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# The non-existence of $\mathbb{F}_q$ -translation spreads of $H(q^2)$

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**ABSTRACT** Using the connection between translation spreads of the generalized hexagon  $H(q)$  and linear sets (see Cardinali et al. in *Eur J Comb* 23:367–376, 2002; Lunardon and Polverino in *J Algebraic Comb* 18:255–262, 2003), the non-existence of  $\mathbb{F}_q$ -translation spreads of  $H(q^2)$  when  $p > 3$ ,  $q = p^h$ , and  $q$  is large enough is proven. This answers to a question posed in [Marino and Polverino in *J Algebraic Comb* 42:725–744, 2015].

## 1. INTRODUCTION AND MAIN RESULTS

The *generalized hexagon*  $H(q)$  is defined by Tits in [18] starting from a triality of a non-degenerate hyperbolic quadric of  $\mathbb{P}^7(\mathbb{F}_q)$ . In the same paper, Tits describes a natural embedding of  $H(q)$  in a parabolic quadric  $Q(6, q)$ . Precisely, if  $Q(6, q)$  is the parabolic quadric of  $\mathbb{P}^6(\mathbb{F}_q)$  with equation  $X_3^2 = X_0X_4 + X_1X_5 + X_2X_6$ , the points of  $H(q)$  are all the points of  $Q(6, q)$ , the lines of  $H(q)$  are the lines of  $Q(6, q)$  whose Grassmann coordinates satisfy the equations  $p_{34} = p_{12}$ ,  $p_{35} = p_{20}$ ,  $p_{36} = p_{01}$ ,  $p_{03} = p_{56}$ ,  $p_{13} = p_{64}$  and  $p_{23} = p_{45}$ , and the incidence is simply that inherited from  $Q(6, q)$ . Two elements of  $H(q)$  are *opposite* if they are at distance 6 in the incidence graph of  $H(q)$ . A *spread* of  $H(q)$  is a set of  $q^3 + 1$  mutually opposite lines of  $H(q)$ . Let  $\ell$  be a fixed line of  $H(q)$  and denote by  $E^\ell$  the group of the automorphisms of  $H(q)$  generated by all collineations fixing  $\ell$  pointwise and stabilizing all the lines through some point of  $\ell$ . The group  $E^\ell$  has order  $q^5$  and acts regularly on the set of the lines of  $H(q)$  at distance 6 from  $\ell$  (see e.g. [19]). A spread  $\mathcal{S}$  of  $H(q)$  containing  $\ell$  is a *translation spread*, with respect to  $\ell$ , if for each  $x \in \ell$  there is a subgroup of  $E^\ell$  that preserves  $\mathcal{S}$  and acts transitively on the lines of  $\mathcal{S}$  at distance 4 from  $m$ , for all lines  $m$  of  $H(q)$  incident with  $x$  and different from  $\ell$  (see [2]). Let  $Q^-(5, q)$  be an elliptic quadric intersection of  $Q(6, q)$  with a 5-dimensional space  $T$  and let  $\mathcal{S}$  be the set of the lines of  $H(q)$  contained in  $Q^-(5, q)$ . Then  $\mathcal{S}$  is a translation spread with respect to all its lines and is called the *Hermitian spread* (see e.g. [17, 12]).

If  $\mathcal{S}$  is a translation spread of  $H(q)$  with respect to the line  $\ell$  then, for each point  $x$  in  $\ell$  the spread  $\mathcal{S}$  gives rise to a translation ovoid  $\mathcal{O}_x(\mathcal{S})$  of a parabolic quadric  $Q(4, q)$  by a process called *projection along the reguli* of  $\mathcal{S}$  from the point  $x$  (see [2]). If  $\mathcal{O}_x(\mathcal{S})$  is the classical ovoid of  $Q(4, q)$  (that is, an elliptic quadric) for all  $x$  in  $\ell$ , then  $\mathcal{S}$  is called *semi-classical*.

Also, any translation spread  $\mathcal{S}$  with respect to a line  $\ell$  of  $H(q)$  defines a subfield of  $\mathbb{F}_q$ , called the *kernel* of  $\mathcal{S}$  and such a kernel is contained in the kernel of  $\mathcal{O}_x(\mathcal{S})$ ,

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for each  $x \in \ell$  ([15, proof of Theorem 7]). If  $\mathcal{S}$  is a translation spread of  $H(q)$  with kernel  $\mathbb{F}_{q'} \leq \mathbb{F}_q$  ( $q = (q')^n$ ), we say that  $\mathcal{S}$  is an  $\mathbb{F}_{q'}$ -translation spread of  $H(q)$ . By [15, Theorem 7],  $\mathcal{S}$  is semi-classical if and only if it has maximal kernel, i.e.  $\mathbb{F}_{q'} = \mathbb{F}_q$ . From [2, 11] semi-classical spreads of  $H(q)$  are classified, up to isomorphisms, and they are either the *Hermitian spreads* (Thas [17]) or the spreads  $\mathcal{S}_{[9]}$  constructed by Bloemen, Thas and Van Maldeghem in [2] for  $q \equiv 1 \pmod{3}$ ,  $q$  odd, or the spreads  $\mathcal{S}_l$  constructed independently by Cardinali, Lunardon, Polverino and Trombetti in [6] and by Offer in [15] for  $q \equiv 1 \pmod{3}$ ,  $q$  even. Also, a sufficient condition for a spread of  $H(q)$  to be semi-classical is the following.

**THEOREM 1.1** ([13, Theorem 1.1]). *Let  $\mathcal{S}$  be a translation spread with respect to a line  $\ell$ . If there exists a point  $x \in \ell$  such that  $\mathcal{O}_x(\mathcal{S})$  is a classical ovoid, then  $\mathcal{S}$  is semi-classical.*

The only known examples of non-semi-classical translation spreads are the  $\mathbb{F}_3$ -translation spreads  $\mathcal{S}_\beta$  of  $H(3^n)$ ,  $n > 1$ , constructed in [2]. The ovoids obtained projecting the spreads  $\mathcal{S}_\beta$  along the reguli are all isomorphic to the Roman ovoid of Thas and Payne (see [2, Theorem 22]).

Finally, the following results are known.

- If  $q$  is even, then  $\mathcal{S}$  is semi-classical ([6, Corollary 1]).
- A translation spread of  $H(3^n)$  is either hermitian or isomorphic to an  $\mathcal{S}_\beta$  ([14]).

Hence, the main question on translation spreads of  $H(q)$  concerns the existence of non-semi-classical translation spreads in characteristic different from 2 and 3. Also, by Theorem 1.1, if  $\mathcal{S}$  is a non-semi-classical translation spread of  $H(q)$ , then all the ovoids  $\mathcal{O}_x(\mathcal{S})$  are non-classical ovoids. So, it is clear that the existence of non-semi-classical translation spreads of  $H(q)$  depends on the existence of non-classical translation ovoids of  $Q(4, q)$ . However translation ovoids of  $Q(4, q)$  are very rare objects. Indeed, by [3] and [10], there is only one family of translation ovoids of  $Q(4, q^n)$ , with kernel  $\mathbb{F}_q$ , containing examples for each value of  $q$  odd and for each  $n > 1$ : the ovoids of Kantor-Knuth type <sup>(1)</sup>.

Also, by [13, Theorem 1.2], the existence of translation spreads of  $H(q^n)$ , with  $n > 2$ , producing, by projection along reguli, all ovoids of Kantor-Knuth type has been excluded.

It follows that

**THEOREM 1.2** ([13, Theorem 1.3]). *If  $q > 2$  is prime,  $n > 2$  and  $q \geq 4n^2 - 8n + 2$  or  $q > 2n^2 - (4 - 2\sqrt{3})n + (3 - 2\sqrt{3})$  then  $H(q^n)$  does not admit  $\mathbb{F}_q$ -translation spreads.*

Two main problems remain open.

The first is to face the existence of  $\mathbb{F}_q$ -translation spreads of  $H(q^2)$  ( $p > 3, q = p^h$ ). The second problem is to find examples of  $\mathbb{F}_q$ -translation spreads of  $H(q^n)$ ,  $n > 2$  ( $p > 3, q = p^h$ ) with  $q$  small enough with respect to  $n$  or to extend to any  $q$  the non-existence result stated in Theorem 1.2.

In this paper we want to focus on the first problem and our main result is the following.

**MAIN THEOREM.** *Let  $1039891 \leq q \not\equiv 0 \pmod{3}$ ,  $q$  odd. Then  $H(q^2)$  does not admit  $\mathbb{F}_q$ -translation spreads.*

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<sup>(1)</sup>For the description of these ovoids see [9].

## 2. LINEAR SETS AND TRANSLATION SPREADS IN $H(q)$

A point set  $L$  of a projective space  $\Omega = \mathbb{P}^{r-1}(\mathbb{F}_{q^n}) = \mathbb{P}(V)$ ,  $V = V(r, q^n)$  ( $q = p^h$ ,  $p$  prime) is said to be an  $\mathbb{F}_q$ -linear set of  $\Omega$  of rank  $t$  if it is defined by the non-zero vectors of a  $t$ -dimensional  $\mathbb{F}_q$ -vector subspace  $U$  of  $V$ , i.e.

$$L = L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}. \quad (2)$$

The connection between translation spreads of the classical generalized hexagon  $H(q)$ , linear sets, and the twisted cubic of  $\mathbb{P}^3(\mathbb{F}_q)$  is the first step to prove the main theorem. Most of the material regarding the twisted cubic can be found in [7, Section 21].

Consider the twisted cubic of  $\mathbb{P}^3(\mathbb{F}_q)$ ,  $q = p^s$ , in the canonical form

$$(1) \quad \mathcal{C} := \{P_t = \langle (t^3, t^2, t, 1) \rangle : t \in \mathbb{F}_q\} \cup \{P_\infty = \langle (1, 0, 0, 0) \rangle\}.$$

Let  $\bar{\mathcal{C}}$  be the twisted cubic of  $\mathbb{P}^3(\mathbb{F})$ , where  $\mathbb{F}$  is the algebraic closure of  $\mathbb{F}_q$ . A line of  $\mathbb{P}^3(\mathbb{F}_q)$  is a *chord* of  $\mathcal{C}$  if it contains two points of  $\mathcal{C}$ . There are three possibilities: the two points are distinct and belong to  $\mathcal{C}$ , or they are coincident, or they are conjugated over  $\mathbb{F}_{q^2}$ . The line is called a *real chord*, a *tangent* or an *imaginary chord*, respectively.

If  $p \neq 3$ , the tangents to  $\mathcal{C}$  are self-polar lines of a non-singular symplectic polarity  $\omega$  of  $\mathbb{P}^3(\mathbb{F})$ . An *axis* of  $\mathcal{C}$  is a line  $\ell$  of  $\mathbb{P}^3(\mathbb{F})$  whose polar line with respect to  $\omega$  is a chord. We say that  $\ell$  is a *real axis* or an *imaginary axis* when its polar line is a real chord or an imaginary chord, respectively (for more details, see [7, Section 21]).

Every point off the cubic  $\mathcal{C}$  lies on exactly one chord (see e.g. [7, Theorem 21.1.9]). For each  $t \in \mathbb{F}_q$ , the plane  $\pi_t : X_0 = 3tX_1 - 3t^2X_2 + t^3X_3$  is the *osculating plane* of  $\mathcal{C}$  at the point  $P_t$  and the plane  $\pi_\infty : X_3 = 0$  is the osculating plane of  $\mathcal{C}$  at  $P_\infty$  <sup>(3)</sup>. Hence  $\pi_t$ , with  $t \in \mathbb{F}_q \cup \{\infty\}$ , is the unique plane with the maximum intersection multiplicity in  $P_t$ . The union of the osculating planes forms the *osculating developable*  $\Gamma$  to  $\mathcal{C}$ . If  $q \geq 5$ , the linear automorphism group  $\mathcal{G}$  of  $\mathcal{C}$  is isomorphic to  $\text{PGL}(2, q)$  and so acts three-transitively on the points of the cubic.

The connection mentioned at the beginning of this section has been described for the first time in [6]. It arises by the construction of  $H(q^n)$  as a coset geometry (see e.g. [1, 19]), pointing out a one-to-one correspondence between the points of the translation line  $\ell$  of  $H(q)$  and the points of the twisted cubic  $\mathcal{C}$ , in such a way that the action of the automorphism group of  $H(q)$  fixing  $\ell$  corresponds to the action of the automorphism group of  $\mathcal{C}$ .

In the paper,  $E_{[\infty]}$  denotes the subgroup of the collineation group of the classical generalized hexagon  $H(q)$  fixing the line  $[\infty]$  pointwise and stabilizing all lines through a point of  $[\infty]$ . This group has order  $q^5$  and acts regularly on the set of lines of  $H(q)$  at distance 6 from  $[\infty]$  in the incidence graph.

Elements of  $E_{[\infty]}$  are expressed via the notation  $\theta(a, b, c, d, e)$ , representing certain upper-triangular collineation matrices of  $\mathbb{P}^6(q)$  that preserve the incidence structure of  $H(q)$ .

By factoring out the central subgroup  $Z \subset E_{[\infty]}$ , the quotient  $\bar{E} = E_{[\infty]}/Z$  becomes a 4-dimensional vector space over  $\mathbb{F}_q$ , identified with  $\mathbb{P}^3(q)$ . A subgroup  $G \leq E_{[\infty]}$  of order  $q^3$ , containing  $Z$ , corresponds to a linear set  $\bar{G} = G/Z \subset \mathbb{P}^3(q)$ . This setup allows translation spreads of  $H(q)$  to be studied via the geometry of their associated linear sets in  $\mathbb{P}^3(q)$ .

<sup>(2)</sup>In what follows, for simplicity, we will write  $\langle \mathbf{u} \rangle$  to denote the  $\mathbb{F}_{q^n}$ -vector space generated by  $\mathbf{u}$ .

<sup>(3)</sup>For the general definition of osculating hyperplane of a curve at one of its points, see e.g. [8, Section 7.6].

**THEOREM 2.1** ([6, 11]). *Each translation spread  $\mathcal{S}$  of  $H(q^n)$  with respect to a line  $\ell$ , whose kernel contains  $\mathbb{F}_q$ , defines an  $\mathbb{F}_q$ -linear set  $L(\mathcal{S})$  of  $\mathbb{P}^3(\mathbb{F}_{q^n})$  of rank  $2n$  whose points belong to imaginary chords of a twisted cubic  $\mathcal{C}$  of  $\mathbb{P}^3(\mathbb{F}_{q^n})$ , and conversely. Also, the automorphism group of  $H(q^n)$  fixing the line  $\ell$  induces on  $\mathbb{P}^3(\mathbb{F}_{q^n})$  the collineation group of  $PGL(4, q^n)$  fixing the twisted cubic  $\mathcal{C}$ .*

Also, by [6], if  $\mathbb{F}_q$  is the kernel of the translation spread  $\mathcal{S}$  of  $H(q^n)$ , then  $\mathbb{F}_q$  is the maximal subfield of linearity of  $L(\mathcal{S})$  (it is not linear over  $\mathbb{F}_{q^s}$  for every  $s > 1$ ).

Recall the following results.

- (R<sub>1</sub>) If  $\ell$  is a line of  $\mathbb{P}^3(\mathbb{F}_{q^n})$  whose points belong to imaginary chords of  $\mathcal{C}$ , then either  $\ell$  is an imaginary chord or  $q^n \equiv 1 \pmod{3}$  and  $\ell$  is an imaginary axis [11, Theorem 1].
- (R<sub>2</sub>) The translation spread  $\mathcal{S}$  of  $H(q^n)$  is semi-classical if and only if the  $\mathbb{F}_{q^n}$ -linear set  $L(\mathcal{S})$  is a line of  $\mathbb{P}^3(\mathbb{F}_{q^n})$  whose points belong to imaginary chords of  $\mathcal{C}$  [6, Corollary 3].
- (R<sub>3</sub>) The translation spread  $\mathcal{S}$  of  $H(q^n)$  is a Hermitian spread if and only if  $L(\mathcal{S})$  is an imaginary chord of  $\mathcal{C}$  [6, Theorem 5].
- (R<sub>4</sub>) The translation spread  $\mathcal{S}$  of  $H(q^n)$  is either the spread  $\mathcal{S}_{[9]}$  (if  $q$  is odd) or the spread  $\mathcal{S}_l$  mentioned in the Introduction (if  $q$  is even) if and only if  $q^n \equiv 1 \pmod{3}$  and  $L(\mathcal{S})$  is an imaginary axis of  $\mathcal{C}$  [6].

**2.1. LINEAR SETS AND THE TWISTED CUBIC.** By using the notation introduced in [7, Section 21], we can fix the twisted cubic  $\mathcal{C}$  of  $\mathbb{P} = \mathbb{P}^3(\mathbb{F}_q)$  with canonical form (1). The points on the tangents to  $\mathcal{C}$  form a quartic surface  $\Omega$  with equation

$$(2) \quad \Omega : F(X_0, X_1, X_2, X_3) = X_0^2 X_3^2 - 3X_1^2 X_2^2 - 6X_0 X_1 X_2 X_3 + 4X_3 X_1^3 + 4X_2^3 X_0 = 0$$

(see e.g. [7, Lemma 21.1.10]). In the sequel we will denote by  $\mathcal{G}$  the collineation group of  $PGL(3, q)$  fixing  $\mathcal{C}$ .

Let  $L$  be an  $\mathbb{F}_q$ -linear set of  $\mathbb{P} = \mathbb{P}^3(\mathbb{F}_{q^n})$  of rank  $2n$  satisfying the following property:

- (P) all points of  $L$  belong to imaginary chords of the twisted cubic  $\mathcal{C}$  of  $\mathbb{P}$ .

In particular,  $L$  is disjoint from any tangent line to  $\mathcal{C}$  and hence  $\Omega \cap L = \emptyset$ .

A point  $P$  of  $\mathbb{P}^3(\mathbb{F})$ , with  $p \neq 2$ , belongs to an imaginary chord of  $\mathcal{C}$  if and only if  $P$  lies on a line with coordinate vector

$$(\lambda_2^2, \lambda_1^2, \lambda_1^2 - \lambda_2, \lambda_2, \lambda_2, -1, 1),$$

where  $\lambda_1, \lambda_2 \in \mathbb{F}_{q^n}$  and  $\lambda_1^2 - 4\lambda_2$  is a non-square in  $\mathbb{F}_{q^n}$  (see [7, Section 21, p. 231]).

Now, by Lemma 15.2.3 of [7], we easily get that  $P = (a_0, a_1, a_2, a_3)$  belongs to an imaginary chord of  $\mathcal{C}$  if and only if  $F(a_0, a_1, a_2, a_3)$  is a non-square in  $\mathbb{F}_{q^n}$ .

Let  $L$  be an  $\mathbb{F}_q$ -linear set of  $\mathbb{P}$  of rank  $2n$  whose points belong to imaginary chords of  $\mathcal{C}$  and suppose that  $\mathbb{F}_q$  is the maximal subfield of  $\mathbb{F}_{q^n}$  with respect to which  $L$  is a linear subset. If  $(a_0, a_1, a_2, a_3)$  and  $(a_0, a'_1, a'_2, a_3)$  are distinct points of  $L$ , then  $(0, a_1 - a'_1, a_2 - a'_2, 0) \in L$  and hence  $F(0, a_1 - a'_1, a_2 - a'_2, 0) = -3(a_2 - a'_2)^2(a_1 - a'_1)^2$  is a non-square in  $\mathbb{F}_{q^n}$ . Therefore, if  $-3$  is a square in  $\mathbb{F}_{q^n}$ , i.e. if  $q^n \equiv 1 \pmod{3}$ , there are no distinct points of  $L$  of type  $(a_0, a_1, a_2, a_3)$  and  $(a_0, a'_1, a'_2, a_3)$ . This implies that, if  $q^n \equiv 1 \pmod{3}$ , there exist two  $\mathbb{F}_q$ -linear functions  $f(x, y), g(x, y) : \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$  such that

$$L = \{(x, f(x, y), g(x, y), y) : (x, y) \in \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}, (x, y) \neq (0, 0)\}.$$

Note that  $\mathbb{F}_q$  is the maximal subfield of  $\mathbb{F}_{q^n}$  with respect to which  $f$  and  $g$  are both linear. Also, if  $p \neq 2$ , since the points of  $L$  belong to imaginary chords of  $\mathcal{C}$ , we have that  $F(x, f(x, y), g(x, y), y)$  is a non-square for all  $(x, y) \neq (0, 0)$ . Let  $L_\infty = L \cap \pi_\infty$

and let  $f_1(x) = f(x, 0)$  and  $g_1(x) = g(x, 0)$ . Since  $\pi_\infty$  has equation  $X_3 = 0$ , we can write  $L_\infty = \{(x, f_1(x), g_1(x), 0) : x \in \mathbb{F}_{q^n}^*\}$ .

Also, from [3] and [4, Prop.6.1], the following result holds true.

PROPOSITION 2.2. *If  $q^n \equiv 1 \pmod{3}$ ,  $q \geq 4n^2 - 8n + 2$ , then either  $L_\infty$  is a point or, without loss of generality, we can suppose  $L_\infty = \{(x, \gamma x, \frac{m}{4}x^\tau + \frac{3}{4}\gamma^2x, 0) : x \in \mathbb{F}_{q^n}^*\}$ , where  $\gamma \in \mathbb{F}_{q^n}$ ,  $\tau = q^{h'}$ ,  $0 < h' < n$ , and  $m$  is a non-square in  $\mathbb{F}_{q^n}$ .*

Let  $L_0 = L \cap \pi_0$  and let  $f_2(y) = f(0, y)$  and  $g_2(y) = g(0, y)$ . Since  $\pi_0$  has equation  $X_0 = 0$ , we can write  $L_0 = \{(0, f_2(y), g_2(y), y) : y \in \mathbb{F}_{q^n}^*\}$ .

From [3] and [4, Prop.6.2], the following holds.

PROPOSITION 2.3. *If  $q^n \equiv 1 \pmod{3}$ ,  $q \geq 4n^2 - 8n + 2$ , then one of the following occurs:*

- $L_0$  is a point;
- $L_0 = \{(0, \frac{3}{4}\rho^2y + \frac{m'}{4}y^\sigma, \rho y, y) : y \in \mathbb{F}_{q^n}^*\}$ , with  $\sigma = q^{h'}$ ,  $0 < h' < n$ ,  $\rho \in \mathbb{F}_{q^n}^*$ , and  $m'$  is a non-square in  $\mathbb{F}_{q^n}$ ;
- $L_0 = \{(0, \alpha g_2(y) + \beta y, g_2(y), y) : y \in \mathbb{F}_{q^n}^*\}$ , with  $\alpha^2 + 3\beta$  a non-zero square of  $\mathbb{F}_{q^n}$  and  $g_2(y)$  a suitable  $\mathbb{F}_q$ -linear map over  $\mathbb{F}_{q^n}$  (which is not  $\mathbb{F}_{q^n}$ -linear).

### 3. THE MAIN RESULT

Let start by the following result.

LEMMA 3.1. *The  $\mathbb{F}_q$ -linear set of rank 4 corresponding to a translation spread of  $H(q^2)$  always forms a Baer subgeometry.*

*Proof.* Let  $\mathcal{S}$  be a translation spread of  $H(q^2)$  with respect to a line  $\ell$ , whose kernel is  $\mathbb{F}_q$ , and let  $L = L(\mathcal{S})$  be the corresponding  $\mathbb{F}_q$ -linear set of rank 4 of  $\mathbb{P} = \mathbb{P}^3(\mathbb{F}_{q^2})$  whose points satisfy Property (P). Also, the hypothesis of Proposition 2.2 are satisfied. Then either  $L$  is contained in a plane of  $\mathbb{P}$ , or it is a Baer subgeometry. In the former case, by [13, Property 2.4],  $L$  is a union of  $q + 1$  lines through one of its points. By Result  $(R_1)$ , each of these lines is either an imaginary chord or an imaginary axis, but this contradicts the fact that two imaginary chords (and two imaginary axes) are disjoint [7, Theorem 21.1.9]. The result follows.  $\square$

The Baer subgeometry  $L = L(\mathcal{S})$  of  $\mathbb{P}^3(\mathbb{F}_{q^n})$  arising from a translation spread of  $H(q^2)$  with respect to a line  $\ell$  intersects a plane of  $\mathbb{P}^3(\mathbb{F}_{q^n})$  in either a subline or a subplane of  $L$ . Thus, by Propositions 2.2 and 2.3, the  $\mathbb{F}_q$ -linear sets  $L_0$  or  $L_\infty$  cannot be a point.

Since  $q^n \equiv 1 \pmod{3}$  and  $q \geq 4n^2 - 8n + 2$  holds true for  $n = 2$  and  $q$  odd, we get that the  $\mathbb{F}_q$ -linear set  $L$  satisfies Property (P) if and only if for each  $P = (a_0, a_1, a_2, a_3) \in L$  we have that  $F(a_0, a_1, a_2, a_3)$  is a non-square in  $\mathbb{F}_{q^2}$ . Also, applying the previous arguments, without loss of generality, we can assume that

$$L = \{(x, f(x, y), g(x, y), y)\},$$

where  $f, g$  are two  $\mathbb{F}_q$ -linear functions over  $\mathbb{F}_{q^2}$  and one of the following cases occurs:

$$(3) \quad \boxed{\text{CASE 1:}} \quad \begin{aligned} f(x, y) &= \gamma x + 3\rho^2y/4 + ny^q/4 & \text{and} \\ g(x, y) &= mx^q/4 + 3\gamma^2x/4 + \rho y, \end{aligned}$$

with  $\gamma, \rho \in \mathbb{F}_{q^2}$ ,  $m$  and  $n$  non-squares in  $\mathbb{F}_{q^2}$ , or

$$(4) \quad \boxed{\text{CASE 2:}} \quad \begin{aligned} f(x, y) &= \gamma x + \alpha(b_0y + b_1y^q) + \beta y & \text{and} \\ g(x, y) &= mx^q/4 + 3\gamma^2x/4 + b_0y + b_1y^q, \end{aligned}$$

with  $\alpha, \beta, \gamma, b_0, b_1 \in \mathbb{F}_{q^2}$ ,  $b_1 \neq 0$ ,  $m$  and  $\alpha^2 + 3\beta$  non-zero squares of  $\mathbb{F}_{q^2}$ .

We can consider a general case, combining (3) and (4):

$$(5) \quad \boxed{\text{GENERAL CASE:}} \quad \begin{aligned} f(x, y) &= \gamma x + a_0 y + a_1 y^q \quad \text{and} \\ g(x, y) &= 3\gamma^2 x/4 + mx^q/4 + b_0 y + b_1 y^q, \end{aligned}$$

with  $\gamma \in \mathbb{F}_{q^2}$ ,  $m$  non-square in  $\mathbb{F}_{q^2}$ .

In the following we will prove our main result:

**MAIN THEOREM.** *Let  $1039891 \leq q \not\equiv 0 \pmod{3}$ ,  $q$  odd. Then  $H(q^2)$  does not admit  $\mathbb{F}_q$ -translation spreads.*

In order to prove our main result we will prove the existence of triples  $(x, y, z) \in \mathbb{F}_{q^2}^3$ , with  $(x, y) \neq (0, 0)$ , such that

$$(6) \quad x^2 y^2 - 3(f(x, y)g(x, y))^2 - 6xyf(x, y)g(x, y) + 4y(f(x, y))^3 + 4x(g(x, y))^3 = z^2,$$

for any  $f, g$  satisfying (5).

Our strategy involves demonstrating that a specific  $\mathbb{F}_q$ -rational variety associated with Equation (5) is absolutely irreducible. We will then utilize the Lang-Weil Theorem [5] to conclude that this variety contains an appropriate number of  $\mathbb{F}_q$ -rational points corresponding to triples  $(x, y, z) \in \mathbb{F}_{q^2}^3$ , with  $(x, y) \neq (0, 0)$  satisfying Equation (5).

**3.1. ALGEBRAIC VARIETIES OVER FINITE FIELDS.** In this subsection, we summarize several concepts and results related to algebraic varieties. We use the notations  $\mathbb{P}^r(\mathbb{F}_q)$  and  $\mathbb{A}^r(\mathbb{F}_q)$  (or simply  $\mathbb{F}_q^r$ ) to denote the projective and the affine space of dimension  $r \in \mathbb{N}$  over the finite field  $\mathbb{F}_q$ , respectively. Let  $\overline{\mathbb{F}_q}$  denote the algebraic closure of  $\mathbb{F}_q$ .

A variety  $\mathcal{V}$  is defined as the set of common zeros for a finite collection of polynomials. Specifically, an affine  $\mathbb{F}_q$ -rational variety (or an affine variety defined over  $\mathbb{F}_q$ ) is a set  $\mathcal{V} \subset \mathbb{A}^r(\overline{\mathbb{F}_q})$  for which there exist polynomials  $F_1, \dots, F_s$  in the polynomial ring  $\mathbb{F}_q[X_1, \dots, X_r]$  such that:

$$\mathcal{V} = \{(a_1, \dots, a_r) \in \mathbb{A}^r(\overline{\mathbb{F}_q}) \mid F_i(a_1, \dots, a_r) = 0 \text{ for all } i = 1, \dots, s\}.$$

This set is also denoted as  $V(F_1, \dots, F_s)$ . Similarly, an  $\mathbb{F}_q$ -rational projective variety (or a projective variety defined over  $\mathbb{F}_q$ ) in  $\mathbb{P}^r(\overline{\mathbb{F}_q})$  is defined using polynomials  $F_1, \dots, F_s \in \mathbb{F}_q[X_0, X_1, \dots, X_r]$ , with the additional requirement that each polynomial  $F_i$  must be homogeneous. The set of  $\mathbb{F}_q$ -rational points of an  $\mathbb{F}_q$ -rational variety  $\mathcal{V}$  is given by the intersection  $\mathcal{V} \cap \mathbb{A}^r(\mathbb{F}_q)$  or  $\mathcal{V} \cap \mathbb{P}^r(\mathbb{F}_q)$ , and it is usually denoted by  $\mathcal{V}(\mathbb{F}_q)$ . A hypersurface is a variety defined by a single polynomial. We say that a variety  $\mathcal{V}$  is *absolutely irreducible* if it cannot be expressed as the union of two proper subvarieties defined over the algebraic closure  $\overline{\mathbb{F}_q}$ . That is, there are no varieties  $\mathcal{V}'$  and  $\mathcal{V}''$  defined over  $\overline{\mathbb{F}_q}$  and different from  $\mathcal{V}$  such that  $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$ . In the specific case of a hypersurface  $\mathcal{V} = V(F) \subset \mathbb{A}^r(\overline{\mathbb{F}_q})$  (or projective space), it is *absolutely irreducible* if and only if its defining polynomial  $F$  is irreducible over the algebraic closure. That is, there are no non-constant polynomials  $G, H \in \overline{\mathbb{F}_q}[X_1, \dots, X_r]$  (or homogeneous polynomials in  $\overline{\mathbb{F}_q}[X_0, \dots, X_r]$ ) such that  $F = GH$ . The dimension of a variety can be defined as the maximal integer  $s$  for which there exists a chain of distinct, nonempty, absolutely irreducible varieties contained in  $\mathcal{V}$ :

$$\emptyset = \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \dots \subsetneq \mathcal{V}_{s+1} = \mathcal{V}.$$

We say that an  $s$ -dimensional projective variety  $\mathcal{V}$  has degree  $d$ , written  $\text{deg}(\mathcal{V}) = d$ , if  $d$  is the number of intersection points with a general projective subspace of

complementary dimension. That is,

$$d = \#(\mathcal{V} \cap H),$$

where  $H \subseteq \mathbb{P}^r(\overline{\mathbb{F}}_q)$  is a general projective subspace of dimension  $r - s$ . Algebraic varieties  $\mathcal{V} \subset \mathbb{A}^r(\overline{\mathbb{F}}_q)$  (or  $\mathcal{V} \subset \mathbb{P}^r(\overline{\mathbb{F}}_q)$ ) of dimension 1, 2, and  $r - 1$  are called curves, surfaces, and hypersurfaces, respectively.

Determining the degree of a variety is generally not straightforward; however, an upper bound to  $\deg(\mathcal{V})$  is given by  $\prod_{i=1}^s \deg(F_i)$ . We also recall that the Frobenius map  $\Phi_q : x \mapsto x^q$  is an automorphism of  $\mathbb{F}_{q^k}$  and generates the group  $Gal(\mathbb{F}_{q^k}/\mathbb{F}_q)$  of automorphisms of  $\mathbb{F}_{q^k}$  that fixes  $\mathbb{F}_q$  pointwise. The Frobenius automorphism also induces a collineation of  $\mathbb{A}^r(\overline{\mathbb{F}}_q)$  and an automorphism of  $\overline{\mathbb{F}}_q[X_1, \dots, X_r]$ .

From now on, with a slight abuse of notation, we will write  $\mathcal{V} \subset \mathbb{A}^r(\mathbb{F}_q)$  or  $\mathcal{V} \subset \mathbb{P}^r(\mathbb{F}_q)$  to indicate that the variety  $\mathcal{V}$  is  $\mathbb{F}_q$ -rational.

A plane curve  $\mathcal{C}$  is a hypersurface in  $\mathbb{A}^2(\mathbb{K})$ ,  $\mathbb{K}$  a field, defined by a unique polynomial  $F(X, Y) \in \mathbb{K}[X, Y]$ .

Let  $P = (u, v) \in \mathbb{A}^2(\mathbb{K})$  be a point in the affine plane, and write

$$F(X + u, Y + v) = F_0(X, Y) + F_1(X, Y) + F_2(X, Y) + \dots,$$

where  $F_i$  is either zero or homogeneous (in  $X, Y$ ) of degree  $i$ . The *multiplicity* of  $P \in \mathcal{C}$ , written as  $m_P(\mathcal{C})$ , is the smallest integer  $m$  such that  $F_m \neq 0$  and  $F_i = 0$  for  $i < m$ . The point  $P$  is on the curve  $\mathcal{C}$  if and only if  $m_P(\mathcal{C}) \geq 1$ . If  $P$  is on  $\mathcal{C}$ , then  $P$  is a *simple* point of  $\mathcal{C}$  if  $m_P(\mathcal{C}) = 1$ , otherwise  $P$  is a *singular* point of  $\mathcal{C}$ . If  $m_P(\mathcal{C}) \geq 1$ , the polynomial  $F_m(X - u, Y - v)$  defines the *tangent cone* of  $\mathcal{C}$  at  $P$ . A linear divisor of the tangent cone defines a *tangent* of  $\mathcal{C}$  at  $P$ . It is possible to define in a similar way the multiplicity of an ideal point of  $\mathcal{C}$ , that is a point of the curve lying on the line at infinity.

Given two plane curves  $\mathcal{A}$  and  $\mathcal{B}$  and a point  $P$  in the plane, the *intersection number*  $I(P, \mathcal{A} \cap \mathcal{B})$  of  $\mathcal{A}$  and  $\mathcal{B}$  at the point  $P$  can be defined by seven axioms. We do not include its precise and long definition here. For more details, we refer to [8] where the intersection number is defined equivalently in terms of local rings and in terms of resultants, respectively.

For our purposes, it is helpful to introduce some notation for the function field associated with a variety. For a variety  $\mathcal{V}$  and a field  $\mathbb{F}$ , let  $\mathbb{F}(\mathcal{V})$  denote the function field of  $\mathcal{V}$ , which consists of a set of rational functions defined over  $\mathbb{F}$ . For an absolutely irreducible curve  $\mathcal{C} : f(X, Y) = 0$  over  $\mathbb{F}_q$ , a non-singular  $\overline{\mathbb{F}}_q$ -rational point  $Q$  on  $\mathcal{C}$ , and a rational function  $\alpha = \frac{H(X, Y) + (f)}{G(X, Y) + (f)}$ , we define the order at  $Q$  as  $ord_Q(\alpha) = \mathcal{I}(Q, \mathcal{C} \cap (H(X, Y) = 0)) - \mathcal{I}(Q, \mathcal{C} \cap (G(X, Y) = 0))$ .

Finally, we will utilize the following result, a special case of [16, Corollary 3.7.4].

**THEOREM 3.2.** *Let  $q = p^h$ ,  $m$  a positive integer with  $\gcd(m, p) = 1$ ,  $g(X, Y) \in \mathbb{F}_q[X, Y]$  be such that there exists a non-singular  $\overline{\mathbb{F}}_q$ -rational point  $Q$  of  $\mathcal{C} : f(X, Y) = 0$ ,  $f(X, Y) \in \mathbb{F}_q[X, Y]$ , satisfying  $\gcd(ord_Q(g(x, y)), m) = 1$ . Let*

$$\mathcal{C}' \subset \mathbb{P}^3(\mathbb{F}_q) : \begin{cases} f(X, Y) = 0 \\ Z^m = g(X, Y). \end{cases}$$

*Then  $\mathcal{C}'$  is an absolutely irreducible curve defined over  $\mathbb{F}_q$  and it is called a Kummer cover of  $\mathcal{C}$ .*

**3.2. COMPUTATIONS.** We aim to associate a suitable  $\mathbb{F}_q$ -rational variety with Equation (5). First, note that Equation (5) is defined over  $\mathbb{F}_{q^2}$ , which naturally describes an  $\mathbb{F}_{q^2}$ -rational surface in  $\mathbb{P}^3(\mathbb{F}_{q^2})$ . Let  $\mathcal{X}_{a_0, a_1, b_0, b_1, \gamma, m} \subset \mathbb{P}^3(\mathbb{F}_{q^2})$  be the surface defined

by

$$(7) \quad \begin{aligned} F_{a_0, a_1, b_0, b_1, \gamma, m}(X, Y, Z) &:= X^2Y^2 - 3(f(X, Y))^2(g(X, Y))^2 - 6XYf(X, Y)g(X, Y) \\ &+ 4Y(f(X, Y))^3 + 4X(g(X, Y))^3 - Z^2 = 0. \end{aligned}$$

PROPOSITION 3.3. *There exists a pair  $(x, y) \in \mathbb{F}_{q^2} \setminus \{(0, 0)\}$  such that the first member of (6) is a square in  $\mathbb{F}_{q^2}$  if and only if the surface  $\mathcal{X}_{a_0, a_1, b_0, b_1, \gamma, m}$  defined in (7) possesses an  $\mathbb{F}_{q^2}$ -rational point  $(x_0, y_0, z_0)$  with  $(x_0, y_0) \neq (0, 0)$ .*

Let  $i \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$  be such that  $i^q = -i$ . Then each element  $\bar{x} \in \mathbb{F}_{q^2}$  can be expressed as a unique pair  $(\bar{x}_1, \bar{x}_2) \longleftrightarrow \bar{x}_1 + \bar{x}_2i$ , with  $(\bar{x}_1, \bar{x}_2) \in \mathbb{F}_q^2$ .

Consider the system

$$(8) \quad \begin{cases} F_{a_0, a_1, b_0, b_1, \gamma, m}(X_0 + iX_1, Y_0 + iY_1, Z_0 + iZ_1) = 0 \\ F_{a_0^q, a_1^q, b_0^q, b_1^q, \gamma^q, m^q}(X_0 - iX_1, Y_0 - iY_1, Z_0 - iZ_1) = 0. \end{cases}$$

Although the coefficients in (8) are in  $\mathbb{F}_{q^2}$ , System (8) defines a variety over  $\mathbb{F}_q$ , since the ideal generated by the polynomials  $F_{a_0, a_1, b_0, b_1, \gamma, m}(X_0 + iX_1, Y_0 + iY_1, Z_0 + iZ_1)$  and  $F_{a_0^q, a_1^q, b_0^q, b_1^q, \gamma^q, m^q}(X_0 - iX_1, Y_0 - iY_1, Z_0 - iZ_1)$  is generated by polynomials with coefficients in  $\mathbb{F}_q$ . Thus System (8) defines an  $\mathbb{F}_q$ -rational variety  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m} \subset \mathbb{P}^6(\mathbb{F}_q)$ . The connection between  $\mathcal{X}_{a_0, a_1, b_0, b_1, \gamma, m}$  and  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$  is highlighted in the following proposition.

PROPOSITION 3.4. *There exists a pair  $(x, y) \in \mathbb{F}_{q^2} \setminus \{(0, 0)\}$  such that the first member of (6) is a square in  $\mathbb{F}_{q^2}$  if and only if the 4-dimensional variety  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$  defined in (8) possesses an  $\mathbb{F}_q$ -rational point  $(x_0, x_1, y_0, y_1, z_0, z_1)$  with  $(x_0, x_1, y_0, y_1) \neq (0, 0, 0, 0)$ .*

Clearly,  $\mathbb{F}_q$ -rational points  $(x_0, x_1, y_0, y_1, z_0, z_1)$  of  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$  correspond to  $\mathbb{F}_{q^2}$ -rational point  $(x_0 + x_1i, y_0 + y_1i, z_0 + z_1i)$  of  $\mathcal{X}_{a_0, a_1, b_0, b_1, \gamma, m}$  and this intrinsic connection between  $\mathcal{X}_{a_0, a_1, b_0, b_1, \gamma, m}$  and  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$  yields an effective method to provide a lower bound on the number of  $\mathbb{F}_{q^2}$ -rational points of  $\mathcal{X}_{a_0, a_1, b_0, b_1, \gamma, m}$ . The absolute irreducibility of  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$  together with the following result by Cafure and Matera [5] will provide the existence of suitable  $(x, y) \in \mathbb{F}_{q^2}^2$  for which the first member of (6) is square in  $\mathbb{F}_{q^2}$ .

THEOREM 3.5 ([5]). *Let  $\mathcal{V} \subset \mathbb{A}^n(\mathbb{F}_q)$  be an  $\mathbb{F}_q$ -irreducible variety of dimension  $n$  and degree  $d$ . If  $q > 2(n + 1)d^2$  then*

$$|\#\mathcal{V}(\mathbb{F}_q) - q^n| \leq (d - 1)(d - 2)q^{n-1/2} + 5d^{1/3}q^{n-1}.$$

To demonstrate the absolute irreducibility of  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$ , we will utilize several targeted studies of polynomial equations. Specifically, it will be more advantageous to examine the absolute irreducibility of another variety that is projectively equivalent (over the algebraic closure) to  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$ .

In particular, consider the projectivity  $\varphi(X_0, X_1, Y_0, Y_1, Z_0, Z_1) = (X_0 + X_1i, X_0 - X_1i, Y_0 + Y_1i, Y_0 - Y_1i, Z_0 + Z_1i, Z_0 - Z_1i)$  and the variety  $\mathcal{Z}_{a_0, a_1, b_0, b_1, \gamma, m} \subset \mathbb{P}^6(\mathbb{F}_{q^2})$   $\mathbb{F}_{q^2}$ -birationally equivalent to  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$  via  $\varphi$ .

Now  $\mathcal{Z}_{a_0, a_1, b_0, b_1, \gamma, m}$  is defined by

$$(9) \quad \begin{cases} H_{a_0, a_1, b_0, b_1, \gamma, m}(X_0, X_1, Y_0, Y_1) = Z_0^2 \\ H_{a_0^q, a_1^q, b_0^q, b_1^q, \gamma^q, m^q}(X_0, X_1, Y_0, Y_1) = Z_1^2 \end{cases}$$

and has degree 16. We will prove that  $\mathcal{Z}_{a_0, a_1, b_0, b_1, \gamma, m}$  (and thus  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$ ) is absolutely irreducible in Theorem 3.9, by proving that the intersection

$$\mathcal{D} : \begin{cases} G(X_0, X_1) = 0 \\ -(4b_1^q + 3\gamma^{2q}X_1 + m^qX_0)^2(3a_1^{2q} + 6a_1^q\gamma^qX_1 - 4b_1^qX_1 - m^qX_0X_1) = 16Z_1^2, \end{cases}$$

where  $G(X_0, Y_0)$  is defined below in (10), between  $\mathcal{Z}_{a_0, a_1, b_0, b_1, \gamma, m}$  and the 3-space of  $\mathbb{P}^6(\mathbb{F}_{q^2})$  defined by  $Y_0 - 1 = Y_1 = Z_0 - 1 = 0$  is an absolutely irreducible curve of degree 16.

First we focus on the absolute irreducibility of  $\mathcal{D}$ .

To this end, let consider the polynomial

$$(10) \quad \begin{aligned} G(X, Y) := & A_{3,1}X^3Y + A_{2,2}X^2Y^2 + A_{1,3}XY^3 \\ & + A_{3,0}X^3 + A_{2,1}X^2Y + A_{1,2}XY^2 \\ & + A_{2,0}X^2 + A_{1,1}XY + A_{0,2}Y^2 \\ & + A_{1,0}X + A_{0,1}Y + A_{0,0} = 0, \end{aligned}$$

where

$$\begin{aligned} A_{0,0} &:= 64a_0^3 - 48a_0^2b_0^2 - 16, \\ A_{0,1} &:= -24a_0^2b_0m, \\ A_{0,2} &:= -3a_0^2m^2, \\ A_{1,0} &:= -72a_0^2b_0\gamma^2 + 192a_0^2\gamma - 96a_0b_0^2\gamma - 96a_0b_0 + 64b_0^3, \\ A_{1,1} &:= -18a_0^2\gamma^2m - 48a_0b_0\gamma m - 24a_0m + 48b_0^2m, \\ A_{1,2} &:= -6a_0\gamma m^2 + 12b_0m^2, \\ A_{1,3} &:= m^3, \\ A_{2,0} &:= -27a_0^2\gamma^4 - 144a_0b_0\gamma^3 + 120a_0\gamma^2 + 96b_0^2\gamma^2 - 96b_0\gamma + 16, \\ A_{2,1} &:= -36a_0\gamma^3m + 48b_0\gamma^2m - 24\gamma m, \\ A_{2,2} &:= 6\gamma^2m^2, \\ A_{3,0} &:= -54a_0\gamma^5 + 36b_0\gamma^4 - 8\gamma^3, \\ A_{3,1} &:= 9\gamma^4m. \end{aligned}$$

LEMMA 3.6. *Let  $q$  be odd and  $\gamma m \neq 0$ . Then the curve  $\mathcal{C} : G(X, Y) = 0$  is absolutely irreducible.*

*Proof.* The homogenous part of the highest degree in  $G(X, Y)$  is  $m(3\gamma^2X + mY)^2XY$  and therefore the only possible linear components of  $\mathcal{C}$  can be  $X = k$ , or  $Y = k$ , or  $3\gamma^2X + mY + k = 0$  for some  $k \in \overline{\mathbb{F}_q}$ .

- (1) If  $X = k$  is a component of  $\mathcal{C}$  then  $G(k, Y)$  is the zero polynomial and in particular  $k = 0$ , which forces  $0 = -64a_0^3 + 48a_0^2b_0^2 + 16 = 24a_0^2b_0m = 3a_0^2m^2$ , yielding  $a_0 = 0$  and thus  $16 = 0$ , a contradiction.
- (2) If  $Y = k$  is a component of  $\mathcal{C}$  then  $G(X, k)$  is the zero polynomial and in particular  $-(A_{3,0} + A_{3,1}) = \gamma^3(54a_0\gamma^2 - 36b_0\gamma - 9\gamma mk + 8) = 0$ . Since  $\gamma m \neq 0$ ,  $k = (54a_0\gamma^2 - 36b_0\gamma + 8)/(9\gamma m)$ . This yields  $m^2\gamma^2(27a_0\gamma^2 - 4) = 0$ . So  $a_0 = 4/(27\gamma^2)$  and finally, considering again  $G(X, k)$ , this provides  $m^2\gamma^{10} = 0$ , a contradiction.
- (3) If  $3\gamma^2X + mY + k = 0$  is a component of  $\mathcal{C}$  then the resultant  $R(Y)$  between  $G(X, Y)$  and  $3\gamma^2X + mY + k$  with respect to  $X$  must vanish. This provides an immediate contradiction since the coefficient of  $Y^3$  in  $R(Y)$  is  $64\gamma^3m^3 \neq 0$ .

The above argument shows that  $G(X, Y)$  does not possess linear factors. Thus, if it is not absolutely irreducible, it must contain degree-2 factors of the type either  $XY + \alpha X + \beta Y + \delta$  or  $X(3\gamma^2X + mY) + \alpha X + \beta Y + \delta$  for some  $\alpha, \beta, \delta \in \overline{\mathbb{F}_q}$ , since if  $Y(3\gamma^2X + mY) + \alpha X + \beta Y + \delta$  divides  $G(X, Y)$  then the other factor of  $G(X, Y)$  is of type  $X(3\gamma^2X + mY) + \alpha X + \beta Y + \delta$ .

- (1) If  $(XY + \alpha X + \beta Y + \delta) \mid G(X, Y)$  then the resultant  $R(Y)$  between the two polynomials with respect to  $X$  must be the zero polynomial in  $Y$ . Since  $R(Y) = (m^3\beta)Y^6 + (3a_0^2m^2 - 6a_0\gamma m^2\beta + 12b_0m^2\beta - 6\gamma^2m^2\beta^2 + 2m^3\alpha\beta + m^3\delta)Y^5 + \dots$ ,  $\beta = 0$  and  $\delta = -3a_0^2/m$ . Looking at the coefficient of  $Y^4$  one gets  $a_0(6a_0\gamma - 4b_0 + m\alpha) = 0$ . We distinguish two cases.
- $a_0 = 0$ . In this case it is readily seen that  $R(Y)$  does not vanish.
  - $a_0 \neq 0$  and  $\alpha = -6a_0\gamma/m + 4b_0/m$ . The coefficient of  $Y^3$  in  $R(Y)$  vanishes only if  $a_0^3 + 2 = 0$  and in this case the coefficient of  $Y^2$  is not zero, a contradiction.
- (2) Since the resultant between  $X(3\gamma^2X + mY) + \alpha X + \beta Y + \gamma$  and  $G(X, Y)$  with respect to  $X$  is not the zero polynomial in  $Y$  (the coefficient of  $Y^6$  is  $-64\gamma^3m^3$ ), the polynomial  $X(3\gamma^2X + mY) + \alpha X + \beta Y + \gamma$  cannot be a factor of  $G(X, Y)$ . □

LEMMA 3.7. *Let  $q$  be odd and  $\gamma = 0, m \neq 0$ . Then the affine plane curve  $\mathcal{C} : G(X, Y) = 0$  is absolutely irreducible.*

*Proof.* The proof is similar to the case  $\gamma \neq 0$ . The homogeneous part of the highest degree in  $G(X, Y)$  is  $m^3XY^3$  and therefore the only possible linear components of  $\mathcal{C}$  can be  $X = k$  or  $Y = k$  for some  $k \in \overline{\mathbb{F}}_q$ .

- (1) If  $X = k$  is a component of  $\mathcal{C}$  then

$$G(k, Y) = m^3kY^3 + 3m^2(-a_0^2m^2 + 4b_0k)Y^2 - 24m(a_0^2b_0 + a_0k - 2b_0^2k)Y + (64a_0^3 - 48a_0^2b_0^2 - 96a_0b_0k + 64b_0^3k + 16k^2 - 16)$$

must vanish. So  $k = 0$  and  $a_0 = 0$ , but in this case  $G(0, Y) = -16$ , a contradiction.

- (2) If  $Y = k$  is a component of  $\mathcal{C}$  then  $G(X, k)$  is the zero polynomial, but the coefficient of  $X^3$  in  $G(X, k)$  is 16, a contradiction.

The above argument shows that  $G(X, Y)$  does not possess linear factors. Thus, if it is not absolutely irreducible, it must contain degree-2 factors of the type  $XY + \alpha X + \beta Y + \delta$  for some  $\alpha, \beta, \delta \in \overline{\mathbb{F}}_q$ .

- (1) If  $(XY + \alpha X + \beta Y + \delta) \mid G(X, Y)$  then the resultant  $R(Y)$  between the two polynomials with respect to  $X$  must be the zero polynomial in  $Y$ . Since

$$R(Y) = (-m^3\beta)Y^5 + (-3a_0^2m^2 - 12b_0m^2\beta - m^3\alpha\beta - m^3\delta)Y^4 + \dots,$$

$\beta = 0$  and  $\delta = -3a_0^2/m$ . Looking at the coefficient of  $Y^3$  one gets  $a_0(-4b_0 + m\alpha) = 0$ . We distinguish two cases.

- $a_0 = 0$  In this case it is readily seen that  $R(Y)$  does not vanish.
- $a_0 \neq 0$  and  $\alpha = 4b_0/m$ . The coefficient of  $Y$  in  $R(Y)$  vanishes only if  $a_0^3 + 2 = 0$  and in this case  $R(Y) \neq 0$ . □

We now focus on a curve  $\mathcal{D}'$  birationally equivalent to  $\mathcal{D}$  and we prove its absolute irreducibility.

LEMMA 3.8. *Let  $q$  be odd and  $m \neq 0$ . The affine space curve*

$$\mathcal{D}' : \begin{cases} G(X, Y) = 0 \\ U^2 = 3a_1^{2q} + 6a_1^q\gamma^q Y - 4b_1^q Y - m^q XY \end{cases}$$

*is absolutely irreducible in the following cases:*

- (1)  $\gamma \neq 0$  and
- $a_1b_1 = 0$ ; or
  - $a_1b_1 \neq 0$  and  $3a_0b_1^2 + a_1^2 - 3a_1b_0b_1 \neq 0$ .

(2)  $\gamma = 0$  and  $b_1 \neq 0$ .

*Proof.* It is enough to prove that  $u = 3a_1^{2q} + 6a_1^q\gamma^q Y - 4b_1^q Y - m^q XY$  is not a square in  $\overline{\mathbb{F}_q}(\mathcal{C})$ , the function field of the curve  $\mathcal{C} : G(X, Y) = 0$ . To this end we will show the existence of a non-singular point  $P$  of  $\mathcal{C}$  for which  $\text{ord}_P(u)$  is odd and then apply Theorem 3.2.

(1) If  $\gamma \neq 0$  then  $G(X, Y)$  possesses two non-singular points on the line at infinity  $\ell_\infty$ , namely  $P_\infty = (0 : 1 : 0)$  and  $Q_\infty = (0 : 0 : 1)$ . Now,

$$\text{ord}_{P_\infty}(u) = \mathcal{I}(P_\infty, \mathcal{C} \cap (3a_1^{2q} + 6a_1^q\gamma^q Y - 4b_1^q Y - m^q XY = 0)) - 2\mathcal{I}(P_\infty, \mathcal{C} \cap \ell_\infty)$$

and

$$\text{ord}_{Q_\infty}(u) = \mathcal{I}(Q_\infty, \mathcal{C} \cap (3a_1^{2q} + 6a_1^q\gamma^q Y - 4b_1^q Y - m^q XY = 0)) - 2\mathcal{I}(Q_\infty, \mathcal{C} \cap \ell_\infty).$$

Note that  $\mathcal{I}(Q_\infty, \mathcal{C} \cap \ell_\infty) = \mathcal{I}(P_\infty, \mathcal{C} \cap \ell_\infty) = 1$ . For the conic  $3a_1^{2q} + 6a_1^q\gamma^q Y - 4b_1^q Y - m^q XY = 0$  the tangents at  $P_\infty$  and  $Q_\infty$  are  $Y = 0$  and  $m^q X = 6a_1^q\gamma^q - 4b_1^q$ , whereas for  $G(X, Y) = 0$  the tangents at  $P_\infty$  and  $Q_\infty$  are  $9\gamma^4 m Y - 54a_0\gamma^5 + 36b_0\gamma^4 - 8\gamma^3 = 0$  and  $X = 0$  respectively.

This means that  $\text{ord}_{P_\infty}(u) = -1$  or  $\text{ord}_{Q_\infty}(u) = -1$  unless the two pairs of tangents coincide, namely

$$\gamma^3(-54a_0\gamma^2 + 36b_0\gamma - 8) = 0 = 6a_1^q\gamma^q - 4b_1^q = 6a_1\gamma - 4b_1,$$

that is  $6a_1\gamma - 4b_1 = 0$  and  $-54a_0\gamma^2 + 36b_0\gamma - 8 = 0$ . Note that if  $a_1 = 0$  or  $b_1 = 0$  then both  $a_1$  and  $b_1$  vanish and  $-54a_0\gamma^2 + 36b_0\gamma - 8 = -8 \neq 0$ , a contradiction. So when  $a_1 b_1 \neq 0$ , if  $3a_0 b_1^2 + a_1^2 - 3a_1 b_0 b_1 \neq 0$ ,  $\mathcal{D}'$  is absolutely irreducible.

(2) If  $\gamma = 0$  then  $Q_\infty$  is a simple point of  $G(X, Y) = 0$  and the tangent at  $Q_\infty$  is  $X = 0$  for  $G(X, Y) = 0$  and  $m^q X = -4b_1^q$  for  $3a_1^{2q} - 4b_1^q Y - m^q XY = 0$ . Thus

$$\text{ord}_{Q_\infty}(u) = \mathcal{I}(Q_\infty, \mathcal{C} \cap (3a_1^{2q} - 4b_1^q Y - m^q XY = 0)) - 2\mathcal{I}(Q_\infty, \mathcal{C} \cap \ell_\infty) = -1,$$

unless  $b_1 = 0$ . The claim follows.  $\square$

Finally, we can establish our main result regarding the absolute irreducibility of  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$ .

**THEOREM 3.9.** *Let  $q$  be an odd prime,  $q \not\equiv 0 \pmod{3}$ . The variety  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$  is absolutely irreducible.*

*Proof.* As already mentioned,  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$  and  $\mathcal{Z}_{a_0, a_1, b_0, b_1, \gamma, m}$  as in (9) are  $\mathbb{F}_{q^2}$ -projectively equivalent via  $\varphi(X_0, X_1, Y_0, Y_1, Z_0, Z_1) = (X_0 + X_1 i, X_0 - X_1 i, Y_0 + Y_1 i, Y_0 - Y_1 i, Z_0 + Z_1 i, Z_0 - Z_1 i)$ .

Also, for  $\gamma \neq 0$ , consider the intersection  $\mathcal{D}$  of  $\mathcal{Z}_{a_0, a_1, b_0, b_1, \gamma, m}$  with the 3-space  $\mathcal{S}$  with equation  $Y_0 - 1 = Y_1 = Z_0 - 1 = 0$ . As already mentioned,

$$\mathcal{D} : \begin{cases} G(X_0, X_1) = 0 \\ -(4b_1^q + 3\gamma^{2q} X_1 + m^q X_0)^2 (3a_1^{2q} + 6a_1^q\gamma^q X_1 - 4b_1^q X_1 - m^q X_0 X_1) = 16Z_1^2, \end{cases}$$

where  $G(X, Y)$  is as in (10). Note that  $\mathcal{D}$  is a curve of degree 16 and therefore its absolute irreducibility yields the absolute irreducibility of the variety  $\mathcal{Z}_{a_0, a_1, b_0, b_1, \gamma, m}$ , of degree 16. Now, the curve  $\mathcal{D}$  is birationally equivalent to the curve  $\mathcal{D}'$  defined by  $G(X_0, X_1) = 0$  and  $U^2 = 3a_1^{2q} + 6a_1^q\gamma^q X_1 - 4b_1^q X_1 - m^q X_0 X_1$  and by Lemma 3.8 it is absolutely irreducible when Conditions (a) or (b) hold.

i. Suppose we are in case (3). If  $\gamma \neq 0$  then, since  $b_1 = 0$ , Condition (a) holds and  $\mathcal{Z}_{a_0, a_1, b_0, b_1, \gamma, m}$  is absolutely irreducible.

ii. Suppose we are in case (4). If  $\gamma = 0$  then  $b_1 \neq 0$  by assumption and Condition (b) holds. If  $\gamma \neq 0$ , since  $a_0 = \alpha b_0 + \beta$ ,  $a_1 = \alpha b_1$ , we have that

$$3a_0b_1^2 + a_1^2 - 3a_1b_0b_1 = 3(\alpha b_0 + \beta)b_1^2 + (\alpha b_1)^2 - 3(\alpha b_1)b_0b_1 = b_1^2(\alpha^2 + 3\beta) \neq 0,$$

and therefore Condition (a) holds and  $\mathcal{Z}_{a_0, a_1, b_0, b_1, \gamma, m}$  is absolutely irreducible.

Thus the only remaining case is when in (3) we assume  $\gamma = 0$ . By symmetry, interchanging the role of  $x$  and  $y$  and applying Lemmas 3.6, 3.7, and 3.8, arguing as above it can be readily seen that  $\rho \neq 0$  yields the absolute irreducibility of  $\mathcal{Z}_{a_0, a_1, b_0, b_1, \gamma, m}$ .

This means that the unique case left is  $\rho = \gamma = 0$  in (3). So,  $f(x, y) = ny^q/4$ ,  $g(x, y) = mx^q/4$ , for some  $m, n$  non-squares in  $\mathbb{F}_{q^2}$ .

In this case, intersecting  $\mathcal{Z}_{a_0, a_1, b_0, b_1, \gamma, m}$  with the 3-space  $\mathcal{S}$  with equation  $Y_0 = Y_1 - 1 = Z_0 - 1 = 0$ , we obtain the curve

$$\mathcal{E} : \begin{cases} -3m^2n^2X_1^2 + 16n^3X_0X_1^3 - 256 = 0 \\ n^{3q}X_0^3X_1 + 16X_1^2 = 16Z_1^2. \end{cases}$$

It is readily seen that the curve  $\mathcal{Q} : -3m^2n^2X_1^2 + 16n^3X_0X_1^3 - 256 = 0$  is absolutely irreducible (it is of the type  $h_1(X_1)X_0 + h_2(X_1)$ , with  $\gcd(h_1, h_2) = 1$ ). Also,  $n^{3q}X_0^3X_1 + 16X_1^2 = X_1(n^{3q}X_0^3 + 16X_1)$  is not a square in  $\overline{\mathbb{F}_q}(\mathcal{Q})$ . To see this it is enough to observe that  $X_1 = 0$  intersects  $\mathcal{Q}$  only at  $P_\infty = (0 : 1 : 0)$ , whereas  $n^{3q}X_0^3 + 16X_1 = 0$  intersects  $\mathcal{Q}$  at  $Q_\infty = (0 : 0 : 1)$  and ten (non-singular) affine points satisfying  $n^{9q+3}X_0^{10} + 3m^2n^2n^{6q}X_0^6 + 65536 = 0$ . Since  $n^{9q+3}X_0^{10} + 3m^2n^2n^{6q}X_0^6 + 65536$  is not a square as polynomial in  $X_0$ , there exists at least one non-singular point  $P$  of  $\mathcal{Q}$  for which

$$\text{ord}_P(n^{3q}X_0^3X_1 + 16X_1^2) = \mathcal{I}(P, \mathcal{Q} \cap (n^{3q}X_0^3X_1 + 16X_1^2 = 0)) - 4\mathcal{I}(P, \mathcal{Q} \cap \ell_\infty)$$

is odd and thus  $n^{3q}X_0^3X_1 + 16X_1^2$  is not a square in  $\overline{\mathbb{F}_q}(\mathcal{Q})$  and  $\mathcal{E}$  is absolutely irreducible. The claim follows.  $\square$

We are now ready to prove our main result.

**THEOREM 3.10.** *Let  $q$  be an odd prime,  $q \not\equiv 0 \pmod{3}$ ,  $q \geq 1039891$ . Then there exists  $(x, y) \neq (0, 0) \in \mathbb{F}_{q^2}^2$  for which the first member of (6) is square in  $\mathbb{F}_{q^2}$ .*

*Proof.* By Theorem 3.9,  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$  is absolutely irreducible and thus from Theorem 3.5 we deduce

$$(11) \quad \begin{aligned} \#\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}(\mathbb{F}_q) &\geq -(15)(14)q^{2-\frac{1}{2}} - 5(16)^{\frac{13}{3}}q + q^2 \\ \#\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}(\mathbb{F}_q) &\geq -210q^{2-\frac{1}{2}} - 5(16)^{\frac{13}{3}}q + q^2. \end{aligned}$$

The right value in (11) is greater than 1 for every  $q \geq 1039845$  (the first prime greater than that is 1039891), and so we have the existence of more than one  $\mathbb{F}_q$ -rational point in  $\mathcal{Y}_{a_0, a_1, b_0, b_1, \gamma, m}$ . The claim is derived from Proposition 3.4.  $\square$

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