ALGEBRAIC COMBINATORICS

Michael Gekhtman, Zachary Greenberg & Daniel Soskin

Multiplicative Inequalities in Cluster Algebras of Finite Type

Volume 9, issue 1 (2026), p. 51-74.
https://doi.org/10.5802/alco.463

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 9, issue 1 (2026), p. 51–74
https://doi.org/10.5802/alco.463

Multiplicative Inequalities in Cluster Algebras of Finite Type

Michael Gekhtman, Zachary Greenberg & Daniel Soskin

Abstract Generalizing the notion of a multiplicative inequality among minors of a totally positive matrix, we describe, over full rank cluster algebras of finite type, the cone of Laurent monomials in cluster variables that are bounded as real-valued functions on the positive locus of the cluster variety. We prove that the extreme rays of this cone are the u-variables of the cluster algebra. Using this description, we prove that all bounded ratios are bounded by 1 and give a sufficient condition for all such ratios to be subtraction free. This allows us to show in $\mathrm{Gr}(2,n)$, $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, and $\mathrm{Gr}(3,8)$ that every bounded Laurent monomial in Plücker coordinates factors into a positive integer combination of so-called primitive ratios. In $\mathrm{Gr}(4,8)$ this factorization does not exists, but we provide the full list of extreme rays of the cone of bounded Laurent monomials in Plücker coordinates.

1. Introduction

Totally nonnegative matrices are matrices in which each minor is nonnegative. These matrices appear in a wide range of mathematical areas including higher Teichmüller theory and the representation theory of quantum groups. In fact, the search for minimal sets of minors to guarantee a matrix is totally nonnegative was one of the problems that inspired the theory of cluster algebras [12]. This search was also related to describing the *dual canonical bases* in the representation theory of quantum groups. Lusztig has shown that the specialization of elements of the dual canonical basis at q=1 are totally nonnegative polynomials, which are polynomials in matrix entries attaining nonnegative values on totally nonnegative matrices [22]. To this end, there is an interest in determinantal inequalities which are a natural source of such polynomials.

Definition 1.1. **Determinantal inequalities** *are inequalities in real linear combinations of products of minors which hold over all totally nonnegative matrices.*

Determinantal inequalities have been studied for years, starting with classical results by Hadamard, Fischer and Koteljanskii [17], [8], [19], [20]. A determinantal inequality is called multiplicative when it compares two products of minors. The problem of describing the set of multiplicative determinantal inequalities for totally

Manuscript received 25th September 2024, revised 1st September 2025, accepted 7th October 2025.

Keywords. Cluster algebras, total positivity, root systems, Grassmannians.

Acknowledgements. M.G. was partially supported by NSF grant DMS #2100785. He is also grateful to the Max Planck Institute for Mathematics in the Sciences for its hospitality during the June 2024 research visit. Z.G. was supported by the European Research Council under ERC-Advanced Grant 101018839. D.S. was supported by the National Science Foundation under Grant No. DMS-1926686.

positive matrices was stated by S. Fallat and C. Johnson in [7] in terms of bounded ratios of products of minors.

Problem A. *Describe ratios R of products of minors bounded as a real-valued function on the locus of totally positive elements in GL_n, where R is of the form*

$$\det(A_{I_1,I_1'})\det(A_{I_2,I_2'})...\det(A_{I_p,I_p'})/\det(A_{J_1,J_1'})\det(A_{J_2,J_2'})...\det(A_{J_q,J_q'}), \tag{1}$$

where $I_k, I_k', J_k, J_k' \subseteq \{1,2,\ldots,n\}$ with $|I_k| = |I_k'|$ and $|J_k| = |J_k'|$.

In [6], necessary and sufficient conditions were given for a ratio of products of two principal minors to be bounded over totally positive matrices.

This result was generalized to non-principal minors in [25] by M. Skandera, whose approach then allowed A. Boocher and B. Froehle to restate the problem in terms of ratios of products of Plücker coordinates bounded over the totally positive Grassmannian [2]. The necessary condition in [6] was generalized in [2] to the case of non-principal minors and an explicit factorization of ratios of products of two minors into products of so-called *primitive ratios* was constructed. It has been conjectured that all bounded ratios can be factored into products of nonnegative integer powers of primitive ratios [6]. Recently, it was proved in [26] that for any $n \times n$ matrix the set of bounded ratios is finitely generated. Along with that, some examples of non-primitive generators were discovered, which disprove the conjecture on factorization into primitive ratios for matrices of order $n \geqslant 4$ stated in [6]. Other conjectures on bounded ratios stated in [2] remain open.

Conjecture 1.2 ([2]). *A ratio given by Equation* (1) *is bounded if and only if it is bounded by 1.*

Both totally positive matrices and elements of totally positive Grassmannian can be parameterized using the *face weights* of a weighted planar network [12, 23]. In this parametrization, any minor (resp. Plücker coordinate) is a polynomial in the face weights. Furthermore, in the framework of cluster algebras the face weights of [23] are Y-coordinates of the cluster algebra associated to the totally positive Grassmannian.

Definition 1.3. *A ratio $\frac{p}{q}$ is called* subtraction-free (in positive weights) *if $q - p$ is a polynomial function in face weights with all positive coefficients.*

Conjecture 1.4 ([2]). *A ratio given by eq.* (1) *is bounded if and only if it is subtraction free.*

Recall that every Plücker coordinate belongs to the set of cluster variables in the cluster structure on a Grassmannian [24, 15]. On the other hand, the totally positive locus is defined for any cluster algebra of geometric type. Thus, it is natural to extend problem A to ratios in all cluster variables in a given cluster algebra.

Problem B. *Describe the set of ratios of products of cluster variables bounded as real-valued functions on the totally positive locus.*

For brevity, we will simply call a ratio of products of cluster variables (equivalently, a Laurent monomial in cluster variables) which is bounded on the totally positive locus *a bounded ratio.*

In this note, we characterize the cone of bounded ratios in problem B for full rank cluster algebras of finite type.

Theorem 1.5. *Let $\mathcal{A}$ be a full rank finite type cluster algebra associated to a Dynkin diagram $\mathfrak{D}$. The generators of the cone of bounded ratios are of the form*

$$\frac{\prod\limits_{\gamma\to\omega} x_\omega}{x_\gamma x'_\gamma},$$

where x_γ is the variable at the source of a Dynkin type bipartite quiver that mutates to x'_γ and the product is taken over all arrows in the quiver with γ as a source.

Corollary 1.6. *The generators of the bounded cone correspond exactly to the u-variables of the cluster algebra defined in* [1].

This classification allows us to prove the analogue of Conjecture 1.2 for full rank cluster algebras of finite type.

Corollary 1.7. *Every bounded ratio in cluster variables is bounded by* 1.

The analogue of Definition 1.3 for cluster variables is:

Definition 1.8. *A ratio $\frac{p}{q}$ is* **subtraction free (in cluster variables)** *if $q-p$ can be expressed as a polynomial function with positive coefficients in the full set of cluster variables. Note that this implies that when $q-p$ is expressed as a Laurent polynomial in the cluster variables from a fixed cluster, the coefficients are all positive.*

We provide an explicit counter example showing the following natural analogue of Conjecture 1.4 is false in general (Example 3.20):

Conjecture 1.9. *Every bounded ratio is subtraction free in cluster variables.*

However we show the following:

Corollary 1.10. *Every* ***extreme ray*** *of the bounded cone is subtraction free when expressed as a polynomial in all cluster variables.*

This proves that every bounded ratio that can be factored into integer powers of extreme rays is subtraction free in cluster variables. Thus the only counter examples are when the integer powers of extreme rays miss some bounded integer ratios.

In type A_{n-3} every cluster variable can be interpreted as a Plücker coordinate in $\mathrm{Gr}(2,n)$. However in other types there are other kinds of cluster variables. Thus we provide an algorithm to reduce the full bounded cone to the bounded cone generated by any finite subset of the cluster variables. Using this we study the cone of Plücker bounded ratios in $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, $\mathrm{Gr}(3,8)$. We show:

Theorem 1.11. *Every bounded ratio of Plücker coordinates in* $\mathrm{Gr}(2,n)$*,* $\mathrm{Gr}(3,6)$*,* $\mathrm{Gr}(3,7)$ *and* $\mathrm{Gr}(3,8)$ *is bounded by 1 and is subtraction free in face weights. Furthermore every such ratio can be factored into a positive integer combination of primitive ratios.*

As discussed in [26] not every Plücker bounded ratio can be factored into a positive combination of primitive ratios. In Figure 11 we provide the full list of extreme rays of the cone. We checked that every such ratio is subtraction free in positive weights and thus every bounded ratio in Plücker coordinates on $\mathrm{Gr}(4,8)$ is bounded by 1. It remains open in this case if every such ratio can be factored as a positive *integer* combination of these extreme rays.

The paper proceeds as follows. In Section 2 we review basic facts from cluster algebra theory. In Section 3 we introduce bounded ratios in cluster variables and torus action which suggests a necessary condition for a ratio to be bounded. Furthermore, we show that generators of the cone of bounded ratios in cluster variables are in correspondence with u-variables. In Section 4, we discuss bounded ratios in any fixed subset of cluster variables along with examples for $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$ and $\mathrm{Gr}(3,8)$.

2. Cluster Algebras

In this section we review the definitions of a cluster algebra. For a complete introduction see [10].

Definition 2.1. *A square matrix B is* **skew symmetrizable** *if there is a diagonal matrix W with positive integer entries such that WB is skew symmetric.*

Definition 2.2. *A* **seed** *of a cluster algebra consists of a pair $(\widehat{B}, \mathbf{x})$ where $\widehat{B}$ is a $(n+m) \times (n+m)$ skew symmetrizable matrix called the* **full exchange matrix** *and $\mathbf{x} = \{x_1, \dots, x_n, f_1, \dots, f_m\}$ is a collection of $n+m$ commuting variables called* **cluster variables**. *The first n variables $x_1, \dots, x_n$ are called* **mutable** *or* **unfrozen** *and the last m variables $f_1, \dots, f_m$ are called* **frozen**.

We consider the rows and columns of $\widehat{B}$ to be indexed by the cluster variables. Thus we call an index i frozen or unfrozen if the corresponding cluster variable is frozen or unfrozen respectively.

It is convenient to represent $\widehat{B}$ with a quiver Q. When $\widehat{B}$ is skew symmetric, Q has a node for each row/column and an arrow of weight $\widehat{B}_{ij}$ from i to j when $\widehat{B}_{i,j} > 0$. When $\widehat{B}$ is skew symmetrizable with matrix W, the associated quiver has weighted nodes. See Section 3 of [27] for full details. We will often borrow terminology of quivers to refer to exchange matrices. For example, we refer to indices of $\widehat{B}$ as ***sources/sinks*** if the node of the associated quiver is a source or sink of the subquiver of mutable nodes.

Definition 2.3. *The* **mutable part** *of a quiver/matrix is the subgraph/submatrix indexed by mutable variables. This corresponds to a matrix B consisting of the top left $n \times n$ block of $\widehat{B}$. We also consider the* **extended exchange matrix**, *$\widetilde{B}$, which is the rectangular $n \times (n+m)$ submatrix of the mutable rows of $\widehat{B}$.*

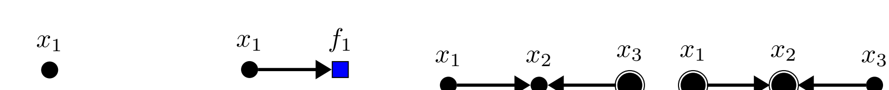
Definition 2.4. *A Y*-**seed**, *$(\widehat{B}, \mathbf{y})$, is a pair that consists of a skew symmetrizable matrix and a list of variables $\mathbf{y} = \{y_1, \dots, y_n\}$, called Y-variables. Note that a Y-seed only has a variable for each mutable index and thus only depends on B.*

Definition 2.5. *We say a seed is* **full rank** *if* $\operatorname{Rank}(\widetilde{B}) = n$.

We note that the property of being full rank depends on the frozen nodes. In fact every seed can be made full rank by "framing" each mutable node with a corresponding frozen node attached out from the mutable node.

Example 2.6. In Figure 1 we see a variety of seeds represented as quivers and exchange matrices. The two seeds in Figure 1a and Figure 1b have the same mutable part. As such we refer to these seeds as type A_1. Although they are the same type the seed in Figure 1a is not full rank, while the seed in Figure 1b is. In Figure 1c and Figure 1d we see two examples with skew symmetrizable matrices. In quivers we use a large node with an extra ring to represent nodes of weight 2.

$$\begin{array}{cc} & x_1 \\ x_1 & [\,0\,] \end{array} \qquad \begin{array}{c} \\ x_1 \\ f_1 \end{array}\!\!\begin{array}{c} \begin{matrix} x_1 & f_1 \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$$



(a) A_1 (b) A_1 (c) B_3 (d) C_3

Figure 1. Seeds as Matrices and Quivers

Example 2.7. Our key running example will be cluster algebras of type A_3. In Figure 2a we see the version with only mutable nodes. This seed is also not full rank. To make a full rank seed with the same mutable part it suffices to add one frozen node attached to either x_1 or x_3 breaking the symmetry. However it will be convenient to add more frozen nodes to obtain a quiver for cluster algebra associated to $\mathrm{Gr}(2,6)$ as in Figure 2b.

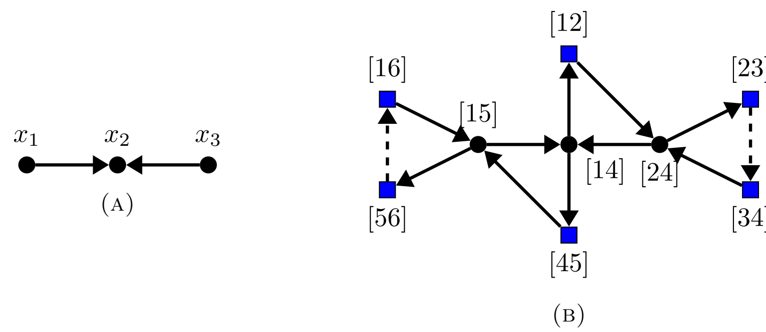


Figure 2. Seeds of Type A_3

Example 2.8. Our other key example will be seeds corresponding to the Grassmannian $\mathrm{Gr}(k,n)$. In [11] Section 6.7 they give a construction of an initial seed with $(k-1)(n-k-1)$ mutable nodes and n frozen nodes arranged in an $k \times (n-k)$ grid. In Figure 3 we give an example for $\mathrm{Gr}(3,6)$. On this seed the cluster variables can be identified with Plücker coordinates $[I]$ living in the coordinate ring of the affine cone. See [24, 15, 5, 11] for full details.

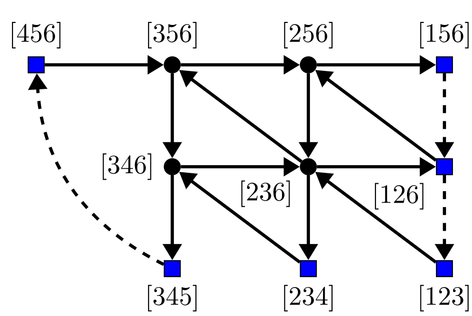


Figure 3. Grassmannian Seed for $Gr(3,6)$

Definition 2.9. *Given a mutable index k, we produce a new matrix* $\mu_k(\widehat{B})$ *via* **matrix mutation***:*

$$\mu_k(\widehat{B})_{i,j} = \begin{cases} -\widehat{B}_{i,j} & i = k \text{ or } j = k \\ \widehat{B}_{i,j} + [\widehat{B}_{i,k}]_+[\widehat{B}_{k,j}]_+ - [\widehat{B}_{i,k}]_-[\widehat{B}_{k,j}]_- & \text{otherwise}, \end{cases} \tag{2}$$

where $[x]_+ = \max(x,0)$ *and* $[x]_- = \min(x,0)$.

Definition 2.10. *A* **(cluster) seed pattern** *is an n regular tree whose vertices are labeled by seeds and edges are labeled* $1,\dots,n$. *The seeds* $(\widehat{B},\mathbf{x}) \overset{k}{\leftrightarrow} (\widehat{B}',\mathbf{x}')$ *are related* *via* $\widehat{B}' = \mu_k(\widehat{B})$ *and*

$$x'_\ell = \mu_k(x_\ell) = \begin{cases} x_\ell & \ell \neq k \\ \frac{1}{x_k}\left(\prod_{i|B_{ki}<0} x_i^{|B_{ki}|} + \prod_{j|B_{kj}>0} x_j^{B_{kj}}\right) & \ell = k\,. \end{cases} \tag{3}$$

Given a seed $S_0 = (\widehat{B}, \mathbf{x})$ we produce a seed pattern by starting at S_0 and performing all possible sequences of mutations. Note that $\mu_k(\mu_k(S)) = S$ so it makes sense to consider the edges of the n regular tree to be unoriented.

Definition 2.11. *A Y*-**seed pattern** *is similarly an n regular tree with vertices labeled by Y-seeds. The key difference is the relation between two seeds on an edge, $(\widehat{B}, \mathbf{y}) \overset{k}{\leftrightarrow} (\widehat{B}', \mathbf{y}')$. The exchange matrices are still related by matrix mutation, but the variables are now related by* Y-**mutation**:

$$y'_i = \mu_k(y_i) = \begin{cases} y_k^{-1} & i = k \\ y_i(1 + y_k^{-1})^{-b_{ik}} & b_{ik} > 0, i \neq k \\ y_i(1 + y_k)^{-b_{ik}} & b_{ik} \leqslant 0, i \neq k\,. \end{cases} \tag{4}$$

Remark 2.12. This definition of Y-variable mutation agrees with [1], and [9, p. 9]. However it is the opposite convention to [10, Definition 3.5.2] whose Y-variables correspond to the inverse of the Y-variables defined above.

Definition 2.13. *The* **cluster algebra** $\mathcal{A}_Q$ *generated by a seed $(Q, \mathbf{x})$ is the subalgebra of the ring of rational functions on $\mathbf{x}$ generated by all cluster variables obtained by performing all possible mutations. Let Π be the set indexing all mutable cluster variables and Π_f be the set of frozen variables. Then $\mathcal{A} = \mathbb{Z}[x_i \mid i \in \Pi_f][x_\gamma \mid \gamma \in \Pi]/I$ where I is the ideal generated by the X-exchange relation for every possible mutation. One similarly defines the* Y-**algebra**, *$\mathcal{A}_Q = \mathbb{Z}[y]/J$ where y ranges over every Y-variable in the Y-seed pattern and J is generated by Y-exchange relations.*

Matrix mutation preserves the rank of $\widetilde{B}$ and thus we consider full rank a property of a cluster algebra/seed pattern.

Proposition 2.14. *Let $\mathcal{A}$ and $\mathcal{Y}$ be cluster and Y-algebras for the same initial seed. There is a map $p \colon \mathcal{Y} \to \mathcal{A}$ given on each seed by*

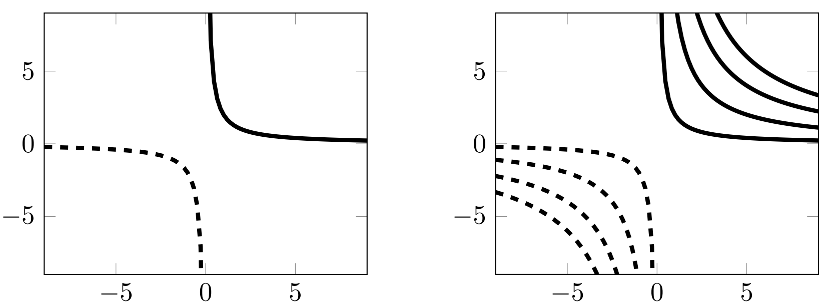
$$y_i \mapsto \prod x_j^{\widetilde{B}_{ij}}. \tag{5}$$

Proof. To check the map is well defined one must verify it commutes with X and Y mutation. See [16],[9] or [10] for the details of the proof. ☐

Definition 2.15. *The* **localized cluster algebra** $\mathcal{A}[\mathbf{x}^{-1}]$ *is obtained by localizing the set of mutable cluster variables.*

Definition 2.16. *The* R-**points** *of a cluster algebra in a semifield R is the set $\mathrm{Hom}(\mathcal{A}[\mathbf{x}^{-1}], R)$. Each element can be identified with a choice of element in R for each cluster variable satisfying the exchange relations.*

Example 2.17. In type A_1 there are only two distinct seeds. There are two mutable cluster variables $x_{-\gamma}$ and x_γ. Thus $\mathcal{A} = \mathbb{Z}[x_{-\gamma}, x_\gamma]/\langle x_{-\gamma}x_\gamma = 2\rangle$. In Figure 4a we plot the real points. The positive points are the hyperbola in the first quadrant. In this somewhat trivial case the algebra is already localized at x_γ and $x_{-\gamma}$ so the real points of $\mathcal{A}$ and $\mathcal{A}[\mathbf{x}^{-1}]$ are the same. In Figure 4b we see a contour plot for the A_1 cluster algebra with one frozen variable f_1. We note that the frozen variable allows $x_{-\gamma}$ and x_γ to take values at any pair of positive real numbers.



(A) No Frozen (B) Full Rank

Figure 4. Real Points of A_1 Cluster Algebra

Let R be a semifield with the additional property that the sum of two nonzero elements is nonzero (and thus invertible). Then any mutation applied to a seed with a nonzero element of R chosen for each cluster variable produces a new seed with the same property. Thus, for every seed a choice of nonzero elements in R for each cluster variable extends to the unique map $\mathcal{A}[\mathbf{x}^{-1}] \to R$. In particular, one can specify an $\mathbb{R}_{>0}$-point of a cluster algebra by choosing a positive number for each cluster variable in any seed. We call the $\mathbb{R}_{>0}$ points of a cluster algebra the ***positive points*** and write $\mathcal{X}_{>0}(\mathcal{A})$ or $\mathcal{X}_{>0}$ if the choice of $\mathcal{A}$ is understood from the context.

2.1. Finite Type. Let $\mathfrak{D}$ be a Dynkin diagram of finite type.

Definition 2.18. *A quiver is of* **Dynkin type** *if its mutable part is an orientation of a Dynkin diagram with every node either a source or a sink. We say a cluster algebra $\mathcal{A}$ is of type $\mathfrak{D}$ if $\mathcal{A}$ contains a seed with quiver isomorphic to an orientation of $\mathfrak{D}$. A* **Dynkin seed** *is a seed of $\mathcal{A}$ whose quiver is of Dynkin type.*

Recall that a Coxeter element of the Weyl group associated to a Dynkin diagram is the product of all the simple reflections. All Coxeter elements have the same order which is called the ***Coxeter number***.

Proposition 2.19. *Let $\mathfrak{D}$ be a connected Dynkin diagram and h the associated Coxeter number. The cluster algebra of type $\mathfrak{D}$ contains $h+2$ Dynkin seeds. Each mutable cluster variable appears at a unique source and a unique sink in this set of seeds.*

Proof. This follows from [13]. As a consequence, the set of mutable cluster variables is in bijection with the set of almost positive roots $\Phi_{\geqslant -1}$ of the associated root system, i.e the set of all positive roots and negative simple roots. □

The set of Dynkin seeds can be obtained by starting at any Dynkin seed and then repeatedly mutating at the full set of sources[13]. This sequence of seeds can be assembled into the Auslander-Reiten quiver of the associated cluster category ([18]).

While there are typically many more Y-variables than cluster variables, in a finite type cluster algebra there is a canonical choice of Y-variable for each cluster variable, y_γ. We take y_γ to be the Y-variable at the node associated to x_γ in the Dynkin seed where x_γ is a source. These Y-variables were used by Fomin and Zelivnsky to prove Zamolodchikov's conjecture [14]. As a consequence of this proof they established a Laurent phenomenon analogous to the Laurent phenomenon for cluster variables for this subset of Y-variables. In particular:

Theorem 2.20. *Every y_γ is of the form $\frac{N_\gamma}{\mathbf{p}^\gamma}$ where N_γ is a polynomial in p_i with constant term 1 and $\mathbf{p}^\gamma = \prod p_i^{\gamma_i}$ and $\gamma = \sum \gamma_i e_i$ for $\{e_i\}$ the set of simple roots.*

Proof. This is a rephrasing of [14, Theorem 1.5]. □

The parameters p_i are exactly the values of the Y-variables at negative simple roots $-e_i$. These roots occur in two distinct Dynkin seeds. Usually, the algebra is parameterized by choosing initial values from a single cluster. To obtain the same formulas as above from a single Dynkin seed we take the sources to be p_i and the sinks to be p_j^{-1}. Then mutation at each sink produces the Y-variable p_i associated to the root $-e_i$ as needed.

Example 2.21. In type A_3 there are 6 Dynkin seeds as seen in Figure 5 containing 9 distinct cluster variables. The values of each cluster variable and Y-variable are given in Figure 6. The exchange matrix for the initial seed without any frozen variables is $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. This matrix is not full rank, it has a null vector $(1,0,-1)$. However one can add frozen nodes to make the extended matrix full rank. For example, the initial Dynkin seed for the cluster algebra associated to $\mathrm{Gr}(2,6)$ is

$$\begin{bmatrix} 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix}.$$

We observe that the map from Y-variables to X-variables fails to be injective in the first case. Here the two Y-variables at the sources are:

$$y_{[-1,0,0]} = y_{[0,0,-1]} = x_{[0,-1,0]}.$$

In the Grassmannian case these variables are distinct:

$$y_{[-1,0,0]} = y_{[24]} = \frac{[14][23]}{[12][34]} \qquad \text{and} \qquad y_{[0,0,-1]} = y_{[15]} = \frac{[14][56]}{[16][45]}.$$

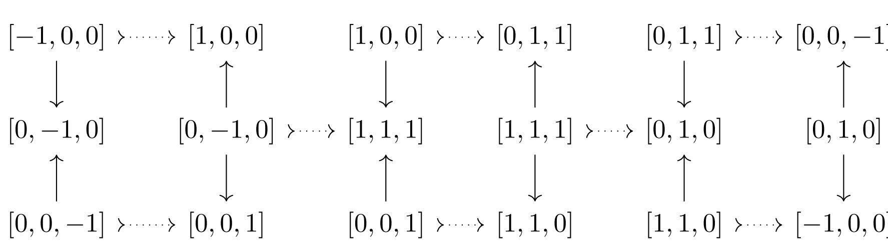


Figure 5. Auslander-Reiten Walk in Type A_3

Definition 2.22. *The* **compatibility degree** *of two distinct almost positive roots γ, ω, written $\epsilon(\gamma,\omega)$, is given by choosing a new set of simple roots such that $-\gamma$ is simple and ω is positive. Then $\epsilon(\gamma,\omega)$ is the coefficient of $-\gamma$ when ω is expressed as a linear combination of these simple roots. We define $\epsilon(\gamma,\gamma)=0$. The* **compatibility degree** *of two cluster variables, $\epsilon(x_\omega, x_\gamma)$, is the compatibility degree of the two roots. When $\gamma \neq \omega$, this corresponds to the power of x_ω in the denominator of the Laurent polynomial expression of x_γ using the Dynkin seed containing x_ω as initial seed.*

This definition of compatibility degree is due to [4], who prove it is equivalent to the definition first given in [14, Section 3.1]. In particular a new choice of simple roots always exists and the compatibility degree is independent of this choice.

Example 2.23. The cluster algebra of type C_2 is full rank with no frozen variables. It has six Dynkin seeds containing 6 distinct cluster variables. If we take the initial seed to be $x_1 \to x_2$ with x_1 on the large node, the variables are:

$$x_1 = x_{[-1,0]} = x_1 \qquad x_3 = x_{[1,0]} = \frac{1+x_2}{x_1} \qquad x_5 = x_{[1,1]} = \frac{1+x_2+x_1^2}{x_1x_2}$$
$$x_2 = x_{[0,-1]} = x_2 \qquad x_4 = x_{[2,1]} = \frac{(1+x_2)^2+x_1}{x_1^2x_2} \qquad x_6 = x_{[0,1]} = \frac{1+x_1^2}{x_2}\,.$$

The variables with odd indices are associated to the node with weight 2, while the variables with even indices are associated to the node with weight 1. We compute the compatibility degree of $x_1 = [-1,0]$ with the other five roots:

$$\epsilon(x_1,x_2)=0 \quad \epsilon(x_1,x_3)=1 \quad \epsilon(x_1,x_4)=2 \quad \epsilon(x_1,x_5)=1 \quad \epsilon(x_1,x_6)=0\,.$$

Note that the compatibility degree is only symmetric for simply-laced Dynkin diagrams. The non symmetry arises when comparing cluster variables associated to nodes with different weights. For example $\epsilon(x_4,x_1) = 1 \neq 2 = \epsilon(x_1,x_4)$.

3. Bounded Ratios

Definition 3.1. *The* **ratio space** $V_r \subset \mathcal{A}[\mathbf{x}^{-1}]$ *is the set of Laurent monomials in mutable and frozen variables. We identify this space with an integer lattice of a real vector space V of dimension $N+m$ where N is the number of mutable cluster variables in the cluster algebra. The integer vector $(v_1,\dots,v_{N+m})$ corresponds to the ratio $\mathbf{x}^{\mathbf{v}} = \prod x_i^{v_i}$.*

We are interested in the subset of bounded ratios.

Definition 3.2. *The* **cone of bounded ratios**, C, *is subset of V where the corresponding ratio is bounded over positive points of the cluster algebra. Formally*

$$C = \operatorname{span}_{\mathbb{R}+}\{\mathbf{v} \in V_r \mid \exists L < \infty : \forall p \in \mathcal{X}_{>0} : \mathbf{x}^{\mathbf{v}}(p) < L\}\,.$$

Example 3.3. Consider the cluster algebra for A_1 with one frozen variable (Figure 1b). We identify V with $\mathbb{R}^3$ using the basis $\{x_{-\gamma}, x_\gamma, f_1\}$. We translate several

Roots	$[-1,0,0]$	$[0,0,-1]$	$[0,-1,0]$
Cluster Variables	x_1	x_3	x_2
Y-Variables	p_1	p_3	p_2
Roots	$[1,0,0]$	$[0,0,1]$	$[1,1,1]$
Cluster Variables	$\frac{1+x_2}{x_1}$	$\frac{1+x_2}{x_3}$	$\frac{1+x_1x_3+2x_2+x_2^2}{x_1x_2x_3}$
Y-Variables	$\frac{1+p_2}{p_1}$	$\frac{1+p_2}{p_3}$	$\frac{1+p_1+2p_2+p_3+p_1p_2+p_1p_3+p_2p_3+p_2^2}{p_1p_2p_3}$
Roots	$[0,1,1]$	$[1,1,0]$	$[0,1,0]$
Cluster Variables	$\frac{1+x_1x_3+x_2}{x_2x_3}$	$\frac{1+x_1x_3+x_2}{x_1x_2}$	$\frac{1+x_1x_3}{x_2}$
Y-Variables	$\frac{1+p_1+p_2+p_3+p_1p_3}{p_2p_3}$	$\frac{1+p_1+p_2+p_3+p_1p_3}{p_1p_2}$	$\frac{1+p_1+p_3+p_1p_3}{p_2}$

Figure 6. Cluster Variables and Y-Variables in A_3

vectors into their corresponding ratios:

$$(-1,-1,0) = x_{-\gamma}^{-1}x_{\gamma}^{-1} = \frac{1}{1+f_1} \qquad (-1,-1,1) = x_{-\gamma}^{-1}x_{\gamma}^{-1}f_1^1 = \frac{f_1}{1+f_1}$$
$$(1,0,1) = x_{-\gamma}f_1 \qquad (1,1,0) = x_{-\gamma}^1 x_{\gamma}^1 = 1+f_1\,.$$

We observe the vectors in the first row are bounded and thus in C. The vectors in the second row are unbounded, and thus not in C.

3.1. Torus Action. We now explore a simple necessary condition for a ratio to be bounded.

Each vector $\alpha \in \ker(\widetilde{B})$ and $z \in R^{\times}$ induce an automorphism $\phi_{\alpha,z}$ of $\mathcal{A} \otimes R$ that sends $x_i \mapsto z^{\alpha_i} x_i$ for each initial mutable and frozen cluster variable. By [15, Lemma 5.3] when $\alpha \in \ker(\widetilde{B})$, the induced action on all other cluster variables is multiplication by a power of z as well. We write $\mathrm{wt}_\alpha(x_\gamma)$ for this power. The weight extends to the full ratio space in the natural way, $\mathrm{wt}_\alpha(\mathbf{v}) = \sum \mathrm{wt}_\alpha(x_\gamma)v_\gamma$.

As $\phi_{\alpha,z}$ is an automorphism of $\mathcal{A} \otimes R$, it induces an action on the R points of the algebra. In particular, if $z \in \mathbb{R}_{>0}$, this action preserves the set of positive points.

Finally we compute the action of $\phi_{\alpha,z}$ on the pairing between any ratio vector $\mathbf{v}$ and R-point p:

$$\mathbf{v}(\phi_{\alpha,z}p) = (\phi_{\alpha,z}\mathbf{v})(p) = z^{\mathrm{wt}_\alpha(\mathbf{v})}\mathbf{v}(p)\,. \tag{6}$$

Remark 3.4. This action can be viewed as an algebraic torus T acting by characters on the ratio space. See [1, 21] for more details.

Lemma 3.5. *If there is a vector $\alpha \in \ker(\widetilde{B})$ such that $\mathrm{wt}_\alpha(\mathbf{v}) \neq 0$ then $\mathbf{v}$ is not a bounded ratio.*

Proof. If $\mathrm{wt}_\alpha(\mathbf{v}) > 0$ then take $z = 2$ otherwise take $z = \frac{1}{2}$. Then

$$\mathbf{v}(\phi_{\alpha,z}^t(p)) = (z^{\mathrm{wt}_\alpha(\mathbf{v})})^t \cdot \mathbf{v}(p).$$

As $t \to \infty$, $(z^{\mathrm{wt}_\alpha(\mathbf{v})})^t \to \infty$ and thus $\mathbf{v}$ is unbounded. □

Corollary 3.6. *A bounded ratio has weight 0 for any vector $\alpha \in \ker(\widetilde{B})$.*

Example 3.7. In A_1 with one frozen variable the vector $\alpha = (-1,0)$ generates the kernel of the exchange matrix (Figure 1b). The induced map on the cluster algebra is

$$(x_{-\gamma}, x_\gamma, f_1) \xmapsto{\phi_{\alpha,z}} (z^{-1}x_{-\gamma}, z^1 x_\gamma, z^0 f_1)\,,$$

and the induced weight vector is $(-1,1,0)$. We observe that the orbits of this action are the hyperbolas in Figure 4b.

Furthermore, from Corollary 3.6, the cone of bounded ratios is contained in the subspace with equal powers of $x_{-\gamma}$ and x_γ. To illustrate this containment, we compute the weights of the ratios in Example 3.3:

$$\mathrm{wt}_\alpha(-1,-1,0) = 0 \quad \mathrm{wt}_\alpha(-1,-1,1) = 0 \quad \mathrm{wt}_\alpha(1,0,1) = -1 \quad \mathrm{wt}_\alpha(1,1,0) = 0.$$

We observe the first two bounded ratios have zero weight as expected and the third unbounded ratio has nonzero weight. However the final ratio is unbounded despite having zero weight.

Remark 3.8. In the Grassmannian cluster algebra Corollary 3.6 can be rephrased using the natural algebraic torus action of $(\mathbb{C}^{\times})^n$ on $\mathrm{Gr}(k,n)$ [2]. This ***ST0 condition*** states that each column index appears an equal number of times in the numerator and denominator. This corresponds to a character given by assigning to every variable weight α_i where α_i is the number of times column i appears when expressed as a polynomial in Plücker coordinates.

Note that all exchange relations corresponding to the standard initial cluster of $\mathrm{Gr}(k,n)$ are either short Plücker relations

$$[ikS][j\ell S] = [ijS][k\ell S] + [i\ell S][jkS]\,,$$

where S is a $k-2$ element subset of $[1,n]$ and $i,j,k,\ell \notin S$, or are of the form

$$[ijkS]\;\mu([ijkS]) = [ikfS]\;[ijdS]\;[jkeS] + [ikdS]\;[ijeS]\;[jkfS],$$

where S is $k-3$ element subset of $[1,n]$, such that $i,d,j,e,k,f \notin S$. In each such relation both monomials in RHS have the same multiplicities for each index i. Thus the vector α_i described above is in the kernel of the exchange matrix.

Lemma 3.9. *When the extended exchange matrix has full rank, the cone of bounded ratios is contained in a space of dimension equal to number of cluster variables, N.*

Proof. Since the extended exchange matrix has full rank n, $\dim(\ker(\widetilde{B})) = m$. Thus for a ratio to have weight 0 with respect to the entire kernel, $\mathbf{v}$ must satisfy m linearly independent conditions. Thus the cone of bounded ratios is contained in a space of dimension $N+m-m=N$. ☐

3.2. U Variables. In [1], they define the cluster configuration space associated to a finite Dynkin diagram $\mathfrak{D}$. These spaces recover the classical configuration space of n distinct points on the projective line, $M_{0,n}$, by taking $\mathfrak{D}$ to be a Dynkin diagram of type A_n. We recall the definitions here.

Definition 3.10. *The* **cluster configuration algebra**, *$\mathcal{U}_{\mathfrak{D}}$, associated to a Dynkin diagram $\mathfrak{D}$ is the ring $\mathbb{Z}[u_\gamma^{\pm 1}]/I_{\mathfrak{D}}$ where $I_{\mathfrak{D}}$ is the ideal generated by the equations of the form*

$$u_\gamma + \prod_{\omega \neq \gamma} u_\omega^{\epsilon(\omega,\gamma)} = 1\,, \tag{7}$$

for each mutable cluster variable $\gamma \in \Pi$ of the associated cluster algebra $\mathcal{A}_{\mathfrak{D}}$.

As with the cluster algebra, we can discuss R points of the cluster configuration algebra. Following [1] we write $\mathcal{M}_{\mathfrak{D}}$ for the $\mathbb{C}$ points of $\mathcal{U}_{\mathfrak{D}}$ and $\mathcal{M}_{\mathfrak{D}}^{>0}$ for the $\mathbb{R}_{>0}$ points.

Surprisingly, the space $\mathcal{M}_{\mathfrak{D}}^{>0}$ is n dimensional where n is the number of nodes in $\mathfrak{D}$. In fact there is a map from $\mathcal{X}_{>0}$ to $\mathcal{M}_{\mathfrak{D}}^{>0}$ given by

$$(x_\gamma)_{\gamma\in\Pi} \mapsto (v_\gamma)_{\gamma\in\Pi} = \left(\frac{y_\gamma}{1+y_\gamma}\right) = \left(\frac{\prod x_\omega}{x_\gamma x'_\gamma}\right)\,, \tag{8}$$

where y_γ is the Y-coordinate in the Dynkin seed at x_γ where γ is a source. The final equality uses the map from Y-coordinates to X-coordinates and the product in the numerator is taken over all ω that are out neighbors of γ and x'_γ is the X-coordinate obtained by mutating at x_γ in this seed.

Proposition 3.11. *When $\mathcal{A}$ is of full rank, the map defined in eq.* (8) *is a bijection $\mathcal{X}_{>0} /\!\!/ T \to \mathcal{M}_{\mathfrak{D}}^{>0}$. Given this bijection, one defines an isomorphism $\mathcal{U}_{\mathfrak{D}} \to \mathcal{A}_{\mathfrak{D}}^T$ given on generators by $u_\gamma \mapsto v_\gamma$.*

Proof. This follows from Theorem 4.2 of [1]. They describe this isomorphism on the level of rings and on the level of $\mathbb{C}$ points. Since the torus action restricts to the set of positive points, the isomorphism works in this context as well. ☐

Example 3.12. The cluster configuration algebra of type A_1 is defined by the degenerate set of equations

$$u_{-\gamma} + u_\gamma = 1 \qquad u_\gamma + u_{-\gamma} = 1\,.$$

In Figure 7 we see $\mathcal{M}_{\mathfrak{D}}^{>0}$. Here we can observe the requirement that $\mathcal{A}$ is of full rank. If we take the A_1 cluster algebra with no frozen nodes the map from $\mathcal{X}_{>0}$ to $\mathcal{M}_{\mathfrak{D}}^{>0}$ is of the form

$$(x_{-\gamma}, x_\gamma) \mapsto \left(\frac{1}{x_{-\gamma}x_\gamma}, \frac{1}{x_{-\gamma}x_\gamma}\right) = \left(\frac{1}{2}, \frac{1}{2}\right),$$

and thus does not fill the entire space of positive points. However when we add a frozen variable to make a full rank cluster algebra the map is now of the form:

$$(x_{-\gamma}, x_\gamma, f_1) \mapsto \left(\frac{1}{x_{-\gamma}x_\gamma}, \frac{f_1}{x_{-\gamma}x_\gamma}\right) = \left(\frac{1}{1+f_1}, \frac{f_1}{1+f_1}\right).$$

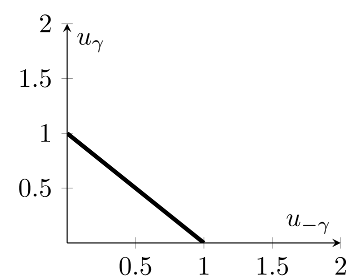


Figure 7. Positive Points of the Cluster Configuration Algebra of type A_1

Example 3.13. The cluster configuration algebra of type C_2 is defined by the orbit of the equations

$$u_1 + u_3u_4u_5 = 1 \qquad u_2 + u_4u_5^2u_6 = 1$$

under increasing the indices by 2 modulo 6. Using eq. (8) the solutions are parameterized by

$$v_1 = \frac{x_2}{x_1x_3} \quad v_3 = \frac{x_4}{x_3x_5} \quad v_5 = \frac{x_6}{x_5x_1} \qquad v_2 = \frac{x_3^2}{x_2x_4} \quad v_4 = \frac{x_5^2}{x_4x_6} \quad v_6 = \frac{x_1^2}{x_6x_2}.$$

Proposition 3.14. *Each variable u_γ is bounded over $\mathcal{M}_{\mathfrak{D}}^{>0}$.*

Proof. This is clear from the form of eq. (7) over the positive reals, each u_γ is 1 minus a positive number and thus is bounded above by 1. ☐

Theorem 3.15. *When $\mathcal{A}$ is of full rank and finite type, there is a choice of initial cluster variables parameterized by $t \in \mathbb{R}_{>0}$ such that $\lim_{t\to 0} v_\gamma = 0$ and for $\omega \neq \gamma$, $0 < \lim_{t\to 0} v_\omega < \infty$.*

Proof. Consider the Dynkin seed where x_γ is a source at vertex i. We consider this seed to be the initial seed and thus can express all other cluster variables in terms of the initial variables $x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}$.

Assume the initial cluster variables are of the form $x_j = t^{\beta_j}$. Then the initial Y-variables are of the form $y_j = \prod x_k^{\widetilde{B}_{jk}} = \prod t^{\widetilde{B}_{jk}\beta_k}$. Thus the vector of powers of initial Y-variables as a function of t is $\mathbf{y} = \widetilde{B}\beta$. As $\widetilde{B}$ is of full rank, the equation $e_i = \widetilde{B}\beta$ has a solution. We use this β to define the initial cluster variables. By construction, the other initial Y-variables are constant with respect to t. By Theorem 2.20 all non-initial Y-variables are Laurent polynomials in variables $p_1, \dots, p_n$ whose numerator has constant term 1. By our choice of initial cluster variables we have that $p_i = t$ and

$p_j = 1$ for $j \neq i$, since $p_i = y_i$ and $p_j = y_j^{\pm 1}$ depending on if j is a source or sink. We then have

$$\lim_{t\to 0} y_\omega = \lim_{t\to 0} \frac{C_\omega + o(t)}{t^{\epsilon(\gamma,\omega)}} = \begin{cases} C_\omega & \epsilon(\gamma,\omega) = 0 \\ \infty & \epsilon(\gamma,\omega) > 0\,, \end{cases} \tag{9}$$

with C_ω a positive constant. As $v_\omega = \frac{y_\omega}{1+y_\omega}$ we see that $\lim_{t\to\infty} v_\omega$ is a positive constant if ω and γ are compatible and equal to 1 otherwise. □

Corollary 3.16. *The ratios v_γ correspond to independent vectors in the root space. In other words there is no vector of powers λ such that $v_\gamma = \prod_{\omega\neq\gamma} v_\omega^{\lambda_\omega}$.*

Proof. Assume for contradiction that such a product exists. At least one power λ_ω is nonzero, otherwise v_γ would be identically 1 for all choices of initial cluster variables. However by Theorem 3.15 there is a choice of initial seed with v_γ going to 0 at t goes to 0.

Now consider the choice of initial seed given by Theorem 3.15 such that only v_ω goes to 0 while each other variable is bounded away from 0. Since the set of ratios is finite, there is a global lower bound $\lim_{t\to\infty} v_\eta > L > 0$ on all other ratios. Note that as all ratios are also bounded above by 1 (Proposition 3.14) the other product terms stay finite. If $\lambda_\omega > 0$ then the entire product is sent to 0. On the other hand v_γ is bounded below by L, contradicting the equality. Similarly if $\lambda_\omega < 0$ the product is sent to infinity contradicting the fact that v_γ is bounded (Proposition 3.14). □

Theorem 3.17. *The cone of bounded ratios C is generated by the set $\{v_\gamma \mid \gamma \in \Pi\}$.*

Proof. From Lemma 3.9 we know that C is contained in a vector space of dimension N, the space of weight 0 ratios. Corollary 3.16 states the the set $\{v_\gamma\}$ is a set of N linearly independent ratios. As each v_γ has weight 0, this gives a basis of the subspace of weight 0 ratios. Therefore every ratio of weight 0 has the form $\sum \lambda_\gamma v_\gamma$ for some vector λ. We will show that if λ contains any negative entries the corresponding ratio is unbounded.

Assume λ is a vector with $\lambda_\gamma < 0$. Then consider the ray from Theorem 3.15 sending v_γ to 0 while keeping all other $v_\omega > L$ for some $L > 0$. Then $v_\gamma^{\lambda_\gamma}$ goes to infinity while the rest of the ratio is bounded away from 0. Thus the whole ratio is unbounded along this ray and thus does not belong to the cone of bounded ratios. Therefore the cone of bounded ratios is exactly the positive linear combinations of v_γ as claimed. □

Corollary 3.18. *The cone of bounded ratios in the cluster configuration algebra for a finite Dynkin diagram $\mathfrak{D}$ is the positive orthant.*

Proof. We can always find a cluster algebra of type $\mathfrak{D}$ with full rank. In this case each coordinate u_γ of the cluster configuration algebra is exactly parameterized by a generating ratio v_γ. □

Corollary 3.19. *A ratio in cluster variables is bounded if and only if it is bounded by 1. If such a ratio has integer coefficients when expressed in the basis v_γ it is also subtraction free i.e. denominator minus numerator is a polynomial with no negative signs in cluster variables (not necessarily from the same cluster).*

Proof. The extreme rays v_γ clearly have both properties from eq. (8). By the cluster relation we see the denominator minus numerator is simply the product of frozen variables coming into node γ. This inductively implies every integer linear combination is subtraction free as well by the following computation:

$$\frac{a}{b}\frac{c}{d} \mapsto bd - ac = b(d-c) + c(b-a)\,. \qquad \square$$

Example 3.20. There are bounded ratios of integer powers of cluster variables that are not subtraction free. In Example 3.13 we computed the extreme rays of the cone of bounded ratios in type C_2. The product $\sqrt{v_2 v_4 v_6} = \frac{x_1 x_3 x_5}{x_2 x_4 x_6}$ is an integer combination of cluster variables. However if we express this ratio in the initial cluster $x_1 \to x_2$ we obtain

$$\begin{aligned} x_2x_4x_6 - x_1x_3x_5 = \;& x_2\left(\frac{(1+x_2)^2+x_1}{x_1^2x_2}\right)\left(\frac{1+x_1^2}{x_2}\right) - x_1\left(\frac{1+x_2}{x_1}\right)\left(\frac{1+x_2+x_1^2}{x_1x_2}\right) \\ = \;& \frac{(1+2x_1^2+x_1^4+2x_2+2x_1^2x_2+x_2^2+x_1^2x_2^2) - \left(x_1+x_1^3+2x_1x_2+x_1^3x_2+x_1x_2^2\right)}{x_1^2x_2}\,. \end{aligned}$$

If this ratio were subtraction free in cluster variables, then specializing to the initial cluster should produce a Laurent polynomial with positive coefficients. Since this did not happen, the ratio cannot be subtraction free in cluster variables as claimed.

Lemma 3.21. *Consider the $(N+m)\times N$ matrix U, where $U_{i,j}$ is the power of the cluster variable x_i in the extreme ratio v_{γ_j}. If there is a subset of rows I such that* $\det(U_I) = \pm 1$ *then every bounded ratio is uniquely expressed as a positive integer combination of the v_γ.*

Proof. Consider a bounded ratio represented by the vector $\mathbf{v}$. By construction, the solution to the equation $U\lambda = \mathbf{v}$ provides the coefficients λ needed to express $\mathbf{v}$ as a combination of v_γ. If we restrict to the I rows we see $U_I\lambda = v_I$. Furthermore by assumption $\det(U_I) = \pm 1$ and thus by Cramer's rule U_I^{-1} is an integer matrix. Therefore $\lambda = U_I^{-1}v_I$ is integer as needed. Since the set of $\mathbf{v}_\gamma$ form a basis, λ is unique. $\square$

Corollary 3.22. *The bounded ratios on the cluster algebras for* $\mathrm{Gr}(3,6), \mathrm{Gr}(3,7)$ *and* $\mathrm{Gr}(3,8)$ *are subtraction free.*

Proof. All three of these cluster algebras satisfy the condition of Lemma 3.21 by explicit computation. $\square$

Example 3.23. We now finish our running example of the A_1 cluster algebra with one frozen variable. From eq. (8) we have the two generating ratios are

$$v_{-\gamma} = \frac{1}{x_{-\gamma}x_\gamma} \qquad v_\gamma = \frac{f_1}{x_{-\gamma}x_\gamma}\,.$$

From Example 3.7 we know the cone of bounded ratios is contained in the subspace where the power of $x_{-\gamma}$ equals the power of x_γ. Thus we can draw the cone of bounded ratios in Figure 8.

Furthermore every bounded ratio is subtraction free by Lemma 3.21 since the matrix U is

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}\,.$$

4. Bounded Ratios on Subsets

The original study of totally nonnegative matrices was concerned with inequalities of determinants of matrix minors. When translated to Grassmannians, $\mathrm{Gr}(k,n)$, these are inequalities among the Plücker coordinates. The most basic such ratio is the *primitive ratio* [26, 2] of the form

$$\frac{[i(j+1)S][j(i+1)S]}{[ijS][(i+1)(j+1)S]},$$

where $i < i+1 < j < j+1$ are cyclically ordered and S is a $k-2$ element subset of $[n]$ disjoint from $i, i+1, j, j+1$. It was conjectured in [6] that all bounded ratios on the Grassmannian can be factored into positive integer combinations of primitive ratios. However in $\mathrm{Gr}(4,8)$, [26] found extreme ratios that are not primitive ratios.

Using the results of this paper we prove that the bounded ratios in Plücker coordinates on Grassmannians with finite type cluster structure ($\mathrm{Gr}(2,n)$, $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, and $\mathrm{Gr}(3,8)$) are generated by primitive ratios. To accomplish this we give a concrete algorithm to reduce the full cluster bounded cone to a bounded cone on a subset of the cluster variables.

Definition 4.1. *Let $S \subset \Pi$. The* **S-ratio space**, *V_r^S, is the subset of V_r generated by the variables in S. The* **cone of bounded S-ratios**, *C_S, is the cone of bounded ratios in V^S, where V^S is a linear subspace of V spanned by V_r^S.*

Proposition 4.2. *The cone of bounded S-ratios C_S is the intersection of the full cone of bounded ratios with V^S. Moreover, when $\mathcal{A}$ is of full rank, the extreme rays of C_S are the minimal positive linear combinations of the extreme rays of the full cone that are contained in V^S.*

Proof. Since every bounded S-ratio is a usual bounded ratio the first statement is clear. In the full rank case every ratio is uniquely expressed as product of v_γ. Thus if no strict subproduct lands in V^S the corresponding ratio cannot be factored in C_S. □

Given Proposition 4.2 it is then straightforward to compute the extreme rays of C_S. Let U be the matrix whose columns are indexed by Π and rows are indexed by the full set of cluster variables without S with $U_{i,j}$ the power of x_i in the extreme ratio v_j.

Proposition 4.3. *Let $\lambda \in \ker(M)$. Then $\prod v_\gamma^{\lambda_\gamma}$ is in V^S. Furthermore if every element of λ is nonnegative the corresponding ratio is in C_S.*

Proof. By construction the entries of $M\lambda$ are the power of corresponding the cluster variable not in S in the resulting ratio. Thus if $\lambda \in \ker(M)$ all variables not in S vanish. If the coefficients of λ are all nonnegative the ratio is bounded and thus lies in C_S. □

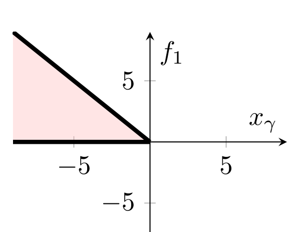


Figure 8. Cone of Bounded Ratios in A_1

Therefore C_S is defined by the equations $M\lambda = 0$ and $\lambda_\gamma \geqslant 0$. Such a system of linear inequalities can be easily solved by programs like Normaliz [3].

We now return our focus to the Grassmannian and specialize S to the set of Plücker variables. In $\mathrm{Gr}(2,n)$ every cluster variable is a Plücker coordinate. Thus the full cone is the same as the Plücker cone and we have the following:

Proposition 4.4. *A ratio of Plücker coordinates on* $\mathrm{Gr}(2,n)$ *is bounded if and only if it can be factored into a positive integer combination of primitive ratios.*

Proof. The cluster structure on $\mathrm{Gr}(2,n)$ is of type A_{n-3}. We can see the generator v_{ij} given by Theorem 3.17 is the primitive ratio indexed by $\{(i-1),(j-1)\}$ since the Dynkin seeds correspond to the zig-zag triangulations of an n-gon. ☐

Example 4.5. Consider the cluster algebra $\mathrm{Gr}(2,n)$. Let S be the set of Plücker coordinates with indices in $\{1,\dots,n-1\}$, i.e. the set of cluster variables in $\mathrm{Gr}(2,n-1)$. Using Proposition 4.2 we can compute the extreme rays of $\mathrm{Gr}(2,n-1)$ which we call $u_{ij}(n-1)$ using the extreme rays of $\mathrm{Gr}(2,n)$, $u_{ij}(n)$. Explicitly we have

$$u_{1j}(n-1) = u_{1j}(n)\cdot u_{jn}(n) \qquad \text{and} \qquad u_{ij}(n-1) = u_{ij}(n) \text{ for } i>1\,.$$

This follow from the following simple computation

$$u_{1j}(n-1) = \frac{[j(n-1)][1(j-1)]}{[1j][(j-1)(n-1)]} = \frac{[1(j-1)][jn]}{[1j][(j-1)n]}\frac{[j(n-1)][(j-1)n]}{[jn][(j-1)(n-1)]}\,,$$

and the observation that $u_{ij}(n)\in V^S$ if $1<i$ and $j<n$.

This inductive structure of the extreme rays allows us to strengthen Proposition 4.4 to the following theorem:

Theorem 4.6. *Every bounded ratio of Plücker coordinates on* $\mathrm{Gr}(2,n)$ *can be factored as an integer combination of primitive ratios. As such, every bounded ratio is subtraction free and bounded by 1.*

Proof. By Lemma 3.21 it suffices to find a minor of the matrix of U-variables with determinant ± 1. In [26, Theorem 4.2] it is proved that such a minor exists for any $\mathrm{Gr}(k,n)$. We give an explicit construction for $\mathrm{Gr}(2,n)$ here.

Consider the minor whose rows are indexed by cluster variables of the form $[ij]$ for $i<j-1$ and $j\geqslant 4$. By adding the column indexed by $[jn]$ to the column $[1j]$ for each $3\leqslant j<n-1$ the column $[1j]$ becomes the column for $\mathrm{Gr}(2,n-1)$ and thus has no entries in any row labeled by $[jn]$. The remaining entries of these rows form an upper triangular matrix of the form

$$\begin{array}{c} \\ {[(n-2)n]} \\ {[(n-3)n]} \\ \vdots \\ {[2n]} \\ {[1n]} \end{array} \begin{array}{c} \begin{array}{ccccc} [1(n-1)] & [(n-2)n] & [(n-3)n] & \dots & [2n] \end{array} \\ \begin{bmatrix} -1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ & & & & 1 \end{bmatrix} \end{array}$$

Continuing inductively this minor reduces to an upper triangular matrix with ± 1 on the diagonal. Each column operation leaves the determinant unchanged and so the minor has determinant ± 1 as needed. ☐

Remark 4.7. The bounded ratios in $\mathrm{Gr}(2,n)$ are both subtraction free as polynomials in all cluster variables and in the face weights of a weighted planar network. This is because every cluster variable in $\mathrm{Gr}(k,n)$ is a positive function of the face weights.

Example 4.8. In $\mathrm{Gr}(3,6)$ there are 16 mutable cluster variables and 6 frozen variables. Two of the mutable variables are degree two polynomials that we call $[124|356]$ and $[135|246]$ using the tableaux indexing of [5]. To find the bounded Plücker cone we must find positive linear combinations of the extreme ratios (Figure 9) that eliminate these variables. If we order the Plücker variables by reverse lexicographic order and then append $e_1 = [124|356], e_2 = [135|246]$ we obtain the following matrix for U:

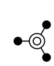
$$\begin{array}{cccccccccccccccc} [356] & [346] & [256] & [246] & [245] & [236] & [235] & [146] & [145] & [136] & [135] & [134] & [125] & [124] & e_1 & e_2 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

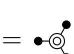
Clearly the ratios associated to $[346], [256], [235], [145], [136]$ and $[124]$ are already extreme rays of the bounded Plücker cone. We obtain 12 new extreme rays given by multiplying the extreme ray at the center of a D_4 with one of its three neighbors. In this way we recover the set of 18 primitive Plücker ratios.

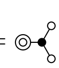
Proposition 4.9. *Every Plücker bounded ratio on* $\mathrm{Gr}(3,6)$ *is subtraction free. Furthermore such a ratio can be factored as a positive integer combination of primitive ratios.*

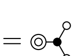
Proof. By Corollary 3.22 we also know that every Plücker bounded ratio is subtraction free as a function of cluster variables. Since each cluster variable (including the exotic variables) is a positive function in face weights this shows Conjecture 1.4 holds as well.

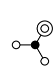
The factorization into primitive ratios is inductive. Consider a minimal subproduct of the factorization in v_γ such that the result is a Plücker ratio. By Proposition 4.2 this is an extreme ray of the Plücker bounded cone and thus is a primitive ratio. The remaining product is smaller and so can be inductively factored. ☐

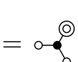
$$v_{[124|356]} = \frac{[346][256][124]}{[246][124|356]} =$$ 

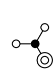
$$v_{[246]} = \frac{[245][236][146]}{[246][135|246]} =$$ 

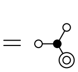
$$v_{[356]} = \frac{[124|356]}{[356][124]} =$$ 

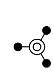
$$v_{[124]} = \frac{[246][123]}{[236][124]} =$$ 

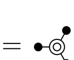
$$v_{[134]} = \frac{[124|356]}{[256][134]} =$$ 

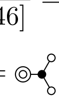
$$v_{[256]} = \frac{[246][156]}{[256][146]} =$$ 

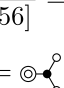
$$v_{[125]} = \frac{[124|356]}{[346][125]} =$$ 

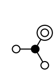
$$v_{[346]} = \frac{[345][246]}{[346][245]} =$$ 

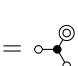
$$v_{[135|246]} = \frac{[235][145][136]}{[135][135|246]} =$$ 

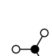
$$v_{[135]} = \frac{[356][134][125]}{[135][124|356]} =$$ 

$$v_{[236]} = \frac{[135|246]}{[236][145]} =$$ 

$$v_{[145]} = \frac{[456][135]}{[356][145]} =$$ 

$$v_{[146]} = \frac{[135|246]}{[235][146]} =$$ 

$$v_{[235]} = \frac{[234][135]}{[235][134]} =$$ 

$$v_{[245]} = \frac{[135|246]}{[245][136]} =$$ 

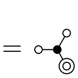
$$v_{[136]} = \frac{[135][126]}{[136][125]} =$$ 

Figure 9. Extreme Rays in $\mathrm{Gr}(3,6)$

Example 4.10. In $\mathrm{Gr}(3,7)$ there are 14 non-Plücker variables of the 42 mutable variables. Via an analogous computation to $\mathrm{Gr}(3,6)$, we compute the cone of bounded Plücker ratios. We can classify the extreme rays of this cone into 6 orbits under the action of rotation. One representative of each orbit is factored into extreme ratios of the full cone in Figure 10. As each extreme ray appears at most once, we can graphically represent the product by circling the node of the Dynkin seed corresponding to each ratio. In Figure 10 we observe that the rays of each factorization are adjacent in the sources-sink walk through the Dynkin seeds.

The same proof as for $\mathrm{Gr}(3,6)$ works in $\mathrm{Gr}(3,7)$ to show:

Proposition 4.11. *Every Plücker bounded ratio on* $\mathrm{Gr}(3,7)$ *is subtraction free. Furthermore such a ratio can be factored as a positive integer combination of primitive ratios.*

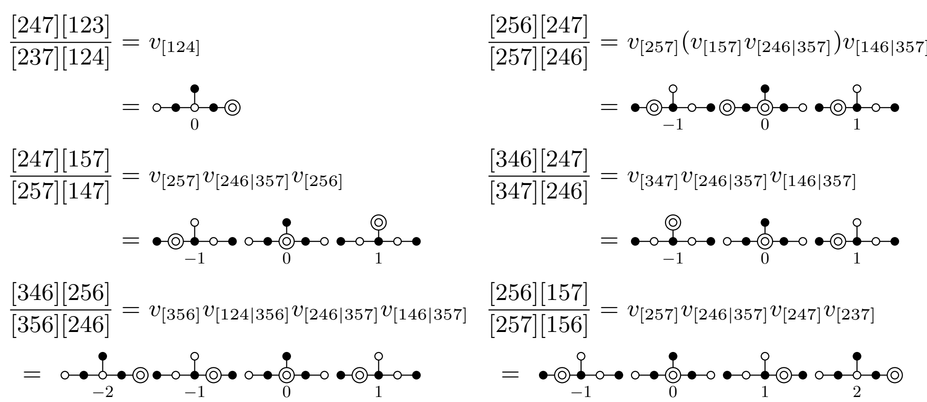


Figure 10. Representatives of Each Orbit of Extreme Plücker Ratios in $\mathrm{Gr}(3,7)$

Example 4.12. In $\mathrm{Gr}(3,8)$ there are 80 non-Plücker variables of the 128 mutable variables. Using the computation described above we obtain 10 orbits of 8 extreme ratios under the cluster modular group. We provide the list of orbit representatives in Appendix A.1. One orbit contains the 8 ratios in the full cone that are already Plücker. The other nine orbits consist of products of 6 or 11 extreme rays. As in $\mathrm{Gr}(3,7)$ these products are adjacent on the sources-sink walk.

Once again, the same proof as for $\mathrm{Gr}(3,6)$ extends to $\mathrm{Gr}(3,8)$ to show:

Proposition 4.13. *Every Plücker bounded ratio on* $\mathrm{Gr}(3,8)$ *is subtraction free. Furthermore such a ratio can be factored as a positive integer combination of primitive ratios.*

Example 4.14. There is a further refinement of the cluster variables in $\mathrm{Gr}(3,8)$. The 80 non-Plücker variables can be divided into the 56 degree 2 polynomials in Plücker coordinates and the 24 degree 3 polynomials. If we let S be the set of cluster variables of degree less than or equal to 2, we obtain a cone C_S between the Plücker cone and the full cone of bounded ratios. Here we find 56 extreme rays of the full cone already belong to V^S. There are 112 new extreme rays consisting of products of 3, 4 or 5 extreme ratios of the full cone. The list of orbit representatives is given in Appendix A.2.

$$\frac{[5678][1467]}{[4678][1567]} \qquad \frac{[3578][1347]}{[3478][1357]} \qquad \frac{[3568][2578]}{[3578][2568]} \qquad \frac{[4578][3678]}{[4678][3578]}$$

$$\frac{[4578][1357]}{[3578][1457]} \qquad \frac{[3578][1356]}{[3568][1357]} \qquad \frac{[3568][3478]}{[3578][3468]} \qquad \frac{[4678][1457]}{[4578][1467]}$$

$$\frac{[3568][3478][2578][1346][1247]}{[3578][3468][2478][1347][1256]}$$

$$\frac{[3578][3468][2478][1347][1256]}{[3568][3478][2468][1357][1247]}$$

$$\frac{[3568][3478][2578][1346][1256]}{[3578][3468][2568][1356][1247]}$$

$$\frac{[4578][3678][2458][2368][2357][1467][1358]}{[4678][3578][2468][2457][2358][1368][1357]}$$

$$\frac{[4578][3678][2458][2368][2357][1467][1358]}{[4678][3578][2468][2367][2358][1458][1357]}$$

$$\frac{[4678][4568][3578][2678][2456][2357][2348][2346][1457][1367][1356][1347][1258]}{[4578][3678][3568][2578][2457][2358][2356][2347][1467][1456][1348][1346][1267]}$$

$$\frac{[4678][4568][3578][2678][2456][2357][2348][2346][1457][1367][1356][1347][1258][1258]}{[4578][3678][3568][2578][2457][2358][2356][2347][1467][1456][1357][1346][1268][1248]}$$

$$\frac{[4678][4568][3578][3468][2578][2456][2357][2348][2346][1457][1358][1356][1356][1347][1347][1267]^2[1248][1246]}{[4578][3568][3478][2678][2468][2457][2358][2356][2347][1467][1456][1357]^2[1348][1346]^2[1268][1256][1247]}$$

$$\frac{[4678][4568][3578][3468][2578][2456][2357][2348][2346][1457][1358][1356][1356][1347][1347][1267]^2[1248][1246]}{[4578][3568]^2[2678][2468][2457][2358][2356][2347][1467][1456][1357]^2[1348][1346]^2[1268][1247][1247]}$$

$$\Bigg(\frac{[4678][4568][3578][3468][2578][2467][2458][2456][2357][2357][2348][2346]}{[4578][3568][3478][2678][2468]^2[2457]^2[2367][2358][2356][2347]}$$
$$\frac{[1457]^2[1368][1358][1356][1356][1347][1347][1267]^2[1248][1246]}{[1467][1458][1456][1357]^3[1348][1346]^2[1268][1256][1247]}\Bigg)$$

$$\Bigg(\frac{[4678][4568][3578][3468][2578][2467][2458][2456][2357][2357][2348][2346]}{[4578][3568]^2[2678][2468]^2[2457]^2[2358]^2[2356]}$$
$$\frac{[1457][1457][1368][1358][1356][1356][1347][1347][1267]^2[1248][1246]}{[2347][1467]^2[1456][1357]^3[1348][1346]^2[1268][1247]^2}\Bigg)$$

$$\Bigg(\frac{[4678][4568][3578][3468][2578][2467][2458][2456][2357][2357][2348]}{[4578][3568][3478][2678][2468]^2[2457]^2[2358]^2[2356][2347]}$$
$$\frac{[2346][1457]^2[1368][1358][1356]^2[1347]^2[1267]^2[1248][1246]}{[1467]^2[1456][1357]^3[1348][1346]^2[1268][1256][1247]}\Bigg)$$

FIGURE 11. Extreme Plücker Rays in $\mathrm{Gr}(4,8)$

The next biggest Grassmannian is $\mathrm{Gr}(4,8)$. The corresponding cluster algebra is not of finite type. It was shown in [26] that the Plücker bounded cone in this case is finitely generated but has extreme rays that are not primitive. In Figure 11 we give a full list of extreme rays up to the dihedral action and duality $I \mapsto [8] \smallsetminus I$. We have verified that each ratio in Figure 11 is subtraction-free in the sense of Definition 1.3, which proves the following statement:

THEOREM 4.15.

(1) *Every bounded ratio in Plücker coordinates of* $\mathrm{Gr}(4,8)$ *is bounded by 1.*
(2) *A bounded ratio in Plücker coordinates of* $\mathrm{Gr}(4,8)$ *which is a product of positive integer powers of generating ratios listed in Figure 11 is subtraction-free in the sense of Definition 1.3.*

COROLLARY 4.16. *Let A be a totally positive* 4×4 *matrix. If a multiplicative determinantal inequality*

$$\det(A_{I_1,I_1'})\det(A_{I_2,I_2'})...\det(A_{I_p,I_p'}) \leqslant C\cdot \det(A_{J_1,J_1'})\det(A_{J_2,J_2'})...\det(A_{J_q,J_q'})\,,$$

where $I_k, I'_k, J_k, J'_k \subseteq \{1,2,3,4\}$ with $|I_k| = |I'_k|$ and $|J_k| = |J'_k|$ holds for some constant $C > 0$ then this inequality holds with $C = 1$.

While the sets of U-variables for cluster algebras of infinite type are less understood, the ratios of the form $\frac{y}{1+y}$ are naturally bounded quantities on any cluster algebra. We observe that some of these new extreme rays can be factored into the simpler ratios of the form $\frac{y}{1+y}$ for y in $\mathrm{Gr}(4,8)$. For example

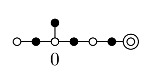
$$\frac{[1368][1458][1467][2367][2358]}{[1358][1367][1468][2368][2457]} = \frac{[1467][1458][1368]}{[1468][1357|1468]}\frac{[1357|2467]}{[2457][1367]}\frac{[1357|1357|2468]}{[1358][1357|2467]}$$
$$\frac{[2367][2358][1368]}{[2368][1357|2368]}\frac{[1357|2368][1357|1468]}{[1368][1357|1357|2468]}.$$

Acknowledgements. The authors would like to thank Thomas Lam for productive discussions and in particular for his crucial insight on the connection between bounded ratios and u-variables discussed in [1]. We would also like to thank the anonymous reviewer for their helpful comments.

Appendix A. Extreme Ratios in Gr(3, 8)

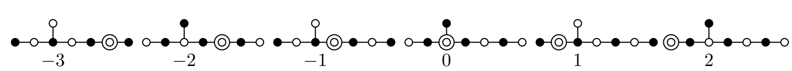
A.1. Extreme Plücker Ratios in Gr(3, 8). Here we give a representative of each orbit of extreme rays of the Plücker cone in $\mathrm{Gr}(3,8)$ (Example 4.12). For each representative we give the factorization into extreme rays of the full bounded cone (U-variables) and give a graphical representation of the location of the U-variable in the Dynkin seeds.

$$\frac{[456][357]}{[457][356]} = v_{[457]} =$$



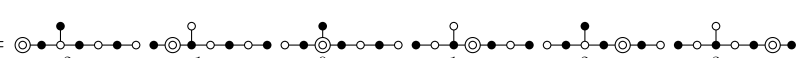
$$\frac{[258][168]}{[268][158]} = v_{[268]}v_{[257|468]}v_{[257|368]}v_{[124|257|368]}v_{[247|368]}v_{[267]}$$

$=$



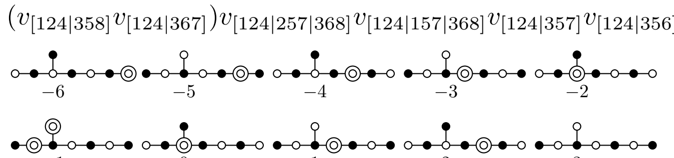
$$\frac{[258][124]}{[248][125]} = v_{[125]}v_{[124|358]}v_{[124|257|368]}v_{[124|157|368]}v_{[124|357]}v_{[124|356]}$$

$=$



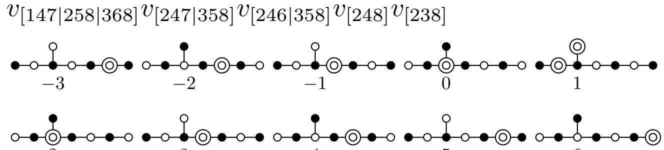
$$\frac{[348][124]}{[248][134]} = v_{[134]}v_{[124|378]}v_{[124|368]}v_{[124|357|468]}v_{[124|357|368]}$$
$$(v_{[124|358]}v_{[124|367]})v_{[124|257|368]}v_{[124|157|368]}v_{[124|357]}v_{[124|356]}$$

$=$



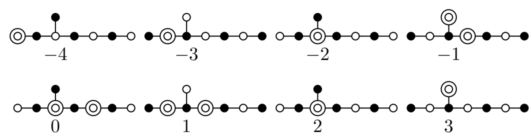
$$\frac{[267][168]}{[268][167]} = v_{[268]}v_{[257|468]}v_{[257|368]}v_{[124|257|368]}(v_{[247|368]}v_{[258]})$$
$$v_{[147|258|368]}v_{[247|358]}v_{[246|358]}v_{[248]}v_{[238]}$$

$=$



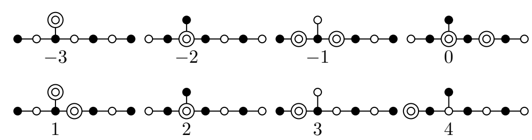
$$\frac{[357][267]}{[367][257]} = v_{[367]}v_{[125|367]}v_{[124|357|368]}(v_{[257|368]}v_{[124|367]})(v_{[124|257|368]}v_{[157|368]})(v_{[247|368]}v_{[124|157|368]})v_{[147|258|368]}v_{[147|368]}$$

=



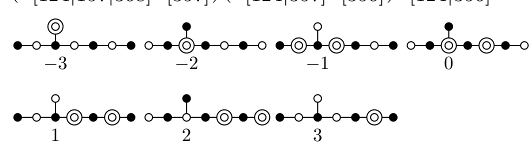
$$\frac{[357][348]}{[358][347]} = v_{[358]}v_{[124|357|368]}(v_{[124|358]}v_{[257|368]})(v_{[124|257|368]}v_{[157|368]})(v_{[124|157|368]}v_{[258]})v_{[147|258|368]}v_{[157|268]}v_{[158]}$$

=



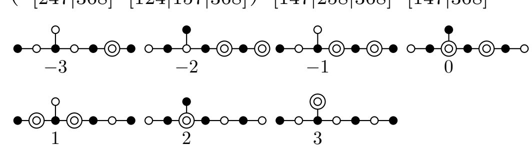
$$\frac{[348][258]}{[358][248]} = v_{[358]}v_{[124|357|368]}(v_{[124|358]}v_{[257|368]})(v_{[124|257|368]}v_{[157|368]})(v_{[124|157|368]}v_{[357]})(v_{[124|357]}v_{[356]})v_{[124|356]}$$

=



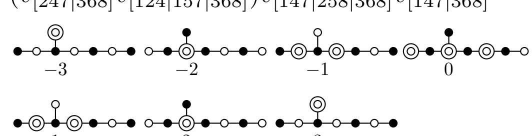
$$\frac{[267][258]}{[268][257]} = v_{[268]}(v_{[257|468]}v_{[168]})(v_{[257|368]}v_{[157|468]})(v_{[124|257|368]}v_{[157|368]})(v_{[247|368]}v_{[124|157|368]})v_{[147|258|368]}v_{[147|368]}$$

=



$$\frac{[357][258]}{[358][257]} = v_{[358]}v_{[124|357|368]}(v_{[124|358]}v_{[257|368]})(v_{[348]}v_{[124|257|368]}v_{[157|368]})(v_{[247|368]}v_{[124|157|368]})v_{[147|258|368]}v_{[147|368]}$$

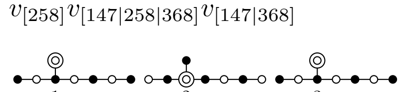
=



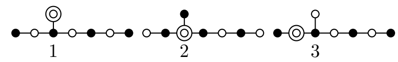
A.2. Extreme Ratios in Cluster Coordinates of Degree $\leqslant 2$ in $\mathrm{Gr}(3,8)$. Next we provide the same analysis for the cone of cluster variables of weight less than 3 in $\mathrm{Gr}(3,8)$ (Example 4.14).

$$\frac{[247|358][157|268]}{[258][257][147|368]} = v_{[258]}v_{[147|258|368]}v_{[147|368]}$$

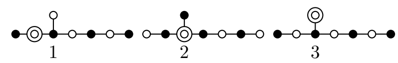
=



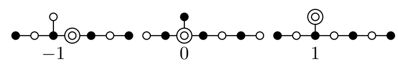
$$\frac{[158][247|358]}{[258][147|358]} = v_{[258]}v_{[147|258|368]}v_{[157|268]}$$

$=$ 

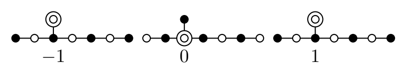
$$\frac{[267][247|358]}{[257][247|368]} = v_{[247|368]}v_{[147|258|368]}v_{[147|368]}$$

$=$ 

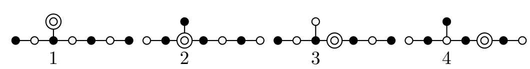
$$\frac{[247|368][157|368]}{[257|368][147|368]} = v_{[257|368]}v_{[124|257|368]}v_{[258]}$$

$=$ 

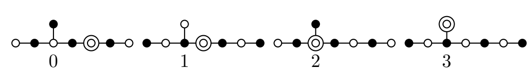
$$\frac{[247|368][124|357]}{[247|358][124|367]} = v_{[124|367]}v_{[124|257|368]}v_{[124|157|368]}$$

$=$ 

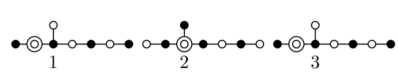
$$\frac{[248][157|268]}{[258][147|268]} = v_{[258]}v_{[147|258|368]}v_{[247|358]}v_{[246|358]}$$

$=$ 

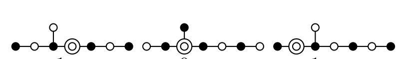
$$\frac{[357][157|268]}{[257][157|368]} = v_{[157|368]}v_{[124|157|368]}v_{[147|258|368]}v_{[147|368]}$$

$=$ 

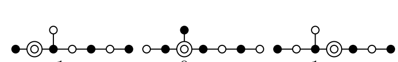
$$\frac{[267][158][247|358][147|368]}{[247|368][147|358][157|268]} = v_{[247|368]}v_{[147|258|368]}v_{[157|268]}$$

$=$ 

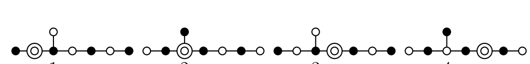
$$\frac{[267][258][157|368]}{[257|368][157|268]} = v_{[257|368]}v_{[124|257|368]}v_{[247|368]}$$

$=$ 

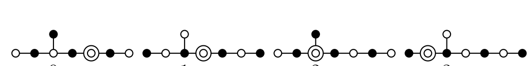
$$\frac{[348][258][124|357]}{[247|358][124|358]} = v_{[124|358]}v_{[124|257|368]}v_{[124|157|368]}$$

$=$ 

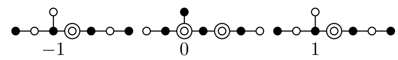
$$\frac{[267][248][147|368]}{[247|368][147|268]} = v_{[247|368]}v_{[147|258|368]}v_{[247|358]}v_{[246|358]}$$

$=$ 

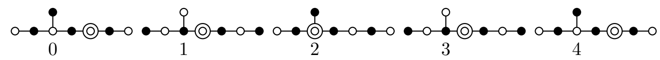
$$\frac{[357][158][147|368]}{[157|368][147|358]} = v_{[157|368]}v_{[124|157|368]}v_{[147|258|368]}v_{[157|268]}$$

$=$ 

$$\frac{[357][258][247|368]}{[257|368][247|358]} = v_{[257|368]}\big(v_{[157|368]}v_{[124|257|368]}\big)v_{[124|157|368]}$$

$=$ 

$$\frac{[357][248][147|368][157|268]}{[247|358][157|368][147|268]} = v_{[157|368]}v_{[124|157|368]}v_{[147|258|368]}v_{[247|358]}v_{[246|358]}$$

$=$ 

References

[1] Nima Arkani-Hamed, Song He, and Thomas Lam, *Cluster configuration spaces of finite type*, SIGMA Symmetry Integrability Geom. Methods Appl. **17** (2021), article no. 092 (41 pages).

[2] Adam Boocher and Bradley Froehle, *On generators of bounded ratios of minors for totally positive matrices*, Linear Algebra Appl. **428** (2008), no. 7, 1664–1684.

[3] W. Bruns, B. Ichim, C. Söger, and U. von der Ohe, *Normaliz. Algorithms for rational cones and affine monoids*, `https://www.normaliz.uni-osnabrueck.de`.

[4] Cesar Ceballos and Vincent Pilaud, *Denominator vectors and compatibility degrees in cluster algebras of finite type*, Trans. Amer. Math. Soc. **367** (2015), no. 2, 1421–1439.

[5] Wen Chang, Bing Duan, Chris Fraser, and Jian-Rong Li, *Quantum affine algebras and Grassmannians*, Math. Z. **296** (2020), no. 3-4, 1539–1583.

[6] Shaun M. Fallat, Michael I. Gekhtman, and Charles R. Johnson, *Multiplicative principal-minor inequalities for totally nonnegative matrices*, Adv. in Appl. Math. **30** (2003), no. 3, 442–470.

[7] Shaun M. Fallat and Charles R. Johnson, *Determinantal inequalities: ancient history and recent advances*, Algebra and its applications (Athens, OH, 1999), Contemp. Math., vol. 259, Amer. Math. Soc., Providence, RI, 2000, pp. 199–212.

[8] E. Fischer, *Über den Hadamardschen Determinantensatz*, Arch. Math. (Basel) **13** (1908), 32–40.

[9] Vladimir V. Fock and Alexander B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 6, 865–930.

[10] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky, *Introduction to Cluster Algebras. Chapters 1-3*, 2021, `https://arxiv.org/abs/1608.05735`.

[11] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky, *Introduction to Cluster Algebras. Chapters 6*, 2021, `https://arxiv.org/abs/2008.09189`.

[12] Sergey Fomin and Andrei Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), no. 2, 335–380.

[13] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. II. Finite type classification*, Invent. Math. **154** (2003), no. 1, 63–121.

[14] Sergey Fomin and Andrei Zelevinsky, *Y-systems and generalized associahedra*, Ann. of Math. (2) **158** (2003), no. 3, 977–1018.

[15] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Poisson geometry*, Mathematical Surveys and Monographs, vol. 167, American Mathematical Society, Providence, RI, 2010.

[16] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Poisson geometry*, Mathematical Surveys and Monographs, vol. 167, American Mathematical Society, Providence, RI, 2010.

[17] Jacques Hadamard, *Resolution d'une question relative aux determinants*, Bull. des sciences math. **2** (1893), 240–246.

[18] Bernhard Keller, *Cluster algebras, quiver representations and triangulated categories*, in Triangulated categories, London Math. Soc. Lecture Note Ser., vol. 375, Cambridge Univ. Press, Cambridge, 2010, pp. 76–160.

[19] D. M. Koteljanskiĭ, *The theory of nonnegative and oscillating matrices*, Amer. Math. Soc. Transl. (2) **27** (1963), 1–8.

[20] D. M. Kotelyanskiĭ, *On the theory of nonnegative and oscillating matrices*, Ukrain. Mat. Žurnal **2** (1950), no. 2, 94–101.

[21] Thomas Lam and David E. Speyer, *Cohomology of cluster varieties I: Locally acyclic case*, Algebra Number Theory **16** (2022), no. 1, 179–230.

[22] George Lusztig, *Total positivity and canonical bases*, in Algebraic groups and Lie groups, Austral. Math. Soc. Lect. Ser, vol. 9, Cambridge Univ. Press, 1997, pp. 281–295.

[23] Alexander Postnikov, *Total positivity, Grassmannians, and networks*, 2006, `https://arxiv.org/abs/math/0609764`.

[24] Jeanne S. Scott, *Grassmannians and cluster algebras*, Proc. London Math. Soc. (3) **92** (2006), no. 2, 345–380.

[25] Mark Skandera, *Inequalities in products of minors of totally nonnegative matrices*, J. Algebraic Combin. **20** (2004), no. 2, 195–211.

[26] Daniel Soskin and Michael Gekhtman, *On bounded ratios of minors of totally positive matrices*, Linear Algebra Appl. **715** (2025), 46–67.

[27] Christian K. Zickert, *Fock-Goncharov coordinates for rank two Lie groups*, Math. Z. **294** (2020), no. 1-2, 251–286.

Michael Gekhtman, The University of Notre Dame, 255 Hurley Bldg, 46556 Notre Dame, Indiana, USA
E-mail : mgekhtma@nd.edu

Zachary Greenberg, Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig, Germany
E-mail : greenberg@mis.mpg.de

Daniel Soskin, Institute for Advanced Study, 1 Einstein Drive, 08540 Princeton, New Jersey, USA
E-mail : dsoskin@ias.edu