ALGEBRAIC COMBINATORICS

Jack H. Koolen, Mamoon Abdullah, Brhane Gebremichel & Jae-Ho Lee

Towards a classification of 1-homogeneous distance-regular graphs with positive intersection number a_1

Volume 8, issue 6 (2025), p. 1529-1546.

https://doi.org/10.5802/alco.454

© The author(s), 2025.

This article is licensed under the
Creative Commons Attribution (CC-BY) 4.0 License.
http://creativecommons.org/licenses/by/4.0/

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 8, issue 6 (2025), p. 1529–1546
https://doi.org/10.5802/alco.454

Towards a classification of 1-homogeneous distance-regular graphs with positive intersection number a_1

Jack H. Koolen, Mamoon Abdullah, Brhane Gebremichel
& Jae-Ho Lee

Abstract Let Γ be a graph with diameter at least two. Then Γ is said to be 1-homogeneous (in the sense of Nomura) whenever for every pair of adjacent vertices x and y in Γ, the distance partition of the vertex set of Γ with respect to both x and y is equitable, and the parameters corresponding to equitable partitions are independent of the choice of x and y. Assume that Γ is 1-homogeneous distance-regular with intersection number $a_1 > 0$ and diameter $D \geqslant 5$. Define $b = b_1/(\theta_1 + 1)$, where b_1 is the intersection number and θ_1 is the second largest eigenvalue of Γ. We show that if intersection number c_2 is at least 2, then $b \geqslant 1$ and one of the following (i)–(vi) holds: (i) Γ is a regular near $2D$-gon, (ii) Γ is a Johnson graph $J(2D, D)$, (iii) Γ is a halved ℓ-cube with $\ell \in \{2D, 2D+1\}$, (iv) Γ is a folded Johnson graph $\bar{J}(4D, 2D)$, (v) Γ is a folded halved $4D$-cube, (vi) the valency of Γ is bounded by a function of b. Using this result, we characterize 1-homogeneous graphs with classical parameters and $a_1 > 0$, as well as tight distance-regular graphs.

1. Introduction

In this paper, we study distance-regular graphs that have the 1-homogeneous property in the sense of Nomura [21]. To motivate our results, we recall some preliminaries and background on 1-homogeneous distance-regular graphs. For more details, refer to [2, 6, 21].

Throughout this paper, let Γ denote a finite, undirected, connected, and simple graph. Let $V(\Gamma)$ denote the vertex set of Γ. For two vertices $x, y \in V(\Gamma)$, the *distance* $d(x, y)$ is the length of a shortest path from x to y in Γ. The *diameter* of Γ is the maximum value of $d(x, y)$ for all pairs of $x, y \in V(\Gamma)$. Let D denote the diameter of Γ. For an integer $0 \leqslant i \leqslant D$ and a vertex $x \in V(\Gamma)$, let $\Gamma_i(x)$ denote the set of vertices in Γ at distance i from x. Abbreviate $\Gamma(x) = \Gamma_1(x)$. The subgraph of Γ induced on the set $\Gamma(x)$ is called the *local graph* of Γ at x. The graph Γ is called *locally* $\mathcal{P}$ whenever every local graph of Γ has the property $\mathcal{P}$ (or belongs to the family $\mathcal{P}$). For example, we might say that a graph is locally connected or locally a strongly regular graph. For a pair of vertices $x, y \in V(\Gamma)$ with $d(x, y) = 2$, the subgraph of Γ induced on the

Manuscript received 6th January 2025, revised 18th June 2025, accepted 22nd September 2025.

Keywords. distance-regular graph, 1-homogeneous, local graph, classical parameters, tight graph.

Acknowledgements. J.H. Koolen is partially supported by the National Key R. and D. Program of China (No. 2020YFA0713100), the National Natural Science Foundation of China (No. 12071454), and the Anhui Initiative in Quantum Information Technologies (No. AHY150000). M. Abdullah is supported by the Chinese Scholarship Council at USTC, China. B. Gebremichel is supported by the National Key R. and D. Program of China (No. 2020YFA0713100) and the Foreign Young Talents Program (No. QN2022200003L). J.-H. Lee was partially supported by the Mathematics Department at Pohang University of Science and Technology (POSTECH).

set $\Gamma(x) \cap \Gamma(y)$ is called the *$\mu(x,y)$-graph* of Γ. If this graph does not depend on the choice of x and y (up to isomorphism), then we simply call it the μ-graph of Γ. For an integer $k \geqslant 0$, we say that Γ is *regular with valency k* (or *k-regular*) if $|\Gamma(x)| = k$ for every $x \in V(\Gamma)$. For an integer $0 \leqslant i \leqslant D$ and for a pair $x, y \in V(\Gamma)$ with $d(x,y) = i$ we define
(1.1)

$$C_i(x,y) := \Gamma_{i-1}(x) \cap \Gamma(y), \quad A_i(x,y) := \Gamma_i(x) \cap \Gamma(y), \quad B_i(x,y) := \Gamma_{i+1}(x) \cap \Gamma(y),$$

where $C_0(x,y) := \varnothing$ and $B_D(x,y) := \varnothing$. Observe that $\Gamma(y)$ is the disjoint union of the vertex sets $C_i(x,y)$, $A_i(x,y)$, $B_i(x,y)$. We say Γ is *distance-regular* whenever the cardinalities

$$c_i = |C_i(x,y)|, \qquad a_i = |A_i(x,y)|, \qquad b_i = |B_i(x,y)| \qquad (0 \leqslant i \leqslant D) \tag{1.2}$$

are constants and do not depend on the choice of x and y. Note that $c_0 = a_0 = b_D = 0$, and $c_1 = 1$. Additionally, Γ is regular with valency $k = b_0$, and $a_i + b_i + c_i = k$ $(0 \leqslant i \leqslant D)$. The numbers a_i, b_i, c_i in (1.2) are called the *intersection numbers* of Γ, and the array $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the *intersection array* of Γ. We note that a distance-regular graph with diameter D has exactly $D+1$ distinct eigenvalues [6, Proposition 2.6].

Notation 1.1. Unless otherwise specified, whenever we denote Γ as a distance-regular graph, we use the following notation: Γ has diameter D, valency k, and distinct eigenvalues $\theta_0 > \theta_1 > \ldots > \theta_D$. Moreover, the intersection numbers of Γ are denoted by $\{c_i\}_{i=1}^{D}$, $\{a_i\}_{i=0}^{D}$, $\{b_i\}_{i=0}^{D-1}$, as shown in (1.2).

Next, we recall the notion of the i-homogeneous property as introduced by Nomura [21]. Let Γ be a connected graph. A partition $\pi = \{C_1, C_2, \ldots, C_p\}$ of $V(\Gamma)$ is called *equitable* whenever, for all $1 \leqslant i, j \leqslant p$, the number of neighbors of a vertex $x \in C_i$ in the set C_j is independent of the choice of x. In other words, for each pair of subsets C_i and C_j in π, the number $c_{ij} := |\Gamma(x) \cap C_j|$ is constant for all $x \in C_i$. These numbers $\{c_{ij}\}_{1 \leqslant i,j \leqslant p}$ are called the *parameters* of π. We say Γ has the *i-homogeneous* property whenever, for every pair of vertices x and y at distance i, the partition of $V(\Gamma)$ according to the path-length distance to both x and y is equitable, and the parameters corresponding to equitable partitions are independent of the choice of x and y; see Section 5. Graphs with the i-homogeneous property are simply said to be i-homogeneous. Note that Γ is 0-homogeneous if and only if it is a distance-regular graph. Moreover, if Γ is 1-homogeneous, then it is a distance-regular graph and also locally strongly regular.

In this paper, we focus on 1-homogeneous distance-regular graphs. Examples of such graphs include the Johnson graphs $J(2D, D)$, the bipartite distance-regular graphs, and the regular near $2D$-gons. We have some comments about the history of 1-homogeneous distance-regular graphs. Over the years, the 1-homogeneous property has received considerable attention and has been used in the study of distance-regular graphs across various contexts, including tight distance-regular graphs [14, 17], distance-regular graphs which support a spin model [4], the Terwilliger algebras [5], and Q-polynomial distance-regular graphs [19]. Jurišić and Koolen [8, 11, 13] explored 1-homogeneous graphs whose μ-graphs are a complete multipartite graph $K_{t\times n}$, $n \geqslant 1$ (i.e., the complement of t copies of the complete graph K_n). In [8], they introduced the CAB property to study the local structures of distance-regular graphs and used this property to characterize 1-homogeneous graphs with $a_1 > 0$. Also, they classified 1-homogeneous graphs with $c_2 \geqslant 2$ whose μ-graphs are $K_{t\times 1}$, i.e., 1-homogeneous Terwilliger graphs. In [11], they classified 1-homogeneous graphs when $n = 2$, i.e., when the μ-graphs are $K_{t\times 2}$ (Cocktail Party graphs). In their subsequent study [12], they extended this work to distance-regular graphs whose μ-graphs are $K_{t\times n}$ ($n \geqslant 2$). Jurišić, Munemasa, and Tagami [15] investigated a more general case, namely, graphs (not necessarily distance-regular) whose μ-graphs are $K_{t\times n}$. Moreover, several studies examined distance-regular graphs whose μ-graphs are complete multipartite; see [9, 13, 17]. These studies have contributed to the research on classifying 1-homogeneous distance-regular graphs with complete multipartite μ-graphs, which is an important problem.

In this paper, one significance of our result lies in making substantial progress towards a classification of 1-homogeneous distance-regular graphs with $a_1 > 0$. This result extends to a broader context, covering 1-homogeneous graphs whose μ-graphs are complete multipartite, as discussed in the preceding paragraph. We now present the main result of this paper.

Theorem 1.2. *Let Γ be a 1-homogeneous distance-regular graph with diameter $D \geqslant 5$ and $a_1 > 0$. Define $b = b_1/(\theta_1 + 1)$. Then, either $c_2 = 1$, or $b \geqslant 1$ and one of the following holds:*

(i) *Γ is a regular near $2D$-gon.*
(ii) *Γ is a Johnson graph $J(2D, D)$.*
(iii) *Γ is a halved ℓ-cube with $\ell \in \{2D, 2D + 1\}$.*
(iv) *Γ is a folded Johnson graph $\bar{J}(4D, 2D)$.*
(v) *Γ is a folded halved $4D$-cube.*
(vi) *The valency k of Γ is bounded by a function $F(b)$ of b,* i.e., *$k \leqslant F(b)$, where*

$$F(b) = 16b^{10} + 80b^9 + 192b^8 + 256b^7 + 192b^6 + 72b^5 + 20b^4 + 24b^3 + 8b^2 + 1. \tag{1.3}$$

The proof of this theorem appears in Section 5.

Remark 1.3. (i) We comment on the cases where the diameter is 3 or 4 for 1-homogeneous distance-regular graphs with $a_1 > 0$. For $D = 3$, the Taylor graphs are examples of 1-homogeneous graphs with $a_1 > 0$, as they are tight distance-regular graphs. For $D = 4$, examples include the Patterson graph and the family $AT_4(p, q, r)$ of antipodal tight graphs with parameters p, q, r; see [10, 13]. It remains an open question whether there exist infinitely many such graphs of diameter 4.

(ii) In the case where Γ is a regular near $2D$-gon as in Theorem 1.2, we can further refine the classification under additional conditions: if $c_2 \geqslant 3$, then Γ is a dual polar graph; if $c_2 = 2$ and $c_3 = 3$, then Γ is a Hamming graph; see [6, Theorem 9.11].

Remark 1.4. In [17], Koolen et al. proposed a conjecture stating that for a tight distance-regular graph with $D \geqslant 3$ and $b = b_1/(1 + \theta_1) \geqslant 2$, the diameter D is bounded by a function of b; see [17, Conjecture 7.5]. We prove this conjecture in Section 7 using Theorem 1.2.

This paper is organized as follows. In Section 2, we review strongly regular graphs and their properties. We discuss a classification of strongly regular graphs with smallest eigenvalue $\leqslant -2$. In Section 3, we discuss distance-regular graphs that are locally strongly regular. We establish a bound on the intersection number c_2 for those graphs. We also show that when a distance-regular graph is locally a conference graph, it is a Taylor graph. In Section 4, we recall the CAB_i property of distance-regular graphs. We focus on the CAB_2 property and examine distance-regular graphs that possess this property. In Section 5, we discuss 1-homogeneous distance-regular graphs with $a_1 > 0$. We prove our main result, Theorem 1.2. In Section 6, we discuss 1-homogeneous distance-regular graphs with classical parameters and $a_1 > 0$. Finally, we conclude the paper in Section 7 with some comments on tight distance-regular graphs.

2. Strongly regular graphs with smallest eigenvalue $-m$

In this section, we review properties of strongly regular graphs with smallest eigenvalue $-m$, where $m > 0$, and discuss their classification. First, we recall the definition of a strongly regular graph. Let Γ be a k-regular graph with v vertices. The graph Γ is called *strongly regular* with parameters (v, k, λ, μ) if each pair of distinct adjacent (resp. non-adjacent) vertices has exactly λ (resp. μ) common neighbors. Suppose Γ is a strongly regular graph with smallest eigenvalue s. It is well known that Γ satisfies $s \leqslant -2$, except in the cases where Γ is a disjoint union of cliques (with $s = -1$) or a pentagon (with $s = (-1-\sqrt{5})/2$) [3, Section 1.1.10].

Let Γ be a strongly regular graph with parameters (v, k, λ, μ) and diameter two. We denote the eigenvalues of Γ as $k > r > s$. It is known that k, r, s are integers except when Γ is a *conference* graph, i.e., a strongly regular graph with parameters $(4\mu+1, 2\mu, \mu-1, \mu)$ [7, Lemma 10.3.3]. The parameters v, k, λ of Γ can be expressed in terms of r, s, and μ as follows:

$$v = \frac{(k-r)(k-s)}{\mu}, \qquad k = \mu - rs, \qquad \lambda = \mu + r + s, \tag{2.1}$$

cf. [2, Theorem 1.3.1]. We present two examples of strongly regular graphs that will be used in this paper.

Example 2.1. A transversal design $\mathrm{TD}(m;n)$ is a partial linear space with mn points and $m+n^2$ lines, with m lines (called groups) of size n forming a partition of the point set, and n^2 lines (called blocks) of size m, each meeting every group in a single point; cf. [3, Section 8.4.1]. The line graph of a transversal design $\mathrm{TD}(m;n)$ with $2 \leqslant m \leqslant n$ is called a *Latin square* graph $\mathrm{LS}_m(n)$; cf. [3, Section 8.4.2]. Note that $\mathrm{LS}_m(n)$ is isomorphic to the block graph of an orthogonal array $\mathrm{OA}(m,n)$. A Latin square graph $\mathrm{LS}_m(n)$ is strongly regular with parameters

$$(n^2, \quad m(n-1), \quad (m-1)(m-2)+n-2, \quad m(m-1)) \tag{2.2}$$

and eigenvalues $m(n-1) > n-m > -m$.

Example 2.2. A Steiner system $S(2,m,n)$ is a 2-$(n,m,1)$ design, that is, a collection of m-subsets of a n-set in which each pair of elements is contained in exactly one m-set. In this context, the elements of the n-set are referred to as points, and the m-sets are referred to as blocks of the system. The *block graph of a Steiner system* $S(2,m,n)$ is defined as the graph whose vertices are the blocks of the system, where two vertices are adjacent whenever they intersect at exactly one point. The block graph of a Steiner system $S(2,m,n)$ with $n > m \geqslant 2$ is strongly regular with parameters

$$\left(\frac{n(n-1)}{m(m-1)}, \quad \frac{m(n-m)}{m-1}, \quad (m-1)^2 + \frac{n-1}{m-1} - 2, \quad m^2\right). \tag{2.3}$$

The eigenvalues of this graph are $\frac{m(n-m)}{m-1} > \frac{n-m^2}{m-1} > -m$. In particular, the block graph of a Steiner system $S(2, m, mn+m-n)$ is called a *Steiner graph* $S_m(n)$. Note that a Steiner graph $S_m(n)$ has parameters

$$\left(\frac{(m+n(m-1))(n+1)}{m}, \quad mn, \quad m^2-2m+n, \quad m^2\right) \tag{2.4}$$

and eigenvalues $mn > n-m > -m$.

Next, we recall some known results on the classification of strongly regular graphs whose smallest eigenvalue is at most -2. For the rest of this section, let Γ be a strongly regular graph with parameters (v, k, λ, μ) and integral eigenvalues $k > r > s$. For our purposes, we set

$$m := -s, \qquad n = r - s \qquad (m \geqslant 2). \tag{2.5}$$

If Γ is primitive, that is, both Γ and its complement are connected, the parameter μ is bounded above by a function of m:

$$\mu \leqslant m^3(2m-3). \tag{2.6}$$

We call (2.6) the μ-bound; see [20, Theorem 3.1]. Let $f(m,\mu) = \frac{1}{2}m(m-1)(\mu+1) + m - 1$. Then, by [20, Theorem 4.7] (cf. [3, Theorem 8.6.3]), the following statements (i)–(iii) hold:

(i) If $\mu = m(m-1)$ and $n > f(m,\mu)$, then Γ is a Latin square graph $\mathrm{LS}_m(n)$.
(ii) If $\mu = m^2$ and $n > f(m,\mu)$, then Γ is a Steiner graph $S_m(n)$.
(iii) If $\mu \neq m(m-1)$ and $\mu \neq m^2$, then

$$n \leqslant f(m,\mu) = \frac{1}{2}m(m-1)(\mu+1) + m - 1. \tag{2.7}$$

We call (2.7) the *claw bound.* As a consequence of the μ-bound and the claw bound, the strongly regular graphs with integral smallest eigenvalue $\leqslant -2$ are characterized as follows.

LEMMA 2.3 (Sims, cf. [3, Theorem 8.6.4]). *Let Γ be a strongly regular graph with parameters (v, k, λ, μ) with integral smallest eigenvalue $-m$, where $m \geqslant 2$. Then Γ belongs to one of the following* (i)–(iv)*:*

(i) *complete multipartite graphs with classes of size m;*
(ii) *Latin square graphs $\mathrm{LS}_m(n)$;*
(iii) *Steiner graphs $S_m(n)$;*
(iv) *finitely many further graphs.*

We give a comment on Lemma 2.3. Assume that Γ is none of a complete multipartite graph, a Latin square graph, or a Steiner graph.

By Lemma 2.3 and the claw bound (2.7), Γ satisfies $n \leqslant \frac{1}{2}m(m-1)(\mu+1) + m - 1$. Since $n = r + m$, it follows

$$r \leqslant \frac{1}{2}m(m-1)(\mu+1) - 1. \tag{2.8}$$

Note that $n \neq m$ since Γ is not a complete multipartite graph. This implies $r \neq 0$, that is, $r \geqslant 1$. From the first equation in (2.1), we have $v = (k-r)(k+m)/\mu$. Substitute k with $\mu + rm$ and simplify the result to obtain

$$v = \mu + m - r + 2rm + \frac{rm(m-1)(1+r)}{\mu}. \tag{2.9}$$

Applying inequalities (2.6) and (2.8) to the right-hand side of (2.9) and expressing the result in terms of m, we obtain

(2.10)

$$v \leqslant m^3(2m-3) + m + \left(2m - 1 + m^2(m-1)^2\right)\left(\frac{m(m-1)}{2}\left(m^3(2m-3)+1\right) - 1\right).$$

Let $\varphi(m)$ denote the right-hand side of (2.10). Simplify the expression for $\varphi(m)$ to obtain

$$\varphi(m) = m^{10} - \frac{9}{2}m^9 + \frac{15}{2}m^8 - \frac{7}{2}m^7 - 4m^6 + 4m^5 + m^4 - \frac{1}{2}m^3 - \frac{5}{2}m^2 - \frac{1}{2}m + 1. \tag{2.11}$$

Note that $\varphi(m) < m^{10}$ for all $m \geqslant 2$. By these comments, we restate Lemma 2.3 as follows:

Corollary 2.4. *Let Γ be a strongly regular graph with parameters (v, k, λ, μ) with integral smallest eigenvalue $-m$, where $m \geqslant 2$. Then one of the following holds:*

(i) Γ *is a complete multipartite graph with class of size* m,
(ii) Γ *is a Latin square graph* $\mathrm{LS}_m(n)$,
(iii) Γ *is a Steiner graph* $S_m(n)$,
(iv) *The number of vertices of* Γ *is bounded by a function in* m, *i.e.,* $v \leqslant \varphi(m)$, *where* $\varphi(m)$ *is from* (2.11).

3. Distance-regular graphs that are locally strongly regular

In this section, we discuss distance-regular graphs whose local graphs are strongly regular. For such graphs, we give a bound on their intersection number c_2. We also show that if a distance-regular graph is locally a conference graph, then it is a Taylor graph. We begin by recalling some known results about distance-regular graphs that will be used, along with references for further discussion.

Lemma 3.1 (cf. [2, Theorem 4.4.3]). *Let Γ be a distance-regular graph of diameter $D \geqslant 3$ with eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$ and intersection number b_1. Let $b = b_1/(\theta_1 + 1)$. Then, $b > 0$. Moreover, for each vertex x in Γ, its local graph has the smallest eigenvalue $\geqslant -1 - b$.*

Let Γ be a distance-regular graph with diameter at least two. Recall the definition of the μ-graph of Γ. Observe that each μ-graph has c_2 vertices and that all are isomorphic. For such Γ, the μ-graph is also called the *c_2-graph* of Γ. If each c_2-graph is a regular graph with valency κ, then we say that Γ is *c_2-graph-regular* with parameter κ. The graph Γ is called a *Terwilliger graph* when it is c_2-graph-regular and every c_2-graph of Γ is complete.

Lemma 3.2 (cf. [9, Theorem 3.1]). *Let Γ be a distance-regular graph which is locally strongly regular with parameters (v', k', λ', μ'). Then Γ is c_2-graph-regular with parameter μ'. Moreover, $c_2 \geqslant \mu' + 1$, with equality if and only if Γ is a Terwilliger graph.*

A *clique* in Γ is a subset of $V(\Gamma)$ such that every pair of distinct vertices is adjacent. A clique of size p is referred to as a complete graph K_p. A *coclique* of Γ is a subset of $V(\Gamma)$ such that no two vertices are adjacent. A *complete bipartite graph $K_{p,q}$* is a graph whose vertex set can be partitioned into two cocliques, say a p-set V_1 and a q-set V_2, where each vertex in V_1 is adjacent to all vertices in V_2. A *complete multipartite graph $K_{t\times p}$* is a graph whose vertex set can be partitioned into cocliques $\{V_i\}_{i=1}^{t}$ of size p, where each vertex in V_i is adjacent to all vertices in V_j $(1 \leqslant j \neq i \leqslant t)$.

Lemma 3.3 (cf. [22, Lemma 3.7]). *Let Γ be a distance-regular graph with valency k, diameter $D \geqslant 5$ and second largest eigenvalue θ_1. Assume that Γ contains an induced subgraph $K_{2,t}$ for some $t \geqslant 2$. Let $b = b_1/(\theta_1 + 1)$. Then $t \leqslant 4b^2 + 1$.*

We now establish a bound on the intersection number c_2 for distance-regular graphs that are locally strongly regular.

Proposition 3.4. *Let Γ be a distance-regular graph with diameter $D \geqslant 5$, valency k, intersection numbers b_1 and c_2, and second largest eigenvalue θ_1. Assume Γ is locally strongly regular with parameters (v', k', λ', μ'). Let $b = b_1/(\theta_1 + 1)$. Then*

$$c_2 \leqslant (4b^2 + 1)(\mu' + 1). \tag{3.1}$$

Proof. We assume $c_2 \geqslant 2$; otherwise, it is trivial. Suppose that Γ is a Terwilliger graph. Then, by Lemma 3.2, we have $c_2 = \mu' + 1 < (4b^2 + 1)(\mu' + 1)$. Now, assume that Γ is not a Terwilliger graph. Since $c_2 \geqslant 2$, Γ contains an induced subgraph $K_{2,t}$

for some $t \geqslant 2$. Applying Lemma 3.3 to Γ, we obtain the bound $t \leqslant 4b^2+1$. Next, as Γ is locally strongly regular, by Lemma 3.2, Γ is c_2-graph-regular with parameter μ'. Thus, we find that the number of vertices of the c_2-graph is at most $t(\mu'+1)$. By these comments, it follows that $c_2 \leqslant t(\mu'+1) \leqslant (4b^2+1)(\mu'+1)$. □

Recall that a conference graph is a strongly regular graph with parameters $(4\mu+1, 2\mu, \mu-1, \mu)$, where $\mu > 0$. We consider distance-regular graphs that are locally a conference graph. First, we recall the classification of distance-regular graphs that satisfy $a_1 \geqslant k/2-1$, as shown by Koolen and Park [18].

LEMMA 3.5 ([18, Theorem 16]). *Let Γ be a distance-regular graph with diameter $D \geqslant 3$ and valency k. If $a_1 \geqslant \frac{1}{2}k-1$, then one of the following holds:*

(i) Γ *is a polygon,*
(ii) Γ *is the line graph of a Moore graph,*
(iii) Γ *is the flag graph of a regular generalized D-gon of order (s,s) for some s,*
(iv) Γ *is a Taylor graph,*
(v) Γ *is the Johnson graph $J(7,3)$,*
(vi) Γ *is the halved 7-cube.*

PROPOSITION 3.6. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$. If Γ is locally a conference graph, then it is a Taylor graph.*

Proof. Let Δ denote a local graph of Γ. Then Δ is a conference graph, and we denote its parameters by $(4\mu+1, 2\mu, \mu-1, \mu)$, where $\mu > 0$. Thus, Γ has valency $k = 4\mu+1$ and intersection number $a_1 = 2\mu$. We observe that k is odd, a_1 is nonzero and even, and Γ satisfies $a_1 > \frac{1}{2}k-1$. By Lemma 3.5, Γ falls into one of the graphs (i)–(vi) listed therein. However, we observe that:

- In case (i), a polygon has $k=2$ and $a_1 = 0$;
- In case (ii), the line graph of a Moore graph has the intersection array $\{2k'-2, k'-1, k'-2; 1, 1, 4\}$, where $k' \in \{3, 7, 57\}$. This implies that $a_1 = k'-2$, which is an odd number;
- In case (iii), the flag graph of a regular generalized D-gon of order (s,s), a generalized $2D$-gon of order $(s,1)$, has $k=2s$ (an even number);
- In case (v), the Johnson graph $J(7,3)$ has $\{12, 6, 2; 1, 4, 9\}$ as its intersection array, so $k=12$ (even) and $a_1 = 5$ (odd).

From these observations, it follows that none of the graphs (i), (ii), (iii), (v) is locally a conference graph. Moreover, as the halved 7-cube has the intersection array $\{21, 10, 3; 1, 6, 15\}$ its local graph has 21 vertices and valency 10, but such a conference graph does not exist. Therefore, Γ is a Taylor graph. □

We finish this section with a comment. The line graph of $K_{p,q}$ is called the $(p\times q)$-*grid*. Note that the square $(p \times p)$-grid, where $p \geqslant 2$, is a strongly regular graph and is isomorphic to the Latin square graph $\mathrm{LS}_2(p)$.

LEMMA 3.7 (cf. [11, Proposition 4.1]). *Let Γ be a distance-regular graph with diameter $D \geqslant 2$. If Γ is locally the $(p\times q)$-grid and $c_2 = 4$, then Γ is the Johnson graph $J(p+q, p)$, or $p = q$ and Γ is the folded Johnson graph $\bar{J}(2p, p)$.*

4. THE CAB$_i$ PROPERTY

In this section, we recall the definition of the CAB$_i$ property for distance-regular graphs and review related known results. We then focus on distance-regular graphs satisfying the CAB$_2$ property.

Let Γ be a distance-regular graph with diameter $D \geqslant 2$. For $0 \leqslant i \leqslant D$ and for $x, y \in V(\Gamma)$ with $d(x,y) = i$, we recall the subsets $C_i(x,y)$, $A_i(x,y)$, $B_i(x,y)$

of $V(\Gamma)$, as defined in (1.1). We also recall the intersection numbers c_i, a_i, b_i of Γ from (1.2). Assume $a_1 > 0$. Let $\Delta(y)$ denote the local graph of Γ at y. Note that $\Delta(y)$ has $k = b_0$ vertices and is regular with valency a_1. We observe that the set $\{C_i(x,y), A_i(x,y), B_i(x,y)\}$ partitions the vertex set of $\Delta(y)$. We denote this partition by $\mathrm{CAB}_i(x,y)$ and call it the $\mathrm{CAB}_i(x,y)$ *partition* of $\Delta(y)$. Observe that the $\mathrm{CAB}_0(x,y)$ partition is $\{\Gamma(x)\}$, and the $\mathrm{CAB}_D(x,y)$ partition is $\{C_D(x,y), A_D(x,y)\}$ if $a_D \neq 0$ and $\{C_D(x,y)\}$ if $a_D = 0$. For $1 \leqslant j \leqslant D$, we say that Γ has the CAB_j *property* if, for each $i \leqslant j$ and for every pair of vertices $x, y \in V(\Gamma)$ with $d(x,y) = i$, the partition $\mathrm{CAB}_i = \mathrm{CAB}_i(x,y)$ is equitable, and its parameters do not depend on the choice of vertices x and y. If Γ has the CAB_D property, then we simply say that it has the *CAB property*.

Lemma 4.1 (cf. [8, Proposition 2.1]). *Let Γ be a distance-regular graph with $a_1 > 0$ and the CAB_1 property. Then all local graphs of Γ are either*

(i) *connected strongly regular graphs with the same parameters, or*
(ii) *disjoint unions of (a_1+1)-cliques.*

We say the (triple) intersection number γ of Γ *exists* if, for every triple of vertices (x,y,z) of Γ such that x and y are adjacent and z is at distance 2 from both x and y, the number of common neighbors of x, y, and z is constant and equal to γ. To avoid the degenerate case, we assume that there exists at least one such triple (x,y,z) in Γ (i.e., $a_2 \neq 0$) when we say γ exists. We note that if Γ has the CAB_2 property and $a_2 \neq 0$, then the intersection number γ exists in Γ. In this case, the parameter α_2, defined as the number of neighbors in $C_2(x,y)$ of a vertex in $A_2(x,y)$, is equal to γ (see Figure 1).

We recall the definition of a regular near polygon. Let Γ be a distance-regular graph with diameter D. For integers s and t, we say that Γ is *of order* (s,t) if it is locally the disjoint union of $t+1$ cliques of size s. The graph Γ of order (s,t) is called a *regular near polygon* if $a_i = c_i a_1$ for all $1 \leqslant i \leqslant D-1$. If $a_D = c_D a_1$, we call Γ a *regular near $2D$-gon*; otherwise it is called a *regular near $(2D+1)$-gon*. We note that if a distance-regular graph with diameter $D \geqslant 2$ and $a_1 > 0$ is locally disconnected and has the CAB property, then it is a regular near $2D$-gon; see [8, Theorem 2.3].

Lemma 4.2 (cf. [6, Theorem 9.11]). *Let Γ be a regular near $2D$-gon with $D \geqslant 4$. If $c_2 \geqslant 3$ or $c_i = i$ $(i = 2,3)$, then Γ is either a dual polar graph or a Hamming graph.*

Let Γ be a distance-regular graph with diameter $D \geqslant 2$ and $a_1 > 0$. Suppose Γ has the CAB_j property for some $1 \leqslant j \leqslant D$. For each $1 \leqslant i \leqslant j$, $i \neq D$, consider the parameters of the CAB_i partition (see Figure 1). Using these parameters, we define the matrix Q_i by

$$Q_i = \begin{pmatrix} \gamma_i & a_1 - \gamma_i & 0 \\ \alpha_i & a_1 - \beta_i - \alpha_i & \beta_i \\ 0 & \delta_i & a_1 - \delta_i \end{pmatrix} \qquad (1 \leqslant i \leqslant j,\ i \neq D). \tag{4.1}$$

We call Q_i the *quotient matrix* associated with the CAB_i partition; cf. [8, Theorem 2.4]. Since Γ has the CAB_j property, it follows from construction and Lemma 4.1 that Γ is locally strongly regular with the same parameters. If Γ is locally connected, then the eigenvalues of Q_i coincide with those of a local graph of Γ; cf. [8, Lemma 2.6].

Lemma 4.3 (cf. [8, Theorem 2.7]). *Let Γ be a locally connected distance-regular graph with $D \geqslant 2$, $a_1 \geqslant 2$, and the CAB_j property for some $1 \leqslant j < D$. Let $\delta_0 = 0$. Then, for $1 \leqslant i \leqslant j$ the parameters a_i, b_i, α_i, β_i, γ_i, δ_i of the CAB_i partition are expressed in terms of the eigenvalues $a_1 > r > s$ of a local graph of Γ and the parameters δ_{i-1}*

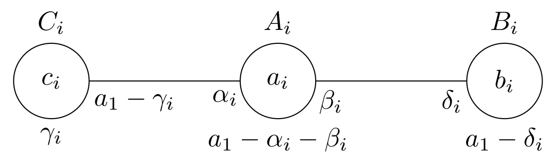

FIGURE 1. The CAB$_i$ partition and its parameters for $1 \leqslant i \leqslant D-1$

and c_i as follows:

(4.2)

$$\gamma_i = \delta_{i-1}, \qquad \alpha_i = \frac{c_i(a_1 - \delta_{i-1})}{k - c_i - b_i}, \qquad \beta_i = \frac{\mu' b_i}{a_1 - \delta_{i-1}}, \qquad \delta_i = \frac{\mu'(k - c_i)}{a_1 - \delta_{i-1}} - \beta_i,$$

$$b_i = k - c_i - \frac{c_i(a_1 - \delta_{i-1})^2}{(a_1 - \delta_{i-1})(a_1 - r - s + \delta_{i-1}) - \mu'(k - c_i)}, \tag{4.3}$$

and $a_i = k - b_i - c_i$, where $k = (a_1 - r)(a_1 - s)/\mu'$ and $\mu' = a_1 + rs$.

Proof. For (4.2), refer to the proof of [8, Theorem 2.7]. We will show (4.3) only. Observe that Γ has the CAB$_1$ property, so it is locally a strongly regular graph with parameters (k, a_1, λ', μ') and eigenvalues $a_1 > r > s$. Consider the quotient matrix Q_i associated with the CAB$_i$ partition for $1 \leqslant i \leqslant j < D$. From (4.1), we have the trace $\operatorname{tr}(Q_i) = 2a_1 - \alpha_i - \beta_i - \delta_i + \gamma_i$. Moreover, since the eigenvalues of Q_i equal the eigenvalues of a local graph of Γ, it follows that $\operatorname{tr}(Q_i) = a_1 + r + s$. From these two equations for the trace of Q_i, we obtain

$$a_1 - r - s = \alpha_i + \beta_i + \delta_i - \gamma_i. \tag{4.4}$$

Eliminate α_i, β_i, δ_i, γ_i in (4.4) using (4.2) and solve for b_i. Simplify the result to obtain (4.3). □

Now, we discuss distance-regular graphs satisfying the CAB$_2$ property. Let Γ be a locally connected distance-regular graph with diameter $D \geqslant 2$ and $a_1 \geqslant 2$. We assume that Γ satisfies the CAB$_2$ property. By Lemma 4.1, every local graph of Γ is a connected strongly regular graph with the same parameters (v', k', λ', μ') and eigenvalues $a_1 > r > s$. Note that $v' = k$ and $k' = a_1$. Consider the CAB$_2$ partition. By construction, we have $\delta_1 = \mu'$. Using this and $\lambda' = \mu' + r + s$, evaluate equations (4.2), (4.3) at $i = 2$. Then the parameters of the CAB$_2$ partition are

(4.5)

$$\gamma_2 = \delta_1 = \mu', \qquad \alpha_2 = \frac{c_2(a_1 - \mu')}{k - c_2 - b_2}, \qquad \beta_2 = \frac{\mu' b_2}{a_1 - \mu'}, \qquad \delta_2 = \frac{\mu'(k - c_2 - b_2)}{a_1 - \mu'},$$

and

$$b_2 = k - c_2 - \frac{c_2(a_1 - \mu')^2}{(a_1 + 2\mu' - \lambda')(a_1 - \mu') - \mu'(k - c_2)}. \tag{4.6}$$

With reference to the above discussion, we will express the parameters of the CAB$_2$ partition in terms of $m = -s$ and $n = r - s$ when Γ is locally a Latin square graph or a Steiner graph. For our convenience, we will use the following notation.

NOTATION 4.4. Let $b \geqslant 1$. We define the polynomial G in the variable b as follows:

$$G(b) = 16^{10} + 32b^9 + 48b^8 + 32b^7 + 16b^6 + 8b^5 + 8b^4 + 8b^3 + 1. \tag{4.7}$$

Recall $F(b)$ from (1.3). Note that $G(b) < F(b)$ for all $b \geqslant 1$.

Recall the intersection number b_1 and the second largest eigenvalue θ_1 of Γ.

Proposition 4.5. *Let Γ be a distance-regular graph with diameter $D \geqslant 5$, valency k, $a_1 > 0$, and satisfying the* CAB_2 *property. Let $b = b_1/(\theta_1 + 1)$. Assume that $k > G(b)$, where $G(b)$ is from (4.7). If Γ is locally a Latin square graph $\mathrm{LS}_m(n)$ with $m \geqslant 2$, then we have*

$$\alpha_2 = m, \qquad \beta_2 = (m-1)(n-m^2+m), \qquad \delta_2 = m^2(m-1), \tag{4.8}$$

$$a_2 = m^2(n-m), \qquad b_2 = (n-m)(n-m^2+m), \qquad c_2 = m^2. \tag{4.9}$$

Proof. From (2.2), we observe that the parameters (k, a_1, λ', μ') of $\mathrm{LS}_m(n)$ are

$$k = n^2, \qquad a_1 = m(n-1), \qquad \lambda' = (m-1)(m-2)+n-2, \qquad \mu' = m(m-1). \tag{4.10}$$

First, we find the parameters α_2 and c_2. In equation (4.4) at $i = 2$, eliminate β_2 and δ_2 using (4.5), then solve the resulting expression for α_2 to obtain $\alpha_2 = a_1 - \lambda' + 2\mu' - \mu'(k - c_2)/(a_1 - \mu')$. In this equation, eliminate the parameters k, a_1, λ', μ' using (4.10) and express the result in terms of n, m, c_2 to have

$$\alpha_2 = m + \frac{(m-1)(c_2 - m^2)}{n-m}. \tag{4.11}$$

Since α_2 is an integer, we have either $(m-1)(c_2 - m^2) = 0$ or $n - m$ divides $(m-1)(c_2 - m^2)$. Suppose $(m-1)(c_2-m^2) \neq 0$. Since $n-m$ divides $(m-1)(c_2-m^2)$, we have

$$n - m \leqslant (m-1)(c_2 - m^2). \tag{4.12}$$

Apply inequality (3.1) to c_2 in (4.12) and substitute $\mu' = m(m-1)$ into the result to get

$$n - m \leqslant (m-1)\big((4b^2+1)(m(m-1)+1) - m^2\big). \tag{4.13}$$

Simplify inequality (4.13) to get

$$n \leqslant 4b^2(m-1)(m(m-1)+1) - m(m-1) + 2m - 1. \tag{4.14}$$

By Lemma 3.1, we have $m \leqslant 1 + b$. Apply this inequality to (4.14) to obtain

$$n \leqslant 4b^3(b^2+b+1) - m(m-1) + 2m - 1. \tag{4.15}$$

Moreover, since $m \geqslant 2$ and $m - 1 > 0$, we have $-m(m-1) \leqslant -2m + 2$. Apply this inequality to (4.15) to obtain

$$n \leqslant 4b^3(b^2+b+1) + 1 = 4b^5 + 4b^4 + 4b^3 + 1. \tag{4.16}$$

Recall that $k = n^2$ from (4.10). Squaring both sides of (4.16) gives $k \leqslant (4b^5 + 4b^4 + 4b^3 + 1)^2 = G(b)$. This contradicts the assumption $k > G(b)$. Therefore, we must have $(m-1)(c_2 - m^2) = 0$, which implies that $c_2 = m^2$. From (4.11), we conclude that $\alpha_2 = m$.

Next, we find β_2. In the third equation in (4.5), eliminate b_2 using (4.6) to obtain

$$\beta_2 = \frac{\mu'}{a_1 - \mu'}\left(k - c_2 - \frac{c_2(a_1-\mu')^2}{(a_1+2\mu'-\lambda')(a_1-\mu') - \mu'(k-c_2)}\right) \tag{4.17}$$

Express the right-hand side of (4.17) in terms of m and n using (4.10) and $c_2 = m^2$. Simplify the result to obtain $\beta_2 = (m-1)(n-m^2+m)$. To find b_2, evaluate the right-hand side of (4.6) using (4.10) and $c_2 = m^2$. Simplify the result to obtain $b_2 = (n-m)(n-m^2+m)$. For δ_2, solve the fourth equation in (4.5) using (4.10) and $b_2 = (n-m)(n-m^2+m)$ to obtain $\delta_2 = m^2(m-1)$. Finally, for a_2, evaluate $k - b_2 - c_2$ using $k = n^2$, $b_2 = (n-m)(n-m^2+m)$, $c = m^2$ to get $a_2 = m^2(n-m)$. The proof is now complete. □

Proposition 4.6. *Let Γ be a distance-regular graph with diameter $D \geqslant 5$, valency k, $a_1 > 0$, and satisfying the* CAB_2 *property. Let $b = b_1/(\theta_1+1)$. Assume that $k > F(b)$, where $F(b)$ is from* (1.3)*. If Γ is locally a Steiner graph $S_m(n)$ with $m \geqslant 2$, then we have*

$$\alpha_2 = m+1, \qquad \beta_2 = (m-1)(n-m^2+1), \qquad \delta_2 = m^3, \tag{4.18}$$

$$a_2 = m^2(n-m), \quad b_2 = (m-1)(n-m)(n-m^2+1)/m, \quad c_2 = m(m+1). \tag{4.19}$$

Proof. From (2.4), we observe that the parameters (k, a_1, λ', μ') of $S_m(n)$ are

$$k = \frac{(m+n(m-1))(n+1)}{m}, \qquad a_1 = mn, \qquad \lambda' = m^2 - 2m + n, \qquad \mu' = m^2. \tag{4.20}$$

In a similar manner to the proof of Proposition 4.5, we have

$$\alpha_2 = m + 1 + \frac{m(c_2 - m(m+1))}{n-m}. \tag{4.21}$$

Since α_2 is an integer, we have either $m(c_2 - m(m+1)) = 0$ or $n-m$ divides $m(c_2 - m(m+1))$. If $c_2 \neq m(m+1)$, then

$$n - m \leqslant m(c_2 - m(m+1)). \tag{4.22}$$

Apply inequality (3.1) to c_2 in (4.22) and substitute $\mu' = m^2$ into the result to get

$$n - m \leqslant m\big((4b^2+1)(m^2+1) - m(m+1)\big). \tag{4.23}$$

Simplify inequality (4.23) to get

$$n \leqslant m\big(4b^2(m^2+1) - m + 2\big). \tag{4.24}$$

Note that $2 \leqslant m \leqslant 1+b$. Apply this to (4.24) to obtain

$$n \leqslant m(4b^4 + 8b^3 + 8b^2). \tag{4.25}$$

Recall that $k = (m+n(m-1))(n+1)/n$ from (4.20). Apply inequality (4.25) to this equation, along with $2 \leqslant m \leqslant 1+b$, to obtain

$$k = \left(\frac{m+n(m-1)}{m}\right)(n+1) \leqslant \big(1+b(4b^4+8b^3+8b^2)\big)\big(1+(1+b)(4b^4+8b^3+8b^2)\big) = F(b). \tag{4.26}$$

This contradicts the given assumption that $k > F(b)$. Therefore, we must have $m(c_2 - m(m+1)) = 0$, which implies that $c_2 = m(m+1)$. From (4.21), we conclude that $\alpha_2 = m+1$. The remaining parameters in (4.18) and (4.19) are obtained similarly to Proposition 4.5. The proof is complete. □

Remark 4.7. (i) Let Γ be locally a Latin square graph $LS_2(n)$ with $n \geqslant 3$. Observe that $LS_2(n)$ has smallest eigenvalue -2 and is isomorphic to the square $(n \times n)$-grid. By Proposition 4.5, we have $c_2 = 4$. Therefore, by construction the c_2-graph of Γ is a quadrangle.

(ii) Let Γ be locally a Steiner graph $S_2(n)$ with $n \geqslant 3$. Observe that $S_2(n)$ has smallest eigenvalue -2 and is isomorphic to the triangular graph $T(n+2)$. By Proposition 4.6, we have $c_2 = 6$. Therefore, by construction the c_2-graph of Γ is a 4-regular graph on 6 vertices, i.e., an octahedron.

5. The 1-homogeneous distance-regular graphs

In this section, we discuss distance-regular graphs that have the 1-homogeneous property. We then prove our main result, Theorem 1.2. We begin by recalling the definition of i-homogeneous graphs and some known results concerning these graphs.

Let Γ be a connected graph with diameter D. For $0 \leqslant i \leqslant D$, fix a pair of vertices x and y in Γ with $d(x,y) = i$. For $0 \leqslant h, j \leqslant D$, we define the subset $D_j^h(x,y)$ of $V(\Gamma)$ by

$$D_j^h(x,y) := \Gamma_j(x) \cap \Gamma_h(y). \tag{5.1}$$

Let $\pi(x,y)$ denote the collection of nonempty sets $D_j^h(x,y)$ for $0 \leqslant h, j \leqslant D$. Observe that $\pi(x,y)$ forms a distance partition of $V(\Gamma)$. The graph Γ is said to be *i-homogeneous* whenever the partition $\pi(x,y)$ is equitable, and the parameters of $\pi(x,y)$ are independent of the choice of vertices x and y; see [21]. Our discussion centers on the 1-homogeneous distance-regular graphs. We note that if Γ is a 1-homogeneous distance-regular graph with $D \geqslant 2$ and $a_2 \neq 0$, then the intersection number γ exists. Jurišić and Koolen [8] studied these graphs in detail using the CAB property and characterized them as follows.

Lemma 5.1 (cf. [8, Theorem 3.1]). *Let Γ be a distance-regular graph with diameter D and $a_1 > 0$. Then Γ is 1-homogeneous if and only if it has the* CAB *property.*

We consider 1-homogeneous distance-regular graphs that are locally disconnected.

Corollary 5.2 (cf. [8, Corollary 3.3]). *Let Γ be a locally disconnected 1-homogeneous distance-regular graph with diameter $D \geqslant 2$ and $a_1 > 0$. Then Γ is a regular near $2D$-gon.*

Proof. Since Γ is 1-homogeneous, it has the CAB property by Lemma 5.1. As Γ is also locally disconnected with $a_1 > 0$, it follows from [8, Theorem 2.3] that Γ is a regular near $2D$-gon. □

We have a comment. Suppose that Γ is a locally disconnected 1-homogeneous distance-regular graph with $D \geqslant 4$ and $a_1 > 0$, and assume that either $c_2 \geqslant 3$ or $c_i = i$ for $i = 2, 3$. Then, by Corollary 5.2, Γ is a regular near $2D$-gon. It follows from Lemma 4.2 that Γ is either a dual polar graph or a Hamming graph.

We now discuss locally connected 1-homogeneous distance-regular graphs. There are some studies on 1-homogeneous distance-regular graphs whose c_2-graphs are complete multipartite; cf. [11], [15]. In particular, Jurišić and Koolen [11] classified 1-homogeneous distance-regular graphs whose c_2-graphs are the Cocktail Party graph.

Theorem 5.3 ([11, Theorem 1.1]). *Let Γ be a 1-homogeneous distance-regular graph with diameter $D \geqslant 2$, whose c_2-graph is a connected Cocktail Party graph. Then Γ is one of the following graphs:*

(i) *a Johnson graph $J(2D, D)$,*
(ii) *a halved ℓ-cube with $\ell \in \{2D, 2D+1\}$,*
(iii) *a folded Johnson graph $\bar{J}(4D, 2D)$,*
(iv) *a folded halved $4D$-cube,*
(v) *a Cocktail Party graph $K_{t\times 2}$ with $t \geqslant 3$.*
(vi) *the Schläfli graph with intersection array $\{16, 5; 1, 8\}$,*
(vii) *the Gosset graph with intersection array $\{27, 10, 1; 1, 10, 27\}$.*

Recently, Koolen et al. [17] studied 1-homogeneous distance-regular graphs that are locally block graphs of Latin square graphs or Steiner systems. The main result they showed is as follows:

Lemma 5.4 (cf. [17, Theorem 1.2]). *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency k, and $a_1 > 0$ satisfying the* CAB_2 *property.*

(i) *If Γ is locally a Latin square graph with smallest eigenvalue $-m$ for $m \geqslant 3$ and $k > m^2$, then $c_2 \neq m^2$.*

(ii) *If Γ is locally the block graph of a Steiner system $S(2, m, n)$ with smallest eigenvalue $-m$ for $m \geqslant 3$ and $k > m(m+1)$, then $c_2 \neq m(m+1)$.*

Remark 5.5. (i) Let Γ be a distance-regular graph that is locally a Latin square graph $\mathrm{LS}_m(n)$ (resp. the block graph of a Steiner system $S(2, m, n)$). In [17], it is shown that if $c_2 = m^2$ (resp. $c_2 = m(m+1)$), then the c_2-graph of Γ is a complete m-partite (resp. $(m+1)$-partite) graph. This result, along with the findings of [15], is used to complete the proof of Lemma 5.4.

(ii) The icosahedron has $b_1 = 2$, $\theta_1 = \sqrt{5}$ and hence $b = \frac{b_1}{\theta_1+1} = \frac{1}{2}(-1+\sqrt{5}) < 1$. The only locally connected distance-regular graph with the CAB_2 property that satisfies $b < 1$ is the icosahedron. This is because the only connected strongly regular graph with the smallest eigenvalue greater than -2 is the pentagon, and the only graph that is locally a pentagon is the icosahedron (cf. [2, p. 35]).

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We assume $c_2 \geqslant 2$. Our proof is divided into two cases: Γ is locally disconnected, and Γ is locally connected. First, we assume that Γ is locally disconnected. Since Γ contains a quadrangle, by the result of [23] (cf. [2, p. 170]), we obtain $\theta_1 \leqslant b_1 - 1$, which implies $b \geqslant 1$. Moreover, by Corollary 5.2 Γ is a regular near $2D$-gon. Therefore, we have (i).

Next, we consider the case where Γ is locally connected. Since Γ satisfies the CAB_1 property, every local graph of Γ is connected strongly regular with the same parameters. Let $-m$ be the smallest eigenvalue of a local graph of Γ. Note that a connected strongly regular graph satisfies $-m < -1$ and the only strongly regular graph satisfying $-m > -2$ is a pentagon. Furthermore, the only graph that is locally pentagonal is the icosahedron. However, an icosahedron has diameter 3, which is less than 5. Therefore, we have $-m \leqslant -2$. By Lemma 3.1, it follows that $b \geqslant 1$.

Now, we show that one of the following (ii)–(vi) holds. We assume that $k > F(b)$; otherwise, we fall into the case (vi). Since Γ is locally a connected strongly regular graph with eigenvalues $a_1 > r > s$, it follows that either Γ is locally a conference graph or r and s are integers. If Γ is locally a conference graph, then it must be a Taylor graph by Proposition 3.6. However, Γ has $D \geqslant 5$ while a Taylor graph has diameter three. Therefore, a local graph of Γ is not a conference graph and hence has integral eigenvalues r and s. We put $s = -m$ and $r = n - m$, where $m \geqslant 2$. According to Corollary 2.4, a local graph of Γ satisfies one of (i)–(iv) listed therein. However, Case (iv) of Corollary 2.4 is ruled out as $b \geqslant m - 1 \geqslant 1$, and, by assumption, we have $k > F(b) > m^{10} > \varphi(m)$. Therefore, Γ is locally either a complete multipartite graph with classes of size m, a Latin square graph $\mathrm{LS}_m(n)$, or a Steiner graph $S_m(n)$. Now, we consider each case.

First, consider the case where Γ is locally a complete multipartite graph with classes of size m, denoted as $K_{t\times m}$. By [2, Proposition 1.1.5], Γ is $K_{(t+1)\times m}$. This contradicts the assumption $D \geqslant 5$ since a complete multipartite graph has diameter two.

Next, consider the case where Γ is locally a Latin square graph $\mathrm{LS}_m(n)$. Since $k > F(b) > G(b)$, by Proposition 4.5 we have $c_2 = m^2$, where $m \geqslant 2$. However, by Lemma 5.4(i), c_2 cannot be equal to m^2 for $m \geqslant 3$. By these comments, it follows $m = 2$. Thus, Γ is locally a Latin square graph $\mathrm{LS}_2(n)$ with smallest eigenvalues -2 and $c_2 = 4$. Observe that $\mathrm{LS}_2(n)$ is isomorphic to the $(n \times n)$-grid. By Lemma 3.7, Γ is the Johnson graph $J(2n, n)$ or a folded Johnson graph $\bar{J}(2n, n)$.

Lastly, consider the case where a local graph of Γ is a Steiner graph $S_m(n)$. Since $k > F(b)$, by Proposition 4.6 we have $c_2 = m(m+1)$, where $m \geqslant 2$. However, by Lemma 5.4(ii) c_2 cannot be equal to $m(m+1)$ for $m \geqslant 3$. Consequently, it follows $m = 2$. Thus, Γ is locally a Steiner graph $S_2(n)$ with smallest eigenvalue -2 and $c_2 = 6$. By Remark 4.7(ii), the c_2-graph of Γ is a 4-regular graph on 6 vertices, that is a Cocktail Party graph $K_{3\times 2}$. Hence, by Theorem 5.3 (also refer to the proof of [11, Theorem 5.3]) Γ is a halved ℓ-cube with $\ell \in \{2D, 2D+1\}$ or a folded halved $4D$-cube. The proof is complete. □

6. Distance-regular graphs with classical parameters

In this section, we discuss 1-homogeneous distance-regular graphs with classical parameters. For integers b and i, we recall the Gaussian binomial coefficient

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}_b = \begin{cases} i & \text{if} \quad b = 1; \\ (b^i - 1)/(b-1) & \text{if} \quad b \neq 1. \end{cases}$$

Let Γ be a distance-regular graph with diameter $D \geqslant 3$. We say that Γ has *classical parameters* (D, b, α, β) if its intersection numbers $\{b_i\}_{i=0}^{D-1}$ and $\{c_i\}_{i=1}^{D}$ satisfy

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right), \qquad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right). \tag{6.1}$$

Note that the parameter b is an integer, excluding 0 or -1, since $b_2 \neq 0$ and $c_2 \neq 0$. Also, from $a_i + b_i + c_i = k = b_0$ $(0 \leqslant i \leqslant D)$, it follows

$$a_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(\beta - 1 + \alpha \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \right) \qquad (0 \leqslant i \leqslant D). \tag{6.2}$$

Observe that $a_1 = 0$ if and only if $\beta = 1 - \alpha b \left[\begin{smallmatrix} D-1 \\ 1 \end{smallmatrix}\right]$. The eigenvalues of Γ are

$$\theta_i = \begin{bmatrix} D-i \\ 1 \end{bmatrix} \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) - \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad (0 \leqslant i \leqslant D), \tag{6.3}$$

cf. [2, Corollary 8.4.2]. We note that the eigenvalues are in the natural ordering $\theta_0 > \theta_1 > \cdots > \theta_D$ if $b > 0$. We recall a lower bound on the parameter β.

Lemma 6.1 ([16, Proposition 1]). *Let Γ be a distance-regular graph with classical parameters (D, b, α, β) and $D \geqslant 3$. If $b > 0$, then $\beta \geqslant 1 + \alpha \left[\begin{smallmatrix} D-1 \\ 1 \end{smallmatrix}\right]$, with equality if and only if $a_D = 0$.*

We now consider 1-homogeneous distance-regular graphs with classical parameters and $a_1 > 0$, and obtain the following result.

Theorem 6.2. *Let Γ be a 1-homogeneous distance-regular graph with classical parameters (D, b, α, β) and $a_1 > 0$, where $D \geqslant 5$ and $b \geqslant 1$. Then one of the following holds:*

(i) $\alpha = 0$,
(ii) Γ *is a Johnson graph* $J(2D, D)$,
(iii) Γ *is a halved* ℓ*-cube with* $\ell \in \{2D, 2D+1\}$,
(iv) Γ *is a folded Johnson graph* $\bar{J}(4D, 2D)$,
(v) Γ *is a folded halved* $4D$*-cube,*
(vi) $D \leqslant 9$, $\alpha > 0$, $b \geqslant 2$.

Proof. Let $\{\theta_i\}_{i=0}^{D}$ be the eigenvalues of Γ as in (6.3). Since $b \geqslant 1$, it follows $\theta_0 > \theta_1 > \cdots > \theta_D$. We observe that $b = b_1/(\theta_1 + 1)$ by evaluating b_i in (6.1) and θ_i in (6.3) at $i = 1$. Moreover, $c_2 \geqslant 2$ since every distance-regular graph with classical parameters and diameter at least three satisfies $c_i < c_{i+1}$ for $0 \leqslant i \leqslant D-1$ (cf. [2, Theorem 6.1.2]). Therefore, by Theorem 1.2, Γ is either a regular near $2D$-gon or falls into one of (ii)–(vi) therein.

We consider the parameter α. Evaluating c_i in (6.1) at $i = 2$ gives $c_2 = (1+b)(1+\alpha)$, which implies that $\alpha(1 + b)$ is an integer. Next, evaluating c_i in (6.1) at $i = 3$ gives $c_3 = (1+b+b^2)(1+\alpha(1+b))$. Since $1+b+b^2 > 0$ and $c_3 > 0$, it follows $1+\alpha(1+b) > 0$. As $\alpha(1 + b)$ is an integer, we have $\alpha(1 + b) \geqslant 0$. Since $1 + b > 0$, we obtain $\alpha \geqslant 0$.

We consider the case where $\alpha > 0$; otherwise, we obtain (i). Since $b \geqslant 1$, we divide the argument into two cases: either $b = 1$ or $b \geqslant 2$. If $b = 1$, all distance-regular graphs with classical parameters $(D, 1, \alpha, \beta)$ are known: the Gosset graph (with $\alpha = 4$), the Johnson graphs (with $\alpha = 1$) and the halved ℓ-cubes, where $\ell \in \{2D, 2D + 1\}$ (with $\alpha = 2$), the folded Johnson graphs (with $\alpha = 1$), and the folded halved $4D$-cubes (with $\alpha = 2$); cf [2, Sections 6.1, 6.3]. However, we rule out the Gosset graph since it has diameter $3 < 5$. Therefore, by Theorem 1.2, Γ is a Johnson graph $J(2D, D)$, a halved ℓ-cube with $\ell \in \{2D, 2D + 1\}$, a folded Johnson graph $\bar{J}(4D, 2D)$, or a folded halved $4D$-cube.

Next, consider the case where $b \geqslant 2$ with $\alpha > 0$. Since $\alpha(1 + b)$ is a positive integer, it follows that $\alpha \geqslant 1/(1 + b)$. Using this together with Lemma 6.1, we have $\beta > \alpha\left[{D-1 \atop 1}\right] \geqslant \frac{1}{1+b}\left[{D-1 \atop 1}\right]$. Note that $k = \beta\left[{D \atop 1}\right]$. By these comments, we have

$$k = \beta \begin{bmatrix} D \\ 1 \end{bmatrix} > \frac{1}{1+b} \begin{bmatrix} D \\ 1 \end{bmatrix} \begin{bmatrix} D-1 \\ 1 \end{bmatrix} = \frac{(b^D - 1)(b^{D-1} - 1)}{(b+1)(b-1)^2}.$$

Using a computer, we find that $\frac{(b^D-1)(b^{D-1}-1)}{(b+1)(b-1)^2} > F(b)$ for $D \geqslant 10$, where $F(b)$ is from (1.3)[(1)]. In other words, for $D \geqslant 10$ we have $k > F(b)$. However, by Theorem 1.2, the inequality $k \leqslant F(b)$ must hold. Therefore, we have $D \leqslant 9$. The proof is complete. □

Remark 6.3. (i) We give a comment on the case (i) of Theorem 6.2. In a subsequent paper, we will show that if Γ is a 1-homogeneous distance-regular graph with $a_1 > 0$ and classical parameters $(D, b, 0, \beta)$ where $D \geqslant 5$, then Γ is locally a disjoint union of cliques. Therefore, by Lemma 5.2, Γ is a regular near $2D$-gon. Consequently, according to [2, Theorem 9.4.4], Γ is either a dual polar graph or a Hamming graph.

(ii) The unitary dual polar graphs $U(2D, r)$[(2)] have two distinct types of classical parameters:

$$(D, b, \alpha, \beta) = \left(D, q^2, 0, q\right), \tag{6.4}$$

$$(D, b, \alpha, \beta) = \left(D, -q, \frac{q(1+q)}{1-q}, \frac{q\left(1 + (-q)^D\right)}{1-q}\right), \tag{6.5}$$

where q is a prime power and $r = q^2$; see [2, Section 6.2]. The graphs $U(2D, r)$ are 1-homogeneous with $a_1 = q - 1 > 0$, and as such, when $D \geqslant 5$, they correspond to instances of case (i) of Theorem 6.2 under the parametrization (6.4). The graph $U(2D, r)$ with classical parameters (6.5) corresponds to none of the cases of Theorem 6.2 because $\alpha \neq 0$ and $b < 0$.

7. Tight distance-regular graphs

Let Γ be a distance-regular graph with $D \geqslant 3$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Jurišić, Koolen and Terwilliger established the following so-called fundamental bound

(1) We note that the inequality $\frac{(b^D-1)(b^{D-1}-1)}{(b+1)(b-1)^2} > F(b)$ also holds for $D = 8$ and $b \geqslant 6$, and for $D = 9$ and $b \geqslant 3$.

(2) This family of dual polar graphs is also denoted by ${}^2A_{2D-1}(r)$.

for Γ:

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_D + \frac{k}{a_1+1}\right) \geqslant -\frac{ka_1b_1}{(a_1+1)^2}, \tag{7.1}$$

cf. [14, Theorem 6.2]. The graph Γ is said to be *tight* whenever Γ is not bipartite and equality holds in (7.1). We note that if Γ is tight, then $a_1 > 0$; cf. [14, Corollary 6.3]. We recall some characterizations of tight distance-regular graphs.

Lemma 7.1 (cf. [14, Theorem 11.7, Theorem 12.6]). *Let Γ be a distance-regular graph with diameter $D \geqslant 3$. Then the following are equivalent.*

(i) Γ *is tight.*
(ii) Γ *is 1-homogeneous with $a_1 > 0$ and $a_D = 0$.*
(iii) *Every local graph of Γ is connected strongly regular with eigenvalues $a_1 > r > s$, where*

$$r = -1 - \frac{b_1}{\theta_D + 1}, \qquad s = -1 - \frac{b_1}{\theta_1 + 1}.$$

In the following result, we characterize tight distance-regular graphs with diameter $D \geqslant 5$.

Theorem 7.2. *Let Γ be a tight distance-regular graph with diameter $D \geqslant 5$. Let $b = b_1/(\theta_1 + 1)$. Then $b \geqslant 1$ and one of the following holds:*

(i) Γ *is a Johnson graph $J(2D, D)$,*
(ii) Γ *is a halved $2D$-cube,*
(iii) Γ *is locally connected with $k \leqslant F(b)$, where $F(b)$ is from* (1.3).

Proof. By Lemma 7.1, the tight graph Γ is locally connected, which implies $c_2 \geqslant 2$. Moreover, Γ is 1-homogeneous with $a_1 > 0$ and $a_D = 0$. Therefore, by Theorem 1.2, it follows that $b \geqslant 1$, and Γ is either a regular near $2D$-gon or falls into one of cases (ii)–(vi) in Theorem 1.2. However, Γ cannot be a regular near $2D$-gon since $a_1 > 0$ and $a_D = 0$. Thus, Γ belongs to one of cases (ii)–(vi) therein.

Assume that $k > F(b)$; otherwise, it leads to case (iii). By Lemma 7.1, the tight graph Γ is locally a connected strongly regular graph. We note that Γ has $a_D = 0$. In a similar manner to the proof of Theorem 1.2, we find that Γ is locally either a Latin square graph or a Steiner graph.

If Γ is locally a Latin square graph, by Theorem 1.2, Γ is either a Johnson graph $J(2D, D)$ or a folded Johnson graph $\bar{J}(4D, 2D)$. However, we rule out the case of the graph $\bar{J}(4D, 2D)$ since it has $a_D \neq 0$. Therefore, Γ is a Johnson graph $J(2D, D)$.

If Γ is locally a Steiner graph, then by Theorem 1.2 Γ is either a halved ℓ-cube with $\ell \in \{2D, 2D+1\}$ or a folded halved $4D$-cube. The folded halved $4D$-cube cannot occur since it satisfies $a_D \neq 0$, and a tight distance-regular graph must have $a_D = 0$. Similarly, the case $\ell = 2D + 1$ is excluded because the halved $(2D + 1)$-cube also satisfies $a_D \neq 0$. Therefore, the only remaining possibility is that Γ is a halved $2D$-cube. This completes the proof. □

Remark 7.3. In cases (i) and (ii) of Theorem 7.2, we have $b = 1$. Also, note that, except for the halved $2D$-cubes and the Johnson graphs $J(2D, D)$, all known tight distance-regular graphs have diameter $D \leqslant 4$.

(ii) Suppose Γ is tight. In [17, Theorem 1.3], Koolen et al. showed that if a local graph of Γ is neither the block graph of an orthogonal array nor the block graph of a Steiner system, then the valency k (and hence diameter D) of Γ is bounded by a function in b, where $b = b_1/(1 + \theta_1) \geqslant 2$. They then proposed a conjecture that generalizes this result: if Γ is a tight distance-regular graph with $b \geqslant 2$, then the diameter D of Γ is bounded by a function in b; see [17, Conjecture 28]. Since the diameter of a distance-regular graph is bounded in terms of its valency (cf. [1, Section 4]), it follows that Theorem 7.2(iii) proves this conjecture.

We finish this section with the following corollary.

Corollary 7.4. *Let Γ be a tight distance-regular graph of diameter $D \geqslant 5$ with classical parameters (D, b, α, β). Then $b \geqslant 1$ and one of the following holds:*

(i) *Γ is a Johnson graph $J(2D, D)$,*
(ii) *Γ is a halved $2D$-cube,*
(iii) *$D \leqslant 9$, $\alpha > 0$, and $b \geqslant 2$.*

Proof. By Theorems 6.2 and 7.2. □

Acknowledgements. The authors thank the anonymous referees for the valuable comments. J.-H. Lee completed part of this work during his sabbatical visit to POSTECH and expresses deep gratitude for their hospitality.

References

[1] S. Bang, A. Dubickas, J. H. Koolen, and V. Moulton, *There are only finitely many distance-regular graphs of fixed valency greater than two*, Adv. Math. **269** (2015), 1–55.

[2] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular graphs*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 18, Springer-Verlag, Berlin, 1989.

[3] Andries E. Brouwer and H. Van Maldeghem, *Strongly regular graphs*, Encyclopedia of Mathematics and its Applications, vol. 182, Cambridge University Press, Cambridge, 2022.

[4] Brian Curtin and Kazumasa Nomura, *Homogeneity of a distance-regular graph which supports a spin model*, J. Algebraic Combin. **19** (2004), no. 3, 257–272.

[5] Brian Curtin and Kazumasa Nomura, *1-homogeneous, pseudo-1-homogeneous, and 1-thin distance-regular graphs*, J. Combin. Theory Ser. B **93** (2005), no. 2, 279–302.

[6] Edwin R. van Dam, Jack H. Koolen, and Hajime Tanaka, *Distance-regular graphs*, Electron. J. Combin. **DS22** (2016), 1–156.

[7] Chris Godsil and Gordon Royle, *Algebraic graph theory*, Graduate Texts in Mathematics, vol. 207, Springer-Verlag, New York, 2001.

[8] Aleksandar Jurišić and Jack Koolen, *A local approach to 1-homogeneous graphs*, Des. Codes Cryptogr. **21** (2000), no. 1-3, 127–147.

[9] Aleksandar Jurišić and Jack Koolen, *Nonexistence of some antipodal distance-regular graphs of diameter four*, European J. Combin. **21** (2000), no. 8, 1039–1046.

[10] Aleksandar Jurišić and Jack Koolen, *Krein parameters and antipodal tight graphs with diameter 3 and 4*, Discrete Math. **244** (2002), no. 1-3, 181–202.

[11] Aleksandar Jurišić and Jack Koolen, *1-homogeneous graphs with cocktail party μ-graphs*, J. Algebraic Combin. **18** (2003), no. 2, 79–98.

[12] Aleksandar Jurišić and Jack Koolen, *Distance-regular graphs with complete multipartite μ-graphs and AT4 family*, J. Algebraic Combin. **25** (2007), no. 4, 459–471.

[13] Aleksandar Jurišić and Jack Koolen, *Classification of the family* AT4(qs, q, q) *of antipodal tight graphs*, J. Combin. Theory Ser. A **118** (2011), no. 3, 842–852.

[14] Aleksandar Jurišić, Jack Koolen, and Paul Terwilliger, *Tight distance-regular graphs*, J. Algebraic Combin. **12** (2000), no. 2, 163–197.

[15] Aleksandar Jurišić, Akihiro Munemasa, and Yuki Tagami, *On graphs with complete multipartite μ-graphs*, Discrete Math. **310** (2010), no. 12, 1812–1819.

[16] Aleksandar Jurišić and Janoš Vidali, *Restrictions on classical distance-regular graphs*, J. Algebraic Combin. **46** (2017), no. 3-4, 571–588.

[17] Jack H. Koolen, Jae-Ho Lee, Shuang-Dong Li, Yun-Han Li, Xiaoye Liang, and Ying-Ying Tan, *On the (non-)existence of tight distance-regular graphs: a local approach*, Electron. J. Combin. **31** (2024), no. 2, article no. 2.25 (20 pages).

[18] Jack H. Koolen and Jongyook Park, *Distance-regular graphs with a_1 or c_2 at least half the valency*, J. Combin. Theory Ser. A **119** (2012), no. 3, 546–555.

[19] Štefko Miklavič, *Q-polynomial distance-regular graphs with $a_1 = 0$*, European J. Combin. **25** (2004), no. 7, 911–920.

[20] A. Neumaier, *Strongly regular graphs with smallest eigenvalue $-m$*, Arch. Math. (Basel) **33** (1979/80), no. 4, 392–400.

[21] K. Nomura, *Homogeneous graphs and regular near polygons*, J. Combin. Theory Ser. B **60** (1994), no. 1, 63–71.

[22] Ying-Ying Tan, Jack H. Koolen, Meng-Yue Cao, and Jongyook Park, *Thin Q-polynomial distance-regular graphs have bounded* c_2, Graphs Combin. **38** (2022), no. 6, article no. 175 (18 pages).

[23] Paul Terwilliger, *Distance-regular graphs with girth* 3 *or* 4. *I*, J. Combin. Theory Ser. B **39** (1985), no. 3, 265–281.

Jack H. Koolen, School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026 (PR China)
CAS Wu Wen-Tsun Key Laboratory of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026 (PR China)
E-mail : koolen@ustc.edu.cn

Mamoon Abdullah, School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026 (PR China)
E-mail : mamoonabdullah@hotmail.com

Brhane Gebremichel, Department of Mathematics, Adigrat University, Adigrat, Tigray, 7040 (Ethiopia)
E-mail : brhaneg220@gmail.com

Jae-Ho Lee, Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224 (USA)
E-mail : jaeho.lee@unf.edu