ALGEBRAIC COMBINATORICS

Ron M. Adin, Pál Hegedűs & Yuval Roichman

Descent set distribution for permutations with cycles of only odd or only even lengths

Volume 9, issue 1 (2026), p. 161-182.
https://doi.org/10.5802/alco.471

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 9, issue 1 (2026), p. 161–182
https://doi.org/10.5802/alco.471

Descent set distribution for permutations with cycles of only odd or only even lengths

Ron M. Adin, Pál Hegedűs & Yuval Roichman

ABSTRACT It is known that the number of permutations in the symmetric group S_{2n} with cycles of odd lengths only is equal to the number of permutations with cycles of even lengths only. We prove a refinement of this equality, involving descent sets: the number of permutations in S_{2n} with a prescribed descent set and all cycles of odd lengths is equal to the number of permutations with the complementary descent set and all cycles of even lengths. There is also a variant for S_{2n+1}. The proof uses generating functions for character values and applies a new identity on higher Lie characters.

1. INTRODUCTION

For a positive even integer n, let $OC(n)$ be the set of all permutations $\pi \in S_n$ with cycles of odd lengths only, and let $EC(n)$ be the set of all permutations $\pi \in S_n$ with cycles of even lengths only. It is known that, for every positive even n,

$$|OC(n)| = |EC(n)| = (n-1)!!^2,$$

where the double factorial $(n-1)!! := (n-1) \cdot (n-3) \cdot (n-5) \cdots$. See e.g. [2, Theorem 6.24] and [9, A001818].

In this paper we prove the following remarkable refinement.

First, let us extend the above definitions to odd values of n. For a positive odd integer n, let $OC(n)$ be the set of all permutations $\pi \in S_n$ with cycles of odd lengths only, and let $EC(n)$ be the set of all permutations $\pi \in S_n$ with cycles of even lengths only, plus a single fixed point. The *descent set* of a permutation $\pi = [\pi_1, \ldots, \pi_n] \in S_n$ is

$$\mathrm{Des}(\pi) := \{1 \leqslant i \leqslant n-1 \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n-1],$$

where $[m] := \{1, 2, \ldots, m\}$.

THEOREM 1.1. *For any positive integer n and subset $J \subseteq [n-1]$,*

$$|\{\pi \in OC(n) \,:\, \mathrm{Des}(\pi) = J\}| = |\{\pi \in EC(n) \,:\, \mathrm{Des}(\pi) = [n-1] \setminus J\}|.$$

Manuscript received 7th April 2025, revised and accepted 2nd December 2025.

KEYWORDS. permutation, descent set, higher Lie character, character generating function.

ACKNOWLEDGEMENTS. The second author was partially supported by the National Research, Development and Innovation Office - NKFIH (K138596).

ISSN: 2589-5486

Theorem 1.1 is equivalent to an apparently new identity on higher Lie characters, Theorem 1.2. For any partition $\lambda \vdash n$, let $\psi^\lambda_{S_n}$ be the corresponding higher Lie character of S_n (for a definition see Subsection 2.1). For any positive integer n, let $OP(n)$ be the set of all partitions of n with odd parts only. For a positive even integer n, let $EP(n)$ be the set of all partitions of n with even parts only; and, for a positive odd integer n, let $EP(n)$ be the set of all partitions of n with all parts even, except for a single part of size 1.

Theorem 1.2. *For any positive integer n,*

$$\sum_{\lambda \in OP(n)} \psi^\lambda_{S_n} = \mathrm{sign} \otimes \sum_{\lambda \in EP(n)} \psi^\lambda_{S_n},$$

where sign *is the sign character of S_n.*

This result follows, in turn, from two explicit generating functions, Theorem 1.3 and Theorem 1.4. For a partition $\nu \vdash n$, let b_j be the number of parts of size j in ν $(\forall\, j \geqslant 1)$. Then

$$|Z_\nu| = \frac{n!}{\prod_j b_j! j^{b_j}}$$

is the size of the centralizer Z_ν of any element of cycle type ν in S_n. Let $\underline{t} = (t_j)_{j\geqslant 1}$ be a countable set of indeterminates, consider the ring $\mathbb{C}[[\underline{t}]]$ of formal power series in these indeterminates, and denote $\underline{t}^{c(\nu)} := \prod_j t_j^{b_j}$.

Theorem 1.3.

$$\sum_{n\geqslant 0} \sum_{\lambda \in OP(n)} \sum_{\nu \vdash n} \psi^\lambda_{S_n}(\nu) \frac{\underline{t}^{c(\nu)}}{|Z_\nu|} = \prod_{p\geqslant 0} \left(\frac{1+t_{2^p}}{1-t_{2^p}} \right)^{1/2^{p+1}}.$$

Theorem 1.4.

$$\sum_{n\geqslant 0} \sum_{\lambda \in EP(n)} \sum_{\nu \vdash n} \mathrm{sign}(\nu) \psi^\lambda_{S_n}(\nu) \frac{\underline{t}^{c(\nu)}}{|Z_\nu|} = \prod_{p\geqslant 0} \left(\frac{1+t_{2^p}}{1-t_{2^p}} \right)^{1/2^{p+1}}.$$

In particular, setting $t_{2^p} = 0$ for all $p \geqslant 1$, we obtain the following result.

Corollary 1.5. *For any integer $n \geqslant 2$,*

$$\sum_{\lambda \in OP(n)} \dim \psi^\lambda_{S_n} = \sum_{\lambda \in EP(n)} \dim \psi^\lambda_{S_n} = \begin{cases} (n-1)!!^2, & \text{if } n \text{ is even;} \\ n!! \cdot (n-2)!!, & \text{if } n \text{ is odd.} \end{cases}$$

Note that the RHS of the formula in Corollary 1.5 is the number of permutations of odd order in S_n (see [9, A000246]), namely the cardinality of $OC(n)$. For further discussion see Section 6.

The group of signed permutations B_n can be viewed as the centralizer, in S_{2n}, of a fixed-point-free involution (a permutation of cycle type $(2,\dots,2)$). Noting that its index $|S_{2n}|/|B_n| = (2n-1)!!$, Corollary 1.5 suggests that the sum of higher Lie characters of S_{2n} over the partitions in $OP(2n)$ is induced from a character of B_n. We prove that this is, indeed, the case (with an analogue for S_{2n+1}). Specifically, denote

$$\eta_{B_n} := \sum_{\lambda \vdash n} \psi^{(\lambda,\varnothing)}_{B_n},$$

the sum of (type B) higher Lie characters of B_n corresponding to conjugacy classes with positive cycles only (for a definition see Subsection 7.2).

Theorem 1.6. *For any integer $n \geqslant 0$,*

$$\sum_{\lambda \in OP(n)} \psi^{\lambda}_{S_n} = \eta_{B_{\lfloor n/2 \rfloor}} \uparrow^{S_n}_{B_{\lfloor n/2 \rfloor}}.$$

See Theorem 7.4 below.

The rest of the paper is organized as follows. Higher Lie characters are defined in Subsection 2.1, and the equivalence of Theorems 1.1 and 1.2 is explained in Subsection 2.2. In Section 3 we compute a generating function (Theorem 3.1) for the values of the higher Lie characters of S_n. In Section 4 we state a signed analogue, Theorem 4.1, and outline its proof. In Section 5 we deduce Theorem 1.3 (from Theorem 3.1) and Theorem 1.4 (from Theorem 4.1), thus proving Theorem 1.2. In Section 6 we study related character identities for odd root enumerators. Finally, in Section 7 we prove Theorem 1.6.

2. Preliminaries

In this section we define higher Lie characters (in S_n), and use basic properties of quasisymmetric functions to explain the equivalence of Theorems 1.1 and 1.2.

2.1. Higher Lie characters.

In this subsection we define a higher Lie character for any element (in fact, for any conjugacy class) in the symmetric group S_n. Let us first recall the well-known description of the centralizer of an arbitrary element of S_n.

We write $\lambda \vdash n$ to denote that λ is a partition of a positive integer n.

Lemma 2.1. (Centralizers in S_n) *Let $x \in S_n$ be an element of cycle type $\lambda \vdash n$ where, for each $i \geqslant 1$, the partition λ has a_i parts of size i. Write*

$$x = \prod_{i \geqslant 1} x_i \,,$$

where x_i has a_i cycles of length i ($i \geqslant 1$); of course, only finitely many factors here are nontrivial. Then the centralizer of x in S_n, $Z_x = Z_{S_n}(x)$, satisfies

$$Z_{S_n}(x) = \bigtimes_{i \geqslant 1} Z_{S_{ia_i}}(x_i) \cong \bigtimes_{i \geqslant 1} G_i \wr S_{a_i},$$

where G_i, isomorphic to the cyclic group of order i, is the centralizer in S_i of a cycle of length i. By convention, $G_i \wr S_0$ is the trivial group while $G_i \wr S_1 \cong G_i$.

Remark 2.2. We use G_i, rather than C_i or $\mathbb{Z}_i$, to denote the cyclic group of order n, since C and Z are intensively used here for other purposes.

Definition 2.3. (Higher Lie characters in S_n) *Let x be an element of cycle type λ in S_n, as in Lemma* 2.1.

(a) *For each $i \geqslant 1$, let ω_i be the linear character on $G_i \wr S_{a_i}$ which is equal to a primitive irreducible character on the cyclic group G_i, and trivial on the wreathing group S_{a_i}. Let*

$$\omega^x := \bigotimes_{i \geqslant 1} \omega_i,$$

a linear character on Z_x.

(b) *Define the corresponding* higher Lie character *to be the induced character*

$$\psi^x_{S_n} := \omega^x \uparrow^{S_n}_{Z_x}.$$

(c) *It is easy to see that $\psi^x_{S_n}$ depends only on the conjugacy class C (equivalently, the cycle type λ) of x, and can therefore be denoted $\psi^C_{S_n}$ or $\psi^{\lambda}_{S_n}$.*

2.2. Quasisymmetric functions and descents. The *descent set* of a permutation $\pi = [\pi_1, \ldots, \pi_n]$ in the symmetric group S_n on n letters is

$$\mathrm{Des}(\pi) := \{1 \leqslant i \leqslant n-1 \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n-1],$$

where $[m] := \{1, 2, \ldots, m\}$.

Definition 2.4. *For each subset $D \subseteq [n-1]$ define the* fundamental quasisymmetric function

$$\mathcal{F}_{n,D}(\mathbf{x}) := \sum_{\substack{i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Given any subset $A \subseteq S_n$, define the quasisymmetric function

$$\mathcal{Q}(A) := \sum_{\pi \in A} \mathcal{F}_{n,\mathrm{Des}(\pi)}.$$

In their seminal paper [5], Gessel and Reutenauer prove the following.

Theorem 2.5 ([5, Theorem 3.6]). *For every partition $\lambda \vdash n$, let C_λ be the conjugacy class of permutations in S_n of cycle type λ. Then*

$$\mathcal{Q}(C_\lambda) = \mathrm{ch}\left(\psi^\lambda_{S_n}\right),$$

where ch *is the Frobenius characteristic map and ψ^λ is the higher Lie character defined in Subsection* 2.1*.*

The remarks preceding [5, Theorem 4.1] imply the following variant.

Corollary 2.6. *For every partition $\lambda \vdash n$,*

$$\mathrm{ch}\left(\mathrm{sign} \otimes \psi^\lambda_{S_n}\right) = \sum_{\pi \in C_\lambda} \mathcal{F}_{n,[n-1] \setminus \mathrm{Des}(\pi)}.$$

Corollary 2.7. *Theorem* 1.1 *and Theorem* 1.2 *are equivalent.*

Proof. Recall from [12, Ch. 7] that the fundamental quasisymmetric functions $\{\mathcal{F}_{n,D} \mid D \subseteq [n-1]\}$ form a basis of the vector space QSym_n of quasisymmetric functions in n variables. Theorem 1.1 is therefore equivalent to

$$\sum_{\pi \in OC(n)} \mathcal{F}_{n,\mathrm{Des}(\pi)} = \sum_{\pi \in EC(n)} \mathcal{F}_{n,[n-1] \setminus \mathrm{Des}(\pi)}.$$

By Theorem 2.5, this happens if and only if

$$\begin{aligned}\sum_{\lambda \in OP(n)} \psi^\lambda_{S_n} &= \mathrm{ch}^{-1}\left(\sum_{\lambda \in OP(n)} \mathcal{Q}(C_\lambda)\right) = \mathrm{ch}^{-1}\left(\sum_{\pi \in OC(n)} \mathcal{F}_{n,\mathrm{Des}(\pi)}\right) \\ &= \mathrm{ch}^{-1}\left(\sum_{\pi \in EC(n)} \mathcal{F}_{n,[n-1] \setminus \mathrm{Des}(\pi)}\right) = \mathrm{sign} \otimes \sum_{\lambda \in EP(n)} \psi^\lambda_{S_n}. \qquad \square\end{aligned}$$

3. A generating function for higher Lie characters

In this section we state and prove an explicit generating function (Theorem 3.1) for the values of all the higher Lie characters of the symmetric group S_n. This formula is an S_n-version of a similar formula, recently proved for the higher Lie characters of the hyperoctahedral group B_n [1, Theorem 4.3]. The main result is stated in Subsection 3.1, and proved in the following subsections.

3.1. Main result. Let λ and ν be two partitions of n. For each integer $i \geqslant 1$, let a_i be the number of parts of size i in the partition λ. Similarly, for each integer $j \geqslant 1$, let b_j be the number of parts of size j in the partition ν. Thus

$$\sum_i i a_i = \sum_j j b_j = n.$$

Let $\underline{s} = (s_i)_{i\geqslant 1}$ and $\underline{t} = (t_j)_{j\geqslant 1}$ be two countable sets of indeterminates. Consider the ring $\mathbb{C}[[\underline{s}, \underline{t}]]$ of formal power series in these indeterminates, and denote $\underline{s}^{c(\lambda)} := \prod_i s_i^{a_i}$ and $\underline{t}^{c(\nu)} := \prod_j t_j^{b_j}$. The main result of this section is the following.

Theorem 3.1.

$$\sum_{n\geqslant 0} \sum_{\lambda,\nu\vdash n} \psi^{\lambda}_{S_n}(\nu) \frac{\underline{s}^{c(\lambda)}\underline{t}^{c(\nu)}}{|Z_\nu|} = \exp\left(\sum_{i,j\geqslant 1} \sum_{e\mid \gcd(i,j)} \mu(e) \frac{s_i^{j/e} t_j^{i/e}}{ij/e} \right),$$

where $\mu(\cdot)$ is the classical Möbius function.

We prove this result in the following subsections.

3.2. Values of induced characters. Let us start our computations by writing a general formula (Lemma 3.2) for the values of each higher Lie character, as an induced character.

For an element $x \in S_n$ denote the conjugacy class and the centralizer of x by C_x and Z_x, respectively. Recall the definitions of ω^x and $\psi^x_{S_n}$ from Subsection 2.1.

Lemma 3.2. *If $x \in S_n$ then*

$$\psi^x_{S_n}(g) = \frac{|C_x|}{|C_g|} \sum_{z\in C_g\cap Z_x} \omega^x(z) \qquad (\forall g \in S_n).$$

Proof. Let G be a group, and χ a character of a subgroup H of G. Define a function $\chi^0 : G \to \mathbb{C}$ by

$$\chi^0(g) := \begin{cases} \chi(g), & \text{if } g \in H; \\ 0, & \text{if } g \in G \setminus H. \end{cases}$$

By [7, (5.1)], an explicit formula for the induced character $\chi\uparrow^G_H$ is

$$\chi\uparrow^G_H(g) = \sum_{G=\cup_a aH} \chi^0(a^{-1}ga) = \frac{1}{|H|} \sum_{a\in G} \chi^0(a^{-1}ga) \qquad (\forall g \in G).$$

The mapping $f : G \to C_g$ defined by $f(a) := a^{-1}ga$ ($\forall a \in G$) is surjective, and satisfies: $f(a_1) = f(a_2)$ if and only if $a_1 a_2^{-1} \in Z_g$. Hence

$$\chi\uparrow^G_H(g) = \frac{|Z_g|}{|H|} \sum_{z\in C_g} \chi^0(z) = \frac{|Z_g|}{|H|} \sum_{z\in C_g\cap H} \chi(z).$$

Consider now an element $x \in S_n$, and apply the above formula with $G = S_n$, $H = Z_x = Z_{S_n}(x)$, and $\chi = \omega^x$, the linear character on the centralizer Z_x described in Definition 2.3(a). Then

$$\psi^x_{S_n}(g) = \omega^x\uparrow^{S_n}_{Z_x}(g) = \frac{|Z_g|}{|Z_x|} \sum_{z\in C_g\cap Z_x} \omega^x(z) \qquad (\forall g \in S_n).$$

Recalling that $|Z_x| = |S_n|/|C_x|$ and $|Z_g| = |S_n|/|C_g|$ completes the proof. □

3.3. The structure of a single cycle. We want to study the S_n-structure of elements $z \in C_g \cap Z_x$. Our main initial result is Corollary 3.4, describing the S_n-structure of a single cycle of z.

Assume that $x \in S_n$ has cycle type λ. Decompose

$$x = \prod_{i \geqslant 1} x_i \,,$$

where each x_i is a product of a_i disjoint cycles of length i ($i \geqslant 1$), and only finitely many of the factors are nontrivial. Then, by Lemma 2.1,

$$Z_{S_n}(x) = \bigtimes_{i \geqslant 1} Z_{S_{ia_i}}(x_i) \cong \bigtimes_{i \geqslant 1} G_i \wr S_{a_i},$$

where G_i is the cyclic group of order i.

Assume that $g \in S_n$ has cycle type ν, with b_j cycles of length j ($j \geqslant 1$). Let $z \in C_g \cap Z_x$, and decompose it as

$$z = \prod_{i \geqslant 1} z_i \,,$$

where $z_i \in Z_{S_{ia_i}}(x_i) \cong G_i \wr S_{a_i}$ ($i \geqslant 1$). Using a finer decomposition, assume that the underlying permutation of z_i, as an element of S_{ia_i}, has $m_{i,j}$ cycles of length j ($j \geqslant 1$). Of course,

$$\sum_j j m_{i,j} = i a_i \quad \text{ and } \quad \sum_i m_{i,j} = b_j.$$

Assume first that, as an element of $G_i \wr S_{a_i}$, z_i has a single cycle c of length ℓ and G_i-class γ. What is the structure of c as an element of S_{ia_i}?

Let $\zeta_i \in \mathbb{C}$ be a primitive complex i-th root of 1, generating the cyclic group G_i interpreted as the group of complex i-th roots of 1.

Lemma 3.3. *Fix $i \geqslant 1$ and $m \geqslant 0$, and let $x_i \in S_{im}$ be a product of m cycles of length i. Let $c \in Z_{S_{im}}(x_i) \cong G_i \wr S_m$ be a single cycle of length ℓ (in S_m) and class $\gamma = \zeta_i^k \in G_i$. Then, as an element of S_{im}, c is a product of $d := \gcd(k, i)$ disjoint cycles, each of length $j := \ell i / d$.*

Proof. Following Lemma 2.1, and after an appropriate conjugation in S_{im}, we can assume that we have the following scenario:

- There is an $\ell \times i$ rectangular array $(p_{s,t})$ of distinct integers in $[im]$ (say, $p_{s,t} = (s-1)i + t$ for all $1 \leqslant s \leqslant \ell$ and $1 \leqslant t \leqslant i$). Each of the ℓ rows corresponds to a cycle (of length i) of x_i. As an element of S_{im},

$$x_i : \quad p_{s,t} \mapsto p_{s,t+1} \qquad (1 \leqslant s \leqslant \ell,\ 1 \leqslant t \leqslant i),$$

  where $p_{s,i+1}$ is interpreted as $p_{s,1}$. On the other elements of $[im]$, x_i acts as the identity.
- As an element of S_{im}, $c \in Z_{x_i}$ permutes the ℓ rows cyclically, and specifically acts as follows: if $\gamma = \zeta_i^k \in G_i$ ($0 \leqslant k < i$), then

$$\begin{aligned} c : \quad & p_{s,t} \mapsto p_{s+1,t} && (1 \leqslant s \leqslant \ell - 1,\ 1 \leqslant t \leqslant i), \\ & p_{\ell,t} \mapsto p_{1,t+k} && (1 \leqslant t \leqslant i), \end{aligned}$$

  with $p_{1,t+k}$ interpreted as $p_{1,t+k-i}$ if $t + k > i$.

Since c permutes the ℓ rows cyclically, each cycle of c (as an element of S_{im}) has length divisible by ℓ. In fact,

$$c^\ell : \quad p_{s,t} \mapsto p_{s,t+k} \qquad (\forall s, t),$$

with $p_{s,t+k}$ interpreted as $p_{s,t+k-i}$ if $t+k>i$. Defining $d:=\gcd(k,i)$, we have $\gcd(k/d,i/d)=1$. The smallest multiple of k which is divisible by i is thus $(i/d)\cdot k = i\cdot(k/d)$. It follows that all the cycles of c have length $\ell i/d$, and their number is d. ☐

Denote, in the statement of Lemma 3.3, $e := i/d$ $(= j/\ell)$. Then e is a common divisor of i and j. We can thus restate Lemma 3.3 as follows.

Corollary 3.4. *Fix $i,j \geqslant 1$ and $m \geqslant 0$, and let $x_i \in S_{im}$ be a product of m cycles of length i. Let $c \in Z_{S_{im}}(x_i) \cong G_i \wr S_m$ be a single cycle of length ℓ (in S_m). Assume that, as an element of S_{im}, c is a product of d disjoint cycles of length j. Then there exists a common divisor e of i and j such that $d = i/e$ and $\ell = j/e$. Moreover, as an element of $G_i \wr S_m$, the cycle c has (length ℓ and) class $\gamma = \zeta_i^k \in G_i$, for some integer k satisfying* $\gcd(k,i)=d$.

3.4. Summation on a single cycle. We now want to compute the sum in Lemma 3.2 on a certain small subset of $C_g \cap Z_x$, when g and x have special cycle types, with all cycles of the same length. The main result here is Lemma 3.6, addressing summation over elements with a single underlying cycle.

As a computational tool, recall the following well-known fact regarding the classical *Möbius function* $\mu : \mathbb{N} \to \{-1,0,1\}$, defined by $\mu(1):=1$, $\mu(n):=(-1)^k$ if n is a product of $k \geqslant 1$ distinct primes, and $\mu(n):=0$ otherwise (namely, if n is not square-free).

Lemma 3.5. *For any positive integer n,*

$$\sum_{\substack{0\leqslant k<n\\ \gcd(k,n)=1}} \zeta_n^k = \mu(n)\,.$$

The following result performs the summation in Lemma 3.2 only on the elements $z \in C_g \cap Z_x$ corresponding to one specific cycle in S_{a_i}; both g and x are assumed to have cycle types with all cycles of the same length.

Lemma 3.6. *Fix $i,j \geqslant 1$ and $m \geqslant 0$, and let $x = x_i \in S_{im}$ be a product of m cycles of length i. Let e be a common divisor of i and j, and denote $d := i/e$ and $\ell := j/e$. Let $\sigma \in S_m$ be a permutation which has a single cycle of length ℓ, and is the identity outside the support of this cycle. Let $R_{i,j}(\sigma,e)$ be the set of all the elements $z \in Z_x \cong G_i \wr S_m$ corresponding to the underlying permutation (cycle) σ, with suitable G_i-classes, such that, as elements of $S_{i\ell} = S_{jd}$, they are products of d disjoint cycles of length j. Denote*

$$K_{i,j}(\sigma,e) := \frac{1}{i^{\ell-1}} \sum_{z\in R_{i,j}(\sigma,e)} \omega^x(z)\,.$$

Then $K_{i,j}(\sigma,e)$ actually depends only on e:

$$K_{i,j}(\sigma,e) = \mu(e),$$

where $\mu(\cdot)$ is the classical Möbius function.

Proof. By Corollary 3.4, the set of possible G_i-classes γ of elements $z \in R_{i,j}(\sigma,e)$ depends on i and $d=i/e$, but not on j or σ (as long as j is a multiple of e). Denote this set by $C(i,d)$:

$$C(i,d) = \{\zeta_i^k \,:\, \gcd(k,i)=d\}.$$

The number of elements $z \in R_{i,j}(\sigma,e)$ with any specific G_i-class is $|G_i|^{\ell-1} = i^{\ell-1}$. Denote

$$K_{i,j}(\sigma,e) := \frac{1}{i^{\ell-1}} \sum_{z\in R_{i,j}(\sigma,e)} \omega^x(z)\,.$$

It follows, by Definition 2.3(a) of ω^x, that

$$K_{i,j}(\sigma, e) = \sum_{\gamma \in C(i,d)} \gamma = \sum_{\substack{0 \leqslant k < i \\ \gcd(k,i)=d}} \zeta_i^k .$$

Denoting $k' := k/d$ and using Lemma 3.5, we conclude that

$$\sum_{\substack{0 \leqslant k < i \\ \gcd(k,i)=d}} \zeta_i^k = \sum_{\substack{0 \leqslant k' < i/d \\ \gcd(k',i/d)=1}} \zeta_{i/d}^{k'} = \mu(i/d)\,,$$

namely

$$K_{i,j}(\sigma, e) = \mu(e),$$

as claimed. □

3.5. Proof of Theorem 3.1. Extending the previous result, we shall now sum ω^x on the whole set $C_g \cap Z_x$, with increasing generality of the cycle types of g and x. This will lead to a proof of Theorem 3.1, providing a generating function for the values of higher Lie characters $\psi^\lambda_{S_n} = \psi^x_{S_n}$.

Definition 3.7. *For positive integers i and j, let*

$$E(i,j) := \{e \geqslant 1 \,:\, e \text{ divides both } i \text{ and } j\}.$$

Note that $E(i,j)$ is never empty, since it always contains $e = 1$.

For an indeterminate s, let $\mathbb{C}[[s]]$ be the ring of formal power series in s over the field $\mathbb{C}$. We now extend Lemma 3.6, and compute the sum in Lemma 3.2 on the whole set $C_g \cap Z_x$, still restricting g and x to have cycle types with all cycles of the same length.

Lemma 3.8. *Fix $i, j \geqslant 1$. For any integer $m \geqslant 0$, let $x = x_i(m) \in S_{im}$ be a product of m disjoint cycles of length i. Let $R_{i,j}(m)$ be the set of all elements $z \in Z_{S_{im}}(x_i(m)) \cong G_i \wr S_m$ which, as elements of S_{im}, are products of im/j disjoint cycles of length j. (Of course, necessarily j divides im.) Then, in $\mathbb{C}[[s]]$,*

$$\sum_{m \geqslant 0} \sum_{z \in R_{i,j}(m)} \omega^x(z) \frac{s^m}{m!} = \exp\left(\sum_{e \in E(i,j)} \mu(e) \, \frac{(is)^{j/e}}{ij/e} \right).$$

Proof. Assume that $E(i,j) = \{e_1, \ldots, e_q\}$, and define $\ell_k := j/e_k$ $(1 \leqslant k \leqslant q)$. For each $m \geqslant 0$, let

$$N_{i,j}(m) := \{(n_1, \ldots, n_q) \in \mathbb{Z}^q_{\geqslant 0} \,:\, n_1\ell_1 + \ldots n_q\ell_q = m\}.$$

By Corollary 3.4, the possible cycle lengths of elements of $R_{i,j}(m)$, viewed as elements of $G_i \wr S_m$, are $\ell_1, \ldots, \ell_q$. If such an element has n_k cycles of length ℓ_k $(1 \leqslant k \leqslant q)$, then clearly $(n_1, \ldots, n_q) \in N_{i,j}(m)$. The number of permutations in S_m with this cycle structure is

$$\frac{m!}{n_1! \cdots n_q! \cdot \ell_1^{n_1} \cdots \ell_q^{n_q}}.$$

By Lemma 3.6, for each common divisor e of i and j and each specific cycle $\sigma \in S_m$ of length $\ell = j/e$,

$$\sum_{z \in R_{i,j}(\sigma, e)} \omega^x(z) = i^{\ell-1} \cdot \mu(e).$$

The linearity of the character ω^x thus implies that

$$\begin{aligned}\sum_{z\in R_{i,j}(m)} \omega^x(z) &= \sum_{(n_1,\dots,n_q)\in N_{i,j}(m)} \frac{m!}{\prod_{k=1}^q n_k!\ell_k^{n_k}} \cdot \prod_{k=1}^q \left(i^{\ell_k-1}\mu(e_k)\right)^{n_k} \\ &= m! \sum_{(n_1,\dots,n_q)\in N_{i,j}(m)} \prod_{k=1}^q \frac{1}{n_k!}\left(\mu(e_k)\,\frac{i^{\ell_k}}{i\ell_k}\right)^{n_k}.\end{aligned}$$

Let s be an indeterminate, and consider the ring $\mathbb{C}[[s]]$ of formal power series in s over the field $\mathbb{C}$. By the definition of $N_{i,j}(m)$ and the above computation, it follows that the number

$$\frac{1}{m!}\sum_{z\in R_{i,j}(m)} \omega^x(z)$$

is the coefficient of s^m in the product

$$\begin{aligned}\prod_{k=1}^q \sum_{n_k=0}^\infty \frac{1}{n_k!}\left(\mu(e_k)\,\frac{(is)^{\ell_k}}{i\ell_k}\right)^{n_k} &= \prod_{k=1}^q \exp\left(\mu(e_k)\,\frac{(is)^{\ell_k}}{i\ell_k}\right) \\ &= \prod_{e\in E(i,j)} \exp\left(\mu(e)\,\frac{(is)^{j/e}}{ij/e}\right).\end{aligned}$$

In other words,

$$\begin{aligned}\sum_{m\geqslant 0}\sum_{z\in R_{i,j}(m)} \omega^x(z)\,\frac{s^m}{m!} &= \prod_{e\in E(i,j)} \exp\left(\mu(e)\,\frac{(is)^{j/e}}{ij/e}\right) \\ &= \exp\left(\sum_{e\in E(i,j)} \mu(e)\,\frac{(is)^{j/e}}{ij/e}\right).\end{aligned}$$ □

Now let s be an indeterminate and $\{t_j \,:\, j\geqslant 1\}$ be a countable set of indeterminates, denoted succinctly by $\underline{t}$. Consider the ring of formal power series $\mathbb{C}[[s,\underline{t}]]$. We extend Lemma 3.8 and compute the sum in Lemma 3.2 on the whole set $C_g\cap Z_x$, restricting only x to have a cycle type with all cycles of the same length.

Lemma 3.9. *Fix $i\geqslant 1$. For any integer $m\geqslant 0$, let $x = x_i(m)\in S_{im}$ be a product of m disjoint cycles of length i. Let $R_i(m) := Z_{S_{im}}(x_i(m)) \cong G_i \wr S_m$. As an element of S_{im}, write each $z\in R_i(m)$ as a product of $m_j(z)$ disjoint cycles of length j ($j\geqslant 1$). Then, in $\mathbb{C}[[s,\underline{t}]]$,*

$$\sum_{m\geqslant 0}\sum_{z\in R_i(m)} \omega^x(z)\,\frac{s^m}{m!}\prod_j t_j^{jm_j(z)} = \exp\left(\sum_j\sum_{e\in E(i,j)} \mu(e)\,\frac{(ist_j^i)^{j/e}}{ij/e}\right).$$

Proof. Following Lemma 3.8, fix integers $m_j\geqslant 0$ ($j\geqslant 1$) such that $\sum_j m_j = m$. Dividing the set of m cycles of $x_i(m)$ into subsets of corresponding sizes m_j can be done in

$$\frac{m!}{\prod_j m_j!}$$

ways. On each piece $G_i\wr S_{m_j}$ we would like to consider $R_{i,j}(m_j)$, as in Lemma 3.8; note that, by that result, $R_{i,j}(m_{j,\theta}) = \varnothing$ unless j divides im_j. By the linearity of ω^x,

$$\sum_{z\in R_i(m)} \omega^x(z) = \sum_{\substack{m_j\geqslant 0\\ \sum_j m_j = m}} \frac{m!}{\prod_j m_j!}\prod_j \sum_{z_j\in R_{i,j}(m_j)} \omega^x(z_j)$$

or, equivalently,

$$\sum_{m\geqslant 0}\sum_{z\in R_i(m)} \omega^x(z)\,\frac{s^m}{m!} = \prod_j \sum_{m_j\geqslant 0}\sum_{z_j\in R_{i,j}(m_j)} \omega^x(z_j)\,\frac{s^{m_j}}{m_j!}.$$

Assume now that $z \in R_i(m)$, as an element of S_{im}, is a product of $m_j(z)$ disjoint cycles of length j ($j \geqslant 1$). This yields a subdivision of the set of m cycles (of length i each) of $x_i(m)$ into subsets of sizes $m_j = jm_j(z)/i$, so that

$$\sum_j jm_j(z) = \sum_j im_j = im.$$

In order to keep track of the individual numbers $jm_j(z) = im_j$, let us use additional indeterminates t_j ($j \geqslant 1$). The previous formula turns into

$$\sum_{m\geqslant 0}\sum_{z\in R_i(m)} \omega^x(z)\,\frac{s^m}{m!}\prod_j t_j^{jm_j(z)} = \prod_j \sum_{m_j\geqslant 0}\sum_{z_j\in R_{i,j}(m_j)} \omega^x(z_j)\,\frac{s^{m_j}t_j^{im_j}}{m_j!}.$$

Rewriting the RHS using Lemma 3.8, with s replaced by st_j^i, yields

$$\begin{aligned}\sum_{m\geqslant 0}\sum_{z\in R_i(m)} \omega^x(z)\,\frac{s^m}{m!}\prod_j t_j^{jm_j(z)} &= \prod_j \exp\left(\sum_{e\in E(i,j)} \mu(e)\,\frac{(ist_j^i)^{j/e}}{ij/e}\right)\\ &= \exp\left(\sum_j\sum_{e\in E(i,j)} \mu(e)\,\frac{(ist_j^i)^{j/e}}{ij/e}\right),\end{aligned}$$

as claimed. □

Recalling the higher Lie characters $\psi^\lambda_{S_n}$ from Definition 2.3(c), we can now prove the main result of this section, Theorem 3.1. It extends Lemma 3.9 and computes the sum in Lemma 3.2 on the whole set $C_g \cap Z_x$, for arbitrary g and x.

Proof of Theorem 3.1. Write any $x \in S_n$ as

$$x = \prod_i x_i\,,$$

where each x_i is a product of a_i disjoint cycles of length i. Then, by Lemma 2.1,

$$Z_x \cong \bigtimes_i Z_{x_i}$$

where

$$Z_{x_i} \cong G_i \wr S_{a_i}.$$

By Lemma 3.9, with summation over $m = a_i \geqslant 0$ and $z = z_i \in G_i \wr S_{a_i}$, we have for each $i \geqslant 1$:

$$\sum_{a_i\geqslant 0}\sum_{z_i\in G_i\wr S_{a_i}} \omega^{x_i}(z_i)\,\frac{s^{a_i}}{a_i!}\prod_j t_j^{jm_j(z_i)} = \exp\left(\sum_j\sum_{e\in E(i,j)} \mu(e)\,\frac{(ist_j^i)^{j/e}}{ij/e}\right).$$

Now replace s by s_i/i and t_j^j by t_j ($\forall j \geqslant 1$). Denote

$$\begin{aligned}\Sigma_i &:= \sum_{a_i\geqslant 0}\sum_{z_i\in G_i\wr S_{a_i}} \omega^{x_i}(z_i)\,\frac{s_i^{a_i}}{a_i!\,i^{a_i}}\prod_j t_j^{m_j(z_i)}\\ &= \exp\left(\sum_j\sum_{e\in E(i,j)} \mu(e)\,\frac{s_i^{j/e}t_j^{i/e}}{ij/e}\right).\end{aligned}$$

The product of Σ_i over all i is therefore

$$\prod_i \sum_{a_i \geqslant 0} \sum_{z_i \in Z_{x_i}} \omega^{x_i}(z_i)\, \frac{s_i^{a_i}}{a_i!\, i^{a_i}} \prod_j t_j^{m_j(z_i)} = \prod_i \Sigma_i = \exp\left(\sum_i \sum_j \sum_{e \in E(i,j)} \mu(e)\, \frac{s_i^{j/e} t_j^{i/e}}{ij/e} \right).$$

If $x = \prod_i x_i \in C_\lambda$ then

$$|C_\lambda| = |C_x| = \frac{|S_n|}{|Z_x|} = \frac{|S_n|}{\prod_i a_i!\, i^{a_i}}.$$

The above equality can thus be written as

$$\sum_{n \geqslant 0} \frac{1}{|S_n|} \sum_{\lambda \vdash n} \sum_{x \in C_\lambda} \prod_i \left(\sum_{z_i \in Z_{x_i}} \omega^{x_i}(z_i) s_i^{a_i} \prod_j t_j^{m_j(z_i)} \right) = \exp\left(\sum_i \sum_j \sum_{e \in E(i,j)} \mu(e)\, \frac{s_i^{j/e} t_j^{i/e}}{ij/e} \right).$$

Denote $z := \prod_i z_i \in Z_x$, and note that

$$\sum_i m_j(z_i) = b_j(z) \qquad (\forall j \geqslant 1).$$

By the definition of ω^x, the LHS of the equality can thus be written as

$$\mathrm{LHS} = \sum_{n \geqslant 0} \frac{1}{|S_n|} \sum_{\lambda \vdash n} \sum_{x \in C_\lambda} \sum_{z \in Z_x} \omega^x(z) \underline{s}^{c(x)} \underline{t}^{c(z)}.$$

In fact, we can rewrite this as

$$\mathrm{LHS} = \sum_{n \geqslant 0} \frac{1}{|S_n|} \sum_{\lambda \vdash n} \sum_{\nu \vdash n} \sum_{x \in C_\lambda} \sum_{z \in Z_x \cap C_\nu} \omega^x(z) \underline{s}^{c(\lambda)} \underline{t}^{c(\nu)},$$

since $c(x)$ depends only on the conjugacy class of x, and may thus be written as $c(\lambda)$; and similarly for $c(z)$ and $c(\nu)$.

Now, by Lemma 3.2, if $x \in C_\lambda$ then

$$|C_\lambda| \cdot \sum_{z \in Z_x \cap C_\nu} \omega^x(z) = |C_\nu| \cdot \psi^x_{S_n}(\nu).$$

Therefore

$$\mathrm{LHS} = \sum_{n \geqslant 0} \frac{1}{|S_n|} \sum_{\lambda \vdash n} \sum_{\nu \vdash n} |C_\nu|\, \psi^\lambda_{S_n}(\nu)\, \underline{s}^{c(\lambda)} \underline{t}^{c(\nu)} = \sum_{n \geqslant 0} \sum_{\lambda \vdash n} \sum_{\nu \vdash n} \psi^\lambda_{S_n}(\nu)\, \frac{\underline{s}^{c(\lambda)} \underline{t}^{c(\nu)}}{|Z_\nu|}.$$

Regarding the RHS, note that $E(i,j)$ is the set of all common divisors of i and j, namely divisors of $\gcd(i,j)$. We can therefore write the above equality as

$$\sum_{n \geqslant 0} \sum_{\lambda, \nu \vdash n} \psi^\lambda_{S_n}(\nu)\, \frac{\underline{s}^{c(\lambda)} \underline{t}^{c(\nu)}}{|Z_\nu|} = \exp\left(\sum_{i,j \geqslant 1} \sum_{e \mid \gcd(i,j)} \mu(e)\, \frac{s_i^{j/e} t_j^{i/e}}{ij/e} \right).$$

This completes the proof. □

4. A signed version

In this section we outline the proof of the following signed analogue of Theorem 3.1.

Theorem 4.1.

$$\sum_{n\geqslant 0}\sum_{\lambda,\nu\vdash n}\operatorname{sign}(\nu)\psi^{\lambda}_{S_n}(\nu)\,\frac{\underline{s}^{c(\lambda)}\underline{t}^{c(\nu)}}{|Z_\nu|}=\exp\left(\sum_{i,j\geqslant 1}\sum_{e|\gcd(i,j)}(-1)^{i(j-1)/e}\mu(e)\,\frac{s_i^{j/e}t_j^{i/e}}{ij/e}\right),$$

where $\mu(\cdot)$ is the classical Möbius function.

The proof is very similar to the proof of Theorem 3.1, described in Section 3. We state only the slightly different main lemmas, without proof.

Here is a signed analogue of Lemma 3.6.

Lemma 4.2. *Fix $i,j\geqslant 1$, and let $x=x_i\in S_{ia_i}$ be a product of a_i cycles of length i. Let e be a common divisor of i and j, and denote $d:=i/e$ and $\ell:=j/e$. Let $\sigma\in S_{a_i}$ be a permutation which has a single cycle of length ℓ, and is the identity outside the support of this cycle. Let $R_{i,j}(\sigma,e)$ be the set of all the elements $z\in Z_x\cong G_i\wr S_{a_i}$ corresponding to the underlying permutation (cycle) σ, with suitable G_i-classes, such that, as elements of $S_{i\ell}=S_{jd}$, they are products of d disjoint cycles of length j. Denote*

$$K^{-}_{i,j}(\sigma,e):=\frac{1}{i^{\ell-1}}\sum_{z\in R_{i,j}(\sigma,e)}\operatorname{sign}(z)\omega^x(z)\,.$$

Then

$$K^{-}_{i,j}(\sigma,e)=(-1)^{i(j-1)/e}\mu(e).$$

Here is a signed analogue of Lemma 3.8.

Lemma 4.3. *Fix $i,j\geqslant 1$. For any integer $m\geqslant 0$, let $x=x_i(m)\in S_{im}$ be a product of m disjoint cycles of length i. Let $R_{i,j}(m)$ be the set of all elements $z\in Z_{S_{im}}(x_i(m))\cong G_i\wr S_m$ which, as elements of S_{im}, are products of im/j disjoint cycles of length j. (Of course, necessarily j divides im.) Then, in $\mathbb{C}[[s]]$,*

$$\sum_{m\geqslant 0}\sum_{z\in R_{i,j}(m)}\operatorname{sign}(z)\omega^x(z)\frac{s^m}{m!}=\exp\left(\sum_{e\in E(i,j)}(-1)^{i(j-1)/e}\mu(e)\,\frac{(is)^{j/e}}{ij/e}\right).$$

Here is a signed analogue of Lemma 3.9.

Lemma 4.4. *Fix $i\geqslant 1$. For any integer $m\geqslant 0$, let $x=x_i(m)\in S_{im}$ be a product of m disjoint cycles of length i. Let $R_i(m):=Z_{S_{im}(x_i(m))}\cong G_i\wr S_m$. As an element of S_{im}, write each $z\in R_i(m)$ as a product of $m_j(z)$ disjoint cycles of length j ($j\geqslant 1$). Then, in $\mathbb{C}[[s,\underline{t}]]$,*

$$\sum_{m\geqslant 0}\sum_{z\in R_i(m)}\operatorname{sign}(z)\omega^x(z)\,\frac{s^m}{m!}\prod_j t_j^{jm_j(z)}=$$
$$\exp\left(\sum_j\sum_{e\in E(i,j)}(-1)^{i(j-1)/e}\mu(e)\,\frac{(ist_j^i)^{j/e}}{ij/e}\right).$$

5. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Use Theorem 3.1 with the substitution

$$s_i:=\begin{cases}1, & \text{if } i \text{ is odd;}\\ 0, & \text{if } i \text{ is even}\end{cases}$$

to get

$$\sum_{n\geqslant 0}\sum_{\lambda\in OP(n)}\sum_{\nu\vdash n}\psi^{\lambda}_{S_n}(\nu)\,\frac{\underline{t}^{c(\nu)}}{|Z_\nu|}=\exp\left(\sum_{i\text{ odd}}\sum_{j\geqslant 1}\sum_{e\mid\gcd(i,j)}\mu(e)\,\frac{t_j^{i/e}}{ij/e}\right).$$

Replacing (in the exponent) summation over i (odd) and divisors e by summation over e (odd) and $d := i/e$ (odd), it follows that

$$\sum_{n\geqslant 0}\sum_{\lambda\in OP(n)}\sum_{\nu\vdash n}\psi^{\lambda}_{S_n}(\nu)\,\frac{\underline{t}^{c(\nu)}}{|Z_\nu|}=\exp\left(\sum_{j\geqslant 1}\sum_{\substack{e\,\mid\,j\\ e\text{ odd}}}\mu(e)\sum_{d\text{ odd}}\frac{t_j^d}{dj}\right).$$

Writing $j = 2^p\cdot(2q+1)$ for integers $p, q\geqslant 0$, it is clear that

$$\sum_{\substack{e\,\mid\,j\\ e\text{ odd}}}\mu(e)=\sum_{e\,\mid\,(2q+1)}\mu(e)=\delta_{q,0}.$$

Therefore necessarily $j = 2^p$, and using

$$\sum_{d\text{ odd}}\frac{x^d}{d}=\frac{1}{2}\left(\ln(1+x)-\ln(1-x)\right)$$

it follows that

$$\sum_{n\geqslant 0}\sum_{\lambda\in OP(n)}\sum_{\nu\vdash n}\psi^{\lambda}_{S_n}(\nu)\,\frac{\underline{t}^{c(\nu)}}{|Z_\nu|}=\exp\left(\sum_{p\geqslant 0}\sum_{d\text{ odd}}\frac{t_{2^p}^d}{2^p d}\right)=\prod_{p\geqslant 0}\left(\frac{1+t_{2^p}}{1-t_{2^p}}\right)^{1/2^{p+1}}. \qquad\square$$

Proof of Theorem 1.4. We shall deal separately with summation over even n and summation over odd n. For even n, since $EP(n)$ consists of the partitions of n with even parts only, we can use Theorem 4.1 with the substitution

$$s_i := \begin{cases}1, & \text{if } i \text{ is even;}\\ 0, & \text{if } i \text{ is odd}\end{cases}$$

to get

$$\sum_{n\geqslant 0\text{ even}}\sum_{\lambda\in EP(n)}\sum_{\nu\vdash n}\operatorname{sign}(\nu)\psi^{\lambda}_{S_n}(\nu)\,\frac{\underline{t}^{c(\nu)}}{|Z_\nu|}= \exp\left(\sum_{i\text{ even}}\sum_{j\geqslant 1}\sum_{e\mid\gcd(i,j)}(-1)^{i(j-1)/e}\mu(e)\,\frac{t_j^{i/e}}{ij/e}\right).$$

Replace double summation over i (even) and divisors e by double summation over e and $d := i/e$. Distinguishing the cases of e odd (thus d even) and e even (thus d of arbitrary parity), it follows that

$$\sum_{n\geqslant 0\text{ even}}\sum_{\lambda\in EP(n)}\sum_{\nu\vdash n}\operatorname{sign}(\nu)\psi^{\lambda}_{S_n}(\nu)\,\frac{\underline{t}^{c(\nu)}}{|Z_\nu|}=\exp\left(\sum_{j\geqslant 1}\sum_{\substack{e\,\mid\,j\\ e\text{ odd}}}\mu(e)\sum_{d\text{ even}}\frac{t_j^d}{dj}\;+\;\sum_{j\text{ even}}\sum_{\substack{e\,\mid\,j\\ e\text{ even}}}\mu(e)\sum_{d\geqslant 1}(-1)^{d(j-1)}\frac{t_j^d}{dj}\right).$$

Writing $j = 2^p \cdot (2q+1)$ for integers $p, q \geqslant 0$, it is clear that

$$\sum_{\substack{e \mid j \\ e \text{ odd}}} \mu(e) = \sum_{e \mid (2q+1)} \mu(e) = \delta_{q,0}$$

and (for j even, namely $p \geqslant 1$)

$$\sum_{\substack{e \mid j \\ e \text{ even}}} \mu(e) = \sum_{\substack{e_1 \mid 2^p \\ e_1 \neq 1}} \mu(e_1) \cdot \sum_{e_2 \mid (2q+1)} \mu(e_2) = (-1) \cdot \delta_{q,0}.$$

Therefore, in both cases, necessarily $j = 2^p$. Noting that $2^p - 1$ is odd for $p \geqslant 1$, we deduce that

$$\sum_{n \geqslant 0 \text{ even}} \sum_{\lambda \in EP(n)} \sum_{\nu \vdash n} \operatorname{sign}(\nu) \psi^{\lambda}_{S_n}(\nu) \frac{\underline{t}^{c(\nu)}}{|Z_\nu|} = \exp\left(\sum_{p \geqslant 0} \sum_{d \text{ even}} \frac{t_{2^p}^d}{2^p d} - \sum_{p,d \geqslant 1} (-1)^{d(2^p-1)} \frac{t_{2^p}^d}{2^p d} \right)$$
$$= \exp\left(\sum_{d \text{ even}} \frac{t_1^d}{d} + \sum_{p \geqslant 1} \sum_{d \text{ odd}} \frac{t_{2^p}^d}{2^p d} \right).$$

Since

$$\exp\left(\sum_{d \text{ even}} \frac{x^d}{d} \right) = \exp\left(-\frac{1}{2} \ln\left(1 - x^2\right) \right) = \left(1 - x^2\right)^{-1/2},$$

and

$$\exp\left(\sum_{d \text{ odd}} \frac{x^d}{d} \right) = \exp\left(\frac{1}{2} \ln(1+x) - \frac{1}{2} \ln(1-x) \right) = \left(\frac{1+x}{1-x} \right)^{1/2},$$

it follows that

$$\sum_{n \geqslant 0 \text{ even}} \sum_{\lambda \in EP(n)} \sum_{\nu \vdash n} \operatorname{sign}(\nu) \psi^{\lambda}_{S_n}(\nu) \frac{\underline{t}^{c(\nu)}}{|Z_\nu|} = \left(1 - t_1^2\right)^{-1/2} \cdot \prod_{p \geqslant 1} \left(\frac{1 + t_{2^p}}{1 - t_{2^p}} \right)^{1/2^{p+1}}.$$

Now consider the summation over odd n. Since $EP(n)$, in this case, consists of the partitions of n with even parts only, except for a single part of size 1, we can use Theorem 4.1 with the substitution

$$s_i := \begin{cases} 1, & \text{if } i \text{ is even;} \\ s_1, & \text{if } i = 1; \\ 0, & \text{if } i > 1 \text{ is odd} \end{cases}$$

to get that

$$\sum_{n \text{ odd}} \sum_{\lambda \in EP(n)} \sum_{\nu \vdash n} \operatorname{sign}(\nu) \psi^{\lambda}_{S_n}(\nu) \frac{s_1 \cdot \underline{t}^{c(\nu)}}{|Z_\nu|}$$

is the sum of all monomials containing a single s_1 in the expansion of

$$\exp\left(\sum_{j \geqslant 1} \sum_{e \mid \gcd(1,j)} (-1)^{(j-1)/e} \mu(e) \frac{s_1^{j/e} t_j^{1/e}}{j/e} + \sum_{i \text{ even}} \sum_{j \geqslant 1} \sum_{e \mid \gcd(i,j)} (-1)^{i(j-1)/e} \mu(e) \frac{t_j^{i/e}}{ij/e} \right).$$

In the first summation, corresponding to $i = 1$, clearly $e = 1$ and therefore (in order to get a single s_1) also $j = 1$. The only monomial containing a single s_1 in the expansion of $\exp(s_1 t_1)$ is $s_1 t_1$. Therefore, using the computation for even n,

$$\sum_{n \text{ odd}} \sum_{\lambda \in EP(n)} \sum_{\nu \vdash n} \operatorname{sign}(\nu) \psi^{\lambda}_{S_n}(\nu) \frac{t^{c(\nu)}}{|Z_\nu|} = t_1 \cdot \left(1 - t_1^2\right)^{-1/2} \cdot \prod_{p \geqslant 1} \left(\frac{1 + t_{2^p}}{1 - t_{2^p}} \right)^{1/2^{p+1}}.$$

Adding together the summations over even and odd n, and using

$$(1 + t_1) \cdot (1 - t_1^2)^{-1/2} = \left(\frac{1 + t_1}{1 - t_1} \right)^{1/2},$$

completes the proof. □

6. Root enumerators

6.1. The odd root enumerator.

Recall that, for an integer k and a nonnegative integer n, the k-th root enumerator in S_n is defined by

$$\rho_k^{S_n}(g) := |\{x \in S_n \,:\, x^k = g\}| = \sum_{\substack{x \in S_n \\ x^k = g}} 1 \qquad (\forall g \in S_n).$$

For a partition λ of n, denote $\lambda \vdash_k n$ if all the part sizes of λ divide k. The following theorem, connecting root enumerators with higher Lie characters, is due to Scharf.

Theorem 6.1 ([10]). *For any integer k and nonnegative integer n,*

$$\rho_k^{S_n} = \sum_{\lambda \vdash_k n} \psi^{\lambda}_{S_n}.$$

Denote

$$\rho_{odd}^{S_n}(g) := |\{x \in S_n \,:\, x^k = g \ \text{ for some odd } k\}| \qquad (\forall g \in S_n).$$

Corollary 6.2. *For any positive integer n,*

$$\rho_{odd}^{S_n} = \sum_{\lambda \in OP(n)} \psi^{\lambda}_{S_n}.$$

Proof. Let k be any odd multiple of $\operatorname{lcm}\{2i - 1 \,:\, 1 \leqslant i \leqslant (n+1)/2\}$. Clearly, $\rho_{odd}^{S_n} = \rho_k^{S_n}$. Scharf's theorem (Theorem 6.1) completes the proof. □

We deduce

Proposition 6.3. *For any positive integer n,*

$$\rho_{odd}^{S_{2n+1}} = \rho_{odd}^{S_{2n}} \uparrow_{S_{2n}}^{S_{2n+1}}.$$

Proof. By Corollary 6.2 together with Theorem 1.2, for any positive integer n,

$$\rho_{odd}^{S_{2n+1}} = \sum_{\lambda \in OP(2n+1)} \psi^{\lambda}_{S_{2n+1}} = \operatorname{sign} \otimes \sum_{\lambda \in EP(2n+1)} \psi^{\lambda}_{S_{2n+1}},$$

and similarly

$$\rho_{odd}^{S_{2n}} = \sum_{\lambda \in OP(2n)} \psi^{\lambda}_{S_{2n}} = \operatorname{sign} \otimes \sum_{\lambda \in EP(2n)} \psi^{\lambda}_{S_{2n}}.$$

By the definition of higher Lie characters, for any partition $\lambda = (\lambda_1, \ldots, \lambda_t) \vdash 2n$ with no parts of size 1,

$$\psi^{(\lambda_1, \ldots, \lambda_t, 1)}_{S_{2n+1}} = \psi^{(\lambda_1, \ldots, \lambda_t)}_{S_{2n}} \uparrow_{S_{2n}}^{S_{2n+1}}.$$

It follows that

$$\rho_{odd}^{S_{2n+1}} = \mathrm{sign} \otimes \sum_{\lambda \in EP(2n+1)} \psi^{\lambda}_{S_{2n+1}} = \mathrm{sign} \otimes \sum_{\lambda \in EP(2n)} \psi^{\lambda}_{S_{2n}} \uparrow_{S_{2n}}^{S_{2n+1}}$$

$$= \left(\mathrm{sign} \otimes \sum_{\lambda \in EP(2n)} \psi^{\lambda}_{S_{2n}} \right) \uparrow_{S_{2n}}^{S_{2n+1}} = \rho_{odd}^{S_{2n}} \uparrow_{S_{2n}}^{S_{2n+1}}. \qquad \square$$

Remark 6.4. Note that, by Corollary 1.5, for every n, ρ_{odd}^{2n} is not induced from ρ_{odd}^{2n-1}.

6.2. The signed odd root enumerator. Let sign : $S_n \to \{1, -1\}$ be the sign character on S_n.

Definition 6.5. *Let k be an integer and n a nonnegative integer. The* signed k-th root enumerator *in S_n is defined by*

$$\overline{\rho}_k^{S_n}(g) := \sum_{\substack{x \in S_n \\ x^k = g}} \mathrm{sign}(x) \qquad (g \in S_n).$$

Remark 6.6. The following generating function was obtained by Leaños, Moreno and Rivera-Martínez [8]. In our notation, with the obvious modifications, their result is:

$$\sum_{n \geqslant 0} \sum_{\nu \vdash n} \rho_k^{S_n}(\nu) \frac{\underline{t}^{c(\nu)}}{|Z_\nu|} = \exp \left(\sum_{j \geqslant 1} \sum_{\substack{h | k \\ \gcd(h,j)=1}} \frac{t_j^{k/h}}{jk/h} \right).$$

Formally, they use $g = k/h$ instead of our h, and sum over all $g \geqslant 1$ such that $\gcd(gj, k) = g$; see Remark 1 after the proof of [8, Proposition 2]. Their t_j is our t_j/j.

If k is a prime power, this generating function appears already in Wilf's book Generatingfunctionology, Theorem 4.8.3.

Remark 6.7. The corresponding expression for the sign character of the symmetric group was obtained by Chernoff [3] and by Glebsky, Licón and Rivera [6]. In our notation, including the class function $\overline{\rho}_k^{S_n}(g) := \sum_{x \in S_n,\, x^k = g} \mathrm{sign}(x)$ (for $g \in S_n$), namely $\overline{\rho}_k^{S_n}(\nu)$ (for a cycle type $\nu \vdash n$), their result is:

$$\sum_{n \geqslant 0} \sum_{\nu \vdash n} \overline{\rho}_k^{S_n}(\nu) \frac{\underline{t}^{c(\nu)}}{|Z_\nu|} = \exp \left(\sum_{j \geqslant 1} \sum_{\substack{h | k \\ \gcd(h,j)=1}} \frac{(-1)^{1+jk/h} t_j^{k/h}}{jk/h} \right).$$

Definition 6.8. *The* signed odd root enumerator *is defined as*

$$\overline{\rho}_{odd}^{S_n}(g) := \sum_{\substack{x \in S_n \\ x^k = g \text{ for some odd } k}} \mathrm{sign}(x). \qquad (\forall g \in S_n).$$

Definition 6.9. (Twisted higher Lie characters in S_n) *Let x be an element of cycle type λ in S_n, as in Definition* 2.3.

(a) *For each $i \geqslant 1$, let $\overline{\omega}_i$ be the linear character on $G_i \wr S_{a_i}$ which is equal to a primitive irreducible character on the cyclic group G_i, and equal to the* sign character *on the wreathing group S_{a_i}. Let*

$$\overline{\omega}^x := \bigotimes_{i \geqslant 1} \overline{\omega}_i,$$

a linear character on Z_x.

(b) *Define the corresponding* twisted higher Lie character *to be the induced character*

$$\tau^x_{S_n} := \overline{\omega}^x \uparrow^{S_n}_{Z_x}.$$

(c) *It is easy to see that $\tau^x_{S_n}$ depends only on the conjugacy class C (equivalently, the cycle type λ) of x, and can therefore be denoted $\tau^C_{S_n}$ or $\tau^\lambda_{S_n}$.*

The following theorem is due to Scharf.

Theorem 6.10 ([11, 2.4.22 Satz]). *For every partition $\lambda \vdash n$,*

$$\overline{\rho}^{S_n}_k = \sum_{\lambda \vdash_k n} \operatorname{sign}(\lambda) \tau^\lambda_{S_n},$$

where $\tau^\lambda_{S_n}$ is the twisted higher Lie character from Definition 6.9 *and* $\operatorname{sign}(\lambda)$ *is the sign of a permutation of cycle type λ.*

Note that $\operatorname{sign}(\lambda) = 1$ for any $\lambda \in OP(n)$.

Corollary 6.11. *For any positive integer n,*

$$\overline{\rho}^{S_n}_{odd} = \sum_{\lambda \in OP(n)} \tau^\lambda_{S_n}.$$

We deduce the following identity.

Theorem 6.12. *For any positive integer n,*

$$\sum_{\lambda \in OP(n)} \tau^\lambda_{S_n} = \sum_{\lambda \in EP(n)} \psi^\lambda_{S_n},$$

where $\tau^\lambda_{S_n}$ is the twisted higher Lie character from Definition 6.9 *and $\psi^\lambda_{S_n}$ is the standard higher Lie character.*

Proof. By Theorem 1.2, Corollary 6.2, Corollary 6.11, and the fact that the sign of an odd power of x is equal to the sign of x:

$$\sum_{\lambda \in EP(n)} \psi^\lambda_{S_n} = \operatorname{sign} \otimes \sum_{\lambda \in OP(n)} \psi^\lambda_{S_n} = \operatorname{sign} \otimes \rho^{S_n}_{odd} = \overline{\rho}^{S_n}_{odd} = \sum_{\lambda \in OP(n)} \tau^\lambda_{S_n}. \qquad \square$$

7. Induction from a subgroup of type B

The group of signed permutations $B_n = \mathbb{Z}_2 \wr S_n$ may be viewed as the centralizer, in S_{2n}, of a fixed-point-free involution (a permutation of cycle type $(2, \ldots, 2)$). The fact that its index $|S_{2n}|/|B_n| = (2n-1)!!$ divides the degree $(2n-1)!!^2$ of the odd roots enumerator $\rho^{S_{2n}}_{odd}$ raises a natural question: Is the character $\rho^{S_{2n}}_{odd}$ induced from a character of B_n?

In this section we answer this question affirmatively. We show that $\rho^{S_{2n}}_{odd}$ is induced from the sum of higher Lie characters of type B corresponding to conjugacy classes in B_n with positive cycles only. The number of signed permutations in B_n with these cycle types is indeed $(2n-1)!!$, see Proposition 7.1 below. Thus the degree of the induced representation is indeed $(2n-1)!!^2$, which is the number of odd roots of the identity permutation in S_{2n}, in accordance with Corollary 1.5. Theorem 7.4 shows that, furthermore, the two characters are equal.

7.1. Signed permutations with positive cycles only. Denoting $[\pm n] := \{-n, \ldots, -1, 1, \ldots, n\}$, the elements of $B_n = \mathbb{Z}_2 \wr S_n$ are *signed permutations*, namely bijections $\sigma : [\pm n] \to [\pm n]$ which satisfy $\sigma(-i) = -\sigma(i)$ for all i. They are encoded by the window (i.e. sequence of values on $1, \ldots, n$) $[\sigma(1), \ldots, \sigma(n)]$.

Conjugacy classes in $B_n = \mathbb{Z}_2 \wr S_n$ are parametrized by pairs of partitions (λ^+, λ^-) of total size n. A cycle in a signed permutation $\sigma \in B_n$ which contains a letter i as well as $-i$ is called *negative*; otherwise it is called *positive*. The conjugacy class $C_{(\lambda^+,\lambda^-)}$ consists of all the signed permutations whose positive cycle lengths are the parts of λ^+ and the negative cycle lengths, divided by 2, are the parts of λ^-.

Proposition 7.1. *The number of signed permutations in B_n with positive cycles only is* $(2n-1)!!$.

Proof. Consider the set M_{2n} of perfect matchings on $2n$ points, labeled by the letters in $[\pm n]$. Let $m_0 \in M_{2n}$ match i with $-i$, for all $1 \leqslant i \leqslant n$. The superposition of m_0 and any perfect matching $m \in M_{2n}$ is a union of vertex-disjoint cycles of even lengths, with edges alternating between m_0 and m. Choose an orientation for each of the cycles; for concreteness, pick in each cycle the largest positive letter, say i_0, and orient the cycle so that the edge of m_0 connecting i_0 and $-i_0$ is oriented away from i_0. Define a function $\sigma_m : [\pm n] \to [\pm n]$ as follows: locate each element $v \in [\pm n]$ in a cycle, and follow the path of two consecutive edges starting at v, an edge from m followed by an edge from m_0 (or vice versa), according to the orientation of the cycle, to get $\sigma_m(v) \in [\pm n]$. The reader may verify that $\sigma_m \in B_n$, has only positive cycles, and $m \mapsto \sigma_m$ defines a bijection from M_{2n} onto the set of all permutations in B_n with positive cycles. The fact that $|M_{2n}| = (2n-1)!!$ completes the proof. ☐

7.2. Higher Lie characters of type B. Recall from [1] the definition of higher Lie characters of type B, parametrized by conjugacy classes in $B_n \cong \mathbb{Z}_2 \wr S_n$.

Lemma 7.2. (Centralizers in B_n)

(a) *Let $x \in B_n$ be an element of cycle type $\underline{\lambda} = (\lambda^+, \lambda^-)$ where, for each $i \geqslant 1$ and $\varepsilon \in \mathbb{Z}_2$, the partition λ^ε has $a_{i,\varepsilon}$ parts of size i. Write*

$$x = \prod_{i \geqslant 1} x_{i,+} x_{i,-} \,,$$

where $x_{i,\varepsilon}$ has $a_{i,\varepsilon}$ cycles of length i and $\mathbb{Z}_2$-class ε ($i \geqslant 1$, $\varepsilon \in \mathbb{Z}_2$); of course, only finitely many factors here are nontrivial. Then the centralizer $Z_x = Z_{B_n}(x)$ satisfies

$$Z_{B_n}(x) = \bigtimes_{i \geqslant 1} \left(Z_{B_{ia_{i,+}}}(x_{i,+}) \times Z_{B_{ia_{i,-}}}(x_{i,-}) \right),$$

and is therefore isomorphic to the direct product

$$\bigtimes_{i \geqslant 1} \left(G_{i,+} \wr S_{a_{i,+}} \times G_{i,-} \wr S_{a_{i,-}} \right),$$

where $G_{i,\varepsilon}$ is the centralizer in B_i of a cycle of length i and $\mathbb{Z}_2$-class ε. By convention, $G \wr S_0$ is the trivial group while $G \wr S_1 \cong G$.

(b) *Let $x_{i,+} \in B_i$ consist of a single cycle, of length i and class $+1 \in \mathbb{Z}_2$. Then the centralizer $Z_{x_{i,+}} = Z_{B_i}(x_{i,+}) = G_{i,+}$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_i$, where the generator of $\mathbb{Z}_i$ is $x_{i,+}$ and the generator of $\mathbb{Z}_2$ is the central (longest) element $w_0 = [-1, \ldots, -i] \in B_i$.*

(c) *Let $x_{i,-} \in B_i$ consist of a single cycle, of length i and class $-1 \in \mathbb{Z}_2$. Then the centralizer $Z_{x_{i,-}} = Z_{B_i}(x_{i,-}) = G_{i,-}$ is isomorphic to the cyclic group $\mathbb{Z}_{2i}$, with generator $x_{i,-}$; note that $x_{i,-}^i = w_0$.*

Definition 7.3. (Higher Lie characters in B_n) *Let x be an element of cycle type $\underline{\lambda} = (\lambda^+, \lambda^-)$ in B_n, as in Lemma 7.2(a).*

(a) *For each $i \geqslant 1$ and $\varepsilon \in \mathbb{Z}_2$, let $\omega_{i,\varepsilon}$ be the linear character on $G_{i,\varepsilon} \wr S_{a_{i,\varepsilon}}$ defined as follows: If $\varepsilon = +1$ then, by Lemma 7.2(b), $G_{i,+} \cong \mathbb{Z}_2 \times \mathbb{Z}_i$. Let $\omega_{i,+}$ be trivial on $\mathbb{Z}_2$, equal to a primitive irreducible character on the cyclic group $\mathbb{Z}_i$, and trivial on the wreathing group $S_{a_{i,+}}$. If $\varepsilon = -1$ then, by Lemma 7.2(c), $G_{i,-} \cong \mathbb{Z}_{2i}$. Let $\omega_{i,-}$ be equal to a primitive irreducible character on the cyclic group $\mathbb{Z}_{2i}$, and trivial on the wreathing group $S_{a_{i,-}}$. Let*

$$\omega^x := \bigotimes_{i \geqslant 1} (\omega_{i,+} \otimes \omega_{i,-}),$$

a linear character on Z_x.

(b) *Define the corresponding* higher Lie character *to be the induced character*

$$\psi^x_{B_n} := \omega^x \uparrow^{B_n}_{Z_x} .$$

(c) *It is easy to see that $\psi^x_{B_n}$ depends only on the conjugacy class C (equivalently, the cycle type $\underline{\lambda}$) of x, and can therefore be denoted $\psi^C_{B_n}$ or $\psi^{\underline{\lambda}}_{B_n}$.*

7.3. The odd root enumerator as an induced character. The centralizer of any permutation of cycle type (2^n) (i.e. a fixed-point-free involution) in S_{2n} is isomorphic to B_n. Similarly, the centralizer of any permutation of cycle type $(2^n 1)$ in S_{2n+1} is isomorphic to B_n. Denote

$$\eta_{B_n} := \sum_{\lambda \vdash n} \psi^{(\lambda, \varnothing)}_{B_n},$$

the sum of type B higher Lie characters corresponding to conjugacy classes with positive cycles only.

Theorem 7.4. *For any $n \geqslant 0$,*

$$\rho^{S_{2n}}_{odd} = \eta_{B_n} \uparrow^{S_{2n}}_{B_n}$$

and

$$\rho^{S_{2n+1}}_{odd} = \eta_{B_n} \uparrow^{S_{2n+1}}_{B_n}.$$

For the proof of Theorem 7.4, we first need the following general lemma, which is of independent interest.

Lemma 7.5. *Let G be a finite group, H a subgroup of G, χ a character of H, and $\xi = \chi\uparrow^G_H$ the corresponding induced character of G. Let $\{C_G(\ell) : \ell \in L\}$ and $\{C_H(m) : m \in M\}$ be the sets of conjugacy classes in G and H, respectively, where L and M are appropriate indexing sets. Let $|Z_G(\ell)| = |G|/|C_G(\ell)|$ and $|Z_H(m)| = |H|/|C_H(m)|$ be the corresponding sizes of centralizer subgroups. Let $\alpha : M \to L$ be the function defined by*

$$C_H(m) \subseteq C_G(\alpha(m)) \qquad (\forall m \in M).$$

Finally, let $\{q_\ell : \ell \in L\}$ be a set of indeterminates. Then

$$\sum_{\ell \in L} \frac{\xi(\ell) \cdot q_\ell}{|Z_G(\ell)|} = \sum_{m \in M} \frac{\chi(m) \cdot q_{\alpha(m)}}{|Z_H(m)|}.$$

Proof. Define a function $\chi^0 : G \to \mathbb{C}$ by

$$\chi^0(g) := \begin{cases} \chi(g), & \text{if } g \in H; \\ 0, & \text{if } g \in G \setminus H. \end{cases}$$

By [7, (5.1)], an explicit formula for the induced character $\xi = \chi\!\uparrow_H^G$ is

$$\xi(g) = \sum_{G=\cup_a aH} \chi^0(a^{-1}ga) = \frac{1}{|H|}\sum_{a\in G}\chi^0(a^{-1}ga) \qquad (\forall g\in G).$$

The mapping $f : G \to C_G(g)$ defined by $f(a) := a^{-1}ga$ ($\forall a \in G$) is surjective, and satisfies: $f(a_1) = f(a_2)$ if and only if $a_1a_2^{-1} \in Z_G(g)$. Hence

$$\xi(g) = \frac{|Z_G(g)|}{|H|}\sum_{z\in C_G(g)}\chi^0(z) = \frac{|Z_G(g)|}{|H|}\sum_{h\in C_G(g)\cap H}\chi(h) \qquad (\forall g\in G).$$

Assume that the G-conjugacy class of g is $C_G(\ell)$. The intersection $C_G(\ell)\cap H$ is a disjoint union of the H-conjugacy classes $C_H(m)$ for which $\alpha(m) = \ell$. Summing over all $\ell \in L$, it follows that

$$\begin{aligned}\sum_{\ell\in L}\frac{\xi(\ell)\cdot q_\ell}{|Z_G(\ell)|} = \sum_{\ell\in L}\frac{q_\ell}{|H|}\sum_{h\in C_G(\ell)\cap H}\chi(h) &= \sum_{m\in M}\frac{q_{\alpha(m)}}{|H|}\sum_{h\in C_H(m)}\chi(h)\\ &= \sum_{m\in M}\frac{q_{\alpha(m)}\cdot|C_H(m)|}{|H|}\chi(m),\end{aligned}$$

which is equivalent to the required formula. □

The proof of Theorem 7.4 is also based on a generating function for higher Lie characters of the hyperoctahedral group B_n, proved in [1].

Let $\underline{\lambda} = (\lambda^+, \lambda^-)$ and $\underline{\nu} = (\nu^+, \nu^-)$ be two bipartitions of n. For each integer $i \geqslant 1$ and sign $\varepsilon \in \mathbb{Z}_2 = \{+1, -1\}$, let $a_{i,\varepsilon}$ be the number of parts of size i in the partition λ^ε. Similarly, for each integer $j \geqslant 1$ and sign $\theta \in \mathbb{Z}_2 = \{+1, -1\}$, let $b_{j,\theta}$ be the number of parts of size j in the partition ν^θ. Thus

$$\sum_{i,\varepsilon} ia_{i,\varepsilon} = \sum_{j,\theta} jb_{j,\theta} = n.$$

Let $\underline{s} = (s_{i,\varepsilon})_{i\geqslant 1,\varepsilon\in\mathbb{Z}_2}$ and $\underline{t} = (t_{j,\theta})_{j\geqslant 1,\theta\in\mathbb{Z}_2}$ be two countable sets of indeterminates. Denote $\underline{s}^{c(\underline{\lambda})} := \prod_{i,\varepsilon} s_{i,\varepsilon}^{a_{i,\varepsilon}}$ and $\underline{t}^{c(\underline{\nu})} := \prod_{j,\theta} t_{j,\theta}^{b_{j,\theta}}$. Consider the ring $\mathbb{C}[[\underline{s},\underline{t}]]$ of formal power series in these indeterminates.

Remark 7.6. The notation used in [1] is slightly different from the notation used in the current paper: wherever we use $s_{i,\varepsilon}$ and $t_{j,\theta}$, [1] uses $s_{i,\varepsilon}^i$ and $t_{j,\theta}^j$, respectively. The following result is stated in our current notation.

Theorem 7.7 ([1, Theorem 4.3]).

$$\sum_{n\geqslant 0}\sum_{\underline{\lambda}\vdash n}\sum_{\underline{\nu}\vdash n}\psi_{B_n}^{\underline{\lambda}}(\underline{\nu})\,\frac{\underline{s}^{c(\underline{\lambda})}\underline{t}^{c(\underline{\nu})}}{|Z_{B_n}(\underline{\nu})|} = \exp\left(\sum_{i,\varepsilon}\sum_{j,\theta}\sum_{e\mid\gcd(i,j)} K_{\varepsilon,\theta}(e)\,\frac{s_{i,\varepsilon}^{j/e}t_{j,\theta}^{i/e}}{2ij/e}\right).$$

Here, for $\varepsilon, \theta \in \mathbb{Z}_2 = \{+1, -1\}$ and $e \geqslant 1$,

$$K_{\varepsilon,\theta}(e) := \varepsilon\theta\cdot\mu(2e) + \frac{(1+\varepsilon)(1+\theta)}{2}\cdot\mu(e),$$

where $\mu(\cdot)$ is the classical Möbius function.

Proof of Theorem 7.4. Following the definition of η_{B_n} set, in Theorem 7.7,

$$s_{i,\varepsilon} := \begin{cases}1, & \text{if } \varepsilon = +1;\\ 0, & \text{otherwise}\end{cases}$$

for all $i \geqslant 1$. Then

$$\sum_{n\geqslant 0}\sum_{\underline{\nu}\vdash n}\eta_{B_n}(\underline{\nu})\,\frac{\underline{t}^{c(\underline{\nu})}}{|Z_{B_n}(\underline{\nu})|}=\exp\left(\sum_i\sum_{j,\theta}\sum_{e|\,\gcd(i,j)}K_{+,\theta}(e)\,\frac{t_{j,\theta}^{i/e}}{2ij/e}\right).$$

Inducing from B_n to S_{2n} (or to S_{2n+1}), a cycle of length j and sign $+1$ in B_n is a product of two disjoint cycles of length j in S_n, while a cycle of length j and sign -1 in B_n is a cycle of length $2j$ in S_n. We therefore set

$$\alpha(t_{j,+})=t_j^2 \qquad (\forall j\geqslant 1)$$

and

$$\alpha(t_{j,-})=t_{2j} \qquad (\forall j\geqslant 1).$$

Consider first induction to S_{2n}. By Lemma 7.5 for $G=S_{2n}$, $H=B_n$, $\chi=\sum_{\underline{\nu}\vdash n}\eta_{B_n}(\underline{\nu})$ and $\xi_{2n}=\chi\uparrow_{B_n}^{S_{2n}}$,

$$\sum_{n\geqslant 0}\sum_{\nu\vdash 2n}\xi_{2n}(\nu)\,\frac{\underline{t}^{c(\nu)}}{|Z_{S_{2n}}(\nu)|}=\exp\left(\sum_i\sum_j\sum_{e|\,\gcd(i,j)}\frac{K_{+,+}(e)\cdot t_j^{2i/e}+K_{+,-}(e)\cdot t_{2j}^{i/e}}{2ij/e}\right).$$

Using

$$K_{+,+}(e)=\mu(2e)+2\cdot\mu(e)$$

and

$$K_{+,-}(e)=-\mu(2e),$$

and denoting $d:=i/e$, we get

$$\sum_{n\geqslant 0}\sum_{\nu\vdash 2n}\xi_{2n}(\nu)\,\frac{\underline{t}^{c(\nu)}}{|Z_{S_{2n}}(\nu)|}=\exp\left(\sum_{d\geqslant 1}\sum_{j\geqslant 1}\sum_{e|j}\frac{(\mu(2e)+2\cdot\mu(e))\cdot t_j^{2d}-\mu(2e)\cdot t_{2j}^{d}}{2dj}\right).$$

Since

$$\sum_{e|j}\mu(2e)=-\sum_{\substack{e|j\\ e\text{ odd}}}\mu(e)=\begin{cases}-1, & \text{if } j=2^p \text{ for } p\geqslant 0;\\ 0, & \text{otherwise}\end{cases}$$

and

$$\sum_{e|j}(\mu(2e)+2\cdot\mu(e))=\begin{cases}1, & \text{if } j=1;\\ -1, & \text{if } j=2^p \text{ for } p\geqslant 1;\\ 0, & \text{otherwise,}\end{cases}$$

it follows that

$$\begin{aligned}\sum_{n\geqslant 0}\sum_{\nu\vdash 2n}\xi_{2n}(\nu)\,\frac{\underline{t}^{c(\nu)}}{|Z_{S_{2n}}(\nu)|}&=\exp\left(\sum_{d\geqslant 1}\frac{t_1^{2d}+t_2^d}{2d}+\sum_{d\geqslant 1}\sum_{p\geqslant 1}\frac{-t_{2^p}^{2d}+t_{2^{p+1}}^d}{2^{p+1}d}\right)\\ &=\exp\left(\sum_{d\geqslant 1}\frac{t_1^{2d}}{2d}+\sum_{d\geqslant 1}\sum_{p\geqslant 1}\frac{-t_{2^p}^{2d}}{2^{p+1}d}+\sum_{d\geqslant 1}\frac{t_2^d}{2d}+\sum_{d\geqslant 1}\sum_{p\geqslant 1}\frac{t_{2^{p+1}}^d}{2^{p+1}d}\right)\\ &=\exp\left(\sum_{d\geqslant 1}\frac{t_1^{2d}}{2d}+\sum_{d\geqslant 1}\sum_{p\geqslant 1}\frac{-t_{2^p}^{2d}}{2^{p+1}d}+\sum_{d\geqslant 1}\sum_{p\geqslant 1}\frac{t_{2^p}^d}{2^p d}\right)\\ &=(1-t_1^2)^{-1/2}\cdot\prod_{p\geqslant 1}\left(1-t_{2^p}^2\right)^{1/2^{p+1}}\cdot\prod_{p\geqslant 1}\left(1-t_{2^p}\right)^{-1/2^p}\\ &=(1-t_1^2)^{-1/2}\cdot\prod_{p\geqslant 1}\left(\frac{1+t_{2^p}}{1-t_{2^p}}\right)^{1/2^{p+1}}.\end{aligned}$$

A similar application of Lemma 7.5 with $G = S_{2n+1}$, $H = B_n$, $\chi = \sum_{\underline{\nu} \vdash n} \eta_{B_n}(\underline{\nu})$ and $\xi_{2n+1} = \chi \uparrow_{B_n}^{S_{2n+1}}$ yields

$$\sum_{n \geqslant 0} \sum_{\nu \vdash 2n+1} \xi_{2n+1}(\nu) \frac{\underline{t}^{c(\nu)}}{|Z_{S_{2n+1}}(\nu)|} = t_1 \cdot (1 - t_1^2)^{-1/2} \cdot \prod_{p \geqslant 1} \left(\frac{1 + t_{2^p}}{1 - t_{2^p}} \right)^{1/2^{p+1}},$$

so altogether we have

$$\sum_{n \geqslant 0} \sum_{\nu \vdash n} \xi_n(\nu) \frac{\underline{t}^{c(\nu)}}{|Z_{S_n}(\nu)|} = \prod_{p \geqslant 0} \left(\frac{1 + t_{2^p}}{1 - t_{2^p}} \right)^{1/2^{p+1}}.$$

Comparing this formula to the one in Theorem 1.3 and to Corollary 6.2 shows that, indeed,

$$\xi_n = \sum_{\lambda \in OP(n)} \psi_{S_n}^{\lambda} = \rho_{odd}^{S_n}$$

for all $n \geqslant 0$, as claimed. $\square$

Acknowledgements. The authors thank Doron Zeilberger for stimulating discussions, and Stoyan Dimitrov for useful comments and relevant references. Discussions with Sergi Elizalde led him to find an alternative, bijective proof of Theorem 1.1; see [4].

References

[1] Ron M. Adin, Pál Hegedűs, and Yuval Roichman, *Higher Lie characters and root enumeration in classical Weyl groups*, J. Algebra **664** (2025), 26–72.

[2] Miklós Bóna, *A walk through combinatorics*, 5th edition ed., World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2024.

[3] W. W. Chernoff, *Solutions to $x^r = \alpha$ in the alternating group*, Ars Combin. **29** (1990), 226–227, Twelfth British Combinatorial Conference (Norwich, 1989).

[4] Sergi Elizalde, *A bijection for descent sets of permutations with only even and only odd cycles*, European J. Combin. **132** (2026), article no. 104280 (17 pages).

[5] Ira M. Gessel and Christophe Reutenauer, *Counting permutations with given cycle structure and descent set*, J. Combin. Theory Ser. A **64** (1993), no. 2, 189–215.

[6] Lev Glebsky, Melany Licón, and Luis Manuel Rivera, *On the number of even roots of permutations*, Australas. J. Combin. **86** (2023), 308–319.

[7] I. Martin Isaacs, *Character theory of finite groups*, Dover Publications, Inc., New York, 1994.

[8] Jesús Leaños, Rutilo Moreno, and Luis Manuel Rivera-Martínez, *On the number of mth roots of permutations*, Australas. J. Combin. **52** (2012), 41–54.

[9] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, Published electronically at `http://oeis.org`.

[10] Thomas Scharf, *Die Wurzelanzahlfunktion in symmetrischen Gruppen*, J. Algebra **139** (1991), no. 2, 446–457.

[11] Thomas Scharf, *Über Wurzelanzahlfunktionen voller monomialer Gruppen (German) [On root number functions of full monomial groups]*, Bayreuth. Math. Schr. **38** (1991), 99–207, Dissertation, Universität Bayreuth, Bayreuth, 1991.

[12] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.

Ron M. Adin, Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900 (Israel)
E-mail : `radin@math.biu.ac.il`

Pál Hegedűs, Department of Algebra and Geometry, Institute of Mathematics, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest (Hungary)
E-mail : `hegpal@math.bme.hu`

Yuval Roichman, Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900 (Israel)
E-mail : `yuvalr@math.biu.ac.il`