

ALGEBRAIC COMBINATORICS

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Volume 9, issue 1 (2026), p. 21-39.

<https://doi.org/10.5802/alco.466>

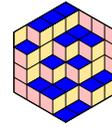
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Algebraic Combinatorics is published by *The Combinatorics Consortium*
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www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486





A tableaux formula for q -rook numbers

Tirtharaj Basu & Aritra Bhattacharya

ABSTRACT We provide a formula for the Garsia-Remmel q -rook numbers as a sum over standard Young tableaux. We connect our formula with the coefficients in q -Whittaker expansion of unicellular LLT functions.

1. INTRODUCTION

The Garsia-Remmel q -rook numbers $R_k(\lambda; q) \in \mathbb{Z}_{\geq 0}[q]$ for $k \in \mathbb{Z}_{\geq 0}$ counts the number of ways to place k non-attacking rooks on a Ferrers board of a partition λ with certain q weight. We provide a tableaux formula for $R_k(\lambda; q)$, which we describe now.

Let π be a Dyck path of semilength n and let $\lambda(\pi)$ denote the partition formed by the shape above π inside the $n \times n$ grid. For $1 \leq i < j \leq n$ let $i <_{\pi} j$ if $(i, j) \notin \text{Area}(\pi)$, i.e, the cell (i, j) is above the Dyck path π . The set SYT_{μ}^{π} is the set of standard Young tableaux of shape μ such that if i is above j in the same column then $i <_{\pi} j$. For $T \in \text{SYT}_{\mu}^{\pi}$ let

$$\gamma(T) = \#\{(b, c) \in \mu \times \mu \mid \text{coleg}(b) > \text{coleg}(c) \text{ and } (T(c), T(b)) \in \text{Area}(\pi)\}$$

be the number of pairs of boxes (b, c) such that c is in some row above b in the Young diagram (English notation) and $T(c) < T(b)$ but $T(c) \not<_{\pi} T(b)$.

We can now state our main result.

THEOREM 1.1. *Let $n \in \mathbb{Z}_{>0}$, $\lambda \in \text{Par}$ and $\pi \in \mathbb{D}_n$ is such that $\lambda(\pi) = \lambda$. Then for $k \in \mathbb{Z}_{\geq 0}$,*

$$(1) \quad R_k(\lambda; q) = \sum_{\substack{\mu \vdash n \\ \mu_1 = n - k}} q^{n(\mu') - \#\text{Area}(\pi)} \sum_{T \in \text{SYT}_{\mu}^{\pi}} q^{\gamma(T)} \prod_{\substack{b \in \mu \\ \text{coleg}(b) > 0}} [\text{arm}_{<_{\pi} T(b)}(\text{up}(b)) + 1]_q.$$

A more detailed explanation of all the notations used above is given in §3.6.

In fact, for $n = N \geq \lambda_1 + \lambda'_1$, the above formula only runs over the partition $(N - k, k)$ and so (Proposition 6.3)

$$R_k(\lambda; q) = q^{|\lambda| - (N - k)k} \sum_{T \in \text{SYT}_{(N - k, k)}^{\pi}} q^{\gamma(T)} \prod_{\substack{b \in \mu \\ \text{coleg}(b) > 0}} [\text{arm}_{<_{\pi} T(b)}(\text{up}(b)) + 1]_q.$$

In [1], [5] and [14] relations between q -rook numbers and symmetric functions appearing in the Macdonald functions universe are explored. We use the formula

Manuscript received 8th July 2025, revised and accepted 24th October 2025.

KEYWORDS. q -rook numbers, unicellular LLT functions, q -Whittaker functions, symmetric functions, Dyck paths.

above to make yet another such connection. By using result of [8], we can relate our formula to the coefficients of unicellular LLT functions $\chi_\pi(q)$ for a Dyck path π in the basis of q -Whittaker functions ($W_\lambda(q) : \lambda \in \text{Par}$). For $\pi \in \mathbb{D}_n$ and partitions $\mu \vdash n$, let $c_{\pi,\mu}(q) \in \mathbb{Q}(q)$ be defined by

$$\chi_\pi(q) = \sum_{\mu \vdash n} (1 - q)^{n - \mu_1} c_{\pi,\mu}(q) W_\mu(q).$$

Then (Corollary 5.2)

$$\sum_{\substack{\mu \vdash n \\ \mu_1 = n - k}} q^{n(\mu') - \#\text{Area}(\pi)} c_{\pi,\mu}(q) = R_k(\lambda(\pi); q).$$

The recent paper [12] obtains another proof of the above identity.

Based on the formula for e -expansion for unicellular LLTs obtained in [2], we also connect the last q -rook number of certain partitions to the e -expansion coefficients in Proposition 7.1.

2. NOTATIONS

2.1. We denote by $[n]$ the integer interval $\{1, \dots, n\}$ for $n \in \mathbb{Z}_{>0}$. This is not to be confused with the q -numbers $[n]_q$ which will always have a q in the subscript, and also should be clear from the context. Unless otherwise mentioned, n is some positive integer in this paper.

2.2. q -NUMBERS. For $n, k \in \mathbb{Z}_{\geq 0}$ with $0 \leq k \leq n$,

$$[n]_q = 1 + \dots + q^{n-1} \quad \text{and} \quad [n]_q! = [n]_q \dots [1]_q.$$

Let $(a; q)_j = (1 - a)(1 - qa) \dots (1 - q^{j-1}a)$, for $j \in \mathbb{Z}_{\geq 0}$. Then

$$(2) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Then

$$(3) \quad \begin{bmatrix} j \\ k \end{bmatrix}_{q^{-1}} = \frac{(q^{-1}; q^{-1})_j}{(q^{-1}; q^{-1})_k (q^{-1}; q^{-1})_{j-k}} = q^{\binom{k}{2} + \binom{j-k}{2} - \binom{j}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q = q^{-k(j-k)} \begin{bmatrix} j \\ k \end{bmatrix}_q.$$

2.3. DYCK PATHS. A Dyck path of semilength n is a lattice path from $(0, 0)$ to (n, n) consisting of unit length north steps N and unit length east steps E such that the path always stays weakly above the diagonal $x = y$. We will write a Dyck path as a word in N and E . We write the cell co-ordinate of each box in the $n \times n$ grid from $(0, 0)$ to (n, n) inside $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ as the co-ordinate of its north-east corner. $\text{Area}(\pi)$ is the set of cells below π above the diagonal. The set of Dyck paths of semilength n is denoted by \mathbb{D}_n , and $\mathbb{D} = \cup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{D}_n$ be the set of all Dyck paths of any semilength. For $\pi \in \mathbb{D}$, let $|\pi|$ denote its semilength. Figure 1 gives an example.

2.4. DYCK PATH TO POSET. For a Dyck path π of semilength n , define a poset on $[n]$ where the strict inequalities are given by $i <_\pi j$ if $i < j$ and $(i, j) \notin \text{Area}(\pi)$, i.e. (i, j) is above the Dyck path.

For the Dyck path in Figure 1, $1 <_\pi 4 <_\pi 6$, $1 <_\pi 5$, $2 <_\pi 4$, $2 <_\pi 5$, and $3 <_\pi 4$, $3 <_\pi 5$. Note that if $1 \leq i <_\pi j \leq k \leq n$ implies that $i <_\pi k$.

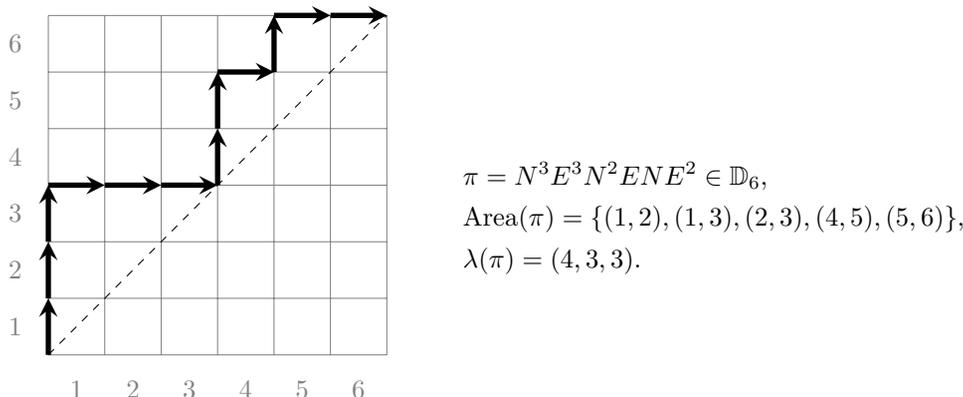


FIGURE 1. Example of a Dyck path

2.5. PARTITIONS. The set of all integer partitions is denoted by Par . We think of the Young diagram in the English convention, as in Macdonald’s book [13], and follow Macdonald’s definition and convention throughout the paper concerning partitions. In particular, for a partition λ , its conjugate is denoted λ' , the weighted size

$$n(\lambda') = \sum_{i \geq 1} \binom{\lambda_i}{2}.$$

The arm, leg, coarm, coleg of a box in the Young diagram is denoted a, l, a', l' respectively in [13]. In particular, the cell co-ordinates of a box b is $(\text{coleg}(b)+1, \text{coarm}(b)+1)$ and

$$\text{arm}(b) + \text{coarm}(b) + 1 = \lambda_{\text{coleg}(b)+1}, \quad \text{and} \quad \text{leg}(b) + \text{coleg}(b) + 1 = \lambda'_{\text{coarm}(b)+1}.$$

For a box $b \in \lambda$, we denote by $\text{up}(b)$ the box directly above it in the previous row, if $\text{coleg}(b) > 0$. So,

$$\text{coarm}(\text{up}(b)) = \text{coarm}(b) \quad \text{and} \quad \text{coleg}(\text{up}(b)) + 1 = \text{coleg}(b).$$

Let $\lambda \in \text{Par}$. Denote by $\lambda \pm \varepsilon_i = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i \pm 1, \lambda_{i+1}, \dots)$ the composition obtained by adding or removing a box in the i th row of λ . If $\lambda_{i-1} > \lambda_i$ then $\lambda + \varepsilon_i \in \text{Par}$ and if $\lambda_i > \lambda_{i+1}$ then $\lambda - \varepsilon_i \in \text{Par}$.

2.6. DYCK PATH TO PARTITION. The boxes in $n \times n$ grid above $\pi \in \mathbb{D}_n$ is the shape of a partition, read row-by-row from top to bottom, which we denote by $\lambda(\pi)$. It is contained inside the staircase shape partition $\rho_n = (n - 1, \dots, 0)$. Figure 1 gives an example.

Then $i <_{\pi} j$ if $j > n - \lambda(\pi)'_i$.

2.7. π -TABLEAUX. The set of standard Young tableaux of some partition shape λ will be denoted by SYT_{λ} . This is the set of fillings $T : \lambda \rightarrow [|\lambda|]$ such that the value increases left-to-right along a row and top-to-bottom along a column.

For $\pi \in \mathbb{D}_n$ and $\mu \vdash n$ let

$$(4) \quad \text{SYT}_{\mu}^{\pi} = \{T \in \text{SYT}_{\mu} \mid T(\text{up}(b)) <_{\pi} T(b) \text{ for all } b \in \mu \text{ with } \text{coleg}(b) > 0\},$$

i.e, SYT_{μ}^{π} is the set of standard Young tableaux of shape μ such that if the number i appears above j in the same column then $i <_{\pi} j$.

For the path in Figure 1,
 (5)

$$\text{SYT}_{(3,3)}^\pi = \left\{ \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \right\}, \quad \text{SYT}_{(3,2,1)}^\pi = \left\{ \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline 6 & & \\ \hline \end{array} \right\}, \quad \text{and} \quad \text{SYT}_{(3,1^3)}^\pi = \emptyset.$$

2.8. DYCK PATH TO HESSENBERG FUNCTIONS. Let $n \in \mathbb{Z}_{>0}$. A Hessenberg function $\mathbf{m} : [n] \rightarrow [n]$ is a non-decreasing function such that $\mathbf{m}(i) \geq i$ for every $i \in [n]$. For a Dyck path $\pi \in \mathbb{D}_n$, define a Hessenberg function $\mathbf{m}(\pi) : [n] \rightarrow [n]$ by

$$\mathbf{m}(\pi)(i) = n - \lambda(\pi)_i, \quad \text{for } i \in [n],$$

i.e, the value of i is the distance between the $(n - i)$ th E step and the line $y = n$, or in other words, $\mathbf{m}(\pi)$ in reverse is the complementary partition of $\lambda(\pi)$ in the $n \times n$ square.

For the path in Figure 1, $\mathbf{m}(\pi) = (2, 3, 3, 6, 6, 6)$, where the i th component denotes the value at i .

3. q -ROOK NUMBERS

In this section, we recall the definition and recursion of q -rook numbers as defined by Garsia and Remmel in [6]. We then provide a proof of our main result, a standard tableaux formula for the q -rook numbers.

3.1. DEFINITION OF q -ROOK NUMBERS. Given a partition λ and $k \in \mathbb{Z}_{\geq 0}$, a *rook placement* with k rooks on λ is the the number of ways to select k cells called *rooks* from the Young diagram of λ , such that no two rooks lie in the same row or column. Denote the set of rook placements with k rooks on λ by $\mathcal{C}_k(\lambda)$. Given such a rook placement $C \in \mathcal{C}_k(\lambda)$, [6] defines $\text{inv}(C)$ to be the number of cells remaining after cancelling all the cells in the same column above and in the same row to the left of the rooks. Then

$$(6) \quad R_k(\lambda; q) = \sum_{C \in \mathcal{C}_k(\lambda)} q^{\text{inv}(C)}.$$

Figure 2 gives an example of a rook placement with the inv statistic.

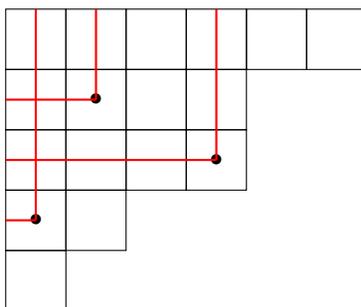


FIGURE 2. $C \in \mathcal{C}_3((6, 4, 4, 2, 1))$ with $\text{inv}(C) = 6$

Since conjugation interchanges cells in the same column above with cells in the same row to the left of a given cell,

$$(7) \quad R_k(\lambda; q) = R_k(\lambda'; q).$$

Because two rooks can not lie in the same row or in the same column, $R_k(\lambda; q) = 0$ for $k > \lambda_1$ or $k > \ell(\lambda)$.

3.2. RECURSIONS FOR q -ROOK NUMBERS. For $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots) \in \text{Par}$, let $\tilde{\lambda} = (\lambda_2, \lambda_3, \dots)$ be the partition obtained by removing the first row. The q -rook numbers $R_k(\lambda; q)$ for $0 \leq k \leq \lambda_1$ are determined by the recursions [6, Theorem 1.1]

$$(8) \quad R_k(\lambda; q) = q^{\lambda_1 - k} R_k(\tilde{\lambda}; q) + [\lambda_1 - k + 1]_q R_{k-1}(\tilde{\lambda}; q),$$

with initial conditions $R_0(\lambda; q) = q^{|\lambda|}$.

3.3. q -STIRLING NUMBERS. Let $n \in \mathbb{Z}_{\geq 0}$ and $\rho_n = (n - 1, n - 2, \dots, 1, 0)$ be the staircase partition. Then

$$(9) \quad R_{n-k}(\rho_n; q) = S_q(n, k) \quad \text{for } 0 \leq k \leq n$$

are the q -Stirling numbers of second kind [6, (I.9)]. They satisfy the recursions

$$S_q(n, k) = q^{k-1} S_q(n-1, k-1) + [k]_q S_q(n-1, k) \text{ for } 0 \leq k \leq n,$$

and $S_q(0, 0) = 1, S_q(n, k) = 0$ for $k < 0$ or $k > n$.

3.4. RECTANGULAR q -ROOK NUMBERS. Let $a, b \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq \min(a, b)$. [5, Proposition 2.15] gives

$$(10) \quad R_i((b^a); q) = q^{(a-i)(b-i)} \frac{[a]_q!}{[a-i]_q!} \begin{bmatrix} b \\ i \end{bmatrix}_q$$

3.5. $\ell(\lambda)$ TH q -ROOK NUMBERS. For $\lambda \in \text{Par}$ with $\ell(\lambda) = \ell$, [5, Proposition 2.2] says

$$(11) \quad R_\ell(\lambda; q) = \prod_{i=1}^{\ell} [\lambda_{\ell-i+1} - i + 1]_q.$$

3.6. TABLEAUX FORMULA FOR q -ROOK NUMBERS. Let $\pi \in \mathbb{D}_n$ for some $n \in \mathbb{Z}_{>0}$ and $\mu \vdash n$. Let $T \in \text{SYT}_\mu^\pi$. Recall that

$$(12) \quad \gamma(T) = \#\{(b, c) \in \mu \times \mu \mid \text{coleg}(b) > \text{coleg}(c) \text{ and } (T(c), T(b)) \in \text{Area}(\pi)\},$$

and for a box $b \in \mu$, let

$$(13) \quad \gamma(T, b) = \#\{c \in \mu \mid \text{coleg}(b) > \text{coleg}(c) \text{ and } (T(c), T(b)) \in \text{Area}(\pi)\},$$

then

$$\gamma(T) = \sum_{b \in \mu} \gamma(T, b).$$

For a box $b \in \mu$, denote by $\text{arm}_{<_\pi j}(b)$ the number of boxes c in the right of b in the same row such that $T(c) <_\pi j$, i.e.,

$$(14) \quad \text{arm}_{<_\pi j}(b) = \#\{c \in \mu \mid \text{coleg}(c) = \text{coleg}(b), \text{coarm}(c) > \text{coarm}(b), T(c) <_\pi j\}.$$

Let

$$(15) \quad \text{wt}(T; q) = q^{n(\mu') - \#\text{Area}(\pi) + \gamma(T)} \prod_{\substack{b \in \mu \\ \text{coleg}(b) > 0}} [\text{arm}_{<_\pi T(b)}(\text{up}(b)) + 1]_q,$$

where recall that $\#\text{Area}(\pi)$ is the number of cells below π strictly above the diagonal and $n(\mu') = \sum_i \binom{\mu_i}{2}$.

We restate Theorem 1.1 from the introduction here.

THEOREM 3.1. *Let $\lambda \in \text{Par}$ and $\pi \in \mathbb{D}_n$ is such that $\lambda(\pi) = \lambda$. Then for $k \geq 0$,*

$$(16) \quad R_k(\lambda; q) = \sum_{\substack{\mu \vdash n \\ \mu_1 = n - k}} \sum_{T \in \text{SYT}_\mu^\pi} \text{wt}(T; q).$$

3.7. EXAMPLE. For the Dyck path in Figure 1, $n = 6$, and let $k = 3$. Then by (5), the sum in (16) runs only over two tableaux,

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \in \text{SYT}_{(3,3)}^\pi \quad \text{and} \quad S = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline 6 & & \\ \hline \end{array} \in \text{SYT}_{(3,2,1)}^\pi,$$

with

$$\gamma(T) = 0 \quad \text{and} \quad \gamma(S) = 1,$$

and using $\#\text{Area}(\pi) = 5$, the weights are

$$\text{wt}(T; q) = q[2]_q[3]_q \quad \text{and} \quad \text{wt}(S; q) = q^{-1}q[2]_q[3]_q = [2]_q[3]_q.$$

So,

$$R_3((4, 3, 3); q) = q[2]_q[3]_q + [2]_q[3]_q = [3]_q[2]_q^2.$$

3.8. PROOF OF THEOREM 3.1. We now prove Theorem 3.1 by showing that the right hand side of (16) satisfies the recursions (8) for the q -rook numbers.

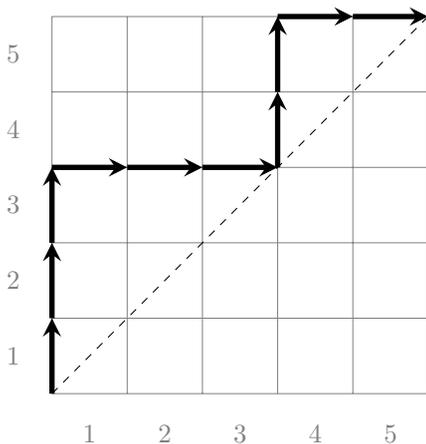
LEMMA 3.2. Let $\pi \in \mathbb{D}_n$. If $(i, n) \in \text{Area}(\pi)$ then i is a maximal element with respect to $<_\pi$ order, i.e, there is no $j \in [n]$ such that $i <_\pi j$.

Proof. If $(i, k) \in \text{Area}(\pi)$ then $(i, j) \in \text{Area}(\pi)$ for all $j \in \{i + 1, \dots, k\}$. In particular, $(i, n) \in \text{Area}(\pi)$ means that $(i, j) \in \text{Area}(\pi)$ for all $j \in \{i + 1, \dots, n\}$. \square

LEMMA 3.3. Let $\pi \in \mathbb{D}_n$ and $\mu \vdash n$. Let $T \in \text{SYT}_\mu^\pi$. If $b \in \mu$ is such that $(T(b), n) \in \text{Area}(\pi)$ then $\text{leg}(b) = 0$.

Proof. By Lemma 3.2, $T(b)$ is maximal with respect to $<_\pi$. Hence there can be no box below b , so $\text{leg}(b) = 0$. \square

Let $\pi \in \mathbb{D}_n$ with $\lambda(\pi) = \lambda$ and $\pi' \in \mathbb{D}_{n-1}$ be the path obtained by removing the first row of π , i.e, π' is obtained from π by removing the last occurrence of NE in π . For the path π from Figure 1, the path π' is shown in Figure 3.



$$\begin{aligned} \pi' &= N^3 E^3 N^2 E^2 \in \mathbb{D}_5, \\ \text{Area}(\pi') &= \{(1, 2), (1, 3), (2, 3), (4, 5)\}, \\ \lambda(\pi') &= (3, 3). \end{aligned}$$

FIGURE 3. Removal of last occurrence of NE from π

Then $\lambda(\pi')$ is obtained by removing the first row of $\lambda(\pi)$, which we denote by $\tilde{\lambda} = (\lambda_2, \dots)$. If $T \in \text{SYT}_\mu^\pi$ for some $\mu \vdash n$ then if we remove the box with entry n from T we obtain an element of $\text{SYT}_{\mu - \varepsilon_i}^{\pi'}$, where i is the row of the box with entry n and $\mu - \varepsilon_i = (\mu_1, \dots, \mu_{i-1}, \mu_i - 1, \mu_{i+1}, \dots)$.

Note that

$$\#\text{Area}(\pi) - \#\text{Area}(\pi') = n - 1 - \lambda_1.$$

Let $T \in \text{SYT}_\nu^{\pi'}$ for some $\nu \vdash n - 1$. Let $T^{+n,i}$ be the tableau obtained by adding a box with entry n in the i th row of T , if $\nu + \varepsilon_i \in \text{Par}$ and $T^{+n,i}$ is a valid tableau in $\text{SYT}_{\nu+\varepsilon_i}^\pi$.

LEMMA 3.4. Let $T \in \text{SYT}_\nu^{\pi'}$ and i is such that $T^{+n,i} \in \text{SYT}_{\nu+\varepsilon_i}^\pi$. For $j \geq 1$, let

$$(17) \quad N(j) = \#\{b \in \nu \mid \text{coleg}_\nu(b) = j - 1, \text{leg}_\nu(b) = 0 \text{ and } T(b) <_\pi n\},$$

and $N(0) = 0$. Then

$$(18) \quad \frac{\text{wt}(T^{+n,i}; q)}{\text{wt}(T; q)} = q^{\nu_1 - n + 1 + \lambda_1} \cdot q^{-(N(1) + \dots + N(i-1))} [N(i-1)]_q.$$

Proof. Using Lemma 3.3,

$$(19) \quad \begin{aligned} N(j) &= \#\{b \text{ in row } j \text{ of } T \text{ with } \text{leg}_\nu(b) = 0 \text{ and } T(b) <_\pi n\} \\ &= \text{arm}_{<_\pi n}((j, \nu_{j+1} + 1)) + 1 \\ &= \nu_j - \nu_{j+1} - \#\{b \text{ in row } j \text{ of } T \text{ with } (T(b), n) \in \text{Area}(\pi)\}. \end{aligned}$$

Then

$$\begin{aligned} \gamma(T^{+n,i}) - \gamma(T) &= \gamma(T^{+n,i}, (i, \nu_i + 1)) = \sum_{j=1}^{i-1} (\nu_j - \nu_{j+1} - N(j)) \\ &= \nu_1 - \nu_i - (N(1) + \dots + N(i-1)). \end{aligned}$$

Using

$$\#\text{Area}(\pi) - \#\text{Area}(\pi') = n - 1 - \lambda_1, \quad \text{and} \quad n((\nu + \varepsilon_i)') - n(\nu') = \nu_i,$$

then

$$\begin{aligned} &(n((\nu + \varepsilon_i)') - \#\text{Area}(\pi) + \gamma(T^{+n,i})) - (n(\nu') - \#\text{Area}(\pi') + \gamma(T)) \\ &= \nu_i - (n - 1 - \lambda_1) + \gamma(T^{+n,i}, (i, \nu_i + 1)) \\ &= \nu_i - (n - 1 - \lambda_1) + \nu_1 - \nu_i - (N(1) + \dots + N(i-1)) \\ &= \nu_1 - (n - 1 - \lambda_1) - (N(1) + \dots + N(i-1)), \end{aligned}$$

and

$$\begin{aligned} &\frac{\prod_{\substack{b \in \nu + \varepsilon_i \\ \text{coleg}_{\nu + \varepsilon_i}(b) > 0}} [\text{arm}_{<_\pi T^{+n,i}(b)}(\text{up}(b)) + 1]_q}{\prod_{\substack{b \in \nu \\ \text{coleg}_\nu(b) > 0}} [\text{arm}_{<_{\pi'} T(b)}(\text{up}(b)) + 1]_q} = [\text{arm}_{<_\pi T^{+n,i}((i, \nu_i + 1))}(\text{up}((i, \nu_i + 1))) + 1]_q \\ &= [\text{arm}_{<_\pi n}((i-1, \nu_i + 1)) + 1]_q = [N(i-1)]_q. \quad \square \end{aligned}$$

LEMMA 3.5. Let $T \in \text{SYT}_\nu^{\pi'}$ where $\pi' \in \mathbb{D}_{n-1}$ is obtained from $\pi \in \mathbb{D}_n$ by deleting the rightmost occurrence of NE . Then with the notations from above,

$$(20) \quad \sum_{i>1} \frac{\text{wt}(T^{+n,i}; q)}{\text{wt}(T; q)} = [\lambda_1 - n + \nu_1 + 1]_q, \quad \text{and} \quad \frac{\text{wt}(T^{+n,1}; q)}{\text{wt}(T; q)} = q^{\nu_1 - (n-1-\lambda_1)}.$$

Proof. We use the same notation as before from (17). Using (19),

$$\begin{aligned} \sum_{j \geq 1} N(j) &= \sum_{j \geq 1} (\nu_j - \nu_{j+1} - \#\{b \text{ in row } j \text{ of } T \text{ with } (T(b), n) \in \text{Area}(\pi)\}) \\ &= \nu_1 - \#\{b \in \nu \text{ with } (T(b), n) \in \text{Area}(\pi)\} \\ &= \nu_1 - \#\{1 \leq j \leq n-1 \text{ with } (j, n) \in \text{Area}(\pi)\} \\ &= \nu_1 - (n-1) + \lambda_1. \end{aligned}$$

Then using (18),

$$\begin{aligned} \sum_{i > 1} \frac{\text{wt}(T^{+n,i}; q)}{\text{wt}(T; q)} &= q^{\nu_1 - n + 1 + \lambda_1} \sum_{i > 1} q^{-(N(1) + \dots + N(i-1))} [N(i-1)]_q \\ &= q^{\nu_1 - n + 1 + \lambda_1} \sum_{i > 1} q^{-(N(1) + \dots + N(i-1))} \frac{1 - q^{N(i-1)}}{1 - q} \\ &= q^{\nu_1 - n + 1 + \lambda_1} \sum_{i > 1} \frac{q^{-(N(1) + \dots + N(i-1))} - q^{-(N(1) + \dots + N(i-2))}}{1 - q} \\ &= q^{\nu_1 - n + 1 + \lambda_1} \frac{q^{-\sum_{j \geq 1} N(j)} - 1}{1 - q} = q^{\nu_1 - n + 1 + \lambda_1} \frac{q^{-\lambda_1 + n - \nu_1 - 1} - 1}{1 - q} \\ &= [\lambda_1 - n + \nu_1 + 1]_q. \end{aligned}$$

This gives the first statement of (20). (18) for $i = 1$ gives the second statement of (20). \square

Now we can finish the proof of Theorem 3.1. Denote by $R'_k(\lambda; q)$ the right hand side of (16). To show that $R'_k(\lambda; q) = R_k(\lambda; q)$, we show that $R'_k(\lambda; q)$ satisfies the determining recursions from §3.2.

Let $\pi' \in \mathbb{D}_{n-1}$ be the path obtained by removing the first row of π . Then $\lambda(\pi') = \tilde{\lambda}$ is obtained by removing the first row of $\lambda(\pi)$. Then by Lemma 3.5,

$$\begin{aligned} &R'_k(\lambda; q) \\ &= \sum_{\substack{\nu \vdash n-1 \\ \nu_1 = n-k-1}} q^{\nu_1 - (n-1 - \lambda_1)} \sum_{T \in \text{SYT}_{\nu'}^{\pi'}} \text{wt}(T; q) + \sum_{\substack{\nu \vdash n-1 \\ \nu_1 = n-k}} [\lambda_1 - n + \nu_1 + 1]_q \sum_{T \in \text{SYT}_{\nu'}^{\pi'}} \text{wt}(T; q) \\ &= q^{n-k-1 - (n-1 - \lambda_1)} R'_k(\tilde{\lambda}; q) + [\lambda_1 - n + n - k + 1]_q R'_{k-1}(\tilde{\lambda}; q) \\ &= q^{\lambda_1 - k} R'_k(\tilde{\lambda}; q) + [\lambda_1 - k + 1]_q R'_{k-1}(\tilde{\lambda}; q), \end{aligned}$$

which matches with the Garsia-Remmel recursions (8). When $k = 0$, the sum in the right hand side of (16) only runs over the partition $\mu = (n)$. There is only one tableau in $\text{SYT}_{(n)}^{\pi}$, which is $T = \boxed{1 \mid \dots \mid n}$. Since $\#\text{Area}(\pi) = \binom{n}{2} - |\lambda| = n((n)') - |\lambda|$, then $\text{wt}(T; q) = q^{n((n)') - \#\text{Area}(\pi)} = q^{|\lambda|}$. This proves that $R'_0(\lambda; q) = q^{|\lambda|}$. Thus $R'_k(\lambda; q)$ satisfies the recursions with the initial conditions, hence $R'_k(\lambda; q) = R_k(\lambda; q)$.

4. UNICELLULAR LLT FUNCTIONS

In this section we recall the definition and some basic properties and examples of unicellular LLT functions, following the exposition in [4]. We follow standard notations and conventions regarding symmetric functions and plethysm. In particular, for a symmetric function f , $f[X]$ denotes $f(x_1, x_2, \dots)$.

4.1. DYCK PATH SYMMETRIC FUNCTIONS. Let $\pi \in \mathbb{D}_n$. For any word $w = (w_1, \dots, w_n) \in \mathbb{Z}_{>0}^n$, let

$$\text{inv}(\pi, w) = \#\{(i, j) \in \text{Area}(\pi) \mid w_i > w_j\}.$$

The unicellular LLT symmetric function corresponding to π is a symmetric function denoted $\chi_\pi(q)$, defined by

$$\chi_\pi(q)[X] = \sum_{w \in \mathbb{Z}_{>0}^n} q^{\text{inv}(\pi, w)} x_w,$$

where for a word w as above, let $x_w = \prod_i x_{w_i}$. A proof of symmetry of $\chi_\pi(q)$ can be found in [4, Proposition 3.2]. [4, Remark 3.6] also explains the connection with the definition of unicellular LLT functions using skew diagrams and diagonal inversions.

The maximum value of $\text{inv}(\pi, w)$ for $w \in \mathbb{Z}_{>0}^n$ is obtained when all boxes in $\text{Area}(\pi)$ contributes 1, in which case it equals to $\#\text{Area}(\pi)$. This means the highest power of q in $\chi_\pi(q)$ is $\#\text{Area}(\pi)$. The reverse polynomial is denoted $\tilde{\chi}_\pi(q)$, defined by

$$(21) \quad \tilde{\chi}_\pi(q) = q^{\#\text{Area}(\pi)} \chi_\pi(q^{-1}).$$

4.2. EXAMPLES. Let $n \in \mathbb{Z}_{>0}$.

(1) For $\pi \in \mathbb{D}_n$,

$$\chi_\pi(1) = h_1^n.$$

(2) Suppose $\pi \in \mathbb{D}_n$ and π does not touch the diagonal line from $(0, 0)$ to (n, n) except at the two ends. Then

$$\chi_\pi(0)[X] = \sum_{\substack{w \in \mathbb{Z}_{>0}^n \\ w_1 \leq \dots \leq w_n}} x_w = h_n[X].$$

Hence, if $\pi \in \mathbb{D}$ meets the line $x = y$ at points $(\alpha_1 + \dots + \alpha_i, \alpha_1 + \dots + \alpha_i)$ for $i \in \mathbb{Z}_{>0}$ for some composition α , then,

$$\chi_\pi(0) = h_\alpha.$$

(3) Let $\pi = (NE)^n \in \mathbb{D}_n$. Then $\text{Area}(\pi) = \emptyset$ and so

$$\chi_{(NE)^n}(q) = e_1^n.$$

(4) Let $\pi = N^n E^n \in \mathbb{D}_n$. Then $\text{Area}(\pi) = \{(i, j) \mid 1 \leq i < j \leq n\}$ is the maximum possible. Then for any word $w \in \mathbb{Z}_{>0}^n$, $\text{inv}(\pi, w) = \text{inv}(w)$ is the usual number of inversions of the word, and since inv is Mahonian ([11, Theorem 1.3]),

$$\chi_{N^n E^n}(q) = \sum_{\mu \vdash n} \begin{bmatrix} n \\ \mu \end{bmatrix}_q m_\mu = W_{(n)}(q),$$

where the right hand side denotes the q -Whittaker functions.

(5) Let $\text{rev} : \mathbb{D} \rightarrow \mathbb{D}$ be the map that takes a Dyck path to its reverse, i.e, the path obtained by reading the Dyck path from right to left and interchanging the N and E steps. [4, Proposition 3.3] says

$$(22) \quad \chi_\pi(q) = \chi_{\text{rev}(\pi)}(q).$$

(6) Suppose $\pi, \eta \in \mathbb{D}$ and let $\pi \cdot \eta$ denote their concatenation, then

$$\chi_{\pi \cdot \eta}(q) = \chi_\pi(q) \cdot \chi_\eta(q).$$

4.3. CHROMATIC QUASISYMMETRIC FUNCTIONS. For a Dyck path $\pi \in \mathbb{D}_n$, let $X_\pi(q)$ be the chromatic quasisymmetric function of the graph with vertex set $[n]$ and edge set

$$\left\{ \{i, j\} \mid 1 \leq i < j \leq n \text{ and } (i, j) \in \text{Area}(\pi) \right\}.$$

The chromatic quasisymmetric function $X_\pi(q)$ is in fact a symmetric function given by

$$X_\pi(q)[X] = \sum_{\substack{w \in \mathbb{Z}_{>0}^n \\ (i,j) \in \text{Area}(\pi) \Rightarrow w_i \neq w_j}} q^{\text{inv}(\pi, w)} x_w.$$

[4, Proposition 3.5] says

$$(23) \quad \chi_\pi(q)[X] = (q-1)^n X_\pi(q) \left[\frac{X}{q-1} \right],$$

where the plethystic substitution on the right hand side can be computed via

$$\sum_{\lambda \in \text{Par}} \alpha_\lambda p_\lambda \left[\frac{X}{q-1} \right] = \sum_{\lambda \in \text{Par}} \alpha_\lambda \frac{p_\lambda[X]}{\prod_{i=1}^{\ell(\lambda)} (q^{\lambda_i} - 1)}, \quad \text{for } \alpha_\lambda \in \mathbb{Q}(q).$$

4.4. ω -INVOLUTION. Recall from [13, Chapter I] that the ring of symmetric functions has the involution ω that interchanges the elementary and complete homogeneous symmetric functions. [4, Proposition 3.4] says that

$$(24) \quad \omega(\chi_\pi(q)) = q^{\#\text{Area}(\pi)} \chi_\pi(q^{-1}) = \tilde{\chi}_\pi(q).$$

5. W -EXPANSION OF UNICELLULAR LLT

In this section, we connect our formula for q -rook numbers with the results of [8], showing in Proposition 5.1 that our tableaux weights are essentially the same as certain specializations of the tableaux weights from [8], thus they give the coefficients of unicellular LLT functions in the q -Whittaker basis. Therefore, it follows in Corollary 5.2 that the q -rook numbers are sum of W -coefficients (for a fixed first row length) of the unicellular LLTs.

5.1. Let $\tilde{H}_\lambda(q, t)$ for $\lambda \in \text{Par}$ be the modified Macdonald functions, with notations same as [9]. The q -Whittaker functions are [3]

$$W_\lambda(q) = q^{n(\lambda')} \omega \tilde{H}_\lambda(q^{-1}, 0) \quad \text{for } \lambda \in \text{Par}.$$

Let

$$(25) \quad \tilde{W}_\lambda(q) = q^{n(\lambda')} W_\lambda(q^{-1}),$$

then

$$(26) \quad \omega(\tilde{H}_\lambda(q, 0)) = \tilde{W}_\lambda(q).$$

The first equation of §4 of [8] says

$$(27) \quad Q_{\lambda'}(q^{-1}) \left[\frac{X}{q-1} \right] = q^{-|\lambda| - n(\lambda')} \tilde{H}_\lambda(q, 0)[X],$$

where $Q_{\lambda'}(q)$ is the same as in [13, Chapter III].

5.2. For $\pi \in \mathbb{D}_n$ and partitions $\mu \vdash n$, let $c_{\pi,\mu}(q) \in \mathbb{Q}(q)$ be defined by

$$(28) \quad \chi_\pi(q) = \sum_{\mu \vdash n} (1-q)^{n-\mu_1} c_{\pi,\mu}(q) W_\mu(q).$$

Let

$$(29) \quad \tilde{c}_{\pi,\mu}(q) = q^{\#\text{Area}(\pi)-n(\mu')} c_{\pi,\mu}(q^{-1}).$$

Recall from (21) and (25) that

$$\tilde{\chi}_\pi(q) = q^{\#\text{Area}(\pi)} \chi_\pi(q^{-1}) \quad \text{and} \quad \widetilde{W}_\lambda(q) = q^{n(\lambda')} W_\lambda(q^{-1}).$$

Then

$$(30) \quad \tilde{\chi}_\pi(q) = \sum_{\mu \vdash n} (1-q^{-1})^{n-\mu_1} \tilde{c}_{\pi,\mu}(q) \widetilde{W}_\mu(q),$$

or applying ω , using (24) and (26) and the fact that ω is an involution,

$$(31) \quad \chi_\pi(q) = \sum_{\mu \vdash n} (1-q^{-1})^{n-\mu_1} \tilde{c}_{\pi,\mu}(q) \tilde{H}_\mu(q, 0).$$

PROPOSITION 5.1. For $\pi \in \mathbb{D}_n$ and $\mu \vdash n$,

$$(32) \quad \begin{aligned} c_{\pi,\mu}(q) &= q^{-n(\mu')+\#\text{Area}(\pi)} \cdot \sum_{T \in \text{SYT}_\mu^\pi} \text{wt}(T; q) \\ &= \sum_{T \in \text{SYT}_\mu^\pi} q^{\gamma(T)} \prod_{\substack{b \in \mu \\ \text{coleg}(b) > 0}} [\text{arm}_{<\pi T}(b)(\text{up}(b)) + 1]_q. \end{aligned}$$

In particular, $c_{\pi,\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$.

Proof. The proposition is just a restatement of [8, Theorem 4.1]. We now explain the changes required from their notations to our notation. Note that their χ_π is our X_π and our χ_π is their F_π .

First, since $\chi_\pi(q) = \chi_{\text{rev}(\pi)}(q)$, we have changed their order \ll to our $<_\pi$, where in the notation of [8], for $1 \leq i < j \leq n$, $i \ll j$ if $(i, j) \notin \text{Area}(\text{rev}(\pi))$ and in our notation $i <_\pi j$ if $(i, j) \notin \text{Area}(\pi)$.

In [8], the authors use another partial order \prec on $[n]$, defined by $i \prec j$ if $(i, j) \in \text{Area}(\text{rev}(\pi))$, which with our conventions then translate to $i \prec j$ if $(i, j) \in \text{Area}(\pi)$.

For a tableau $T \in \text{SYT}_\mu^\pi$, let T_1 denote the fillings in the first row of T , for $1 \leq i \leq n$, let $T_{\prec i}$ be the skew shape with fillings $\prec i$, and $T_{< i}$ be the skew shape with fillings $< i$. Let i appear in row $s + 1$ of T and $d(T, i) = \text{sh}(T_{\prec i})_s - \text{sh}(T_{\prec i})_{s+1}$, $L(T, i)$ is the coarm of the leftmost element in row s of $T_{\prec i}$, $m(T, i)$ is the number of elements that are at least 2 rows above the box with filling $\prec i$.

Let

$$\widetilde{\text{wt}}(T; q) = \prod_{i \notin T_1} q^{-m(T,i)-d(T,i)} \prod_{\substack{i \notin T_1 \\ (T_{\prec i})_s = \emptyset}} [d(T, i)]_q \prod_{i: (T_{\prec i})_s \neq \emptyset} [L(T, i) - \text{sh}(T_{\prec i})_{s+1}]_q,$$

where the notations assume that i appears in row $s + 1$ of T . Then [8, Theorem 4.1] says that

$$X_\pi(q) = \sum_{\mu \vdash n} \left(\sum_{T \in \text{SYT}_\mu^\pi} \widetilde{\text{wt}}(T; q) \right) \frac{q^{\binom{n+1}{2} - |\lambda(\pi)|}}{(q-1)^{\mu_1}} Q_{\mu'}(q^{-1}).$$

Note that $|\lambda(\pi)| = \binom{n}{2} - \#\text{Area}(\pi)$. Then using (23) and (27),

$$\begin{aligned} \chi_\pi(q) &= \sum_{\mu \vdash n} \left(\sum_{T \in \text{SYT}_\mu^\pi} \widetilde{\text{wt}}(T; q) \right) \frac{q^{n+\#\text{Area}(\pi)}}{(q-1)^{\mu_1}} \cdot (q-1)^n Q_{\mu'}(q^{-1}) \left[\frac{X}{q-1} \right] \\ &= \sum_{\mu \vdash n} \left(\sum_{T \in \text{SYT}_\mu^\pi} \widetilde{\text{wt}}(T; q) \right) q^{n+\#\text{Area}(\pi)} \cdot (q-1)^{n-\mu_1} Q_{\mu'}(q^{-1}) \left[\frac{X}{q-1} \right] \\ &= \sum_{\mu \vdash n} \left(\sum_{T \in \text{SYT}_\mu^\pi} \widetilde{\text{wt}}(T; q) \right) q^{n+\#\text{Area}(\pi)} \cdot (q-1)^{n-\mu_1} q^{-n} q^{-n(\mu')} \widetilde{H}_\mu(q, 0)[X] \\ &= \sum_{\mu \vdash n} \left(\sum_{T \in \text{SYT}_\mu^\pi} \widetilde{\text{wt}}(T; q) \right) q^{\#\text{Area}(\pi)-n(\mu')} \cdot q^{n-\mu_1} (1-q^{-1})^{n-\mu_1} \widetilde{H}_\mu(q, 0)[X]. \end{aligned}$$

Comparing with (31),

$$\widetilde{c}_{\pi, \mu}(q) = q^{\#\text{Area}(\pi)-n(\mu')+n-\mu_1} \sum_{T \in \text{SYT}_\mu^\pi} \widetilde{\text{wt}}(T; q),$$

or, using (29),

$$(33) \quad c_{\pi, \mu}(q) = q^{-n+\mu_1} \sum_{T \in \text{SYT}_\mu^\pi} \widetilde{\text{wt}}(T; q^{-1}).$$

Suppose that i appears in row $s+1$ in the box $T^{(i)}$. If $(T_{<i})_s = \emptyset$ then $d(T, i)$ is one plus the number of boxes c in row s in the arm of $\text{up}(T^{(i)})$ whose value $T(c) < i$. Since $T(c) \not\prec i$, this means $T(c) <_\pi i$. On the other hand, if $T(c) <_\pi i$, then $T(c) < i$ and $T(c) \not\prec i$. So

$$d(T, i) = \text{arm}_{<_\pi i}(\text{up}(T^{(i)})) + 1, \quad \text{if } (T_{<i})_s = \emptyset.$$

Suppose $(T_{<i})_s \neq \emptyset$. If $j \prec i$ then $(j, i) \in \text{Area}(\pi)$ and if $k \geq j$, then (k, i) has to be below the path as well, so $k \not\prec_\pi i$. Then $L(T, i) - \text{coarm}(T^{(i)})$ is one plus the number of boxes c in the arm of $\text{up}(T^{(i)})$ such that $T(c) <_\pi i$. So

$$L(T, i) - \text{sh}(T_{<i})_{s+1} = \text{arm}_{<_\pi i}(\text{up}(T^{(i)})) + 1, \quad \text{if } (T_{<i})_s \neq \emptyset.$$

Then

$$\widetilde{\text{wt}}(T; q) = \prod_{i \notin T_1} q^{-m(T, i) - d(T, i)} \prod_{i \notin T_1} [\text{arm}_{<_\pi i}(\text{up}(T^{(i)})) + 1]_q$$

Note that

$$d(T, i) = \text{arm}_{< i}(\text{up}(T^{(i)})) + 1$$

and

$$\begin{aligned} \widetilde{\text{wt}}(T; q^{-1}) &= \prod_{i \notin T_1} q^{m(T, i) + d(T, i)} [\text{arm}_{<_\pi i}(\text{up}(T^{(i)})) + 1]_{q^{-1}} \\ &= \prod_{i \notin T_1} q^{m(T, i) + \text{arm}_{< i}(\text{up}(T^{(i)})) + 1 - \text{arm}_{<_\pi i}(\text{up}(T^{(i)}))} [\text{arm}_{<_\pi i}(\text{up}(T^{(i)})) + 1]_q, \end{aligned}$$

and $\text{arm}_{< i}(\text{up}(T^{(i)})) - \text{arm}_{<_\pi i}(\text{up}(T^{(i)}))$ is the number of boxes in row s with fillings $\prec i$. So $m(T, i) + \text{arm}_{< i}(\text{up}(T^{(i)})) - \text{arm}_{<_\pi i}(\text{up}(T^{(i)}))$ is the number of boxes with fillings $\prec i$ which are above i (such a filling must occur to the right of i). Then $m(T, i) + \text{arm}_{< i}(\text{up}(T^{(i)})) - \text{arm}_{<_\pi i}(\text{up}(T^{(i)})) = \gamma(T, T^{(i)})$.

Then comparing with (15),

$$\widetilde{\text{wt}}(T; q^{-1}) = q^{n-\mu_1} \prod_{i \notin T_1} q^{\gamma(T, T^{(i)})} [\text{arm}_{<_\pi i}(\text{up}(T^{(i)})) + 1]_q = q^{n-\mu_1-n(\mu')+\#\text{Area}(\pi)} \text{wt}(T; q).$$

Then by (33),

$$c_{\pi,\mu}(q) = q^{-n(\mu')+\#\text{Area}(\pi)} \cdot \sum_{T \in \text{SYT}_\mu^\pi} \text{wt}(T; q). \quad \square$$

COROLLARY 5.2. *Let $\pi \in \mathbb{D}_n$ with $\lambda(\pi) = \lambda$. Suppose*

$$\chi_\pi(q) = \sum_{\mu \vdash n} (1-q)^{n-\mu_1} c_{\pi,\mu}(q) W_\mu(q) \quad \text{and} \quad \tilde{\chi}_\pi(q) = \sum_{\mu \vdash n} (1-q^{-1})^{n-\mu_1} \tilde{c}_{\pi,\mu}(q) \tilde{W}_\mu(q).$$

Then

$$R_k(\lambda; q) = \sum_{\substack{\mu \vdash n \\ \mu_1 = n-k}} q^{n(\mu')-\#\text{Area}(\pi)} c_{\pi,\mu}(q) \quad \text{and} \quad R_k(\lambda; q^{-1}) = \sum_{\substack{\mu \vdash n \\ \mu_1 = n-k}} \tilde{c}_{\pi,\mu}(q).$$

The recent paper [12] obtains another proof of Corollary 5.2.

6. ABELIAN DYCK PATHS

In this section we first provide a condition for which partitions appear in (16). Then we focus our attention to Abelian Dyck paths, which are paths π such that if $\lambda = \lambda(\pi)$ then $|\pi| \geq \lambda_1 + \lambda'_1$. We show that in the case of Abelian Dyck paths the sum in (16) only runs over a single partition. We then provide another proof of a result of Guay-Paquet that says in the case of Abelian Dyck paths, the unicellular LLT functions are sum of unicellular LLT functions with rectangle shapes, where the coefficients are given by certain q -hit numbers, which are closely related with the q -rook numbers.

6.1. A CONDITION FOR $\text{SYT}_\mu^\pi \neq \emptyset$. Recall that a subset of a poset P is a chain if any two elements are comparable, and it is an anti-chain if any two distinct elements are incomparable.

For $\pi \in \mathbb{D}_n$ let $P(\pi) \vdash n$ denote the Greene shape of the poset determined by $<_\pi$ on $[n]$, i.e. $P(\pi)_1 + \dots + P(\pi)_k$ is the maximum number of elements in a union of k anti-chains in $[n]$ with respect to $<_\pi$. By [7, Theorem 1.5], $P(\pi)'_1 + \dots + P(\pi)'_k$ is the maximum number of elements in a union of k chains in $[n]$ with respect to $<_\pi$, where $P(\pi)'$ denotes the conjugate partition of $P(\pi)$.

For the path π from Figure 1, $P(\pi) = (3, 2, 1)$. For example, the sets $\{1, 2, 3\}$, $\{4, 5\}$, and $\{6\}$ are antichains of length 3, 2, and 1, respectively, and the sets $\{1, 4, 6\}$, $\{2, 5\}$ and $\{3\}$ are chains of length 3, 2, and 1, respectively.

LEMMA 6.1. *Let $\pi \in \mathbb{D}_n$ and $\mu \vdash n$ be such that $\text{SYT}_\mu^\pi \neq \emptyset$. Then $\mu \geq P(\pi)$ in the dominance order.*

Proof. Since entries in each column increase according to $<_\pi$, the entries in each column consists of a chain in $[n]$, therefore $\mu'_1 + \dots + \mu'_k \leq$ the maximum number of elements in a union of k chains with respect to $<_\pi = P(\pi)'_1 + \dots + P(\pi)'_k$. So, $\mu' \leq P(\pi)'$, or $\mu \geq P(\pi)$ in dominance order. \square

6.2. ABELIAN DYCK PATHS. For $n, m \in \mathbb{Z}_{\geq 0}$, denote by (m^n) the rectangular partition (m, \dots, m) with n rows with all parts equal to m . A partition $\mu \subseteq (m^n)$ if $\mu_1 \leq m$ and $\mu'_1 \leq n$.

For $\mu \subseteq (m^n)$, denote by $\pi^{n,m}(\mu) \in \mathbb{D}_{n+m}$ the path with $\lambda(\pi^{n,m}(\mu)) = \mu$.

PROPOSITION 6.2. *Let $\lambda \in \text{Par}$, $\lambda \subseteq (m^n)$ and $\pi = \pi^{n,m}(\lambda)$. Then $P(\pi)'_1 \leq 2$.*

Proof. Suppose $1 \leq i <_\pi j <_\pi k \leq m+n$. Then $i < j < k$ and $(i, j) \notin \text{Area}(\pi)$ and $(j, k) \notin \text{Area}(\pi)$. Now, $(i, j) \notin \text{Area}(\pi)$ implies that $j > m+n-\lambda'_i \geq m+n-n = m$, but then for any $k > j$, $(j, k) \in \text{Area}(\pi)$, a contradiction. Then the maximum length of an chain in $([m+n], <_\pi)$ has to be ≤ 2 . Therefore, Lemma 6.1 proves the claim. \square

PROPOSITION 6.3. *Let $\lambda \in \text{Par}$ and let $N \geq \lambda_1 + \lambda'_1$. Suppose $\pi \in \mathbb{D}_N$ is such that $\lambda(\pi) = \lambda$. Then*

$$(34) \quad R_k(\lambda; q) = q^{|\lambda| - (N-k)k} c_{\pi, (N-k, k)}(q).$$

Proof. Taking $m \geq \lambda_1$ and $n \geq \lambda'_1$ such that $m+n = N$ in Proposition 6.2, $P(\pi)'_1 \leq 2$ and if $\text{SYT}_\mu^\pi \neq \emptyset$ then $\mu' \leq P(\pi)'$, thus $\mu'_1 \leq 2$. Then the summands in $R_k(\lambda; q)$ from (16) runs over $\mu \vdash N$ with $\mu_1 = N - k$, there is only one possibility of μ , namely, $\mu = (N - k, k)$. Using $\#\text{Area}(\pi) = \binom{N}{2} - |\lambda|$, and

$$n((N - k, k)') - \#\text{Area}(\pi) = \binom{N - k}{2} + \binom{k}{2} - \binom{N}{2} + |\lambda| = |\lambda| - (N - k)k$$

in Corollary 5.2 gives the statement. □

6.3. PROOF OF [5, THEOREM 1.3]. In this subsection we provide another proof of [5, Theorem 1.3], where it is attributed to Guay-Paquet’s unpublished work. It says that for abelian Dyck paths, the unicellular LLT functions are a sum of the corresponding functions for rectangle shaped paths, with coefficients q -hit numbers.

Let $\lambda \subseteq (m^n)$ be a partition with $n \leq m$. Recall from [5, Definition 2.3] the q -hit numbers of λ are defined for $k \in \mathbb{Z}_{\geq 0}$, by

$$(35) \quad H_k^{m, n}(\lambda; q) = \frac{q^{\binom{k}{2} - |\lambda|}}{[m - n]_q!} \sum_{i=k}^n R_i(\lambda; q) [m - i]_q! \begin{bmatrix} i \\ k \end{bmatrix}_q (-1)^{i+k} q^{mi - \binom{i}{2}},$$

and the reverse relation is [5, (2.3)]

$$(36) \quad R_k(\lambda; q) = q^{|\lambda| - mk} \frac{[m - n]_q!}{[m - k]_q!} \sum_{j=k}^n H_j^{m, n}(\lambda; q) \begin{bmatrix} j \\ k \end{bmatrix}_{q^{-1}}.$$

PROPOSITION 6.4 ([5, THEOREM 1.3]). *Let $\lambda \subseteq (m^n)$ with $n \leq m$. Let $\pi = \pi(\lambda) = \pi^{n, m}(\lambda) \in \mathbb{D}_{n+m}$ be the Dyck path such that $\lambda(\pi) = \lambda$, and for $0 \leq j \leq n$, let $\pi(m^j) \in \mathbb{D}_{n+m}$ be the Dyck paths for which $\lambda(\pi(m^j)) = (m^j)$. Then*

$$\chi_{\pi(\lambda)}(q) = \frac{[m - n]_q!}{[m]_q!} \sum_{j=0}^n H_j^{m, n}(\lambda; q) \cdot \chi_{\pi(m^j)}(q).$$

Note that the version in [5] is about chromatic symmetric functions, which is equivalent to the statement above by using (23).

Proof. Using Proposition 6.3 and (10),

$$(37) \quad \begin{aligned} c_{\pi((m^j)), (m+n-k, k)}(q) &= q^{-mj + (m+n-k)k} R_k((m^j); q) \\ &= q^{-mj + (m+n-k)k} \cdot q^{(j-k)(m-k)} \frac{[j]_q!}{[j - k]_q!} \begin{bmatrix} m \\ k \end{bmatrix}_q \\ &= q^{(n-j)k} \frac{[j]_q!}{[j - k]_q!} \begin{bmatrix} m \\ k \end{bmatrix}_q. \end{aligned}$$

In particular, $c_{\pi((m^j)), (m+n-k, k)} = 0$ if $k > j$. By (28) and Proposition 6.2,

$$(38) \quad \chi_{\pi(m^j)}(q) = \sum_{k=0}^j (1 - q)^k c_{\pi((m^j)), (m+n-k, k)}(q) W_{(m+n-k, k)}(q).$$

Using Proposition 6.3, (36), (3), (2) and (37),

$$\begin{aligned}
 c_{\pi(\lambda), (m+n-k, k)}(q) &= q^{-|\lambda|+(m+n-k)k} R_k(\lambda; q) \\
 &= q^{-|\lambda|+(m+n-k)k} \cdot q^{|\lambda|-mk} \frac{[m-n]_q!}{[m-k]_q!} \sum_{j=k}^n H_j^{m,n}(\lambda; q) \left[\begin{matrix} j \\ k \end{matrix} \right]_{q^{-1}} \\
 &= q^{(n-k)k} \cdot \frac{[m-n]_q!}{[m-k]_q!} \sum_{j=k}^n H_j^{m,n}(\lambda; q) \left[\begin{matrix} j \\ k \end{matrix} \right]_{q^{-1}} \\
 &= q^{(n-k)k} \cdot \frac{[m-n]_q!}{[m-k]_q!} \sum_{j=k}^n H_j^{m,n}(\lambda; q) q^{-k(j-k)} \left[\begin{matrix} j \\ k \end{matrix} \right]_q \\
 &= q^{(n-k)k} \cdot \frac{[m-n]_q!}{[m-k]_q!} \sum_{j=k}^n H_j^{m,n}(\lambda; q) q^{-k(j-k)} \frac{[j]_q!}{[k]_q! [j-k]_q!} \\
 &= q^{(n-k)k} \cdot \frac{[m-n]_q!}{[m]_q!} \sum_{j=k}^n H_j^{m,n}(\lambda; q) q^{-k(j-k)} \frac{[j]_q!}{[k]_q! [j-k]_q!} \frac{[m]_q!}{[m-k]_q!} \\
 &= \frac{[m-n]_q!}{[m]_q!} \sum_{j=k}^n H_j^{m,n}(\lambda; q) q^{(n-j)k} \frac{[j]_q!}{[j-k]_q!} \left[\begin{matrix} m \\ k \end{matrix} \right]_q \\
 &= \frac{[m-n]_q!}{[m]_q!} \sum_{j=k}^n H_j^{m,n}(\lambda; q) c_{\pi((m^j)), (m+n-k, k)}(q).
 \end{aligned}$$

Therefore, using (38),

$$\begin{aligned}
 \chi_{\pi(\lambda)}(q) &= \sum_{k=0}^{m+n} (1-q)^k c_{\pi(\lambda), (m+n-k, k)} W_{(m+n-k, k)}(q) \\
 &= \sum_{k=0}^{m+n} (1-q)^k \left(\frac{[m-n]_q!}{[m]_q!} \sum_{j=k}^n H_j^{m,n}(\lambda; q) c_{\pi((m^j)), (m+n-k, k)}(q) \right) W_{(m+n-k, k)}(q) \\
 &= \sum_{k=0}^n (1-q)^k \left(\frac{[m-n]_q!}{[m]_q!} \sum_{j=k}^n H_j^{m,n}(\lambda; q) c_{\pi((m^j)), (m+n-k, k)}(q) \right) W_{(m+n-k, k)}(q) \\
 &= \frac{[m-n]_q!}{[m]_q!} \sum_{j=0}^n H_j^{m,n}(\lambda; q) \cdot \sum_{k=0}^j (1-q)^k c_{\pi((m^j)), (m+n-k, k)}(q) W_{(m+n-k, k)}(q) \\
 &= \frac{[m-n]_q!}{[m]_q!} \sum_{j=0}^n H_j^{m,n}(\lambda; q) \cdot \chi_{\pi((m^j))}(q). \quad \square
 \end{aligned}$$

7. *n*TH ROOK NUMBERS FROM *e*-COEFFICIENTS

In this section, we show that if $\lambda \subseteq (n^n)$ is a partition with $n - i + 1 \leq \lambda_i \leq n$ for every $i \in [n]$, then the *n*-th *q*-rook number $R_n(\lambda; q)$ can be obtained from taking sums of coefficients from the *e*-expansion of certain unicellular LLT functions.

For $\pi \in \mathbb{D}_n$ and $\mu \vdash n$, define $b_{\pi, \mu}(q) \in \mathbb{Q}(q)$ by

$$(39) \quad \chi_{\pi}(q) = \sum_{\mu \vdash n} (q-1)^{n-\ell(\mu)} b_{\pi, \mu}(q) e_{\mu}.$$

A formula for $b_{\pi, \mu}(q)$ is obtained in [2, Theorem 1.3], where it is also shown that $b_{\pi, \mu}(q) \in \mathbb{Z}_{\geq 0}[q]$ for all π, μ . We do not need the precise formulas here. The following is our main result in this section.

PROPOSITION 7.1. Let $\pi \in \mathbb{D}_n$ with $\lambda(\pi) = \lambda$ and let $\lambda^c = \lambda(\pi)^c = (n - \lambda_n, n - \lambda_{n-1}, \dots, n - \lambda_1)$ denote the complimentary partition of λ inside (n^n) . Then

$$(40) \quad \sum_{\mu \vdash n} q^{n-\ell(\mu)} b_{\pi, \mu}(q) = \prod_{j=1}^n [n - \lambda_j - j + 1]_q = R_n(\lambda^c; q).$$

(11) is the second equality above. To show the first equality, we prove that both sides of (40) are multiplicative and satisfy the modular laws of Abreu and Nigro from [1] and [2]. [1, Theorem 1.2] says that such functions are completely determined by their values on the paths $N^n E^n$ for $n \in \mathbb{Z}_{\geq 0}$, and we show that two sides of the first equality in (40) are equal for these paths.

For $\pi, \eta \in \mathbb{D}$, let $\pi \cdot \eta$ denote the concatenation of two Dyck paths. For an algebra A , a function $f : \mathbb{D} \rightarrow A$ is multiplicative if $f(\pi \cdot \eta) = f(\pi) \cdot f(\eta)$ for any $\pi, \eta \in \mathbb{D}$.

Taking the generators $y_n = q^{-1} p_n$ for $n \in \mathbb{Z}_{>0}$ of the ring of symmetric functions in [2, Definition 3.1] we get

$$\text{IF}(\pi) = \sum_{\mu \vdash n} b_{\pi, \mu}(q) q^{-\ell(\mu)} p_\mu \quad \text{for } \pi \in \mathbb{D}_n.$$

Taking specialization at $(1, 0, \dots)$, and multiplying by $q^{|\pi|}$,

$$(41) \quad q^n \text{IF}(\pi)[1] = \sum_{\mu \vdash n} q^{n-\ell(\mu)} b_{\pi, \mu}(q) \quad \text{for } \pi \in \mathbb{D}_n,$$

which is the left hand side of (40). By [2, Proposition 3.3, 3.4] $\pi \mapsto \text{IF}(\pi)$ is multiplicative and satisfies the modular laws. Then so is $\pi \mapsto q^n \text{IF}(\pi)[1]$.

Lemma 7.2 says that the two sides of the first equality in (40) agree on paths $N^n E^n$ for $n \in \mathbb{Z}_{>0}$ and Lemma 7.3 and the above discussion says that both of them are multiplicative and satisfy modular law. Hence their equality is proved by [1, Theorem 1.2].

We now provide the details concerning the product side of (40).

A similar result has been proved in [2, Proposition 3.8], which says for $\pi \in \mathbb{D}_n$ and $\lambda = \lambda(\pi)$,

$$\sum_{\mu \vdash n} b_{\pi, \mu}(q) = \prod_{i=1}^n (1 + [n - \lambda_i - i]_q).$$

7.1. q -STIRLING NUMBERS OF FIRST KIND. The q -Stirling numbers of first kind $s_q(n, k)$ are defined by [2, page 4]

$$x(x - [1]_q) \dots (x - [n - 1]_q) = \sum_{k=1}^n (-1)^{n-k} s_q(n, k) x^k.$$

Putting $x = -z^{-1}$ and multiplying by $(-z)^n$, we get

$$(42) \quad \sum_{k=1}^n s_q(n, k) z^{n-k} = \prod_{i=0}^{n-1} (1 + [i]_q z).$$

In other words,

$$s_q(n, k) = e_{n-k}([0]_q, \dots, [n - 1]_q).$$

7.2. EQUALITY FOR $N^n E^n$. Now we prove (40) when $\pi = N^n E^n$.

LEMMA 7.2. For $n \in \mathbb{Z}_{>0}$,

$$(43) \quad \sum_{\lambda \vdash n} q^{n-\ell(\lambda)} b_{N^n E^n, \lambda} = [n]_q! = R_n((n^n); q)$$

Proof. By (11),

$$R_n((n^n); q) = [n]_q!$$

[2, Corollary 3.7] says that

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} b_{N^n E^n, \lambda} = s_q(n, k).$$

Then by (42),

$$\sum_{\lambda \vdash n} q^{n-\ell(\lambda)} b_{N^n E^n, \lambda} = \sum_{k=1}^n q^{n-k} s_q(n, k) = [n]_q!. \quad \square$$

7.3. MULTIPLICATIVITY AND MODULAR LAW.

LEMMA 7.3. Define $G : \mathbb{D} \rightarrow \mathbb{Z}[q]$ by

$$G(\pi) = \prod_{j=1}^n [n - \lambda(\pi)_j - j + 1]_q \text{ for } \pi \in \mathbb{D}_n.$$

Then G is multiplicative and satisfies the modular law.

Proof. For showing that G is multiplicative, let $\pi \in \mathbb{D}_n, \eta \in \mathbb{D}_m$ and $\pi \cdot \eta$ denotes the concatenation of those two Dyck paths. Then

$$\lambda(\pi \cdot \eta)_j = \begin{cases} \lambda(\pi)_j & \text{for } 1 \leq j \leq n, \\ \lambda(\eta)_{j-n} + n & \text{for } n + 1 \leq j \leq n + m. \end{cases}$$

Then

$$\begin{aligned} G(\pi) \cdot G(\eta) &= \prod_{i=1}^n [n - \lambda(\pi)_i - i + 1]_q \cdot \prod_{j=1}^m [m - \lambda(\eta)_j - j + 1]_q \\ &= \prod_{i=1}^n [n + m - \lambda(\pi)_i - (i + m) + 1]_q \cdot \prod_{j=1}^m [m + n - (\lambda(\eta)_j + n) - j + 1]_q \\ &= \prod_{j=1}^{m+n} [n + m - \lambda(\pi \cdot \eta)_j - j + 1]_q = G(\pi \cdot \eta). \end{aligned}$$

Hence, G is multiplicative.

Let π^0, π^1, π^2 be Dyck paths satisfying conditions (1) or (2) of modular law of [1, Definition 2.1] with $\lambda(\pi^{(i)}) = \lambda^{(i)}$ for $i \in \{0, 1, 2\}$. Using the transformation given by §2.8, one of the following is true

(1) $\lambda^{(0)} = \lambda^{(1)} + \varepsilon_s$, and $\lambda^{(2)} = \lambda^{(1)} - \varepsilon_s$, for some $s \in \mathbb{Z}_{>0}$,

or,

(2) $\lambda^{(2)} = \lambda^{(1)} - \varepsilon_r, \lambda^{(0)} = \lambda^{(1)} + \varepsilon_{r+1}$, and $\lambda_r^{(1)} - \lambda_{r+1}^{(1)} = 1$ for some $r \in \mathbb{Z}_{>0}$,

To show that modular law holds we need to show that for any three Dyck paths π^0, π^1, π^2 with associated partitions $\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}$ satisfying (1) or (2) as above,

$$(1 + q)G(\pi^{(1)}) = qG(\pi^{(0)}) + G(\pi^{(2)}).$$

Both of these cases are proved by using

$$(44) \quad (1 + q)[a]_q = q[a - 1]_q + [a + 1]_q, \quad \text{for } a \in \mathbb{Z}_{\geq 0}.$$

Assume that condition (1) is true. In this case, since the difference is only on one component s , taking

$$a = n - \lambda_s^{(1)} - s + 1$$

gives

$$n - \lambda_s^{(0)} - s + 1 = a - 1, \text{ and } n - \lambda_s^{(2)} - s + 1 = a + 1,$$

so using (44) shows that modular law holds true in this case.

Next, assume that condition (2) is true. Let

$$a = n - \lambda_r^{(1)} - r + 1 = n - \lambda_{r+1}^{(1)} - (r + 1) + 1 \text{ and } b = \prod_{j \neq r, r+1} [n - \lambda_j^{(1)} - j + 1]_q.$$

Then using

$$\begin{aligned} n - \lambda_r^{(2)} - r + 1 &= n - (\lambda_r^{(1)} - 1) - r + 1 = a + 1, \\ n - \lambda_{r+1}^{(2)} - (r + 1) + 1 &= n - \lambda_{r+1}^{(1)} - r = n - \lambda_r^{(1)} - r + 1 = a, \\ n - \lambda_r^{(0)} - r + 1 &= n - \lambda_r^{(1)} - r + 1 = a, \\ n - \lambda_{r+1}^{(0)} - (r + 1) + 1 &= n - (\lambda_{r+1}^{(1)} + 1) - (r + 1) + 1 = a - 1, \end{aligned}$$

we get,

$$\begin{aligned} qG(\pi^{(0)}) + G(\pi^{(2)}) &= b \cdot ([a + 1]_q [a]_q + q[a]_q [a - 1]_q) = b \cdot (1 + q)[a]_q [a]_q \\ &= \prod_{j \neq r, r+1} [n - \lambda_j^{(1)} - j + 1]_q \cdot (1 + q)[n - \lambda_r^{(1)} - r + 1]_q [n - \lambda_{r+1}^{(1)} - (r + 1) + 1]_q \\ &= (1 + q)G(\pi^{(1)}). \end{aligned}$$

Hence, G satisfies the modular law. □

8. FURTHER COMMENTS

8.1. A FORMULA FOR THE q -STIRLING NUMBERS. Recall from (9) that the q -Stirling numbers of second kind are $S_q(n, k) = R_{n-k}(\rho_n; q)$, where $\rho_n = (n - 1, \dots, 0)$ is the staircase partition. Theorem 3.1 in this case becomes a sum over the usual standard Young tableaux.

PROPOSITION 8.1. *Let $n, k \in \mathbb{Z}_{>0}$, the q -Stirling numbers of second kind has the formula*

$$S_q(n, k) = \sum_{\substack{\mu \vdash n \\ \mu_1 = k}} q^{n(\mu')} \sum_{T \in \text{SYT}_\mu} \prod_{\substack{b \in \mu \\ \text{coleg}(b) > 0}} [\text{arm}_{<T(b)}(\text{up}(b)) + 1]_q.$$

Proof. For $n \in \mathbb{Z}_{>0}$, let $\lambda = \rho_n = (n - 1, n - 2, \dots, 1, 0)$. Let $\pi = \pi(\lambda) \in \mathbb{D}_n$. Then $\#\text{Area}(\pi) = 0$ and $i, j \in [n]$, $i <_\pi j$ if and only if $i < j$. So for $\mu \vdash n$, $\text{SYT}_\mu^\pi = \text{SYT}_\mu$ and for $T \in \text{SYT}_\mu$ and $b \in \mu$, $\gamma(T, b) = 0$. Then

$$\text{wt}(T; q) = q^{n(\mu')} \prod_{\substack{b \in \mu \\ \text{coleg}(b) > 0}} [\text{arm}_{<T(b)}(\text{up}(b)) + 1]_q,$$

so Theorem 3.1 gives the result. □

8.2. MATRIX COUNTING OVER \mathbb{F}_q . Let $\pi \in \mathbb{D}_n$ and $\lambda = \lambda(\pi)$, $P_k(\pi; q)$ be the number of $n \times n$ matrices over \mathbb{F}_q of rank k such that all non-zero entries appear above π . Then by [10, Theorem 1]

$$P_k(\pi; q) = (q - 1)^k q^{|\lambda| - k} R_k(\lambda; q^{-1}) = (1 - q^{-1})^k q^{|\lambda|} R_k(\lambda; q^{-1}).$$

Then using Corollary 5.2,

$$(45) \quad P_k(\pi; q) = q^{|\lambda|} (1 - q^{-1})^k \sum_{\substack{\mu \vdash n \\ \mu_1 = n - k}} \tilde{c}_{\pi, \mu}(q) = q^{|\lambda|} \sum_{\substack{\mu \vdash n \\ \mu_1 = n - k}} [\widetilde{W}_\mu(q)] \tilde{\chi}_\pi(q),$$

where $[\widetilde{W}_\mu(q)]\widetilde{\chi}_\pi(q)$ denotes the coefficient of $\widetilde{W}_\mu(q)$ in the \widetilde{W} -expansion of $\widetilde{\chi}_\pi(q)$.

Since $\sum_{k \geq 0} P_k(\pi; q)$ is the total number of matrices such that all non-zero entries lie above π , which is simply $q^{|\lambda|}$, we get

$$\sum_{\mu \vdash n} [\widetilde{W}_\mu(q)]\widetilde{\chi}_\pi(q) = 1.$$

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