

ALGEBRAIC COMBINATORICS

Abeer Al Ahmadieh, May Cai & Josephine Yu

Real and Positive Tropicalizations of Symmetric Determinantal Varieties

Volume 9, issue 1 (2026), p. 1-19.

<https://doi.org/10.5802/alco.462>

© The author(s), 2026.



This article is licensed under the
CREATIVE COMMONS ATTRIBUTION (CC-BY) 4.0 LICENSE.
<http://creativecommons.org/licenses/by/4.0/>

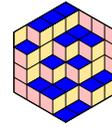


*Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing*

www.tccpublishing.org www.centre-mersenne.org

e-ISSN: 2589-5486





Real and Positive Tropicalizations of Symmetric Determinantal Varieties

Abeer Al Ahmadieh, May Cai & Josephine Yu

ABSTRACT We study real and positive tropicalizations of the varieties of low rank symmetric matrices over real or complex Puiseux series. We show that real tropicalization coincides with complex tropicalization for rank two and corank one cases. We also show that the two notions of positive tropicalization introduced by Speyer and Williams coincide for symmetric rank two matrices, but they differ for symmetric corank one matrices.

1. INTRODUCTION

Let $\mathcal{C} = \mathbb{C}\{\{t\}\}$ and $\mathcal{R} = \mathbb{R}\{\{t\}\}$ be the fields of Puiseux series over complex and real numbers respectively. The tropical semiring $(\mathbb{R}, \oplus, \odot)$ is the set of real numbers with the tropical addition and the tropical multiplication given by $x \oplus y = \min(x, y)$ and $x \odot y = x + y$ respectively. Then the Puiseux series fields admit a valuation to the tropical semiring

$$\text{val} : \mathcal{C} \text{ or } \mathcal{R} \rightarrow \mathbb{R} \text{ given by } \text{val}(a) = \deg(\text{lt}(a))$$

where $\text{lt}(a)$ is the leading term, or the lowest degree term, of a Puiseux series a .

For an algebraic variety X in \mathcal{C}^n or $(\mathcal{C}^*)^n$, its (complex) tropicalization is defined to be

$$\text{trop}_{\mathcal{C}}(X) = \overline{\{\text{val}(x) \mid x \in X \cap (\mathcal{C}^*)^n\}} \subset \mathbb{R}^n$$

where the valuation is taken coordinatewise and the closure is under Euclidean topology. The Fundamental Theorem and the Structure Theorem hold for these tropical varieties [16]. Analogously, we can consider points in X with real coordinates and define its *real tropicalization*:

$$\text{trop}_{\mathcal{R}}(X) = \overline{\{\text{val}(x) \mid x \in X \cap (\mathcal{R}^*)^n\}} \subset \mathbb{R}^n.$$

Let \mathcal{C}^+ and \mathcal{R}^+ be the subsets of \mathcal{C} and \mathcal{R} respectively with positive and real leading terms. Following [6], we define the *positive part* of the tropical variety to be

$$\text{trop}_{\mathcal{C}^+}(X) = \overline{\{\text{val}(x) \mid x \in X \cap (\mathcal{C}^+)^n\}} \subset \mathbb{R}^n.$$

and the *really positive part* to be

$$\text{trop}_{\mathcal{R}^+}(X) = \overline{\{\text{val}(x) \mid x \in X \cap (\mathcal{R}^+)^n\}} \subset \mathbb{R}^n.$$

The real tropicalization $\text{trop}_{\mathcal{R}}(X)$ and the really positive tropicalization $\text{trop}_{\mathcal{R}^+}(X)$ arise as real tropicalizations of real semialgebraic sets and satisfy the Fundamental

Manuscript received 26th February 2025, revised 5th October 2025, accepted 7th October 2025.

KEYWORDS. ranks of tropical matrices, positive tropicalization, symmetric matrices, determinantal varieties, bicolored trees.

Theorem of Real Tropical Geometry [15]. Their real tropical bases were studied in [23]. When X is defined over \mathbb{R} they coincide with the logarithmic limit sets of (the positive part of) the real variety.

On the other hand, the positive tropicalization $\text{trop}_{\mathcal{C}^+}(X)$ is a subcomplex of the Gröbner complex of the ideal defining X , corresponding to the initial ideals with positive real roots. The positive and really positive parts coincide for Grassmannians under the Plücker embedding [22] and for linear spaces, but they may differ in general [24]. Some sufficient conditions for a point in the positive tropicalization to lie in the really positive tropicalization can be found in [24, 20]. Note that our notations are different from those used in [22], [24], and [20].

Our main goal here is to study real and positive tropicalization of varieties of low rank symmetric matrices. There have been several prior works on the tropical geometry of low rank matrices. Develin, Santos, and Sturmfels [11] defined three notions of ranks for matrices over the tropical semiring. The Kapranov rank is the one compatible with the tropicalization process described above.

The (complex) tropicalization of the variety of $d \times n$ matrices of rank at most 2 was studied by Markwig and the third author in [17]. It coincides with the space of bicolored trees, is shellable, and therefore has nice topology. Motivated by matrix completion problems, Bernstein described the algebraic matroid of this rank 2 variety, as well as the skew-symmetric counterpart, in [4]. Hultman and Jonsson described the topology of the space of tropical matrices of Barvinok rank at most two in [14].

For symmetric matrices, Cartwright and Chan defined three notions of rank [10]. Zwick provided two additional notions, the symmetric tropical rank and the symmetric Kapranov rank, and showed that they coincide in the rank 2 case and in the corank 1 case. Again, the tropical Kapranov rank is the one arising from the tropicalization process. The space of symmetric tropical rank 2 matrices coincides with the space of symmetric bicolored trees and is shellable [9]. We will see later in Corollary 3.7 a characterization of symmetric Barvinok rank 2 matrices using the bicolored trees.

All the works mentioned above have been on complex tropicalization. The positive tropicalization of the space of rank 2 matrices is described as the space of Barvinok rank 2 matrices by Ardila in [3] and Brandenburg, Loho, and Sinn in [6]. (See Remark 2.4.) The latter paper also gave some criterion for positivity in rank 3 case and described the positive tropicalization of the determinantal hypersurface.

In this paper we describe the real, positive, and really positive tropicalization of the varieties of *symmetric* $n \times n$ matrices of rank at most r , where $r = 2$ or $r = n - 1$. In particular, we compare their real versus complex tropicalizations and positive versus really positive tropicalizations. We also fill in some gaps in the literature for the general (nonsymmetric) cases. A summary of our results is shown in Table 1.

TABLE 1. A summary of results

	$\text{trop}_{\mathcal{R}} \stackrel{?}{=} \text{trop}_{\mathcal{C}}$	$\text{trop}_{\mathcal{R}^+} \stackrel{?}{=} \text{trop}_{\mathcal{C}^+}$
rank 2	Yes ([11, Theorem 6.5])	Yes (Theorem 2.3)
symmetric rank 2	Yes (Theorem 3.3)	Yes (Theorem 3.5)
singular	Yes (Corollary 4.2)	Yes (Theorem 4.1)
symmetric singular	Yes (Theorem 5.6)	No (Example 5.4, Theorem 5.5)

2. RANK 2

For positive integers d, n, r , let $\mathcal{M}_{d,n}^r \subset \mathcal{C}^{d \times n}$ be the variety of $d \times n$ matrices over \mathcal{C} of rank at most r . It is defined by vanishing of $(r+1) \times (r+1)$ minors of a $d \times n$ matrix of indeterminates. Our goal is to study real, positive, and really positive tropicalization of these varieties.

We now recall a few different notions of ranks for tropical matrices introduced in [11]. For a tropical matrix A in $\mathbb{R}^{n \times d}$, its *Kapranov rank* over a valued field \mathcal{K} is defined to be the smallest integer r such that A lies in $\text{trop}_{\mathcal{K}}(\mathcal{M}_{d,n}^r)$. This is equivalent to A being the image under entrywise valuation of a rank $\leq r$ matrix over \mathcal{K} . In this case we say that A has a rank r *lift* to \mathcal{K} .

A tropical matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ is *tropically singular* if the minimum in

$$\text{val}(\det(A)) = \bigoplus_{\sigma \in S_n} (a_{1\sigma(1)} \odot \cdots \odot a_{n\sigma(n)}) = \min_{\sigma \in S_n} (a_{1\sigma(1)} + \cdots + a_{n\sigma(n)})$$

is attained at least twice. The *tropical rank* of a matrix A is the size of the largest tropically nonsingular submatrix of A . The *Barvinok rank* of a $d \times n$ tropical matrix A is the smallest integer r such that $A = B \odot C$ where B is $d \times r$ and C is $r \times n$. It was shown in [11] that for any matrix we have

$$\text{tropical rank} \leq \text{Kapranov rank over } \mathcal{K} \leq \text{Barvinok rank}$$

where the first inequality is valid for any field \mathcal{K} with nontrivial valuation, and the second inequality is valid when \mathcal{K} is a Puiseux series field over an infinite field. The inequalities can be both strict in general. However if a matrix has tropical rank 2, then it also has Kapranov rank 2 over Puiseux series with coefficients in an infinite field [11, Theorem 6.5]. This immediately implies the following.

LEMMA 2.1. *For any $d, n \geq 2$, the real and complex tropicalizations coincide for the variety of $d \times n$ rank 2 matrices:*

$$\text{trop}_{\mathcal{R}}(\mathcal{M}_{d,n}^2) = \text{trop}_{\mathcal{C}}(\mathcal{M}_{d,n}^2).$$

This is not true for rank 3 matrices.

LEMMA 2.2. *The real and complex tropicalizations differ for the variety of 9×12 matrices of rank at most 3:*

$$\text{trop}_{\mathcal{R}}(\mathcal{M}_{9,12}^3) \subsetneq \text{trop}_{\mathcal{C}}(\mathcal{M}_{9,12}^3).$$

Proof. Let us recall the following results from [11, Section 7]. For a matroid M the cocircuit matrix $C(M)$ has rows indexed by the ground set of M and columns indexed by the cocircuits of M . It has a 0 in entry (i, j) if the i -th element is in the j -th cocircuit and a 1 otherwise. Then the tropical rank of $C(M)$ is the rank of the matroid M . If the Kapranov rank of $C(M)$ over Puiseux series with a ground field \mathbf{k} is equal to the rank of M , then M is representable over \mathbf{k} . If \mathbf{k} is an infinite field, then the converse also holds.

The ternary affine plane is a rank 3 matroid on a 9 element ground set with 12 cocircuits which is representable over \mathbb{C} but not over \mathbb{R} [18]. Thus its cocircuit matrix has Kapranov rank = 3 over \mathcal{C} but Kapranov rank > 3 over \mathcal{R} . In other words, it lies in $\text{trop}_{\mathcal{C}}(\mathcal{M}_{9,12}^3)$ but not in $\text{trop}_{\mathcal{R}}(\mathcal{M}_{9,12}^3)$. \square

The space of $d \times n$ matrices of tropical rank 2 has a simplicial fan structure as the space of *bicolored trees* with $d + n$ leaves, which are trees with d red leaves and n blue leaves such that removing any internal edge partitions the leaves into two parts with both colors in each part [17]. The construction is as follows. For each $d \times n$ tropical matrix A of tropical rank 2, the tropical convex hull of its columns, modulo tropical scaling, is a contractible one dimensional polyhedral complex, or a tree. This

tree can be made balanced by attaching one infinite ray in each of the coordinate directions e_1, \dots, e_d in a unique way, making it into a tropical line. To obtain the desired bicolored tree, attach a blue leaf labeled i at the position of the i -th column of A in the tree, and a red leaf j to where the infinite ray e_j is attached. The length of each internal edge is the length of the corresponding edge in the tropical convex hull, under the tropical Hilbert metric

$$d(x, y) = \max_i(x_i - y_i) - \min_i(x_i - y_i).$$

The leaf edges do not have lengths (or are considered to have length ∞). Conversely, every bicolored metric tree uniquely determines a tropical rank 2 matrix modulo tropically scaling rows and columns. For example, the following matrix corresponds to the bicolored tree of Figure 1:

$$(1) \quad \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad a, b, c > 0.$$

Using the bicolored trees, we will describe the positive tropicalizations of rank two matrices.

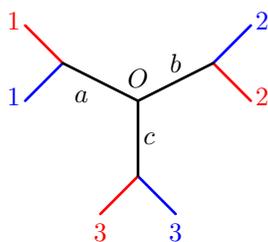


FIGURE 1. The bicolored tree corresponding to the matrix (1), which has tropical rank 2 and symmetric tropical rank 3. Any bicolored tree that is not a caterpillar must contain this as a subtree. The origin is marked as O , and a, b, c are positive real numbers.

THEOREM 2.3. *For any integers $d, n \geq 2$, the positive and really positive tropicalization of $\mathcal{M}_{d,n}^2$ coincide and are equal to the set of $d \times n$ matrices of Barvinok rank at most 2.*

Proof. Let $B_{d,n}^2$ be the set of $d \times n$ tropical matrices of Barvinok rank at most 2. We will show that

$$\text{trop}_{\mathcal{C}^+}(\mathcal{M}_{d,n}^2) \subseteq B_{d,n}^2 \subseteq \text{trop}_{\mathcal{R}^+}(\mathcal{M}_{d,n}^2).$$

The reverse containment $\text{trop}_{\mathcal{R}^+}(\mathcal{M}_{d,n}^2) \subseteq \text{trop}_{\mathcal{C}^+}(\mathcal{M}_{d,n}^2)$ follows from the definition.

We first show the inclusion on the right. If A is a $d \times n$ matrix of Barvinok rank 2, then there exists matrices $B \in \mathbb{R}^{d \times 2}$ and $C \in \mathbb{R}^{2 \times n}$ satisfying $A = B \odot C$. There are no conditions on B and C , so any lift of B and C to \tilde{B} and \tilde{C} respectively over \mathcal{R}^+ (for instance, simply lifting m to t^m) induces a matrix $\tilde{A} = \tilde{B}\tilde{C}$. Because matrix multiplication is subtraction-free, and \tilde{B} and \tilde{C} are positive real Puiseux series matrices, no cancellation occurs in the product $\tilde{B}\tilde{C}$ so \tilde{A} has entries in \mathcal{R}^+ and $\text{trop}(\tilde{A}) = A$. The matrix \tilde{A} has rank at most 2 because it is a product of a $d \times 2$ matrix and a $2 \times n$ matrix.

Now we show that any matrix in $\text{trop}_{\mathcal{C}^+}(\mathcal{M}_{d,n}^2)$ has Barvinok rank 2. From [17], we have a bijection between $\mathcal{M}_{d,n}^2$ modulo lineality, which correspond to tropically scaling of rows and columns of the matrices, and bicolored metric trees with d leaves of one color and n leaves of the other color. Furthermore, it follows from [11, Proposition 6.1]

that a tropical rank 2 matrix has Barvinok rank 2 if and only if its bicolored tree is a “caterpillar” tree: all of its internal vertices lie along a path. We will now show that any non-caterpillar phylogenetic tree cannot be in $\text{trop}_{\mathcal{C}^+}(M_{d,n}^2)$.

If a tree is not caterpillar, then its internal vertices do not all lie along a path. Up to permutation of rows and columns (corresponding to simultaneous permutation of the leaf labels within each color class), it contains Figure 1 as a subtree. By tropically rescaling the rows and columns of the corresponding matrix, which preserves all notions of rank, we obtain the submatrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad a, b, c > 0$$

inside our matrix. The tropical determinant of this submatrix achieves the minimum among the two all-zero monomials. Both of these monomials have positive sign in the 3×3 determinant polynomial, so plugging in elements of \mathcal{C}^+ does not make the determinant vanish. Thus any matrix containing this submatrix cannot lie in $\text{trop}_{\mathcal{C}^+}(M_{d,n}^2)$, as desired. \square

Caterpillar trees also arise as Stiefel tropical lines in [13]. A matrix of Barvinok rank 2 gives rise to a Stiefel tropical line as follows. We can factor a $d \times n$ matrix A of Barvinok rank 2 as $A = B \odot C$ where B is $d \times 2$ and C is $2 \times n$. As in the proof above, if we lift B and C to any matrices \tilde{B}, \tilde{C} over \mathcal{R}^+ , then there is no cancellation of leading terms in the matrix product $\tilde{B}\tilde{C}$. So the columns of A lie in the tropicalization of the column space of \tilde{B} . Entries of \tilde{B} can be chosen generically enough so that there is no cancellation of leading terms in their 2×2 minors, then the tropicalization of the column space of \tilde{B} is the Stiefel tropical linear space of the matrix B . In this case the bicolored tree associated to A coincides with the caterpillar tree arising from the Stiefel tropical line, with some additional leaves added to mark the locations of columns of A .

REMARK 2.4. The result in Theorem 2.3 has appeared in the literature [3, 6]. However the proof in [3], which was made more explicit in [6, Theorem 5.2], relies on [19, Theorem 2] for which there is a gap in the proof. The hypothesis of [19, Theorem 2] states that the map under consideration is *surjectively positive* over \mathbb{R}^+ , but the proof relies on [22, Proposition 2.5] which assumes an analogous condition over \mathcal{C}^+ . (Note that the notation \mathcal{R}^+ in [22] refers to our \mathcal{C}^+ , and \mathcal{T}^+ in [19] refers to our $\text{trop}_{\mathcal{C}^+}$.) In general, we do not know whether being “surjectively positive over \mathbb{R}^+ ” implies being “surjectively positive over \mathcal{C}^+ .” The proof in [3, 6] can be fixed by changing the hypothesis in [19, Theorem 2] from \mathbb{R}^+ to \mathcal{C}^+ and using the new result from [21] that the matrix multiplication map of $d \times 2$ and $2 \times n$ matrices is surjectively positive over \mathcal{C}^+ .

The result in Theorem 2.3 is also implied by Corollary 2.14 in [6], but there is a gap in the proof of Proposition 2.13 that the corollary relies upon. See the corrigendum [7]. A concrete counterexample is given in [21]. \square

3. SYMMETRIC RANK 2

The *symmetric tropical rank* of an $n \times n$ symmetric matrix $A = (A_{i,j}) \in \mathbb{R}$ is the smallest number $r \leq n$ such that A is not in the tropical prevariety formed by the $r \times r$ minors of an $n \times n$ symmetric matrix of indeterminates. That is, the symmetric tropical rank of A is $< r$ if for all the $r \times r$ tropical minors, considered as tropical polynomials in the symmetric variables $a_{ij} = a_{ji}$, substituting the entries a_{ij} attains the minimum

at least twice. For example, the tropical determinant of the 3×3 matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$ is $(a_{11} \odot a_{22} \odot a_{33}) \oplus (a_{12} \odot a_{13} \odot a_{23}) \oplus (a_{12} \odot a_{12} \odot a_{33}) \oplus (a_{13} \odot a_{13} \odot a_{22}) \oplus (a_{23} \odot a_{23} \odot a_{11})$, so the classical identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has symmetric Kapranov rank 3 since the minimum in the symmetric determinant is attained uniquely at the tropical monomial $a_{12} \odot a_{13} \odot a_{23}$, although it has tropical rank 2.

Let $\mathcal{S}_n^r \subset \mathcal{C}^{\binom{n+1}{2}}$ be the variety of $n \times n$ symmetric matrices of rank at most r . The symmetric Kapranov rank over \mathcal{K} of an $n \times n$ symmetric tropical matrix A is defined to be the smallest r such that A lies in $\text{trop}_{\mathcal{K}}(\mathcal{S}_n^r)$, for $\mathcal{K} = \mathcal{C}$ or \mathcal{R} .

The symmetric Barvinok rank of an $n \times n$ symmetric matrix A is the smallest integer r such that A can be factored as $A = B \odot B^\top$ where B is an $n \times r$ matrix. We have

symmetric tropical rank \leq symmetric Kapranov rank \leq symmetric Barvinok rank where the symmetric Kapranov rank is over any valued field, and the inequalities can be strict. However if a symmetric tropical matrix has symmetric tropical rank 2, then it has symmetric Kapranov rank 2 over \mathcal{C} [25, Theorem 6]. We will show that such a matrix has symmetric Kapranov rank 2 over \mathcal{R} as well. We need the following lemma.

LEMMA 3.1. *Let $M = (m_{ij})_{1 \leq i, j \leq n}$ be an $n \times n$ symmetric matrix of indeterminates. Fix $i, j \in [n]$ and consider the determinant of M as a quadratic polynomial in m_{ij} . Then the discriminant of $\det(M)$ with respect to m_{ij} is given by*

$$\text{Disc}_{m_{ij}}(\det(M)) = 4M_i M_j$$

where M_i is the determinant of the submatrix of M obtained by deleting the i^{th} row and the i^{th} column.

Proof. Without loss of generality, let $i = 1$ and $j = 2$, and write $m_{12} = x$. Denote by $M_{S,T}$ the determinant of the submatrix of M obtained by deleting the rows indexed by S and columns indexed by T . Let $P(x) = \det(M) = Ax^2 + Bx + C$. Ax^2 consists of all terms that include m_{12} twice, Bx of all terms that include m_{12} once, and C all the terms of the determinant that avoid m_{12} . Thus, $A = M_{\{1,2\},\{1,2\}}$, $B = 2(M_{\{2\},\{1\}})|_{x=0}$, and $C = P(0)$. Thus

$$\text{Disc}_{12}(P) = B^2 - 4AC = 4(M_{\{2\},\{1\}}|_{x=0})^2 - 4M_{\{1,2\},\{1,2\}}P(0) = 4(M_1 M_2)|_{x=0}$$

where the last equality is the Dodgson condensation [12]. Since M_1 and M_2 are independent of x , then the result follows. \square

The next two results were proved in [25] over \mathcal{C} , and we now extend them to \mathcal{R} . The following Lemma is an extension of [25, Lemma 2]. Zwick’s proof in fact does not require C (or B_2 there) to have all positive entries. The argument goes through with the weaker assumption on C that we use here.

LEMMA 3.2. *Let A be an $n \times n$ tropical symmetric matrix of symmetric tropical rank two of the form*

$$A = \begin{pmatrix} B & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C \end{pmatrix}$$

where B is symmetric with positive diagonal entries and C is symmetric with nonnegative entries and having no zero rows or columns. If \tilde{B} and \tilde{C} are symmetric rank 2 lifts of the submatrices

$$\begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C \end{pmatrix}$$

over \mathcal{C} , then A has a symmetric rank 2 lift \tilde{A} over \mathcal{C} . Moreover, if \tilde{B} and \tilde{C} are over \mathcal{R} and \tilde{C} contains a nonpositive 2×2 principal minor, then A has a symmetric rank 2 lift over \mathcal{R} .

Proof. By scaling the rows and columns we may assume that the bottom right entry of \tilde{B} and the top left entry of \tilde{C} are equal to one. Let m and r denote the dimensions of B and C respectively. Thus $n = m + r + 1$. Consider the principal submatrix $M = A[m, m + 1, m + 2]$ of A where $A[S]$ denote the submatrix of A indexed on the rows and columns by S . Thus

$$M = \begin{pmatrix} a_{m,m} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{m+2,m+2} \end{pmatrix}$$

with $a_{m,m}$ is the bottom right entry of B and $a_{m+2,m+2}$ is the top left entry of C . Using the lifts \tilde{B} and \tilde{C} we let

$$\tilde{M} = \begin{pmatrix} \tilde{a}_{m,m} & \tilde{a}_{m,m+1} & x \\ \tilde{a}_{m,m+1} & 1 & \tilde{a}_{m+1,m+2} \\ x & \tilde{a}_{m+1,m+2} & \tilde{a}_{m+2,m+2} \end{pmatrix}$$

where x is an indeterminate. We want this matrix to be symmetric and singular. Setting its determinant to zero gives a quadratic equation $x^2 + bx + c = 0$ where $b = 2\tilde{a}_{m,m+1}\tilde{a}_{m+1,m+2}$, so b has valuation zero. The term c has valuation at least zero, since $a_{m,m} > 0$ and $a_{m+2,m+2} \geq 0$. The product of the two roots is c , and the sum of the two roots is $-b$, so one of the roots must have valuation zero and the other has nonnegative valuation. Let x be a root with valuation zero.

Over \mathcal{R} , we want the discriminant of this quadratic equation to be nonnegative. Thus, using Lemma 3.1, we want the product of the determinants of the 2×2 submatrices $\tilde{M}[1, 2]$ and $\tilde{M}[2, 3]$ to be nonnegative. By permuting rows and columns of \tilde{C} and \tilde{C} , if necessary, we may assume that $\tilde{M}[2, 3]$ is the negative 2×2 submatrix of \tilde{C} , so by assumption $\det \tilde{M}[2, 3] \leq 0$. And we have $\det \tilde{M}[1, 2] = \tilde{a}_{m,m} - \tilde{a}_{m,m+1}^2$. Since the valuation of $\tilde{a}_{m,m}$ is positive and the valuation of $\tilde{a}_{m,m+1}$ is zero, so the leading term of $\det \tilde{M}[1, 2] < 0$ comes from $-\tilde{a}_{m,m+1}^2$ and is negative. This implies that the discriminant is nonnegative. Thus, we get a symmetric rank 2 lift \tilde{M} of M over \mathcal{R} .

The remaining entries of the matrix can then be completed using linear combinations of existing rows and columns, as in the proof of Lemma 3 in [25]. \square

THEOREM 3.3. *Let A be an $n \times n$ tropical symmetric matrix. Then A has symmetric Kapranov rank two over \mathcal{R} if and only if it has symmetric tropical rank two.*

Proof. We will follow the proof of Theorem 6 in [25]. While Zwick proved the theorem over \mathcal{C} , most parts of the proof also work over \mathcal{R} , except for some details that we will identify and adjust to make the proof valid over \mathcal{R} .

The forward direction (\Rightarrow) follows from the facts that the symmetric tropical rank is at most equal to the symmetric Kapranov rank and that if A has symmetric tropical rank one, then we can find a symmetric lift of A over \mathcal{R} with rank one.

For the other direction (\Leftarrow), the main idea of the proof is that any symmetric matrix A with symmetric tropical rank two can be written in the following form, after possibly a tropical scaling and diagonal permutation:

$$\begin{pmatrix} B_1 & 0 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C \\ 0 & 0 & 0 & C^\top & 0 \end{pmatrix}$$

the matrices B_1 and B_2 are symmetric and positive, and the matrix C is non-negative and has no zero columns. The matrix C^\top represents the transpose of C and each $\mathbf{0}$ represents a zero matrix of the appropriate size. The matrix A does not need to contain any zero rows or columns, and some of the blocks B_1 , B_2 , and C may be empty. But we may assume that A does not contain only one positive block and that it is not the zero matrix, [25, Lemma 4].

By [25, Lemma 5], adding rows/columns of zeroes to a symmetric matrix does not affect the property of having symmetric Kapranov rank two over \mathcal{R} . It is easy to see that Zwick’s proof works over \mathcal{R} as it only requires taking generic linear combinations to avoid cancellation of leading terms. Removing rows/columns does not increase the symmetric Kapranov, by definition.

Thus we may assume that A contains exactly one zero row/column. Using [25, Lemmas 1 and 2], we construct symmetric rank 2 lifts to \mathcal{R} for the two submatrices:

$$A_1 = \begin{pmatrix} B_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C \\ \mathbf{0} & C^\top & \mathbf{0} \end{pmatrix}$$

denoted by \tilde{A}_1 and \tilde{A}_2 , respectively. We can adjust the proof of Lemma 1 in [25] to ensure that \tilde{A}_2 contains a 2×2 principal submatrix with negative determinant (the construction of the lift is done by choosing generic entries, we require one of these generic choices satisfies this inequality, so it is still generic). Now using Lemma 3.2, we conclude that A has the desired lift \tilde{A} . \square

The theorem above, combined with [25, Theorem 6], which says that symmetric tropical rank 2 matrices have symmetric Kapranov rank 2 over \mathcal{C} , gives us the following.

COROLLARY 3.4. *The real and complex tropicalizations coincide for symmetric matrices of rank at most 2. That is,*

$$\text{trop}_{\mathcal{R}}(\mathcal{S}_n^2) = \text{trop}_{\mathcal{C}}(\mathcal{S}_n^2).$$

Now let us consider the positive and really positive parts of \mathcal{S}_n^2 . As in the nonsymmetric case, we will use the notion of bicolored trees to classify the possible cases. Inspired by the case of nonsymmetric rank 2 matrices, Cai, Lee, and Yu [9] proved that the polyhedral fan $\text{trop}_{\mathcal{C}}\mathcal{S}_n^2$ has a simplicial fan structure given by a special type of bicolored trees called symmetric bicolored trees, or a *symbic trees*. The symbic trees are bicolored trees on n blue leaves and n red leaves which are invariant under swapping the colors, with the additional assumption that color-swapping the leaves induces an automorphism on the whole tree whose fixed points form a path (possibly a single point).

For example, the bicolored tree in Figure 1 is not symbic as its fixed points do not form a path, although it is symmetric under swapping colors. On the other hand, the matrix

$$A = \begin{pmatrix} 0 & a & b \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, \quad a, b > 0$$

has symmetric rank 2, and its symbic tree is shown in Figure 3(a). See [9] for more details.

THEOREM 3.5. *The positive and really positive parts of the tropical variety \mathcal{S}_n^2 coincide, and are equal to the set of $n \times n$ symmetric matrices of Barvinok rank at most 2.*

Here we are using the usual Barvinok rank and not the symmetric Barvinok rank; that is, we do not require the tropical matrix factorization to be symmetric. Symmetric Barvinok rank 2 matrices will be discussed in Corollary 3.7.

Proof. Let SB_n^2 be the set of symmetric tropical matrices with Barvinok rank at most 2. As in the proof of Theorem 2.3, since $\text{trop}_{\mathcal{R}^+}(\mathcal{S}_n^2) \subseteq \text{trop}_{\mathcal{C}^+}(\mathcal{S}_n^2)$, we proceed by proving that

$$\text{trop}_{\mathcal{C}^+}(\mathcal{S}_n^2) \subseteq SB_n^2 \subseteq \text{trop}_{\mathcal{R}^+}(\mathcal{S}_n^2).$$

We will first prove the inclusion on the right. We cannot use the Barvinok rank 2 factorization directly as we did in the proof of Theorem 2.3 because the factorization is not always symmetric. Instead we will construct explicit lifts to rank 2 matrices over \mathcal{R}^+ for a symmetric matrix with Barvinok rank 2. Having Barvinok rank 2 still is equivalent to the bicolored trees being caterpillar. There are two types of caterpillar symmetric trees, shown in Figure 2.

First we construct an explicit lift for matrices of the type in Figure 2(a). For this tree type, we denote by i the internal vertex adjacent to leaves i and i' . As matrices preserve symmetric tropical rank and Barvinok rank under simultaneous permutations of rows and columns (and their corresponding trees simultaneously permute the leaves of each color), we can consider without loss of generality that the leaves are ordered $1, n, \dots, 3, 2$. We denote by d_i the length along the internal vertices from 1 to i . In particular, we have $0 \leq d_n \leq d_{n-1} \leq \dots \leq d_2$. Up to simultaneous tropical scaling of rows and columns, this caterpillar symmetric tree corresponds to a unique matrix

$$M = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & d_2 & d_3 & \dots & d_n \\ 0 & d_3 & d_3 & \dots & d_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & d_n & d_n & \dots & d_n \end{bmatrix}.$$

In particular,

$$M_{ij} = \begin{cases} 0 & \text{if } i = 1 \text{ or } j = 1 \\ d_{\max(i,j)} & \text{otherwise.} \end{cases}.$$

This matrix has explicit symmetric rank 2 lift \tilde{M} given by

$$(2) \quad M_{ij} = t^{d_i} + t^{d_j}$$

where $d_1 = 0$. It is clearly symmetric and has the correct valuation. It has rank two because it is the sum of two (non-symmetric) rank one matrices which are transposes of each other.

Now we show that the second type of tree also has a lift. The second type, in Figure 2(b), is not unique up to simultaneous row and column permutation for $n > 3$, so we will have to examine it more closely. For $n = 3$, the only tree up to simultaneous row and column permutation is Figure 3(a). Let d_i be the distance from the fixed point O to the internal vertex attached to the leaf i . We have $0 \leq d_3 \leq d_2 = d_1$. Then the matrix associated with Figure 3(a) is

$$M = \begin{bmatrix} 0 & d_2 & d_3 \\ d_2 & 0 & 0 \\ d_3 & 0 & 0 \end{bmatrix}.$$

Then M has a symmetric factorization into $M = M_1 \odot M_1^\top$ where $M_1 = \begin{bmatrix} 0 & d_2 \\ d_2 & 0 \\ d_3 & 0 \end{bmatrix}$.

Any lift of M_1 into \mathcal{R}^+ gives us a symmetric rank 2 lift of M to \mathcal{R}^+ . More concretely,

we can take

$$\tilde{M}_1 = \begin{bmatrix} 1 & t^{d_2} \\ t^{d_2} & 1 \\ t^{d_3} & 1 \end{bmatrix}, \quad \tilde{M} = \tilde{M}_1 \odot \tilde{M}_1^\top = \begin{bmatrix} 1 + t^{2d_2} & 2t^{d_2} & t^{d_3} + t^{d_2} \\ 2t^{d_2} & 1 + t^{2d_2} & 1 + t^{d_2+d_3} \\ t^{d_3} + t^{d_2} & 1 + t^{2d_2} & 1 + t^{2d_3} \end{bmatrix}.$$

For $n > 3$, we can assume that, after simultaneous row and column permutation, as in Figure 3 the outer two leaves are 1 and 2, and that 2 and 3 share the same color on each “side” of the tree, and the other leaves are $4, \dots, n$ in order from outside in. Then, up to these assumptions, the only kinds of symbic trees of this type are determined entirely by whether or not the color of leaf i , $i > 3$, shares the color of the leaf 1 or the leaf 2 on each side.

Defining distances as in the $n = 3$ case, a tree of this type has corresponding matrix

$$M_{ij} = \begin{cases} 0 & \text{if the leaves } i, j \text{ have the same color on each side of the tree,} \\ \min(d_i, d_j) & \text{otherwise.} \end{cases}$$

We claim that every tree of this type has a matrix M with a symmetric factorization into $M_1 \odot M_1^\top$. In particular, the i -th row of M_1 is $[0 \ d_i]$ if leaf i and leaf 1 share the same color on each side of the tree, and $[d_i \ 0]$ otherwise. If i and j share the same color on each side of the tree, then without loss of generality $M_{ij} = [0 \ d_i] \odot \begin{bmatrix} 0 \\ d_j \end{bmatrix} = \min(0, d_i + d_j) = 0$, and if they have different colors, then without loss of generality $M_{ij} = [d_i \ 0] \odot \begin{bmatrix} 0 \\ d_j \end{bmatrix} = \min(d_i, d_j)$ as desired. Thus, as in the $n = 3$ case, any positive lift of M_1 yields a symmetric rank 2 positive lift of M .

For example, the trees in Figure 3(b) and Figure 3(c) have, respectively, tropical symmetric matrix factorizations into

$$M_{1(b)} = \begin{bmatrix} 0 & d_1 \\ d_2 & 0 \\ d_3 & 0 \\ d_4 & 0 \end{bmatrix} \quad \text{and} \quad M_{1(c)} = \begin{bmatrix} 0 & d_1 \\ d_2 & 0 \\ d_3 & 0 \\ 0 & d_4 \end{bmatrix}.$$

Now that we have shown that the set of Barvinok rank at most 2 symmetric matrices are contained in $\text{trop}_{\mathcal{R}^+}(\mathcal{S}_n^2)$, we need to show that $\text{trop}_{\mathcal{C}^+}(\mathcal{S}_n^2)$ is contained in the set of Barvinok rank 2 matrices and then we are done. As in the proof of Theorem 2.3, any matrix that isn't Barvinok rank 2 must contain the matrix associated with the tree in Figure 1 as a subtree. For symmetric matrices, this must appear as some nonsimultaneous permutation of rows and columns, but it still yields a tropical subdeterminant where the minimum is achieved at two monomials of the same sign, and so any matrix that is not Barvinok rank 2 must not lie in $\text{trop}_{\mathcal{C}^+}(\mathcal{S}_n^2)$, as desired. \square

This result can be restated using the tropical generators introduced in [6].

COROLLARY 3.6. *The 3×3 minors of the symmetric $n \times n$ matrix forms a set of positive and really-positive tropical generators for \mathcal{S}_n^2 .*

In other words, it suffices to check that the minimum in every 3×3 tropical minor is achieved at two monomials of opposite signs to see that a $n \times n$ symmetric rank 2 matrix is in the positive, or really-positive, parts of the variety.

Proof. From the proof of Theorem 3.5, if a matrix has Barvinok rank > 2 , then it has some 3×3 submatrix where the minimums in its tropical determinant are achieved at only monomials of the same sign, which is the contrapositive of the corollary. \square

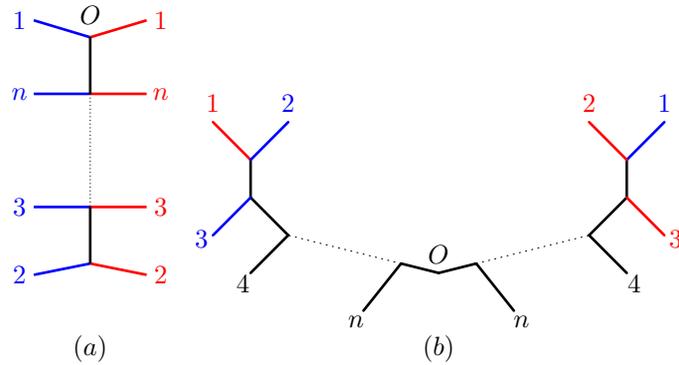


FIGURE 2. The two types of Barvinok rank 2 symmetric tropical matrices, as used in the proof of Theorem 3.5. In type (b), the black leaves $4, \dots, n$ have indeterminate, opposite colors—every choice of color pair determines a different combinatorial type. We suppress edge lengths. In (a), internal edges can be any nonnegative number, while in (b) edge lengths must be symmetric about the midpoint. Without loss of generality, we translate the trees so that O marks the origin.

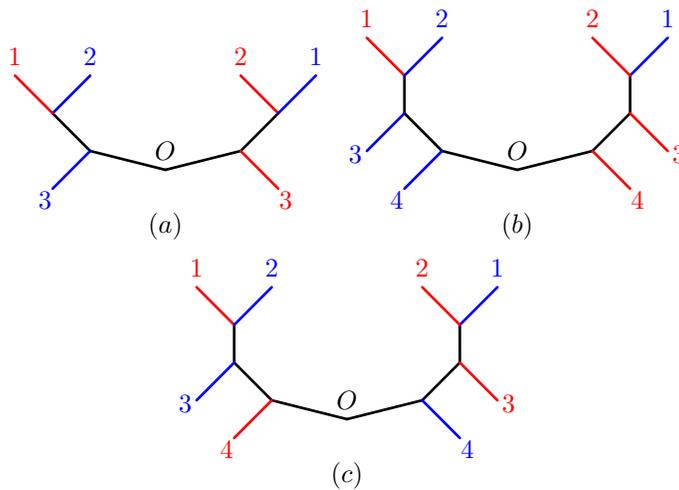


FIGURE 3. (a) is the only symbyc tree of its combinatorial type (i.e., up to simultaneous permutations of red and blue leaves) for $n = 3$. (b) and (c) are the only two symbyc trees of this combinatorial type for $n = 4$. Each tree is translated so that the origin is at the point marked O .

COROLLARY 3.7. *A symmetric tropical rank 2 matrix has symmetric Barvinok rank 2 if and only if its symbyc tree is caterpillar with only one fixed point.*

Proof. The “if” direction follows from the proof of the theorem above, where an explicit symmetric factorization is given. The other direction goes as follows. Let A be an $n \times n$ symmetric tropical rank 2 matrix, and assume we have a symmetric factorization $A = B \odot B^\top$, where B is an $n \times 2$ matrix. By permuting and tropically rescaling rows and columns of A and B , we may assume that B has all nonnegative

entries, with at least one 0 in each row. We may assume B does not have a zero column; otherwise $B \odot B$ would be the zero matrix. Thus, up to row and column permutations, the first three rows of B appear as $\begin{bmatrix} 0 & b_1 \\ b_2 & 0 \\ b_3 & 0 \end{bmatrix}$, and the remaining rows are either $[0 \ b_i]$ or $[b_i \ 0]$ with $b_i \geq 0$.

Furthermore, we can assume that $b_3 \leq b_1$ and $b_3 \leq b_2$, and for $i \geq 4$, $b_i \leq b_{i-1}$. Note that b_1 or b_2 can only appear in $A = B \odot B^\top$ at $A_{1,2}$ and $A_{2,1}$, and in particular $A_{1,2} = A_{2,1} = \min(b_1, b_2)$. Thus we may replace B with B' by replacing b_1, b_2 with $\min(b_1, b_2)$, without affecting A . This B' is precisely the matrix factorization of a matrix whose symbic tree is a caterpillar of only one fixed point from the proof of Theorem 3.5.

In other words, from following the proof of Theorem 3.5 backwards, any symmetric Barvinok rank 2 matrix has symbic trees as in Figure 3, with caterpillars of only one fixed point, and leaf colors determined by whether row i has the zero on the first or second column. \square

4. SINGULAR MATRICES

The variety $\mathcal{M}_{n,n}^{n-1}$ of singular $n \times n$ matrices is a hypersurface defined by the determinant of an $n \times n$ matrix of indeterminates. The *Newton polytope* of a polynomial f is the convex hull of the exponent vectors of its monomial terms. The Newton polytope of the determinant is the Birkhoff polytope, whose vertices are $n \times n$ permutation matrices. It consists of matrices with nonnegative entries, all of whose row sums and column sums are equal to 1. The maximal cones of the tropical hypersurface $\text{trop}_{\mathcal{C}}(\mathcal{M}_{n,n}^{n-1})$ are normal cones of the edges of the Birkhoff polytope. The lineality space of this hypersurface consists of matrices of tropical rank one. Two permutations σ_1 and $\sigma_2 \in S_n$ form an edge in the Birkhoff polytope when $\sigma_1\sigma_2^{-1}$ is a cycle [5, Corollary 2.1]. A maximal cone of $\text{trop}_{\mathcal{C}}(\mathcal{M}_{n,n}^{n-1})$ dual to the edge formed by σ_1 and σ_2 is in the positive part $\text{trop}_{\mathcal{C}^+}(\mathcal{M}_{n,n}^{n-1})$ exactly when σ_1 and σ_2 have opposite signs [6, Proposition 3.1].

THEOREM 4.1. *If the minimum of the tropical determinant of a $n \times n$ matrix M is achieved at two monomials σ_1 and σ_2 with opposite signs, then that matrix has a lift to an $n \times n$ singular matrix of real positive Puiseux series entries. That is,*

$$\text{trop}_{\mathcal{C}^+}(\mathcal{M}_{n,n}^{n-1}) = \text{trop}_{\mathcal{R}^+}(\mathcal{M}_{n,n}^{n-1}).$$

This follows from [24, Lemma 2.6]. But we give a constructive proof here.

Proof. Since the really positive part of the tropical variety $\text{trop}_{\mathcal{R}^+}(\mathcal{M}_{n,n}^{n-1})$ is a closed subset of the tropical variety [2, Theorem 4.1], it suffices to consider only the interiors of maximal cones and $\text{trop}_{\mathcal{C}^+}(\mathcal{M}_{n,n}^{n-1})$ and assume that the minimum in the tropical determinant is achieved exactly at two distinct permutations σ_1 and σ_2 .

By the lineality space of the tropical hypersurface, we can add multiples of the all-ones vector to any column, so without loss of generality we may assume that $M_{i\sigma_1(i)} = 0$ for every $i \in [n]$. Thus the tropical monomial σ_1 has valuation 0, and because the minimum is achieved at σ_1 and σ_2 , σ_2 also has valuation 0 and any other monomial in the tropical hypersurface must be strictly positive.

Because σ_1 and σ_2 are different monomials, $\sigma_1(i) \neq \sigma_2(i)$ for some i . Let us make a preliminary lift of M to $\tilde{M} \in \mathcal{R}_+^{n \times n}$ by arbitrarily choosing any positive real Puiseux series of the right valuation for every entry except the $(i, \sigma_1(i))$ entry, and let the $(i, \sigma_1(i))$ entry be an indeterminate variable x . We need M to be a singular matrix,

so we need

$$\det(\tilde{M}) = Ax + B = 0.$$

It remains show that $x = -B/A$ is a positive real Puiseux series of valuation 0.

The determinant of \tilde{M} is the sum of all permutation monomials. Thus B consists of the sum of the monomials that do not contain $\tilde{M}_{i\sigma_1(i)}$. By construction, this contains the monomial σ_2 , which has valuation 0. Every other monomial that it contains has strictly higher valuation, so cancellation cannot occur and the leading term of B is precisely the leading term of the (usual) monomial σ_2 in \tilde{M} . In particular, the sign of this monomial is determined exactly by the sign of the permutation σ_2 because the entries of $\tilde{M}_{i\sigma_2(i)}$ all have positive leading term, and $\text{val}(B) = 0$.

The part Ax consists of permutations of S_n that does include $\tilde{M}_{i\sigma_1(i)}$, and in particular this includes the monomial σ_1 . Since every $\tilde{M}_{j\sigma_1(j)}$ has valuation 0 by construction (and every other monomial has higher valuation by assumption), the leading term of Ax comes from the leading term of σ_1 . As above, the sign of this monomial is the sign of the permutation σ_1 , and $\text{val}(A) = 0$. Thus, $x = -B/A$ has valuation $0 - 0 = 0$, and it is positive because B and A have differing signs, and so this yields a lift of M to \tilde{M} , a singular matrix of positive real Puiseux series, as desired. \square

COROLLARY 4.2. *The real tropicalization $\text{trop}_{\mathcal{R}}(\mathcal{M}_{n,n}^{n-1})$ is equal to the tropicalization $\text{trop}_{\mathcal{C}}(\mathcal{M}_{n,n}^{n-1})$.*

Proof. This follows exactly from the proof of Theorem 4.1, except we are no longer concerned with σ_1 and σ_2 having different sign, nor with the signs of A , B , and x . \square

More generally, it was shown in [24, Example: Harmony] that for any $2 \leq d \leq n$, the real and complex tropicalization coincide for the variety of $d \times n$ matrices of rank less than d .

5. SYMMETRIC SINGULAR MATRICES

The variety \mathcal{S}_n^{n-1} of singular symmetric $n \times n$ matrices is defined by the $n \times n$ symmetric determinant, which is a polynomial in $\binom{n+1}{2}$ variables. We can express the exponent vector of each monomial in the symmetric determinant as an upper triangular matrix M of 0s, 1s, and 2s. The symmetrization map $M \mapsto \frac{1}{2}(M + M^T)$ is a linear isomorphism from the vector space of upper triangular matrices to the vector space of symmetric matrices, and we can study our Newton polytope in either of these vector spaces. In the space of symmetric matrices, the Newton polytope is the polytope of $n \times n$ symmetric doubly stochastic matrices. It is the image of the Birkhoff polytope under the symmetrization map.

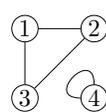
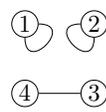
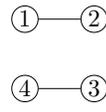
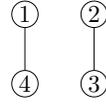
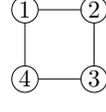
The vertices of this Newton polytope correspond to permutations without cycles of even length at least 4, up to reversing cycles [8, Lemma 1]. Monomial terms of the symmetric determinant can be represented by certain semisimple graphs on n vertices: for a permutation σ let G_σ on vertices $[n]$ with an edge between i and $\sigma(i)$ for every i . For a transposition (ij) we only put one edge between i and j . If an element i is fixed by a permutation σ , then the graph has a loop at i . The connected components of G_σ are simply the cycles in the disjoint cycle decomposition of σ . Then the vertices of the Newton polytope of the symmetric determinant are in bijection with semisimple graphs whose connected components consist only of loops (fixed points of a permutation), isolated edges (transpositions), and cycles of odd length, up

to reversing those cycles. By [1, Corollary 1.4⁽¹⁾], the edges of the Newton polytope are pairs of vertices (u, v) where the edge sets of their graphs $G_u = ([n], E_u)$ and $G_v = ([n], E_v)$ satisfy $|E_u \cup E_v| \leq n + 1$, and $G_{u,v} = ([n], E_u \cup E_v)$ contains at most one cycle of even length ≥ 4 .

EXAMPLE 5.1. Some examples of monomial terms of symmetric 4×4 determinant are shown in Table 2. All monomials in the table except the last one are vertices in the Newton polytope. The last monomial is the midpoint of the edge formed by the two transpositions. Every pair of vertices in the table forms an edge except $2x_{12}x_{13}x_{23}x_{44}$ and $-x_{11}x_{22}x_{34}^2$, whose the graph union has 7 edges. The edges between $2x_{12}x_{13}x_{23}x_{44}$ and the monomials that are the product of two disjoint transpositions is not in $\text{trop}_{C^+}(\mathcal{S}_4^3)$, because the monomials have the same sign.

The lattice length of an edge is one less than the number of lattice points in the edge. The edges between $-x_{11}x_{22}x_{34}^2$ and the product-of-transpositions monomials have lattice length 1 and are in $\text{trop}_{C^+}(\mathcal{S}_4^3)$. The edge formed by the two product-of-transpositions monomials has lattice length 2, with a 4-cycle as the midpoint.

TABLE 2. Several monomials of the 4×4 symmetric determinant and equivalent ways to represent them.

Monomial	Permutation(s)	Upper Triangular	Symmetric Matrix	Graph
$2x_{12}x_{13}x_{23}x_{44}$	$(123)(4) \equiv (132)(4)$	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 0 \\ & & & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	
$-x_{11}x_{22}x_{34}^2$	$(1)(2)(34)$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 0 & 2 \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & & 0 & 0 & 1 & 0 \end{pmatrix}$	
$x_{12}^2x_{34}^2$	$(12)(34)$	$\begin{pmatrix} 0 & 2 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 2 \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 0 & 0 & 1 \\ & & & 0 & 0 & 1 & 0 \end{pmatrix}$	
$x_{14}^2x_{23}^2$	$(14)(23)$	$\begin{pmatrix} 0 & 0 & 0 & 2 \\ & 0 & 2 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \end{pmatrix}$	
$-2x_{12}x_{14}x_{23}x_{34}$	$(1243) \equiv (3421)$	$\begin{pmatrix} 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$	

LEMMA 5.2. *The edges of the Newton polytope of the symmetric determinant have lattice length either 1 or 2. The edges of lattice length 2 are the convex hull of two vertices*

⁽¹⁾The phrasing of this Corollary is confusing. To verify a d -dimensional face, it is not enough to just count the number of vertices and edges of the graphs; one also needs to check that the polytope has the correct dimension. Here we have rephrased the result for the edges.

that differ along k transpositions for $k \geq 2$. That is, in their corresponding graphs the isolated edges $(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)$ in one graph are replaced by new isolated edges $(b_k, a_1), (b_1, a_2), \dots, (b_{k-1}, a_k)$ to obtain the other graph. Replacing those isolated edges with a $2k$ -cycle $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ gives the graph corresponding to the midpoint of the edge.

Proof. If u and v are vertices and (u, v) is an edge in the Newton polytope of the determinant, denoted by $\mathcal{N}(\det(M))$, then considering $G_u = ([n], E_u)$ and $G_v = ([n], E_v)$, we have that $E_u \cup E_v$ contains at most one cycle of even length ≥ 4 .

We consider whether or not the union $E_u \cup E_v$ contains a cycle of even length, and we will see that these determine the two types of edges of the polytope. Considering a_u and a_v as the exponent vectors of u and v respectively, u and v form an edge that contains a lattice point on the interior if and only if the gcd of the nonzero entries of the vector $a_u - a_v$ is greater than one. Since the entries of a_u and a_v can only lie in $\{0, 1, 2\}$, the vector $a_u - a_v$ has entries in $\{0, \pm 1, \pm 2\}$, so the gcd > 1 only when $u - v$ contains only entries in $\{0, 2, -2\}$, and if so, the only possible interior lattice point is the midpoint $\frac{1}{2}(a_u + a_v)$. The exponent 2 in the monomial corresponds to a transposition in the permutation (i.e. a connected component consisting of an isolated edge.) In terms of graphs we have $G_{\frac{1}{2}(u+v)} = ([n], E_u \cup E_v)$: every edge in the graphs of G_u or G_v is also an edge in $G_{\frac{1}{2}(u+v)}$. Thus, an isolated edge that belongs to G_u but not to G_v , or vice versa, becomes an edge of a cycle.

These new cycles cannot have odd length: we can bipartition them based on whether they come from G_u or G_v , and cycles of odd length are not 2-edge-colorable. Furthermore, if there are multiple new cycles of even length at least 4, then u and v did not form an edge, so this cycle must be unique and of even length ≥ 4 . \square

The positive tropicalization of a variety X defined by the vanishing of an ideal $I \subset \mathcal{C}[x_1, \dots, x_n]$ has the following characterization

$$\text{trop}_{\mathcal{C}^+}(X) = \{w \in \mathbb{R}^n \mid \text{the initial ideal } in_w(I) \text{ has a positive real zero}\}.$$

When I is generated by a single polynomial $f = \sum_{a \in A} c_a x^a$, with all $c_a \neq 0$, we have that $in_w(I) = \langle in_w f \rangle$ where $in_w f = \sum c_a x^a$ with the sum taken over all $\{a \in A \mid a \cdot w \leq b \cdot w \forall b \in A\}$. Thus the positive tropicalization is a subfan of the (inner) normal fan of the Newton polytope of f .

THEOREM 5.3. *The normal cone of an edge of the Newton polytope lies in the positive tropical hypersurface $\text{trop}_{\mathcal{C}^+}(\mathcal{S}_n^{n-1})$ if and only if one of the following holds:*

- (1) *The edge has lattice length 1, and the monomials terms in the symmetric determinant corresponding to the two vertices have opposite signs.*
- (2) *The edge has lattice length 2, and the unique cycle of even length ≥ 4 on its midpoint monomial has length divisible by 4.*

In other words, $\text{trop}_{\mathcal{C}^+}(\mathcal{S}_n^{n-1})$ is the union of normal cones to edges containing a positive and negative monomial, or equivalently a positive and negative permutation.

Proof. If the edge has lattice length 1, then the initial ideal is, for some integer c , $\langle cx^u - x^v \rangle$ if the permutations that form u and v have opposite signs and is $\langle cx^u + x^v \rangle$ if they have the same sign. It has a positive real root only in the case of opposite signs.

If the edge has lattice length 2, then u and v have the same sign, as their symmetric difference consists of an even number of transpositions. Their midpoint $\frac{1}{2}(u + v)$ consists of a $2k$ cycle instead of the k transpositions. Trading k transpositions for a cycle amounts to changing the sign $k + 1$ times. When k is odd, the midpoint has the same sign as u and v , so the initial ideal cannot have a positive zero. When k is even,

the midpoint has opposite sign, and the initial ideal has form $\langle c(x^u - 2x^{\frac{1}{2}(u+v)} + x^v) \rangle = \langle c(x^{\frac{u}{2}} - x^{\frac{v}{2}})^2 \rangle$ where c is a monomial, so it has positive real root. \square

Now we consider $\text{trop}_{\mathcal{R}^+}(\mathcal{S}_n^{n-1})$. We will first show that $\text{trop}_{\mathcal{R}^+}(\mathcal{S}_n^{n-1}) \neq \text{trop}_{\mathcal{C}^+}(\mathcal{S}_n^{n-1})$ for $n \geq 4$ in an example, and then give a description of $\text{trop}_{\mathcal{R}^+}(\mathcal{S}_n^{n-1})$ in Theorem 5.5.

EXAMPLE 5.4. Consider the following matrix

$$M = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}.$$

It has minimum achieved at the cycle (1234), and therefore also at the pairs of transpositions (12)(34), in red, and (14)(23), in blue. By Theorem 5.3, it is in $\text{trop}_{\mathcal{C}^+}(\mathcal{S}_4^3)$.

Any lift of M into \mathcal{R}^+ must take the form

$$\tilde{M} = \begin{bmatrix} c_{11}t^2 & x & c_{13}t & c_{14} \\ x & c_{22}t^2 & c_{23} & c_{24}t^2 \\ c_{13}t & c_{23} & c_{33}t^2 & c_{34} \\ c_{14} & c_{24}t^2 & c_{34} & c_{44}t \end{bmatrix},$$

where the c_{ij} are all elements of \mathcal{R}^+ with valuation zero. Further, this lift must make \tilde{M} singular. As in the proof of Theorem 4.1, we examine the determinant of \tilde{M} as a quadratic polynomial in x . We want a root of this polynomial in \mathcal{R}^+ with valuation zero. So we want its discriminant to be nonnegative for some choice of c_{ij} s.

By explicit expansion of the determinant, and using the fact that the c_{ij} 's and x must have valuation 0, we can check that the discriminant with respect to x of the determinant of \tilde{M} has the leading term $-8c_{13}c_{14}c_{23}^2c_{34}c_{44}t^2$. No choice of (positive) real Puiseux series for the c_{ij} s with valuation 0 can make this term positive, so the discriminant must be negative, and there is no lift of M into \mathcal{R}^+ , so $M \notin \text{trop}_{\mathcal{R}^+}(\mathcal{S}_4^3)$.

The lineality space of the tropical variety $\text{trop}_{\mathcal{R}^+}(\mathcal{S}_n^{n-1})$ is spanned by matrices $\mathcal{M}^{(k)} \in \mathbb{R}^{n \times n}$ of the following form, for every $k \in [n]$,

$$M_{ij}^{(k)} = \begin{cases} 2 & i = j = k, \\ 1 & i = k \text{ or } j = k, i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

which represents simultaneous tropical scaling of row k and column k . The really positive tropicalization $\text{trop}_{\mathcal{R}^+}(\mathcal{S}_n^{n-1})$ is a subset of the positive tropicalization $\text{trop}_{\mathcal{C}^+}(\mathcal{S}_n^{n-1})$ described in Theorem 5.3, but it is not a subfan of the normal fan of the Newton polytopes. The positive tropical cones of type (1) in Theorem 5.3 are also really positive, but additional conditions are required for matrices in type (2) cones to be really positive.

THEOREM 5.5. *An $n \times n$ symmetric tropical matrix M lies in the really positive tropical variety $\text{trop}_{\mathcal{R}^+}(\mathcal{S}_n^{n-1})$ if and only if the minimum among the tropical monomials of the symmetric tropical determinant on M satisfies one of the following:*

- (1) *the minimum is achieved at two monomials forming an edge of lattice length 1 in the Newton polytope, whose coefficients have opposite signs in the symmetric determinant.*
- (2) *the minimum is achieved at three monomials u , v , and $\frac{1}{2}(u+v)$ where u and v differ along $2k$ transpositions for some $k \geq 1$, and $\frac{1}{2}(u+v)$ differs from u and v along a $4k$ -cycle, and for any pair of adjacent elements i, j on that cycle, the*

minimums of the tropical minors of M_i and M_j are achieved on monomials of the same sign, where M_i is M with the i -th row and i -th column deleted.

In case (2), it is sufficient to check a single adjacent pair i, j on the cycle.

Proof. We first consider the case where the minimum in the monomials of the symmetric tropical determinant of M is achieved on two monomials of opposite sign forming an edge of lattice length 1.

The normal vectors to the edges of lattice length 1 gives the initial ideal $\langle cx^u - x^v \rangle$, which, after monomial change of basis, is equivalent to $\langle cx_1 - x_2 \rangle$, for some $c > 0$. By [24, Lemma 2.6], since this is everywhere smooth, the cone dual to this edge is in $\text{trop}_{\mathcal{R}}(\mathcal{S}_n^{n-1})$. Furthermore, any solution to $cx_2 - x_1$ is (after possibly multiplying by -1 , and modifying $x_i, i \geq 3$) a positive solution, so the dual cone is in $\text{trop}_{\mathcal{R}^+}(\mathcal{S}_n^{n-1})$.

Now we consider the edges of lattice length 2. By relabeling, we can assume that u and v agree everywhere except on the $4k$ -cycle $(1\ 2\ 3 \dots 4k)$ (up to reversing the cycle). Again assume without loss of generality that u contains the transpositions $(12)(34) \dots ((4k-1)(4k))$, and v contains the transpositions $(23)(45) \dots ((4k)1)$. Via tropical scaling, we can also assume that $M_{12} = M_{21} = 0$.

Again let us consider an arbitrary (generic) lift of the entries of M to a symmetric matrix of positive real Puiseux series, \tilde{M} , leaving indeterminate $x = \tilde{M}_{12} = \tilde{M}_{21}$. We view the determinant $\det(\tilde{M})$ as a quadratic equation in x , and we will describe when x has a positive real Puiseux series lift of valuation 0.

We have $\det(\tilde{M})(x) = Ax^2 + Bx + C$. Again let c be the minimum valuation of monomials in $\det(\tilde{M})$. As before, the term Ax^2 captures all monomials of the symmetric determinant that includes both M_{12} and M_{21} . This includes the vertex u , and the leading term of Ax^2 is the leading term of the monomial corresponding to u , so $\text{val}(A) = c$ and A shares sign with u . The term Bx captures the midpoint $\frac{1}{2}(u+v)$, which contains the $4k$ -cycle $(1\ 2\ 3 \dots 4k)$. Thus, $\text{val}(B) = c$. The term C includes v , so $\text{val}(C) = c$ and A and C share signs with u and v and each other, and opposite signs with B .

As $\det(\tilde{M})(x)$ is a quadratic equation in x , its roots in \mathcal{C} have the form

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

These solutions are in \mathcal{R} when the discriminant is nonnegative. Furthermore, if the discriminant is positive, since we see that A and C share sign, $|B| > \sqrt{B^2 - 4AC} > 0$. Therefore, both $-B \pm \sqrt{B^2 - 4AC}$ have the same sign as $-B$, and, based on the sign of $-B$, at least one of $\pm\sqrt{B^2 - 4AC}$ has the same sign as $-B$, and so adding it to $-B$ cannot cause cancellation of the leading term. Thus, without loss of generality, $\text{val}(-B + \sqrt{B^2 - 4AC}) = \text{val}(-B) = \text{val}(B)$. Thus, there is a choice of x so that $\text{val}(x) = \text{val}(B) - \text{val}(2A) = c - c = 0$, as desired, and as $-B$ and A have the same sign, x is positive.

It remains to show that the discriminant is positive if and only if the minimum of $\text{trop det}(M_1)$ and $\text{trop det}(M_2)$ are achieved by monomials of the same sign. From Lemma 3.1, the discriminant $\text{Disc}_{m_{12}}(\det(\tilde{M}))$ is equal to $\det(\tilde{M}_1)\det(\tilde{M}_2)$. We can choose lifts of rows 1 and 2 of \tilde{M} so that the signs of $\det \tilde{M}_1$ and $\det \tilde{M}_2$ are equal to signs of any choice of permutations attaining the minimum in $\text{trop det}(M_1)$ and $\text{trop det}(M_2)$ respectively. In particular, we can choose them to have the same sign, so that the discriminant $\text{Disc}_{m_{12}}(\det(\tilde{M}))$ is positive, and \tilde{M} is a lift of M as desired.

If $\text{trop det}(M_1)$ and $\text{trop det}(M_2)$ have their minimums attained only on monomials of opposite signs, then any lifts of M_1 and M_2 must have $\det(\tilde{M}_1)$ and $\det(\tilde{M}_2)$ have

opposite signs because cancellation cannot occur. Thus, $\text{Disc}_{m_{12}}(\det(\tilde{M}))$ must be negative. \square

THEOREM 5.6. *For the variety \mathcal{S}_n^{n-1} of symmetric singular $n \times n$ matrices, the real and complex tropicalizations coincide, that is,*

$$\text{trop}_{\mathcal{R}}(\mathcal{S}_n^{n-1}) = \text{trop}_{\mathcal{C}}(\mathcal{S}_n^{n-1}).$$

Proof. The edges of lattice length 1 have initial ideal $\langle cx^u + x^v \rangle$ for $c \in \mathbb{R}$, so after monomial change of basis we have $\langle cx_1 + x_2 \rangle$. The normal cone to this edge is in $\text{trop}_{\mathcal{R}}(\mathcal{S}_n^{n-1})$ by [24, Lemma 2.6]. For edges of lattice length 2, we follow the same argument as in the proof of Theorem 5.5. However now the sign of $\tilde{M}_1\tilde{M}_2$ will always be positive for appropriate choice of the signs of \tilde{M}_{ij} : every monomial in $\det \tilde{M}_1$ contains some element \tilde{M}_{2j} from the second column of \tilde{M} , and no monomial in $\det \tilde{M}_2$ contains any, so we can simply flip the sign of \tilde{M}_{2j} if $\tilde{M}_1\tilde{M}_2$ is negative. For edges of lattice length 2 where the cycle of even length in $\frac{1}{2}(u+v)$ has length $4k+2$, the same argument works directly, only the sign of x may change, which doesn't matter for $\text{trop}_{\mathcal{R}}(\mathcal{S}_n^{n-1})$. \square

Acknowledgements. We are grateful to Marie-Charlotte Brandenburg, Georg Loho, and Rainer Sinn for discussions about the paper [6] and related topics, and to Felipe Rincón and Cynthia Vinzant for discussions about real tropicalizations. We thank Jonathan Leake for the observation of Lemma 3.1 and Changxin Ding for pointing us to the example in the proof of Lemma 2.2. We are grateful to the referee for a close reading and many detailed comments which helped us improve the paper, and for providing the lift (2) in the proof of Theorem 3.5 which is much simpler than our original one. MC and JY were partially supported by NSF grant #1855726. JY was partially supported by NSF grant #2348701.

REFERENCES

- [1] Maya Mohsin Ahmed, *Polytopes of magic labelings of graphs and the faces of the Birkhoff polytope*, Ann. Comb. **12** (2008), no. 3, 241–269.
- [2] Xavier Allamigeon, Stéphane Gaubert, and Mateusz Skomra, *Tropical spectrahedra*, Discrete Comput. Geom. **63** (2020), no. 3, 507–548.
- [3] Federico Ardila, *A tropical morphism related to the hyperplane arrangement of the complete bipartite graph*, 2004, <https://arxiv.org/abs/math/0404287>.
- [4] Daniel Irving Bernstein, *Completion of tree metrics and rank 2 matrices*, Linear Algebra Appl. **533** (2017), 1–13.
- [5] Louis J. Billera and A. Sarangarajan, *The combinatorics of permutation polytopes*, in Formal power series and algebraic combinatorics (New Brunswick, NJ, 1994), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 24, Amer. Math. Soc., Providence, RI, 1996, pp. 1–23.
- [6] Marie-Charlotte Brandenburg, Georg Loho, and Rainer Sinn, *Tropical positivity and determinantal varieties*, Algebr. Comb. **6** (2023), no. 4, 999–1040.
- [7] Marie-Charlotte Brandenburg, Georg Loho, and Rainer Sinn, *Corrigendum to “Tropical Positivity and Determinantal Varieties”*, Algebr. Comb. **7** (2024), no. 3, 749–751.
- [8] Richard A. Brualdi and Lei Cao, *Symmetric, Hankel-symmetric, and centrosymmetric doubly stochastic matrices*, Acta Math. Vietnam. **43** (2018), no. 4, 675–700.
- [9] May Cai, Kisun Lee, and Josephine Yu, *The tropical variety of symmetric rank 2 matrices*, Linear Algebra Appl. **720** (2025), 50–71.
- [10] Dustin Cartwright and Melody Chan, *Three notions of tropical rank for symmetric matrices*, Combinatorica **32** (2012), no. 1, 55–84.
- [11] Mike Develin, Francisco Santos, and Bernd Sturmfels, *On the rank of a tropical matrix*, in Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ., vol. 52, Cambridge Univ. Press, Cambridge, 2005, pp. 213–242.
- [12] Charles Lutwidge Dodgson, *Condensation of determinants, being a new and brief method for computing their arithmetical values*, Proc. R. Soc. Lond. **15** (1867), 150–155.
- [13] Alex Fink and Felipe Rincón, *Stiefel tropical linear spaces*, J. Combin. Theory Ser. A **135** (2015), 291–331.

- [14] Axel Hultman and Jakob Jonsson, *The topology of the space of matrices of Barvinok rank two*, *Beiträge Algebra Geom.* **51** (2010), no. 2, 373–390.
- [15] Philipp Jell, Claus Scheiderer, and Josephine Yu, *Real tropicalization and analytification of semialgebraic sets*, *Int. Math. Res. Not. IMRN* **2022** (2022), no. 2, 928–958, [Initially appeared as 2021, no. 24, 19178–19208].
- [16] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, *Graduate Studies in Mathematics*, vol. 161, American Mathematical Society, Providence, RI, 2015.
- [17] Hannah Markwig and Josephine Yu, *The space of tropically collinear points is shellable*, *Collect. Math.* **60** (2009), no. 1, 63–77.
- [18] James Oxley, *Matroid theory*, second ed., *Oxford Graduate Texts in Mathematics*, vol. 21, Oxford University Press, Oxford, 2011.
- [19] Lior Pachter and Bernd Sturmfels, *Tropical geometry of statistical models*, *Proc. Natl. Acad. Sci. USA* **101** (2004), no. 46, 16132–16137.
- [20] Kemal Rose and Máté L. Telek, *Computing positive tropical varieties and lower bounds on the number of positive roots*, *J. Symbolic Comput.* **132** (2026), article no. 102477 (26 pages).
- [21] Yaroslav Shitov, *On the nonnegative ranks of matrices in Puiseux series fields*, *Adv. in Appl. Math.* **170** (2025), article no. 102916 (6 pages).
- [22] David Speyer and Lauren Williams, *The tropical totally positive Grassmannian*, *J. Algebraic Combin.* **22** (2005), no. 2, 189–210.
- [23] Luis Felipe Tabera, *On real tropical bases and real tropical discriminants*, *Collect. Math.* **66** (2015), no. 1, 77–92.
- [24] Cynthia Vinzant, *Real radical initial ideals*, *J. Algebra* **352** (2012), 392–407.
- [25] Dylan Zwick, *Symmetric Kapranov and symmetric tropical ranks*, 2021, <https://arxiv.org/abs/2112.14945>.

ABEER AL AHMADIEH, School of Mathematics, American University of Sharjah, Sharjah 26666, UAE
E-mail : aalahmadieh@aus.edu

MAY CAI, School of Mathematics, Georgia Tech, Atlanta, GA 30332, USA
E-mail : mcai@gatech.edu

JOSEPHINE YU, School of Mathematics, Georgia Tech, Atlanta, GA 30332, USA
E-mail : jyu@math.gatech.edu