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Hyperoctahedral group characters and a type-BC analog of graph coloring

Mark A. Skandera

Abstract We state combinatorial formulas for hyperoctahedral group ($\mathfrak{B}_n$) character evaluations of the form $\chi(\widetilde{C}^{\mathsf{BC}}_w(1))$ where $\widetilde{C}^{\mathsf{BC}}_w(1) \in \mathbb{Z}[\mathfrak{B}_n]$ is a type-BC Kazhdan–Lusztig basis element, with $w \in \mathfrak{B}_n$ corresponding to simultaneously smooth type-B and C Schubert varieties. We also extend the definition of symmetric group codominance to elements of $\mathfrak{B}_n$ and show that for each element $w \in \mathfrak{B}_n$ as above, there exists a BC-codominant element $v \in \mathfrak{B}_n$ satisfying $\chi(\widetilde{C}^{\mathsf{BC}}_w(1)) = \chi(\widetilde{C}^{\mathsf{BC}}_v(1))$ for all $\mathfrak{B}_n$-characters χ. Combinatorial structures and maps appearing in these formulas are type-BC extensions of planar networks, unit interval orders, indifference graphs, poset tableaux, and colorings. Using the ring of type-BC symmetric functions, we introduce natural generating functions $Y(\widetilde{C}^{\mathsf{BC}}_w(1))$ for the above evaluations. These provide a new type-BC analog of Stanley's chromatic symmetric functions [Adv. Math. **111** (1995) pp. 166–194].

1. Introduction

Let W be a Coxeter group, $H = H(W)$ its Hecke algebra, and $\mathcal{T}(H)$ the space of Hecke algebra *traces*, linear functionals $\theta_q : H \to \mathbb{Z}[q, q^{-1}]$ satisfying $\theta_q(DD') = \theta_q(D'D)$ for all $D, D' \in H$. Included in $\mathcal{T}(H)$ are the H-characters, which encode much of the structure of H in a condensed form. Since traces are linear, one might hope to solve the following problem for particular bases $\mathcal{D} = \{D_w \,|\, w \in W\}$ of H and $\Theta = \{\theta_q^{(i)} \,|\, i = 1, \dots, p\}$ of $\mathcal{T}(H)$.

Problem 1.1. *Find combinatorial formulas for each of the trace evaluations* $\{\theta_q^{(i)}(D_w) \,|\, \theta_q^{(i)} \in \Theta, D_w \in \mathcal{D}\}$.

Unfortunately, trace evaluation is not always easy, even in type A, when W is the symmetric group $\mathfrak{S}_n$ with Hecke algebra $H = H_n(q)$. (See e.g. [18, §1].) Type-A solutions were given in [19, 39], using the induced sign character basis of $\mathcal{T}(H_n(q))$, and bases consisting of products of simple elements of the (modified, signless) Kazhdan–Lusztig basis $\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n\}$ of $H_n(q)$. It would be interesting to solve Problem 1.1 for other pairs of type-A bases as well, as these evaluations are related to facts and conjectures concerning nonnegativity, graph coloring, and Hessenberg varieties. (See e.g. [33, Lem. 1.1], [33, Conj. 2.1], [59, Conj. 4.9], [68, Conj. 5.5].)

Partial type-A solutions to Problem 1.1 were given in [18, 62] for various bases of $\mathcal{T}(H_n(q))$, and the subset

$$\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n \text{ avoids the patterns } 3412 \text{ and } 4231\} \qquad (1)$$

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of the Kazhdan–Lusztig basis of $H_n(q)$. By [61, Thm. 4.3], we have that for w avoiding the patterns 3412 and 4231, there exists a planar network $F = F(w)$ which serves as a combinatorial interpretation for $\widetilde{C}_w(q)$. By [18, Thm. 7.4], there also exist a poset $P = P(w)$ and graph $G = G(w)$ such that evaluations $\theta_q(\widetilde{C}_w(q))$ may be computed combinatorially by

(1) filling Young diagrams with paths in F,
(2) filling Young diagrams with elements of P,
(3) coloring vertices of G,
(4) orienting edges of G,

while obeying certain rules in each case. (See also [9, 60, 64].) While (1) is only a subset of the Kazhdan–Lusztig basis of $H_n(q)$, it is conjectured [4, Conj. 1.9], [33, Conj.3̇.1] that an even smaller subset

(2) $$\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n \text{ is codominant, i.e. avoids the pattern 312}\}$$

explains trace evaluations at the entire Kazhdan–Lusztig basis. It is known [18, Thm. 4.6] that for each element $\widetilde{C}_w(q)$ of (1) there exists an element $\widetilde{C}_v(q)$ of (2) with the property that $P(w) \cong P(v)$ and therefore that $\theta_q(\widetilde{C}_w(q)) = \theta_q(\widetilde{C}_v(q))$ for all traces $\theta_q \in \mathcal{T}(H_n(q))$.

One could also answer Problem 1.1 from the point of view of symmetric functions. Let Λ_n be the $\mathbb{Z}$-module of homogeneous, degree-n symmetric functions. Since the ranks of Λ_n and $\mathcal{T}(H_n(q))$ are equal, it is possible to define a generating function in Λ_n for evaluations of traces at any fixed element $D \in H_n(q)$. Following [62, §2], we use the induced sign character basis $\{\epsilon_q^\lambda\}$ of $\mathcal{T}(H_n(q))$ and monomial symmetric function basis $\{m_\lambda\}$ of Λ_n to define

$$Y_q(D) := \sum_\lambda \epsilon_q^\lambda(D) m_\lambda \in \mathbb{Z}[q, q^{-1}] \otimes \Lambda_n.$$

A certain pairing of six natural bases of $\mathcal{T}(H_n(q))$ and six natural bases of Λ_n then guarantees that for each pair $(\{\theta_q^\lambda\}, \{g_\lambda\})$, we have

$$Y_q(D) = \sum_\lambda \theta_q^\lambda(D) g_\lambda.$$

Thus $Y_q(D)$ is in fact a generating function for the evaluation of all elements of these six trace bases at D. (See e.g. [62, Prop. 2.1].) Conveniently, the combinatorial computations mentioned above also guarantee that a certain *chromatic (quasi-)symmetric function* $X_{G,q}$, defined in terms of the proper colorings of G [59], satisfies $X_{G,q} = Y_q(\widetilde{C}_w(q))$ [18, Thm. 7.4]. Thus for $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231, the graph G essentially encodes all trace evaluations of the form $\theta_q^\lambda(\widetilde{C}_w(q))$ for $\{\theta_q^\lambda\}$ one of the six natural bases of $\mathcal{T}(H_n(q))$.

Some of the above results from [18, 61, 62] have type-BC analogs, i.e. extensions to the hyperoctahedral group $\mathfrak{B}_n$ and its Hecke algebra $H_n^{\mathsf{BC}}(q)$. In Sections 2 – 4, we present these algebras, their Kazhdan–Lusztig bases, and their trace spaces. In Section 5, we define type-BC analogs of type-A planar networks, and use these to graphically represent the subset

(3) $$\{\widetilde{C}_w^{\mathsf{BC}}(1) \,|\, w \in \mathfrak{B}_n \text{ avoids the patterns 3412 and 4231}\}$$

of the Kazhdan–Lusztig basis of $\mathbb{Z}[\mathfrak{B}_n]$. In Section 6 we use immanants and total nonnegativity to interpret trace evaluations at (3) in terms of paths in the type-BC planar networks. In Sections 7 – 8, we define type-BC analogs $Q(w)$ and $\Gamma(w)$ of the type-A posets and graphs associated to planar networks. We define a type-BC analog of codominant permutations and show that the posets correspond bijectively to the proper subset

$$\{\widetilde{C}_w^{\mathsf{BC}}(1) \,|\, w \in \mathfrak{B}_n \text{ } \mathsf{BC}\text{-codominant}\} \tag{4}$$

of (3). We use the above networks, posets, and graphs in Section 9 to state and prove our main results on the combinatorial computation of type-BC trace evaluations, and in Section 10 to show that for each element $\widetilde{C}_w^{\mathsf{BC}}(1)$ of (3) there exists an element $\widetilde{C}_v^{\mathsf{BC}}(1)$ of (4) with the property that $Q(w) \cong Q(v)$ and therefore that $\theta(\widetilde{C}_w^{\mathsf{BC}}(1)) = \theta(\widetilde{C}_v^{\mathsf{BC}}(1))$ for all traces $\theta \in \mathcal{T}(\mathfrak{B}_n)$. Formulas in Section 9 lead to natural type-BC analogs of type-A chromatic symmetric functions in Section 11. We finish in Section 12 with open problems concerning Hessenberg varieties.

2. The symmetric and hyperoctahedral groups

The hyperoctahedral group $\mathfrak{B}_n$ is closely related to the symmetric groups on n and $2n$ letters. To describe these relationships, we will use subintervals of the set

$$[\overline{n}, n] := \{-n, \dots, n\} \smallsetminus \{0\},$$

where we define $\overline{a} = -a$ for all $a \in [\overline{n}, n]$. We call any subset $[h, l] := \{h, \dots, l\} \smallsetminus \{0\}$ of $[\overline{n}, n]$ an *interval*, even if $h < 0 < l$. Let $\mathfrak{S}_{[h,l]}$ denote the group of permutations of letters in the interval $[h, l]$. The group $\mathfrak{B}_n$ is naturally related both to $\mathfrak{S}_{[\overline{n},n]}$ and $\mathfrak{S}_n = \mathfrak{S}_{[1,n]}$. To illustrate these relationships and prepare for our main results, we will consider the groups' presentations, conjugacy classes, Bruhat orders, and pattern-avoidance definitions.

2.1. $\mathfrak{B}_n$ as a subgroup of $\mathfrak{S}_{[\overline{n},n]}$.

The $2n$th symmetric group $\mathfrak{S}_{[\overline{n},n]}$ is the Coxeter group (see e.g. [15]) of type A_{2n-1}, with generators $s_{\overline{n-1}}, \dots, s_{\overline{1}}, s_0, s_1, \dots, s_{n-1}$ and relations

$$\begin{aligned} s_i^2 &= e && \text{for } i = \overline{n-1}, \dots, n-1, \\ s_i s_j &= s_j s_i && \text{for } |i-j| \geqslant 2, \\ s_i s_j s_i &= s_j s_i s_j && \text{for } |i-j| = 1. \end{aligned}$$

If an expression $s_{i_1} \cdots s_{i_\ell}$ for $w \in \mathfrak{S}_{[\overline{n},n]}$ is as short as possible, then call it *reduced* and call $\ell = \ell(w)$ the *length* of w. Define a (left) action of $\mathfrak{S}_{[\overline{n},n]}$ on rearrangements $w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n$ of the word $\overline{n} \cdots \overline{1} 1 \cdots n$ by

$$\begin{cases} s_i \text{ swaps letters in positions } i, i+1 & \text{for } i = 1, \dots, n-1, \\ s_{\overline{i}} \text{ swaps letters in positions } \overline{i}, \overline{i+1} & \text{for } i = 1, \dots, n-1, \\ s_0 \text{ swaps letters in positions } \overline{1}, 1, \end{cases} \tag{5}$$

and define the *one-line notation* of $w = s_{i_1} \cdots s_{i_r} \in \mathfrak{S}_{[\overline{n},n]}$ to be

$$w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n = s_{i_1}(s_{i_2}(\cdots(s_{i_r}(\overline{n} \cdots \overline{1} 1 \cdots n)) \cdots)). \tag{6}$$

For example, when $n = 4$, the element $s_{\overline{1}} s_0 s_1$ has one-line notation

$$s_{\overline{1}}(s_0(s_1(\overline{4321}1234))) = s_{\overline{1}}(s_0(\overline{4321}2134)) = s_{\overline{1}}(\overline{432}2\overline{1}134) = \overline{43}2\overline{21}134.$$

(By our definition, the right action of s_i swaps the letters $i, i+1$, wherever they are.) It follows that w_i^{-1} is the index j satisfying $w_j = i$. It is known that $\ell(w)$ equals the number of *inversions* in w:

$$\textsc{inv}(w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n) := \{(j, i) \,|\, j > i \text{ and } j \text{ appears before } i \text{ in } w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n\}.$$

Thus we have $\ell(\overline{43}2\overline{21}134) = \textsc{inv}(\overline{43}2\overline{21}134) = |\{(2, \overline{2}), (2, \overline{1}), (2, 1)\}| = 3$.

For $[a, b] \subseteq [h, l]$ let $s_{[a,b]}^{[h,l]}$ be the permutation in $\mathfrak{S}_{[h,l]}$ whose one-line notation has the form

$$h \cdots (a-1) \cdot b(b-1) \cdots (a+1)a \cdot (b+1) \cdots l.$$

When the interval $[h,l]$ is clear from context, we will simply write $s_{[a,b]}$. Call such an element a (type-A) *reversal*. Observe that the standard generators of $\mathfrak{S}_{[\overline{n},n]}$ are all reversals: $s_0 = s_{[-1,1]}$, and for $i \geqslant 1$ we have $s_i = s_{[i,i+1]}$, $s_{\overline{i}} = s_{[\overline{i+1},\overline{i}]}$. Also observe that each trivial reversal $s_{[a,a]}$ is equal to the identity element e, and that two reversals $s_{[a,b]}$, $s_{[c,d]}$ commute if their intervals $[a,b]$, $[c,d]$ do not intersect. Let $\mathfrak{B}_n$ be the Coxeter group of type $\mathsf{C}_n = \mathsf{B}_n$, i.e. the hyperoctahedral group. We may view $\mathfrak{B}_n$ as the subgroup of $\mathfrak{S}_{[\overline{n},n]}$ generated by elements

$$t \text{ (also written } s'_0) := s_0, \qquad s'_i := s_i s_{\overline{i}}, \text{ for } i = 1, \dots, n-1,$$

which satisfy the relations

$$\begin{aligned} {s'_i}^2 &= e && \text{for } i = 0, \dots, n-1, \\ ts'_1ts'_1 &= s'_1ts'_1t, && \\ s'_is'_j &= s'_js'_i && \text{for } i,j \geqslant 0 \text{ and } |i-j| \geqslant 2, \\ s'_is'_js'_i &= s'_js'_is'_j && \text{for } i,j \geqslant 1 \text{ and } |i-j| = 1. \end{aligned}$$

The one-line notation for elements of $\mathfrak{B}_n$ is inherited from that of $\mathfrak{S}_{[\overline{n},n]}$. For example, when $n=4$, the element $ts'_1s'_2 \in \mathfrak{B}_4$ has one-line notation

$$t(s'_1(s'_2(\overline{4321}1234))) = ts'_1(\overline{4231}1324)) = t(\overline{4213}3124) = \overline{4213}\overline{3}124.$$

If an expression $s'_{i_1} \cdots s'_{i_\ell}$ for $w \in \mathfrak{B}_n$ is as short as possible, then call it *reduced* and call $\ell = \ell(w)$ the *length* of w. Let $\ell_t(w)$ be the number of occurrences of t in any (equivalently, every) reduced expression for w. Analogously, let $\ell_s(w) = \ell(w) - \ell_t(w)$ be the number of occurrences of $s'_1, \dots, s'_{n-1}$. It is easy to see that one-line notations of elements of $\mathfrak{B}_n$ are precisely the set of permutations $w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n \in \mathfrak{S}_{[\overline{n},n]}$ which satisfy $w_{\overline{i}} = \overline{w_i}$, i.e. each is completely determined by the n-letter subword $w_1 \cdots w_n$. Call these words the *long* and *short* one-line notations of w, respectively. We can read $\ell(w)$, $\ell_t(w)$, $\ell_s(w)$ from the short one-line notation of w by

$$\ell(w) = \textsc{inv}(w_1 \cdots w_n) + \sum_{\substack{i>0 \\ w_i<0}} |w_i|, \qquad \ell_t(w) = \#\{i > 0 \,|\, w_i < 0\},$$

and $\ell_s(w) = \ell(w) - \ell_t(w)$. Thus we have $\ell(\overline{3}124) = |3| = 3$, $\ell_t(\overline{3}124) = 1$, and $\ell_s(\overline{3}124) = 2$.

Define *type*-BC *reversals* to be those elements of $\mathfrak{B}_n$ having the forms

$$\begin{aligned} s'_{[\overline{a},a]} &:= s_{[\overline{a},a]}, \text{ for } 1 \leqslant a \leqslant n, \\ s'_{[a,b]} &:= s_{[\overline{b},\overline{a}]} s_{[a,b]}, \text{ for } 1 \leqslant a \leqslant b \leqslant n, \end{aligned} \tag{7}$$

where the elements $s_{[a,b]}$ are reversals in $\mathfrak{S}_{[\overline{n},n]}$.

2.2. Conjugacy classes, partitions, tableaux, bipartitions, bitableaux. Conjugacy classes of $\mathfrak{S}_n$ correspond to *(integer) partitions* of n, weakly decreasing positive integer sequences $\lambda = (\lambda_1, \dots, \lambda_\ell)$ satisfying $\lambda_1 + \cdots + \lambda_\ell = n$. The $\ell = \ell(\lambda)$ components of a partition λ are called its *parts* and we let the expressions $|\lambda| = n$ and $\lambda \vdash n$ denote that λ is a partition of n. Sometimes we use the notation k^{a_k} to denote a sequence of a_k copies of the letter k. Given $\lambda \vdash n$, we define the *transpose* partition $\lambda^\top = (\lambda^\top_1, \dots, \lambda^\top_{\lambda_1})$ by $\lambda^\top_i = \#\{j \,|\, \lambda_j \geqslant i\}$. Thus $n^\top = 1^n$. We call λ *self-transpose* if $\lambda^\top = \lambda$ and we define the empty sequence $\varnothing$ to be the unique partition of the integer 0. Generalizing integer partitions of n are *compositions* $\alpha = (\alpha_1, \dots, \alpha_r)$ of n, which are simply positive integer sequences summing to n. We let the notation $\alpha \vDash n$ denote that α is a composition of n. (See e.g. [66, §1.2].)

The conjugacy class of $\mathfrak{S}_n$ corresponding to $\lambda \vdash n$ is the set of all permutations having cycle type λ. We write $\mathrm{ctype}(w) = \lambda$. Letting a_k be the multiplicity of k in

λ, for $k = 1, \dots, n$, we may express the cardinality of the λ-conjugacy class of $\mathfrak{S}_n$ as $n!/z_\lambda$, where

$$z_\lambda = 1^{a_1} \cdots n^{a_n} a_1! \cdots a_n!. \tag{8}$$

Conjugacy classes of $\mathfrak{B}_n$ correspond to *integer bipartitions of* n, pairs (λ, μ) of integer partitions with $|\lambda| + |\mu| = n$. We let $(\lambda, \mu) \vdash n$ denote that (λ, μ) is a bipartition of n. To explicitly describe the conjugacy classes of $\mathfrak{B}_n$, we define the homomorphism

$$\begin{aligned} \varphi : \mathfrak{B}_n &\to \mathfrak{S}_n \\ s_i' &\mapsto s_i, \quad i = 1, \dots, n-1, \\ t &\mapsto e, \end{aligned} \tag{9}$$

which replaces letters in the short one-line notation of $v \in \mathfrak{B}_n$ by their absolute values. For each element $v \in \mathfrak{B}_n$ and each cycle $C = (c_1, \dots, c_k = c_0)$ of $\varphi(v) \in \mathfrak{S}_n$, define the *signed cycle* $\hat{C} = (\hat{c}_1, \dots, \hat{c}_k)$ of v by

$$\hat{c}_i = \begin{cases} c_i & \text{if } v(c_{i-1}) = c_i, \\ \overline{c}_i & \text{if } v(c_{i-1}) = \overline{c_i}. \end{cases}$$

Call $\hat{C}$ *positive* if it has an even number of negative letters, and *negative* otherwise. The conjugacy class of $\mathfrak{B}_n$ corresponding to $(\lambda, \mu) \vdash n$ is precisely the set of elements whose *signed cycle type*, the bipartition of positive cycle cardinalities and negative cycle cardinalities, is equal to (λ, μ). We write $\mathrm{sct}(w) = (\lambda, \mu)$. We may express the cardinality of the (λ, μ) conjugacy class of $\mathfrak{B}_n$ as $2^n n!/(z_\lambda z_\mu 2^{\ell(\lambda)+\ell(\mu)})$.

To each integer partition $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ we associate a *Young diagram of shape* λ, an arrangement of n boxes into ℓ left-justified rows with λ_i boxes in row i. By the French convention, row 1 appears on the bottom. A Young diagram filled with elements of a set S is called a *tableau* or more specifically an *S-tableau*. If $S \subseteq \mathbb{Z}$ we also call it a *Young tableau*. Repeated elements are permitted. Given a bipartition $(\lambda, \mu) \vdash n$, we define a Young *bidiagram* of shape (λ, μ) to be an ordered pair of Young diagrams of shapes λ and μ. We define *bitableaux* similarly.

2.3. The Bruhat order.

For any Coxeter group W, the *Bruhat order on* W is the poset defined by declaring $v \leqslant_W w$ if some (equivalently, every) reduced expression for w contains a reduced expression for v. Ehresmann [26] showed that the Bruhat order on $\mathfrak{S}_{[\overline{n},n]}$ is isomorphic to the (dual of the) componentwise order on tableaux $\{A(w) \,|\, w \in \mathfrak{S}_{[\overline{n},n]}\}$ of shape $(2n, 2n-1, \dots, 1)$ defined by placing the increasing rearrangement of $w_i \cdots w_n$ in row i, for $i = \overline{n}, \dots, n$. For example, the type-A Bruhat order comparison $\overline{3}21\overline{1}\overline{2}3 \leqslant_{\mathfrak{S}_{[\overline{n},n]}} \overline{3}213\overline{2}\overline{1}$ may be verified by the componentwise inequality $A(\overline{3}21\overline{1}\overline{2}3) \geqslant A(\overline{3}213\overline{2}\overline{1})$,

$$\begin{array}{|c|c|c|c|c|c|} \hline 3 \\ \cline{1-1} \overline{2} & 3 \\ \cline{1-2} \overline{2} & \overline{1} & 3 \\ \cline{1-3} \overline{2} & \overline{1} & 1 & 3 \\ \cline{1-4} \overline{2} & \overline{1} & 1 & 2 & 3 \\ \cline{1-5} \overline{3} & \overline{2} & \overline{1} & 1 & 2 & 3 \\ \hline \end{array} \quad \geqslant \quad \begin{array}{|c|c|c|c|c|c|} \hline \overline{1} \\ \cline{1-1} \overline{2} & \overline{1} \\ \cline{1-2} \overline{2} & \overline{1} & 3 \\ \cline{1-3} \overline{2} & \overline{1} & 1 & 3 \\ \cline{1-4} \overline{2} & \overline{1} & 1 & 2 & 3 \\ \cline{1-5} \overline{3} & \overline{2} & \overline{1} & 1 & 2 & 3 \\ \hline \end{array} \, .$$

Proctor [55, Thm. 5BC] showed that the Bruhat order on $\mathfrak{B}_n$ is isomorphic to a similar order on tableaux $\{B(w) \,|\, w \in \mathfrak{B}_n\}$ of shape $(n, n-1, \dots, 1)$ defined by placing the increasing rearrangement of $w_i \cdots w_n$ in row i, for $i = 1, \dots, n$. For example, the type-BC Bruhat order comparison $\overline{3}21\overline{12}3 \leqslant_{\mathfrak{B}_n} 12\overline{3}3\overline{21}$ may be verified by the componentwise inequality $B(\overline{3}21\overline{12}3) \geqslant B(12\overline{3}3\overline{21})$,

$$\begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline \overline{2} & \multicolumn{1}{c|}{3} & \\ \hline \overline{2} & \multicolumn{1}{c|}{\overline{1}} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \geqslant \quad \begin{array}{|c|cc}\hline \overline{1} & \multicolumn{2}{c}{} \\ \hline \overline{2} & \multicolumn{1}{c|}{\overline{1}} & \\ \hline \overline{2} & \multicolumn{1}{c|}{\overline{1}} & \multicolumn{1}{c|}{3} \\ \hline \end{array}\ .$$

It is not difficult to show that the Bruhat order on $\mathfrak{B}_n$ is an induced subposet of the Bruhat order on $\mathfrak{S}_{[\overline{n},n]}$.

Proposition 2.1. *For $v, w \in \mathfrak{B}_n \subset \mathfrak{S}_{[\overline{n},n]}$, we have $v \leqslant_{\mathfrak{B}_n} w$ if and only if $v \leqslant_{\mathfrak{S}_{[\overline{n},n]}} w$.*

Proof. Consider $v, w \in \mathfrak{B}_n \subset \mathfrak{S}_{[\overline{n},n]}$. If we have $v \leqslant_{\mathfrak{B}_n} w$, then there is a reduced $\mathfrak{B}_n$-expression $s'_{i_1} \cdots s'_{i_k}$ for v which is a subword of a reduced $\mathfrak{B}_n$-expression $s'_{j_1} \cdots s'_{j_k}$ for w. Then the recipe

$$s'_i \mapsto \begin{cases} s_{\overline{i}} s_i & \text{if } i > 0 \\ t & \text{if } i = 0 \end{cases}$$

produces a reduced $\mathfrak{S}_{[\overline{n},n]}$-expression for v which is a subword of a reduced $\mathfrak{S}_{[\overline{n},n]}$-expression for w. Now suppose that $v \leqslant_{\mathfrak{S}_{[\overline{n},n]}} w$. By Ehresmann's criterion, we have the componentwise tableau inequality $A(v) \geqslant A(w)$. But the upper n rows of these tableaux give the inequality $B(v) \geqslant B(w)$. This is precisely Proctor's criterion for $v \leqslant_{\mathfrak{B}_n} w$. □

2.4. Pattern avoidance.

Given a word $u = u_1 \cdots u_k$ in $\mathfrak{S}_k$, and a word $y = y_1 \cdots y_k$ having k distinct letters, we say that *y matches the pattern u* if the letters of y appear in the same relative order as those of u; that is, if we have $u_i < u_j$ if and only if $y_i < y_j$ for all $i, j \in [k]$. On the other hand, given a word $w = w_1 \dots w_m$ having distinct letters, e.g. $w \in \mathfrak{S}_n$ or $w \in \mathfrak{B}_n$, we say that *w avoids the pattern u* if no subword of w matches the pattern u. In $\mathfrak{B}_n$, a second notion of pattern avoidance involves signed letters and short one-line notation. Let $v = v_1 \cdots v_k$ be the short one-line notation of an element of $\mathfrak{B}_k$, i.e. a word in letters $[\overline{k}, k]$ with $|v_1| \cdots |v_k| \in \mathfrak{S}_k$. Let $y = y_1 \cdots y_k$ be a word in $[\overline{n}, n]$ such that $|y_1| \cdots |y_k|$ has no repeated letters. Say that *y matches the signed pattern v* if

(1) for $i = 1, \dots, k$, the letters v_i and y_i have the same sign,
(2) for all i, j, $|v_i| < |v_j|$ if and only if $|y_i| < |y_j|$.

Say that *$w \in \mathfrak{B}_n$ avoids the signed pattern v* if no subword of the short one-line notation of w matches the signed pattern v. (See [12, p. 108].)

Many properties of elements of $\mathfrak{B}_n$ can be expressed in terms of signed pattern avoidance.

Lemma 2.2. *The element $w \in \mathfrak{B}_n$ avoids the signed patterns $1\overline{2}$ and $\overline{2}1$ if and only if the set of negative letters in $w_1 \cdots w_n$ is empty or forms an interval $[\overline{b}, \overline{1}]$ for some $b \geqslant 1$.*

Proof. If all letters in $w_1 \cdots w_n$ are positive, then the claim is true.

($\Rightarrow$) Suppose therefore that the negative letters in this word do not form an interval of the desired form. Then for some i, j, we have $w_j \leqslant \overline{1}$ and $1 \leqslant w_i < |w_j|$. It follows that $w_i w_j$ matches the signed pattern $1\overline{2}$ or $w_j w_i$ matches the signed pattern $\overline{2}1$.

($\Leftarrow$) If some subword $w_i w_j$ with $i < j$ matches the signed pattern $1\overline{2}$ or $\overline{2}1$, then clearly the negative letters in $w_1 \cdots w_n$ do not form the desired interval. □

Avoidance of signed patterns can also imply the avoidance of ordinary patterns.

Lemma 2.3. *If $w \in \mathfrak{B}_n$ avoids the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$, then w avoids the unsigned patterns* 3412 *and* 4231.

Proof. First we claim that if w contains a subword $cdab$ matching the unsigned pattern 3412, then it must contain a subword matching one of the five signed patterns. Suppose that $cdab$ or just dab is a subword of $w_1 \cdots w_n$. If $b > 0$ then dab matches the signed pattern 312 or $3\overline{1}2$. If $b < 0$ then ab matches the signed pattern $\overline{21}$. Now suppose that cd appears in $w_{\overline{n}} \cdots w_{\overline{1}}$ and ab in $w_1 \cdots w_n$. If $b < 0$ then ab matches the signed pattern $\overline{21}$. If $b > 0$ then $\overline{c}$ and $\overline{d}$ appear in $w_1 \cdots w_n$ without $\overline{b}$, contradicting Lemma 2.2. Now suppose that $cdab$ or just cda appears in $w_{\overline{n}} \cdots w_{\overline{1}}$. Then $\overline{badc}$ or $\overline{adc}$ is a subword of $w_1 \cdots w_n$, matching the unsigned pattern 3412 or 412. By the first case, $w_1 \cdots w_n$ has a subword matching one of the signed patterns 312, $3\overline{1}2$, $\overline{21}$.

Now we claim that if w contains a subword $dbca$ matching the unsigned pattern 4231, then it must contain a subword matching one of the five signed patterns. Suppose that $dbca$ is a subword of $w_1 \cdots w_n$. If $c > 0$ then dbc matches the signed pattern 312 or $3\overline{1}2$. If $c < 0$ then bc matches the signed pattern $\overline{21}$. Now suppose that d appears in $w_{\overline{n}} \cdots w_{\overline{1}}$ and bca in $w_1 \cdots w_n$. If $c > 0$ then $\overline{d}$ appears in $w_1 \cdots w_n$, contradicting Lemma 2.2. If $c < 0$ then bc matches the signed pattern $\overline{21}$. Now suppose that db appears in $w_{\overline{n}} \cdots w_{\overline{1}}$ and ca in $w_1 \cdots w_n$. If $c > 0$ then $\overline{d}$ (< 0) appears in $w_1 \cdots w_n$, contradicting Lemma 2.2. If $c < 0$ then $\overline{b}$ (> 0) appears in $w_1 \cdots w_n$, also contradicting Lemma 2.2. Finally, suppose that $dbca$ or just dbc appears in $w_{\overline{n}} \cdots w_{\overline{1}}$. Then $\overline{acbd}$ or $\overline{cbd}$ is a subword of $w_1 \cdots w_n$, matching the unsigned pattern 4231 or 423. By the first and second cases above, $w_1 \cdots w_n$ has a subword matching one of the five signed patterns. □

3. Schubert varieties and Hecke algebras

Our main results (Theorem 9.6 – Theorem 9.8) partially answer Problem 1.1 using a linearly independent set in $\mathbb{Z}[\mathfrak{B}_n]$. This set is best described in terms of a special basis of the Hecke algebra $H(\mathfrak{B}_n)$ and smoothness of certain Schubert varieties.

3.1. Schubert varieties.

Let G be a complex connected semisimple algebraic group, choose a Borel subgroup B of G, and consider the quotient G/B, called a *flag variety.* The action of B on G/B by left multiplication partitions it into orbits often written $\mathrm{B}\dot{w}\mathrm{B}$, which are parametrized by elements w of the corresponding Weyl group W. The Zariski closure Ω_w of $\mathrm{B}\dot{w}\mathrm{B}$ in G/B is called the *Schubert variety indexed by w*. We have $\Omega_v \supseteq \Omega_w$ if and only if $v \leqslant w$ in the Bruhat order. (See e.g. [12, § 4.7].) Standard choices of G are $\mathrm{SL}_n(\mathbb{C})$ (type A), $\mathrm{SO}_{2n+1}(\mathbb{C})$ (type B), and $\mathrm{SP}_n(\mathbb{C})$ (type C). The corresponding Weyl groups are $\mathfrak{S}_n$ (type A), and $\mathfrak{B}_n$ (types B and C).

Call the Schubert variety Ω_w *rationally smooth* if its ordinary cohomology $\mathrm{H}^*(\Omega_w)$ and intersection cohomology $\mathrm{IH}^*(\Omega_w)$ coincide. (See [12, Ch. 6].) Call Ω_w *smooth* if the tangent space at every point has dimension equal to the dimension of the variety. It is known that every smooth Schubert variety is rationally smooth. Let Ω_w^{A} ($w \in \mathfrak{S}_n$), Ω_w^{B}, Ω_w^{C} ($w \in \mathfrak{B}_n$) denote the type-A, B, and C Schubert varieties, respectively. Elements $w \in \mathfrak{S}_n$ for which Ω_w^{A} is rationally smooth or smooth are characterized by pattern avoidance [47].

Proposition 3.1. *For $w \in \mathfrak{S}_n$, the Schubert variety Ω_w^{A} is smooth, equivalently rationally smooth, if and only if w avoids the patterns* 3412 *and* 4231.

Smoothness and rational smoothness of type-B and C Schubert varieties are characterized by more intricate pattern avoidance. The conditions on $w \in \mathfrak{B}_n$ which imply rational smoothness of Ω_w^{B} are the same as those which imply rational smoothness of

Ω^{C}_w [11, Thm. 4.2]: w must avoid the twenty-five patterns listed in [12, Eq. (13.3.5)]. On the other hand, the conditions which imply smoothness of the two Schubert varieties are different [12, Thm. 8.3.17]: Ω^{B}_w is smooth if and only if it is rationally smooth and w avoids the additional pattern 3412; Ω^{C}_w is smooth if and only if it is rationally smooth and w avoids the additional pattern 4231.

We will be interested in those elements $w \in \mathfrak{B}_n$ for which Ω^{B}_w and Ω^{C}_w are simultaneously smooth. These are precisely the elements $w \in \mathfrak{B}_n \subset \mathfrak{S}_{[\overline{n},n]}$ for which Ω^{A}_w is smooth when $\mathrm{G} = \mathrm{SL}_{2n}(\mathbb{C})$.

Proposition 3.2. *For $w \in \mathfrak{B}_n$, the Schubert varieties Ω^{B}_w and Ω^{C}_w are simultaneously smooth if and only if w avoids the patterns* 3412 *and* 4231*.*

Proof. If Ω^{B}_w and Ω^{C}_w are both smooth, then by the above discussion, w avoids the patterns 3412 and 4231. Suppose that w avoids the patterns 3412 and 4231. It is straightforward to check that each of the twenty-five patterns listed in [12, Eq. (13.3.5)] contains 3412 and/or 4231. Thus w avoids these twenty-five patterns as well, and Ω^{B}_w, Ω^{C}_w are both smooth. □

3.2. Hecke algebras.

Given Coxeter group W with generator set S, define the *Hecke algebra* $H(W)$ of W to be the $\mathbb{Z}[q, q^{-1}]$-span of $\{T_w \,|\, w \in W\}$ with multiplicative unit T_e and multiplication defined by

$$T_sT_w = \begin{cases} qT_{sw} + (q-1)T_w & \text{if } sw <_W w, \\ T_{sw} & \text{if } sw >_W w, \end{cases}$$

where $s \in S$, $w \in W$, and $<_W$ is the Bruhat order on W. This formula guarantees that for $w \in W$ and any reduced expression $s_{i_1} \cdots s_{i_\ell}$ for w, we have $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$. Call $\{T_w \,|\, w \in W\}$ the *natural basis* of $H(W)$. It is easy to see that the specialization of $H(W)$ at $q = 1$ is isomorphic to $\mathbb{Z}[W]$.

A second basis [41] of H is the (modified, signless) *Kazhdan–Lusztig basis* $\{\widetilde{C}_w(q) \,|\, w \in W\}$, related to the natural basis by

$$\widetilde{C}_w(q) = \sum_{v \leqslant_W w} P_{v,w}(q) T_v,$$

where $\{P_{v,w}(q) \,|\, v, w \in W\} \subseteq \mathbb{Z}[q]$ are the *Kazhdan–Lusztig polynomials* whose recursive definition appears in [41]. Coefficients of these polynomials may be interpreted in terms of intersection cohomology $\mathrm{IH}^*(\Omega_w)$ [42]. Specifically, when Ω_w is rationally smooth, all polynomials $\{P_{v,w}(q) \,|\, v \leqslant_{\mathfrak{S}_n} w\}$ are identically 1 [41, Thm. A.2]. Thus we have the following.

Proposition 3.3. *For W equal to $\mathfrak{S}_n$ or $\mathfrak{B}_n$ and $w \in W$ avoiding the patterns* 3412 *and* 4231*, the Kazhdan–Lusztig basis element $\widetilde{C}_w(q)$ of $H(W)$ satisfies*

$$\widetilde{C}_w(q) = \sum_{v \leqslant_W w} T_v.$$

Proof. This follows from Propositions 3.1, 3.2. □

In Sections 4 – 12 we will find it convenient to define

$$H_n(q) := H(\mathfrak{S}_n), \qquad H_{[h,l]}(q) := H(\mathfrak{S}_{[h,l]}), \qquad H^{\mathsf{BC}}_n(q) := H(\mathfrak{B}_n),$$

and to let $\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n\}$ and $\{\widetilde{C}^{\mathsf{BC}}_w(q) \,|\, w \in \mathfrak{B}_n\}$ denote the Kazhdan–Lusztig bases of $H_n(q)$ and $H^{\mathsf{BC}}_n(q)$, respectively.

4. Trace spaces

Given Coxeter group W and Hecke algebra $H = H(W)$, let $\mathcal{T}(H)$ be the $\mathbb{Z}[q, q^{-1}]$-module of H-traces, those linear functionals $\theta_q : H \to \mathbb{Z}[q, q^{-1}]$ satisfying $\theta_q(DD') = \theta_q(D'D)$ for all $D, D' \in H$. This is the $\mathbb{Z}[q, q^{-1}]$-span of all H-characters. Let $\mathcal{T}(W) = \mathcal{T}(\mathbb{Z}[W])$ be the specialization of $\mathcal{T}(H)$ at $q = 1$. That is, for each H-trace $\theta_q \in \mathcal{T}(H)$ satisfying $\theta_q(T_w) = f_w(q)$ for all $w \in W$, define the W-trace $\theta = \theta_1 \in \mathcal{T}(\mathbb{Z}[\mathfrak{S}_n])$ by $\theta(w) = f_w(1)$ for all $w \in W$. (See e.g. [29].)

The ranks of $\mathcal{T}(H)$ and $\mathcal{T}(W)$ are both equal to the number of conjugacy classes of W. We consider six bases of $\mathcal{T}(H_n(q))$ and $\mathcal{T}(\mathfrak{S}_n)$, and eleven bases of $\mathcal{T}(H_n^{\mathsf{BC}}(q))$ and $\mathcal{T}(\mathfrak{B}_n)$.

4.1. The trace spaces $\mathcal{T}(H_n(q))$ and $\mathcal{T}(\mathfrak{S}_n)$.

The rank of $\mathcal{T}(H_n(q))$ equals the number of partitions of n. Three commonly used bases consist of $H_n(q)$-characters. These are the bases of irreducible characters $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$, induced trivial characters $\{\eta_q^\lambda \,|\, \lambda \vdash n\}$, and induced sign characters $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$, where

$$\eta_q^\lambda = \mathrm{triv}_q\big\uparrow_{H_\lambda(q)}^{H_n(q)}, \qquad \mathrm{triv}_q(T_{s_i}) = q, \qquad \epsilon_q^\lambda = \mathrm{sgn}_q\big\uparrow_{H_\lambda(q)}^{H_n(q)}, \qquad \mathrm{sgn}_q(T_{s_i}) = -1, \tag{10}$$

and $H_\lambda(q)$ is the Young subalgebra of $H_n(q)$ generated by

$$\{T_{s_1}, \dots, T_{s_{n-1}}\} \smallsetminus \{T_{s_{\lambda_1}}, T_{s_{\lambda_1+\lambda_2}}, \dots, T_{s_{n-\lambda_\ell}}\}.$$

All $H_n(q)$-characters in $\mathcal{T}(H_n(q))$ belong to $\mathrm{span}_{\mathbb{N}[q]}\{\chi_q^\lambda \,|\, \lambda \vdash n\}$. Three more non-character bases of $\mathcal{T}(H_n(q))$ consist of traces called *power sum traces* $\{\psi_q^\lambda \,|\, \lambda \vdash n\}$, *monomial traces* $\{\phi_q^\lambda \,|\, \lambda \vdash n\}$, and *forgotten traces* $\{\gamma_q^\lambda \,|\, \lambda \vdash n\}$, defined by

$$\psi_q^\lambda = \sum_\mu \chi^\mu(\lambda)\chi_q^\mu, \qquad \phi_q^\lambda = \sum_\mu K^{-1}_{\lambda,\mu}\chi_q^\mu, \qquad \gamma_q^\lambda = \sum_\mu K^{-1}_{\lambda,\mu^\top}\chi_q^\mu,$$

where $\chi^\mu(\lambda) := \chi^\mu(w)$ for any $w \in \mathfrak{S}_n$ with $\mathrm{ctype}(w) = \lambda$, and $\{K^{-1}_{\lambda,\mu} \,|\, \lambda, \mu \vdash n\}$ are the *inverse Kostka numbers*. (See [67, §7].) The specialization of the power sum trace basis at $q = 1$ is essentially an indicator basis for conjugacy classes of $\mathfrak{S}_n$,

$$\psi^\lambda(w) = \begin{cases} z_\lambda & \text{if } \mathrm{ctype}(w) = \lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

For few traces $\theta_q \in \mathcal{T}(H_n(q))$ do we have cancellation-free formulas for all evaluations of the form $\{\theta_q(T_w) \,|\, w \in \mathfrak{S}_n\}$. Two examples are the trivial and sign characters in (10): for all $w \in \mathfrak{S}_n$ we have

$$\chi_q^n(T_w) = \eta_q^n(T_w) = q^{\ell(w)}, \qquad \chi_q^{1^n}(T_w) = \epsilon_q^n(T_w) = (-1)^{\ell(w)}.$$

4.2. The trace spaces $\mathcal{T}(\mathfrak{B}_n)$ and $\mathcal{T}(H_n^{\mathsf{BC}}(q))$.

The rank of $\mathcal{T}(H_n^{\mathsf{BC}}(q))$ equals the number of bipartitions of n. Ten commonly used bases can be constructed from pairs of type-A Hecke algebra trace bases, i.e. bases of

$$\bigoplus_{k=0}^{n} \mathcal{T}(H_k(q)) \otimes \mathcal{T}(H_{n-k}(q)),$$

and from the Young subalgebra $H_{k,n-k}^{\mathsf{BC}}(q)$ of $H_n^{\mathsf{BC}}(q)$ generated by

$$\{T_t, T_{s'_1}, \dots, T_{s'_{k-1}}\} \cup \{T_{t_k}, T_{s'_{k+1}}, \dots, T_{s'_{n-1}}\},$$

where $t_k = s'_k \cdots s'_1 t s'_1 \cdots s'_k$. Specifically, given bases

$$\{\zeta_q^\lambda \,|\, \lambda \vdash k\} \subseteq \mathcal{T}(H_k(q)), \qquad \{\xi_q^\mu \,|\, \mu \vdash n-k\} \subseteq \mathcal{T}(H_{n-k}(q)), \tag{12}$$

we define traces $q\zeta_q^\lambda \in \mathcal{T}(H_k^{\mathsf{BC}}(q))$, $\delta\xi_q^\mu \in \mathcal{T}(H_{n-k}^{\mathsf{BC}}(q))$ by

$$q\zeta_q^\lambda(T_w) := q^{\ell_t(w)}\zeta_q^\lambda(T_{\varphi(w)}), \qquad \delta\xi_q^\mu(T_w) := (-1)^{\ell_t(w)}\xi_q^\mu(T_{\varphi(w)}),$$

where ℓ_t, φ are defined as in Subsections 2.1 – 2.2. (When ζ_q^λ and ξ_q^μ are $H_n(q)$-characters, i.e. traces of matrix representations, the modifications $q\zeta_q^\lambda$ and $\delta\xi_q^\mu$ correspond to type-A matrix representations extended by the definitions $T_t \mapsto qI$ and $T_t \mapsto -I$, respectively.) Then we create a basis $\{(\zeta\xi)_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}$ of $\mathcal{T}(H_n^{\mathsf{BC}}(q))$ by inducing

$$(\zeta\xi)_q^{\lambda,\mu} := (q\zeta_q^\lambda \otimes \delta\xi_q^\mu)\big\uparrow_{H_{k,n-k}^{\mathsf{BC}}(q)}^{H_n^{\mathsf{BC}}(q)}. \tag{13}$$

This construction of the irreducible characters

$$\{(\chi\chi)_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}$$

of $H_n^{\mathsf{BC}}(q)$ can be deduced from Hoefsmit [37, §2.2]. (See also [25], [29, §5.5].) One then verifies that other trace bases are related to the irreducible character basis by matrices described in [10, §3]. When the bases in (12) are type-A character bases, the definition (13) gives a character basis of $\mathcal{T}(H_n^{\mathsf{BC}}(q))$. More examples are the induced one-dimensional characters,

$$\{(\eta\eta)_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \quad \{(\eta\epsilon)_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\},$$
$$\{(\epsilon\eta)_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \quad \{(\epsilon\epsilon)_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}.$$

Five bases of $\mathcal{T}(H_n^{\mathsf{BC}}(q))$ which do not consist of characters are formed from the definition (13) and type-A power sum, monomial, and forgotten traces,

$$\{(\psi\psi)_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \quad \{(\phi\phi)_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \quad \{(\phi\gamma)_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\},$$
$$\{(\gamma\phi)_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \quad \{(\gamma\gamma)_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}.$$

An eleventh basis of $\mathcal{T}(H_n^{\mathsf{BC}}(q))$,

$$\{\iota_q^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\},$$

may be defined in terms of irreducible characters by

$$\iota_q^{\lambda,\mu} = \sum_{(\alpha,\beta)\vdash n} (\chi\chi)^{\alpha,\beta}(\lambda,\mu)(\chi\chi)_q^{\alpha,\beta}, \tag{14}$$

where we define $(\chi\chi)^{\alpha,\beta}(\lambda,\mu) := (\chi\chi)^{\alpha,\beta}(w)$ for any $w \in \mathfrak{B}_n$ having signed cycle type (λ,μ). (See e.g. [6].) The specialization of this basis at $q=1$ is essentially an indicator basis for conjugacy classes of $\mathfrak{B}_n$,

$$\iota^{\lambda,\mu}(w) = \begin{cases} z_\lambda z_\mu 2^{\ell(\lambda)+\ell(\mu)} & \text{if } \mathrm{sct}(w) = (\lambda,\mu), \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Unsurprisingly, there are few traces $\theta_q \in \mathcal{T}(H_n^{\mathsf{BC}}(q))$ for which we have cancellation-free formulas for $\{\theta_q(T_w) \,|\, w \in \mathfrak{B}_n\}$. Four examples are the one-dimensional characters constructed from (10) and (13): for all $w \in \mathfrak{B}_n$ we have

$$(\chi\chi)_q^{(n,\varnothing)}(T_w) = (\eta\eta)_q^{(n,\varnothing)}(T_w) = (\eta\epsilon)_q^{(n,\varnothing)}(T_w) = q^{\ell(w)},$$
$$(\chi\chi)_q^{(1^n,\varnothing)}(T_w) = (\epsilon\eta)_q^{(n,\varnothing)}(T_w) = (\epsilon\epsilon)_q^{(n,\varnothing)}(T_w) = q^{\ell_t(w)}(-1)^{\ell_s(w)},$$
$$(\chi\chi)_q^{(\varnothing,n)}(T_w) = (\eta\eta)_q^{(\varnothing,n)}(T_w) = (\epsilon\eta)_q^{(\varnothing,n)}(T_w) = (-1)^{\ell_t(w)}q^{\ell_s(w)},$$
$$(\chi\chi)_q^{(\varnothing,1^n)}(T_w) = (\eta\epsilon)_q^{(\varnothing,n)}(T_w) = (\epsilon\epsilon)_q^{(\varnothing,n)}(T_w) = (-1)^{\ell(w)},$$

where ℓ_s is defined as in Subsection 2.1.

5. Planar networks

Several partial solutions to the type-A case of Problem 1.1 involve the subset

$$\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n \text{ avoids the patterns 3412 and 4231}\} \tag{16}$$

of the Kazhdan–Lusztig basis of $H_n(q)$. The graphical representation of these elements by planar networks called type-A *zig-zag networks* [61, §3] allows for simple combinatorial interpretation of certain trace evaluations [61, §5–10]. Moreover, the subset

$$\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n \text{ avoids the pattern 312}\} \tag{17}$$

of (16) and its graphical representation by the subset of zig-zag networks called *descending star networks* [61] captures much of the same information.

We will extend the above type-A results to types B and C by defining *type*-BC *zig-zag networks* to graphically represent the subset

$$\{\widetilde{C}^{\mathsf{BC}}_w(q) \,|\, w \in \mathfrak{B}_n \text{ avoids the patterns 3412 and 4231}\} \tag{18}$$

of the Kazhdan–Lusztig basis of $H^{\mathsf{BC}}_n(q)$, and *type*-BC *descending star networks* to graphically represent the subset

$$\{\widetilde{C}^{\mathsf{BC}}_w(q) \,|\, w \in \mathfrak{S}_n \text{ avoids the signed patterns } 1\overline{2},\ \overline{2}1,\ \overline{21},\ 312,\ 3\overline{1}2\} \tag{19}$$

of (18). These graphical representations facilitate simple combinatorial interpretation of certain trace evaluations (Section 9), when we specialize at $q = 1$.

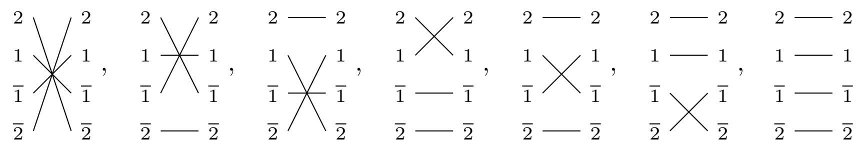
5.1. Type-A planar networks and factorization.

Define a *type*-A *planar network with boundary vertices indexed by the interval* $[h, l]$ to be a directed, planar, acyclic multigraph which can be embedded in a disc so that $2|[h, l]|$ boundary vertices can be labeled clockwise as *source* $h, \dots,$ *source* l, *sink* $l, \dots,$ *sink* h. We will allow edges (x, y) to be marked by a positive integer multiplicity k and will say that such an edge contributes k to the outdegree of x and to the indegree of y. We will assume all sources to have indegree 0 and outdegree 1, and all sinks to have indegree 1 and outdegree 0. Let $\mathcal{F}^{\mathsf{A}}([h, l])$ denote the set of such networks.

For each subinterval $[a, b]$ of $[h, l]$ we define a *simple star network* $F^{[h,l]}_{[a,b]} \in \mathcal{F}^{\mathsf{A}}([h, l])$ by

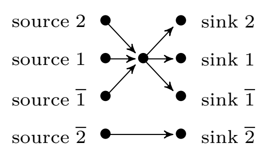
1. Sources $h, \dots, l$ lie on a vertical line to the left; sinks $h, \dots, l$ lie on a vertical line to the right. Both are labeled from bottom to top.
2. An interior vertex lies between the sources and sinks.
3. For $i = h, \dots, a-1$ and $i = b+1, \dots, l$, a directed edge begins at source i and terminates at sink i.
4. For $i = a, \dots, b$, a directed edge begins at source i and terminates at the interior vertex, and another directed edge begins at the interior vertex and terminates at sink i.
5. All edges have multiplicity 1.

When the set of source and sink labels is clear, we omit the superscript $[h, l]$ and write $F_{[a,b]}$. For zero- and one-element subintervals we define the trivial network $F_\varnothing = F_{[h,h]} = \cdots = F_{[l,l]}$ to have no interior vertex, and $|[h, l]|$ horizontal edges, each from source i to sink i, for $i = h, \dots, l$. For example, the (infinite) set $\mathcal{F}^{\mathsf{A}}([\overline{2}, 2])$ contains seven simple star networks:

(20) 

$F_{[\overline{2},2]}$ $F_{[\overline{1},2]}$ $F_{[\overline{2},1]}$ $F_{[1,2]}$ $F_{[\overline{1},1]}$ $F_{[\overline{2},\overline{1}]}$ $F_{\varnothing}$

where $F_{\varnothing} = F_{[\overline{2},\overline{2}]} = F_{[\overline{1},\overline{1}]} = F_{[1,1]} = F_{[2,2]}$. In figures, all edges in planar networks should be understood to be oriented from left to right, with vertices at both ends of all line segments, and additional vertices at the centers of the stars formed from crossing line segments. Thus $F_{[\overline{1},2]}$ above can be more completely drawn as



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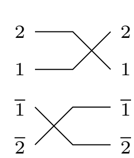
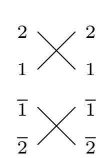
For economy, we will omit edge orientations and vertices from drawings of planar networks. When there is no danger of confusion, we will omit source and sink labels as well.

Given networks $E, F \in \mathcal{F}^{\mathsf{A}}([h,l])$, in which all sources have outdegree 1 and all sinks have indegree 1, define the concatenation $E \circ F$ of E and F as follows. For $i = h, \dots, l$, do

(1) remove sink i of E and source i of F,
(2) merge each edge $(x, \text{sink } i)$ in E with each edge $(\text{source } i, y)$ in F to form a single edge (x,y) in $E \circ F$.

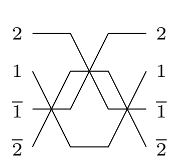
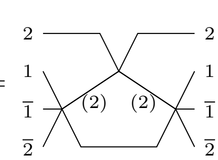
Observe that for nonintersecting intervals $[c_1,d_1]$, $[c_2,d_2]$, the concatenations $F_{[c_1,d_1]} \circ F_{[c_2,d_2]}$ and $F_{[c_2,d_2]} \circ F_{[c_1,d_1]}$ are isomorphic as directed graphs. Observe also that sometimes in a concatenation $E \circ F$, there may exist vertices x in E, y in F with $m(x,y) > 1$ multiplicity-1 edges incident upon both. Define the *condensed concatenation* $E \bullet F$ to be the subdigraph of $E \circ F$ obtained by removing, for all such pairs (x,y), all but one of the $m(x,y)$ edges incident upon both, and by marking this edge with the multiplicity $m(x,y)$. For example, in $\mathcal{F}^{\mathsf{A}}([\overline{2},2])$ we have the isomorphic graphs

(21)

$F_{[\overline{2},\overline{1}]} \circ F_{[1,2]} = F_{[\overline{2},\overline{1}]} \bullet F_{[1,2]} =$  $\cong$  $\cong$ $F_{[1,2]} \circ F_{[\overline{2},\overline{1}]} = F_{[1,2]} \bullet F_{[\overline{2},\overline{1}]}$,

and the nonisomorphic graphs

(22)

$F_{[\overline{2},1]} \circ F_{[\overline{1},2]} \circ F_{[\overline{2},1]} =$  $\ncong$ $F_{[\overline{2},1]} \bullet F_{[\overline{1},2]} \bullet F_{[\overline{2},1]} =$ 

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in which two pairs of edges are replaced by two single edges marked with multiplicity 2. We refer to all iterations of concatenations and condensed concatenations of simple star networks as *star networks*. In fact, each element of $\mathcal{F}^{\mathsf{A}}([h,l])$ is isomorphic to a star network, so we may think of $\mathcal{F}^{\mathsf{A}}([h,l])$ as a set of star networks.

Given planar network $F \in \mathcal{F}^{\mathsf{A}}([h,l])$, define its *path matrix* $A = A(F) = (a_{i,j})_{i,j\in[h,l]}$ by

$$a_{i,j} = \#\text{ paths in } F \text{ from source } i \text{ to sink } j, \tag{23}$$

ignoring multiplicities. For instance, the star networks in (21) – (22) have path matrices

$$\begin{bmatrix} a_{\overline{2},\overline{2}} & a_{\overline{2},\overline{1}} & a_{\overline{2},1} & a_{\overline{2},2} \\ a_{\overline{1},\overline{2}} & a_{\overline{1},\overline{1}} & a_{\overline{1},1} & a_{\overline{1},2} \\ a_{1,\overline{2}} & a_{1,\overline{1}} & a_{1,1} & a_{1,2} \\ a_{2,\overline{2}} & a_{2,\overline{1}} & a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} 1&1&0&0 \\ 1&1&0&0 \\ 0&0&1&1 \\ 0&0&1&1 \end{bmatrix}, \begin{bmatrix} 5&5&5&2 \\ 5&5&5&2 \\ 5&5&5&2 \\ 2&2&2&1 \end{bmatrix}, \begin{bmatrix} 2&2&2&1 \\ 2&2&2&1 \\ 2&2&2&1 \\ 1&1&1&1 \end{bmatrix},$$

respectively.

Definition 5.1. *Define $\mathcal{S}^{\mathsf{A}}([h,l])$ to be set of all type-*A *planar networks of the form*

$$F = F_{[c_1,d_1]} \bullet \cdots \bullet F_{[c_t,d_t]}, \tag{24}$$

and call these type-A condensed star networks *(with boundary vertices indexed by $[h,l]$).*

We will be interested in two subclasses of these, which we define as follows.

Definition 5.2. *Call a type-*A *condensed star network F* (24) *a* type-A zig-zag network *if we have $F = F_\varnothing$ or*

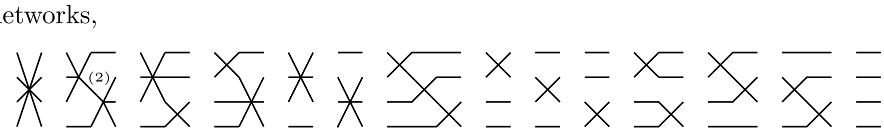
1. *the intervals $[c_1,d_1],\dots,[c_t,d_t]$ are distinct and pairwise nonnesting,*
2. *for all triples $i<j<k$ satisfying $[c_i,d_i]\cap[c_j,d_j]\neq\varnothing$ and $[c_j,d_j]\cap[c_k,d_k]\neq\varnothing$, we have $c_i<c_j<c_k$ (and $d_i<d_j<d_k$) or $c_i>c_j>c_k$ (and $d_i>d_j>d_k$).*

Let $\mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([h,l])$ denote the set of type-A zig-zag networks with boundary vertices indexed by $[h,l]$.

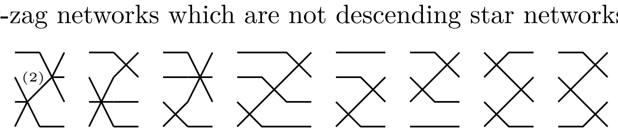
Definition 5.3. *Call a type-*A *condensed star network F* (24) *a* type-A descending star network *if we have $F = F_\varnothing$ or*

1. *the intervals $[c_1,d_1],\dots,[c_t,d_t]$ are distinct and pairwise nonnesting,*
2. *for all pairs $i<j$ satisfying $[c_i,d_i]\cap[c_j,d_j]\neq\varnothing$ we have $c_i>c_j$ (and $d_i>d_j$).*

Let $\mathcal{S}^{\mathsf{A}}_{\mathrm{D}}([h,l])$ denote the set of type-A descending star networks with boundary vertices indexed by $[h,l]$. Thus we have $\mathcal{S}^{\mathsf{A}}_{\mathrm{D}}([h,l]) \subseteq \mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([h,l])$. To illustrate, let us fix boundary vertices indexed by any interval of cardinality 4. Then we have 14 descending star networks,

(25) 

and 8 more zig-zag networks which are not descending star networks,

(26) 

The result [18, Lem. 3.5] describes intersections of paths in a descending star network.

Lemma 5.4. *Let π_{i_1}, π_{i_2} be paths in a descending star network F from sources $i_1<i_2$ to sinks m_1, m_2, respectively. Then the two paths intersect if and only if there exists a path in F from i_1 to sink m_2.*

By [61, Thm. 3.5, Lem. 5.3] and [18, Thm. 3.6], the sets $\mathcal{S}^{\mathsf{A}}_{\mathrm{D}}([h,l])$, $\mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([h,l])$ are related to pattern avoidance in $\mathfrak{S}_{[h,l]}$.

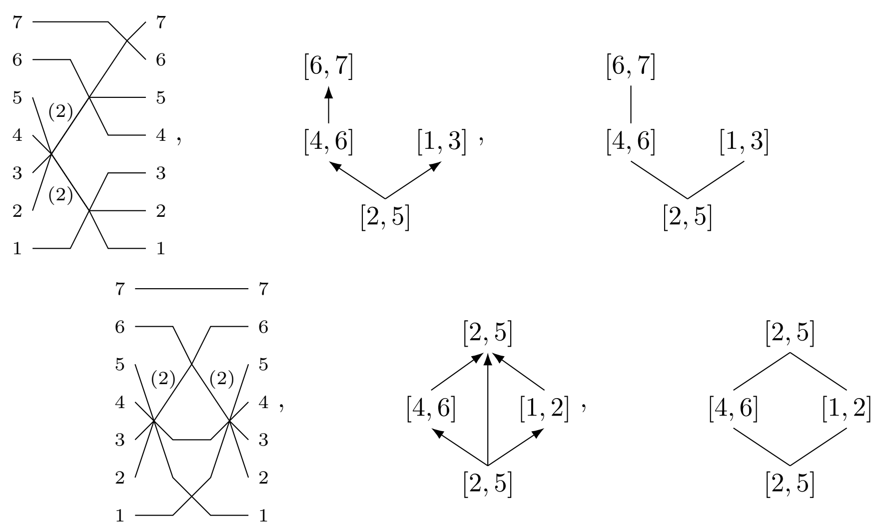
Proposition 5.5. *There is a natural bijection $F \mapsto w(F)$ from $\mathcal{S}^{\mathsf{A}}_{\mathsf{Z}}([h,l])$ to* 3412-*avoiding,* 4231-*avoiding permutations in $\mathfrak{S}_{[h,l]}$, which restricts to a bijection from $\mathcal{S}^{\mathsf{A}}_{\mathsf{D}}([h,l])$ to* 312-*avoiding permutations in $\mathfrak{S}_{[h,l]}$.*

To describe the bijection explicitly we define a relation $\prec\!\!\cdot$ on the set of intervals appearing in (24) by declaring

$$[c_i, d_i] \prec\!\!\cdot\ [c_j, d_j] \tag{27}$$

if $i < j$ and $[c_i, d_i] \cap [c_j, d_j] \smallsetminus ([c_{i+1}, d_{i+1}] \cup \cdots \cup [c_{j-1}, d_{j-1}]) \neq \varnothing$. The relation $\prec\!\!\cdot$ may be viewed as an acyclic directed graph on the intervals. The transitive, reflexive closure of $\prec\!\!\cdot$ is a partial order $\preceq$. For $F \in \mathcal{S}^{\mathsf{A}}_{\mathsf{Z}}([h,l])$, the directed graph is the Hasse diagram of the partial order; for other $F \in \mathcal{S}^{\mathsf{A}}([h,l])$ this is not the case. For example, the networks $F_{[2,5]} \bullet F_{[1,3]} \bullet F_{[4,6]} \bullet F_{[6,7]} \in \mathcal{S}^{\mathsf{A}}_{\mathsf{Z}}([1,7])$ and $F_{[2,5]} \bullet F_{[1,2]} \bullet F_{[4,6]} \bullet F_{[2,5]} \in \mathcal{S}^{\mathsf{A}}([1,7])$ and their corresponding interval digraphs and posets are

(28)



The bijection $F \mapsto w(F)$, stated in [61, §3], is given by the following algorithm.

Algorithm 5.6. *Given F as in* (24)*, do*

1. *Initialize the sequence of reversals $S := (s_{[c_1,d_1]}, \dots, s_{[c_t,d_t]})$.*
2. *For all pairs (i,j) with $[c_i,d_i] \prec\!\!\cdot\ [c_j,d_j]$ and $|[c_i,d_i] \cap [c_j,d_j]| > 1$,*
   (a) *Update S by inserting $s_{[c_i,d_i]\cap[c_j,d_j]}$ immediately after $s_{[c_i,d_i]}$.*
3. *Define $w(F)$ to be the product of reversals in S, from left to right.*

We call the final sequence of reversals a *zig-zag factorization* of $w(F)$. For example, let F be the first star network in (28). This zig-zag network F gives the reversal sequence $(s_{[2,5]}, s_{[1,3]}, s_{[4,6]}, s_{[6,7]})$ which we update by inserting $s_{[2,5]\cap[1,3]} = s_{[2,3]}$ after $s_{[2,5]}$, and then $s_{[2,5]\cap[4,6]} = s_{[4,5]}$ after $s_{[2,5]}$ to obtain the permutation

$$w = w(F) = s_{[2,5]} s_{[4,5]} s_{[2,3]} s_{[1,3]} s_{[4,6]} s_{[6,7]} = 3752146.$$

The inverse of the map $F \mapsto w(F)$, which we write

$$w \mapsto F_w, \tag{29}$$

is a bit intricate and is given in [61, §3]. It turns out that the network F above is $F_{3752146}$. In (25), if we label sources and sinks $1, 2, 3, 4$ from bottom to top, the descending star networks are

$$F_{4321}, F_{3421}, F_{2431}, F_{3241}, F_{1432}, F_{3214}, F_{2341}, F_{1243}, F_{1324}, F_{2134}, F_{2143}, F_{1342}, F_{2314}, F_{1234}, \tag{30}$$

respectively. In (26), the remaining zig-zag networks are

$$F_{4312}, F_{4213}, F_{4132}, F_{4123}, F_{3124}, F_{1423}, F_{1423}, F_{3142}, F_{2413}. \tag{31}$$

The restriction of the map (29) to 312-avoiding elements of $\mathfrak{S}_n$ is in fact rather simple. Given word $w = w_1 \cdots w_n$ with distinct letters, say that w has a *record* at position j if $w_j = \max\{w_1, \dots, w_j\}$.

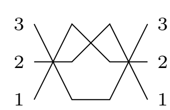
Algorithm 5.7. *Given* $w = w_1 \cdots w_n \in \mathfrak{S}_n$ *avoiding the pattern* 312*, do*

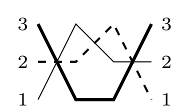
(1) *Let* w *have records at positions* $1 = j_1, \dots, j_k$.
(2) *Define* $F_w = F_{[j_k, w_{j_k}]} \bullet \cdots \bullet F_{[j_1, w_{j_1}]}$.

The bijection $F \mapsto w(F)$ is closely related to families of source-to-sink paths in F, and also to Kazhdan–Lusztig basis elements of the Hecke algebra of $\mathfrak{S}_{[h,l]}$. Given $F \in \mathcal{F}^{\mathsf{A}}([h,l])$, call a sequence $\pi = (\pi_h, \dots, \pi_l)$ of paths in F a *path family of type* $w = w_h \cdots w_l$ if for $i = h, \dots, l$, path π_i begins at source i and ends at sink w_i. Say that a path family π *covers* F if every edge of F appears in at least one path of π, and define the sets

$$\begin{gathered} \Pi(F) = \{\pi \,|\, \pi \text{ covers } F\}, \\ \Pi_w(F) = \{\pi \in \Pi(F) \,|\, \mathrm{type}(\pi) = w\}. \end{gathered} \tag{32}$$

For example, the star network and path family

(33) $F = F_{[1,3]} \circ F_{[2,3]} \circ F_{[1,3]} =$ 

, $\pi =$ 

belong to $\mathcal{F}^{\mathsf{A}}([1,3])$ and $\Pi_{s_1}(F) \subset \Pi(F)$, respectively. When F is a zig-zag network, we may characterize $w(F)$ in terms of $\Pi(F)$ as follows [61, Lem. 5.3].

Proposition 5.8. *For* $F \in \mathcal{S}^{\mathsf{A}}_{\mathsf{Z}}([h,l])$*,* $w(F)$ *is the unique permutation of maximum length in* $\{\mathrm{type}(\pi) \,|\, \pi \in \Pi(F)\} \subseteq \mathfrak{S}_{[h,l]}$.

For all $F \in \mathcal{F}^{\mathsf{A}}([h,l])$, the set $\Pi(F)$ associates an element of $\mathbb{Z}[\mathfrak{S}_{[h,l]}]$ to F: we say that F *graphically represents*

$$\sum_{\pi \in \Pi(F)} \mathrm{type}(\pi) \tag{34}$$

as an element of $\mathbb{Z}[\mathfrak{S}_{[h,l]}]$. For example, the network F in (33) can be covered by 72 different path families: 12 of each type $w \in \mathfrak{S}_3$. Thus it graphically represents $12\,\widetilde{C}_{s_1 s_2 s_1}(1)$ as an element of $\mathbb{Z}[\mathfrak{S}_3]$.

Again for all $F \in \mathcal{F}^{\mathsf{A}}([h,l])$, the set $\Pi(F)$ also associates an element of $H_{[h,l]}(q)$ to F. To describe this element explicitly, we first assume that F is formed by some iteration of ordinary or condensed concatenation of simple star networks $F_{[c_1,d_1]}, \dots, F_{[c_t,d_t]}$ with internal vertices $z_1, \dots, z_t$. Observe that the intersection of two source-to-sink paths π_i, π_j in F must be a disjoint union of the above internal vertices of F and paths between these. We say that π_i and π_j *meet* at a central vertex z_k if both paths contain z_k, and enter it via different edges. Given a path family π covering F, define a *defect* of π to be a triple (π_i, π_j, k) with

(1) $i < j$,
(2) π_i and π_j meet at vertex z_k of F after having crossed an odd number of times.

Let $\mathrm{dfct}(\pi)$ denote the number of defects of π. (This definition from [19] generalizes those of [14, 24].) We say that F *graphically represents*

$$\sum_{\pi \in \Pi(F)} q^{\mathrm{dfct}(\pi)} T_{\mathrm{type}(\pi)} \tag{35}$$

as an element of $H_{[h,l]}(q)$. For example, the path family π in (33) satisfies $\mathrm{dfct}(\pi) = 3$: the defects are $(\pi_1, \pi_2, 2)$, $(\pi_1, \pi_3, 3)$, $(\pi_2, \pi_3, 3)$. It is possible to show that the network F in (33) graphically represents $(1+q)^2(1+q+q^2)\widetilde{C}_{s_1s_2s_1}(q)$ as an element of $H_3(q)$.

It is clear that if F graphically represents $D(q) \in H_{[h,l]}(q)$ as an element of $H_{[h,l]}(q)$, then it graphically represents $D(1)$ as an element of $\mathbb{Z}[\mathfrak{S}_{[h,l]}]$. It is possible to show that all star networks graphically represent products of Kazhdan–Lusztig basis elements (possibly divided by integers or polynomials in q). In particular, the result [14, Thm. 1] shows that sometimes such a product consists of a single Kazhdan–Lusztig basis element, and that the star network is a *wiring diagram*, i.e. all intervals $[c_i, d_i]$ satisfy $d_i = c_i + 1$.

Proposition 5.9. *Let* $s_{c_1} \cdots s_{c_t}$ *be a reduced expression for* $w \in \mathfrak{S}_{[h,l]}$ *avoiding the patterns* 321*,* 56781234*,* 56718234*,* 46781235*,* 46718235*. Then the star network* $F_{[c_1,c_1+1]} \bullet \cdots \bullet F_{[c_t,c_t+1]}$ *graphically represents* $\widetilde{C}_w(q)$ *as an element of* $H_{[h,l]}(q)$*.*

The result [61, Lem. 5.3] shows that zig-zag networks give graphical representations of other Kazhdan–Lusztig basis elements.

Proposition 5.10. *For* $w \in \mathfrak{S}_{[h,l]}$ *avoiding the patterns* 3412 *and* 4231*, the zig-zag network* F_w *graphically represents* $\widetilde{C}_w(q)$ *as an element of* $H_{[h,l]}(q)$*.*

This fact has the following consequence.

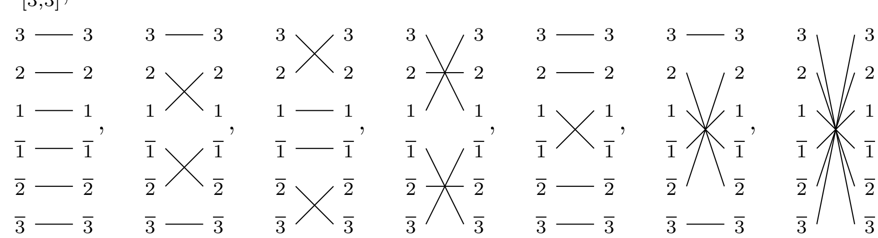
Corollary 5.11. *For* $v, w \in \mathfrak{S}_{[h,l]}$ *with* w *avoiding the patterns* 3412 *and* 4231*, the number of path families of type* v *covering* F_w *is* 1 *if* $v \leqslant_{\mathfrak{S}_{[h,l]}} w$*, and is* 0 *otherwise.*

5.2. Type-BC planar networks and factorization.

For fixed n, define *type-*BC *simple star networks with boundary vertices indexed by* $[\overline{n}, n]$ to be the type-A star networks

$$\begin{aligned} F'_{[a,b]} &:= F_{[a,b]} \bullet F_{[\overline{a},\overline{b}]} = F_{[a,b]} \circ F_{[\overline{a},\overline{b}]}, && 1 \leqslant a \leqslant b \leqslant n, \\ F'_{[\overline{a},a]} &:= F_{[\overline{a},a]}, && 1 \leqslant a \leqslant n, \end{aligned} \tag{36}$$

which correspond naturally to the type-BC reversals (7). For example the seven type-BC simple star networks $F'_{\varnothing} = F'_{[1,1]} = F'_{[2,2]} = F'_{[3,3]}$ and $F'_{[1,2]}$, $F'_{[2,3]}$, $F'_{[1,3]}$, $F'_{[\overline{1},1]}$, $F'_{[\overline{2},2]}$, $F'_{[\overline{3},3]}$,

(37)


correspond to the reversals $s'_{\varnothing}$, $s'_{[1,2]}$, $s'_{[2,3]}$, $s'_{[1,3]}$, $s'_{[\overline{1},1]}$, $s'_{[\overline{2},2]}$, $s'_{[\overline{3},3]}$. We refer to all iterations of concatenations and condensed concatenations of type-BC simple star networks as *type-*BC *star networks*, and let $\mathcal{F}^{\mathsf{BC}}([\overline{n}, n])$ denote the set of these having boundary vertices indexed by $[\overline{n}, n]$. We will be interested in three subsets of these formed by condensed concatenation of type-BC simple star networks.

Definition 5.12. *Define* $\mathcal{S}^{\mathsf{BC}}([\overline{n}, n])$ *to be the set of all type-*A *condensed star networks of the form*

$$F = F'_{[c_1,d_1]} \bullet \cdots \bullet F'_{[c_t,d_t]} \tag{38}$$

and call these type-BC condensed star networks (with boundary vertices indexed by $[\overline{n}, n]$).

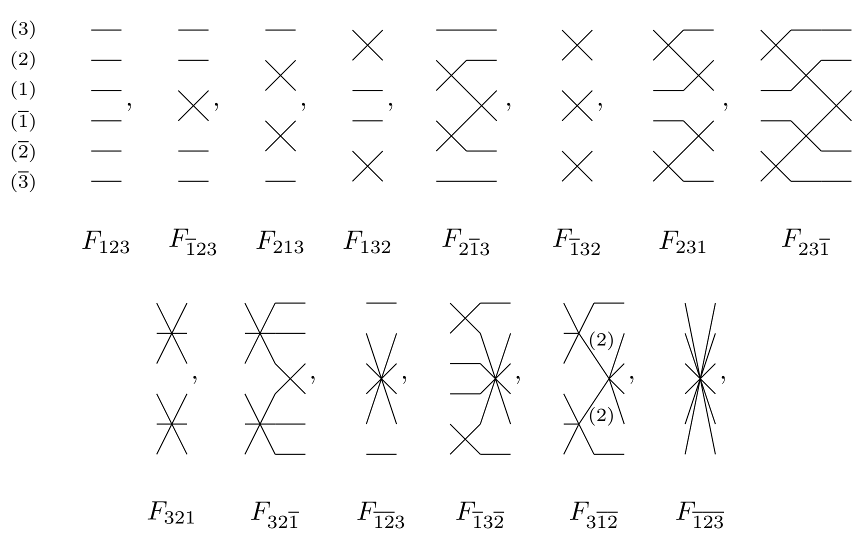
Definition 5.13. *Call a type-*BC *condensed star network* F (38) *a* type-BC zig-zag network *if the intervals* $[c_1, d_1], \dots, [c_t, d_t]$ *satisfy the conditions of Definition* 5.2*, i.e. if the type-*A *star network* $F_{[c_1,d_1]} \bullet \cdots \bullet F_{[c_t,d_t]}$ *is a type-*A *zig-zag network with boundary vertices indexed by* $[\min\{1, c_1, \dots, c_t\}, n]$*. Let* $\mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{n}, n])$ *denote the set of type-*BC *zig-zag networks with boundary vertices labeled by* $[\overline{n}, n]$*.*

Definition 5.14. *Call a type-*BC *star network* F (38) *a* type-BC descending star network *if the intervals* $[c_1, d_1], \dots, [c_t, d_t]$ *satisfy the conditions of Definition* 5.3*, i.e. if the type-*A *star network* $F_{[c_1,d_1]} \bullet \cdots \bullet F_{[c_t,d_t]}$ *is a type-*A *descending star network with boundary vertices indexed by* $[\min\{1, c_1, \dots, c_t\}, n]$*. Let* $\mathcal{S}^{\mathsf{BC}}_{\mathsf{D}}([\overline{n}, n])$ *denote the set of type-*BC *descending star networks with boundary vertices labeled by* $[\overline{n}, n]$*.*

Thus we have $\mathcal{S}^{\mathsf{BC}}_{\mathsf{D}}([\overline{n}, n]) \subseteq \mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{n}, n]) \subseteq \mathcal{S}^{\mathsf{A}}_{\mathsf{Z}}([\overline{n}, n])$, and each zig-zag network of type BC is F_w for some $w \in \mathfrak{S}_{[\overline{n},n]}$. By the symmetry of these networks, we necessarily have $w \in \mathfrak{B}_n$.

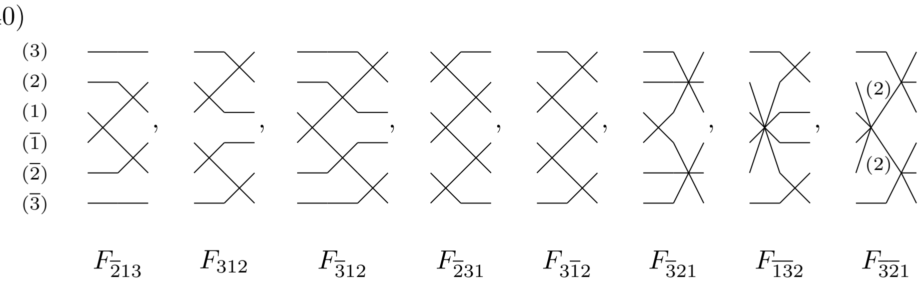
To illustrate, consider the set $\mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{3}, 3])$ of twenty-two type-BC zig-zag networks. Fourteen of these are type-BC descending star networks

(39)

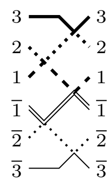
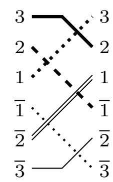


and eight are not,

(40)



By Corollary 5.11 and the containment $\mathfrak{B}_n \subseteq \mathfrak{S}_{[\overline{n},n]}$, we have that for all $F_w \in \mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{n}, n])$ and $v \in \mathfrak{B}_n$, at most one path family π of type v covers F_w. For $i > 0$, paths π_i and $\pi_{\overline{i}}$ in this family are necessarily mirror images of one another, and we call π_i *grounded* if it intersects path $\pi_{\overline{i}}$. For example consider $F_{3\overline{1}2}$ in (40) and the path families π of type 123 and σ of type $3\overline{1}2$ covering it,

(41) $\pi =$  , $\sigma =$  .

The path π_1 is grounded while π_2 and π_3 are not; the path σ_2 is grounded while σ_1 and σ_3 are not.

By (36) the intervals appearing in the construction of the type-BC star network G (38) are roughly half of those that appear in the type-A construction of the same network. The subposet of $\preceq$ induced by these intervals satisfies the following.

Proposition 5.15. *For $F'_{[a_1,b_1]} \bullet \cdots \bullet F'_{[a_t,b_t]}$ a type-BC zig-zag network, there is at most one interval $[a_i, b_i]$ satisfying $a_i < 0$, $b_i = \overline{a_i}$. Furthermore, this interval is maximal or minimal (or both) in the poset $\preceq$ on $\{[a_1, b_1], \dots, [a_t, b_t]\}$.*

Proof. Condition (1) of Definition 5.2 requires that the intervals $[a_1, b_1], \dots, [a_t, b_t]$ be distinct and form a nonnesting set. Thus at most one of these intervals satisfies $b_i = \overline{a_i}$. Let $[a_j, \overline{a_j}]$ be such an interval ($a_j < 0$) and suppose that it is neither maximal nor minimal in the partial order $\preceq$. Then there are indices i, k with $i < j < k$ and $[a_i, b_i] \cap [a_j, \overline{a_j}] \neq \varnothing$, $[a_k, b_k] \cap [a_j, \overline{a_j}] \neq \varnothing$. By Condition (2) of Definition 5.2, we must have $a_i < a_j < a_k$ or $a_i > a_j > a_k$. But this implies that $a_i < 0$ or $a_k < 0$, and therefore that $[a_j, \overline{a_j}]$ is properly contained in $[a_i, \overline{a_i}]$ or $[a_k, \overline{a_k}]$, contradicting Condition (1). $\square$

As a consequence, the cardinalities of $\mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n])$ and $\mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n}, n])$ are related to their type-A analogs, with the second cardinality equal to a Catalan number.

Theorem 5.16. *For all n we have*

(1) $|\mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n])| = |\mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([1, n+1])|$,
(2) $|\mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n}, n])| = |\mathcal{S}^{\mathsf{A}}_{\mathrm{D}}([1, n+1])| = \frac{1}{n+2}\binom{2n+2}{n+1}$.

Proof. (1) Define a map $\Upsilon : \mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n]) \to \mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([1, n+1])$ by $\Upsilon(F'_{[a_i,b_i]}) = F_{[\max\{a_i+1,1\}, b_i+1]}$ and

$$\Upsilon(F'_{[a_1,b_1]} \bullet \cdots \bullet F'_{[a_t,b_t]}) = \Upsilon(F'_{[a_1,b_1]}) \bullet \cdots \bullet \Upsilon(F'_{[a_t,b_t]}),$$

so that all positive endpoints of intervals increase by one and all negative endpoints are replaced by 1. To see that Υ is well defined, recall that by Proposition 5.15 at most one of the intervals $[a_i, b_i]$ satisfies $a_i < 0$. Thus the conditions of Definition 5.2 are satisfied and $\Upsilon(F)$ belongs to $\mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([1, n+1])$. Furthermore, for $F \in \mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n}, n])$, the inequalities $a_1 > \cdots > a_t$ imply that we have $a_1 + 1 > \cdots > \max\{a_t + 1, 1\}$ and $\Upsilon(F) \in \mathcal{S}^{\mathsf{A}}_{\mathrm{D}}([1, n+1])$. To see that Υ is bijective, observe that we have

$$\begin{aligned} [a_i, b_i] \subseteq [1, n] &\iff [a_i + 1, b_i + 1] \subseteq [2, n+1], \\ a_i = b_i \in [1, n] &\iff a_i + 1 = b_i + 1 \in [2, n+1], \\ [\overline{b_i}, b_i] \subseteq [\overline{n}, n] &\iff [1, b_i + 1] \subseteq [1, n+1]. \end{aligned}$$

Finally, by [18, Thm. 3.6] we have $|\mathcal{S}^{\mathsf{A}}_{\mathrm{D}}([1, n])| = \frac{1}{n+1}\binom{2n}{n}$. $\square$

In Theorems 5.18 – 5.19 we will characterize $\mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n])$ and $\mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n}, n])$ as subsets of $\{F_w \in \mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([\overline{n}, n]) \,|\, w \in \mathfrak{B}_n\}$ defined by w avoiding certain patterns. In order to do so, we decompose certain elements of $\mathfrak{B}_n$ into pairs $(u, v) \in \mathfrak{B}_k \times \mathfrak{S}_{n-k}$, and certain zig-zag networks in $\mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{n},n])$ into pairs of components in $\mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{k},k]) \times \mathcal{S}^{\mathsf{A}}_{\mathsf{Z}}([1,n-k])$. Define the map

$$\begin{aligned} \oplus : \mathfrak{B}_k \times \mathfrak{S}_{[1,n-k]} &\to \mathfrak{B}_n \\ (u,v) &\mapsto u \oplus v = w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n \end{aligned} \tag{42}$$

by

$$w_i = \begin{cases} u_i & \text{if } i \in [\overline{k},k], \\ v_i + k & \text{if } i > k, \\ \overline{v_i + k} & \text{if } i < \overline{k}. \end{cases}$$

For example, the elements $u = \overline{1} \in \mathfrak{B}_1$, $v = 231 \in \mathfrak{S}_3$ give $u \oplus v = \overline{243}1\overline{1}342 \in \mathfrak{B}_4$. Observe that to make sense of the more general expression $u \oplus v^{(1)} \oplus \cdots \oplus v^{(p)}$, we must interpret it as $(\cdots((u \oplus v^{(1)}) \oplus v^{(2)}) \oplus \cdots \oplus v^{(p-1)}) \oplus v^{(p)}$, with $u \in \mathfrak{B}_k$, $v^{(i)} \in \mathfrak{S}_{[1,j_i]}$ for some $k, j_1, \ldots, j_p$. We will say that any element $w \in \mathfrak{B}_n$ which can be written as $w = u \oplus v$ is $\oplus$-*decomposable*. Equivalently, $w \in \mathfrak{B}_n$ is $\oplus$-decomposable if there is some index k such that

$$\{|w_1|, \ldots, |w_k|\} = [1,k], \qquad \{w_{k+1}, \ldots, w_n\} = [k+1,n].$$

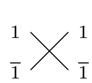
We define a similar map

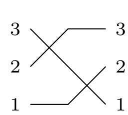
$$\begin{aligned} \oplus : \mathcal{S}^{\mathsf{BC}}([\overline{k},k]) \times \mathcal{S}^{\mathsf{A}}([1,n-k]) &\to \mathcal{S}^{\mathsf{BC}}([\overline{n},n]) \\ (E,F) &\mapsto E \oplus F \end{aligned} \tag{43}$$

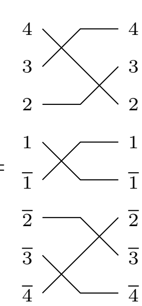
as follows.

(1) Create $F^+ \in \mathcal{S}^{\mathsf{A}}([k+1,n])$ by adding k to the indices of all sources and sinks of F.
(2) Create $F^- \in \mathcal{S}^{\mathsf{A}}([\overline{n},\overline{k+1}])$ by drawing F^+ upside-down and by multiplying each source and sink index by -1.
(3) Vertically arrange the sources and sinks of these networks and E in order $(\overline{n},\ldots,n)$, so that we have F^+ above E above F^-.

For example, to construct the network $F'_{[\overline{1},1]} \oplus (F_{[2,3]} \bullet F_{[1,2]})$, we place $F'_{[\overline{1},1]}$ between two copies of $F_{[2,3]} \bullet F_{[1,2]}$, one upside-down, to obtain

$F'_{[\overline{1},1]} =$  ,

$F_{[2,3]} \bullet F_{[1,2]} =$  ,

$F'_{[\overline{1},1]} \oplus (F_{[2,3]} \bullet F_{[1,2]}) =$  $\cong F'_{[\overline{1},1]} \bullet F'_{[2,3]} \bullet F'_{[1,2]}$.

Lemma 5.17. *For elements $u \in \mathfrak{B}_k$, $v \in \mathfrak{S}_{n-k}$ and zig-zag networks $F_u \in \mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{k},k])$, $F_v \in \mathcal{S}^{\mathsf{A}}_{\mathsf{Z}}([1,n-k])$, we have the following.*

(1) *$F_u \oplus F_v \in \mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{n},n])$ is a zig-zag network satisfying $w(F_u \oplus F_v) = u \oplus v$.*
(2) *If F_u and F_v are descending star networks, then so is $F_u \oplus F_v$.*

Proof. (1) To see that $F_u \oplus F_v$ belongs to $\mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{n},n])$, write

$$F_u = F'_{[c_1,d_1]} \bullet \cdots \bullet F'_{[c_t,d_t]}, \qquad F_v = F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_r,b_r]}.$$

By the definition (43) of the map $\oplus$ we have

$$F_u \oplus F_v = F'_{[c_1,d_1]} \bullet \cdots \bullet F'_{[c_t,d_t]} \bullet F'_{[a_1+k,b_1+k]} \bullet \cdots \bullet F'_{[a_r+k,b_r+k]}.$$

It is easy to see that the set $\{[a_1{+}k, b_1{+}k], \dots, [a_r{+}k, b_r{+}k]\}$ satisfies the conditions of Definition 5.2, and since each interval $[c_i, d_i]$ is disjoint from each interval $[a_j{+}k, b_j{+}k]$, the union $\{[c_1, d_1], \dots, [c_t, d_t], [a_1{+}k, b_1{+}k], \dots, [a_r{+}k, b_r{+}k]\}$ satisfies the conditions of Definition 5.2 as well.

Now let $w = w(F_u \oplus F_v) \in \mathfrak{B}_n$ and let $y = u \oplus v \in \mathfrak{B}_n$. To see that $w = y$, recall by Proposition 5.8 that w is the permutation in $\mathfrak{S}_{[\overline{n},n]}$ which maximizes $\textsc{inv}(z)$ over all $z \in \mathfrak{S}_{[\overline{n},n]}$ for which there is a path family of type z covering $F_u \oplus F_v$. By the disconnectedness of $F_u \oplus F_v$, we have

$$\{w_{\overline{k}}, \dots, w_k\} = [\overline{k}, k], \qquad \{w_{k+1}, \dots, w_n\} = [k{+}1, n], \qquad \{w_{\overline{n}}, \dots, w_{\overline{k+1}}\} = [\overline{n}, \overline{k+1}].$$

Since $w(F_u) = u$, it is clear that w has as many inversions as possible among entries $\overline{k}, \dots, k$ when $w_{\overline{k}} \cdots w_k = u_{\overline{k}} \cdots u_k$. Similarly, w has as many inversions as possible among entries $k{+}1, \dots, n$ when $w_{k+1} \cdots w_n$ matches the pattern $v_1 \cdots v_{n-k}$, i.e. when $w_{k+i} = v_i{+}k$ for $i = 1, \dots, n{-}k$. In this case, we also have $w_{\overline{n}} \cdots w_{\overline{k+1}} = \overline{w_n} \cdots \overline{w_{k+1}}$. Thus we have $w = u \oplus v$.

(2) The fact that $F_u \oplus F_v$ belongs to $\mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n}, n])$ follows immediately from Definition 5.14. ☐

Now we may characterize $\mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n])$ and $\mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n}, n])$ in terms of pattern avoidance.

Theorem 5.18. *Elements of* $\mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n])$ *correspond bijectively to* 3412*-avoiding,* 4231*-avoiding elements of* $\mathfrak{B}_n$*. Specifically we have*

$$\mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n]) = \{F_w \in \mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([\overline{n}, n]) \,|\, w \in \mathfrak{B}_n\}. \tag{44}$$

Proof. ($\subseteq$) Consider $F \in \mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n]) \subset \mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([\overline{n}, n])$. By [61, §3], F has the form F_w for some $w \in \mathfrak{S}_{[\overline{n},n]}$ avoiding the patterns 3412 and 4231, and factors as in (38) and Definition 5.2. By Proposition 5.15, at most one of the intervals $[c_i, d_i]$ appearing in (38) satisfies $c_i = \overline{d_i}$. If such an interval exists, then we may assume that it appears first or last. Thus we may factor F as

$$F'_{[c_0,d_0]} \bullet (F'_{[c_1,d_1]} \bullet F'_{[\overline{d_1},\overline{c_1}]}) \bullet \cdots \bullet (F'_{[c_t,d_t]} \bullet F'_{[\overline{d_t},\overline{c_t}]}) \bullet F'_{[c_{t+1},d_{t+1}]},$$

with one or both of the intervals $[c_0, d_0]$ and $[c_{t+1}, d_{t+1}]$ satisfying $c_i = d_i$, and at most one of these satisfying $c_i = \overline{d_i}$. Algorithm 5.6 then gives a reversal factorization of w. This factorization consists of the subsequence of reversals

$$(s_{[c_0,d_0]}, s_{[c_1,d_1]}, s_{[\overline{d_1},\overline{c_1}]}, \dots, s_{[c_t,d_t]}, s_{[\overline{d_t},\overline{c_t}]}, s_{[\overline{d_{t+1}},\overline{c_{t+1}}]}), \tag{45}$$

and more pairs of reversals

$$s_{[c_i,d_i]\cap[c_j,d_j]}, \quad s_{[\overline{d_i},\overline{c_i}]\cap[\overline{d_j},\overline{c_j}]} \tag{46}$$

inserted between these. Since the only intervals appearing in (45)) – (46)) which can contain both positive and negative integers are $[c_0, d_0]$, $[c_{t+1}, d_{t+1}]$, we may reorder the sequence of reversals to place each pair $s_{[a,b]}$ and $s_{[\overline{b},\overline{a}]}$ consecutively. Thus w equals a product of type-BC reversals of the forms $s_{[c_0,d_0]}$, $s'_{[a,b]} = s_{[a,b]} s_{[\overline{b},\overline{a}]}$, $s_{[c_{t+1},d_{t+1}]}$, and belongs to $\mathfrak{B}_n$.

($\supseteq$) We claim that for each element $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231, we have $F_w \in \mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n])$. This is true when $n = 1$ because $\mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([\overline{1}, 1]) = \{F_\varnothing, F_{[\overline{1},1]}\} = \mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{1}, 1])$. Now suppose that the statement is true for $w \in \mathfrak{B}_1, \dots, \mathfrak{B}_{n-1}$ and consider $w \in \mathfrak{B}_n$.

If w is $\oplus$-decomposable then we can write $w = u \oplus v$ for $u \in \mathfrak{B}_k$, $v \in \mathfrak{S}_{n-k}$, and $1 \leqslant k < n$. Then by induction we have $F_u \in \mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{k}, k])$ and $F_v \in \mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([1, n-k])$. By Lemma 5.17 the network $F_w \in \mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([\overline{n}, n])$ satisfies $F_w = F_u \oplus F_v \in \mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n])$.

If w is not $\oplus$-decomposable, then we may apply [61, Obs. 3.3] to find a zig-zag factorization of w (as in the paragraph following (28)) and to obtain an expression

(24) for F_w which satisfies the conditions of Definition 5.2. In particular, we compare the lengths ℓ, m of the longest decreasing prefixes of w and w^{-1} respectively,

$$w_{\overline{n}} > \cdots > w_{\overline{n-\ell+1}}, \qquad (w^{-1})_{\overline{n}} > \cdots > (w^{-1})_{\overline{n-m+1}}.$$

If $\ell = m$, then $w = s_{[\overline{n},n]}$ and F_w is the type-BC zig-zag network $F_{[\overline{n},n]}$. If $\ell < m$, then w has a type-A zig-zag factorization beginning with

$$s_{[\overline{n},\overline{n-\ell+1}]}s_{[\overline{n-m},\overline{n-\ell+1}]}s_{[\overline{n-m},k]}$$

for some $k > \overline{n-m}$, and the interval $[\overline{n}, \overline{n-\ell+1}]$ is $\prec$-minimal. By the $\mathfrak{B}_n$-skew-symmetry of w, it also has a type-A zig-zag factorization beginning with

$$s_{[n-\ell+1,n]}s_{[n-\ell+1,n-m]}s_{[\overline{k},n-m]},$$

and the interval $[n-\ell+1, n]$ is also $\prec$-minimal. In other words, we can write $w = s'_{[n-\ell+1,n]}w'$ for some $w' \in \mathfrak{B}_n$ satisfying $w'_i = i$ for $i = \overline{n}, \dots, \overline{n-m+1}, n-m+1, \dots, n$, i.e.

$$w' = w'_{\overline{n-m}} \cdots w'_{n-m} \oplus 1 \cdots m.$$

It follows that we have $F_w \cong F'_{[n-\ell+1,n]} \bullet F_{w'}$. By induction $F_{w'}$ is a type-BC zig-zag network, and so is F_w. □

Theorem 5.19. *Elements of $\mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n},n])$ correspond bijectively to elements of $\mathfrak{B}_n$ avoiding the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$. Specifically we have*

$$(47)\quad \mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n},n]) = \{F_w \in \mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([\overline{n},n]) \,|\, w \in \mathfrak{B}_n \textit{ avoids the signed patterns } 1\overline{2},\, \overline{2}1,\, \overline{21},\, 312,\, 3\overline{1}2\}.$$

Proof. First we observe that by Lemma 2.3, avoidance of the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$ implies avoidance of the unsigned patterns 3412 and 4231. Thus the right-hand side of (47) includes one zig-zag network F_w for every element $w \in \mathfrak{B}_n$ avoiding the five signed patterns. Next, consider the subset of $\mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n},n])$ consisting of networks F_w factoring as

$$(48)\qquad F'_{[c_1,d_1]} \bullet \cdots \bullet F'_{[c_t,d_t]}$$

with $c_1 > \cdots > c_t > 0$. By Proposition 5.5, these networks are precisely

$$\{F_u \in \mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([\overline{n},n]) \,|\, u \in \mathfrak{B}_n, u_1 \cdots u_n \text{ nonnegative and avoiding the pattern } 312\}.$$

Therefore we may prove the proposition by proving (47), restricting our attention on the right-hand-side to networks F_u with $u \in \mathfrak{B}_n$ having at least one negative letter in the subword $u_1 \cdots u_n$, and on the left-hand-side to networks $F_w \in \mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n},n])$ factoring as (48) with $c_1 > \cdots > c_t = \overline{d_t}$. Such networks F_w correspond bijectively to networks $F_v \in \mathcal{S}^{\mathsf{A}}_{\mathrm{D}}([c_t,n])$ factoring as

$$(49)\qquad F_v := F_{[c_1,d_1]} \bullet \cdots \bullet F_{[c_t,d_t]},$$

with v and w satisfying

$$(50)\qquad \begin{gathered} v_{\overline{d_t}} \cdots v_{\overline{2}}v_{\overline{1}} = d_t \cdots 21, \qquad v_1 \cdots v_n = w_1 \cdots w_n, \\ \{w_1, \dots, w_n\} = \{\overline{1}, \dots, \overline{d_t}, d_t+1, \dots, n\}. \end{gathered}$$

($\subseteq$) Consider $F_w \in \mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n},n]) \subset \mathcal{S}^{\mathsf{A}}_{\mathrm{Z}}([\overline{n},n])$ factoring as (48) with $c_1 > \cdots > c_t = \overline{d_t}$. Since the related network F_v (49) belongs to $\mathcal{S}^{\mathsf{A}}_{\mathrm{D}}([c_t,n])$, the element $v = v_{c_t} \cdots v_{\overline{1}}v_1 \cdots v_n \in \mathfrak{S}_{[c_t,n]}$ avoids the ordinary pattern 312. Thus $v_1 \cdots v_n = w_1 \cdots w_n$ avoids the signed patterns 312 and $3\overline{1}2$. By [61, Obs. 3.2] we have that $\overline{1} \cdots \overline{d_t}$ is a subword of $w_1 \cdots w_n$. Thus $w_1 \cdots w_n$ contains a negative letter and also avoids the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$.

($\supseteq$) Consider F_w on the right-hand side of (47) with $w_1 \cdots w_n$ containing a negative letter. By (44) we have $F_w \in \mathcal{S}_{\mathsf{Z}}^{\mathsf{BC}}([\overline{n}, n])$. By Proposition 5.15, there exists a factorization (48) of F_w in which exactly one interval $[c_i, d_i]$ satisfies $c_i = \overline{d_i}$, and this interval must be maximal or minimal (or both) in the partial order $\preceq$.

Assume that this interval is minimal and that $i = 1$. By [61, Obs. 3.2] we have

$$w_{\overline{d_i}} > \cdots w_{\overline{1}} > w_1 > \cdots > w_{d_i}, \tag{51}$$

with $w_{\overline{1}} > 0 > w_1$ since $w \in \mathfrak{B}_n$. If some positive letter $j \leqslant d_i$ does not appear in these positions, then $w_1 \cdots w_n$ contains either the subword $w_{d_i} j$ which matches the pattern $\overline{2}1$, or the subword $w_{d_i}\overline{j}$ which matches the pattern $\overline{21}$. This contradicts our choice of F_w. Thus the $2d_i$ letters in these positions must be $d_i > \cdots > 1 > \overline{1} > \cdots > \overline{d_i}$, and the interval $[\overline{d_i}, d_i]$ is both minimal and maximal. It follows that we have $w = s_{[\overline{d_i}, d_i]} \oplus u$ for some $u \in \mathfrak{S}_{[d_i+1,n]}$. Since w avoids the signed pattern 312, we have that u avoids the ordinary pattern 312, F_u belongs to $\mathcal{S}_{\mathsf{D}}^{\mathsf{A}}([1, n-d_i])$, and $F_w = F'_{[\overline{d_i}, d_i]} \oplus F_u$ belongs to $\mathcal{S}_{\mathsf{D}}^{\mathsf{BC}}([\overline{n}, n])$.

Now assume that the interval $[c_i, d_i] = [\overline{d_i}, d_i]$ in the factorization (48) of F_w is maximal with $i = t$. Define $F_v \in \mathcal{S}_{\mathsf{Z}}^{\mathsf{A}}([c_t, n])$ as in (49). We claim that $v_{c_t} \cdots v_{\overline{1}} v_1 \cdots v_n \in \mathfrak{S}_{[c_t,n]}$ avoids the ordinary pattern 312. To obtain a contradiction, assume that some subword $v_{j_1} v_{j_2} v_{j_3}$ matches the ordinary pattern 312. Suppose first that $j_1 \geqslant 1$. Then $w_{j_1} w_{j_2} w_{j_3}$ matches one of the signed patterns 312, $3\overline{1}2$, $3\overline{21}$, $\overline{132}$ and this contradicts the containment of F_w on the right-hand side of (47). Now suppose that $j_1 \leqslant \overline{1}$ and $j_2 \geqslant 1$. Then the letter v_{j_1} is positive by (50), and letters $\overline{1}, \ldots, \overline{v_{j_1}}$ appear in $v_1 \cdots v_n = w_1 \cdots w_n$. Since w avoids the signed pattern $\overline{21}$, it is impossible for letters $v_{j_2} v_{j_3}$ in $v_1 \cdots v_n$ to complete the ordinary pattern 312. Now suppose that $j_1 < j_2 \leqslant \overline{1}$. Since v_{j_1} and v_{j_2} are both positive, all of the letters $v_{j_2} + 1, \ldots, v_{j_1} - 1$ appear between these two letters, none can complete the pattern 312. Thus no subword of $v_{c_t} \cdots v_{\overline{1}} v_1 \cdots v_n$ matches the pattern 312, and F_w belongs to $\mathcal{S}_{\mathsf{D}}^{\mathsf{BC}}([\overline{n}, n])$. ☐

It is easy to see that the list of signed patterns in Theorem 5.19 cannot be shortened. For $w \in \mathfrak{B}_n \subset \mathfrak{S}_{[\overline{n},n]}$, failure to avoid the signed pattern $1\overline{2}$ or $\overline{21}$ implies failure to avoid the ordinary pattern 3412 or 4231, which implies that F_w is not a type-BC zig-zag network. Furthermore, inspection of $F_{\overline{2}1}$ ($F_{\overline{2}13}$ with highest and lowest edges removed), F_{312}, and $F_{3\overline{1}2}$ in (40) shows that these are not type-BC descending star networks.

Since any element $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231 can be viewed as a permutation in $\mathfrak{S}_{[\overline{n},n]}$ and any zig-zag network in $\mathcal{S}_{\mathsf{Z}}^{\mathsf{BC}}([\overline{n}, n])$ can be viewed as a zig-zag network in $\mathcal{S}_{\mathsf{Z}}^{\mathsf{A}}([\overline{n}, n])$, the bijection $F \mapsto w(F)$ guaranteed by Theorems 5.18 – 5.19 can be realized by Algorithm 5.6. The inverse $w \mapsto F_w$ of the map can be realized as in [61, §3], or as follows in the special case that w avoids the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$.

Algorithm 5.20. *Given $w \in \mathfrak{B}_n$ avoiding the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$, do*

(1) *Set*

$$v = \begin{cases} |h| \cdots 1 w_1 \cdots w_n & \text{if } w_1 \ldots w_n \text{ contains negative letters } h \cdots \overline{1}, \\ w_1 \cdots w_n & \text{if } w_1 \cdots w_n \text{ contains no negative letters.} \end{cases}$$

(2) *Apply Algorithm 5.7 to v to obtain $F_{[c_1,d_1]} \bullet \cdots \bullet F_{[c_t,d_t]}$.*
(3) *Set $F_w = F'_{[c_1,d_1]} \bullet \cdots \bullet F'_{[c_t,d_t]}$.*

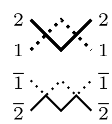
Theorems 5.18 and 5.19 suggest defining type-BC analogs of path families and graphical representation (34), (35). Given $F \in \mathcal{F}^{\mathsf{BC}}([\overline{n}, n])$, and $\pi =$

$(\pi_{\overline{n}}, \dots, \pi_{\overline{1}}, \pi_1, \dots, \pi_{\overline{n}})$ covering F, call π a BC-*path family* if for each factor $F'_{[a,b]}$ of F and each index $i \in [1, n]$, there exist indices j, k, such that paths π_i and $\pi_{\overline{i}}$ enter $F'_{[a,b]}$ via sources j, $\overline{j}$ and exit $F'_{[a,b]}$ via sinks k, $\overline{k}$, respectively. In other words, $\pi_{\overline{i}}$ must be a reflection of π_i. For $F \in \mathcal{F}^{\mathsf{BC}}([\overline{n}, n])$ and $u \in \mathfrak{B}_n$, define the sets

$$\Pi^{\mathsf{BC}}(F) = \{\pi \mid \pi \text{ a } \mathsf{BC}\text{-path family covering } F\},$$
$$\Pi^{\mathsf{BC}}_u(F) = \{\pi \in \Pi^{\mathsf{BC}}(F) \mid \mathrm{type}(\pi) = u\}. \tag{52}$$

For example, the two path families in (41) belong to $\Pi^{\mathsf{BC}}(F_{3\overline{1}2})$ with $\pi \in \Pi^{\mathsf{BC}}_e(F_{3\overline{1}2})$, $\sigma \in \Pi^{\mathsf{BC}}_{3\overline{1}2}(F_{3\overline{1}2})$. On the other hand, the path family

(53)



is not a BC-path family, even though it has type $e \in \mathfrak{B}_2$.

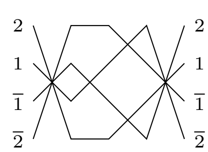
The set $\Pi^{\mathsf{BC}}(F)$ associates elements of $\mathbb{Z}[\mathfrak{B}_n]$ and $H^{\mathsf{BC}}_n(q)$ to F. Specifically, we say that F *graphically represents*

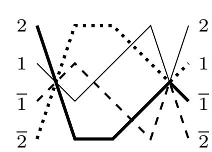
$$\sum_{\pi \in \Pi^{\mathsf{BC}}(F)} \mathrm{type}(\pi) \tag{54}$$

as an element of $\mathbb{Z}[\mathfrak{B}_n]$. To describe the corresponding element of $H^{\mathsf{BC}}_n(q)$ we first extend the definition of *type-*A *defects* from Subsection 5.1 (and [14, 19, 24]). Assume that F is formed by some iteration of ordinary or condensed concatenatation of simple star networks $F'_{[c_1,d_1]}, \dots, F'_{[c_t,d_t]}$. Each factor $F'_{[c_k,d_k]}$ contributes a single internal vertex if $c_k = \overline{d_k}$, and two such vertices otherwise. Given a BC-path family π covering F, define a *type-*BC *defect* of π to be a triple (π_i, π_j, k) with

(1) $|i| \leqslant j$,
(2) π_i and π_j meet at one of the internal vertices of $F'_{[c_k,d_k]}$ after having crossed an odd number of times.

(The first condition prevents the double-counting of path meetings which occur in pairs when $|i| \neq |j|$.) Let $\mathrm{dfct}^{\mathsf{BC}}(\pi)$ denote the number of type-BC defects of π. For example, consider the star network and path family

(55) $F'_{[\overline{2},2]} \circ F'_{[\overline{1},1]} \circ F'_{[1,2]} \circ F'_{[\overline{2},2]} =$ 

, $\quad \pi =$ 

.

The defects of π are $(\pi_{\overline{1}}, \pi_1, 2)$, $(\pi_{\overline{1}}, \pi_2, 3)$, $(\pi_1, \pi_2, 4)$, $(\pi_{\overline{2}}, \pi_2, 4)$, and we have $\mathrm{dfct}^{\mathsf{BC}}(\pi) = 4$. We say that F *graphically represents*

$$\sum_{\pi \in \Pi^{\mathsf{BC}}(F)} q^{\mathrm{dfct}^{\mathsf{BC}}(\pi)} T_{\mathrm{type}(\pi)} \tag{56}$$

as an element of $H^{\mathsf{BC}}_n(q)$. Specializing at $q = 1$, we see that if F graphically represents $D(q)$ as an element of $H^{\mathsf{BC}}_n(q)$, then it graphically represents $D(1)$ as an element of $\mathbb{Z}[\mathfrak{B}_n]$.

In the special case that $F = F_w \in \mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{n}, n])$, it graphically represents a Kazhdan–Lusztig basis element.

Theorem 5.21. *For* $w \in \mathfrak{B}_n$ *avoiding the patterns* 3412 *and* 4231*, the zig-zag network* F_w *represents* $\widetilde{C}^{\mathsf{BC}}_w(q)$ *as an element of* $H^{\mathsf{BC}}_n(q)$*.*

Proof. By [61, Lem. 5.3], we have that for all $u \in \mathfrak{S}_{[\overline{n},n]}$, there exists exactly one path family of type u covering F_w if $u \leqslant_{\mathfrak{S}_{[\overline{n},n]}} w$ and no such path family otherwise. In particular, this is true for $u \in \mathfrak{B}_n \subset \mathfrak{S}_{[\overline{n},n]}$. But by Proposition 2.1 we have $u \leqslant_{\mathfrak{B}_n} w$ if and only if $u \leqslant_{\mathfrak{S}_{[\overline{n},n]}} w$. Since w avoids the patterns 3412 and 4231, the network F_w belongs to $\mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{n},n])$, and every path family $\pi \in \Pi^{\mathsf{BC}}(F_w)$ satisfies $\mathrm{dfct}(\pi) = 0$. Thus the sum (56) becomes

$$\sum_{u \leqslant_{\mathfrak{B}_n} w} T_u,$$

which by (3.3) is $\widetilde{C}^{\mathsf{BC}}_w(q)$. □

Corollary 5.22. *For $v, w \in \mathfrak{B}_n$ with w avoiding the patterns 3412 and 4231, the number of BC-path families of type v covering F_w is 1 if $v \leqslant_{\mathfrak{B}_n} w$, and is 0 otherwise.*

6. Immanants and total nonnegativity

In order to use Section 5 to produce partial solutions to Problem 1.1 for the subsets (16) – (19) of the Kazhdan–Lusztig bases, we rely heavily upon methods borrowed from the study of total nonnegativity and upon trace generating functions in a ring $\mathbb{Z}[\mathbf{x}]$ where $\mathbf{x} = (\mathbf{x}_{i,j})_{i,j \in [\overline{n},n]}$ is viewed as the $2n \times 2n$ matrix

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_{\overline{n},\overline{n}} & \cdots & \mathbf{x}_{\overline{n},\overline{1}} & \mathbf{x}_{\overline{n},1} & \cdots & \mathbf{x}_{\overline{n},n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbf{x}_{\overline{1},\overline{n}} & \cdots & \mathbf{x}_{\overline{1},\overline{1}} & \mathbf{x}_{\overline{1},1} & \cdots & \mathbf{x}_{\overline{1},n} \\ \mathbf{x}_{1,\overline{n}} & \cdots & \mathbf{x}_{1,\overline{1}} & \mathbf{x}_{1,1} & \cdots & \mathbf{x}_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbf{x}_{n,\overline{n}} & \cdots & \mathbf{x}_{n,\overline{1}} & \mathbf{x}_{n,1} & \cdots & \mathbf{x}_{n,n} \end{bmatrix}. \tag{57}$$

For subsets $I, J \subseteq [\overline{n}, n]$ we define the submatrix $\mathbf{x}_{I,J} := (\mathbf{x}_{i,j})_{i \in I, j \in J}$. To economize notation, we abbreviate

$$[n] := [1, n]. \tag{58}$$

Thus $\mathbf{x}_{[n],[n]}$ denotes the submatrix of positively indexed entries of $\mathbf{x}$. Given polynomial $p(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$, and $2n \times 2n$ matrix $A = (a_{i,j})_{i,j \in [\overline{n},n]}$, we define $p(A)$ to be the expression obtained by evaluating $p(\mathbf{x})$ at $\mathbf{x}_{i,j} = a_{i,j}$, for all $i, j \in [\overline{n}, n]$.

6.1. Type-A immanants.

For certain $\theta \in \mathcal{T}(\mathfrak{S}_n)$ and for all $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231, combinatorial formulas for $\theta(\widetilde{C}_w(1))$ depend upon generating functions which are polynomials in entries of the submatrix $\mathbf{x}_{[n],[n]}$ of $\mathbf{x}$. Following Littlewood [51] and Stanley [65], we define the *(type-*A*) θ-immanant* to be

$$\mathrm{Imm}^{\mathfrak{S}_n}_\theta(\mathbf{x}_{[n],[n]}) := \sum_{w \in \mathfrak{S}_n} \theta(w) \mathbf{x}_{1,w_1} \cdots \mathbf{x}_{n,w_n} \in \mathbb{Z}[\mathbf{x}_{1,1}, \mathbf{x}_{1,2}, \dots, \mathbf{x}_{n,n}]. \tag{59}$$

When θ is an induced one-dimensional character η^λ or ϵ^λ with $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, we may neatly express its corresponding immanant in terms of permanents or determinants, and *ordered set partitions of type* λ, i.e. sequences $(J_1, \dots, J_r)$ of subsets of $[n]$ with

1. $J_1 \uplus \cdots \uplus J_r = [n]$,
2. $|J_i| = \lambda_i$ for $i = 1, \dots, r$.

In particular, we have the Littlewood–Merris–Watkins identities [51, 53],

$$\begin{aligned}\mathrm{Imm}^{\mathfrak{S}_n}_{\epsilon^\lambda}(\mathbf{x}_{[n],[n]}) &= \sum_{(J_1,\dots,J_r)} \det(\mathbf{x}_{J_1,J_1})\cdots\det(\mathbf{x}_{J_r,J_r}),\\ \mathrm{Imm}^{\mathfrak{S}_n}_{\eta^\lambda}(\mathbf{x}_{[n],[n]}) &= \sum_{(J_1,\dots,J_r)} \mathrm{per}(\mathbf{x}_{J_1,J_1})\cdots\mathrm{per}(\mathbf{x}_{J_r,J_r}),\end{aligned} \tag{60}$$

where the sums are over ordered set partitions $(J_1,\dots,J_r)$ of $[n]$ of type λ. (See [45, Thm. 2.1] for a q-analog.) We also have

$$\mathrm{Imm}^{\mathfrak{S}_n}_{\psi^\lambda}(\mathbf{x}_{[n],[n]}) = z_\lambda \sum_{\substack{w\\ \mathrm{ctype}(w)=\lambda}} \mathbf{x}_{1,w_1}\cdots\mathbf{x}_{n,w_n}, \tag{61}$$

where z_λ is defined as in (8). (See [31] for work on $\mathrm{Imm}^{\mathfrak{S}_n}_{\chi^\lambda}(\mathbf{x}_{[n],[n]})$.)

Immanants and trace evaluations of the form $\theta(\widetilde{C}_w(1))$ are connected by the following identity [18, Eqn. (3.5)].

Theorem 6.1. *Fix $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231 with corresponding zig-zag network F_w having path matrix $A = A(w)$. Then for any linear functional $\theta : \mathbb{C}[\mathfrak{S}_n] \to \mathbb{C}$ we have*

$$\theta(\widetilde{C}_w(1)) = \mathrm{Imm}^{\mathfrak{S}_n}_\theta(A), \tag{62}$$

where $\mathrm{Imm}^{\mathfrak{S}_n}_\theta(A)$ should be interpreted as $\mathrm{Imm}^{\mathfrak{S}_n}_\theta(\mathbf{x}_{[n],[n]})$ evaluated at $\mathbf{x}_{i,j} = a_{i,j}$.

Thus each combinatorial interpretation of $\mathrm{Imm}^{\mathfrak{S}_n}_\theta(A)$ yields a combinatorial interpretation of $\theta(\widetilde{C}_w(1))$. To produce such combinatorial interpretations, we appeal to methods of total nonnegativity, namely, Lindstrom's Lemma [40, 50] and some simple extensions.

Proposition 6.2. *Fix $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231 with corresponding zig-zag network F_w having path matrix $A = A(w)$. We have*

$$\mathrm{Imm}^{\mathfrak{S}_n}_{\epsilon^n}(A) = \det(A) = \#\{\pi \in \Pi_e(F_w) \,|\, \pi_1,\dots,\pi_n \text{ pairwise nonintersecting}\}, \tag{63}$$

$$\mathrm{Imm}^{\mathfrak{S}_n}_{\eta^n}(A) = \mathrm{per}(A) = \#\Pi(F_w), \tag{64}$$

$$\mathrm{Imm}^{\mathfrak{S}_n}_{\psi^n}(A) = n \cdot \#\{\pi \in \Pi_u(F_w) \,|\, u \in \mathfrak{S}_n, \mathrm{ctype}(u) = n\}. \tag{65}$$

Proposition 6.2 implies simple interpretations of $\mathrm{Imm}^{\mathfrak{S}_n}_{\epsilon^\lambda}(A)$, $\mathrm{Imm}^{\mathfrak{S}_n}_{\eta^\lambda}(A)$, $\mathrm{Imm}^{\mathfrak{S}_n}_{\psi^\lambda}(A)$ as well, for $\lambda \vdash n$ arbitrary. We will return to these in Subsection 9.1. For q-analogs, see [18].

6.2. Type-BC immanants.

To create a generating function for $\theta \in \mathcal{T}(\mathfrak{B}_n)$, we define the *(type-BC) θ-immanant* to be

$$\mathrm{Imm}^{\mathfrak{B}_n}_\theta(\mathbf{x}) := \sum_{w\in\mathfrak{B}_n} \theta(w)\mathbf{x}_{\overline{n},w_{\overline{n}}}\cdots\mathbf{x}_{\overline{1},w_{\overline{1}}}\mathbf{x}_{1,w_1}\cdots\mathbf{x}_{n,w_n} \in \mathbb{Z}[\mathbf{x}]. \tag{66}$$

This is a special case of the *wreath product immanant* defined in [63, Eqns. (26)–(27)], and generalizes the Littlewood – Stanley immanant (59). When θ is an induced character of the form $(\zeta\otimes\delta\xi)\big\uparrow^{\mathfrak{B}_n}_{\mathfrak{B}_m\times\mathfrak{B}_{n-m}}$ for symmetric group characters ζ, ξ, as in (13)) with $q=1$, then we may neatly express its corresponding immanant in terms of type-A immanants and $n\times n$ matrices $\mathbf{x}^+ = (\mathbf{x}^+_{i,j})_{i,j\in[n]}$, $\mathbf{x}^- = (\mathbf{x}^-_{i,j})_{i,j\in[n]}$ defined in terms of the $2n\times 2n$ matrix $\mathbf{x}$ (57) by

$$\mathbf{x}^+_{i,j} = \mathbf{x}_{i,j}\mathbf{x}_{\overline{i},\overline{j}} + \mathbf{x}_{i,\overline{j}}\mathbf{x}_{\overline{i},j}, \qquad \mathbf{x}^-_{i,j} = \mathbf{x}_{i,j}\mathbf{x}_{\overline{i},\overline{j}} - \mathbf{x}_{i,\overline{j}}\mathbf{x}_{\overline{i},j}. \tag{67}$$

For $I, J \subseteq [n]$, we let $\mathbf{x}^+_{I,J} := (\mathbf{x}^+)_{I,J}$ and $\mathbf{x}^-_{I,J} := (\mathbf{x}^-)_{I,J}$ denote the I, J submatrices of these. For example,

$$\mathbf{x}^+_{12,14} = \begin{bmatrix} \mathbf{x}_{1,1}\mathbf{x}_{\overline{1},\overline{1}} + \mathbf{x}_{1,\overline{1}}\mathbf{x}_{\overline{1},1} & \mathbf{x}_{1,4}\mathbf{x}_{\overline{1},\overline{4}} + \mathbf{x}_{1,\overline{4}}\mathbf{x}_{\overline{1},4} \\ \mathbf{x}_{2,1}\mathbf{x}_{\overline{2},\overline{1}} + \mathbf{x}_{2,\overline{1}}\mathbf{x}_{\overline{2},1} & \mathbf{x}_{2,4}\mathbf{x}_{\overline{2},\overline{4}} + \mathbf{x}_{2,\overline{4}}\mathbf{x}_{\overline{2},4} \end{bmatrix}.$$

By [63, Thm. 3.1] we have for bipartitions $(\lambda, \mu) \vdash n$ with $|\lambda| = k$ that

$$\mathrm{Imm}^{\mathfrak{B}_n}_{(\epsilon\epsilon)^{\lambda,\mu}}(\mathbf{x}) = \sum_I \mathrm{Imm}^{\mathfrak{S}_k}_{\epsilon^\lambda}(\mathbf{x}^+_{I,I}) \mathrm{Imm}^{\mathfrak{S}_{n-k}}_{\epsilon^\mu}(\mathbf{x}^-_{[n]\smallsetminus I,[n]\smallsetminus I}), \tag{68}$$

where the sum is over all m-element subsets I of $[n]$. More generally we have the following formula, which is a type-BC analog of [70, Prop. 2.4].

Lemma 6.3. *Given symmetric group traces $\zeta \in \mathcal{T}(\mathfrak{S}_k)$, $\xi \in \mathcal{T}(\mathfrak{S}_{n-k})$, and hyperoctahedral group trace $\theta \in \mathcal{T}(\mathfrak{B}_n)$ satisfying $\theta = (\zeta \otimes \delta\xi)\big\uparrow^{\mathfrak{B}_n}_{\mathfrak{B}_{k,n-k}}$, we have*

$$\mathrm{Imm}^{\mathfrak{B}_n}_\theta(\mathbf{x}) = \sum_{\substack{I \subseteq [n] \\ |I| = k}} \mathrm{Imm}^{\mathfrak{S}_k}_\zeta(\mathbf{x}^+_{I,I}) \mathrm{Imm}^{\mathfrak{S}_{n-k}}_\xi(\mathbf{x}^-_{[n]\smallsetminus I,[n]\smallsetminus I}). \tag{69}$$

Proof. Expand ζ, ξ in the induced sign character bases of $\mathcal{T}(\mathfrak{S}_k)$, $\mathcal{T}(\mathfrak{S}_{n-k})$ as

$$\zeta = \sum_{\lambda \vdash k} a_\lambda \epsilon^\lambda, \qquad \xi = \sum_{\mu \vdash n-k} b_\mu \epsilon^\mu.$$

Then we have

$$\theta = \Big(\sum_{\lambda \vdash k} a_\lambda \epsilon^\lambda \otimes \delta \sum_{\mu \vdash n-k} b_\mu \epsilon^\mu\Big)\big\uparrow^{\mathfrak{B}_n}_{\mathfrak{B}_{k,n-k}} = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} a_\lambda b_\mu (\epsilon^\lambda \otimes \delta\epsilon^\mu)\big\uparrow^{\mathfrak{B}_n}_{\mathfrak{B}_{k,n-k}} = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} a_\lambda b_\mu (\epsilon\epsilon)^{\lambda,\mu},$$

and the left-hand side of (69) is

$$\sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} a_\lambda b_\mu \mathrm{Imm}^{\mathfrak{B}_n}_{(\epsilon\epsilon)^{\lambda,\mu}}(\mathbf{x}).$$

But by (68), this is

$$\sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} a_\lambda b_\mu \sum_{\substack{I \subseteq [n] \\ |I| = k}} \mathrm{Imm}^{\mathfrak{S}_k}_{\epsilon^\lambda}(\mathbf{x}^+_{I,I})\, \mathrm{Imm}^{\mathfrak{S}_{n-k}}_{\epsilon^\mu}(\mathbf{x}^-_{[n]\smallsetminus I,[n]\smallsetminus I})$$
$$= \sum_{\substack{I \subseteq [n] \\ |I| = k}} \sum_{\lambda \vdash k} a_\lambda \mathrm{Imm}^{\mathfrak{S}_k}_{\epsilon^\lambda}(\mathbf{x}^+_{I,I}) \sum_{\mu \vdash n-k} b_\mu \mathrm{Imm}^{\mathfrak{S}_{n-k}}_{\epsilon^\mu}(\mathbf{x}^-_{[n]\smallsetminus I,[n]\smallsetminus I}),$$

which is the right-hand side of (69). □

Evaluating the immanants (66) at path matrices of type-BC zig-zag networks

$$\{F_w \,|\, w \in \mathfrak{B}_n \text{ avoids the patterns 3412 and 4231}\}$$

gives the following type-BC analog of (62) which allows us to use type-BC immanants to compute trace evaluations of the form $\theta(\widetilde{C}^{\mathsf{BC}}_w(1))$.

Theorem 6.4. *Let $w \in \mathfrak{B}_n$ avoid the patterns 3412 and 4231, and let zig-zag network F_w have path matrix A. Then for any linear functional $\theta : \mathfrak{B}_n \to \mathbb{C}$ we have*

$$\theta(\widetilde{C}^{\mathsf{BC}}_w(1)) = \mathrm{Imm}^{\mathfrak{B}_n}_\theta(A). \tag{70}$$

Proof. The right-hand side of (70) is

$$\sum_{v \in \mathfrak{B}_n} \theta(v) a_{\overline{n},v_{\overline{n}}} \cdots a_{\overline{1},v_{\overline{1}}} a_{1,v_1} \cdots a_{n,v_n}. \tag{71}$$

Since F_w is a type-BC zig-zag network of order $2n$, it is also a type-A zig-zag network of order $2n$. By [61, Lem. 5.3], the product $a_{\overline{n},v_{\overline{n}}} \cdots a_{\overline{1},v_{\overline{1}}} a_{1,v_1} \cdots a_{n,v_n}$ is 1 when $v \leqslant_{\mathfrak{S}_{[\overline{n},n]}} w$ and is 0 otherwise. Thus by Proposition 2.1 it is 1 when $v \leqslant_{\mathfrak{B}_n} w$ and is 0 otherwise, and the sum (71) is

$$\sum_{v \leqslant_{\mathfrak{B}_n} w} \theta(v) = \theta\Big(\sum_{v \leqslant_{\mathfrak{B}_n} w} v\Big) = \theta(\widetilde{C}^{\mathsf{BC}}_w(1)). \qquad \square$$

Thus each combinatorial interpretation of $\mathrm{Imm}^{\mathfrak{B}_n}_\theta(A)$ yields a combinatorial interpretation of $\theta(\widetilde{C}^{\mathsf{BC}}_w(1))$. Taking the special cases of Lemma 6.3 corresponding to ζ, ξ equal to triv, sgn, or ψ^n and evaluating $\mathfrak{S}_n$-immanants at A^+ and A^-, we have the following type-BC analogs of the sets of path families appearing in Proposition 6.2.

Proposition 6.5. *Fix $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231, let $F_w \in \mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{n},n])$ have path matrix A, and define A^+, A^- as in (67). We have*

$$\mathrm{per}(A^+) = \#\Pi^{\mathsf{BC}}(F_w), \tag{72}$$

$$\begin{aligned}\mathrm{per}(A^-) &= \#\{\pi \in \Pi^{\mathsf{BC}}(F_w) \,|\, \pi_i, \pi_j \text{ may share a vertex only if } i,j<0 \text{ or } i,j>0\} \\ &= \begin{cases} \#\Pi^{\mathsf{BC}}(F_w) & \text{if } \ell_t(w)=0, \\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{73}$$

$$\begin{aligned}\det(A^+) &= \#\{\pi \in \Pi^{\mathsf{BC}}(F_w) \,|\, \pi_i, \pi_j \text{ may share a vertex only if } -1 \leqslant i,j \leqslant 1\} \\ &= \begin{cases} 2^{\ell(w)} & \text{if } w \in \{e,t\}, \\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{74}$$

$$\begin{aligned}\det(A^-) &= \#\{\pi \in \Pi^{\mathsf{BC}}(F_w) \,|\, \pi_i, \pi_j \text{ are vertex-disjoint for all } i \neq j\} \\ &= \begin{cases} 1 & \text{if } w=e, \\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{75}$$

$$\mathrm{Imm}^{\mathfrak{S}_n}_{\psi^n}(A^+) = n \cdot \#\{\pi \in \Pi^{\mathsf{BC}}_u(F_w) \,|\, u \in \mathfrak{B}_n, \mathrm{ctype}(\varphi(u)) = n\}, \tag{76}$$

$$\mathrm{Imm}^{\mathfrak{S}_n}_{\psi^n}(A^-) = \begin{cases} n \cdot \#\{\pi \in \Pi^{\mathsf{BC}}_u(F_w) \,|\, u \in \mathfrak{B}_n, \mathrm{ctype}(\varphi(u)) = n\}, & \text{if } \ell_t(w)=0, \\ 0 & \text{otherwise.}\end{cases} \tag{77}$$

Proof. Define ℓ_t, ℓ_s, φ as in Subsections 2.1 – 2.2. Observe that for $v \in \mathfrak{S}_n$ we have

$$a^+_{1,v_1} \cdots a^+_{n,v_n} = \sum_{\substack{u \in \mathfrak{B}_n \\ \varphi(u)=v}} a_{\overline{n},u_{\overline{n}}} \cdots a_{\overline{1},u_{\overline{1}}} a_{1,u_1} \cdots a_{n,u_n} = \sum_{\substack{u \in \mathfrak{B}_n \\ \varphi(u)=v}} |\Pi^{\mathsf{BC}}_u(F_w)|,$$

$$a^-_{1,v_1} \cdots a^-_{n,v_n} = \sum_{\substack{u \in \mathfrak{B}_n \\ \varphi(u)=v}} (-1)^{\ell_t(u)} a_{\overline{n},u_{\overline{n}}} \cdots a_{\overline{1},u_{\overline{1}}} a_{1,u_1} \cdots a_{n,u_n} = \sum_{\substack{u \in \mathfrak{B}_n \\ \varphi(u)=v}} (-1)^{\ell_t(u)} |\Pi^{\mathsf{BC}}_u(F_w)|.$$

Thus we have

(78)

$$\mathrm{per}(A^+) = \sum_{u \in \mathfrak{B}_n} |\Pi^{\mathsf{BC}}_u(F_w)|, \qquad \mathrm{per}(A^-) = \sum_{u \in \mathfrak{B}_n} (-1)^{\ell_t(u)} |\Pi^{\mathsf{BC}}_u(F_w)|,$$
$$\det(A^+) = \sum_{u \in \mathfrak{B}_n} (-1)^{\ell_s(u)} |\Pi^{\mathsf{BC}}_u(F_w)|, \qquad \det(A^-) = \sum_{u \in \mathfrak{B}_n} (-1)^{\ell(u)} |\Pi^{\mathsf{BC}}_u(F_w)|,$$
$$\mathrm{Imm}^{\mathfrak{S}_n}_{\psi^n}(A^+) = \sum_{\substack{u \in \mathfrak{B}_n \\ \mathrm{ctype}(\varphi(u)) = n}} n |\Pi^{\mathsf{BC}}_u(F_w)|, \qquad \mathrm{Imm}^{\mathfrak{S}_n}_{\psi^n}(A^-) = \sum_{\substack{u \in \mathfrak{B}_n \\ \mathrm{ctype}(\varphi(u)) = n}} (-1)^{\ell_t(u)} n |\Pi^{\mathsf{BC}}_u(F_w)|.$$

By Corollary 5.22 the cardinality $|\Pi^{\mathsf{BC}}_u(F_w)|$ is 1 if $u \leqslant_{\mathfrak{B}_n} w$ and is 0 otherwise.

The interpretations (72), (76) follow from the subtraction-free expressions for $\mathrm{per}(A^+)$ and $\mathrm{Imm}^{\mathfrak{S}_n}_{\psi^n}(A^+)$ in (78).

Now consider the interpretations (73), (77). If $\ell_t(w) = 0$, then all elements $u \leqslant_{\mathfrak{B}_n} w$ also satisfy $\ell_t(u) = 0$. Thus the expressions for $\mathrm{per}(A^-)$ and $\mathrm{Imm}^{\mathfrak{S}_n}_{\psi^n}(A^-)$ in (78) are subtraction-free and have the claimed interpretations. Furthermore, since there is no path in F_w from source 1 to sink $\overline{1}$ (or source $\overline{1}$ to sink 1), in any path family π covering F_w paths π_i and π_j cannot intersect unless $i, j < 0$ or $i, j > 0$. On the other hand if $\ell_t(w) \neq 0$, then F_w has a factorization of the form (36) which begins or ends with $F^{\prime}_{[\overline{k},k]}$ for some k. If the factorization begins with $F^{\prime}_{[\overline{k},k]}$, define an involution on $\Pi^{\mathsf{BC}}(F_w)$ by $\pi \mapsto \pi'$ where π' is obtained from π by swapping paths π_1 and $\pi_{\overline{1}}$ after they touch at the central vertex of $F^{\prime}_{[\overline{k},k]}$. This map satisfies

$$\mathrm{ctype}(\varphi(\mathrm{type}(\pi'))) = \mathrm{ctype}(\varphi(\mathrm{type}(\pi))), \qquad \mathrm{type}(\pi') = t \cdot \mathrm{type}(\pi).$$

Thus the two families contribute to the expressions for $\det(A^-)$ and $\mathrm{Imm}^{\mathfrak{S}_n}_{\psi^n}(A^-)$ in (78), specifically contributing

$$(-1)^{\ell_t(\mathrm{type}(\pi))} + (-1)^{\ell_t(\mathrm{type}(\pi))+1} = 0$$

to each. If the factorization of F_w ends with $F^{\prime}_{[\overline{k},k]}$, form π' from π by swapping the final portions (from the central vertex of $F^{\prime}_{[\overline{k},k]}$ to the end) of paths terminating at sinks 1, $\overline{1}$. Then we have $\mathrm{type}(\pi') = \mathrm{type}(\pi) \cdot t$ and again the two families together contribute 0 to $\mathrm{per}(A^-)$ and to $\mathrm{Imm}_{\psi^n}(A^-)$.

Now consider the interpretation (74). If $\ell_s(w) = 0$, then we have $w \in \{e, t\}$ and the third sum in (78) is subtraction-free. It has two terms equal to 1 if $w = t$, and one such term if $w = e$. On the other hand, if $\ell_s(w) \neq 0$, then F_w has a factorization of the form (36) which contains at least one factor of the form $F'_{[k_1,k_2]}$ with $1 \leqslant k_1 < k_2 \leqslant n$ and with $[k_1, k_2]$ maximal or minimal with respect to $\preceq$. If $[k_1, k_2]$ is minimal, define an involution on $\Pi^{\mathsf{BC}}(F_w)$ by $\pi \mapsto \pi'$ where π' is obtained from π by swapping paths π_{k_1} and π_{k_1+1} (and $\pi_{\overline{k_1}}$ and $\pi_{\overline{k_1+1}}$) after they intersect at the central vertices of $F'_{[k_1,k_2]}$. Then we have $\mathrm{type}(\pi') = s'_{k_1} \cdot \mathrm{type}(\pi)$ and the two families together contribute

$$(-1)^{\ell_s(\mathrm{type}(\pi))} + (-1)^{\ell_s(\mathrm{type}(\pi))\pm 1} = 0$$

to $\det(A^+)$. If $[k_1, k_2]$ is maximal, then form π' from π by swapping the final portions (from the central vertices of $F'_{[k_1,k_2]}$ to the end) of the paths terminating at sinks k_1, $k_1 + 1$ (and $\overline{k_1}$, $\overline{k_1+1}$). Then we have $\mathrm{type}(\pi') = \mathrm{type}(\pi) \cdot s'_{k_1}$ and the two families together contribute 0 to $\det(A^+)$.

Finally consider the interpretation (75). Repeating either of the above arguments with $\ell(w)$ in place of $\ell_t(w)$ or $\ell_s(w)$, we see that any network F_w with $w \neq e$ leads to a bijection in which all pairs of paths families contribute 0. The only path families which are counted by $\det(A^-)$ are those of type e covering the network F_e. □

It would be interesting to define an appropriate noncommutative ring in the variables (57) to extend the above results for $\mathfrak{B}_n$-characters to analogous results for $H_n^{\mathsf{BC}}(q)$-characters.

Problem 6.6. *State and prove q-analogs of Lemma* 6.3 – *Proposition* 6.5.

7. Unit interval orders

More partial solutions to Problem 1.1 for the subsets (16) – (19) of the Kazhdan–Lusztig bases employ posets called *unit interval orders*, those posets for which no induced four-element subposet is isomorphic to a disjoint union of two two-element chains ($\mathbf{2}+\mathbf{2}$) or of a three-element chain and a single element ($\mathbf{3}+\mathbf{1}$).

In type A, a map $w \mapsto P(w)$ from 3412-avoiding, 4231-avoiding permutations in $\mathfrak{S}_n$ to unit interval orders facilitates combinatorial interpretations of trace evaluations [18, §4–10]. The restriction of this map to 312-avoiding permutations is bijective. In types B and C, we define an analogous map $w \mapsto Q(w)$ from 3412-avoiding, 4231-avoiding elements of $\mathfrak{B}_n$ to posets we call *type-*BC *unit interval orders*. The restriction of this map to elements avoiding the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$ is bijective. These graphical representations facilitate combinatorial interpretation of trace evaluations (Section 9) when we specialize at $q = 1$.

7.1. Type-A unit interval orders. Fix $w \in \mathfrak{S}_{[h,n]}$ ($h \in \{\overline{n}, 1\}$) avoiding the patterns 3412 and 4231, and let F_w be the planar network corresponding to w by the bijection following (29), i.e. in [61, §3]. Given path family $\pi = (\pi_h, \dots, \pi_n)$ covering F_w, we define a partial order $P(\pi)$ on these paths by declaring $\pi_i <_{P(\pi)} \pi_j$ if

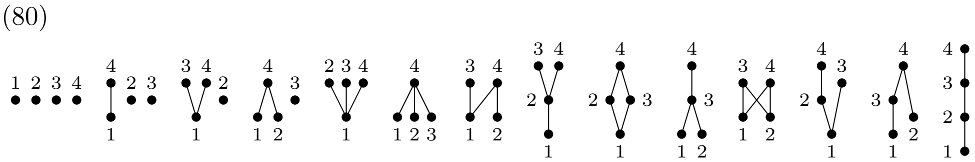
(1) $i < j$ as integers,
(2) π_i does not intersect π_j.

For every zig-zag network F_w, there is a unique path family of type e which covers F_w. If π is this path family, we define

$$P(w) := P(\pi), \tag{79}$$

and we label the elements of $P(w)$ by $h, \dots, n$ rather than by $\pi_h, \dots, \pi_n$. For example, consider the descending star networks (25) in $\mathcal{S}_{\mathrm{D}}^{\mathsf{A}}([1,4])$, labeled $F_{4321}, \dots, F_{1234}$ as in (30). The unit interval orders $P(4321), \dots, P(1234)$ are

(80)



respectively. The map $w \mapsto P(w)$ is a surjection from 3412-avoiding, 4231-avoiding permutations in $\mathfrak{S}_{[h,n]}$ to unit interval orders on $|[h,n]|$ elements. Furthermore, we have the following [18, Thm. 4.4].

Theorem 7.1. *The restriction of the map $w \mapsto P(w)$ to the subset of* 312*-avoiding permutations in $\mathfrak{S}_{[h,n]}$ is a bijection.*

One may construct $P(w)$ directly from w as follows.

Algorithm 7.2. *Given $w = w_h \cdots w_n \in \mathfrak{S}_{[h,n]}$ avoiding the pattern* 312*, do*

(1) *Define the word $m_h \cdots m_n$ by $m_i = \max\{w_h, \dots, w_i\}$.*
(2) *For $i = h, \dots, n$ define $i <_{P(w)} j$ if and only if $j > m_i$.*

The labels which paths in F_w assign to poset elements are redundant in the sense that they are determined up to automorphism by the structure of the poset. Specifically, for each poset element y define

$$\beta(y) = \#\{x \in P \,|\, x \leqslant_P y\} - \#\{z \in P \,|\, z \geqslant_P y\}. \tag{81}$$

It is easy to see that the labels of $P(w)$ inherited from the zig-zag network F_w satisfy $i < j$ (as integers) if $\beta(i) < \beta(j)$. The inverse of Algorithm 7.2 is the following.

Algorithm 7.3. *Given unlabeled unit interval order P on $|[h,n]|$ elements, do*

(1) *For all $y \in P$, compute $\beta(y) := \#\{x \in P \,|\, x \leqslant_P y\} - \#\{z \in P \,|\, z \geqslant_P y\}$.*
(2) *Label the poset elements by $[h,n]$ so that we have $\beta(h) \leqslant \cdots \leqslant \beta(n)$.*
(3) *Define $w = w_h \cdots w_n$ by $w_j = \max(\{i \in [h,n] \,|\, i \not>_P j\} \smallsetminus \{w_h, \dots, w_{j-1}\})$.*

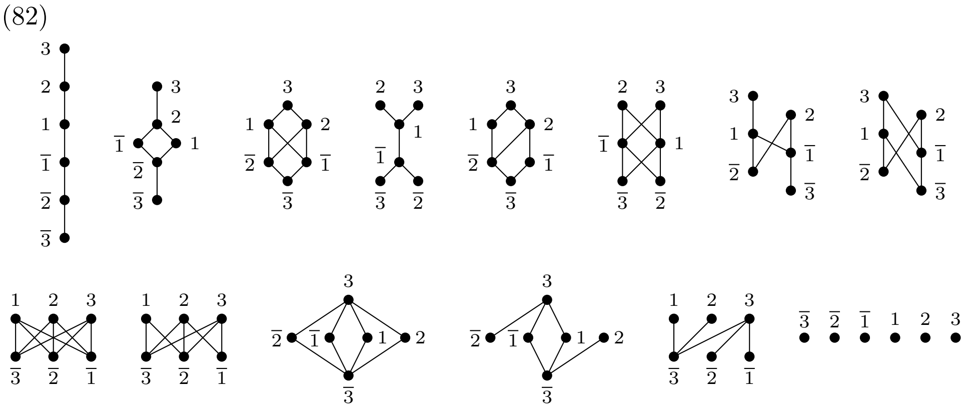
Observe that the path families of type e covering the zig-zag networks (26), which are not descending star networks and which have the form F_w for w containing the pattern 312, form posets isomorphic to posets 2, 4, 3, 7, 13, 12, 7, 7, respectively, in (80). It is straightforward to show that the poset labeling inherited from π (79) guarantees that for some indices i, j, the minimal and maximal elements of $P(w)$ are given by intervals $[h,i]$ and $[j,n]$, respectively. Furthermore we have the following. (See e.g. [28, p. 33], [72, §8.2].)

Proposition 7.4. *Fix $w \in \mathfrak{S}_{[h,n]}$ avoiding the patterns* 3412 *and* 4231 *and define $P = P(w)$.*

(1) *If i, j are incomparable in P with $i < j$ in $\mathbb{Z}$, then $[i,j]$ is an antichain in P.*
(2) *If $i <_P j$ then all elements $h, \dots, i$ are less than all elements $j, \dots, n$ in P.*

7.2. Type-BC unit interval orders. For each element $w \in \mathfrak{B}_n \subseteq \mathfrak{S}_{[\overline{n},n]}$ avoiding the patterns 3412 and 4231, the zig-zag network F_w and poset $P(w)$ are defined as in Subsections 5.1, 7.1. For example, the fourteen posets corresponding to the descending star networks in $\mathcal{S}_{\mathrm{D}}^{\mathsf{BC}}([\overline{3},3])$ (39) are

(82)



respectively. Observe that the path families of type e covering the zig-zag networks (40) which are not descending star networks form posets isomorphic to posets 5, 7, 8, 8, 8, 10, 12, 13, respectively in (82).

The conditions preceding (52), which define BC-path families, guarantee that each such poset $P(w)$ is self-dual with antiautomorphism $i \mapsto \overline{i}$. Thus it belongs to the class of *type-*C *posets* defined in [20, Defn. 10]. Since $P(w)$ is a unit interval order, we also have the following.

Proposition 7.5. *Fix $w \in \mathfrak{B}_n$ avoiding the patterns* 3412 *and* 4231 *and define $P = P(w)$. Let π be the unique path family of type e covering F_w, and let $i+1$ be the smallest element of $[1,n]$ such that π_{i+1} is not grounded. Then we have*

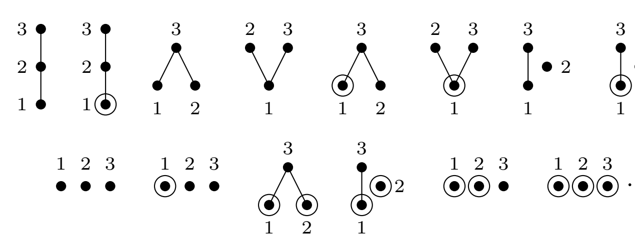
(1) *if* $i > 0$ *then* $[\bar{i}, i]$ *is an antichain in* P,
(2) $\overline{n}, \ldots, \overline{i+1}$ *are less than* $1, \ldots, n$ *in* P,
(3) $\overline{n}, \ldots, \overline{1}$ *are less than* $i+1, \ldots, n$ *in* P.

Proof. (1) Since $\pi_{\bar{i}}$ and π_i intersect in F_w, elements $\bar{i}$ and i are incomparable in P. By Proposition 7.4, $[\bar{i}, i]$ is an antichain in P.

(2),(3) Suppose that $\overline{i+1}$ is incomparable to 1 in P. By symmetry, $\overline{1}$ is incomparable to $i+1$ as well. Then $\pi_{\overline{i+1}}$ and π_1 intersect, as do $\pi_{\overline{1}}$ and π_{i+1}. Factor F_w as in (38) and suppose that paths $\pi_{\overline{i+1}}$, π_1 meet in $F'_{[c_k,d_k]}$. By the definition of BC-path family, paths π_{i+1}, $\pi_{\overline{1}}$ meet there as well. If $c_k \geqslant 1$ then $\pi_{\overline{1}}$, π_1 cross twice, contradicting the uniqueness of π of type e covering F_w. Thus we have that $c_k = \overline{d_k}$. But then $\pi_{\overline{i+1}}$, π_{i+1} meet as well, contradicting the assumption that these paths are not grounded. We conclude that $\overline{i+1} <_P 1$ and $\overline{1} <_P i+1$. Now Proposition 7.4 gives the desired results. ☐

The self-duality $i \mapsto \bar{i}$ of $P(w)$ and [27, Lem. 1.1] show that $P(w)$ is a *signed poset* as defined in [27, 56]. By Proposition 7.5 the information in $P(w)$ can be recorded by the subposet induced by elements $[1, n]$, if we circle elements corresponding to grounded paths of π. (This is not true of signed posets in general.) Call this decorated poset $Q(w)$, and in general, define a *type*-BC *unit interval order* to be a unit interval order decorated by circling a (possibly empty) subset of minimal elements, declared to be *grounded*, with the property that if element i is grounded and j is not, then $\beta(i) \leqslant \beta(j)$, where β is the function defined in Algorithm 7.3. We define an isomorphism of type-BC unit interval orders to be a poset isomorphism which respects circled elements. For example, the 3-element type-BC unit interval orders $Q(w)$ corresponding to the 6-element unit interval orders $P(w)$ in (82) are

(83)



If we remove labels from the map $w \mapsto Q(w)$, we obtain a surjection from 3412-avoiding, 4231-avoiding elements of $\mathfrak{B}_n$ to type-BC unit interval orders. The restriction of this map to the subset of $\mathfrak{B}_n$ avoiding the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$ is a bijection. Equivalently, we have the following.

Proposition 7.6. *The map* $F_w \mapsto Q(w)$ *from* $\mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n}, n])$ *to type*-BC *unit interval orders is bijective.*

Proof. To see that the map is injective, consider $F_v \neq F_w$ in $\mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{n}, n])$. By [18, Thm. 4.4] we have $P(v) \neq P(w)$, since for each fixed unit interval order P on $2n$ elements, the set $\{F_w \in \mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n]) \,|\, P(w) = P\}$ contains exactly one type-BC descending star network: the rearrangement $F'_{[a_{u_1}, b_{u_1}]} \bullet \cdots \bullet F'_{[a_{u_t}, b_{u_t}]}$ of (38) satisfying $a_{u_1} > \cdots > a_{u_t}$ (and $b_{u_1} > \cdots > b_{u_t}$). Now let $P'(v)$, $P'(w)$ be the subposets of $P(v)$ and $P(w)$ induced by elements $\{1, \ldots, n\}$. If $P'(v) \neq P'(w)$ then we clearly have $Q(v) \neq Q(w)$. Suppose therefore that $P'(v) = P'(w)$. Since $P(v) \neq P(w)$, there must be two indices $i \neq j$ such that elements $1, \ldots, i$ of $P'(v)$ are grounded, and elements $1, \ldots, j$ of $P'(w)$ are grounded. Again we have $Q(v) \neq Q(w)$.

To see that the map is surjective, consider a type-BC unit interval order Q on n elements with elements labeled as in Algorithm 7.9 and with a subset $\{1, \ldots, i\}$ of minimal elements circled, for some i. Let $F_u \in \mathcal{S}_{\mathsf{D}}^{\mathsf{A}}([1,n])$, $u \in \mathfrak{S}_n$, be the descending star network corresponding to Q viewed as an ordinary poset, ignoring circles, and write $F_u = F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_t,b_t]}$ as in Definition 5.3. Now construct $F'_{[a_1,b_1]} \bullet \cdots \bullet F'_{[a_t,b_t]} \bullet F'_{[\bar{i},i]}$ in $\mathcal{S}_{\mathsf{D}}^{\mathsf{BC}}([\overline{n},n])$ and call this F_w for $w \in \mathfrak{B}_n$. It is easy to see that we have $F_w \mapsto Q$, i.e. $Q = Q(w)$. □

The bijection $w \mapsto Q(w)$, which we have defined to be the composition

$$w \mapsto F_w \mapsto P(w) \mapsto Q(w) \tag{84}$$

of the three maps described in [61, §3], (79), and before (83), can also be described by the following algorithm.

Algorithm 7.7. *Given $w \in \mathfrak{B}_n$ avoiding the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$, do*

1. *Let b be the least positive letter in $\{w_1, \dots w_n, n+1\}$.*
2. *Define the word $m_1 \cdots m_n$ by $m_j = \max\{b-1, w_1, \dots, w_j\}$.*
3. *For $j = 1, \dots, n-1$, define $j <_{Q(w)} m_j + 1, \dots, n$.*
4. *For $j = 1, \dots, n$, if $w_j < 0$ then circle element $|w_j|$.*

Proposition 7.8. *For $w \in \mathfrak{B}_n$ avoiding the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$, the composition* (84) *agrees with Algorithm* 7.7*.*

Proof. Computing $Q(w)$ via the composition (84), we let F_w be the descending star network given by (29), i.e. [61, §3]. To construct $P(w)$, let

$$\pi = (\pi_{\overline{n}}, \dots, \pi_{\overline{1}}, \pi_1, \dots, \pi_n), \qquad \sigma = (\sigma_{\overline{n}}, \dots, \sigma_{\overline{1}}, \sigma_1, \dots, \sigma_n)$$

be the unique path families of types e and w covering F_w, and for $j = 1, \dots, n-1$ find the elements $k \in P$ satisfying $j <_P k$. First we claim that for i^* maximizing $\{w_i \,|\, i \in [\overline{n}, j]\}$, we have that

$$j \not<_P \overline{n}, \dots, w_{i^*}. \tag{85}$$

By the definition of $P(w)$ we have $j \not<_P \overline{n}, \dots, j$, and by the pigeonhole principle, we have $w_{i^*} \geqslant j$ (as integers). Since the path σ_{i^*} from source i^* to sink w_{i^*} intersects the path π_j from source j to sink j, we have a path from source j to sink w_{i^*}. This path in turn intersects all paths $\pi_{j+1}, \dots, \pi_{w_{i^*}}$, and we have paths from source j to all sinks $j+1, \dots, w_{i^*}$. Since the subnetwork of F covered by paths $\pi_1, \dots, \pi_n$ is isomorphic to a type-A descending star network, we may apply Lemma 5.4 to conclude that π_j intersects $\pi_{j+1}, \dots, \pi_{w_{i^*}}$. Thus we obtain the remaining inequalities $j \not<_P j+1, \dots, w_{i^*}$ in (85).

Now we claim that

$$j <_P w_{i^*} + 1, \dots, n. \tag{86}$$

Consider the paths π_k for $k > w_{i^*}$. Again by Lemma 5.4, paths π_k and π_j do not intersect, since there is no path in F_w from source j to sink k. Thus we have $j <_P k$ as in (86).

Now we define b to be the least positive letter in $\{w_1, \dots, w_n, n+1\}$, and we claim that

$$w_{i^*} = \max\{b-1, w_1, \dots, w_j\}. \tag{87}$$

By Lemma 2.2, the set of positive letters in $w_1 \cdots w_n$ is empty or forms the interval $[b,n]$. If this set is empty, then avoidance of the signed pattern $\overline{21}$ implies that $w_1 \cdots w_n = \overline{1} \cdots \overline{n}$. Thus we have $w_{i^*} = \max\{n, \dots, 1, \overline{1}, \dots, \overline{j}\} = n$, $b = n+1$, and the right-hand-side of (87) is n. Suppose therefore that the positive letters are $[b,n]$. Then the positive letters $[1, b-1]$ appear in $w_{\overline{n}} \cdots w_{\overline{1}}$. This allows us to write

$$w_{i^*} = \max\{w_i \,|\, i \leqslant j\} = \max(\{w_{\overline{n}}, \dots, w_{\overline{1}}\} \cup \{w_1, \dots, w_j\}) = \max\{b-1, w_1, \dots, w_j\}.$$

The subposet of $P(w)$ induced by $[1, n]$, which will become $Q(w)$, now agrees with steps (1) – (3) of Algorithm 7.7. To complete the construction of $Q(w)$ by (84), we circle grounded elements of $P(w)$, if there are any. If no path of π is grounded, then we do nothing. In this case, no path of σ has a source and sink with different signs, all letters in $w_1 \cdots w_n$ are positive, and nothing is done in step (4) of Algorithm 7.7. On the other hand, if some $2k$ paths of π are grounded, then by Proposition 7.5, these paths are $(\pi_{\overline{k}}, \dots, \pi_{\overline{1}}, \pi_1, \dots, \pi_k)$, and we circle elements $1, \dots, k$ of $P(w)$ to form $Q(w)$ by (84). In this case, $F'_{[c_t,d_t]} = F'_{[\overline{k},k]}$ is the last factor in the expression (38) for F_w, and the letters $\overline{1}, \dots, \overline{k}$ appear in $w_1 \cdots w_n$. Thus in step (4) of Algorithm 7.7, elements $1, \dots, k$ are circled. ☐

Like Algorithm 5.6, Algorithm 7.7 is invertible even if labels of the poset Q are not given.

Algorithm 7.9. *Given unlabeled type-*BC *unit interval order Q with p circled elements, do*

1. *For all $y \in Q$, compute $\beta(y) := \#\{x \in Q \mid x \leqslant_Q y\} - \#\{z \in Q \mid z \geqslant_Q y\}$.*
2. *Label the poset elements by $[1, n]$ so that we have $\beta(1) \leqslant \cdots \leqslant \beta(n)$, and so that circled elements form the interval $[1, p]$.*
3. *Define the word $a_1 \cdots a_n = \overline{1} \cdots \overline{p}(p + 1) \cdots n$.*
4. *Define $w = w_1 \cdots w_n$ by $w_j = \max(\{a_i \mid i \not\geqslant_Q j\} \smallsetminus \{w_1, \dots, w_{j-1}\})$.*

To see that Algorithm 7.9 inverts Algorithm 7.7, we consider a close relationship between certain descending star networks of types A and BC.

Lemma 7.10. *Fix $w \in \mathfrak{B}_n$ avoiding the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$ with $p > 0$ negative letters $(\overline{1}, \dots, \overline{p})$ appearing in $w_1 \cdots w_n$, and type-*BC *descending star network*

$$F_w = F'_{[c_1,d_1]} \bullet \cdots \bullet F'_{[c_{t-1},d_{t-1}]} \bullet F'_{[\overline{p},p]},$$

with factors defined as in (36)*. Define $u \in \mathfrak{S}_{[\overline{p},n]}$ to be the* 312*-avoiding permutation corresponding to the type-*A *descending star network*

$$F_u = F_{[c_1,d_1]} \bullet \cdots \bullet F_{[c_{t-1},d_{t-1}]} \bullet F_{[\overline{p},p]},$$

with factors defined as in (§5.1*). Then the one-line notation of u is $p \cdots 21 w_1 \cdots w_n$ and the subposet $P_{[1,n]}$ of $P(u)$ induced by $[1, n]$ satisfies $P_{[1,n]} \cong Q(w)$ (as undecorated posets).*

Proof. Let $\pi' = (\pi'_{\overline{n}}, \dots, \pi'_{\overline{1}}, \pi'_1, \dots, \pi'_n)$ and $\pi = (\pi_{\overline{p}}, \dots, \pi_{\overline{1}}, \pi_1, \dots, \pi_n)$ be the unique path families of type e covering F_w and F_u, respectively. By Definition 5.14 and Proposition 5.15 we have $c_1 > \cdots > c_{t-1} \geqslant 1$. Thus for $1 \leqslant i < j \leqslant n$ we have that π_i intersects π_j if and only if π'_i intersects π'_j. It follows that $P_{[1,n]} \cong Q(w)$.

Now let $\sigma' = (\sigma'_{\overline{n}}, \dots, \sigma'_{\overline{1}}, \sigma'_1, \dots, \sigma'_n)$ and $\sigma = (\sigma_{\overline{p}}, \dots, \sigma_{\overline{1}}, \sigma_1, \dots, \sigma_n)$ be the unique path families of types w and u covering F_w and F_u, respectively. Both families have the property that any two paths which intersect must cross. Thus paths $\sigma_{\overline{p}}, \dots, \sigma_{\overline{1}}$, which intersect only at the central vertex of $F_{[\overline{p},p]}$, have sinks $p, \dots, 1$, respectively. Thus $u_{\overline{p}} \cdots u_{\overline{1}} = p \cdots 1$. Also, paths $\sigma_1, \dots, \sigma_n$ pass through the same stars as $\sigma'_1, \dots, \sigma'_n$, respectively, and have the same sinks. Thus $u_1 \cdots u_n = w_1 \cdots w_n$. ☐

Proposition 7.11. *Algorithm* 7.9 *inverts Algorithm* 7.7*.*

Proof. Fix $w \in \mathfrak{B}_n$. If no negative letters appear in $w_1 \cdots w_n$, then we may interpret this word as an element of $\mathfrak{S}_n$. The applications of Algorithms 7.2 and 7.7 to w agree and produce the poset $P(w) = Q(w)$. Since this poset has no circled elements, the applications of Algorithms 7.3 and 7.9 to it agree, producing w since Algorithm 7.3 inverts Algorithm 7.2. It follows that Algorithm 7.9 inverts Algorithm 7.7 as well.

Now suppose that $p > 0$ negative letters appear in $w_1 \cdots w_n$, and define

$$u = u_{\overline{p}} \cdots u_{\overline{1}} u_1 \cdots u_n = p \cdots 21 w_1 \cdots w_n \in \mathfrak{S}_{[\overline{p},n]}.$$

By Lemma 7.10, u avoids the ordinary pattern 312. Define $P_{[1,n]}$ to be the subposet of $P(u)$ induced by elements $[1,n]$. Applying Algorithm 7.3 to $P(u)$, we obtain u. It follows that for $j = 1, \ldots, n$, we have

$$\begin{aligned} u_j = w_j &= \max(\{i \in [\overline{p}, n] \,|\, i \not>_{P(u)} j\} \smallsetminus \{u_{\overline{p}}, \ldots, u_{j-1}\}) \\ &= \max(\{i \in [\overline{p}, \overline{1}] \cup [p+1, n] \,|\, i \not>_{P(u)} j\} \smallsetminus \{w_1, \ldots, w_{j-1}\}). \end{aligned} \tag{88}$$

Since $[\overline{p}, p] \subseteq P(u)$ is an antichain of minimal elements, each pair $(i,j) \in [\overline{p}, \overline{1}] \times [1,n]$ satisfies $i \not>_{P(u)} j$ if and only if $\overline{i} \not>_{P_{[1,n]}} j$. Thus we may rewrite (88) as

$$w_j = \max(\{i \in [\overline{p}, \overline{1}] \,|\, \overline{i} \not>_{P_{[1,n]}} j\} \cup \{i \in [p+1,n] \,|\, i \not>_{P_{[1,n]}} j\} \smallsetminus \{w_1, \ldots, w_{j-1}\}).$$

On the other hand, applying Algorithm 7.9 to $Q(w)$, we obtain a word $v_1 \cdots v_n$ satisfying

$$v_j = \max(\{i \in [\overline{p}, \overline{1}] \,|\, \overline{i} \not>_{Q(w)} j\} \cup \{i \in [p+1,n] \,|\, i \not>_{Q(w)} j\} \smallsetminus \{v_1, \ldots, v_{j-1}\}).$$

By Lemma 7.10, we have $P_{[1,n]} \cong Q(w)$, and therefore $v_1 \cdots v_n = w_1 \cdots w_n$. Again, Algorithm 7.9 inverts Algorithm 7.7. □

8. Indifference graphs

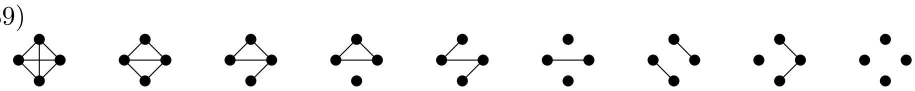
More partial solutions to Problem 1.1 for the subsets (16) – (19) of the Kazhdan–Lusztig bases employ graphs called *indifference graphs*, those graphs whose vertices correspond to elements of a unit interval order P and whose edges correspond to unordered pairs $\{i,j\}$ of poset elements which are incomparable, i.e. $i \not\leq_P j$ and $j \not\leq_P i$.

In type A, we have a map $w \mapsto G(w)$, from 3412-avoiding, 4231-avoiding permutations in $\mathfrak{S}_n$ to indifference graphs whose colorings and edge orientations facilitate simple combinatorial interpretations of trace evaluations [18, §5–10]. In types B and C, we define an analogous map $w \mapsto G(w)$ from 3412-avoiding, 4231-avoiding elements of $\mathfrak{B}_n$ to objects which we call *type-*BC *indifference graphs.* These graphical representations facilitate simple combinatorial interpretation of certain trace evaluations (Section 9), when we specialize at $q = 1$.

8.1. Type-Aindifference graphs, coloring, and orientation.

Given any poset P, we define its *incomparability graph* $\mathrm{inc}(P)$ to be the graph whose vertices are the elements of P and whose edges are the pairs of incomparable elements of P. When $P = P(w)$ is a unit interval order, write $G(w) = \mathrm{inc}(P)$ and call this an *indifference graph.* It is possible to have $P(w) \not\cong P(v)$ and $G(w) \cong G(v)$. For example, the incomparability graphs of the fourteen unit interval orders (80) are the nine nonisomorphic indifference graphs

(89)



In Section 9 we will combinatorially evaluate certain traces at Kazhdan–Lusztig basis elements $\widetilde{C}_w(1) \in \mathbb{Z}[\mathfrak{S}_n]$ with w avoiding the patterns 3412 and 4231 by coloring the vertices of $G(w)$ or by orienting the edges of $G(w)$.

Given any graph $G = (V, E)$ call a map $\kappa : V \to \mathbb{N} \smallsetminus \{0\}$ a *proper coloring of* G if $\{a, b\} \in E$ implies that $\kappa(a) \neq \kappa(b)$. More specifically, say that a proper coloring has *type* $\alpha = (\alpha_1, \ldots, \alpha_r) \vDash n$ if α_k vertices have color k for $k = 1, \ldots, r$. If $G = \mathrm{inc}(P)$ then a proper coloring of $\mathrm{inc}(P)$ of type $\lambda \vdash n$ corresponds to a sequence of pairwise disjoint chains in P having weakly decreasing cardinalities $(\lambda_1, \ldots, \lambda_r)$.

Call a directed graph $O = (V, E')$ an *orientation* of $G = (V, E)$ if O is obtained from G by replacing each undirected edge $\{a, b\} \in E$ with exactly one of the directed edges (a, b) or (b, a). Call O *acyclic* if it has no directed cycles. Acyclic orientations of $G(w)$ correspond to sequences $(v_h, \dots, v_n)$ of elements of $P(w)$ satisfying $v_i \not>_{P(w)} v_{i+1}$ for $i = h, \dots, n-1$. We call these $P(w)$*-descent-free* sequences. (See [9, §4] and references there.)

Proposition 8.1. *For $w \in \mathfrak{S}_{[h,n]}$, acyclic orientations of $G(w)$ correspond bijectively to $P(w)$-descent-free sequences of elements of $P(w)$.*

Specifically, this bijection from acyclic orientations to $P(w)$-descent-free sequences is given by the following algorithm.

Algorithm 8.2. *Given $w \in \mathfrak{S}_{[h,l]}$ and an acyclic orientation O of $G(w)$, do*

1. *Set $O(h) = O$.*
2. *For $i = h, \dots, l$,*
   - (a) *Let j be the least integer appearing as a vertex in $O(i)$ and having indegree 0.*
   - (b) *Set $v_i = j$.*
   - (c) *Form $O(i+1)$ by removing vertex j and its incident edges from $O(i)$.*
3. *Output the sequence $(v_h, \dots, v_l)$.*

The inverse of Algorithm 8.2 is very simple.

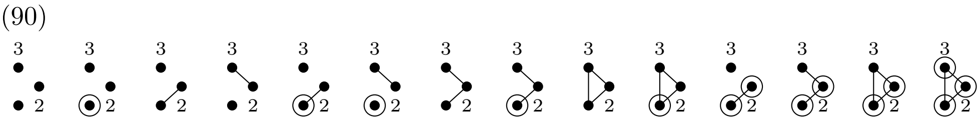
Algorithm 8.3. *Given $w \in \mathfrak{S}_{[h,l]}$, undirected graph $G(w)$ with vertices labeled $\{h, \dots, l\}$, and $P(w)$-descent-free sequence $v = (v_h, \dots, v_l)$, do*

1. *Orient each edge $\{a, b\}$ of $G(w)$ as (a, b) if a appears before b in v, and as (b, a) otherwise.*

8.2. Type-BCndifference graphs, coloring, and orientation.

Given a type-BC unit interval order Q, define its *incomparability graph* $\mathrm{inc}(Q)$ to be the decorated graph whose vertices are the elements of Q, maintaining circles, and whose edges are the pairs of incomparable elements of Q. For $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231, write $\Gamma(w) = \mathrm{inc}(Q(w))$ and call $\Gamma(w)$ a *type-*BC *indifference graph*. We define an isomorphism of type-BC indifference graphs to be a graph isomorphism which respects circled elements. Again, it is possible to have $Q(w) \not\cong Q(v)$ and $\Gamma(w) \cong \Gamma(v)$. For instance, the fourteen (labeled) type-BC indifference graphs on three elements are

(90)



with the third and fourth graphs being isomorphic.

Analogous to type-A indifference graphs, type-BC indifference graphs have colorings and edge orientations which facilitate the evaluation of certain type-BC traces at Kazhdan–Lusztig basis elements $\widetilde{C}_w^{\mathsf{BC}}(1) \in \mathbb{Z}[\mathfrak{B}_n]$ when w avoids the patterns 3412 and 4231. Given a type-BC indifference graph $\Gamma = (V, E)$, define a *marked* BC*-coloring*

$$\kappa = (\kappa_1, \kappa_2) : V \to (\mathbb{Z} \smallsetminus \{0\}) \times \{0, 1\}$$

of Γ to be an assignment of a nonzero color $\kappa_1(b)$ and possibly a star (if $\kappa_2(b) = 1$) to each vertex $b \in V$, with the properties that

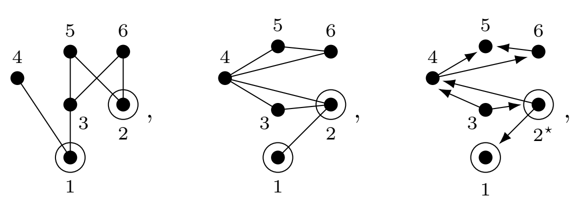
1. for vertex b grounded we have $\kappa_1(b) > 0$,
2. for vertex b not grounded we have $\kappa_2(b) = 0$.

Say that κ has *type* $(\lambda, \mu) = ((\lambda_1, \dots, \lambda_m), (\mu_1, \dots, \mu_k))$ if

(1) λ_i vertices have color i, for $i = 1, \dots, m$,
(2) μ_i vertices have color $\bar{i}$, for $i = 1, \dots, k$,

and that κ is *proper* if $\{a, b\} \in E$ implies that $\kappa_1(a) \neq \kappa_1(b)$. As before, monochromatic sets of $\Gamma(w)$ correspond to chains in $Q(w)$; now each such chain contains at most one grounded element. Thus a proper BC-coloring of Γ of type (λ, μ) may be represented by a pair (U, V) of Q-tableaux in which column i of U $(i = 1, \dots, m)$ contains the color-i chain of Q with at most one grounded element marked with a star, and column i of V $(i = 1, \dots, k)$ contains the color-$\bar{i}$ chain of Q with no grounded elements.

Define a *marked acyclic orientation* of a type-BC indifference graph to be a directed graph O on the same vertices, with some subset of grounded vertices marked by stars, in which each undirected edge $\{a, b\}$ is replaced with one of the directed edges (a, b) or (b, a). For example, the type-BC unit interval order $Q = Q(\bar{1}43652\bar{2})$, its incomparability graph $\mathrm{inc}(Q)$, a marked acyclic orientation of $\mathrm{inc}(Q)$, and a marked coloring of $\mathrm{inc}(Q)$ of type $((2, 1, 1), (2))$ are

(91) 
$$\left(\begin{array}{|c|cc} \hline 5 & & \\ \hline 1^\star & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|} \hline 6 \\ \hline 3 \\ \hline \end{array} \right).$$

To connect acyclic orientations of $\Gamma(w)$ to $Q(w)$-descent-free sequences as in Proposition 8.1, we define *marked $Q(w)$-descent-free* sequences to be those $Q(w)$-descent-free sequences in which some subset of grounded elements is marked.

Proposition 8.4. *Marked acyclic orientations of $\Gamma(w)$ correspond to marked $Q(w)$-descent-free sequences of elements of $Q(w)$.*

Proof. The correspondence is given by Algorithms 8.2 – 8.3, modified so that marked graph vertices correspond to marked poset elements. ☐

For example, the Q-descent-free sequence corresponding to the acyclic orientation in (91) is $(3, 2^\star, 1, 4, 6, 5)$.

We remark that other authors have defined BC-analogs of graphs [35, 56], have associated these to posets generalizing type-BC unit interval orders [21, 56], and have studied their colorings [46, 76]. However, it is not clear that such graphs and colorings are closely related to ours. In particular, the other authors' graphs have edges describing comparability of poset elements rather than incomparability, and their colorings include restrictions on pairs of vertices whose colors can share an absolute value, whereas ours do not.

9. Combinatorial trace evaluations: path tableaux, poset tableaux, and acyclic orientations

Our main results, Theorem 9.6 – Theorem 9.8, combinatorially interpret trace evaluations $\theta(\widetilde{C}^{\mathsf{BC}}_w(1))$ for certain $\theta \in \mathcal{T}(\mathfrak{B}_n)$ and all $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231. Analogous to known type-A results, our new type-BC evaluations use the type-BC unit interval orders and their incomparability graphs defined in Subsections 7.2, 8.2.

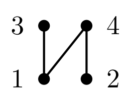
9.1. Type-A trace evaluations.

To state these interpretations, we fill (French) Young diagrams with paths and we call the resulting structures *path tableaux*. If the paths are a family $\pi = (\pi_1, \dots, \pi_n)$ which covers F_w, we will more specifically call the path tableau an *F_w-tableau*, or a *π-tableau*. If π has type $v \in \mathfrak{S}_n$, then we also say that each π-tableau has *type* v. Since π can be viewed as the poset $Q(\pi)$ defined in Subsection 7.1, π-tableaux are special cases of Gessel and Viennot's *poset tableaux* [30], Young diagrams filled with elements of a poset. Thus if π is the unique family of type e covering F_w, then a $P(\pi)$-tableau is a $P(w)$-tableau. For any tableau U, let U_i be the ith row of U, and let $U_{i,j}$ be the jth entry in row i. Let $\mathcal{U}(\pi, \lambda)$ denote the set of all π-tableaux of shape λ, and let $\mathcal{U}(F_w, \lambda)$ denote the set of *all* F_w-tableaux of shape λ, i.e. containing all path families covering F_w,

$$\mathcal{U}(F_w, \lambda) = \bigcup_{\pi \in \Pi(F_w)} \mathcal{U}(\pi, \lambda). \tag{92}$$

If π is the unique path family of type e covering F_w, then define $\mathcal{U}(P(w), \lambda) := \mathcal{U}(\pi, \lambda)$. For example, consider F_{2341}, the seventh zig-zag network in (26), and let $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ be the unique path family of type e covering F_{2341}. Then $P(2341) = Q(\pi)$ is the seventh unit interval order in (80). Labeling each element π_i of $P(2341)$ by i and forming a few $P(2341)$-tableaux, we have

(93)

$P(2341) =$ 

, $\quad S = \begin{array}{|c|ccc} \hline 4 & & & \\ \hline 1 & 2 & 3 & \\ \hline \end{array}, \quad T = \begin{array}{|c|ccc} \hline 2 & & & \\ \hline 1 & 3 & 4 & \\ \hline \end{array}, \quad U = \begin{array}{|c|ccc} \hline 1 & & & \\ \hline 4 & 3 & 2 & \\ \hline \end{array},$

$$V = \begin{array}{|c|c|c|} \hline 3 \\ \hline 4 & 1 & 2 \\ \hline \end{array}, \quad W = \begin{array}{|c|c|c|} \hline 3 \\ \hline 1 & 4 & 2 \\ \hline \end{array}, \quad W_1 = \begin{array}{|c|c|c|} \hline 1 & 4 & 2 \\ \hline \end{array}, \qquad W_{1,2} = 4.$$

The tableaux S, T, U, V, W all belong to $\mathcal{U}(P(2341), 31)$.

Several properties which path-tableaux may possess can be defined for poset tableaux. Let P be any labeled poset and let U be a P-tableau. Call an entry $U_{i,j}$ a *record* in U if it is greater in P than $U_{i,1}, \dots, U_{i,j-1}$. Call a record $U_{i,j}$ *nontrivial* if $j > 1$. Call a row of U *left anchored* (*right anchored*) if its leftmost (rightmost) element is less in $\mathbb{Z}$ than all other elements in the row. Call elements (a, b) a *P-inversion* in U if the elements are incomparable in P with $a < b$ in $\mathbb{Z}$ and b appearing in an earlier column than a. Let $\mathrm{INV}_P(U)$ denote the number of P-inversions in U. Call elements $(U_{i,j}, U_{i,j+1})$ a *P-descent* in U if $U_{i,j} >_P U_{i,j+1}$. Let $\mathrm{des}_P(U)$ denote the number of P-descents in U. Define $\mathrm{sort}(U)$ to be the tableau obtained from U by sorting entries in each row so that labels increase to the right. Define a P-excedance in U to be a position (i,j) such that $U_{i,j} >_P \mathrm{sort}(U)_{i,j}$. Let $\mathrm{exc}_P(U)$ be the number of P-excedances in U.

Call a P-tableau U

1. *column-strict* if the entries of each column satisfy $U_{i,j} <_P U_{i+1,j}$,
2. *descent-free* or *row-semistrict* if $\mathrm{des}_P(U) = 0$,
3. *cyclically row-semistrict* if it is row-semistrict, and if the last entry U_{i,λ_i} of each row satisfies $U_{i,\lambda_i} \not>_P U_{i,1}$,
4. *standard* if it is column-strict and row-semistrict,
5. *excedance-free* if $\mathrm{exc}_P(U) = 0$,
6. *record-free* if no row has a nontrivial P-record,
7. *left anchored* (*right anchored*) if each row is left anchored (right anchored).

For example, we may examine the tableaux in (93) for these properties to obtain the table

	S	T	U	V	W
column-strict	✓				✓
row-semistrict	✓	✓	✓		
cyclically row-semistrict			✓		
standard	✓				
excedance-free	✓	✓			
record-free	✓		✓	✓	
left anchored	✓	✓			✓
right anchored			✓	✓	

,

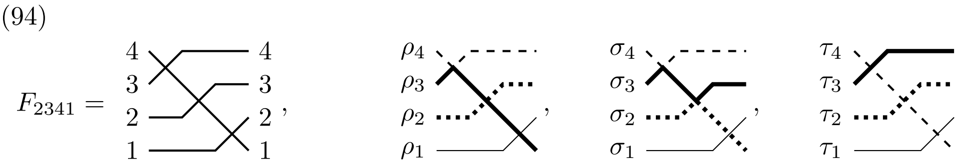
where the row-semistrict tableaux S and T fail to be cyclically row-semistrict because their first rows begin with 1 and end with $3 >_P 1$ and $4 >_P 1$, respectively.

Other properties of path-tableaux depend upon the fact that each path π_j in a path family has a source vertex $\mathrm{src}(\pi_j)$ and a sink vertex $\mathrm{snk}(\pi_j)$. Given a path-tableau U, let $\mathrm{src}(U)$ and $\mathrm{snk}(U)$ denote the Young tableaux of integers obtained from U by replacing paths $\pi_1, \ldots, \pi_n$ with their corresponding source and sink indices, respectively. If U is a path-tableau, call U

(1) *row-closed* if for each index i, $\mathrm{snk}(U_i)$ is a permutation of $\mathrm{src}(U_i)$,
(2) *left row-strict* if entries of $\mathrm{src}(U)$ strictly increase in each row,
(3) *cylindrical* if each row i satisfies $\mathrm{snk}(U_{i,1}, \ldots, U_{i,k}) = \mathrm{src}(U_{i,2}, \ldots, U_{i,k}, U_{i,1})$.

For example consider F_{2341} again, the unique path families ρ, σ, and τ of type 2314, 2134, and 2341 which cover F_{2341},

(94)



and the path tableaux

(95)

$$SS = \begin{array}{|c|cc}\hline \rho_4 & & \\ \hline \rho_2 & \rho_3 & \rho_1 \\ \hline\end{array}, \quad TT = \begin{array}{|c|cc}\hline \rho_4 & & \\ \hline \rho_1 & \rho_2 & \rho_3 \\ \hline\end{array}, \quad UU = \begin{array}{|c|cc}\hline \sigma_4 & & \\ \hline \sigma_1 & \sigma_2 & \sigma_3 \\ \hline\end{array}, \quad VV = \begin{array}{|c|cc}\hline \tau_3 & & \\ \hline \tau_1 & \tau_2 & \tau_4 \\ \hline\end{array},$$

belonging to $\mathcal{U}(\rho, 31)$, $\mathcal{U}(\sigma, 31)$, $\mathcal{U}(\tau, 31)$. To inspect these tableaux for the properties defined above, we replace each path π_j with the ordered pair $(\mathrm{src}(\pi_j), \mathrm{snk}(\pi_j))$,

$$\begin{array}{|c|cc}\hline 44 & & \\ \hline 23 & 31 & 12 \\ \hline\end{array}, \qquad \begin{array}{|c|cc}\hline 44 & & \\ \hline 12 & 23 & 31 \\ \hline\end{array}, \qquad \begin{array}{|c|cc}\hline 44 & & \\ \hline 12 & 21 & 33 \\ \hline\end{array}, \qquad \begin{array}{|c|cc}\hline 34 & & \\ \hline 12 & 23 & 41 \\ \hline\end{array}, \tag{96}$$

and we obtain the summary

	SS	TT	UU	VV
row-closed	✓	✓	✓	
left row-strict		✓	✓	✓
cylindrical	✓	✓		

.

Using Lindstrom's Lemma [40, 50], its permanental analogs [62, Thm. 4.15], and its power sum immanant analogs [62, Thm. 4.16] we may now extend Proposition 6.2 to include more combinatorial interpretations.

Proposition 9.1. *Fix $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231, with corresponding zig-zag network F_w having path matrix A. Let $P = P(w)$ and $G = \mathrm{inc}(P)$ be the corresponding unit interval order and incomparability graph. We have*

$$\mathrm{Imm}^{\mathfrak{S}_n}_{\epsilon^n}(A) = \det(A) = \#\{U \in \mathcal{U}(P, 1^n) \,|\, U \text{ column-strict } \}, \\ = \begin{cases} 1 & \text{if } G \text{ is an independent set } (w = e \text{ and } P \text{ is a chain}), \\ 0 & \text{otherwise.} \end{cases} \tag{97}$$

$$\begin{aligned} \mathrm{Imm}^{\mathfrak{S}_n}_{\eta^n}(A) = \mathrm{per}(A) &= \#\{U \in \mathcal{U}(F_w, n) \,|\, U \text{ left row-strict}\}, \\ &= \#\{U \in \mathcal{U}(P, n) \,|\, U \text{ row-semistrict } \}, \\ &= \#\{U \in \mathcal{U}(P, n) \,|\, U \text{ excedance-free } \}, \\ &= \# \text{ acyclic orientations of } G. \end{aligned} \tag{98}$$

$$\begin{aligned} \mathrm{Imm}^{\mathfrak{S}_n}_{\psi^n}(A) &= \#\{U \in \mathcal{U}(F_w, n) \,|\, U \text{ cylindrical}\}, \\ &= \#\{U \in \mathcal{U}(P, n) \,|\, U \text{ cyclically row-semistrict } \}, \\ &= \#\{U \in \mathcal{U}(P, n) \,|\, U \text{ record-free, row-semistrict } \}, \\ &= n \cdot \#\{U \in \mathcal{U}(P, n) \,|\, U \text{ right-anchored, row-semistrict } \}, \\ &= \# \text{ acyclic orientations of } G \text{ having exactly one source.} \end{aligned} \tag{99}$$

By (62), Proposition 9.1 gives interpretations of $\epsilon^\lambda(\widetilde{C}_w(1))$, $\eta^\lambda(\widetilde{C}_w(1))$, $\psi^\lambda(\widetilde{C}_w(1))$ in the special case that $\lambda = n$. The identities (60) – (61) then lead to results for general λ. (See [18, Thm. 4.7], [62, Thms. 30–31].)

Theorem 9.2. *Fix $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231 with corresponding zig-zag network F_w and unit interval order $P = P(w)$ as in (79). For each partition $\lambda \vdash n$ we have the following.*

- (i-a) $\epsilon^\lambda(C'_w(1)) = \#\{U \in \mathcal{U}(P, \lambda^\top) \,|\, U \text{ column-strict}\}$.
- (i-b) $\epsilon^\lambda(C'_w(1)) = \#$ *colorings of* $\mathrm{inc}(P)$ *of type* λ.
- (ii-a) $\eta^\lambda(C'_w(1)) = \#\{U \in \mathcal{U}(F_w, \lambda) \,|\, U \text{ row-closed, left row-strict}\}$.
- (ii-b) $\eta^\lambda(C'_w(1)) = \#\{U \in \mathcal{U}(P, \lambda) \,|\, U \text{ row-semistrict}\}$.
- (ii-c) $\eta^\lambda(C'_w(1)) = \#\{U \in \mathcal{U}(P, \lambda) \,|\, U \text{ excedance-free}\}$.
- (iii) $\chi^\lambda(C'_w(1)) = \#\{U \in \mathcal{U}(P, \lambda) \,|\, U \text{ standard}\}$.
- (iv-a) $\psi^\lambda(C'_w(1)) = \#\{U \in \mathcal{U}(F_w, \lambda) \,|\, U \text{ cylindrical }\}$.
- (iv-b) $\psi^\lambda(C'_w(1)) = \#\{U \in \mathcal{U}(P, \lambda) \,|\, U \text{ cyclically row-semistrict}\}$.
- (iv-c) $\psi^\lambda(C'_w(1)) = \#\{U \in \mathcal{U}(P, \lambda) \,|\, U \text{ record-free, row-semistrict}\}$.
- (iv-d) $\psi^\lambda(C'_w(1)) = \lambda_1 \cdots \lambda_r \cdot \#\{U \in \mathcal{U}(P, \lambda) \,|\, U \text{ right-anchored, row-semistrict}\}$.

See [62, Thm. 31] for a proof of statement (*ii-c*) and its q-analog; see [18, Thm. 4.7] for proofs of other statements and [16, Cor. 31], [18, §5–9] for proofs of their q-analogs. We may also interpret $\eta^\lambda_q(C'_w(1))$ and $\psi^\lambda_q(C'_w(1))$ in terms of acyclic orientations of sequences of subgraphs of $\mathrm{inc}(P(w))$ [62, Thm. 10, Thm. 13].

Theorem 9.3. *Fix $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231, and define $P = P(w)$ as in Subsection 7.1. For all $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ we have*

1. *$\eta^\lambda(C'_w(1))$ equals the number of acyclic orientations of subgraph sequences*

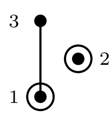
$$(\mathrm{inc}(P_{I_1}), \dots, \mathrm{inc}(P_{I_r})), \tag{100}$$

where $(I_1, \dots, I_r)$ is an ordered set partition of $[n]$ of type λ.

2. *$\psi^\lambda(C'_w(1))$ equals the number of acyclic orientations of subgraph sequences (100) in which each subgraph $\mathrm{inc}(P_{I_j})$ is connected and its orientation has a unique source.*

9.2. Type-BC trace evaluations. It is possible to extend Proposition 6.5 to include interpretations of the functions there in terms of path tableaux, poset tableaux, and acyclic orientations, just as Theorem 9.2 extends Propositions 6.2 and 9.1. To do this, we define BC-analogs of poset tableaux and path tableaux (and use the marked acyclic orientations defined at the end of Subsection 8.2).

Given a type-BC unit interval order Q, define a *marked Q-tableau* to be a Young diagram filled with elements of Q, in which a (possibly empty) subset of grounded elements of Q is marked with stars. Define $\mathcal{U}^{\mathsf{BC}}(Q,\lambda)$ to be the set of marked Q-tableaux of shape λ. The seven properties of P-tableaux stated after (93) carry over in a straightforward way to Q-tableaux. For example, the type-BC unit interval order $Q = Q(\overline{1}3\overline{2})$ and a few row-semistrict marked Q-tableaux of shape 3 are



,

$$\boxed{1}\boxed{2}\boxed{3},\quad \boxed{1^\star}\boxed{2}\boxed{3},\quad \boxed{1}\boxed{2^\star}\boxed{3},\quad \boxed{1^\star}\boxed{2^\star}\boxed{3},$$

$$\boxed{3}\boxed{2}\boxed{1},\quad \boxed{3}\boxed{2^\star}\boxed{1},\quad \boxed{3}\boxed{2}\boxed{1^\star},\quad \boxed{3}\boxed{2^\star}\boxed{1^\star}.$$

Given type-BC zig-zag network $F_w \in \mathcal{S}^{\mathsf{BC}}_{\mathsf{Z}}([\overline{n}, n])$ and a path family

$$\pi = (\pi_{\overline{n}}, \dots, \pi_{\overline{1}}, \pi_1, \dots, \pi_n) \in \Pi^{\mathsf{BC}}(F_w),$$

define an *F_w-tableau*, or more specifically a *π-tableau*, to be a Young diagram filled with paths $(\pi_1, \dots, \pi_n)$. Define $\mathcal{U}^{\mathsf{BC}}(F_w, \lambda)$ to be the set of (unmarked) F_w-tableaux of shape λ. Properties of such tableaux are simple extensions of properties of type-A path tableaux stated before (94), with sink indices replaced by their absolute values. For $U \in \mathcal{U}^{\mathsf{BC}}(F_w, \lambda)$, call U

(1) *row-closed* if $\{|\mathrm{snk}(U_{i,1})|, \dots, |\mathrm{snk}(U_{i,\lambda_i})|\} = \{\mathrm{src}(U_{i,1}), \dots, \mathrm{src}(U_{i,\lambda_i})\}$ for all i,
(2) *left row-strict* if $\mathrm{src}(U_{i,1}) < \cdots < \mathrm{src}(U_{i,\lambda_i})$ for all i,
(3) *cylindrical* if $|\mathrm{snk}(U_{i,j})| = \mathrm{src}(U_{i,j+1})$ for $j = 1, \dots, \lambda_i - 1$ and $|\mathrm{snk}(U_{i,\lambda_i})| = \mathrm{src}(U_{i,1})$.

For example, consider $F_{\overline{1}3\overline{2}}$ in (105) and let $\pi = (\pi_{\overline{3}}, \pi_{\overline{2}}, \pi_{\overline{1}}, \pi_1, \pi_2, \pi_3)$ be the fourth path family shown there. Then $\mathcal{U}^{\mathsf{BC}}(F_w, 3)$ and $\mathcal{U}^{\mathsf{BC}}(F_w, 21)$ contain row-closed tableaux such as

$$\boxed{\pi_1}\boxed{\pi_2}\boxed{\pi_3}, \qquad \begin{array}{l}\boxed{\pi_1}\\ \boxed{\pi_3}\boxed{\pi_2}\end{array},$$

the first of which is left-row strict and the second of which is cylindrical.

Left row-strict F_w-tableaux of shape n correspond bijectively to path families in $\Pi^{\mathsf{BC}}(F_w)$:

$$\boxed{\pi_1} \cdots \boxed{\pi_n} \quad \leftrightarrow \quad (\pi_{\overline{n}}, \dots, \pi_{\overline{1}}, \pi_1, \dots, \pi_n). \tag{101}$$

These tableaux and path families also correspond bijectively to marked acyclic orientations of $\mathrm{inc}(Q(w))$ and to certain subsets of marked $Q(w)$-tableaux. To describe these correspondences, we first define an equivalence relation on $\Pi^{\mathsf{BC}}(F_w)$ by declaring

$$\pi \sim \tau \quad \text{if} \quad \varphi(\mathrm{type}(\pi)) = \varphi(\mathrm{type}(\tau)), \tag{102}$$

where $\varphi : \mathfrak{B}_n \to \mathfrak{S}_n$ is the map defined in (9). In terms of paths in the two families, $\pi \sim \tau$ if $|\mathrm{snk}(\pi_i)| = |\mathrm{snk}(\tau_i)|$ for $i = 1, \dots, n$.

The cardinality of an equivalence class (102) depends on the number of positive sources of F_w from which there exists a path to a negative sink. Specifically, if the related type-BC unit interval order $Q(w)$ has k grounded elements, then each equivalence class consists of 2^k families, with exactly one family τ in each class satisfying

$\ell_t(\mathrm{type}(\tau)) = 0$, i.e. $\mathrm{snk}(\pi_i) > 0$ for $i = 1, \dots, n$. Thus we have the bijection

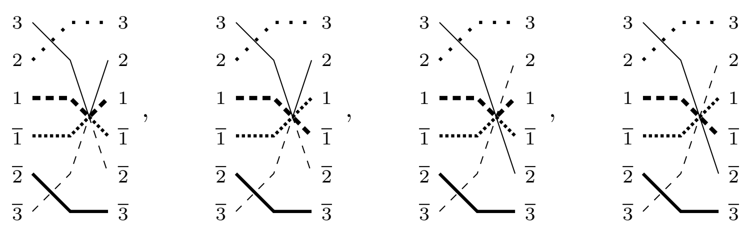
$$\begin{aligned} \Pi^{\mathsf{BC}}(F_w) &\to \{(\tau, K) \in \Pi^{\mathsf{BC}}(F_w) \times 2^{[k]} \mid \ell_t(\mathrm{type}(\tau)) = 0\} \\ \pi &\mapsto (\tau, \{\mathrm{snk}(\pi_{\overline{n}}), \dots, \mathrm{snk}(\pi_{\overline{1}})\} \cap \mathbb{N}). \end{aligned} \tag{103}$$

Since the positively indexed paths $\tau_1, \dots, \tau_n$ of τ cover the upper half of F_w, i.e. the planar network $F_{\varphi(w)} \in \mathcal{S}_{\mathrm{D}}^{\mathsf{A}}([n])$, we also have the bijection

$$\begin{aligned} &\mathcal{U}^{\mathsf{BC}}(F_w, n) \to \mathcal{U}^{\mathsf{A}}(F_{\varphi(w)}, n) \times 2^{[k]} \\ &\boxed{\pi_{u_1}} \cdots \boxed{\pi_{u_n}} \mapsto \big(\boxed{\tau_{u_1}} \cdots \boxed{\tau_{u_n}}, \{\mathrm{snk}(\pi_{\overline{n}}), \dots, \mathrm{snk}(\pi_{\overline{1}})\} \cap \mathbb{N} \big), \end{aligned} \tag{104}$$

which preserves the row-closed, left row-strict, and cylindrical properties of tableaux. For example consider the network $F_{\overline{1}3\overline{2}}$ and the unique path family $\tau \in \Pi^{\mathsf{BC}}(F_{\overline{1}3\overline{2}})$ of type 132. Since the type-BC unit interval order $Q(\overline{1}3\overline{2})$ has 2 grounded elements, the equivalence class of τ consists of four path families encoded by (τ, K) for subsets $K \subseteq \{1, 2\}$,

(105)



$(\tau, \varnothing)$ $\quad$ $(\tau, \{1\})$ $\quad$ $(\tau, \{2\})$ $\quad$ $(\tau, \{1, 2\})$

We can now relate certain sets of F_w-tableaux and marked $Q(w)$-tableaux as follows.

Lemma 9.4. *For $F_w \in \mathcal{S}_{\mathrm{D}}^{\mathsf{BC}}([\overline{n}, n])$, and corresponding $Q = Q(w)$, we have bijections*

(i) $\{U \in \mathcal{U}^{\mathsf{BC}}(F_w, n) \mid U$ *left row-strict*$\} \overset{1-1}{\longleftrightarrow} \{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \mid U$ *descent-free*$\}$,

(ii) $\{U \in \mathcal{U}^{\mathsf{BC}}(F_w, n) \mid U$ *cylindrical*$\} \overset{1-1}{\longleftrightarrow}$
$\{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \mid U$ *cyclically row-semistrict*$\}$.

Proof. Let k be the number of grounded elements of $Q(w)$. By (104), tableaux on the left-hand side of (i) correspond bijectively to pairs

$$\{(U, K) \in \mathcal{U}^{\mathsf{A}}(F_{\varphi(w)}, n) \times 2^{[k]} \mid U \text{ left row-strict}\},$$

and by (98) these correspond bijectively to

$$\{(V, K) \in \mathcal{U}^{\mathsf{A}}(Q(w), n) \times 2^{[k]} \mid V \text{ descent-free}\}.$$

Elements of this set correspond bijectively to tableaux on the right-hand side of (i): simply modify V by marking entries belonging to K. Similarly, tableaux in the first set of (ii) correspond bijectively to pairs

$$\{(U, K) \in \mathcal{U}^{\mathsf{A}}(F_{\varphi(w)}, n) \times 2^{[k]} \mid U \text{ cylindrical}\},$$

and by (99) these correspond bijectively to

$$\{(V, K) \in \mathcal{U}^{\mathsf{A}}(Q(w), n) \times 2^{[k]} \mid V \text{ cyclically row-semistrict}\}.$$

Elements of this set correspond bijectively to tableaux on the right-hand side of (ii): again modify V by marking entries belonging to K. □

Combining these bijections with Proposition 6.5, we obtain the following type-BC analogs of the results in Proposition 9.1.

Proposition 9.5. *Fix $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231 with $F_w \in \mathcal{S}_Z^{\mathsf{BC}}([\overline{n}, n])$ having path matrix A, and define A^+, A^- as in (67). Let $Q = Q(w)$ be the type-*BC *unit interval order defined before Proposition 7.6. We have*

$$\begin{aligned}
\mathrm{per}(A^+) &= \#\{U \in \mathcal{U}^{\mathsf{BC}}(F_w, n) \,|\, U \text{ left row-strict}\} \\
&= \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \,|\, U \text{ descent-free}\} \\
&= \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \,|\, U \text{ excedance-free}\} \\
&= \# \text{ marked acyclic orientations of } \mathrm{inc}(Q),
\end{aligned}$$

$$\begin{aligned}
\mathrm{per}(A^-) &= \#\{U \in \mathcal{U}^{\mathsf{BC}}(F_w, n) \,|\, U \text{ left row-strict with no grounded paths}\} \\
&= \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \,|\, U \text{ descent-free with no grounded elements}\} \\
&= \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \,|\, U \text{ excedance-free with no grounded elements}\} \\
&= \# \text{ acyclic orientations of } \mathrm{inc}(Q) \text{ with no grounded vertices},
\end{aligned}$$

$$\begin{aligned}
\det(A^+) &= \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, 1^n) \,|\, U \text{ column-strict with at most 1 grounded element}\} \\
&= \# \text{ proper marked } \mathsf{BC}\text{-colorings of } \mathrm{inc}(Q) \text{ of type } (n, \varnothing), \\
&= \begin{cases} 2^k & \text{if } Q \text{ is a chain with } k \leqslant 1 \text{ grounded elements}, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned}$$

$$\begin{aligned}
\det(A^-) &= \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, 1^n) \,|\, U \text{ column-strict with no grounded elements}\} \\
&= \# \text{ proper marked } \mathsf{BC}\text{-colorings of } \mathrm{inc}(Q) \text{ of type } (\varnothing, n), \\
&= \begin{cases} 1 & \text{if } Q \text{ is a chain with no grounded elements}, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Imm}_{\psi^n}^{\mathfrak{S}_n}(A^+) &= \#\{U \in \mathcal{U}^{\mathsf{BC}}(F_w, n) \,|\, U \text{ cylindrical}\}, \\
&= \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \,|\, U \text{ cyclically row-semistrict}\}, \\
&= \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \,|\, U \text{ record-free, row-semistrict}\}, \\
&= n \cdot \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \,|\, U \text{ right-anchored, row-semistrict}\}, \\
&= \# \text{ marked acyclic orientations of } \mathrm{inc}(Q) \text{ with one source},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Imm}_{\psi^n}^{\mathfrak{S}_n}(A^-) &= \#\{U \in \mathcal{U}^{\mathsf{BC}}(F_w, n) \,|\, U \text{ cylindrical with no grounded paths}\}, \\
&= \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \,|\, U \text{ cyclically row-semistrict with no grounded elements}\}, \\
&= \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \,|\, U \text{ record-free, row-semistrict with no grounded elements}\}, \\
&= n \cdot \#\{U \in \mathcal{U}^{\mathsf{BC}}(Q, n) \,|\, U \text{ right-anchored, row-semistrict w/no grounded elts.}\}, \\
&= \# \text{ acyclic orientations of } \mathrm{inc}(Q) \text{ with one source and no grounded vertices.}
\end{aligned}$$

Proof. By Proposition 6.5, $\mathrm{per}(A^+)$ counts all families in $\Pi^{\mathsf{BC}}(F_w)$, equivalently (101) all left row-strict tableaux in $\mathcal{U}^{\mathsf{BC}}(F_w, n)$. By Lemma 9.4 this equals the number of descent-free marked Q-tableaux, and by (98) it equals the number of excedance-free marked Q-tableaux. By the algorithms at the end of Subsection 8.1, this also equals the number of marked acyclic orientations of $\mathrm{inc}(Q)$. By Proposition 6.5, $\mathrm{per}(A^-)$ counts the same assuming that $\ell_t(w) = 0$, i.e. that Q has no grounded elements, and is 0 otherwise.

Now let $\sigma = (\sigma_{\overline{n}}, \dots, \sigma_{\overline{1}}, \sigma_1, \dots, \sigma_n)$ be the unique path family in $\Pi_e^{\mathsf{BC}}(F_w)$, so that Q is the poset on $\sigma_1, \dots, \sigma_n$, with grounded elements $\sigma_1, \dots, \sigma_k$ for some k. By Proposition 6.5, $\det(A^+)$ counts the families $\pi \in \Pi^{\mathsf{BC}}(F_w)$ in which only $\pi_{\overline{1}}$ and π_1 may share a vertex. This number is nonzero if and only if Q is an n-element chain. If $\sigma_{\overline{1}}$ and σ_1 do not share a vertex, then σ is the unique such family and $\det(A^+) = 1$.

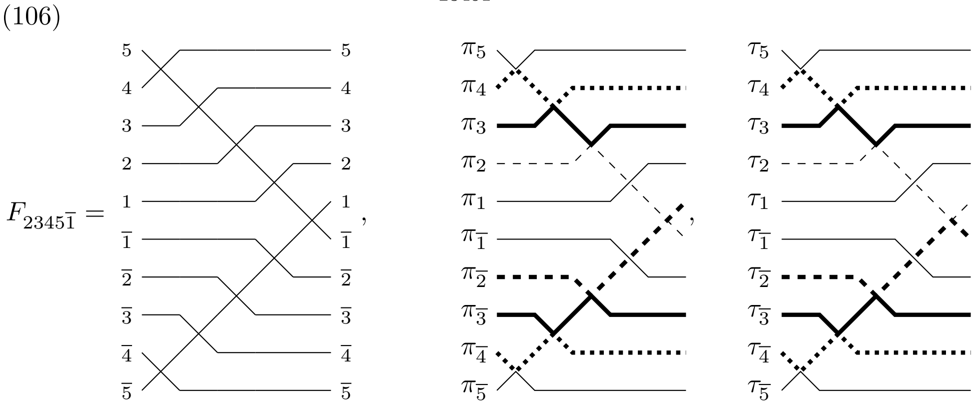
In this case, Q has no grounded element and $|\mathcal{U}^{\mathsf{BC}}(Q,1^n)| = 1$. If on the other hand $\sigma_{\overline{1}}$ and σ_1 do share a vertex, then exactly one other path family $\pi \in \Pi^{\mathsf{BC}}(F_w)$ is counted and we have $\det(A^+) = 2$. In this case, Q has one grounded element and $|\mathcal{U}^{\mathsf{BC}}(Q,1^n)| = 2$ because the element may appear with or without a star in a marked Q-tableau. More simply, $\det(A^-)$ is 1 if $F_w = F_e$ and is 0 otherwise. Equivalently, $\det(A^-)$ is 1 if Q is a chain with no grounded element and is 0 otherwise.

By Proposition 6.5, $\mathrm{Imm}^{\mathfrak{S}_n}_{\psi^n}(A^+)$ equals n times the number of path families π in $\Pi^{\mathsf{BC}}(F_w)$ such that $\varphi(\mathrm{type}(\pi)) \in \mathfrak{S}_n$ is an n-cycle. This is the number of cylindrical tableaux in $\mathcal{U}^{\mathsf{BC}}(F_w, n)$ because each family π can be arranged in the orders $(\pi_j, \pi_{v(j)}, \pi_{v(v(j))}, \dots, \pi_{v^{-1}(j)})$ for $j = 1, \dots, n$ to create n cylindrical π-tableaux. The remaining interpretations follow from (99). $\mathrm{Imm}_{\psi^n}(A^-)$ is the same, assuming that $\ell_t(w) = 0$, i.e. that Q has no grounded elements, and is 0 otherwise. □

To combinatorially interpret evaluations of $\mathfrak{B}_n$-traces, and state type-BC analogs of the results in Theorem 9.2, we will use *pairs* of path-tableaux, poset-tableaux, and acyclic orientations. Define a *Young bidiagram of shape* (λ, μ) to be a pair of Young diagrams of shapes λ and μ. Given a type-BC path family $\pi = (\pi_{\overline{n}}, \dots, \pi_{\overline{1}}, \pi_1, \dots, \pi_n)$ covering a zig-zag network $F_w \in \mathcal{S}^{\mathsf{BC}}_{\mathrm{Z}}([\overline{n}, n])$, we fill Young bidiagram of shape (λ, μ) with the paths $(\pi_1, \dots, \pi_n)$, keeping grounded paths in the left diagram. We call the resulting pair (U, V) of tableaux an F_w*-bitableau* or more specifically, a π*-bitableau* of shape (λ, μ). If π has type $v \in \mathfrak{B}_n$, then we also say that each π-bitableau has *type* v. Let $\mathcal{B}(F_w, \lambda, \mu)$ be the set of all F_w-bitableaux of shape (λ, μ). If (U, V) is a π-bitableau of type v with $\varphi(v) = e$, then we may use (103) to replace paths $\pi_1, \dots, \pi_n$ in (U, V) with the elements of $Q(w)$, marking each grounded element i in U with a star if $\mathrm{snk}(\pi_i) < 0$. We call the resulting structure a *marked* $Q(w)$*-tableau*. Let $\mathcal{B}(Q(w), \lambda, \mu)$ be the set of all marked $Q(w)$-bitableaux of shape (λ, μ).

For example, consider $F_{2345\overline{1}} \in \mathcal{S}^{\mathsf{BC}}_{\mathrm{D}}([\overline{5}, 5])$ and the unique path families π of type $2\overline{1}345$ and τ of type 21345 covering $F_{2345\overline{1}}$,

(106)

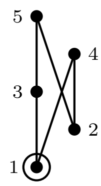


Two π-bitableaux and two τ-bitableaux of shape $(21, 11)$ are

(107)

$$\left(\begin{array}{|c|c|}\hline \pi_5 \\ \cline{1-1} \pi_2 & \pi_1 \\ \hline\end{array}, \begin{array}{|c|}\hline \pi_4 \\ \hline \pi_3 \\ \hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline \pi_5 \\ \cline{1-1} \pi_2 & \pi_3 \\ \hline\end{array}, \begin{array}{|c|}\hline \pi_1 \\ \hline \pi_4 \\ \hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline \tau_5 \\ \cline{1-1} \tau_1 & \tau_2 \\ \hline\end{array}, \begin{array}{|c|}\hline \tau_4 \\ \hline \tau_3 \\ \hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline \tau_4 \\ \cline{1-1} \tau_1 & \tau_2 \\ \hline\end{array}, \begin{array}{|c|}\hline \tau_5 \\ \hline \tau_3 \\ \hline\end{array}\right).$$

Since paths π_2 and τ_2 intersect $\pi_{\overline{2}}$ and $\tau_{\overline{2}}$, respectively, they are grounded and must appear in the left tableaux. The poset $Q(2345\overline{1})$ and four marked $Q(2345\overline{1})$-bitableaux of shape $(21, 11)$ are

(108) 

$$, \quad \left(\begin{array}{|c|c} \hline 3 & \\ \hline 1^\star & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|} \hline 5 \\ \hline 4 \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline \end{array}, \begin{array}{|c|} \hline 5 \\ \hline 2 \\ \hline \end{array} \right),$$
$$\left(\begin{array}{|c|c} \hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array}, \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c} \hline 1^\star & \\ \hline 3 & \multicolumn{1}{c|}{5} \\ \hline \end{array}, \begin{array}{|c|} \hline 4 \\ \hline 2 \\ \hline \end{array} \right).$$

If (U, V) is an F_w-bitableau or $Q(w)$-bitableau, then the component tableaux U, V could have some of the properties enumerated between (93) and Proposition 9.1. Some combinations of these may be used to interpret trace evaluations as in Theorems 9.6 – 9.9.

Theorem 9.6. *Let $w \in \mathfrak{B}_n$ avoid the patterns 3412 and 4231 and let $Q = Q(w)$ be the related type-BC unit interval order. For each bipartition $(\lambda, \mu) \vdash n$ we have*

$$(\epsilon\epsilon)^{\lambda,\mu}(\widetilde{C}^{\mathsf{BC}}_w(1)) = \#\{(U,V) \in \mathcal{B}(Q, \lambda^\top, \mu^\top) \,|\, U, V \textit{ column-strict}\},$$
$$(\epsilon\eta)^{\lambda,\mu}(\widetilde{C}^{\mathsf{BC}}_w(1)) = \#\{(U,V) \in \mathcal{B}(Q, \lambda^\top, \mu) \,|\, U \textit{ column-strict}, V \textit{ row-semistrict}\},$$
$$(\eta\epsilon)^{\lambda,\mu}(\widetilde{C}^{\mathsf{BC}}_w(1)) = \#\{(U,V) \in \mathcal{B}(Q, \lambda, \mu^\top) \,|\, U \textit{ row-semistrict}, V \textit{ column-strict}\},$$
$$(\eta\eta)^{\lambda,\mu}(\widetilde{C}^{\mathsf{BC}}_w(1)) = \#\{(U,V) \in \mathcal{B}(Q, \lambda, \mu) \,|\, U, V \textit{ row-semistrict}\}.$$

Proof. Let $F = F_w$ have path matrix A, fix $(\lambda, \mu) \vdash n$ with $|\lambda| = k$, and let $\theta = (\zeta\xi)^{\lambda,\mu}$ be one of the characters in the theorem. By Lemma 6.3 and Theorem 6.4, we have

$$(\zeta\xi)^{\lambda,\mu}(\widetilde{C}^{\mathsf{BC}}_w(1)) = \mathrm{Imm}^{\mathfrak{B}_n}_{(\zeta\xi)^{\lambda,\mu}}(A) = \sum_{\substack{I \subseteq [n] \\ |I| = k}} \mathrm{Imm}^{\mathfrak{S}_k}_{\zeta^\lambda}(A^+_{I,I}) \mathrm{Imm}^{\mathfrak{S}_{n-k}}_{\xi^\mu}(A^-_{[n] \smallsetminus I, [n] \smallsetminus I}). \tag{109}$$

By (60), each of the type-A immanants above is a sum of products of determinants or permanents of matrices in $\{A^+_{J,J} \,|\, J \subset I\}$ or $\{A^-_{J,J} \,|\, J \subset [n] \smallsetminus I\}$. By Proposition 9.5 each factor counts certain one-row or one-column tableaux and their product counts bitableaux of the required shape. ☐

Corollary 9.7. *Let $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231 have zig-zag network F_w and type-BC unit interval order $Q = Q(w)$. The combinatorial interpretations of trace evaluations in Theorem 9.6 have several valid alternatives. We may replace (marked) column-strict Q-tableaux U, V of shape ν with (marked) colorings of* $\mathrm{inc}(Q)$ *of type ν. We may replace (marked) row-semistrict Q-tableaux U, V of shape ν with either of the following:*

(i) *(marked) excedance-free Q-tableaux of shape ν,*
(ii) *(marked) row-closed, left row-strict F_w-tableaux of shape ν.*

Furthermore, we have that $(\eta\eta)^{\lambda,\mu}(\widetilde{C}^{\mathsf{BC}}_w(1))$ equals

(iii) *the number of marked acyclic orientations of subgraph sequences*

$$(\mathrm{inc}(Q_{I_1}), \dots, \mathrm{inc}(Q_{I_r}), \mathrm{inc}(Q_{J_1}), \dots, \mathrm{inc}(Q_{J_t})) \tag{110}$$

where $(I_1, \dots, I_{\ell(\lambda)}, J_1, \dots, J_{\ell(\mu)})$ varies over all ordered set partitions of $[n]$ of type

$$(\lambda_1, \dots, \lambda_{\ell(\lambda)}, \mu_1, \dots, \mu_{\ell(\mu)}),$$

and all grounded vertices appear in $I_1, \dots, I_{\ell(\lambda)}$.

Proof. By Proposition 9.5, one-rowed tableaux in Theorem 9.6 which are row-semistrict (descent-free) correspond bijectively to one-rowed tableaux with properties (*i*) or (*ii*) and to acyclic orientations (*iii*) of $\mathrm{inc}(Q)$. Thus there is a correspondence of several-rowed tableaux and of acyclic orientations of subgraph sequences. In all cases, we have equality of the indices of paths, poset elements, and vertices which are grounded, so markings correspond in the obvious way. □

For example, consider evaluating traces at $\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1)$ by applying Theorem 9.6 to the poset $Q(2345\bar{1})$ in (108). The first bitableau in (108) contributes to

$$(\epsilon\eta)^{21,11}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1)),\ (\eta\eta)^{21,11}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1)),$$

the second to

$$(\epsilon\epsilon)^{21,2}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1)),\ (\epsilon\eta)^{21,11}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1)),\ (\eta\epsilon)^{21,2}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1)),\ (\eta\eta)^{21,11}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1)),$$

the third to

$$(\eta\eta)^{21,11}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1)),$$

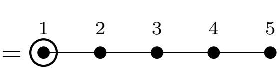
and the fourth to

$$(\eta\epsilon)^{21,2}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1)),\ (\eta\eta)^{21,11}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1)).$$

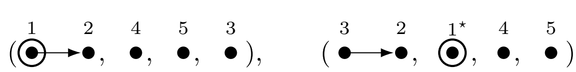
If we modify the criteria of the theorem as in Corollary 9.7 (i), the same is true, since a tableau with two or fewer columns is Q-excedance-free if and only if it is Q-row-semistrict.

If instead we modify the criteria of the theorem as in Corollary 9.7 (ii) we may apply these to path families covering $F_{2345\bar{1}}$ in (106). In this case, we find that the third and fourth tableaux in (107) contribute to $(\eta\eta)^{21,11}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1))$, and that the fourth tableau also contributes to $(\eta\epsilon)^{21,2}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1))$. The tableaux contribute to no additional evaluations listed in Theorem 9.6 because the left tableau of the first bitableau is not left row-semistrict, the left tableau of the second bitableau is not row-closed, and the left tableaux of all four bitableaux fail to have the property that source and sink indices are equal.

Finally, we may modify the criteria of the theorem by applying Corollary 9.7 (iii) to

(111) $\mathrm{inc}(Q(2345\bar{1})) =$ ,

and by marking and acyclically orienting sequences of subgraphs on $(2,1,1,1)$ vertices. Two sequences contributing to $(\eta\eta)^{21,11}(\widetilde{C}_{2345\bar{1}}(1))$ are

(112) 

Theorem 9.8. *Let $w \in \mathfrak{B}_n$ avoiding the patterns* 3412 *and* 4231 *have type-*BC *unit interval order Q, and fix bipartition $(\lambda,\mu) \vdash n$. We have*

$$(\chi\chi)^{\lambda,\mu}(\widetilde{C}^{\mathsf{BC}}_{w}(1)) = \#\{(U,V) \in \mathcal{B}(Q,\lambda,\mu) \,|\, U, V \text{ standard }\}.$$

Proof. Let A be the path matrix of F_w and let χ^λ, χ^μ be the characters of $\mathfrak{S}_k$, $\mathfrak{S}_{n-k}$ satisfying $(\chi\chi)^{\lambda,\mu} = (\chi^\lambda \otimes \delta\chi^\mu)\big\uparrow^{\mathfrak{B}_n}_{\mathfrak{B}_{k,n-k}}$. By Lemma 6.3 and Theorem 6.4 we have

$$(\chi\chi)^{\lambda,\mu}(\widetilde{C}^{\mathsf{BC}}_{w}(1)) = \mathrm{Imm}^{\mathfrak{B}_n}_{(\chi\chi)^{\lambda,\mu}}(A) = \sum_{\substack{I\subset[n]\\|I|=k}} \mathrm{Imm}^{\mathfrak{S}_k}_{\chi^\lambda}(A^+_{I,I})\mathrm{Imm}^{\mathfrak{S}_{n-k}}_{\chi^\mu}(A^-_{[n]\smallsetminus I,[n]\smallsetminus I}).$$

Expanding irreducible character immanants $\mathrm{Imm}^{\mathfrak{S}_k}_{\chi^\lambda}$ and $\mathrm{Imm}^{\mathfrak{S}_{n-k}}_{\chi^\mu}$ in terms of induced trivial character immanants, we obtain

$$\sum_{\substack{I\subset[n]\\|I|=k}} \sum_{\alpha\vdash k} K^{-1}_{\alpha,\lambda}\mathrm{Imm}_{\eta^\alpha}(A^+_{I,I}) \sum_{\beta\vdash n-k} K^{-1}_{\beta,\mu}\mathrm{Imm}_{\eta^\beta}(A^-_{[n]\smallsetminus I,[n]\smallsetminus I}).$$

Now let Q be the type-BC unit interval order corresponding to w. For any subset J of $[n]$ and any partition $\nu \vdash |J|$, let $r(Q_J, \nu)$ be the number of row-semistrict Q_J-tableaux of shape ν, and let $p(J)$ be the number of elements of Q_J which are grounded. By (60) and Proposition 9.5, we have

$$\mathrm{Imm}^{\mathfrak{S}_k}_{\eta^\alpha}(A^+_{I,I}) = 2^{p(I)} r(Q_I, \alpha),$$

and

$$\mathrm{Imm}^{\mathfrak{S}_{n-k}}_{\eta^\beta}(A^-_{[n]\smallsetminus I,[n]\smallsetminus I}) = \begin{cases} r(Q_{[n]\smallsetminus I}, \beta) & \text{if } p([n] \smallsetminus I) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus by [18, Thm. 4.7 (ii-b), (iii)], the sums

$$\sum_{\alpha \vdash k} K^{-1}_{\alpha,\lambda} \mathrm{Imm}^{\mathfrak{S}_k}_{\eta^\alpha}(A^+_{I,I}) = 2^{p(I)} \sum_{\alpha \vdash k} K^{-1}_{\alpha,\lambda} r(Q_I, \alpha),$$

$$\sum_{\beta \vdash n-k} K^{-1}_{\beta,\mu} \mathrm{Imm}^{\mathfrak{S}_{n-k}}_{\eta^\beta}(A^-_{[n]\smallsetminus I,[n]\smallsetminus I}) = \begin{cases} \displaystyle\sum_{\beta \vdash n-k} K^{-1}_{\beta,\mu} r(Q_{[n]\smallsetminus I}, \beta) & \text{if } p([n] \smallsetminus I) = 0, \\ 0 & \text{otherwise} \end{cases}$$

are equal to the numbers of standard Q_I-tableaux of shape λ containing any number of grounded elements which may be circled, and standard $Q_{[n]\smallsetminus I}$-tableaux of shape μ containing no grounded elements, respectively. ☐

For example, consider the evaluation $(\chi\chi)^{21,11}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1))$ and the poset $Q(2345\bar{1})$ in (108). Of the bitableaux shown there, only the second contributes to this evaluation.

Theorem 9.9. *Let $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231 have type-BC unit interval order Q, and fix bipartition $(\lambda, \mu) \vdash n$. We have*

$$(\psi\psi)^{\lambda,\mu}(\widetilde{C}^{\mathsf{BC}}_w(1)) = \#\{(U, V) \in \mathcal{B}(Q, \lambda, \mu) \,|\, U, V \text{ cyclically row-semistrict } \}.$$

Proof. Let A be the path matrix of F_w and let $k = |\lambda|$. By Lemma 6.3 and Theorem 6.4 we have

(113)
$$(\psi\psi)^{\lambda,\mu}(\widetilde{C}^{\mathsf{BC}}_w(1)) = \mathrm{Imm}^{\mathfrak{B}_n}_{(\psi\psi)^{\lambda,\mu}}(A) = \sum_{\substack{I \subset [n] \\ |I| = m}} \mathrm{Imm}^{\mathfrak{S}_k}_{\psi^\lambda}(A^+_{I,I}) \mathrm{Imm}^{\mathfrak{S}_{n-k}}_{\psi^\mu}(A^-_{[n]\smallsetminus I,[n]\smallsetminus I}).$$

By [70, Prop. 2.4] we have

$$\mathrm{Imm}^{\mathfrak{S}_k}_{\psi^\lambda}(\mathbf{x}) = \sum_{(J_1,\ldots,J_r)} \mathrm{Imm}^{\mathfrak{S}_{\lambda_1}}_{\psi^{\lambda_1}}(\mathbf{x}_{J_1,J_1}) \cdots \mathrm{Imm}^{\mathfrak{S}_{\lambda_r}}_{\psi^{\lambda_r}}(\mathbf{x}_{J_r,J_r}).$$

Thus each term in the sum (113) is itself a sum of products of type-A single-cycle power sum trace immanants, evaluated at A^+ or A^-. By Proposition 9.5 each factor counts one-row cyclically row-semistrict tableaux, and their product counts bitableaux of the required shape. ☐

Corollary 9.10. *Let $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231 have zig-zag network F_w and type-BC unit interval order $Q = Q(w)$. In Theorem 9.9 we may replace cyclically row-semistrict Q-tableaux with either of the following:*

(i) *record-free Q-tableaux,*
(ii) *cylindrical F_w-tableaux.*

We also have that $(\psi\psi)^{\lambda,\mu}(\widetilde{C}^{\mathsf{BC}}_w(1))$ equals

(iii) $\lambda_1 \cdots \lambda_r \mu_1 \cdots \mu_s \cdot \#\{(U, V) \in \mathcal{B}(Q, \lambda, \mu) \,|\, U, V$ *left-anchored, row-semistrict* $\}$.
(iv) *the number of marked acyclic orientations of subgraph sequences (110) in which each oriented subgraph has one source, and all grounded vertices appear in $I_1, \ldots, I_r$.*

Proof. By Proposition 9.5, one-rowed tableaux in Theorem 9.9 which are cyclically row-semistrict correspond bijectively to one-rowed tableaux with properties (i) or (ii) and to acyclic orientations (iii) of $\mathrm{inc}(Q)$ which have one source. Thus there is a correspondence of several-rowed tableaux and of acyclic orientations of subgraph sequences. In all cases, we have equality of the indices of paths, poset elements, and vertices which are grounded, so markings correspond in the obvious way. □

For example, consider evaluating $(\psi\psi)^{21,11}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1))$ by applying Theorem 9.9 to the poset $Q(2345\bar{1})$ in (108). Of the tableaux listed there, only the first and third contribute to this trace evaluation. If we modify the criteria of the theorem as in Corollary 9.10 (i), the same is true, since a tableau with two or fewer columns is Q-record-free if and only if it is cyclically Q-row-semistrict. On the other hand, if we modify the criteria of the theorem as in Corollary 9.10 (iii), then the first, second, and fourth bitableaux contribute.

Now suppose that we modify the criteria of the theorem as in Corollary 9.10 (ii). Then we may apply these to path families covering $F_{2345\bar{1}}$ in (106), and we find that the first, third and fourth tableaux in (107) contribute to $(\psi\psi)^{21,11}(\widetilde{C}^{\mathsf{BC}}_{2345\bar{1}}(1))$.

Finally, we may modify the criteria of the theorem by applying Corollary 9.10 (iii) to (111). Both tableaux in (112) contribute, since no component has more than one source.

While Proposition 9.1 – Theorem 9.3 have known q-analogs, no such q-analogs are known for the results in Subsection 9.2.

Problem 9.11. *State and prove q-analogs of the results in Lemma* 9.4 *– Corollary* 9.10

10. Some equivalence relations

Given a Coxeter group W and its Hecke algebra $H = H(W)$, the trace space $\mathcal{T}(H)$ naturally partitions H into equivalence classes via the relation $\approx$ defined by

$$D_1 \approx D_2 \text{ if for all } \theta_q \in \mathcal{T}(H) \text{ we have } \theta_q(D_1) = \theta_q(D_2). \tag{114}$$

When $W = \mathfrak{S}_n$ or $\mathfrak{B}_n$, we may restrict this relation to the subset of Kazhdan-Lusztig basis elements indexed by elements of W avoiding the patterns 3412 and 4231. Two more equivalence relations related to this restricted relation are defined in terms of isomorphism of the posets and graphs described in Sections 7 – 8: $P(v) \cong P(w)$ ($Q(v) \cong Q(w)$), or $G(v) \cong G(w)$. In type A these relations refine the first; in types BC, we conjecture the same to be true and prove a weaker statement.

10.1. Type-A equivalence relations.

The equivalence relation on

$$\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n \text{ avoids the patterns 3412 and 4231}\} \tag{115}$$

defined by poset isomorphism $P(v) \cong P(w)$ refines that defined by graph isomorphism $G(v) \cong G(w)$, which in turn refines the restriction of the relation $\approx$ (114) to this set.

Theorem 10.1. *For $v, w \in \mathfrak{S}_n$ avoiding the patterns* 3412 *and* 4231*, we have the implications*

$$P(v) \cong P(w) \quad \Longrightarrow \quad G(v) \cong G(w) \quad \Longrightarrow \quad \widetilde{C}_v(q) \approx \widetilde{C}_w(q).$$

Proof. The first implication is clear; the second follows from a q-analog of Theorem 9.2 (i-b) in [18, Prop. 7.3 – Thm. 7.4] which states that all of the evaluations $\{\epsilon_q^\lambda(\widetilde{C}_w(q)) \,|\, \lambda \vdash n\}$ are determined by $G(w)$. □

The converse of the first implication above is not true (89); the converse of the second is not known to be true.

Problem 10.2. *For some n find* 3412*-avoiding,* 4231*-avoiding permutations $v, w \in \mathfrak{S}_n$ which satisfy $G(v) \not\cong G(w)$ and $\widetilde{C}_v(q) \approx \widetilde{C}_w(q)$ or show that this is impossible.*

By Theorems 7.1 and 10.1, the problem of evaluating $H_n(q)$-traces at (115) reduces to the problem of evaluating these traces at the subset of (115) indexed by 312-avoiding permutations [18, Thm. 5.6].

Corollary 10.3. *For $w \in \mathfrak{S}_n$ avoiding the patterns* 3412 *and* 4231*, there exists $v \in \mathfrak{S}_n$ avoiding* 312 *such that $\widetilde{C}_w(q) \approx \widetilde{C}_v(q)$.*

Furthermore, some experimentation suggests that the problem of evaluating traces at *all* Kazhdan–Lusztig basis elements reduces to the problem of evaluating traces at the subset of (115) indexed by 312-avoiding permutations. With precise details of such a reduction not yet conjectured, we have the following problem [4, Conj. 1.9], [33, Conj. 3.1].

Problem 10.4. *Show that for each $w \in \mathfrak{S}_n$ there exists a set $S = S(w) \subseteq \mathfrak{S}_n$ of* 312*-avoiding permutations and a set $\{p_{v,w}(q) \,|\, v \in S\} \subset \mathbb{N}[q]$ of polynomials such that we have*

$$\widetilde{C}_w(q) \approx \sum_{v \in S} p_{v,w}(q) \widetilde{C}_v(q).$$

We remark that Haiman [33, §3] introduced the name *codominant* for 312-avoiding permutations because by (5) these have the form vw_0 for w_0 the longest element of $\mathfrak{S}_n$ and v belonging to the dominant (132-avoiding) subset of the vexillary (2143-avoiding) permutations defined in [49].

10.2. Type-BC equivalence relations.

We distinguish between the type-BC case of the equivalence relation (114) and its $q = 1$ specialization. For $D_1, D_2 \in H_n^{\mathsf{BC}}(q)$, define

$$D_1 \approx_q D_2 \text{ if for all } \theta_q \in \mathcal{T}(H_n^{\mathsf{BC}}(q)) \text{ we have } \theta_q(D_1) = \theta_q(D_2). \tag{116}$$

For $D_1, D_2 \in \mathbb{Z}[\mathfrak{B}_n]$, let $D_1 \approx_1 D_2$ be the $q = 1$ specialization of the above. These relations naturally restrict to the Kazhdan–Lusztig bases of $H_n^{\mathsf{BC}}(q)$ and $\mathbb{Z}[\mathfrak{B}_n]$, and to the subsets of these indexed by elements $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231. It is easy to show that the equivalence relation on

$$\{\widetilde{C}_w^{\mathsf{BC}}(q) \,|\, w \in \mathfrak{B}_n \text{ avoiding the patterns 3412 and 4231}\} \tag{117}$$

defined by type-BC unit interval order isomorphism $Q(v) \cong Q(w)$ refines that defined by type-BC indifference graph isomorphism $\Gamma(v) \cong \Gamma(w)$, which in turn refines the relation $\approx_1$.

Theorem 10.5. *For $v, w \in \mathfrak{B}_n$ avoiding the patterns* 3412 *and* 4231*, we have the implications*

$$Q(v) \cong Q(w) \quad \implies \quad \Gamma(v) \cong \Gamma(w) \quad \implies \quad \widetilde{C}_v^{\mathsf{BC}}(1) \approx_1 \widetilde{C}_w^{\mathsf{BC}}(1).$$

Proof. The first implication is clear; the second follows from Corollary 9.7 ☐

The converse of the first implication above is not true (90); the converse of the second is not known to be true. We conjecture that the second implication can be strengthened.

Conjecture 10.6. *For $v, w \in \mathfrak{B}_n$ avoiding the patterns* 3412 *and* 4231*, we have the implication $\Gamma(v) \cong \Gamma(w) \implies \widetilde{C}_v^{\mathsf{BC}}(q) \approx_q \widetilde{C}_w^{\mathsf{BC}}(q)$.*

The converse of this conjectured implication is not known to be true.

Problem 10.7. *For some n find 3412-avoiding, 4231-avoiding permutations $v, w \in \mathfrak{B}_n$ which satisfy $\Gamma(v) \not\cong \Gamma(w)$ and $\widetilde{C}^{\mathsf{BC}}_v(q) \approx_q \widetilde{C}^{\mathsf{BC}}_w(q)$, or show that this is impossible.*

By Theorems 5.19 and 10.5, the problem of evaluating $\mathfrak{B}_n$-traces at

$$\{\widetilde{C}^{\mathsf{BC}}_w(1) \,|\, w \in \mathfrak{B}_n \text{ avoids the patterns 3412 and 4231}\} \tag{118}$$

reduces to the problem of evaluating these traces at the subset of (118) indexed by elements of $\mathfrak{B}_n$ avoiding the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$.

Corollary 10.8. *For $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231, there exists $v \in \mathfrak{B}_n$ avoiding the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$ such that we have $\widetilde{C}^{\mathsf{BC}}_w(1) \approx_1 \widetilde{C}^{\mathsf{BC}}_v(1)$.*

If Conjecture 10.6 is true, then the conclusion of Corollary 10.8 becomes $\widetilde{C}^{\mathsf{BC}}_w(q) \approx_q \widetilde{C}^{\mathsf{BC}}_v(q)$. It would be interesting to discover the extent to which trace evaluations at

$$\{\widetilde{C}^{\mathsf{BC}}_w(q) \,|\, w \in \mathfrak{B}_n \text{ avoids the signed patterns } 1\overline{2},\ \overline{2}1,\ \overline{21},\ 312,\ 3\overline{1}2\}$$

describe trace evaluations at the entire Kazhdan–Lusztig basis of $H^{\mathsf{BC}}_n(q)$, as in Problem 10.4. One might call the above $\mathfrak{B}_n$-elements *codominant* in analogy to codominant permutations in $\mathfrak{S}_n$, although the author is not aware of a definition of dominant elements of $\mathfrak{B}_n$ in the literature. On the other hand, it would be interesting to relate codominant elements of $\mathfrak{B}_n$ to the subsets of vexillary, theta-vexillary, Grassmannian, leading, and amenable elements of $\mathfrak{B}_n$, which appear in [7, 8, 13, 48, 71].

11. Symmetric functions

For $W = \mathfrak{S}_n$ or $\mathfrak{B}_n$, and $H = H(W)$ its Hecke algebra, bases of $\mathcal{T}(W)$ and $\mathcal{T}(H(W))$ are often studied in conjunction with bases of an appropriate module Λ of symmetric functions. When $W = \mathfrak{S}_n$, the symmetric function bases consist of traces of the corresponding Lie group; when $W = \mathfrak{B}_n$ they do not. In either case, we have the equalities

$$\mathrm{rank}(\mathcal{T}(W)) = \mathrm{rank}(\mathcal{T}(H(W))) = \mathrm{rank}(\Lambda),$$

which makes Λ a convenient setting in which to define generating functions for trace evaluations.

11.1. Type-A symmetric functions.

Corresponding to the six commonly used bases of $\mathcal{T}(H_n(q))$ and $\mathcal{T}(\mathfrak{S}_n)$ (§4.1) are six bases of the $\mathbb{Z}$-module

$$\Lambda_n(x) = \Lambda_n(x_1, x_2, \dots),$$

of homogeneous degree-n symmetric functions: the Schur basis $\{s_\lambda \,|\, \lambda \vdash n\}$, elementary basis $\{e_\lambda \,|\, \lambda \vdash n\}$, (complete) homogeneous basis $\{h_\lambda \,|\, \lambda \vdash n\}$, power sum basis $\{p_\lambda \,|\, \lambda \vdash n\}$, monomial basis $\{m_\lambda \,|\, \lambda \vdash n\}$, and forgotten basis $\{f_\lambda \,|\, \lambda \vdash n\}$. (See [67, Ch. 6].) The correspondence of trace bases and symmetric function bases is given explicitly by (the q-extension of) the Frobenius map

$$\mathrm{ch}_q : \mathcal{T}(H_n(q)) \to \Lambda_n(x) \tag{119}$$

$$\mathrm{ch}_q(\theta_q) := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \theta(w) p_{\mathrm{ctype}(w)} \tag{120}$$

$$= \sum_{\mu \vdash n} \frac{1}{z_\mu} \theta(\mu) p_\mu, \tag{121}$$

where $\theta = \theta_1$ as in Section 4, and $\theta(\mu) := \theta(w)$ for any $w \in \mathfrak{S}_n$ of cycle type μ. Specifically, we have

$$\begin{gathered}\mathrm{ch}_q(\chi_q^\lambda) = s_\lambda, \qquad \mathrm{ch}_q(\epsilon_q^\lambda) = e_\lambda, \qquad \mathrm{ch}_q(\eta_q^\lambda) = h_\lambda, \\ \mathrm{ch}_q(\psi_q^\lambda) = p_\lambda, \qquad \mathrm{ch}_q(\phi_q^\lambda) = m_\lambda, \qquad \mathrm{ch}_q(\gamma_q^\lambda) = f_\lambda.\end{gathered} \tag{122}$$

We construct generating functions for $H_n(q)$-trace evaluations as follows. Given element $D \in \mathbb{Q}(q) \otimes H_n(q)$, define the generating function

$$Y_q(D) := \sum_{\lambda \vdash n} \epsilon_q^\lambda(D) m_\lambda \in \mathbb{Q}(q) \otimes \Lambda_n(x)$$

for the evaluation of induced sign characters at D. By [62, Prop. 2.1], this symmetric function is in fact a generating function for the evaluation of *all* the standard traces at D, because it is equal to

$$\sum_{\lambda \vdash n} \eta_q^\lambda(D) f_\lambda = \sum_{\lambda \vdash n} \frac{(-1)^{n-\ell(\lambda)}\psi_q^\lambda(D)}{z_\lambda} p_\lambda = \sum_{\lambda \vdash n} \chi_q^{\lambda^\top}(D) s_\lambda = \sum_{\lambda \vdash n} \phi_q^\lambda(D) e_\lambda = \sum_{\lambda \vdash n} \gamma_q^\lambda(D) h_\lambda.$$

Equivalently, if we let $\omega : \Lambda_n(x) \to \Lambda_n(x)$ be the standard involution mapping

$$s_\lambda \mapsto s_{\lambda^\top}, \qquad e_\lambda \mapsto h_\lambda, \qquad p_\lambda \mapsto (-1)^{n-\ell(\lambda)} p_\lambda, \qquad m_\lambda \mapsto f_\lambda, \tag{123}$$

then we have that $\omega Y_q(D) = \sum_{\lambda \vdash n} \epsilon_q^\lambda(D) f_\lambda$ is equal to

$$\sum_{\lambda \vdash n} \eta_q^\lambda(D) m_\lambda = \sum_{\lambda \vdash n} \frac{\psi_q^\lambda(D)}{z_\lambda} p_\lambda = \sum_{\lambda \vdash n} \chi_q^\lambda(D) s_\lambda = \sum_{\lambda \vdash n} \phi_q^\lambda(D) h_\lambda = \sum_{\lambda \vdash n} \gamma_q^\lambda(D) e_\lambda.$$

It is not difficult to show that every symmetric function in $\mathbb{Q}(q) \otimes \Lambda_n(x)$ is $Y_q(D)$ for some $D \in \mathbb{Q}(q) \otimes H_n(q)$. (See [62, Prop. 3].)

Theorem 9.2 (*i-b*) shows that when $w \in \mathfrak{S}_n$ avoids the patterns 3412 and 4231, the symmetric function $Y_q(\widetilde{C}_w(q))$ is related to colorings of $\mathrm{inc}(P(w))$. More generally, Stanley [64] defined the *chromatic symmetric function* of any simple graph G to be

$$X_G := \sum_\kappa x_1^{|\kappa^{-1}(1)|} x_2^{|\kappa^{-1}(2)|} \cdots , \tag{124}$$

where the sum is over all proper colorings $\kappa : V \to \{1, 2, \dots, \}$ of G (§8.1). Expanding in the monomial basis of Λ_n, we have

$$X_G = \sum_\lambda c_{G,\lambda} m_\lambda,$$

where $c_{G,\lambda}$ is the number of proper colorings of G of type λ.

Shareshian and Wachs [59] defined a quasisymmetric extension $X_{G,q}$ of the symmetric function X_G. Given a proper coloring κ of G, define $\mathrm{INV}_G(\kappa)$ to be the number of pairs $(i, j) \in E$ with $i < j$ and $\kappa(i) > \kappa(j)$. For any composition $\alpha = (\alpha_1, \dots, \alpha_\ell) \vDash n$, define

$$c_{G,\alpha}(q) = \sum_{\substack{\kappa \text{ proper} \\ \mathrm{type}(\kappa) = \alpha}} q^{\mathrm{INV}_G(\kappa)},$$

and let

$$M_\alpha = \sum_{i_1 < \cdots < i_\ell} x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell}$$

be the *monomial quasisymmetric function* indexed by α. Then we have the definition

$$X_{G,q} = \sum_\kappa q^{\mathrm{INV}_G(\kappa)} x_1^{|\kappa^{-1}(1)|} x_2^{|\kappa^{-1}(2)|} \cdots = \sum_{\alpha \vDash n} c_{G,\alpha}(q) M_\alpha, \tag{125}$$

where the first sum is over proper colorings of G. It is easy to see that the $q = 1$ specialization of $X_{G,q}$ satisfies $X_{G,1} = X_G$. When $G = \mathrm{inc}(P)$ for a unit interval order P labeled as in Algorithm 7.3, the quasisymmetric function $X_{\mathrm{inc}(P),q}$ is in fact symmetric [60, Thm. 4.5]. Furthermore, we have the following [18, Thm. 7.4].

THEOREM 11.1. *For $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231, and $P = P(w)$ defined as in Subsection 7.1, we have $Y_q(\widetilde{C}_w(q)) = X_{\mathrm{inc}(P),q}$.*

11.2. TYPE-BC SYMMETRIC FUNCTIONS.

Corresponding to the eleven commonly used bases of $\mathcal{T}(H_n^{\mathsf{BC}}(q))$ and $\mathcal{T}(\mathfrak{B}_n)$ (§4.2) are eleven natural bases of the $\mathbb{Z}$-module of *type-*BC *symmetric functions* of degree n,

$$\Lambda_n(x,y) := \bigoplus_{k=0}^{n} \Lambda_k(x) \otimes \Lambda_{n-k}(y),$$

where $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. These bases of $\Lambda_n(x,y)$ consist of ten *nonplethystic bases* of the form $(og)_{\lambda,\mu} := o_\lambda(x) g_\mu(y)$,

$$\begin{gathered} \{(ss)_{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \quad \{(ee)_{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \quad \{(hh)_{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \\ \{(eh)_{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \quad \{(he)_{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \\ \{(mm)_{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \quad \{(ff)_{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \quad \{(mf)_{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \\ \{(fm)_{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \quad \{(pp)_{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}, \end{gathered} \tag{126}$$

and the *plethystic power sum basis*

$$\{p^+_\lambda p^-_\mu \,|\, (\lambda,\mu) \vdash n\}, \tag{127}$$

defined in terms of ordinary power sum symmetric functions $p_k(x) := x_1^k + x_2^k + \cdots$ by

$$\begin{aligned} p^+_\lambda &:= (p_{\lambda_1}(x) + p_{\lambda_1}(y)) \cdots (p_{\lambda_{\ell(\lambda)}}(x) + p_{\lambda_{\ell(\lambda)}}(y)), \\ p^-_\mu &:= (p_{\mu_1}(x) - p_{\mu_1}(y)) \cdots (p_{\mu_{\ell(\mu)}}(x) - p_{\mu_{\ell(\mu)}}(y)). \end{aligned} \tag{128}$$

The functions (128) often appear in the literature as $p_\lambda[X+Y], p_\mu[X-Y]$, respectively. A correspondence between bases of $\mathcal{T}(H_n^{\mathsf{BC}}(q))$ and the bases (126) – (127) of $\Lambda_n(x,y)$ is given explicitly by the (q-extension of the) plethystic BC-Frobenius map [52, §1, App. B]. (See also [6, Eq. (2.5)].)

$$\mathrm{pch}_q \colon \mathcal{T}(H_n^{\mathsf{BC}}(q)) \to \Lambda_n(x,y) \tag{129}$$

$$\mathrm{pch}_q(\theta_q) = \frac{1}{2^n n!} \sum_{w \in \mathfrak{B}_n} \theta(w) p^+_{\alpha(w)} p^-_{\beta(w)} \tag{130}$$

$$= \sum_{(\lambda,\mu) \vdash n} \frac{\theta(\lambda,\mu)}{z_\lambda z_\mu 2^{\ell(\lambda)+\ell(\mu)}} p^+_\lambda p^-_\mu, \tag{131}$$

where $\alpha(w)$, $\beta(w)$ are the partitions satisfying $\mathrm{sct}(w) = (\alpha(w), \beta(w))$, and where we define $\theta(\lambda,\mu) := \theta(w)$ for any $w \in \mathfrak{B}_n$ having $\mathrm{sct}(w) = (\lambda,\mu)$. Specifically, pch_q maps

$$\begin{gathered} (\eta\eta)_q^{\lambda,\mu} \mapsto (hh)_{\lambda,\mu}, \quad (\eta\epsilon)_q^{\lambda,\mu} \mapsto (he)_{\lambda,\mu}, \quad (\epsilon\eta)_q^{\lambda,\mu} \mapsto (eh)_{\lambda,\mu}, \quad (\epsilon\epsilon)_q^{\lambda,\mu} \mapsto (ee)_{\lambda,\mu}, \\ (\phi\phi)_q^{\lambda,\mu} \mapsto (mm)_{\lambda,\mu}, \quad (\phi\gamma)_q^{\lambda,\mu} \mapsto (mf)_{\lambda,\mu}, \quad (\gamma\phi)_q^{\lambda,\mu} \mapsto (fm)_{\lambda,\mu}, \quad (\gamma\gamma)_q^{\lambda,\mu} \mapsto (ff)_{\lambda,\mu}, \\ (\chi\chi)_q^{\lambda,\mu} \mapsto (ss)_{\lambda,\mu}, \quad (\psi\psi)_q^{\lambda,\mu} \mapsto (pp)_{\lambda,\mu}, \quad \iota_q^{\lambda,\mu} \mapsto p^+_\lambda p^-_\mu. \end{gathered} \tag{132}$$

We extend the involutive homomorphism ω on $\Lambda_n(x)$ (123) to an involutive homomorphsim on $\Lambda_n(x,y)$ in the simplest way: $\omega(o_\lambda(x) g_\mu(y)) := \omega(o_\lambda(x))\omega(g_\mu(y))$. This exchanges the symmetric functions on line 1 of (132) with the corresponding functions on line 2, transposes the index shapes of $(ss)_{\lambda,\mu}$ and multiplies each power sum basis element by $(-1)^{\ell(\lambda)+\ell(\mu)}$. Transition matrices relating the ten nonplethystic bases have entries which are simply products of entries of transition matrices relating type-A symmetric functions, e.g.

$$(ss)_{\lambda,\mu} = \sum_{(\alpha,\beta)\vdash n} K_{\lambda,\alpha} K_{\mu,\beta} (mm)_{\alpha,\beta}. \tag{133}$$

The plethystic power sum basis can be related to the others via the nonplethystic Schur basis,

$$p^+_\lambda p^-_\mu = \sum_{(\alpha,\beta)\vdash n} (\chi\chi)^{\alpha,\beta}(\lambda,\mu)(ss)_{\alpha,\beta}. \tag{134}$$

We construct generating functions for $H^{\mathsf{BC}}_n(q)$-trace evaluations as follows. Given any element $D \in H^{\mathsf{BC}}_n(q)$, define the generating function

$$Y^{\mathsf{BC}}_q(D) = \sum_{\lambda\vdash n} (\epsilon\epsilon)^{\lambda,\mu}_q(D)(mm)_{\lambda,\mu} \in \mathbb{Z}[q,q^{-1}] \otimes \Lambda_n(x,y). \tag{135}$$

This symmetric function is in fact a generating function for the evaluation of *all* the standard traces at D, in the following sense.

Proposition 11.2. *For $D \in H^{\mathsf{BC}}_n(q)$ we have*

$$\begin{aligned}
Y^{\mathsf{BC}}_q(D) &= \sum_{(\lambda,\mu)\vdash n} (\epsilon\epsilon)^{\lambda,\mu}_q(D)(mm)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\epsilon\eta)^{\lambda,\mu}_q(D)(mf)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\eta\epsilon)^{\lambda,\mu}_q(D)(fm)_{\lambda,\mu} \\
&= \sum_{(\lambda,\mu)\vdash n} (\eta\eta)^{\lambda,\mu}_q(D)(ff)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\chi\chi)^{\lambda^\top,\mu^\top}_q(D)(ss)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\phi\phi)^{\lambda,\mu}_q(D)(ee)_{\lambda,\mu} \\
&= \sum_{(\lambda,\mu)\vdash n} (\phi\gamma)^{\lambda,\mu}_q(D)(eh)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\gamma\phi)^{\lambda,\mu}_q(D)(he)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\gamma\gamma)^{\lambda,\mu}_q(D)(hh)_{\lambda,\mu} \\
&= \sum_{(\lambda,\mu)\vdash n} \frac{(-1)^{\ell(\lambda)+\ell(\mu)}(\psi\psi)^{\lambda,\mu}_q(D)}{z_\lambda z_\mu}(pp)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (-1)^{\ell(\lambda)+\ell(\mu)} \iota^{\lambda,\mu}_q(D) p^+_\lambda p^-_\mu.
\end{aligned}$$

Equivalently, $\omega Y^{\mathsf{BC}}_q(D)$ is equal to

$$\begin{aligned}
&\sum_{(\lambda,\mu)\vdash n} (\epsilon\epsilon)^{\lambda,\mu}_q(D)(ff)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\epsilon\eta)^{\lambda,\mu}_q(D)(fm)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\eta\epsilon)^{\lambda,\mu}_q(D)(mf)_{\lambda,\mu} \\
&= \sum_{(\lambda,\mu)\vdash n} (\eta\eta)^{\lambda,\mu}_q(D)(mm)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\chi\chi)^{\lambda,\mu}_q(D)(ss)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\phi\phi)^{\lambda,\mu}_q(D)(hh)_{\lambda,\mu} \\
&= \sum_{(\lambda,\mu)\vdash n} (\phi\gamma)^{\lambda,\mu}_q(D)(he)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\gamma\phi)^{\lambda,\mu}_q(D)(eh)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} (\gamma\gamma)^{\lambda,\mu}_q(D)(ee)_{\lambda,\mu} \\
&= \sum_{(\lambda,\mu)\vdash n} \frac{(\psi\psi)^{\lambda,\mu}_q(D)}{z_\lambda z_\mu}(pp)_{\lambda,\mu} = \sum_{(\lambda,\mu)\vdash n} \iota^{\lambda,\mu}_q(D) p^+_\lambda p^-_\mu.
\end{aligned} \tag{136}$$

Proof. Consider the fourth and fifth sums in (136), in which the symmetric functions and traces satisfy

$$(ss)_{\lambda,\mu} = \sum_{(\alpha,\beta)\vdash n} K_{\lambda,\alpha} K_{\mu,\beta}(mm)_{\alpha,\beta}, \qquad (\eta\eta)^{\alpha,\beta}_q = \sum_{(\lambda,\mu)\vdash n} K_{\lambda,\alpha} K_{\mu,\beta}(\chi\chi)^{\lambda,\mu}_q. \tag{137}$$

Using (137) to expand the fifth sum in the monomial symmetric function basis, we have

$$\sum_{(\lambda,\mu)\vdash n}(\chi\chi)_q^{\lambda,\mu}(D)\sum_{(\alpha,\beta)\vdash n}K_{\lambda,\alpha}K_{\mu,\beta}(mm)_{\alpha,\beta}=\sum_{(\alpha,\beta)\vdash n}\sum_{(\lambda,\mu)\vdash n}K_{\lambda,\alpha}K_{\mu,\beta}(\chi\chi)_q^{\lambda,\mu}(D)(mm)_{\alpha,\beta}$$
$$=\sum_{(\alpha,\beta)\vdash n}(\eta\eta)_q^{\alpha,\beta}(D)(mm)_{\alpha,\beta},$$

i.e. it is equal to the fourth sum. Similarly, for each of the remaining sums of the form $\sum_{(\lambda,\mu)\vdash n}(\zeta\xi)_q^{\lambda,\mu}(D)o_\alpha(x)g_\beta(y)$ in (136), there is a matrix $(M_{(\lambda,\mu),(\alpha,\beta)})_{(\lambda,\mu)\vdash n,(\alpha,\beta)\vdash n}$ and equations

$$(ss)_{\lambda,\mu}=\sum_{(\alpha,\beta)}M_{(\lambda,\mu),(\alpha,\beta)}o_\alpha(x)g_\beta(y),\qquad (\zeta\xi)_q^{\alpha,\beta}=\sum_{(\lambda,\mu)}M_{(\lambda,\mu),(\alpha,\beta)}(\chi\chi)_q^{\lambda,\mu},$$

relating it to the fifth sum. In particular, we have $M_{(\lambda,\mu),(\alpha,\beta)}=K_{\lambda^\top,\alpha}K_{\mu^\top,\beta}$, $K_{\lambda^\top,\alpha}K_{\mu,\beta}$, $K_{\lambda,\alpha}K_{\mu^\top,\beta}$, $K^{-1}_{\alpha,\lambda}K^{-1}_{\beta,\mu}$, $K^{-1}_{\alpha,\lambda}K^{-1}_{\beta,\mu^\top}$, $K^{-1}_{\alpha,\lambda^\top}K^{-1}_{\beta,\mu}$, $K^{-1}_{\alpha,\lambda^\top}K^{-1}_{\beta,\mu^\top}$, $\chi^\lambda(\alpha)\chi^\mu(\beta)$, respectively. (See [10, §3].) Relating the last sum to the fifth sum, we have equations

$$(ss)_{\lambda,\mu}=\sum_{(\alpha,\beta)}\frac{(\chi\chi)^{\lambda,\mu}(\alpha,\beta)}{z_\alpha z_\beta 2^{\ell(\alpha)+\ell(\beta)}}p^+_\alpha p^-_\beta,\qquad \iota^{\alpha,\beta}=\sum_{(\lambda,\mu)}\frac{(\chi\chi)^{\lambda,\mu}(\alpha,\beta)}{z_\alpha z_\beta 2^{\ell(\alpha)+\ell(\beta)}}(\chi\chi)^{\lambda,\mu}.\qquad\square$$

To say that the functions $\{Y_q(D)\,|\,D\in\mathbb{Z}[\mathfrak{B}_n]\}$ arise often in the study of type-BC symmetric functions would be an understatement; in fact, *every* element of $\mathbb{Z}[q]\otimes\Lambda_n(x,y)$ has this form.

Proposition 11.3. *Every symmetric function in $\mathbb{Z}[q]\otimes\Lambda_n(x,y)$ has the form $Y_q(D)$ for some element $D\in\mathbb{Q}(q)[\mathfrak{B}_n]$.*

Proof. Fix a symmetric function in $\Lambda_n(x,y)$ and expand it in the plethystic power sum basis as $\sum_{(\lambda,\mu)\vdash n}a_{\lambda,\mu}(q)p^+_\lambda p^-_\mu$. Then choose one representative $w_{\kappa,\nu}$ of each conjugacy class of $\mathfrak{B}_n$ and consider the trace evaluations (14)

$$\iota_q^{\lambda,\mu}(T_{w_{\kappa,\nu}})=\sum_{(\alpha,\beta)\vdash n}(\chi\chi)^{\alpha,\beta}(\lambda,\mu)(\chi\chi)_q^{\alpha,\beta}(T_{w_{\kappa,\nu}}).$$

Since the matrices $((\chi\chi^{\alpha,\beta}(\lambda,\mu))_{(\lambda,\mu),(\alpha,\beta)}$ and $((\chi\chi_q^{\alpha,\beta}(T_{w_{\kappa,\nu}}))_{(\alpha,\beta),(\kappa,\nu)}$ are both invertible, so is their product $((\iota_q^{\lambda,\mu}(T_{w_{\kappa,\nu}}))_{(\lambda,\mu),(\kappa,\nu)}$. Call the inverse of this product $B=(b_{(\alpha,\beta),(\lambda,\mu)}(q))$ and for each $(\alpha,\beta)\vdash n$ define

$$U_{\alpha,\beta}=\sum_{(\kappa,\nu)\vdash n}b_{(\alpha,\beta),(\kappa,\nu)}(q)T_{w_{\kappa,\nu}}\in\mathbb{Q}(q)\otimes H_n^{\mathsf{BC}}(q).$$

Then we have

$$\iota_q^{\lambda,\mu}(U_{\alpha,\beta})=\begin{cases}1 & \text{if } (\lambda,\mu)=(\alpha,\beta),\\ 0 & \text{otherwise.}\end{cases}$$

Now define the $\mathbb{Q}(q)[\mathfrak{B}_n]$ element

$$D=\sum_{(\alpha,\beta)\vdash n}a_{\alpha,\beta}(q)U_{w_{\alpha,\beta}}.$$

By (14) and (136) we have

$$Y_q(D)=\sum_{(\lambda,\mu)\vdash n}\iota_q^{\lambda,\mu}\Big(\sum_{(\alpha,\beta)\vdash n}a_{\alpha,\beta}(q)U_{\alpha,\beta}\Big)p^+_\lambda p^-_\mu=\sum_{(\lambda,\mu)\vdash n}a_{\lambda,\mu}(q)p^+_\lambda p^-_\mu,$$

as desired. $\square$

For any type-BC incomparability graph $\Gamma = (V, E)$, we define the *type*-BC *chromatic symmetric function of* Γ to be

$$X_G^{\mathsf{BC}} := \sum_{\kappa} (x_1^{|\kappa^{-1}(1)|} x_2^{|\kappa^{-1}(2)|} \cdots)(y_1^{|\kappa^{-1}(-1)|} y_2^{|\kappa^{-1}(-2)|} \cdots), \tag{138}$$

where the sum is over all proper BC-colorings $\kappa : V \to \mathbb{Z} \smallsetminus \{0\}$ of Γ (§8.2).

For the incomparability graph $\mathrm{inc}(Q)$ of a BC-poset Q, we may express the symmetric function $X_{\mathrm{inc}(Q)}^{\mathsf{BC}}$ in terms of decompositions of P into chains. Letting $c_{Q,\lambda,\mu}$ be the number of column-strict marked Q-bitableaux of shape $(\lambda^\top, \mu^\top)$, we have

$$X_{\mathrm{inc}(Q)}^{\mathsf{BC}} = \sum_{\lambda} c_{Q,\lambda,\mu} (mm)_{\lambda,\mu}.$$

Theorem 11.4. *For $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231 and $Q = Q(w)$ defined as in Subsection 7.2, we have $Y^{\mathsf{BC}}(\widetilde{C}_w^{\mathsf{BC}}(1)) = X_{\mathrm{inc}(Q)}^{\mathsf{BC}}$.*

Proof. The coefficient of $(mm)_{\lambda,\mu}$ in $X_{\mathrm{inc}(Q)}^{\mathsf{BC}}$ equals the number of proper BC-colorings of $\mathrm{inc}(Q)$ of type (λ, μ). This is exactly the number of marked column-strict $\mathrm{inc}(Q)$-bitableaux of shape $(\lambda^\top, \mu^\top)$. By Theorem 9.6, this is $(\epsilon\epsilon)^{\lambda,\mu}(\widetilde{C}_w^{\mathsf{BC}}(1))$, and by (135), we have the desired equality. □

It would be interesting to state and prove a q-analog of Theorem 11.4.

Problem 11.5. *Define a statistic* stat *on proper* BC*-colorings κ of type-*BC *interval graphs so that for $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231, $Q = Q(w)$ and $\Gamma = \mathrm{inc}(Q)$, we have a q-chromatic symmetric function of the form*

$$X_{\Gamma,q}^{\mathsf{BC}} = \sum_{\kappa} q^{\mathrm{stat}(\kappa)} (x_1^{|\kappa^{-1}(1)|} x_2^{|\kappa^{-1}(2)|} \cdots)(y_1^{|\kappa^{-1}(-1)|} y_2^{|\kappa^{-1}(-2)|} \cdots)$$

which satisfies the identity $Y_q^{\mathsf{BC}}(\widetilde{C}_w^{\mathsf{BC}}(q)) = X_{\Gamma,q}^{\mathsf{BC}}$.

11.3. Another approach to type-BC symmetric functions.

From the plethystic power sums $\{p_\lambda^+ \,|\, \lambda \vdash k\} \cup \{p_\lambda^- \,|\, \lambda \vdash k\} \subseteq \Lambda_k(x, y)$, one can define other plethystic symmetric functions [10, §3], [69]

$$\{s_\lambda^+ \,|\, \lambda \vdash k\}, \quad \{h_\lambda^+ \,|\, \lambda \vdash k\}, \quad \{e_\lambda^+ \,|\, \lambda \vdash k\}, \quad \{m_\lambda^+ \,|\, \lambda \vdash k\}, \quad \{f_\lambda^+ \,|\, \lambda \vdash k\},$$
$$\{s_\lambda^- \,|\, \lambda \vdash k\}, \quad \{h_\lambda^- \,|\, \lambda \vdash k\}, \quad \{e_\lambda^- \,|\, \lambda \vdash k\}, \quad \{m_\lambda^- \,|\, \lambda \vdash k\}, \quad \{f_\lambda^- \,|\, \lambda \vdash k\}$$

to be the symmetric functions in $\Lambda_k(x, y)$ related to $\{p_\lambda^+ \,|\, \lambda \vdash k\}$ or $\{p_\lambda^- \,|\, \lambda \vdash k\}$ just as $\{s_\lambda \,|\, \lambda \vdash k\}$, $\{h_\lambda \,|\, \lambda \vdash k\}$, $\{e_\lambda \,|\, \lambda \vdash k\}$, $\{m_\lambda \,|\, \lambda \vdash k\}$, $\{f_\lambda \,|\, \lambda \vdash k\}$ in $\Lambda_k(x)$ are related to $\{p_\lambda \,|\, \lambda \vdash k\}$. Such functions often appear in the literature as $s_\lambda[X+Y], \dots, f_\lambda[X+Y]$ and $s_\lambda[X-Y], \dots, f_\lambda[X-Y]$. Certain products of pairs of these form nine more *plethystic* bases of the space $\Lambda_n(x, y)$:

(139)

$$\{s_\lambda^+ s_\mu^- \,|\, (\lambda, \mu) \vdash n\}$$
$$\{h_\lambda^+ h_\mu^- \,|\, (\lambda, \mu) \vdash n\}, \quad \{h_\lambda^+ e_\mu^- \,|\, (\lambda, \mu) \vdash n\}, \quad \{e_\lambda^+ h_\mu^- \,|\, (\lambda, \mu) \vdash n\}, \quad \{e_\lambda^+ e_\mu^- \,|\, (\lambda, \mu) \vdash n\},$$
$$\{m_\lambda^+ m_\mu^- \,|\, (\lambda, \mu) \vdash n\}, \quad \{m_\lambda^+ f_\mu^- \,|\, (\lambda, \mu) \vdash n\}, \quad \{f_\lambda^+ m_\mu^- \,|\, (\lambda, \mu) \vdash n\}, \quad \{f_\lambda^+ f_\mu^- \,|\, (\lambda, \mu) \vdash n\}.$$

Naturally, one may study $\Lambda_n(x, y)$ in terms of these bases and

$$\{p_\lambda^+ p_\mu^- \,|\, (\lambda, \mu) \vdash n\}, \quad \{(pp)_{\lambda,\mu} \,|\, (\lambda, \mu) \vdash n\},$$

instead of using the eleven bases in (126) – (127). It is straightforward to show that matrices relating bases of the forms $\{o_\lambda^+ g_\mu^- \,|\, (\lambda, \mu) \vdash n\}$ to $\{(pp)_{\lambda,\mu} \,|\, (\lambda, \mu) \vdash n\}$ are the same as those relating bases of the forms $\{(og)_{\lambda,\mu} \,|\, (\lambda, \mu) \vdash n\}$ to

$\{p^+_\lambda p^-_\mu/2^{\ell(\lambda)+\ell(\mu)} \,|\, (\lambda,\mu) \vdash n\}$. Formulas for the matrix entries are given in [10, App. A].

A correspondence between the plethystic bases and those of the trace space $\mathcal{T}(H^{\mathsf{BC}}_n(q))$ is given explicitly by the (q-extension of the) *nonplethystic* BC-Frobenius map [10, §3], [69]

$$\text{nch}_q\colon \mathcal{T}(H^{\mathsf{BC}}_n(q)) \to \Lambda_n(x,y) \tag{140}$$

$$\text{nch}_q(\theta_q) = \frac{1}{2^n n!}\sum_{w\in\mathfrak{B}_n} 2^{\ell(\alpha(w))+\ell(\beta(w))}\theta(w)(pp)_{\alpha(w),\beta(w)} \tag{141}$$

$$= \sum_{(\lambda,\mu)\vdash n} \frac{\theta(\lambda,\mu)}{z_\lambda z_\mu}(pp)_{\lambda,\mu}, \tag{142}$$

analogous to (129) – (131). Specifically, nch_q maps

$$(\epsilon\epsilon)^{\lambda,\mu}_q \mapsto e^+_\lambda e^-_\mu, \quad (\epsilon\eta)^{\lambda,\mu}_q \mapsto e^+_\lambda h^-_\mu, \quad (\eta\epsilon)^{\lambda,\mu}_q \mapsto h^+_\lambda e^-_\mu, \quad (\eta\eta)^{\lambda,\mu}_q \mapsto h^+_\lambda h^-_\mu,$$
$$(\phi\phi)^{\lambda,\mu}_q \mapsto m^+_\lambda m^-_\mu, \quad (\phi\gamma)^{\lambda,\mu}_q \mapsto m^+_\lambda f^-_\mu, \quad (\gamma\phi)^{\lambda,\mu}_q \mapsto f^+_\lambda m^-_\mu, \quad (\gamma\gamma)^{\lambda,\mu}_q \mapsto f^+_\lambda f^-_\mu,$$
$$(\chi\chi)^{\lambda,\mu}_q \mapsto s^+_\lambda s^-_\mu, \quad (\psi\psi)^{\lambda,\mu}_q \mapsto p^+_\lambda p^-_\mu, \quad \iota^{\lambda,\mu}_q \mapsto 2^{\ell(\lambda)+\ell(\mu)}(pp)_{\lambda,\mu}.$$

Defining a trace generating function in $\mathbb{Z}[q,q^{-1}]\otimes\Lambda_n(x,y)$ in terms of characters $(\epsilon\epsilon)^{\lambda,\mu}_q$, plethystic monomial symmetric functions $m^+_\lambda m^-_\mu$, and elements $D \in H^{\mathsf{BC}}_n(q)$, we have expansions analogous to those in Proposition 11.2,

$$\sum_{(\lambda,\mu)\vdash n}(\epsilon\epsilon)^{\lambda,\mu}_q(D) m^+_\lambda m^-_\mu = \sum_{(\lambda,\mu)\vdash n}(\epsilon\eta)^{\lambda,\mu}_q(D) m^+_\lambda f^-_\mu = \cdots = \sum_{(\lambda,\mu)\vdash n}(\gamma\gamma)^{\lambda,\mu}_q(D) h^+_\lambda h^-_\mu$$
$$= \sum_{(\lambda,\mu)\vdash n}\frac{(-1)^{\ell(\lambda)+\ell(\mu)}(\psi\psi)^{\lambda,\mu}_q(D)}{z_\lambda z_\mu} p^+_\lambda p^-_\mu = \sum_{(\lambda,\mu)\vdash n}(-1)^{\ell(\lambda)+\ell(\mu)}\iota^{\lambda,\mu}_q(D)(pp)_{\lambda,\mu}.$$

While one could define $Y^{\mathsf{BC}}_q(D)$ to be the above symmetric function instead of that in (135), this would lead to a less natural connection to the BC-chromatic symmetric function (138).

12. Hessenberg varieties

Hessenberg varieties are subvarieties of flag varieties which were first studied [22, 23] in conjunction with questions concerning eigenvalues of linear operators. More recent work reveals connections to other varieties, representation theory, and combinatorics.

The Hessenberg varieties of Coxeter types A, B, C are parametrized by certain vector spaces called Hessenberg spaces. Specifically, given a reductive algebraic group G, Borel subgroup B, and Lie algebras $\mathfrak{g}$, $\mathfrak{b}$ of these, call a subspace $\mathcal{H} \subseteq \mathfrak{g}$ a *Hessenberg space* if it satisfies the Lie algebra containment conditions

$$\mathfrak{b} \subseteq \mathcal{H}, \qquad [\mathfrak{b},\mathcal{H}] \subseteq \mathcal{H}. \tag{143}$$

Hessenberg spaces may be parametrized by appropriate sets of roots in root systems. (See [38] for definitions.) Let Φ be the root system of type A, B or C, let $\Delta \subset \Phi$ be a simple system, let $\Phi^- \subseteq \Phi$ be the set of negative roots, and for $\gamma \in \Phi$ let $\mathfrak{g}_\gamma$ be the root space of $\mathfrak{g}$ corresponding to γ. Define the *root poset* on Φ by $\alpha \leqslant_\Phi \beta$ if $\beta - \alpha \in \text{span}_{\mathbb{N}}\Delta$, and define the *negative root poset* to be the subposet of this induced by Φ^-. Call subset $I \subseteq \Phi^-$ a *dual order ideal* of Φ^- if for all $\alpha,\beta \in \Phi^-$ we have

$$\left.\begin{matrix}\alpha \in I,\\ \alpha \leqslant_{\Phi^-} \beta\end{matrix}\right\} \Rightarrow \beta \in I.$$

In particular, we have the following [22, Lem. 1].

Proposition 12.1. *A bijection between Hessenberg spaces* $\mathcal{H} \subseteq \mathfrak{g}$ *and dual order ideals* I *of the negative root poset on* Φ^- *is given by*

$$I \mapsto \mathcal{H}(I) := \mathfrak{b} \oplus \bigoplus_{\gamma \in I} \mathfrak{g}_\gamma. \tag{144}$$

For Hessenberg space $\mathcal{H}$ and matrix $S \in \mathfrak{g}$, define a subvariety of the flag variety G/B by

$$\mathrm{Hess}(\mathcal{H}) = \mathrm{Hess}(\mathcal{H}, S) := \{g\mathrm{B} \in \mathrm{G}/\mathrm{B} \,|\, g^{-1}Sg \in \mathcal{H}\}, \tag{145}$$

and call this a *Hessenberg variety associated to* $\mathcal{H}$. If S is a regular semisimple element of $\mathfrak{g}$, then call $\mathrm{Hess}(\mathcal{H})$ a *regular semisimple Hessenberg variety.* In this case, its cohomology vanishes in odd degree [22, §3],

$$\mathrm{H}^*(\mathrm{Hess}(\mathcal{H})) = \bigoplus_{j \geqslant 0} \mathrm{H}^{2j}(\mathrm{Hess}(\mathcal{H})). \tag{146}$$

(See also [54], [73].) Tymozcko [74, 75] defined a graded W-module structure on (146), where W is the Weyl group of G. Let

$$\mathrm{ch}(\mathrm{H}^{2j}(\mathrm{Hess}(\mathcal{H}))) \tag{147}$$

be the Frobenius characteristic of the character of the submodule $\mathrm{H}^{2j}(\mathrm{Hess}(\mathcal{H}))$. For regular semisimple Hessenberg varieties of type A, the Frobenius characteristics (147) are closely related to trace evaluations and graph coloring. In types B and C, no such relation is known.

12.1. Regular semisimple Hessenberg varieties of type A.

Define type-A Hessenberg spaces as in (143) with $\mathrm{G} = \mathrm{GL}_n(\mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. In addition to the bijection (144) with dual order ideals, we have the following bijection with codominant elements of $\mathfrak{S}_n$.

Proposition 12.2. *A bijective correspondence between* 312*-avoiding permutations in* $\mathfrak{S}_n$ *and Hessenberg spaces in* $\mathfrak{gl}_n(\mathbb{C})$ *is given by*

$$w \mapsto \mathcal{H}(w) := \{A = (a_{i,j}) \in \mathfrak{gl}_n(\mathbb{C}) \,|\, a_{i,j} = 0 \textit{ for all } j > \max(w_1, \dots, w_i)\}. \tag{148}$$

Proof. Given $w \in \mathfrak{S}_n$ avoiding the pattern 312, define $m_i = \max(w_1, \dots, w_i)$ for $i = 1, \dots, n$. By Theorem 7.1 the map $w \mapsto m_1 \cdots m_n$ is bijective, and it is easy to see that $m_1 \cdots m_n$ satisfies the defining conditions of a *Hessenberg function*:

(1) $i \leqslant m_i \leqslant n$ for $i = 1, \dots, n$,
(2) $m_1 \leqslant \cdots \leqslant m_n$.

Thus the $\frac{1}{n+1}\binom{2n}{n}$ spaces $\mathcal{H}(w)$ in (148) are precisely the Hessenberg spaces usually denoted by $\mathcal{H}(m_1 \cdots m_n)$ in the literature. (See e.g. [1, Eq. (2.2)].) ☐

Let $\mathrm{Hess}^{\mathsf{A}}(\mathcal{H})$ denote a type-A regular semisimple Hessenberg variety, defined as in (145) with $\mathcal{H}$ as above and $S = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are distinct. For $w \in \mathfrak{S}_n$ avoiding the pattern 312, define the generating function

$$\mathrm{Fr}_q^{\mathsf{A}}(\mathcal{H}(w)) := \sum_{j=0}^{\ell(w)} \mathrm{ch}(\mathrm{H}^{2j}(\mathrm{Hess}^{\mathsf{A}}(\mathcal{H}(w))))q^j \tag{149}$$

for the type-A Frobenius characteristics (147). Shareshian and Wachs conjectured that $\omega\mathrm{Fr}_q^{\mathsf{A}}(\mathcal{H}(w))$ is equal to a chromatic symmetric function, viewed as a polynomial in q with coefficients in Λ_n [60, Conj. 10.1]. This was first proved by Brosnan–Chow [16]. (See also [5, 32, 43, 44].) Combining the equality with Theorem 11.1 we obtain the following.

Theorem 12.3. *For $w \in \mathfrak{S}_n$ avoiding the pattern 312 and unit interval order $P = P(w)$, we have*

$$\mathrm{Fr}_q^{\mathsf{A}}(\mathcal{H}(w)) = \omega X_{\mathrm{inc}(P),q} = \omega Y_q(\widetilde{C}_w(q)). \tag{150}$$

We can see in three ways that the function in (150) belongs to $\mathrm{span}_{\mathbb{N}[q]}\{s_\lambda \,|\, \lambda \vdash n\}$:

(1) The character of $\mathrm{H}^{2j}(\mathrm{Hess}^{\mathsf{A}}(\mathcal{H}(w)))$ belongs to $\mathrm{span}_{\mathbb{N}}\{\chi^\lambda \,|\, \lambda \vdash n\}$. Thus its Frobenius characteristic (122) belongs to $\mathrm{span}_{\mathbb{N}}\{s_\lambda \,|\, \lambda \vdash n\}$.
(2) By [60, Thm. 6.3] the coefficient of s_λ in $X_{\mathrm{inc}(P),q}$ belongs to $\mathbb{N}[q]$.
(3) By [33, Lem. 1.1] we have $\chi_q^\lambda(\widetilde{C}_w(q)) \in \mathbb{N}[q]$.

Furthermore, the function (150) is conjectured to belong to $\mathrm{span}_{\mathbb{N}[q]}\{h_\lambda \,|\, \lambda \vdash n\}$. This open problem also can be viewed in three ways.

Problem 12.4. *Prove one of the following.*

(1) [60, Conj. 10.4] *For each permutation $w \in \mathfrak{S}_n$ avoiding the pattern 312 and index $j = 0, \dots, \ell(w)$, the $\mathfrak{S}_n$-module $\mathrm{H}^{2j}(\mathrm{Hess}^{\mathsf{A}}(\mathcal{H}(w)))$ is a permutation module in which each point stabilizer is a Young subgroup.*
(2) [60, Conj. 5.1] *For each unit interval order P labeled as in Algorithm 7.3, the function $X_{\mathrm{inc}(P),q}$ belongs to $\mathrm{span}_{\mathbb{N}[q]}\{e_\lambda \,|\, \lambda \vdash n\}$. (See also* [68, Conj. 5.5]*.)*
(3) [33, Conj. 2.1] *For each permutation $w \in \mathfrak{S}_n$ and each partition $\lambda \vdash n$, we have $\phi_q^\lambda(\widetilde{C}_w(q)) \in \mathbb{N}[q]$. (See also* [70, Conj. 2.1]*.)*

The statements above satisfy the implications (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3). For progress on these problems, see e.g. [3, 34, 36, 57], and references cited in [62, §3.5].

12.2. Regular semisimple Hessenberg varieties of types B and C.
Define type-B Hessenberg spaces as in (143) with $\mathrm{G} = \mathrm{SO}_{2n+1}(\mathbb{C})$, $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$; define type-$\mathsf{C}$ Hessenberg spaces similarly with $\mathrm{G} = \mathrm{SP}_{2n}(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$. Let $\mathcal{M}_n^{\mathsf{B}}$, $\mathcal{M}_n^{\mathsf{C}}$ denote the sets of these spaces, respectively. The cardinalities of these sets are equal to the number of order ideals appearing in (144). By [17, Thm. 3.1], this is $\binom{2n}{n}$. Thus neither collection of spaces corresponds bijectively to the $\frac{1}{n+2}\binom{2n+2}{n+1}$ "codominant" elements of $\mathfrak{B}_n$ avoiding the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$. This suggests the following problem. (See also the type-B and C Hessenberg functions defined in [2, §10].)

Problem 12.5. *State a type-B or C analog of Proposition 12.2 in terms of a subset of $\binom{2n}{n}$ elements of $\mathfrak{B}_n$.*

Let $\mathrm{Hess}^{\mathsf{B}}(\mathcal{H})$ denote the type-B regular semisimple Hessenberg variety defined as in (145) with $\mathcal{H} \in \mathcal{M}_n^{\mathsf{B}}$ and $S \in \mathfrak{so}_{2n+1}(\mathbb{C})$ having distinct eigenvalues $(0, \lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n)$. Similarly, let $\mathrm{Hess}^{\mathsf{C}}(\mathcal{H})$ denote the type-C regular semisimple Hessenberg variety defined as in (145) with $\mathcal{H} \in \mathcal{M}_n^{\mathsf{C}}$ and $S \in \mathfrak{sp}_{2n}(\mathbb{C})$ having distinct eigenvalues $(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n)$.

In analogy to (149) we define the generating functions

$$\mathrm{Fr}_q^{\mathsf{B}}(\mathcal{H}) := \sum_j \mathrm{ch}(\mathrm{H}^{2j}(\mathrm{Hess}^{\mathsf{B}}(\mathcal{H})))q^j, \qquad \mathrm{Fr}_q^{\mathsf{C}}(\mathcal{H}) := \sum_j \mathrm{ch}(\mathrm{H}^{2j}(\mathrm{Hess}^{\mathsf{C}}(\mathcal{H})))q^j. \tag{151}$$

It would be interesting to connect the symmetric functions $\mathrm{Fr}_q^{\mathsf{B}}(\mathcal{H})$, $\mathrm{Fr}_q^{\mathsf{C}}(\mathcal{H})$ to type-BC chromatic symmetric functions and type-BC trace generating functions, i.e. to formulate type-B and C analogs of Theorem 12.3. By Theorem 11.4 the $q = 1$ specializations of the chromatic symmetric functions and trace generating functions are equal; general equality is stated as Problem 11.5. One could also consider interpreting $\mathrm{Fr}_q^{\mathsf{B}}(\mathcal{H})$, $\mathrm{Fr}_q^{\mathsf{C}}(\mathcal{H})$ as trace generating functions. By Proposition 11.3, such interpretations must exist.

Problem 12.6. *Find families* $\{D^{\mathsf{B}}_{\mathcal{H}} \,|\, \mathcal{H} \in \mathcal{M}^{\mathsf{B}}_n\}$, $\{D^{\mathsf{C}}_{\mathcal{H}} \,|\, \mathcal{H} \in \mathcal{M}^{\mathsf{C}}_n\}$ *of elements of* $H^{\mathsf{BC}}_n(q)$ *whose trace generating functions satisfy*

(1) $\omega Y^{\mathsf{BC}}_q(D^{\mathsf{B}}_{\mathcal{H}}) = \mathrm{Fr}^{\mathsf{B}}_q(\mathcal{H})$ *for all* $\mathcal{H} \in \mathcal{M}^{\mathsf{B}}_n$,
(2) $\omega Y^{\mathsf{BC}}_q(D^{\mathsf{C}}_{\mathcal{H}}) = \mathrm{Fr}^{\mathsf{C}}_q(\mathcal{H})$ *for all* $\mathcal{H} \in \mathcal{M}^{\mathsf{C}}_n$.

Similarly, one could try to interpret $\mathrm{Fr}^{\mathsf{B}}_q(\mathcal{H})$, $\mathrm{Fr}^{\mathsf{C}}_q(\mathcal{H})$ in terms of graph coloring [58].

Problem 12.7. *Define families* $\mathcal{G}^{\mathsf{B}} = \{G^{\mathsf{B}}(\mathcal{H}) \,|\, \mathcal{H} \in \mathcal{M}^{\mathsf{B}}_n\}$, $\mathcal{G}^{\mathsf{C}} = \{G^{\mathsf{C}}(\mathcal{H}) \,|\, \mathcal{H} \in \mathcal{M}^{\mathsf{C}}_n\}$ *of graphs and families* $\{X^{\mathsf{B}}_{G,q} \,|\, G \in \mathcal{G}^{\mathsf{B}}\}$, $\{X^{\mathsf{C}}_{G,q} \,|\, G \in \mathcal{G}^{\mathsf{C}}\}$ *of chromatic symmetric functions so that we have*

(1) $\omega X^{\mathsf{B}}_{G^{\mathsf{B}}(\mathcal{H}),q} = \mathrm{Fr}^{\mathsf{B}}_q(\mathcal{H})$ *for all* $\mathcal{H} \in \mathcal{M}^{\mathsf{B}}_n$,
(2) $\omega X^{\mathsf{C}}_{G^{\mathsf{C}}(\mathcal{H}),q} = \mathrm{Fr}^{\mathsf{C}}_q(\mathcal{H})$ *for all* $\mathcal{H} \in \mathcal{M}^{\mathsf{C}}_n$.

In addition, it would be interesting to connect these problems to the Kazhdan–Lusztig basis of $H^{\mathsf{BC}}_n(q)$ and to the type-BC indifference graphs defined in Subsection 8.2.

Problem 12.8. *Decide to what extent the symmetric functions in Problems* 12.6 – 12.7 *can be chosen to simultaneously solve Problem* 11.5*.*

Finally, it would be interesting to study appropriate q-extensions of symmetric functions $X^{\mathsf{BC}}_{\mathrm{inc}(Q)}$ (as in Problem 11.5) in conjunction with type-B and C Lusztig varieties in addition to Hessenberg varieties. (See [5].)

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