \text{Inv}(w)$, so there exists a dominant pairing \mathcal{P} of G_C and $\text{Inv}(w)$ such that $[a, b] \subseteq [c, d]$ for each pair $((a, b), (c, d)) \in \mathcal{P}$. Observe that each pair $(a, b) \in G_C$ corresponds to a linear factor $x_a + x_{a+1} + \dots + x_{b-1}$ of m_C , and likewise each pair $(c, d) \in \text{Inv}(w)$ corresponds to a linear factor $x_c + x_{c+1} + \dots + x_{d-1}$ of $\text{GW}(w)$. As a result, each monomial of m_C is also in $\text{GW}(w)$, so the lemma is shown. \square

THEOREM 3.16. *Every greedy chain of $[\text{id}, w]$ is a dominant chain of D^w , and*

$$\text{supp}(D^w) = \text{supp}(\text{GW}(w)).$$

As a result, for all $w \in S_n$, the dual Schubert polynomial D^w has SCNP.

Proof. The inclusion \subseteq follows from Lemma 3.15 and Proposition 2.8. The reverse inclusion \supseteq follows from Lemma 3.10 and Lemma 3.8. \square

COROLLARY 3.17. *For all $w \in S_n$, the dual Schubert polynomial D^w has SNP.*

Proof. This follows from Proposition 3.5 and Theorem 3.16. \square

Moreover, Theorem 3.16 gives a more precise characterization of $\text{supp}(D^w)$, as stated in Theorem 1.2.

Proof of Theorem 1.2. This follows from Theorem 3.16, Definition 3.9, and Proposition 2.8. \square

3.3. NEWTON POLYTOPES AS GENERALIZED PERMUTAHEDRA.

DEFINITION 3.18. For $1 \leq a < b \leq n$, let $M_{ab} = ([n], \mathcal{B})$ be the matroid on $[n]$ with bases $\mathcal{B} = \{\{a\}, \{a+1\}, \dots, \{b-1\}\}$.

The motivation for defining such a matroid is the following observation about its matroid polytope: letting $e_i \in \mathbb{R}^n$ denote the unit vector with a 1 in the i th coordinate, we have

$$(1) \quad P(M_{ab}) = \text{conv}\{e_a, e_{a+1}, \dots, e_{b-1}\} = \text{Newton}(x_a + x_{a+1} + \dots + x_{b-1}).$$

Now we may apply theorems about matroid polytopes from Section 2.4 to obtain a characterization of $\text{Newton}(D^w)$ as a generalized permutahedron. In the following, let $I \supseteq [a, b]$ denote $I \supseteq \{a, a+1, \dots, b-1\}$, for $I \subseteq [n]$.

THEOREM 3.19. For $w \in S_n$, $\text{Newton}(D^w)$ is a generalized permutahedron $P_n^z(\{z_I\})$ with

$$z_I = \sum_{(a,b) \in \text{Inv}(w)} \mathbf{1}_{I \supseteq [a,b]}$$

for all $I \subseteq [n]$.

Proof. By Definition 2.17, we have for $I \subseteq [n]$ that

$$r_{M_{ab}}([n] \setminus I) = \begin{cases} 0 & \text{if } I \supseteq [a, b] \\ 1 & \text{if } I \not\supseteq [a, b] \end{cases}.$$

Then from Proposition 2.18, we have

$$\begin{aligned} P(M_{ab}) &= P_n^z(\{r_{M_{ab}}([n]) - r_{M_{ab}}([n] \setminus I)\}) \\ &= P_n^z(\{1 - r_{M_{ab}}([n] \setminus I)\}) \\ &= P_n^z(\{\mathbf{1}_{I \supseteq [a,b]}\}). \end{aligned}$$

Recall from Theorem 3.16 and Definition 3.9 that $\text{Newton}(D^w) = \text{Newton}(\text{GW}(w))$. We may then write

$$\begin{aligned} \text{Newton}(D^w) &= \sum_{(a,b) \in \text{Inv}(w)} \text{Newton}(x_a + x_{a+1} + \dots + x_{b-1}) && \text{Prop 2.9} \\ &= \sum_{(a,b) \in \text{Inv}(w)} P(M_{ab}) && \text{Eq 1} \\ &= \sum_{(a,b) \in \text{Inv}(w)} P_n^z(\{\mathbf{1}_{I \supseteq [a,b]}\}) && \text{Def 2.17, Prop 2.18} \\ &= P_n^z\left(\left\{ \sum_{(a,b) \in \text{Inv}(w)} \mathbf{1}_{I \supseteq [a,b]} \right\}\right). && \text{Prop 2.16.} \end{aligned}$$

□

Now we are ready to prove Corollary 1.3.

Proof of Corollary 1.3. This follows from Corollary 3.17, Theorem 3.19, and Proposition 2.20. □

4. VERTICES OF NEWTON POLYTOPES

We characterize the vertices of $\text{Newton}(D^w)$, as described in Corollary 1.4.

Proof of Corollary 1.4. This follows from Theorem 3.16 and [2, Theorem 3.5]. \square

Furthermore, we describe a combinatorial procedure to obtain the vertices of $\text{Newton}(D^w)$ for a given $w \in S_n$.

- *Step 1:* Construct a Young diagram of staircase shape $(n - 1, n - 2, \dots, 1)$, and label the boxes by the following pairs of inversions: in the i th row of the diagram for $1 \leq i \leq n - 1$, label the boxes from left to right by $(i, n), (i, n - 1), \dots, (i, i + 1)$.
- *Step 2:* In each box, write a 1 if the inversion pair is in $\text{Inv}(w)$, and a 0 otherwise.
- *Step 3:* Construct the $\frac{1}{n+1} \binom{2n}{n}$ tilings of the Young diagram by $n - 1$ rectangles [20].
- *Step 4:* For each tiling, sum the entries of each rectangle and write the sum at the bottom right corner of the rectangle. Reading the summands from top to bottom gives a vertex of $\text{Newton}(D^w)$.

COROLLARY 4.1. *The procedure above gives all the vertices of $\text{Newton}(D^w)$.*

Proof. This follows from Theorem 1.2 and [16, Corollary 8.2]. \square

EXAMPLE 4.2. Figure 2 depicts the procedure for obtaining the vertex $(0, 1, 0, 6, 1)$ of $\text{Newton}(D^{253641})$.

EXAMPLE 4.3. Figure 3 illustrates the procedure for obtaining all vertices of $\text{Newton}(D^{4213})$.

5. VANISHING PROBLEM FOR DUAL SCHUBERT POLYNOMIALS

For a Schubert polynomial $\mathfrak{S}_w = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n-1}} c_{\alpha,w} x^\alpha$ with $w \in S_n$, Adve, Robichaux, and Yong give a polynomial-time algorithm for determining if $c_{\alpha,w} = 0$ for a given α [1]. We prove an analogous result for dual Schubert polynomials.

THEOREM 5.1. *Given $w \in S_n$ and $\alpha \in \mathbb{Z}_{\geq 0}^{n-1}$, there is an $O(n^5)$ algorithm to determine whether $\alpha \in \text{supp}(D^w)$.*

Proof. We state an algorithm, show its runtime is $O(n^5)$, and prove its validity.

Given a tuple of nonnegative integers $\alpha = (c_1, \dots, c_{n-1})$ and the dual Schubert polynomial D^w , we first check if $\sum_{i=1}^{n-1} c_i = \ell(w)$. If not, the algorithm terminates and we conclude that x^α vanishes in D^w . If so, we construct a bipartite graph $B = (U, V, E)$ where $U = \text{Inv}(w)$ and V contains c_i copies of x_i for each i . Let E consist of all edges $(a, b) \rightarrow x_i$ with $a \leq i < b$.

We now use the blossom algorithm, specifically the optimized variant due to Micali and Vazirani [13], to compute the maximum cardinality of a matching in B . The maximum matching has ℓ edges if and only if $\alpha \in \text{supp}(D^w)$. The runtime of the algorithm in our case is $O(\sqrt{|U| + |V|}|E|) = O(\sqrt{n^2}n^4) = O(n^5)$. We note that there exist slightly more efficient algorithms, but the point is only to give a polynomial time algorithm.

The algorithm works for the following reason. The maximum matching has size at most ℓ because B is bipartite and $|U| = |V| = \ell$. If the maximum matching has size ℓ , then by selecting edges $(a, b) \rightarrow x_i$, we may identify a summand x_i in each factor $x_a + \dots + x_{b-1}$ of $\text{GW}(w)$ such that the summands x_i multiply to x^α . This indicates

(1,6)	(1,5)	(1,4)	(1,3)	(1,2)
(2,6)	(2,5)	(2,4)	(2,3)	
(3,6)	(3,5)	(3,4)		
(4,6)	(4,5)			
(5,6)				

Step 1: A staircase Young diagram with $n = 6$.

(1,6)	(1,5)	(1,4)	(1,3)	(1,2)
1	0	0	0	0
(2,6)	(2,5)	(2,4)	(2,3)	
1	1	0	1	
(3,6)	(3,5)	(3,4)		
1	0	0		
(4,6)	(4,5)			
1	1			
(5,6)				
1				

Step 2: When $w = 253641$, the boxes are filled with 1's as shown.

(1,6)	(1,5)	(1,4)	(1,3)	(1,2)
1	0	0	0	0
(2,6)	(2,5)	(2,4)	(2,3)	
1	1	0	1	
(3,6)	(3,5)	(3,4)		
1	0	0		
(4,6)	(4,5)			
1	1			
(5,6)				
1				

Step 3: We consider a tiling by $n - 1$ rectangles. Each rectangle is a different color.

(1,6)	(1,5)	(1,4)	(1,3)	(1,2)
1	0	0	0	0
(2,6)	(2,5)	(2,4)	(2,3)	
1	1	0	1	0
(3,6)	(3,5)	(3,4)		
1	0	0		1
(4,6)	(4,5)			
1	1			0
(5,6)				
1		6		
				1

Step 4: $\text{Newton}(D^{253641})$ has a vertex at $(0, 1, 0, 6, 1)$.

FIGURE 2. $\text{Newton}(D^{253641})$ has a vertex at $(0, 1, 0, 6, 1)$.

that x^α has coefficient at least 1 in $\text{GW}(w)$. If the maximum matching has size less than ℓ , then there is no way to select edges $(a, b) \rightarrow x_i$ such that the x_i 's multiply to x^α , so x^α has coefficient 0. \square

EXAMPLE 5.2. Our algorithm builds the graph in Figure 4 to verify that $x^{(1,2,1,3,1)} = x_1 x_2^2 x_3 x_4^3 x_5$ is in $\text{supp}(D^{253641})$.

6. FUTURE DIRECTIONS

We hope to generalize our results to Postnikov–Stanley polynomials.

CONJECTURE 6.1. *Postnikov–Stanley polynomials are M-convex.*

We are also interested in a characterization of when D_u^w has SCNP. Though we experienced some computational limitations due to the large number of covering relations in the Bruhat order on S_n for $n \geq 7$, we have verified the following pattern-containment conjecture using SageMath [19] for S_n with $n \leq 6$.

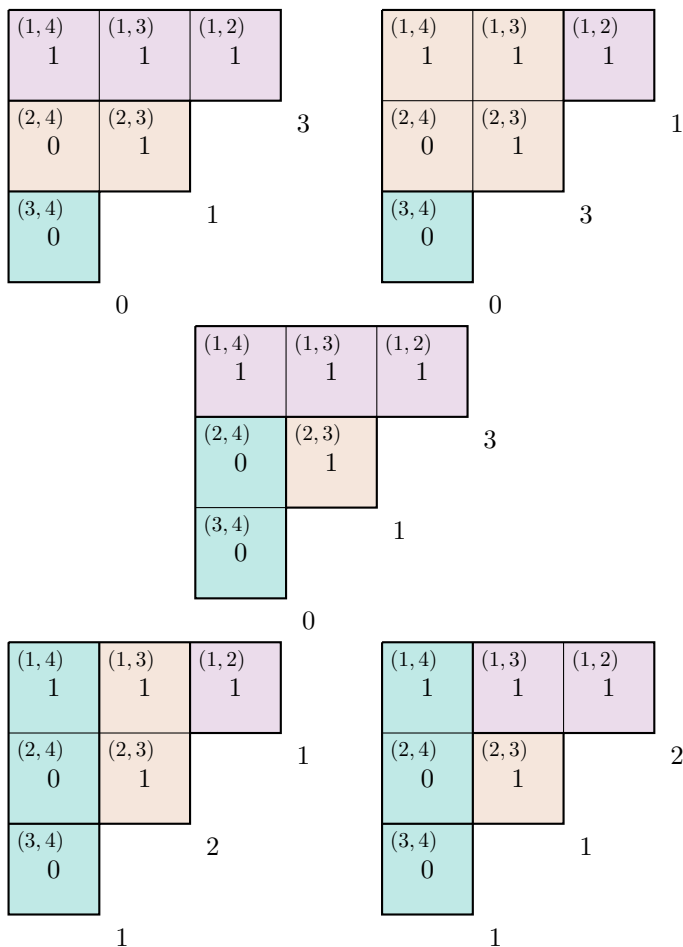


FIGURE 3. $\text{Newton}(D^{4213})$ has vertices $(3, 1, 0)$, $(1, 3, 0)$, $(1, 2, 1)$, $(2, 1, 1)$.

CONJECTURE 6.2. For a permutation $u \in S_n$, there exists a permutation $w \in S_n$ such that D_u^w does not have SCNP if and only if u contains 1324 . Analogously, for a permutation $w \in S_n$, there exists a permutation $u \in S_n$ such that D_u^w does not have SCNP if and only if w contains 4231 .

Acknowledgements. This research was conducted through the University of Minnesota Combinatorics and Algebra REU and supported by NSF grant DMS-2053288 and the D.E. Shaw group. We are grateful to our mentor Shiyun Wang and our TA Meagan Kenney for their guidance, as well as Pavlo Pylyavskyy for his invaluable support throughout the project. We thank Ayah Almousa, Casey Appleton, Shiliang Gao, Yibo Gao, June Huh, Jonas Iskander, Victor Reiner, and Lauren Williams for helpful discussions. We also thank Grant Barkley, Meagan Kenney, Mitchell Lee, Pavlo Pylyavskyy, Victor Reiner, and Shiyun Wang for their feedback on earlier drafts of this paper. Finally, we thank the anonymous referee for their careful reading and useful suggestions, which helped us improve the exposition.

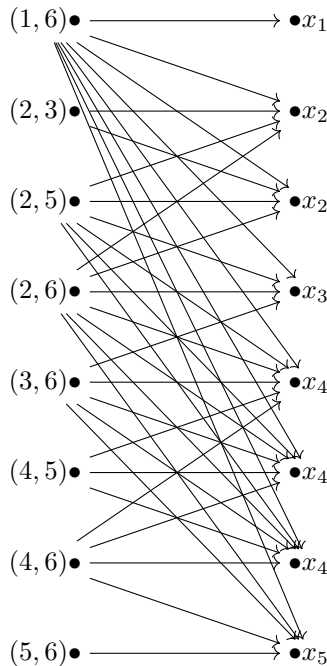


FIGURE 4. A maximum matching of this graph has $8 = \ell(253641)$ edges, so the vertex $(1, 2, 1, 3, 1)$ is in $\text{supp}(D^{253641})$.

REFERENCES

[1] Anshul Adve, Colleen Robichaux, and Alexander Yong, *An efficient algorithm for deciding vanishing of Schubert polynomial coefficients*, Adv. Math. **383** (2021), article no. 107669 (38 pages).
 [2] Geir Aagnarsson and Walter D. Morris, *On Minkowski sums of simplices*, Ann. Comb. **13** (2009), no. 3, 271–287.
 [3] Serena An, Katherine Tung, and Yuchong Zhang, *Postnikov–Stanley polynomials are Lorentzian*, 2024, <https://arxiv.org/abs/2412.02051>.
 [4] Federico Ardila, Carolina Benedetti, and Jeffrey Doker, *Matroid polytopes and their volumes*, Discrete Comput. Geom. **43** (2010), no. 4, 841–854.
 [5] Joseph Bernstein, Israel Gelfand, and Sergei Gelfand, *Schubert cells and cohomology of the spaces G/P* , Russian Mathematical Surveys **28** (1973), 1–26.
 [6] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.
 [7] Federico Castillo, Yairon Cid-Ruiz, Fatemeh Mohammadi, and Jonathan Montaña, *Double Schubert polynomials do have saturated Newton polytopes*, Forum Math. Sigma **11** (2023), article no. e100 (9 pages).
 [8] Alex Fink, Karola Mészáros, and Avery St. Dizier, *Schubert polynomials as integer point transforms of generalized permutahedra*, Adv. Math. **332** (2018), 465–475.
 [9] Yibo Gao, *An involution on RC-graphs and a conjecture on dual Schubert polynomials by Postnikov and Stanley*, Algebr. Comb. **3** (2020), no. 3, 593–602.
 [10] I. M. Gel’fand, M. M. Kapranov, and A. V. Zelevinsky, *Newton polytopes of the classical resultant and discriminant*, Adv. Math. **84** (1990), no. 2, 237–254.
 [11] Zachary Hamaker, *Dual Schubert polynomials via a Cauchy identity*, 2023, <https://arxiv.org/abs/2311.18099>.
 [12] June Huh, Jacob P. Matherne, Karola Mészáros, and Avery St. Dizier, *Logarithmic concavity of Schur and related polynomials*, Trans. Amer. Math. Soc. **375** (2022), no. 6, 4411–4427.
 [13] Silvio Micali and Vijay Vazirani, *An $O(\sqrt{|V||E|})$ algorithm for finding maximum matching in general graphs*, 21st Annual Symposium on Foundations of Computer Science, 1980, pp. 17–27.
 [14] Cara Monical, Neriman Tokcan, and Alexander Yong, *Newton polytopes in algebraic combinatorics*, Selecta Math. (N.S.) **25** (2019), no. 5, article no. 66 (37 pages).

- [15] Kazuo Murota, *Discrete convex analysis*, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.
- [16] Alexander Postnikov, *Permutohedra, associahedra, and beyond*, Int. Math. Res. Not. IMRN (2009), no. 6, 1026–1106.
- [17] Alexander Postnikov and Richard P. Stanley, *Chains in the Bruhat order*, J. Algebraic Combin. **29** (2009), no. 2, 133–174.
- [18] Richard Rado, *An inequality*, J. London Math. Soc. **27** (1952), 1–6.
- [19] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 10.4)*, 2024, <https://www.sagemath.org>.
- [20] Richard Stanley, *Catalan addendum to Enumerative Combinatorics. Vol. 2.*, 2013, <https://math.mit.edu/~rstan/ec/catadd.pdf>.

SERENA AN, Massachusetts Institute of Technology, Department of Mathematics, Cambridge, MA 02139 (USA)
E-mail : anser@mit.edu

KATHERINE TUNG, Harvard University, Department of Mathematics, Cambridge, MA 02138 (USA)
E-mail : katherinetung@college.harvard.edu

YUCHONG ZHANG, University of Michigan, Department of Mathematics, Ann Arbor, MI 48109 (USA)
E-mail : zongxun@umich.edu